UNIVERSITY OF GOTHENBURG

An argument principle for generalised point residues

Master's thesis in Mathematics

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An argument principle for generalised point residues

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Abstract

We define a point residue for any Artinian $\mathscr{O}$-modules via Hermitian free resolutions, generalising the one dimensional residue and the classical multivariate Grothendieck point residue. We consider various definitions of multiplicity for such modules and prove a residue formula connecting the algebraic multiplicity to our residues. Our result can be seen as a generalisation both of the argument principle and a corresponding result for Grothendieck residues. It is also a special case of a recent result for Andersson-Wulcan currents proven by Lärkäng and Wulcan.

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1

Introduction

Consider a function f holomorphic in some neighbourhood of the origin in $\mathbb{C}$. Suppose also that $f^{-1}(0) = \{0\}$. From analytic function theory we know that there is a decomposition $f(z) = z^m g(z)$ where g is holomorphic and $g(0) \neq 0$. The exponent m is called the *multiplicity* of the vanishing point at the origin. Another usual way of defining this number is by using derivatives: m is the least positive integer k such that $f^{(k)}(0) \neq 0$. The *argument principle* connects this number (via the *residue theorem*) to a certain integral

$$m = \frac{1}{2\pi i} \int\limits_{|f|=\varepsilon} \frac{f'(z)dz}{f(z)}.$$

A generalisation of this states that if h is another holomorphic function defined near the origin, then

$$m \cdot h(0) = \frac{1}{2\pi i} \int\limits_{|f|=\varepsilon} h(z)\frac{f'(z)dz}{f(z)}. \tag{1.1}$$

These types of integrals are sometimes called *residue integrals* and indeed one may confirm that m and $m \cdot h(0)$ are the *residues* of the corresponding integrands. A consequence of the argument principle is *Rouché's theorem*, which states that a small enough holomorphic perturbation $\varepsilon(z)$ of f near the origin, may move around the vanishing points of $f + \varepsilon$ but the total number of zeros near the origin remain the same. We will see in the next chapter that this kind of stability can be used to "separate" a multiple root to simple vanishing points counting the multiplicity.
The starting point for this thesis considers instead a tuple $f = (f_1, \ldots, f_N) : U \to \mathbb{C}^N$ of holomorphic functions where $U \subseteq \mathbb{C}^n$ is a neighbourhood of the origin. We assume that $f^{-1}(0) = \{0\}$, so that the origin is an isolated vanishing point of f. We want to obtain a residue formula generalising the argument principle (1.1). There is a "canonical" way of achieving this in the case when the tuple has length $N = n$, given by the so-called *Grothendieck residue* of f. The goal of the thesis is to

recall this construction and cast it in a way that points to a generalisation.

Outline of thesis

In Chapter 2, we begin by recalling various ways of defining the multiplicity for a tuple f. We also introduce the Grothendieck residue and recall how it is connected to the multiplicity in a formula generalising (1.1). In the third chapter we exploit some algebro-geometric properties of the tuple f and use these to reformulate the residue in terms of the *Koszul complex*. Taking this perspective, we present a proof of the *transformation law*, which governs the relation between certain residues. We finish by generalising the approach with Koszul complexes to define a more general residue in the last chapter and prove a residue formula further generalising the argument principle.
The material in Chapter 2 & 3 is mostly well-known and classical, but we have tried to write out some details (mainly in Chapter 2) which are not so explicit in the literature. The main results of the thesis, found in the last chapter, are also mostly well-known and concern special cases of some recent results from the papers [LW18b] & [LW18a] reformulated in more elementary terms. Another difference from these papers is that we use the Koszul complex and a formula proven in [Lär19a] to define our residue "indirectly", similar to the construction in [LM18]. The proofs in Chapter 4 are mostly new, though they incorporate methods and results from the papers [LW18b] & [LW18a] and other sources.

1.1 Preliminaries

We assume the reader is familiar with some complex analysis and very basic generalisations of this theory to several complex variables, corresponding to e.g. [Leb19, Ch. 1.1–3]. We will also be using the multi-index convention given in e.g. [Leb19, p. 12].
The reader is assumed to have some familiarity with commutative algebra and very basic algebraic geometry, e.g. the undergraduate texts by Reid [Rei95] & [Rei88]. Throughout, we will consider the *local ring* $\mathscr{O} = {}_n\mathscr{O}_p$ *of germs of holomorphic functions at a point* $p \in \mathbb{C}^n$. The elements of this ring are equivalence classes of holomorphic functions f defined near p, where two functions are equivalent if they agree on some small neighbourhood of the point. The operations are then given by the equivalence classes of the pointwise operations on representatives. The unique maximal ideal of $\mathscr{O}$ is the ideal generated by the coordinates $\mathfrak{m} := (z_1 - p_1, \dots, z_n - p_n)$ and consists of all germs of functions vanishing at the point. Some of the nice properties of $\mathscr{O}$ are a consequence of it being a local

Noetherian UFD (see e.g. [Leb19, Ch. 6.4]).
In analogy with algebraic geometry, the ideals in $\mathcal{O}$ cut out *germs of analytic sets* at the point, i.e. sets that near the point are defined by the common vanishing of a set of generators of the ideal. As usual the set cut out by an ideal $I \subseteq \mathcal{O}$ is denoted $(\mathcal{V}(I), p)$, where the notation is to remind us that it is the germ of a set at p. The objects are local, but essentially all basic definitions coincide with those in algebraic geometry (see e.g. [Leb19, Ch. 6.5] for some of these) so we do not present them further at this point.
We also assume the reader is somewhat familiar with the basic notions of differential calculus on manifolds, e.g. differential forms and Stokes' theorem. For a reference, see e.g. [Tu11]. A complex manifold is a manifold that locally is biholomorphic to open sets $U \subseteq \mathbb{C}^n$. We recall the basic *sheaves* one encounters in differential calculus on a complex manifold X, namely

i) the sheaf $\mathcal{O}_X$ of holomorphic functions on X,
ii) and the sheaves $\mathcal{E}_X$ and $\mathcal{E}_X^{\cdot}$ of smooth functions and smooth forms on X.

The *sections* of these sheaves will be precisely the holomorphic functions, and the smooth functions and forms, respectively. In general, a sheaf $\mathcal{F}$ assigns (in a "nice" way) to each open set $U \subseteq X$ a set $\mathcal{F}(U)$ of *sections*, which for us will be a family of holomorphic or smooth mappings or smooth differential forms on U. For a proper definition and some basic properties one may consult [Wel08], but the sheaves we mentioned above are the only ones we will work with. To each sheaf $\mathcal{F}$ and each $x \in X$, we define the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x to be germs of the sections of $\mathcal{F}$ at x. So e.g. $\mathcal{O}_{\mathbb{C}^n,0}$ is the local ring $\mathcal{O}$ we have already encountered.
We define a *trivial holomorphic vector bundle* on X to be a manifold of the form $X \times \mathbb{C}^k$, together with the projection $\pi : X \times \mathbb{C}^k \to X$ (see [Wel08] for a general definition). We consider only trivial holomorphic bundles, or bundles (E, π) which are trivial after a change of coordinates. A section of a holomorphic bundle E is a holomorphic mapping $s : X \to E$ such that $\pi s = id_X$. For a trivial bundle $X \times \mathbb{C}^k$, the sections are simply vector valued holomorphic mappings $s : X \to X \times \mathbb{C}^k$ and can hence be identified with holomorphic mappings $X \to \mathbb{C}^k$.
As one might suspect, there is a connection between certain *analytic sheaves* and holomorphic vector bundles. The specific ones we will encounter will be sheaves of the particularly simple form $\mathcal{O}_X^{\oplus k}$. These correspond to trivial vector bundles $X \times \mathbb{C}^k$. Indeed, as already mentioned, for $E = X \times \mathbb{C}^k$, the sheaf $\mathcal{O}_X(E)$ of E-valued holomorphic sections of X is precisely the sheaf $\mathcal{O}_X^{\oplus k}$ of holomorphic mappings with values in $\mathbb{C}^k$. We introduce this here because we will sometimes switch between viewing the object $\mathcal{O}_X^{\oplus k}$ as simply a sheaf and as the sheaf $\mathcal{O}_X(E)$ of E-valued sections for a trivial bundle $E = X \times \mathbb{C}^k$. We will also consider the sheaves $\mathcal{E}_X(E)$ of smooth sections of E, i.e. smooth vector valued mappings in

analogy with $\mathscr{O}_X(E)$, and $\mathscr{E}^\cdot_X(E) = \mathscr{E}^\cdot_X \otimes_{\mathscr{E}_X} \mathscr{E}_X(E)$ of smooth E-valued differential forms.

A bundle morphism from $X \times \mathbb{C}^k$ to $X \times \mathbb{C}^\ell$ is simply a choice of linear maps $\mathbb{C}^k \to \mathbb{C}^\ell$ at each point x, varying holomorphically with x. Similarly, we specialise to sheaf morphisms $\mathscr{O}_X^{\oplus k} \to \mathscr{O}_X^{\oplus \ell}$ given by linear maps $\mathbb{C}^k \to \mathbb{C}^\ell$ at each stalk, varying holomorphically with the point. We say that a sequence of analytic sheaves and morphisms

$$\cdots \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \longrightarrow \cdot \tag{1.2}$$

is *exact at* $\mathscr{G}$ if the induced sequence on stalks

$$\cdots \longrightarrow \mathscr{F}_x \xrightarrow{f_x} \mathscr{G}_x \xrightarrow{g_x} \mathscr{H}_x \longrightarrow \cdots \tag{1.3}$$

is exact at $\mathscr{G}_x$ for any $x \in X$, that is if $\operatorname{im} f_x = \ker g_x$. A sequence of bundles and bundle morphisms

$$\cdots \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow \cdots \tag{1.4}$$

is said to be exact if in addition to exactness $\operatorname{im} f_x = \ker g_x$ at each fibre all the fibres are of the same dimension, i.e. f has constant rank. We can therefore find examples of exact sequences of free analytic sheaves which are not exact when viewed as holomorphic vector bundles.

We finish this section by saying some things about the metrics and connections on our bundles. Suppose $E = X \times \mathbb{C}^k$ is a trivial bundle on X. A (global) *smooth frame* for E is a choice of smooth sections $\xi_1, \dots, \xi_k : X \to \mathbb{C}^k$ such that $\xi_1(x), \dots, \xi_k(x)$ is a basis for $\mathbb{C}^k$ at all points $x \in X$. We consider the frame $e_1, \dots, e_k$ given by the standard basis $e_1(x) = e_1, \dots, e_k(x) = e_k$ for $\mathbb{C}^k$. We can define a *metric* on E via this frame by taking the standard inner product on $\mathbb{C}^k$. A trivial connection D on E with respect to the frame $\{e_1, \dots, e_k\}$ is an extension of the exterior derivative $d : \mathscr{E}^\cdot_X \to \mathscr{E}^\cdot_X$ to a linear map on E-valued forms $\mathscr{E}^\cdot_X(E)$ defined by $Df = df$ for smooth functions $f \in \mathscr{E}_X$ and $De_i = 0$ on the smooth frame for E. Note that a trivial connection gives a way to extend also the holomorphic and antiholomorphic differentials ∂ and $\bar{\partial}$, since $d = \partial + \bar{\partial}$. We also note that even though we may start with a trivial bundle with a trivial connection, there may exist nontrivial *subbundles* (i.e. a bundle which is locally a linear subspace) which are not trivial and where the connection is more complicated.

2

Intersection multiplicities

Given a holomorphic mapping $f : U \to \mathbb{C}^n$ defined on a neighbourhood $U \subseteq \mathbb{C}^n$ of the origin with $f^{-1}(0) = \{0\}$, we want to define the *multiplicity* — or *order* — of the vanishing point. There are several approaches to this that generalises the one-dimensional definition of multiplicity. In this chapter we recall the definition of the *Grothendieck residue* of a holomorphic function with respect to f and see how it is related to different notions of multiplicity via a generalised argument principle.

At present we only consider $\mathbb{C}^n$ but all the constructions are local and hold on a general complex manifold. The type of maps that will be studied in either case are the *equidimensional finite holomorphic mappings*.

Definition 2.0.1. A map $f : U \to V$, where U, V are manifolds of the same dimension, is called *equidimensional*. Sometimes we might say *n-equidimensional* to declare the dimension without needing to specify the domains U, V.

Suppose f is a holomorphic mapping between complex manifolds and assume f is a proper mapping, i.e. $f^{-1}(K)$ is compact for any compact set $K \subseteq V$. If the fibre $f^{-1}(y)$ is a finite set for $y \in V$ we call f a *finite holomorphic mapping*.

2.1 The index of a mapping

We begin by recalling a topological perspective on the multiplicity of a map, based on the so-called *degree* or *winding number*. This will then be used to motivate the definition of the *residue*, which is the main object studied in the thesis.

Recall that for a sphere map $\varphi : S^m \to S^m$ the *topological degree* $\deg \varphi$ of φ is the unique integer d such that the induced homomorphism in homology

$$\varphi_* : H_m(S^m, \mathbb{Z}) \to H_m(S^m, \mathbb{Z}) \cong \mathbb{Z}$$

maps a generator $[S^m]$ to $\varphi_*[S^m] = d[S^m]$. Some basic results about homology, the degree and their applications may be found in [Hat02, Ch. 2]. The degree formalises the intuitive notion of counting how many times φ “wraps” the sphere around itself. A key property that we sometimes use without comment, is the stability of the degree under homotopy. Recall that a homotopy between continuous maps $\varphi, \psi : X \to Y$ is a continuous map $h : X \times [0,1] \to Y$ such that $h(x,0) = \varphi(x)$ and $h(x,1) = \psi(x)$. We sometimes view this as a continuous family of continuous maps $h_t : X \to Y$ for $t \in [0,1]$, the condition is then that $h_0 = \varphi$ and $h_1 = \psi$.

Lemma 2.1.1. *If $\varphi, \psi : S^m \to S^m$ are homotopy equivalent, then* $\deg \varphi = \deg \psi$.

We also write out another important consequence of the functoriality of homology.

Lemma 2.1.2. *If $\varphi, \psi : S^m \to S^m$, then*

$$\deg(\varphi\psi) = (\deg \varphi)(\deg \psi). \tag{2.1}$$

Definition 2.1.3. Suppose $f \in \mathscr{C}^\infty(U, \mathbb{R}^n)$, where $p \in U \subseteq \mathbb{R}^n$ is an isolated zero of f. Let S be a sphere centered at p such that f has no other zeros inside or on S. After identifying the spheres $S \cong S^n$ we define the *index of f at p* to be the integer

$$\operatorname{ind}_p f := \deg \left.\frac{f}{\|f\|}\right|_S .$$

Remark 2.1.4. If f has no zeros inside the sphere S then $\deg \left.\frac{f}{\|f\|}\right|_S = 0$. This is seen by using that in this case $\left.\frac{f}{\|f\|}\right|_S$ extends to a map on the ball so that the induced map on homology factors through the homology of a contractible space. This implies that the degree is 0, as only the trivial endomorphism on $\mathbb{Z}$ factors through the trivial group.

We therefore extend the definition of the index to any point in the domain by defining $\operatorname{ind}_p f = 0$ if $f(p) \neq 0$.

The theory of index and it's connections to intersection multiplicities is well-known. In this section we recall some of the basic results, basing most of the presentation on that given in [DAn93, Ch. 2.1]. We also work out the proofs of some steps which we did not find in the appropriate form, or which were either left out or briefly sketched in the references.

Lemma 2.1.5. *Any invertible matrix A is homotopy equivalent to a diagonal matrix with entries ± 1. In particular, the degree of A, viewed as a map of spheres, is given by the sign of* $\det A$.

Proof. Consider the matrix $E_{k\ell}$ with a 1 on the $k\ell$:th position and 0 in all the other entries. The matrix $(I + \nu E_{k\ell})A$ is the matrix one gets after adding ν times row l of A to row k. Gauss-Jordan elimination is successively doing this, possibly with row-changes, until one gets a diagonal matrix B. Not wanting to change the determinant we must be careful with how we do the row-changes and also be satisfied with ± 1 on the diagonal. Row changes correspond to multiplying with matrices $I + E_{k\ell}$ as is shown in the following calculation on two rows

$$\begin{aligned}\begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ b_1 & b_2 & b_3 & \dots \end{pmatrix} &\sim \begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & \dots \\ b_1 & b_2 & b_3 & \dots \end{pmatrix} \\ &\sim \begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & \dots \\ -a_1 & -a_2 & -a_3 & \dots \end{pmatrix} \\ &\sim \begin{pmatrix} b_1 & b_2 & b_3 & \dots \\ -a_1 & -a_2 & -a_3 & \dots \end{pmatrix}\end{aligned}$$

Hence we can multiply by $I + E_{k\ell}$ to switch two rows if we also switch the sign of one of them. The second operation in Gauss-Jordan elimination, after we have gotten a diagonal matrix, is to scale so we get each diagonal entry to ± 1. This is given by adding a multiple $\mu > -1$ of E_{kk} to the identity. With the Gauss-Jordan algorithm we can thus get a diagonal matrix B with only ± 1 entries from a given invertible A by using $E_{k\ell}, E_{kk}$. Now, the matrix obtained is homotopic to A by the following homotopies

$$\begin{aligned} H(x,t) &= (I + t\nu E_{k\ell})\, Ax \qquad (k \neq l) \\ G(x,t) &= (I + t\mu E_{kk})\, Ax \qquad (\mu > -1). \end{aligned}$$

To see why these are continuous one notes that the paths $A \to (I + t\nu E_{k\ell})\, A$ and $A \to (I + t\mu E_{kk})\, A$ are straight lines in $\mathbb{R}^{n^2}$ which are continuous paths in GL_n so long as the lines are contained in GL_n. This is seen to be the case as $\det (I + t\mu E_{kk})\, A = (1 + t\mu) \det A$, so that for $\mu > -1$ the determinant is always non-zero. Moreover, $\det (I + t\nu E_{k\ell})\, A = \det A$. In particular, these operations do not change the signs of the determinant.
The last statement of the lemma follow from the fact that such a diagonal matrix B is a composition of m reflections of the sphere, where m is the number of -1 entries on the diagonal, and reflections have degree -1. Thus $\det A = \det B = (-1)^m = \deg B$ by (2.1). ∎

Lemma 2.1.6. *Suppose f is a smooth function on $\mathbb{R}^n$ with a regular zero at p. Then in some small sphere S_ε centered at p, the Jacobi matrix $\mathscr{J}_{\mathbb{R}} f(p)$ is homotopy equivalent to f as maps $S_\varepsilon \to \mathbb{R}^n - \{0\}$.*

Proof. Without loss of generality we may assume $p = 0$. By definition of the derivative, inside some small ball B_r at the origin we can write

$$f(x) = \mathscr{J}_{\mathbb{R}} f(0)x + \varphi(x)$$

for some smooth mapping $\varphi : B_r \to \mathbb{R}^n$ with

$$\lim_{x \to 0} \frac{\varphi(x)}{\|x\|} = 0. \tag{2.2}$$

On a small sphere $S_\varepsilon := \partial B_\varepsilon \subseteq B_r$ and for $t \in [0,1]$, define the functions

$$f_t(x) := \mathscr{J}_{\mathbb{R}} f(0)x + t\varphi(x). \tag{2.3}$$

We want to show that on S_ε, this family gives a homotopy $\frac{f_t}{\|f_t\|}$ of sphere maps. The only possible obstruction to this is that for some t the function $f_t(x)$ vanishes at some point $x \in S_\varepsilon$. The condition (2.2) implies that $\|\varphi(x)\| = \|x\|^2 \rho(x) \geqslant 0$ on the ball B_r, for some bounded map ρ. Suppose $\|\rho\|_\infty = C$. Then, on S_ε the inequality $\|\varphi\|_{S_\varepsilon} \leqslant C\varepsilon^2$ holds. By the regularity assumption $\|\mathscr{J}_{\mathbb{R}} f(0)x\| \geqslant C'\varepsilon$ holds on S_ε, where C' does not depend on ε. This is seen by using $\|BAx\| \leqslant \|B\|_{op}\|Ax\|$ for $A = \mathscr{J}_{\mathbb{R}} f(0)$ and $B = A^{-1}$.
Choosing $\varepsilon \in (0, r)$, such that $C\varepsilon < C'$, we get by the reverse triangle inequality

$$\|f_t(x)\| \geqslant |\|\mathscr{J}_{\mathbb{R}} f(0)x\| - \|\varphi(x)\|| \geqslant |C' - C\varepsilon|\varepsilon > 0.$$

Hence, we have a homotopy $f \simeq \mathscr{J}_{\mathbb{R}} f(0) : S_\varepsilon \to \mathbb{R}^n - \{0\}$. ∎

Lemma 2.1.7. *Suppose $f : \overline{B} \to \mathbb{R}^n$ is smooth, where $B \subseteq \mathbb{R}^n$ is some ball such that the zeros $p_1, \dots, p_m$ of f in B are regular, and such that f never vanishes on the sphere ∂B. Then*

$$\deg \left.\frac{f}{\|f\|}\right|_{\partial B} = \sum_{\ell=1}^{m} \operatorname{ind}_{p_\ell} f = \sum_{\ell=1}^{m} \operatorname{sgn}\det \mathscr{J}_{\mathbb{R}} f(p_\ell). \tag{2.4}$$

Proof. The second equality follows by Lemma 2.1.6.
It remains to show that the the degree is the sum of the indices

$$\deg \left.\frac{f}{\|f\|}\right|_{\partial B} = \sum_{\ell=1}^{m} \operatorname{ind}_{p_\ell} f.$$

Recall that the index of f at p_ℓ is the degree of the map $g_\ell = \left.\frac{f}{\|f\|}\right|_{S_{\varepsilon_\ell}}$ on a sphere not containing any of the other vanishing points. Consider the manifold with boundary $B' = \overline{B} \setminus \bigcup_\ell B_{\varepsilon_\ell}$ where B_{ε_ℓ} is a small ball centered at p_ℓ containing no other zero in its closure.

Denote by g the map $\frac{f}{\|f\|}\Big|_{B'}$ and define $S = \partial B$. What we want to prove is that $\deg g|_S = \sum_{\ell=1}^{m} \deg g_\ell$. Denote by $S_\ell = \partial B_{\varepsilon_\ell}$ and let ι_ℓ be homeomorphisms

$$\iota_\ell : S^{n-1} \to S_\ell \subseteq B', \qquad \iota : S^{n-1} \to S \subseteq B'.$$

Then the maps g_ℓ may be construed as precomposing g with ι_ℓ. From this one gets the following description of the degrees

$$\begin{tikzcd}
H_{n-1}(S^{n-1}) \arrow[dr, "\iota_*"] \arrow[drr, bend left, "\deg g|_S"] & & \\
& H_{n-1}(B') \arrow[r, "g_*"] & H_{n-1}(S^{n-1}) \\
\bigoplus_\ell H_{n-1}(S^{n-1}) \arrow[ur, "(\iota_\ell)_*"] \arrow[urr, bend right, "\oplus_\ell \deg g_\ell"'] & &
\end{tikzcd}$$

We note that $\iota[S^{n-1}] = \sum_{\ell=1}^{m} \iota_\ell[S^{n-1}]$ in $H_{n-1}(B')$, which is seen by observing that

$$\partial B' = S - \sum_{\ell=1}^{m} S_\ell.$$

so that the difference between the cycles representing these classes is a boundary. Hence, by the functoriality of homology

$$\begin{aligned}(\deg g)[S^{n-1}] &= (g\iota)_*[S^{n-1}] = g_*\iota_*[S^{n-1}] \\ &= \sum_{\ell=1}^{m} g_*(\iota_\ell)_*[S_\ell] = \sum_{\ell=1}^{m} (g\iota_\ell)_*[S^{n-1}] = \sum_{\ell=1}^{m} (\deg g_\ell)[S^{n-1}]\end{aligned}$$

which proves the proposition. ■

Suppose $f : U \to \mathbb{R}^n$ is a smooth equidimensional mapping with an isolated zero at p. Suppose also that B is a ball centered at p such that $\overline{B}$ contains no other zero. Then we may find a continuous family $f_t : U \to \mathbb{R}^n$ of smooth equidimensional maps such that $f_0 = f$, and f_t does not vanish on ∂B, and such that f_1 has only regular zeros. This family then restricts to homotopy equivalent maps $f_t : S \to \mathbb{R}^n - \{0\}$ on the boundary sphere $S = \partial B$.

Definition 2.1.8. Such a family is called a *good perturbation of f on S*, and we shall sometimes call such a sphere S an *isolating sphere* for the isolated vanishing point p. If $f : U \to \mathbb{R}^{2n} \cong \mathbb{C}^n$ is assumed to be holomorphic, then we require a good perturbation of f to be a family of holomorphic mappings.

The usefulness of a good perturbation stems from the "coalescing property" of the index. This property simplifies the calculations significantly by allowing us to separate a multiple zero to several simple vanishing points.

Corollary 2.1.9 (The Coalescing Property)**.** *Suppose f has an isolated zero p and S is an isolating sphere for p. For a good perturbation f_t of f on S the index satisfies*

$$\operatorname{ind}_p f = \sum_{p_\ell \in f_t^{-1}(0)} \operatorname{ind}_{p_\ell} f_t.$$

Proof. Since a good perturbation on S is a family of homotopic sphere maps $\frac{f}{\|f\|}\Big|_S \simeq \frac{f_t}{\|f_t\|}\Big|_S$ and the degree is stable under homotopy, the corollary follows from the first equality in (2.4). ■

Lemma 2.1.10. *If f has an isolated zero at p, then for any isolating sphere S of p there is a good perturbation of f on S. This is also true in the holomorphic case.*

Proof. By Sard's theorem we can find a path $\gamma : I \to \mathbb{R}^n$ with $\gamma(0) = p$ such that $\gamma(1)$ is a regular value of f. Then $f_t(z) = f(z) - \gamma(t)$ has only regular zeros for $t = 1$. Note that we can choose the regular value $\gamma(1)$ to be arbitrarily close to the point p. Moreover, if f is holomorphic then clearly so is this family of functions. To see that this gives a good perturbation, we mimic the argument in the proof of Lemma 2.1.6. That is, for $\delta = \inf_{z\in S} \|f(z)\|$ we choose a small regular value $\|\gamma(1)\| < \delta$ and a path γ with $\|\gamma(t)\| < \delta$. Then on S we get that

$$\|f_t(z)\| \geqslant |\|f(z)\| - \|\gamma(t)\|| \geqslant |\delta - \|\gamma(t)\|| > 0. \tag{2.5}$$

Thus the family consists of homotopy equivalent maps from S into $\mathbb{R}^n - \{0\}$. This shows that $f_t^{-1}(0)$ is a (nonempty) finite set of points inside S, by continuity. ■

In the sequel we denote by $f - w$, for a regular value w, a good perturbation where it is implied that one has taken a nice path $p \to w$ as in the above construction.

Proposition 2.1.11. *The index of a holomorphic mapping f at an isolated zero p is strictly positive.*

Proof. This follows immediately from the existence of a good perturbation f_t of f on a sphere S isolating the vanishing point. Indeed, as the perturbation is holomorphic

$$\det \mathscr{J}_{\mathbb{R}} f_t = |\det \mathscr{J}_{\mathbb{C}} f_t|^2 \geqslant 0$$

where $\det \mathscr{J}_{\mathbb{C}} f_t$ is the holomorphic Jacobian (see e.g. [DAn93, p.16]). Hence an application of Lemma 2.1.6 and Corollary 2.1.9 yields the result. ■

Corollary 2.1.12. *If f is an equidimensional finite holomorphic mapping, then f is an open mapping.*

Proof. Take a value $f(p)$, which we may assume to be 0, since otherwise we can consider $f - f(p)$. To see why f is an open mapping we show that $f(p) = 0$ is an interior point. We do this by showing that for $\delta > 0$ sufficiently small, any point w with $\|w\| < \delta$ is a value.
Fix a small sphere $S \subseteq U$ isolating p from other vanishing points. Since $\inf_{z \in S} \|f\| > 0$, we can define a mapping $\frac{f}{\|f\|}\Big|_S$ of spheres. Let $\delta < \inf_{z \in S} \|f\|$. For any point w with $\|w\| < \delta$ the perturbation $f - w$ does not vanish on the sphere by the reverse triangle inequality. We therefore get homotopic sphere maps $\frac{f}{\|f\|}\Big|_S \simeq \frac{f-w}{\|f-w\|}\Big|_S$. Note that $\operatorname{ind}_p f > 0$ by Proposition 2.1.11. Therefore, homotopy invariance of the degree (Lemma 2.1.1) implies

$$0 < \operatorname{ind}_p f = \deg \frac{f}{\|f\|}\Big|_S = \deg \frac{f - w}{\|f - w\|}\Big|_S .$$

By Remark 2.1.4 it follows that $f(z) = w$ for some z inside S. This proves that any $\|w\| < \delta$ is a value. ∎

We now prove that the geometric interpretation of multiplicity — namely how many "times" f is 0 — coincide with the topological definition.
Proposition 2.1.13. *Let $f : U \to V$ be a finite equidimensional mapping holomorphic mapping where V is a neighbourhood of the origin in $\mathbb{C}^n$. If B is a ball isolating a vanishing point p, then the index $\operatorname{ind}_p f$ is precisely the generic number of preimages $\# f^{-1}(w) \cap B$ for w close enough to the origin.*
Furthermore, for δ sufficiently small

$$\sum_{p \in f^{-1}(0)} \operatorname{ind}_p f \int_{B_\delta} dV = \int_{f^{-1}(B_\delta)} \det \mathscr{J}_{\mathbb{R}} f \, dV. \tag{2.6}$$

Proof. Generic points w sufficiently close to the origin are regular values and give good perturbations $f - w$ of f on $S = \partial B$. The formula (2.4) then implies

$$\operatorname{ind}_p f = \deg \frac{f - w}{\|f - w\|}\Big|_S = \sum_{p_\ell \in f^{-1}(w) \cap B} \operatorname{ind}_{p_\ell} (f - w) = \# f^{-1}(w) \cap B.$$

For the second part, assume for simplicity that $f^{-1}(0) = \{p\}$. Consider first the case when p is a regular vanishing point. Since any ball isolates p we see that $\# f^{-1}(w) = 1$ for any value w, i.e. f is injective. Moreover, the inverse function theorem implies that f is a diffeomorphism of some neighbourhood B of p to its image. Choose a small neighbourhood $B_\delta(0) \subseteq f(B)$. We get a diffeomorphism $f : f^{-1}(B_\delta) \to B_\delta$ and we may apply the change of variables formula

$$\int_{f^{-1}(B_\delta)} \det \mathscr{J}_{\mathbb{R}} f dV = \int_{B_\delta} dV.$$

More generally, if p is not necessarily a regular vanishing point, consider a small neighbourhood B_δ of values. By Sard's theorem, there is some negligible set N such that $B' = B_\delta \setminus N$ consists of regular values. Since f is proper, $f^{-1}(N)$ is precompact and it follows that this set is also negligible. For a regular value $w \in B'$ we get $\# f^{-1}(w) = \operatorname{ind}_p f$ by the first part of the proposition. Therefore, $f^{-1}(w)$ consists of $\operatorname{ind}_p f$ regular points. For any $w \in B'$ let $B_w \subseteq B_\delta$ be some ball centered at w such that the inverse image consists of $\operatorname{ind}_p f$ disjoint sets diffeomorphic to B_w. Define $B'' = \bigcup_w B_w$ and note that since $B' \subseteq B'' \subseteq B_\delta$ we see that B'' is a conull set in B_δ, and hence $f^{-1}(B'')$ is a conull set in $f^{-1}(B_\delta)$. Let $\{\chi_w\}$ be a partition of unity subordinate to the cover $\{B_w\}$ of B''. By using the change of variable formula on each patch of $f^{-1}(B_w)$ we get

$$\int_{f^{-1}(B_w)} (\chi_w \circ f) \det \mathscr{J}_{\mathbb{R}} f dV = \operatorname{ind}_p f \int_{B_w} \chi_w dV \tag{2.7}$$

and the result follows from the partition of unity. ■

2.2 The Grothendieck residue

Consider the holomorphic mapping $f(z) = (z_1^{\alpha_1}, \ldots, z_n^{\alpha_n})$, where $\alpha_\ell \in \mathbb{Z}_+$. Intuitively, we want to define the *multiplicity* of f at the isolated zero at the origin to be equal to $\alpha_1 \cdots \alpha_n$. Indeed, with the definitions in the previous section, we see that on a large enough sphere S centered at the origin, the perturbation $f - \mathbf{1}$ is good, with isolated regular zeros at points $(u_{\alpha_1}, \ldots, u_{\alpha_n})$ where u_{α_ℓ} is an α_ℓ:th root of unity — and there are precisely $\alpha_1 \cdots \alpha_n$ such points. The analytic multiplicity is defined by considering the following multivariable residue integral

$$\frac{1}{(2\pi i)^n} \int_{\cap\{|z_\ell|=\varepsilon\}} \frac{dz^{\alpha_1}}{z_1^{\alpha_1}} \wedge \cdots \wedge \frac{dz_n^{\alpha_n}}{z^{\alpha_n}} = \frac{1}{(2\pi i)^n} \int_{\cap\{|z_\ell|=\varepsilon\}} \frac{\alpha_1 \cdots \alpha_n}{z} dz = \alpha_1 \cdots \alpha_n$$

which counts the number of zeros of f. We also get a similar formula as in the one-variable argument principle

$$\frac{1}{(2\pi i)^n}\int\limits_{\cap\{|z_\ell|=\varepsilon\}} g\frac{dz_1^{\alpha_1}}{z_1^{\alpha_1}}\wedge\cdots\wedge\frac{dz_n^{\alpha_n}}{z_n^{\alpha_n}} = \alpha_1\cdots\alpha_n\frac{1}{(2\pi i)^n}\int\limits_{\cap\{|z_\ell|=\varepsilon\}}\frac{gdz}{z} = \alpha_1\cdots\alpha_n g(0).$$

For the given function $f(z) = (z_1^{\alpha_1},\cdots,z_n^{\alpha_n})$, we consider therefore the functional Res_f defined by

$$\mathrm{Res}_f[\omega] := \frac{1}{(2\pi i)^n}\int\limits_{\cap\{|z_\ell|=\varepsilon\}}\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}\omega$$

acting on holomorphic n-forms ω defined near the origin. We have just seen that for a holomorphic function g defined near the origin

$$\mathrm{Res}_f[gdf] = \alpha_1\cdots\alpha_n g(0) = g(0)\,\mathrm{ind}_0 f.$$

In the following, we define and study this integral for more general functions f with isolated vanishing points.

Definition 2.2.1 (Grothendieck point residue)**.** Given $f_1,\ldots,f_n\in\mathscr{O} = {}_n\mathscr{O}_p$, set $f=(f_1,\ldots,f_n)$. Suppose p is an isolated zero of f and define the cycle $\Gamma = \bigcap\{z : |f_\ell(z)| = \varepsilon_\ell\}$ for ε_ℓ sufficiently small. For $g\in\mathscr{O}$ we define *the residue of g with respect to f at p* by

$$\mathrm{Res}_f[gdz] := \frac{1}{(2\pi i)^n}\int_\Gamma\frac{gd\zeta}{f}.$$

For $g = \det df$ the integral is sometimes called the residue of f at p.

The set $\Gamma = \cap\{|f_\ell| = \varepsilon_\ell\}$ in the above definition is in fact a smooth cycle of real codimension n for a generic small ε, which can be seen by invoking *Sard's Lemma*. We define an orientation on Γ by considering a smooth regularisation χ of the characteristic function on $[1,\infty[$. For a generic choice $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$ of small ε_ℓ we define $\chi_\ell^\varepsilon = \chi\left(|f_\ell|^2/\varepsilon_\ell^2\right)$. The orientation is then specified by deciding how to integrate an $(n,0)$-form ω

$$\int_\Gamma\omega := \int\bar\partial\chi_1^\varepsilon\wedge\cdots\wedge\bar\partial\chi_n^\varepsilon\wedge\omega. \tag{2.8}$$

Remark 2.2.2. This approach to orienting Γ is taken from [Ber+93], which is easier to work with than that found in [GH78] where they use a form $d\arg f_1\wedge\cdots d\arg f_n$ to define an orientation. In particular, the above definition makes it easy to use Stokes' theorem as we automatically get the correct sign by anticommutativity of the wedge product.

There are several things left to check for Definition 2.2.1 to make sense. First, we will see that fixing a representative of the germ g the above integral does not depend on the cycle Γ, for a generic small $\varepsilon > 0$. This also explains the use of function germs, which a priori does not make sense in the above integral. Indeed, if integrating over smaller cycles gives the same integral, then using different representatives also yield the same integral.

Proposition 2.2.3. *The residue integral* Res_f *is well-defined.*

Proof. Let Γ, Γ' be two cycles, defined by ε and ε' respectively. If these cycles are the same, there is nothing to prove. Otherwise there is some k with $\varepsilon_\ell \neq \varepsilon'_\ell$ and we may assume $\varepsilon'_\ell < \varepsilon_\ell$. Without loss of generality, we assume $\varepsilon'_j = \varepsilon_j$ except for $j = k$, and that $k = 1$. Using the orientation specified in Equation (2.8) together with Stokes' theorem we get

$$\int_\Gamma \omega = \int_{|f_1|=\varepsilon_1} \bar\partial\chi_2^\varepsilon \wedge \cdots \wedge \bar\partial\chi_n^\varepsilon \wedge \omega$$
$$\int_{\Gamma'} \omega = \int_{|f_1|=\varepsilon'_1} \bar\partial\chi_2^\varepsilon \wedge \cdots \wedge \bar\partial\chi_n^\varepsilon \wedge \omega.$$

Now, $\psi = \bar\partial\chi_2^\varepsilon/f_2 \wedge \cdots \wedge \bar\partial\chi_n^\varepsilon/f_n$ is smooth and $\bar\partial$-closed in $\mathbb{C}^n$. Since $\psi \wedge g/f_1 dz$ is a smooth $(n, n-1)$-form on $\varepsilon'_1 \leqslant |f_1| \leqslant \varepsilon_1$ where $g/f_1 dz$ is holomorphic, we get by Stokes'

$$\int_{\partial\{\varepsilon' \leqslant |f| \leqslant \varepsilon\}} \psi \wedge g/f_1 dz = \int_{\{\varepsilon' \leqslant |f| \leqslant \varepsilon\}} \bar\partial(\psi \wedge g/f_1 dz) = 0.$$

This implies that $\int_\Gamma g dz/f = \int_{\Gamma'} g dz/f$ so that Res_f is indeed independent of the choice of cycle. ∎

For $g \in \mathscr{O}$ there is a natural way of multiplying the residue with g

$$g \operatorname{Res}_f[\omega] := \operatorname{Res}_f[g\omega].$$

Define the annihilator of the residue to be the ideal

$$\operatorname{ann} \operatorname{Res}_f = \{g \in \mathscr{O} :\ g \operatorname{Res}_f = 0\}. \tag{2.9}$$

The *Local Duality Theorem*, found e.g. in [GH78, p. 659], states that if f is finite equidimensional then $\operatorname{ann} \operatorname{Res}_f = (f)$. One consequence of this is that the residue gives an analytic condition for ideal membership in the case of ideals given by the component functions of finite equidimensional mappings. We prove only the direction $(f) \subseteq \operatorname{ann} \operatorname{Res}_f$ before ending this section.

Proposition 2.2.4. *If* $g \in (f)$*, then* $g \operatorname{Res}_f = 0$*. Hence,* $(f) \subseteq \operatorname{ann} \operatorname{Res}_f$*.*

Proof. If $g \in (f)$ then $g = \sum_\ell f_\ell h_\ell$ for some $h_\ell \in \mathscr{O}$. So $\mathrm{Res}_f[gdz] = \sum_\ell \mathrm{Res}_f[h_\ell f_\ell]$. So showing that $g = f_\ell h$ satisfies $\mathrm{Res}[gdz] = 0$ yields the result. To this end, consider the forms

$$\psi_\ell = f_\ell dz/f = \frac{dz}{f_1 \cdots f_{\ell-1} f_{\ell+1} \cdots f_n}.$$

By Equation (2.8) and Stokes' theorem we have for any $h \in \mathscr{O}$

$$\mathrm{Res}_f[hf_\ell] = \int_\Gamma h\psi_\ell = (-1)^{\ell-1} \int\limits_{|f_\ell|=\varepsilon} \widehat{\bar\partial \chi_\ell^\varepsilon} \wedge \psi_\ell = (-1)^{\ell-1} \int\limits_{|f_\ell|\leqslant\varepsilon} d(\widehat{\bar\partial \chi_\ell^\varepsilon} \wedge \psi_\ell) = 0$$

where we have used that form $(n, n-1)$-form $\widehat{\bar\partial \chi_\ell^\varepsilon} \wedge h\psi_\ell$ is smooth in $\mathbb{C}^n$ and $\bar\partial$-closed (and hence d-closed) on $|f_\ell| \leqslant \varepsilon$, where it is also compactly supported. ■

2.3 Residues and multiplicity

We will show that the analytic notion of multiplicity captured by the residue and the topological definition using winding numbers coincide. To this end we define two forms that will provide the connection.

Definition 2.3.1. Let $f = (f_1, \ldots, f_n)$ be a finite n-equidimensional holomorphic mapping. We define the differential $(n, n-1)$-forms B_f and K_f by

$$B_f := \frac{(n-1)!}{(2\pi i)^n} \frac{1}{\|f\|^{2n}} \sum_{\ell=1}^n (-1)^{\ell-1} \bar f_\ell \bar\partial \bar f_n \wedge \cdots \wedge \widehat{\bar\partial \bar f_\ell} \wedge \cdots \wedge \bar\partial \bar f_1 \tag{2.10}$$

$$K_f := B_f \wedge df \tag{2.11}$$

We shall refer to these as *Bochner-Martinelli forms* associated to f.

As a consequence of the equality of the integrals in (2.6), we can relate the index and the Bochner-Martinelli forms as follows.

Proposition 2.3.2. *Suppose f is a finite n-equidimensional holomorphic mapping with $\{p\} = f^{-1}(0)$. The Bochner-Martinelli form K_f and the index* $\mathrm{ind}_p f$ *are related by the integral*

$$\int\limits_{\|f\|=\varepsilon} K_f = \mathrm{ind}_p f. \tag{2.12}$$

Proof. By (2.6) we have

$$\mathrm{ind}_p f \int\limits_{B_\varepsilon} dV = \int\limits_{f^{-1}(B_\varepsilon)} \det \mathscr{J}_{\mathbb{R}} f \, dV = \int\limits_{f^{-1}(B_\varepsilon)} |\det \mathscr{J}_{\mathbb{C}} f|^2 \, dV \tag{2.13}$$

where det $\mathscr{J}_{\mathbb{R}} f$ is the real and det $\mathscr{J}_{\mathbb{C}} f$ the complex Jacobian (see e.g. [Leb19] for their relation). On the other hand on $\|f\| = \varepsilon$, K_f coincides with the form

$$K_f^\varepsilon = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{\varepsilon^{2n}} \left(\sum_{\ell=1}^{n} (-1)^{\ell-1} \bar{f}_\ell \bar{\partial} \bar{f}_n \wedge \cdots \wedge \widehat{\bar{\partial} \bar{f}_\ell} \wedge \cdots \wedge \bar{\partial} \bar{f}_1 \right) \wedge df$$

and the exterior derivative of this form is

$$dK_f^\varepsilon = \bar{\partial} K_f^\varepsilon = \frac{n!}{\pi^n} \frac{1}{\varepsilon^{2n}} |\det \mathscr{J}_{\mathbb{C}} f|^2 \, dV. \tag{2.14}$$

Hence, an application of Stokes theorem

$$\begin{aligned} \int_{\|f\|=\varepsilon} K_f = \int_{\|f\|=\varepsilon} K_f^\varepsilon &= \frac{n!}{\pi^n} \varepsilon^{-2n} \int_{f^{-1}(B_\varepsilon)} |\det \mathscr{J}_{\mathbb{C}} f|^2 \, dV \\ &= \frac{n!}{\pi^n} \varepsilon^{-2n} \operatorname{ind}_p f \int_{B_\varepsilon} dV = \operatorname{ind}_p f \end{aligned}$$

where we have used (2.14) together with Equation (2.13), and where the last equality follows from the fact that $\mathrm{Vol}(B_\varepsilon) = \frac{\pi^n}{n!} \varepsilon^{2n}$. ■

To connect $\operatorname{ind}_p f$ and the residue of f at p we also need the following lemma.

Lemma 2.3.3. *For a finite equidimensional holomorphic mapping f vanishing at p, the residue satisfies*

$$\operatorname{Res}_f[gdz] = \int_{\|f\|=\varepsilon} gB_f \wedge dz. \tag{2.15}$$

In particular

$$\operatorname{Res}_f[gdf] = \int_{\|f\|=\varepsilon} gK_f. \tag{2.16}$$

A full proof of this can be found in [GH78, pp. 653ff]. A more elementary proof of Equation (2.16) in the case $n = 2$ and for $g = 1$ can be found in [DAn93, pp. 62f].

We thus get that the residue of det $\mathscr{J}_{\mathbb{C}} f$ at p with respect to f and the index of f at p coincide

$$\operatorname{Res}_f[df] = \operatorname{ind}_p f, \tag{2.17}$$

where we have used the formula $df = (\det \mathscr{J}_{\mathbb{C}} f) dz$.

We define the global residue of a finite equidimensional holomorphic mapping by

$$\operatorname{Res}_f := \sum_{p \in f^{-1}(0)} \operatorname{Res}_f^p$$

where Res_f^p is the residue with respect to f at p. We finish this section by recording the following generalisation of the one-variable argument principle.

Proposition 2.3.4. *For a finite equidimensional holomorphic mapping f and for a holomorphic function g defined near the points of $f^{-1}(0)$ one has*

$$\operatorname{Res}_f[gdf] = \sum_{p \in f^{-1}(0)} g(p) \operatorname{ind}_p f. \tag{2.18}$$

This is a consequence of the continuity of the global residue, since at regular points p the formula can be verified directly by appealing to Cauchy's formula and change of variables. For a discussion on the continuity of the residue see e.g. [GH78, p. 657]. The reader is also referred to [GH78, pp. 663–666] for a different but rather more complete presentation on why the formula (2.18) holds.

Algebraic multiplicity

We end this chapter by recalling an algebraic definition of the intersection number. Let $\mathscr{O} =_n \mathscr{O}_p$ and let $f = f_1, \ldots, f_N \in \mathscr{O}$, and write $(f) = (f_1, \ldots, f_N)$. Define the algebraic multiplicity $\mathbf{D}(f)$ (or *codimension* in [DAn93]) of (f) at p to be the dimension

$$\mathbf{D}(f) := \dim_{\mathbb{C}} \mathscr{O}/(f). \tag{2.19}$$

This definition is really only interesting when $\operatorname{rad}(f) = \mathfrak{m}$, i.e. p is the common vanishing point of the functions. Otherwise we get either $\mathbf{D}(f) = 0$ if $(f) = \mathscr{O}$, or $\mathbf{D}(f) = \infty$ if the vanishing set has positive dimension. Note that in the interesting cases, this definition coincides with the definition of mutliplicity of the fat point at p from algebraic geometry. Note also that, unlike the previous definitions of multiplicity, the algebraic one does not require equidimensionality $N = n$. Indeed, it turns out that this definition generalises the ones previously mentioned.

Proposition 2.3.5. *Let f be a finite n-equidmensional holomorphic mapping vanishing at p. Then the index at p and the algebraic multiplicity at p coincide*

$$\mathbf{D}(f) = \operatorname{ind}_p f. \tag{2.20}$$

We therefore get a residue formula

$$\operatorname{Res}_f[gdf] = \sum_{p \in f^{-1}(0)} \big(\dim_{\mathbb{C}} \mathscr{O}_p/(f)\big)\, g(p). \tag{2.21}$$

See [DAn93, Lemma 1 & 2, pp. 58–59] and the work referenced there for a proof. The main goal of the rest of the thesis is to generalise the Grothendieck residue to non-equidimensional f and show that a formula generalising (2.21) holds also in this case.

3

Residues of complete intersections

For a finite equidimensional holomorphic map $f = (f_1, \dots, f_n)$ with $p \in f^{-1}(0)$ the germs $f_1, \dots, f_n \in \mathcal{O}$ form a so-called *regular sequence* in $\mathcal{O}$. We use this property to deduce that each subcollection of the germs of the hypersurfaces Z_{f_ℓ} form a *complete intersection* at p.
We use regular sequences to study the *Koszul complex* associated to f and see that it gives a free resolution of the $\mathcal{O}$-module $\mathcal{O}/(f)$. Using certain maps in the Koszul complex, we get a different integral representing the residue, which is easier to generalise.
For regular sequences f and g in $\mathcal{O}$ such that $(f) \subseteq (g)$, *the Transformation Law* (Theorem 3.3.1) states the relation between the corresponding residues. A proof of this theorem may be found in [GH78, p. 657], but we present a different proof using the Koszul complex, based on the proof given in [Lär12].

3.1 Regular sequences and complete intersections

We begin by recalling the definition of regular sequences. We restrict to the case of a Noetherian local ring $(A, \mathfrak{m})$, but a more general presentation is found e.g. in [Eis95].
Definition 3.1.1. Let $(A, \mathfrak{m})$ be a local Noetherian ring. An ordered sequence $r = r_1, \dots, r_k$ of elements $r_\ell \in \mathfrak{m}$ is called a *regular sequence* of *length* k if for each $\ell = 1, \dots, k$ the element r_ℓ is not a zerodivisor in the quotient

$$A/(r_1, \dots, r_{\ell-1}).$$

Lemma 3.1.2. *If* $r = r_1, \dots, r_k$ *is a regular sequence in* $(A, \mathfrak{m})$ *then so is any sequence obtained by reordering the elements of* r.
The condition that the ring be local is crucial to prove this lemma, and indeed it generally fails to be true if one extends the definition to more general rings, e.g. one may find counterexamples in $k[x, y, z]$. A nice inductive proof using Nakayama's lemma is given in [DAn93, pp. 68f]. In [DAn93, pp. 69f], we also get

a proof of the following important fact.

Proposition 3.1.3. *If $f = (f_1, \dots, f_n)$ is an equidimensional finite holomorphic map vanishing at p, then the sequence of germs $f = f_1, \dots, f_n \in \mathscr{O}$ is a regular sequence.*

Complete intersections

A consequence of Proposition 3.1.3 is that the analytic sets cut out by any subcollection of the component functions of a finite equidimensional mapping f is a *complete intersection* at the isolated vanishing points f. This means precisely that the *codimension* $n - \dim(\mathcal{V}(f_{\ell_1}, \dots, f_{\ell_k}), p)$ of $(\mathcal{V}(f_{\ell_1}, \dots, f_{\ell_k}), p)$ is precisely k if $f(p) = 0$. To see this, we need to take for granted the following proposition, for a proof see [Dem12].

Proposition 3.1.4. *If $(B, p) \subseteq (A, p)$ are analytic sets and (A, p) is irreducible, then*

$$\dim(B, p) < \dim(A, p) \quad \text{or} \quad (B, p) = (A, p).$$

To prove that the subcollections of the components form complete intersection, we begin by studying how the dimension changes when going from a regular sequence of length k to one of length $k+1$.

Lemma 3.1.5. *Let $f_1, \dots, f_{k+1} \in \mathscr{O}$ be a sequence such that f_{k+1} is a nonzerodivisor in the ring $\mathscr{O}/(f_1, \dots, f_k)$. Then $(Z_{f_{k+1}}, p)$ does not contain any irreducible component of $(\mathcal{V}(f_1, \dots, f_k), p)$.*

Proof. We suppress the point p in the notation in the following proof.
Suppose $A_1, \dots, A_m$ are the irreducible components of $\mathcal{V}(f_1, \dots, f_k)$, where $f_1, \dots, f_k$ is a sequence of elements in the maximal ideal. Then the ideals $\mathfrak{p}_j = \mathcal{I}(A_j)$ are prime and for $I = (f_1, \dots, f_k)$ one has

$$\sqrt{I} = \bigcap_{j=1}^{m} \mathfrak{p}_j.$$

For $\ell = 1, \dots, m$, define new ideals

$$J_\ell := \bigcap_{j \neq \ell} \mathfrak{p}_j.$$

As $\mathscr{O}$ is Noetherian, the ideals are finitely generated $J_\ell = (g_1^\ell, \dots, g_r^\ell)$. The varieties $\mathcal{V}(J_\ell)$ satisfy

$$\mathcal{V}(J_\ell) = \bigcap_{i=1}^{r} Z_{g_i} = \bigcup_{j \neq \ell}^{m} A_j \subsetneq \mathcal{V}(I)$$

as $A_\ell \not\subseteq A_j$ for any $\ell \neq j$. This means that $A_\ell \not\subseteq Z_{g_i}$ for some generator $g_i^\ell =: h_\ell$. Hence, we have for any $\ell = 1, \dots, m$ an element $h_\ell \in J_\ell \setminus \mathfrak{p}_\ell$ such that $h_\ell \mathfrak{p}_\ell \subseteq \sqrt{I}$. To conclude, if $Z_{f_{k+1}}$ contains an irreducible component A_ℓ, then $f_{k+1} \in \mathfrak{p}_\ell$, and hence $h_\ell f_{k+1} \in \sqrt{I}$. This means that $(h_\ell f_{k+1})^N \in I$ for some N, by the Nullstellensatz, and therefore f_{k+1} is a zerodivisor in $\mathscr{O}/I$. This proves the contrapositive. ■

Corollary 3.1.6. *Let $f = (f_1, \dots, f_n)$ be an equidimensional finite analytic mapping. Then at any vanishing point $p \in f^{-1}(0)$*

$$\operatorname{codim}\big(\mathcal{V}(f_{\ell_1}, \dots, f_{\ell_k}), p\big) = k$$

for any indexing $\{\ell_1 \dots, \ell_k\} \subseteq \{1, \dots, n\}$ of k elements. In other words, the variety $\big(\mathcal{V}(f_{\ell_1}, \dots, f_{\ell_k}), p\big)$ is a complete intersection.

Proof. Suppose first that $p \in f^{-1}(0)$. Then the sequence is regular at p by Proposition 3.1.3. Without loss of generality we may assume the indexing is $\ell_r = r$, since regularity is independent of ordering. We get a strictly increasing sequence

$$1 = \operatorname{codim} \mathcal{V}(f_1) < \operatorname{codim} \mathcal{V}(f_1, f_2) < \cdots < \operatorname{codim} \mathcal{V}(f_1, \dots, f_n) = n.$$

Indeed, by Lemma 3.1.5 the vanishing set $Z_{f_{k+1}}$ does not contain any irreducible component A_ℓ of $\mathcal{V}(f_1, \dots, f_k)$, whence the new variety $Z_{f_{k+1}} \cap A_\ell \subseteq A_\ell$ are of strictly lower dimension by Proposition 3.1.4. The result follows now as the above sequence of inequalities forces the codimension to increase by 1 at each step. ■

3.2 Koszul complexes

There are a couple of different ways of defining the Koszul complex. We give two definitions: one because it gives the easiest way of illustrating the kind of inductive argument one applies when dealing with these complexes, and the other because it is more useful for the calculations in the rest of the chapter.

Definition 3.2.1. Let A be a ring. The *Koszul complex $K.(x)$ associated to an element* $x \in A$ is defined to be the chain complex

$$0 \longrightarrow A_1 \xrightarrow{\cdot x} A_0 \longrightarrow 0$$

where $A_1 = A_0 = A$ and the indices are just to keep track of the homological degree. We sometimes include the ring A in the notation and write $K(x, A)$.

For a sequence $x = x_1, \dots, x_n \in A$ one defines inductively the Koszul complex

$K.(x)$ by

$$K.(x) := K.(x_1, \dots, x_{n-1}) \otimes K.(x_n).$$

Writing $x' = x_1, \dots, x_{n-1}$, the chain complex $K.(x)$ consists of modules

$$K_m(x) = \big(K_m(x') \otimes A_0\big) \oplus \big(K_{m-1}(x') \otimes A_1\big) = K_m(x') \oplus K_{m-1}(x')$$

and the tensor product differential d defined by $d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$, where $|a|$ is the homological degree of a. This differential is given explicitly in this realisation by

$$d_m = \begin{pmatrix} d'_m & (-1)^{m-1} x_n \\ 0 & d'_{m-1} \end{pmatrix}.$$

The inductive nature of the proofs that $K.(x)$ gives a resolution of $A/(x)$, for a regular sequence x, rely on the fact that the Koszul complex fits in a short exact sequence of complexes. As a short exact sequence of complexes induces a long exact sequence in homology we can calculate the homology by breaking apart the complex. This is achieved by considering the suspension $\Sigma K.(x')$, which is the complex defined by $(\Sigma K.(x'))_m := K_{m-1}(x')$ with differential $d''_m = -d_{m-1}$. The desired exact sequence is then

$$0 \longrightarrow K(x') \xrightarrow{\iota} K.(x) \xrightarrow{\pi} \Sigma K.(x') \longrightarrow 0$$

By the Snake Lemma, we get a long exact sequence in homology

$$\begin{tikzcd} H.(K.(x')) \arrow[rr, "\iota_*"] & & H.(K.(x)) \arrow[dl, "\pi_*"] \\ & H.(\Sigma K.(x')) \arrow[ul, "\delta"] & \end{tikzcd} \tag{3.1}$$

where δ is the connecting homomorphism lowering the degree.

One can describe δ explicitly by carrying out the diagram chase. Namely, starting with $c \in \ker d''_m = \ker d'_{m-1}$ we want an element $a' \in \ker d'_{m-1}$ such that $\iota(a') = d_m(0, (-1)^{m-1}\, c)$. Using

$$d_m(0, (-1)^{m-1} c) = \Big(x_n c,\ (-1)^{m-1} d'_{m-1} c\Big)$$

we see that the desired element can be represented by $x_n c$. So δ is just the map in homology given by multiplication by x_n.

We now give a second definition, which is useful for our calculations. See e.g. [Eis95, Ch. 17.3] for why the two definitions coincide.

Definition 3.2.2. A concrete realisation of the Koszul complex (which is the one we use in the sequel) is given by identifying it with a complex $(\bigwedge A^{\oplus n}, \delta_x)$, where $K_m(x) = \bigwedge^m A^{\oplus n}$ with differential $d_m = \delta_x$ given by contraction with $x = \sum_\ell x_\ell e_\ell^*$, where e_ℓ is the standard basis of $A^{\oplus m}$. Recall that we get an induces basis

$$\{e_{i_1\cdots i_m} = e_{i_1} \wedge \cdots \wedge e_{i_m} : \ i_1 < \cdots < i_m\}$$

for the m:th exterior power $K_m(x) = \wedge^m A^{\oplus n}$. In this basis δ_x is given by

$$\delta_x(e_{i_1\cdots i_m}) := \sum_\ell (-1)^{\ell-1} x_{i_\ell} \widehat{e}_{i_\ell}$$

where the circumflex indicates the exclusion of the element in the wedge product. Note, that the contraction δ_x is an anti-derivation on the graded algebra $\bigwedge A$, i.e. for homogeneous forms α, β, the contraction satisfies Leibniz' rule

$$\delta_x (\alpha \wedge \beta) = \delta_x \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \delta_x \beta. \tag{3.2}$$

This means in particular that the Koszul complex, thus realised, is also a differential graded algebra. This concrete realisation is much more useful for the work in the sequel.

Lemma 3.2.3. *Let A be a local ring and x a regular sequence of order n. Then the Koszul complex $K.(x)$ is a free resolution of $A/(x)$.*

We also record a result that we need momentarily, when considering the Koszul complex associated to a finite holomorphic mapping as a complex of free analytic sheaves.

Lemma 3.2.4. *If x is a sequence in a ring A such that some element x_ℓ is a unit, then the Koszul complex $K.(x)$ has trivial homology.*

The following consequence of Proposition 3.1.3 now readily follows from the results in this section.

Corollary 3.2.5. *Let $f = (f_1, \ldots, f_n)$ be a finite equidimensional holomorphic map. Then the Koszul complex $K.(f)$ associated to the sequence of germs $f = f_1, \ldots, f_n \in \mathscr{O}_p$ at any point of the domain of f is a free resolution of $\mathscr{O}_p/(f)$.*

Example 3.2.6. Consider the identity function $z = (z_1, z_2)$ on $X = \mathbb{C}^2$. The coordinates z_1, z_2 generate the maximal ideal in the ring of germs $\mathscr{O}_0$ at the origin and the whole ring in $\mathscr{O}_p$ for $p \neq 0$. We get a complex of free analytic sheaves $K.(z, \mathscr{O}_X)$ corresponding to the sequence $z = z_1, z_2 \in \mathfrak{m}_0$, which we also call the

Koszul complex, and which is explicitly given as

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}} \mathscr{O}_X^{\oplus 2} \xrightarrow{\begin{pmatrix} z_1 & z_2 \end{pmatrix}} \mathscr{O}_X \longrightarrow 0.$$

This complex gives a free resolution of the quotient $\mathscr{O}_X/\mathfrak{m}_0$, since this is true stalkwise.

Koszul complexes as resolutions of analytic sheaves

For a finite equidimensional mapping f we saw that the Koszul complex $K.(f)$ Section 3.2 was a free resolution of $\mathscr{O}/(f)$ at all points in the domain of f. We may construe the Koszul complex at each point as coming from a complex of locally free $\mathscr{O}_X$-modules

$$K.(f, \mathscr{O}_X) = \left(\bigwedge \mathscr{O}_X^{\oplus n}, \delta_f\right)$$

which is a free resolution of the finitely supported analytic sheaf $\mathscr{O}/\mathcal{J}(f)$, since this is true stalkwise by 3.2.3 and 3.2.4. Locally free analytic sheaves coincide with holomorphic vector bundles (cf. discussion on sheaves in Section 1.1 and the work referenced there). We may view this as a complex of (trivial) holomorphic bundles instead.

Lemma 3.2.7. *Consider the Koszul complex $K.(f, \mathscr{O}_X)$ of a finite mapping as a complex (E, φ) of (trivial) holomorphic bundles over X. In this setting, the Koszul complex is an exact complex of bundles outside the vanishing set $f^{-1}(0)$.*

Proof. For $z \notin f^{-1}(0)$ we get a Koszul complex $K.(f(z), \mathbb{C})$ over $\mathbb{C}$. Since $f(z) \neq 0$, $f_k(z) \neq 0$ for some k, and this means that $f_k(z)$ is a unit and Lemma 3.2.4 implies exactness. Note also that by the rank-nullity theorem the rank is constant for each φ_k since rank $\varphi_1 = 1$ outside the vanishing set. ∎

When $z \in f^{-1}(0)$, all the maps in the complex at z are trivial and hence the complex cannot be exact since it is a nontrivial complex. In the sequel we therefore mean by (E, φ) the Koszul complex of f restricted to $Y = X \setminus f^{-1}(0)$.

We now relate the Koszul complex of a finite equidimensional mapping to the residue. This is achieved by viewing the Koszul complex as a complex of vector bundles and relating a composition of maps occuring in this complex to the residue via the Bochner-Martinelli form.

Consider the Koszul complex (E, φ) with the standard metric and e the standard frame. Then φ_k is the contraction δ_f with the element $f = \sum_\ell f_\ell e_\ell^*$. The *Moore-*

Penrose inverse σ_k of φ_k is defined by the following conditions

$$\begin{aligned} \varphi_k\sigma_k\varphi_k &= \varphi_k \\ \operatorname{im}\varphi_k^{\perp} &\subseteq \ker\sigma_k \\ \ker\varphi_k &\perp \operatorname{im}\sigma_k. \end{aligned}$$

For the purpose of finding this inverse explicitly, we consider a new frame e' defined via duality by completing

$$(e'_1)^* := \frac{f}{\|f\|}$$

into an orthonormal basis $(e'_1)^*, \dots, (e'_n)^*$ of the dual. In this basis, the contraction δ_f is simply $\delta_f e'_1 = \|f\|$ and $\delta_f e'_\ell = 0$ for $\ell \neq 1$. Hence, on E_k the contraction on the induced orthonormal basis

$$e'_{i_1\cdots i_k} = e'_{i_1} \wedge \cdots \wedge e'_{i_k} \qquad (i_1 < \cdots < i_k)$$

is the map given by

$$\delta_f e'_{i_1\cdots i_k} = \|f\| e_{i_2\cdots i_k}$$

if $i_1 = 1$, and $\delta_f e'_{i_1\cdots i_k} = 0$ otherwise. Hence, on the image one gets that the inverse is given by

$$\sigma_k e'_{i_2\cdots i_k} = \frac{1}{\|f\|} e'_1 \wedge e'_{i_2\cdots i_k}.$$

Writing $s = \frac{e'_1}{\|f\|}$ one observes that $\sigma_k = s \wedge -$ is the desired map as it is seen to satisfy $\operatorname{im}\sigma_k \perp \ker\varphi_k$ and $\ker\sigma_k = \operatorname{im}\varphi_k^{\perp}$. Note that in the original frame

$$s = \frac{1}{\|f\|^2} \sum_\ell \bar{f}_\ell e_\ell.$$

In the sequel, we do some calculations with δ_f and s so we present the the formalism used for this calculus.

Superstructure

Our goal is to equip the Koszul complex with a so-called *superstructure*. This can be viewed as a way to establish a sign convention which simplifies our formulas by appropriately "hiding" signs of the form $(-1)^{\ell-1}$ that would appear otherwise. We base our presentation on [LW18b, Section 2.2].

Suppose we have a complex

$$0 \longrightarrow E_\nu \xrightarrow{\varphi_\nu} E_{\nu-1} \xrightarrow{\varphi_{\nu-1}} \cdots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0$$

of locally free $\mathscr{O}_X$-modules, e.g. a Koszul complex. We equip $E := \bigoplus_k E_k$ with a *superstructure* $E = E^+ \oplus E^-$ splitting E into an even part $E^+ := \bigoplus E_{2k}$ and an odd part $E^- := \bigoplus E_{2k+1}$. We get a superstructure on $\operatorname{End} E$ by taking $\operatorname{End} E^+$ to be the degree-preserving and $\operatorname{End} E^-$ the degree-reversing endomorphisms. An element is said to be homogeneous if it is either odd or even. Let $\mathscr{E}_X$ and $\mathscr{E}_X^{\bullet}$ be the sheaves of smooth functions and forms on X. We consider also the sheaves

$$\mathscr{E}_X^{\bullet}(E) := \mathscr{E}_X^{\bullet} \otimes_{\mathscr{E}_X} \mathscr{E}_X(E) \quad \text{and} \quad \mathscr{E}_X^{\bullet}(\operatorname{End}_X E) := \mathscr{E}_X^{\bullet} \otimes_{\mathscr{E}_X} \mathscr{E}_X(\operatorname{End} E)$$

of E-valued and $\operatorname{End} E$-valued forms. These inherit natural superstructures as follows. Consider first $\gamma = \omega \otimes \eta$, where ω is a form and η is a smooth section of either E or $\operatorname{End} E$, such that ω and η are homogeneous. On such elements, we define the degree by $\deg := \deg_f + \deg_e$, where $\deg_f$ is the degree of the form and $\deg_e$ is the degree of the sections of E or $\operatorname{End} E$, i.e. $\deg \gamma := \deg \omega + \deg \eta$. The sheaves $\mathscr{E}_X^{\bullet}(E)$ and $\mathscr{E}_X^{\bullet}(\operatorname{End} E)$ get natural left $\mathscr{E}_X^{\bullet}$-module structures by construction. We want to make them into right modules, in order to give meaning to multiplications on either side. We endow $\mathscr{E}_X^{\bullet}(\operatorname{End} E)$ with an algebra structure by defining for elements $\gamma = \omega \otimes \eta$ and $\gamma' = \omega' \otimes \eta'$ the product

$$\gamma\gamma' := (-1)^{\deg_e \gamma \deg_f \gamma'} \omega \wedge \omega' \otimes \eta\eta'.$$

By viewing a form ω as a form-valued endomorphism: $\omega \otimes \mathrm{Id}$, we get a right module structure as well. If $\beta = \alpha \otimes \xi \in \mathscr{E}_X^{\bullet}(E)$, then we interpret $\gamma = \omega \otimes \eta \in \mathscr{E}_X^{\bullet}(\operatorname{End} E)$ as the endomorphism on $\mathscr{E}_X^{\bullet}(E)$ defined by

$$\gamma(\beta) = (-1)^{(\deg_e \gamma)(\deg_f \beta)} \omega \wedge \alpha \otimes \eta(\xi). \tag{3.3}$$

If the bundles E_k are equipped with connections D_{E_k} then $D_E = \bigoplus_k D_{E_k}$ is a connection on E. We define the induced connection D_{End} on $\operatorname{End} E$ with the given superstructure to be the connection satisfying

$$D_{End}(\gamma)\xi = D_E(\gamma(\xi)) - (-1)^{\deg \gamma} \gamma(D_E(\xi)). \tag{3.4}$$

One can show that for form-valued endomorphisms γ, γ'

$$D_{End}(\gamma\gamma') = D_{\mathrm{End}}\gamma\gamma' + (-1)^{\deg \gamma} \gamma D_{\mathrm{End}}\gamma'. \tag{3.5}$$

We usually skip the subscripts and simply write D for all the connections except when we want to be extra clear.

Koszul complexes and intersection numbers

We endow the Koszul complex (E, φ) with a superstructure as above. Note that the contraction $\varphi = \delta_f$ is a derivation on $\mathscr{E}_X^{\cdot}(E)$ with the inherited superstructure and that δ_f acts trivially on forms

$$\delta_f(\omega \otimes \xi) = (-1)^{\deg \omega} \omega \otimes \delta_f \xi.$$

Let D be a connection that is trivial with respect to the induced basis on E and that is compatible with the complex structure, i.e. $De_{i_1 \dots i_k} = 0$, and $\bar{\partial}$ is the $(0,1)$-part of D.
Consider the morphisms

$$u_n := \sigma_n \bar{\partial} \sigma_{n-1} \cdots \bar{\partial} \sigma_1 : E_0 \to E_n$$

$$D\varphi := D\varphi_1 \cdots D\varphi_n : E_n \to E_0.$$

Note that $E_n \cong E_0$ are isomorphic to the sheaf $\mathscr{O}_X$ restricted to $Y = X \setminus f^{-1}(0)$. Hence, these form-valued morphisms may be viewed as form-valued scalars and may thus be identified with usual forms. We relate the Koszul complex to the Bochner-Martinelli form via these morphisms in the following proposition (see e.g. [LW18b] for these calculations).

Proposition 3.2.8. *For an equidimensional finite mapping f the Bochner-Martinelli form B_f and the form df satisfy*

$$u_n = (2\pi i)^n B_f \quad \textit{and} \quad D\varphi = n! df.$$

As a consequence, we can represent the residue as integration against forms obtained from a Koszul complex. This perspective is at the foundation for our generalisation of the residue to non-complete zero-dimensional intersections.

Corollary 3.2.9.

$$\operatorname{Res}_f[gdz] = \frac{1}{(2\pi i)^n} \int gdz \wedge u_n.$$

Proof. Since $gdz \wedge u_n = (2\pi i)^n gdz \wedge B_f$ by Proposition 3.2.8, the statement follows immediately from Lemma 2.3.3. ∎

Induced morphisms of Koszul complexes

We finish this section by showing how to relate the complexes $K.(f, \mathscr{O}_X)$ and $K.(g, \mathscr{O}_X)$ for finite equidimensional mappings with $(f) \subseteq (g)$. We begin by noting that the inclusion implies that if $p \in Z_g \subseteq Z_f$, then there is a germ of a holomorphic matrix A defined near p, such that $f = gA$ in $\mathscr{O}_p^{\oplus n}$. The matrix induces a morphism of the Koszul complexes (near the point)

$$K.(A) := \bigwedge A : K.(f, \mathscr{O}_X) \to K.(g, \mathscr{O}_X)$$

defined on each level by $K_m(A)e_{i_1 \cdots i_m} := Ae_{i_1} \wedge \cdots \wedge Ae_{i_m}$. This gives a commutative diagram

$$\begin{tikzcd} K.(f, \mathscr{O}_X) \arrow[r, "\delta_f"] \arrow[d, "K.(A)"] & K.(f, \mathscr{O}_X) \arrow[d, "K.(A)"] \\ K.(g, \mathscr{O}_X) \arrow[r, "\delta_g"] & K.(g, \mathscr{O}_X) \end{tikzcd} \tag{3.6}$$

We use this construction to relate the Bochner-Martinelli forms u_n^f and u_n^g.

3.3 The Transformation Law

Using the calculations in the preceding section we present a proof of a theorem providing the relation between the intersection numbers of two finite equidimensional mappings vanishing at the same point. Throughout this section we view the Koszul complex as the corresponding complex of bundles.

Theorem 3.3.1 (Transformation Law)**.** *Suppose f and g are finite equidimensional maps $X \to \mathbb{C}^n$. Suppose that in the local ring $\mathscr{O}$ at a common vanishing point we have an inclusion $(f) \subseteq (g)$, so that near the point there is a holomorphic matrix A with $f = gA$. Then the residues are related by the formula*

$$\mathrm{Res}_g = (\det A)\, \mathrm{Res}_f \,. \tag{3.7}$$

Note that if $I = (f) = (g)$, then the residues differ only by multiplication with an inveritble holomorphic matrix. This means that the residue essentially only depends on the ideal I. In particular, the formulation of the argument principle in (2.21) shows that

$$\operatorname{Res}_f[hdf] = \operatorname{Res}_g[hdg] = (\dim \mathscr{O}/I)h(p).$$

Remark 3.3.2. The Transformation law reduces the computation of residues into a purely algebraic problem. Indeed, suppose we are given a zero-dimensional complete intersection g in the local ring $\mathscr{O}$ at the origin. Then for large enough N the Nullstellensatz implies $z_\ell^N \in (g)$, and we may therefore choose the regular sequence $f_\ell = z_\ell^N$. Then $f = gA$ and we may compare the residues

$$\operatorname{Res}_g[\xi dz] = \operatorname{Res}_f[(\det A)\xi dz].$$

By Cauchy's integral formula we see that the residue of ξ with respect to g is a derivative

$$\frac{\partial^{Nn}}{\partial z_1^N \cdots \partial z_n^N}\left((\det A)\xi\right)(0).$$

and we observe that the residue is just some weighted combination of derivatives of point masses at the origin.

For a finite equidimensional mapping f, define a morphism u^f on the Koszul complex by

$$u^f := \sum_\ell u_\ell^f = \sum_\ell \sigma^f(\bar\partial\sigma^f)^{\ell-1}.$$

If $f = gA$ we write $u_A^f = K.(A)u^f$, where $K.(A) = \bigwedge A$ is the induced morphism of the Koszul complexes. We define $\nabla_f := \delta_f - \bar\partial$, where $\bar\partial$ is the induced $(0,1)$-part of our trivial hermitian connection D on the space of End-valued forms. Note that ∇_f is an antiderivation on $\mathscr{E}_X^{\cdot}(E)$ and that ∇_f is a differential. The last part can be seen by using that the differentials δ_f and $\bar\partial$ anticommute in the superstructure $\delta_f\bar\partial = -\bar\partial\delta_f$. By using that since $\delta_f\sigma^f = \operatorname{Id}$ by definition, and that $\sigma(\bar\partial\sigma)^n = 0$ since $\bigwedge^{n+1} E = 0$, we see that outside the vanishing set Z_f the equality $\nabla_f u^f = \operatorname{Id}$ holds. Indeed

$$\nabla_f u^f = \nabla_f(u^f + \sigma(\bar\partial\sigma)^n) = \operatorname{Id} - \sigma(\bar\partial\sigma)^n = \operatorname{Id}. \tag{3.8}$$

Note that we also get $(\bar\partial\sigma)^n = 0$, since $(\bar\partial\sigma)^n = \delta_f\sigma(\bar\partial\sigma)^n = 0$.

We claim that if $f = gA$ then

$$\nabla_g\left(u^g \wedge u_A^f\right) = u_A^f - u^g \tag{3.9}$$

holds outside the vanishing set $Z_f \supset Z_g$. By commutativity in (3.6) we have

$$\delta_g u^f_A = \delta_g K.(A) u^f = K.(A)\delta_f u^f.$$

As $K.(A)$ is a holomorphic bundle morphism we get that $\bar\partial K.(A) = K.(A)\bar\partial$. From this together with the fact that ∇_g is a derivation with respect to the superstructure, we see that

$$\nabla_g\left(u^g \wedge u^f_A\right) = \nabla_g u^g \wedge u^f_A + (-1)^{\deg u^g} u^g \wedge K.(A)\nabla_f u^f = u^f_A - u^g \tag{3.10}$$

where we use that u^g is odd with the sign convention in Section 3.2.
To prove the theorem we apply the following representation lemma that we then use to get a straightforward comparison of the residues.
Lemma 3.3.3. *The residue with respect to a finite holomorphic mapping g satisfies*

$$\operatorname{Res}_g[\omega] = \frac{1}{(2\pi i)^n}\int \bar\partial\chi_\varepsilon \wedge \omega \wedge u^g_n. \tag{3.11}$$

Where χ is a smooth function such that $\chi \equiv 1$ on some neighbourhood of $[1,\infty[$ and $\chi \equiv 0$ in some neighbourhood of 0, and where we define $\chi_\varepsilon = \chi(\|g\|^2/\varepsilon)$ for any sufficiently small $\varepsilon > 0$.

Proof of Lemma 3.3.3. Consider the form $\chi_\varepsilon\omega \wedge u^g_n$. This form extends smoothly across the vanishing points Z_g as χ_ε vanishes identically near such points. Hence, since χ_ε is smooth we may apply Stokes' theorem on the integral representation in Corollary 3.2.9 to get

$$\begin{aligned}\operatorname{Res}_g[\omega] = \frac{1}{(2\pi i)^n}\int\limits_{\|g\|^2=\varepsilon} \omega \wedge u^g_n &= \frac{1}{(2\pi i)^n}\int\limits_{\|g\|^2=\varepsilon} \chi_\varepsilon\omega \wedge u^g_n \\ &= \frac{1}{(2\pi i)^n}\int\limits_{\|g\|^2\leqslant\varepsilon} d\left(\chi_\varepsilon\omega \wedge u^g_n\right).\end{aligned}$$

Using that $\chi_\varepsilon\omega \wedge u^f_n$ is of bi-degree $(n, n-1)$, we get that on this form $d = \bar\partial$. Since ω is holomorphic and $\bar\partial u^g_n = 0$ on $\operatorname{supp}\chi_\varepsilon$, we see that the residue is given by integrating $\bar\partial\chi_\varepsilon \wedge \omega \wedge u^g_n$ as claimed. ■

Lemma 3.3.4. *With the notation as in Lemma 3.3.3 the* End*-valued form $\nabla_g\left(u^g \wedge u^f_A\right)$ satisfies*

$$\int \bar\partial\chi_\varepsilon \wedge \omega \wedge \nabla_g\left(u^g \wedge u^f_A\right)_n = 0 \tag{3.12}$$

where $\nabla_g\left(u^g \wedge u^f_A\right)_n$ denotes the part with form-degree $n-1$.

Proof. We first identify the parts with form degree $n-1$. Note that an element u_k^f has form degree $k-1$ and section degree k. Since δ_g lowers the section degree we see that $\delta_g\left(u_k^g \wedge (u_A^f)_\ell\right)$ has form degree $n-1$ precisely if $k+\ell = n+1$. This means that $u_k^g \wedge (u_A^f)_\ell$ has section degree $k+\ell = n+1$. But the Koszul complex E is generated by n elements and hence $\bigwedge^{n+1} E = 0$.
Thus, we may replace ∇_f with $-\bar\partial$ in the integral. The part of form degree $n-1$ in $\nabla_g\left(u^g \wedge u_A^f\right)_n$ is therefore given by taking $-\bar\partial$ of a sum of elements of form-degree $n-2$ and we denote this sum by ψ. We apply Stokes' theorem by first writing

$$\bar\partial\chi_\varepsilon \wedge \omega \wedge \nabla_g\left(u^g \wedge u_A^f\right)_n = \pm d\left(\bar\partial\chi_\varepsilon \wedge \omega \wedge \psi\right)$$

where we have used that $\bar\partial\chi_\varepsilon \wedge \omega \wedge \psi$ is a form with bi-degree $(n, n-1)$, which means $d = \bar\partial$ on this form. Since this form is compactly supported inside $\|g\| \leqslant \varepsilon$ we can use Stokes' theorem

$$\int \bar\partial\chi_\varepsilon \wedge \omega \wedge \nabla_g\left(u^g \wedge u_A^f\right)_n = \pm \int_{\|g\|^2=\varepsilon} \bar\partial\chi_\varepsilon \wedge \omega \wedge \psi = 0$$

where the last equality follows from the fact that $\bar\partial\chi_\varepsilon \equiv 0$ on $\|g\|^2 = \varepsilon$ by construction. ∎

Proof of Theorem 3.3.1. Using the equality $\nabla_g\left(u^f \wedge u_A^f\right) = u_A^f - u^g$ (see (3.9)) outside $Z_f \supset Z_g$ we may apply Lemma 3.3.4 to compare the residues as follows. If we define $\chi'_\varepsilon = \chi(\|f\|^2/\varepsilon)$ the equality in Lemma 3.3.3 still holds if we replace χ_ε with χ'_ε. This follows from the inequality $\|f\|^2 = \|gA\|^2 \leqslant \|A\|^2\|g\|^2$, which implies that possibly after choosing a slightly bigger ε' we see can make sure that $\chi_\varepsilon = \chi'_\varepsilon \equiv 1$ on a set $\|f\|, \|g\| \geqslant \varepsilon'$. By Stokes' theorem we therefore get that

$$\int \left(\bar\partial\chi_\varepsilon - \bar\partial\chi'_\varepsilon\right) \wedge \omega \wedge u_n^g = 0. \tag{3.13}$$

By Lemma 3.3.3, Lemma 3.3.4 and *Equation* (3.9) we therefore get

$$\begin{aligned}\operatorname{Res}_g[\omega] = \frac{1}{(2\pi i)^n}\int \bar\partial\chi'_\varepsilon \wedge \omega \wedge u_n^g &= \frac{1}{(2\pi i)^n}\int \bar\partial\chi'_\varepsilon \wedge \omega \wedge \left(u^g + \nabla_g\left(u^f \wedge u_A^f\right)\right)_n \\ &= \frac{1}{(2\pi i)^n}\int \bar\partial\chi'_\varepsilon \wedge \omega \wedge \left(u_A^f\right)_n.\end{aligned}$$

Since $\left(u_A^f\right)_n = K_n(A)u_n^f = (\det A)\, u^f$, another application of Lemma 3.3.3 shows that

$$\operatorname{Res}_g[\omega] = \frac{1}{(2\pi i)^n}\int \bar\partial\chi'_\varepsilon \wedge (\det A)\omega \wedge u_n^f = \operatorname{Res}_f[(\det A)\omega]$$

i.e. $\mathrm{Res}_g = (\det A)\,\mathrm{Res}_f$, as claimed. ∎

4

Residues of Artinian $\mathscr{O}$-modules

Suppose we are given an ideal $J \subseteq \mathscr{O} := {}_n\mathscr{O}_0$ such that $\mathcal{V}(J) = 0$, i.e. J is generated by functions whose common zero set is the origin. In the case when J is generated by a regular sequence $f = (f_1, \dots, f_n)$ we saw in Proposition 3.2.8 and Corollary 3.2.9 how the Grothendieck residue Res_f can be obtained from the morphisms in the Koszul complex. Hence, the residue formula (2.21) may be written

$$\int \bar{\partial}\chi_\varepsilon \wedge g D\varphi \wedge u_n = (2\pi i)^n n! \,(\dim_{\mathbb{C}} \mathscr{O}/J)\, g(0). \tag{4.1}$$

We want to generalise this to any zero-dimensional J by constructing forms $D\varphi$ and u satisfying Equation (4.1), such that these forms coincide with the Bochner-Martinelli form whenever J is a complete intersection. By considering a free resolution (E, φ) for a more general J and defining $D\varphi$ and u_n as before, one would hope this to be achievable. However, since there is generally no canonical choice of resolution if J is not a complete intersection, the Moore-Penrose inverse σ of φ is not explicit and hence the form u_n is far from explicit. In [Lun12] the author showed how to define a so-called *cohomological residue* for *Cohen-Macaulay ideals* via a form that coincides with u_n in the complete intersection case. In this thesis we take an indirect approach by invoking a comparison theorem proven in [Lär19a]. This comparison theorem generalises the Transformation law to the so-called *Andersson-Wulcan currents* constructed in [AW07] which are vast generalisations of the residues considered here. The comparison theorem allows us to define a residue of J by using a complete intersection $I \subseteq J$. This was done in a more general setting in [LM18] where they give elementary constructions of residue currents associated to Cohen-Macaulay ideals.[1]

The main tool in the proofs of the chapter is homological algebra, and the next section is dedicated to recall the basic terminology and some slightly more advanced algebraic theory that we use to define the residue. The section is included mostly for completeness as we will appeal to some of the main results in passing

[1] Here *elementary* means that they do not use Hironaka's theorem of resolution of singularities to construct their currents.

in the sequel. The reader familiar with some basic terminology from commutative and homological algebra may skip the next section as we refer to the necessary results.

4.1 Free resolutions of Cohen-Macaulay modules

Suppose $I \subseteq \mathscr{O}$ is a complete intersection (of arbitrary dimension) and suppose $J \subseteq \mathscr{O}$ is zero-dimensional. The Koszul complex gives an explicit realisation of a *minimal resolution* of the corresponding cyclic module $\mathscr{O}/I$. Although we may find a resolution for $\mathscr{O}/J$, by very general arguments, there is no natural choice of resolution that can be made as explicit as the Koszul complex. However, because of some properties of $\mathscr{O}/J$ and of the ring $\mathscr{O}$, we have quite a lot of important information about the resolutions. Before recalling the properties of such resolutions, we present some basic definitions and results. We provide slightly more general theory than will actually be used in the sequel. This is done in order to relate to the more general results found in [LW18a]. Indeed, the basic tools needed for the proof we provide in the Artinian case has the same flavour algebraically as in the general case for *Cohen-Macaulay modules*, that are defined below. The main references used for this section are [Sim20] and [Eis95].
Throughout this section, let A be a commutative ring with unity.

Projective resolutions

Definition 4.1.1. An A-module M is said to be *projective* if for any surjective $\psi \in \mathrm{Hom}_A(N, L)$, any $\varphi \in \mathrm{Hom}_A(M, L)$ lifts to a homomorphism $\chi : M \to N$ so that $\varphi = \psi\chi$, i.e. there is a lifting $\chi \in \mathrm{Hom}_A(M, N)$ of φ making the diagram commute

$$\begin{tikzcd} & N \arrow[d, two heads, "\psi"] \\ M \arrow[ur, dotted, "\chi"] \arrow[r, "\varphi"] & L \end{tikzcd}$$

Remark 4.1.2. In the case of local rings the notions of projective and free coincide, but we introduce this concept since we mainly use the projective property of free modules in the constructions.
Definition 4.1.3. Let M be an A-module. A *projective resolution* (E, φ) of M is a complex of projective A-modules inducing an exact sequence

$$\cdots \longrightarrow E_k \xrightarrow{\varphi_k} \cdots \longrightarrow E_1 \xrightarrow{\varphi_1} E_0 \longrightarrow M \longrightarrow 0.$$

Sometimes we denote by φ_0 the induced surjection $E_0 \to M$.
The resolution E is said to be *finite* if there is some n such that $E_k = 0$ for $k > n$. If E is finite then the minimal n satisfying this condition is said to be the *length* of E. The module M is said to have *projective dimension* n over A if n is the minimal length of all finite projective resolutions of M. We denote the projective dimension over A by $\operatorname{pd}_A M$.
We also record this important lemma and its consequence that all resolutions are essentially the same.
Proposition 4.1.4. *Suppose (E, φ) is a chain complex with* $\operatorname{coker} \varphi_1 = M$*, and (F, ψ) is a projective resolution of N. If $\alpha \in \operatorname{Hom}_A(M, N)$ then there is morphism $a : E \to F$ extending α.*
This means that α induces a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_\ell \arrow[r, "\psi_\ell"] & \cdots \arrow[r] & F_1 \arrow[r, "\psi_1"] & F_0 \arrow[r, "\psi_0"] & N \arrow[r] & 0 \\
\cdots \arrow[r] & E_\ell \arrow[r, "\varphi_\ell"] \arrow[u, dotted, "a_\ell"] & \cdots \arrow[r] & E_1 \arrow[r, "\varphi_1"] \arrow[u, dotted, "a_1"] & E_0 \arrow[r, "\varphi_0"] \arrow[u, dotted, "a_0"] & M \arrow[r] \arrow[u, "\alpha"] & 0
\end{tikzcd}$$

Moreover, this morphism is unique up to chain homotopy.
Corollary 4.1.5. *Any two projective resolutions of M are chain homotopic.*
As we saw previously, for a regular sequence $x_1, \ldots, x_k$ defining an ideal I in a local Noetherian ring A, the Koszul complex $K.(x)$ gives a finite resolution of length k of the cyclic module A/I, consisting of finite free modules. More generally, we may always find a resolution of any module over any ring by constructing so-called *syzygy modules*. If we have a Noetherian ring and a finite module, the modules in such a resolutions will also be finite.
Proposition 4.1.6. *If A is Noetherian and M is finite then there is a projective resolution of M consisting of finite free modules.*
We want to say that the $\mathscr{O}$-modules under consideration have finite projective dimension bounded by the dimension n of $\mathbb{C}^n$ as a complex manifold. This is true for the Koszul complex and in general the existence of such resolutions follows from the local analytic version of the Syzygy theorem. We refer the reader to [Tay02, p. 142].
Proposition 4.1.7 (Hilbert's Syzygy Theorem)**.** *If M is a finite module over $\mathscr{O} = {}_n\mathscr{O}_0$ then* $\operatorname{pd}_{\mathscr{O}} M \leqslant n$*.*
Working with the residue we sometimes use the fact that there is the notion of a *minimal free resolution* over a local ring. These will be the syzygy modules and minimality is captured by the intuition that syzygy modules contain no redundant generators. Hence they must have minimal length and minimal rank at

each level.

Definition 4.1.8. Suppose $(A, \mathfrak{m}, k)$ is a local ring and let F, G be complexes of (projective) A-modules.

We say that F is a *minimal complex* if the maps in the complex $F \otimes k$ are all trivial, i.e. $\psi_\ell(F_\ell) \subseteq \mathfrak{m}F_{\ell-1}$ for all ℓ.

If F is a minimal complex that is also a free resolution of a module M, then we say that F is a *minimal free resolution* of M.

We say that G is a *trivial complex* if G is a free resolution of the trivial module.

Proposition 4.1.9. *A free resolution F over a local ring A is minimal precisely if a basis of $F_{\ell-1}$ maps onto a minimal generating set of* coker φ_ℓ *for all ℓ.*

Lemma 4.1.10. *Let M be a finite module over a local ring A. If F is a minimal free resolution of M then any free resolution E of M is isomorphic to a direct sum $E = F \oplus G$, where G is a trivial complex. In particular, up to isomorphism, there is a unique minimal resolution of M.*

The Syzygy theorem can be made into an effective statement (Lemma 4.1.17) giving a relation between the projective dimension, the dimension of our manifold and the *depth* of the module M. This is the main result that we use to compare different resolutions to each other.

Regular sequences, depth and dimension

Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finite A-module.

Definition 4.1.11. An M-sequence (or a *regular sequence on* M) is a sequence $x_1, \ldots, x_k \in A$ such that

i) $(x_1, \ldots, x_k)M \neq M$, and
ii) x_ℓ is a nonzero-divisor on $M/(x_1, \ldots, x_{\ell-1})M$, i.e. $x_\ell m = 0 \mod (x_1, \ldots, x_{\ell-1})M$ only if $m \in (x_1, \ldots, x_{\ell-1})M$, for any $\ell = 1, \ldots, k$.

The case when $M = A$ we simply call an M-sequence a regular sequence (cf. Definition 3.1.1).

The *depth of the module* M is the maximum of all lengths of M-sequences.

The ring $(A, \mathfrak{m})$ is said to be *regular* if the maximal ideal is generated by a regular sequence.

Definition 4.1.12. The *(Krull) dimension* $\dim M$ of the module M is defined as

$$\dim M := \dim\left(A/\operatorname{ann} M\right).$$

Definition 4.1.13. We say that M is *Cohen-Macaulay* if $\dim M = \operatorname{depth} M$.

In [LW18a] Lärkäng and Wulcan consider very general so-called residue currents and proved a theorem generalising Theorem 4.3.4 to Cohen-Macaulay modules[2]. We are only interested in the zero-dimensional case of Artinian modules.

Lemma 4.1.14. *A ring $R \neq 0$ is Artinian precisely if it is Noetherian and* $\dim R = 0$*.*

Corollary 4.1.15. *If M is module over a Noetherian ring then M is Artinian precisely if* $\dim M = 0$*.*

Proposition 4.1.16. *Any Artinian module is Cohen-Macaulay, i.e.* $\operatorname{depth}_A M = 0$*.*

The reason we restrict to Artinian modules is that these correspond to zero-dimensional varieties, i.e. points with multiplicity, and for these we do not need to appeal to the more advanced analytic constructions of residue currents. Indeed, the analytic theory we have already presented, together with some homological algebra, will suffice to generalise the construction of point residues.

The Auslander-Buchsbaum formula

By the Syzygy theorem we saw that the projective dimension of any module over $\mathscr{O}$ was bounded from above by the dimension n of $\mathbb{C}^n$. For a given module this can be made even more specific by invoking the following formula.

Lemma 4.1.17 (Auslander-Buchsbaum)**.** *Let $(A, \mathfrak{m})$ be a local ring and M a finite module of finite projective dimension. Then the projective dimension satisfies*

$$\operatorname{pd}_A M = \operatorname{depth} A - \operatorname{depth} M.$$

In particular, if M is Cohen-Macaulay over a Cohen-Macaulay ring then

$$\operatorname{pd}_A M = \dim A - \dim M. \tag{4.2}$$

Since $\mathscr{O}$ is a regular n-dimensional local ring it is Cohen-Macaulay[3], as $\mathfrak{m}$ is generated by the coordinates. Moreover, by the Syzygy theorem, any finite $\mathscr{O}$-module has finite projective dimension bounded by n. Hence, we may apply the Auslander-Buchsbaum formula for any finite $\mathscr{O}$-module. The case we are interested in is when the module is zero-dimensional.

Corollary 4.1.18. *If M is an Artinian $\mathscr{O}$-module then*

$$\operatorname{pd}_{\mathscr{O}} M = n.$$

In particular, M has a minimal free resolution of length n.

[2] Indeed in [LW18a] the authors study even more generally residues associated to cycles of complexes of vector bundles.

[3] In fact, this is true for any regular local ring and Auslander-Buchsbaums formula holds for any finite module over such a ring.

This last result is very useful to us since it allows us to define all of our residues using resolutions of length precisely n.

4.2 Residues of Artinian modules

Consider an artinian module M over $\mathscr{O}$. By the Auslander-Buchsbaum formula (Corollary 4.1.18, see also the paragraph after Lemma 4.1.17) any minimal free resolution of M has length n. Moreover, by Lemma 4.1.10 any free resolution is a direct sum of a minimal and a trivial resolution. We only consider resolutions of length precisely n. Let (E, φ) be such a resolution. The resolution gives an isomorphism $M \cong \mathscr{O}^{\oplus r}/\mathcal{J}$ where $r = \operatorname{rank} E_0$. We endow $E = \bigoplus E_l$ with a superstructure as in Section 3.2, and choose a basis for E and a trivial connection $D_E = \bigoplus D_{E_l}$ with respect to this basis. Define the form $D\varphi = D\varphi_1 \cdots D\varphi_n$, where D is the induced connection on $\mathscr{E}^\bullet(\operatorname{End} E)$ which is also endowed with a superstructure as before. We call a resolution with such a connection D a *hermitian free resolution*.

We also need the following proposition.

Proposition 4.2.1. *If M is an Artinian $\mathscr{O}$-module, then there is a complete intersection $(f) = (f_1, \ldots, f_n) \subseteq \operatorname{ann} M$.*

Proof. Since $\dim M = \dim(\mathscr{O}/\operatorname{ann} M) = 0$ (by Corollary 4.1.15) we see that $z_k^N \in \operatorname{ann} M$ for large enough N, i.e. $\operatorname{rad}\operatorname{ann} M = \mathfrak{m}$. We can therefore choose e.g. $f = z_1^N, \ldots, z_n^N$. ■

Let $I = (f) \subseteq \operatorname{ann} M$ be a zero-dimensional complete intersection, and let (K, ψ) be the Koszul complex of the regular sequence (f) (cf. Section 3.2). Choose an identification $a_0 : K_0^{\oplus r} \xrightarrow{\cong} E_0$. Since $M \cong \mathscr{O}^r/\mathcal{J}$ we see that $(\operatorname{ann} M)\mathscr{O}^{\oplus r} \subseteq \mathcal{J}$, and hence the inclusion $I \subseteq \operatorname{ann} M$ implies $I\mathscr{O}^{\oplus r} \subseteq \mathcal{J}$. After the identification a_0, the inclusion $I\mathscr{O}^{\oplus r} \subseteq \mathcal{J}$ induces a (unique) surjection $a_{-1} : (\mathscr{O}/I)^{\oplus r} \to M$ which is a factor of the map $\varphi_0 a_0$, by the homomorphism theorem. By Proposition 4.1.4, this surjection extends to a chain morphism $a : K^{\oplus r} \to E$ which is unique up to homotopy and such that a_0 is the identification $K_0^{\oplus r} \cong E_0$ we made above. Note that by construction a_{-1} is determined by the identification a_0, hence we may lift to a from a_0 directly.

With these constructions, we define the residue of M as follows.

Definition 4.2.2. The *residue of $\xi \in \mathscr{O}$ with respect to M* (after choosing the hermitian resolution (E,φ, D), the complete intersection (f) and the morphism a) is defined as

$$\operatorname{Res}_M^{E,f,a}[\xi] := \frac{1}{n!} \operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1} D\varphi \wedge a_n(e_{1\ldots n})\right)\right]. \tag{4.3}$$

Remark 4.2.3. The factor $n!$ is included in the definition to compensate for the fact that in the complete intersection case we have $D\varphi = n!df$ (cf. Proposition 3.2.8). We will, however, ignore this factor at present, and only include it again when proving the residue formula.
In fact, the residue does not depend on anything but the module M. This is not obvious and relies mainly on certain algebraic properties of the morphism a. We begin by showing that a different choice of the morphism a does not change the residue. Then, that it is independent of the regular sequence $f_1, \dots, f_n \in \operatorname{ann} M$. Though nontrivial, these two parts follow from the Transformation Law (Theorem 3.3.1) together with the fact (Proposition 2.2.4) that $(f) = \operatorname{ann} \operatorname{Res}_f$. We then show independence of the hermitian resolution in two steps by using a property of the morphism a and the fact that a minimal resolution is a direct summand of any resolution.
Proposition 4.2.4. *Given an identification $a_0 : K_0^{\oplus r} \xrightarrow{\cong} E_0$, the residue $\operatorname{Res}_M^{E,f,a}$ is independent of the induced morphism $a : K^{\oplus r} \to E$.*

Proof. Suppose that $a, b : K^{\oplus r} \to E$ are liftings such that $a_0 = b_0$, then the surjections coincide $a_{-1} = b_{-1}$. By homotopy uniqueness of liftings (Proposition 4.1.4) we have that $a \simeq b$, so that there is a homotopy $s : K^{\oplus r} \to E[+1]$ such that

$$a - b = \varphi s + s\psi.$$

In particular, $a_n - b_n = \varphi_{n+1} s_n + s_{n-1}\psi_n$. Since $s_n = 0$ as $E_{n+1} = 0$, we get that $a_n - b_n = s_{n-1}\psi_n$. Therefore

$$\left(\operatorname{Res}_M^{E,f,a} - \operatorname{Res}_M^{E,f,b}\right)[\xi] = \operatorname{Res}_f\left[\xi \operatorname{tr}\left(a_0^{-1} D\varphi \wedge s_{n-1}\psi_n(e_{1\dots n})\right)\right] = 0$$

where the last equality follows by Proposition 2.2.4. ■

We therefore replace the a in notation with a_0.
Proposition 4.2.5. *The residue $\operatorname{Res}_M^{E,f,a_0}$ is independent of the choice of complete intersection f.*

Proof. Suppose we have two complete intersections $(f_1, \dots, f_n)$, $(g_1, \dots, g_n) \subseteq \operatorname{ann} M$. We may assume that $(f) \subseteq (g)$, as the Nullstellensatz guarantees that for some large integer N we have $(f_1^N, \dots, f_n^N) \subseteq (g)$ and hence both f and g may be compared with $(f_1^N, \dots, f_n^N)$ using the Tranformation law (Theorem 3.3.1). Let (F, ψ) be the Koszul complex of f and (G, γ) the Koszul complex of g. Let $a : F^{\oplus r} \to E$ and $b : G^{\oplus r} \to E$ be the induced morphisms.
By the assumption $(f) \subseteq (g)$, there is a germ of a holomorphic matrix A such

that $f = gA$. Applying the transformation law $\operatorname{Res}_g = (\det A)\operatorname{Res}_f$, and hence

$$\begin{aligned}\operatorname{Res}_M^{E,\,g,\,b_0}[\xi] &= \operatorname{Res}_g\left[\xi \operatorname{tr}\left(b_0^{-1}D\varphi \wedge b_n(e_{1\dots n})\right)\right] \\ &= \operatorname{Res}_f\left[(\det A)\,\xi \operatorname{tr}\left(b_0^{-1}D\varphi \wedge b_n(e_{1\dots n})\right)\right]\end{aligned} \tag{4.4}$$

By the property $\operatorname{tr} cA = c \operatorname{tr} A$ for a matrix A and a scalar c, we may move the factor $\det A$ inside the trace in (4.4) to get

$$\operatorname{Res}_M^{E,\,g,\,b_0}[\xi] = \operatorname{Res}_f\left[\xi \operatorname{tr}\left((\det A)\, b_0^{-1}D\varphi \wedge b_n(e_{1\dots n})\right)\right] \tag{4.5}$$

The induced morphism $K.(A) : (F, \psi) \to (G, \gamma)$ induces a morphism $K.(A)^{\oplus r}$: $(F^{\oplus r}, \psi^{\oplus r}) \to (G^{\oplus r}, \gamma^{\oplus r})$ and by (4.5) we have

$$\operatorname{Res}_f\left[\xi \operatorname{tr}\left((\det A)\, b_0^{-1}D\varphi \wedge a_n(e_{1\dots n})\right)\right] = \operatorname{Res}_f\left[\xi \operatorname{tr}\left(b_0^{-1}D\varphi \wedge b_n \wedge K_n(A)^{\oplus r}(e_{1\dots n})\right)\right].$$

Finally, $b \wedge K.(A)^{\oplus r}$ is also a lifting of a_{-1} since $K_0(A) : \mathscr{O} \to \mathscr{O}$ is the identity morphism, so that in fact $a_0 = b_0 K_0(A)^{\oplus r} = b_0$. Hence, by appealing to Proposition 4.2.4 we get

$$\operatorname{Res}_f\left[\xi \operatorname{tr}\left(b_0^{-1}D\varphi \wedge b_n \wedge K_n(A)^{\oplus r}(e_{1\dots n})\right)\right] = \operatorname{Res}_f\left[\xi \operatorname{tr}\left(a_0^{-1}D\varphi \wedge a_n(e_{1\dots n})\right)\right]$$

and it follows that $\operatorname{Res}_M^{E,\,g,\,b_0} = \operatorname{Res}_M^{E,\,f,\,a_0}$ as required. ■

We therefore drop the dependence of f in the notation for the residue.

We must now show that the residue is independent of the resolution E of M. In the case when M is cyclic, this is due to the fact that the functional

$$\operatorname{Res}_f[\omega \wedge a_n(e_{1\dots n})] \tag{4.6}$$

defined for n-forms ω is annihilated by $\operatorname{ann} M$, a property similar to Proposition 2.2.4. Proving this claim is a bit more involved and relies on a result in the theory of *linkage*.

Suppose M is a finite module, so that $M \cong \mathscr{O}^{\oplus r}/\mathcal{J}$, for a submodule $\mathcal{J} \subseteq \mathscr{O}^{\oplus r}$. Define $I : \mathcal{J}$ to be the module

$$I : \mathcal{J} = \{g \in \operatorname{Hom}(\mathscr{O}^{\oplus r}, \mathscr{O}) : g(\mathcal{J}) \subseteq I\}$$

which coincides with the usual notion of colon ideal for an ideal $J \subseteq \mathscr{O}$.

Lemma 4.2.6. *Suppose $M = \mathscr{O}^{\oplus r}/\mathcal{J}$ is an Artinian module. Let K be the Koszul complex of a complete intersection $I = (f) \subseteq \operatorname{ann} M$. Let E be a free resolution of M*

and $a : K^{\oplus r} \to E$ be the associated morphism. Then

$$I : K_n \otimes \mathcal{J} = I(K_n^{\oplus r})^* + (E_n^*)a_n.$$

In particular, if $M = \mathscr{O}/J$, then

$$I : J = I + \mathcal{I}(a_n)$$

where $\mathcal{I}(a_n)$ is the ideal generated by the entries of a_n.

Proof. We refer the reader to [Lär19b, Lemma 4.6] for a proof of this result. ■

We indicate briefly how we apply this seemingly technical lemma. In our comparison of residues given by different resolutions, we end up with a form $\operatorname{tr} a_0^{-1}\varphi_1\mu \wedge a_n$. As we have an isomorphism $e_{1\ldots n} \otimes a_0^{-1} : E_0 \xrightarrow{\cong} K_n^{\oplus r}$, we can identify $\operatorname{im} \varphi_1 = \mathcal{J}$ with the submodule $K_n \otimes \mathcal{J}$. The lemma together with $\operatorname{ann} \operatorname{Res}_f = I$ then imply that if $g \in \operatorname{Hom}(E_n, \mathscr{O})$ then

$$\operatorname{Res}_f[g \wedge a_n e_{1\ldots n} a_0^{-1}\varphi_1] = 0$$

and in particular

$$\operatorname{Res}_f\left[\operatorname{tr}\left(\mu \wedge a_n e_{1\ldots n} a_0^{-1}\varphi_1\right)\right] = 0. \tag{4.7}$$

We now have the means to prove that the residue depends only on the module M. First, we show that this holds for two resolutions with the same rank at level 0. Then, we show this in general by relating a free resolution to the minimal part by comparing both to a third resolution. This approach was adopted to accommodate for the resolution constructed in Theorem 4.3.2, which is not minimal at level 0 in the cases of interest. In [LW18a] they showed the corresponding formula in Theorem 4.3.2 by constructing a resolution of length $n+1$ but with minimal rank at level 0.

Lemma 4.2.7. *If E and E' are hermitian resolutions of M such that* $\operatorname{rank} E_0 = \operatorname{rank} E_0'$, *then they define the same residue.*

In particular, the residue does not depend on the connection D on E or the identification $a_0 : K_0^{\oplus r} \xrightarrow{\cong} E_0$.

Proof. For the first part, let (E, φ) and (E', φ') be free resolutions of equal rank on level 0 and (K, ψ) a Koszul complex as before, with a, a' the associated morphisms from the Koszul complex to E and E' respectively. We begin by relating the differential forms $D\varphi$ and $D\varphi'$ as follows. There is an identification $b_0 : E_0' \xrightarrow{\cong} E_0$ such that $b_0 a_0 = a_0'$, defined by $b_0 = a_0' a_0^{-1}$. Then any lifting $b : E \to E'$ of b_0 gives

a morphism $b \wedge a' : K^{\oplus r} \to E$ which is homotopic to a, since they agree on level 0 by construction.
A calculation with the superstructure (see [LW18b, Lemma 4.4]) yields

$$D\varphi \wedge b_n = b_0 \wedge D\varphi' + \varphi_1 \wedge \mu + \nu \wedge \varphi'_n \tag{4.8}$$

for some Hom-valued sections μ and ν.
By using that $\operatorname{tr} AB = \operatorname{tr} BA$ and Equation (4.7) we see that

$$\operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1}\varphi_1 \wedge \mu \wedge a'_n e_{1\ldots n}\right)\right] = \operatorname{Res}_f \left[\xi \operatorname{tr}\left(\mu \wedge a'_n e_{1\ldots n} a_0^{-1}\varphi_1)\right)\right] = 0. \tag{4.9}$$

We also get

$$\operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1}\nu \wedge \varphi'_n a'_n e_{1\ldots n}\right)\right] = \operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1}\nu \wedge a'_{n-1}\psi_n e_{1\ldots n}\right)\right] = 0 \tag{4.10}$$

by Proposition 2.2.4.
Thus, we only have to compare the residues with respect to f of the forms associated to $D\varphi \wedge b_n$ and $b_0 \wedge D\varphi'$. Since $b \wedge a' \simeq a$, we may apply Proposition 4.2.4 to see that

$$\begin{aligned} \operatorname{Res}_M^{E,\,a_0}[\xi] &= \operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1} D\varphi \wedge a_n e_{1\ldots n}\right)\right] \\ &= \operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1} D\varphi \wedge b_n \wedge a'_n e_{1\ldots n}\right)\right]. \end{aligned} \tag{4.11}$$

Furthermore, since $b_0 a'_0 = a_0$ we have $(a'_0)^{-1} = a_0^{-1} b_0$, and it follows that

$$\operatorname{Res}_M^{E',\,a'_0}[\xi] = \operatorname{Res}_f \left[\xi \operatorname{tr}\left(a_0^{-1} b_0 \wedge D\varphi' \wedge a'_n e_{1\ldots n}\right)\right]. \tag{4.12}$$

To conclude, the equalities in (4.8) — (4.12) together imply that $\operatorname{Res}_M^{E,\,a_0} = \operatorname{Res}_M^{E',\,a'_0}$. ■

We therefore drop the dependence of a_0 in the notation.
We now prove that the minimal part of the resolution is the only contributing factor in the residue.

Lemma 4.2.8. *The residue is independent of the free resolution.*

Proof. We prove this by comparing any resolution (E, φ) to a minimal resolution (F, ε) which is a direct summand of E. This is done by creating a new complex H from F which satisfies $\operatorname{rank} H_0 = \operatorname{rank} E_0$, whence $\operatorname{Res}_M^H = \operatorname{Res}_M^E$ by Lemma 4.2.7. We then show that the residue corresponding to the complex H captured completely by the minimal part F.
Suppose therefore $\operatorname{rank} E_0 = r > \operatorname{rank} F_0 = s$ and consider the trivial complex

(G, γ) defined by

$$0 \longrightarrow \mathscr{O}^{\oplus t} \xrightarrow{id} \mathscr{O}^{\oplus t} \longrightarrow 0$$

where $t = r - s$. Let $(H, \eta) = (F \oplus G, \varepsilon \oplus \gamma)$. Then rank $H_0 = s + t = r$ as required and H defines the same residue as E. To compare it with F, endow H with the direct sum connection $D_H = D_F \oplus D_G$, where D_G is the trivial connection on G in the given basis.

Let (K, ψ) be a Koszul complex and $b_0 : K_0^{\oplus s} \overset{\cong}{\to} F_0$ an identification inducing a morphism $b : K^{\oplus s} \to F$. Since the residue does not depend on the morphism $a : K^{\oplus r} \to H$ or the Koszul complex K, we consider a new morphism by choosing the identification $a_0 = b_0 \oplus \alpha : K^{\oplus r} \overset{\cong}{\to} H_0$ where $\alpha : K_0^{\oplus t} \overset{\cong}{\to} \mathscr{O}^{\oplus t}$ is some identification. Then a lifting a of a_0 is given by $a_1 = b_1 \oplus \alpha$ and $a_k = b_k \oplus 0$, for $k \geqslant 2$.

Before calculating $D\gamma = D\gamma_1$, we begin by recalling that the odd endomorphism γ_1 on G induces an endomorphism on $\mathscr{E}_X^{\boldsymbol{\cdot}}(G) = \mathscr{E}_X^{\boldsymbol{\cdot}} \otimes E_X(G)$ with the superstructure as before (cf. Section 3.2) which acts on homogeneous elements $\omega \otimes \xi$ by

$$\gamma_1(\omega \otimes \xi) := (-1)^{(\deg_e \gamma)(\deg \omega)} \omega \otimes \gamma_1(\xi) = (-1)^{\deg \omega} \omega \otimes \xi. \tag{4.13}$$

Since D_G is trivial with respect to the basis

$$D(\gamma_1(\omega \otimes \xi)) = (-1)^{\deg \omega} d\omega \otimes \xi \tag{4.14}$$

$$\gamma_1(D(\omega \otimes \xi)) = (-1)^{\deg d\omega} d\omega \otimes \xi. \tag{4.15}$$

Now, the induced connection on $\mathscr{E}_X^{\boldsymbol{\cdot}}(\operatorname{End} G)$ is then by definition (cf. Section 3.2)

$$D\gamma_1(\omega \otimes \xi) := D(\gamma_1(\omega \otimes \xi)) - (-1)^{\deg \gamma_1} \gamma_1(D(\omega \otimes \xi)) \tag{4.16}$$

$$= (-1)^{\deg \omega} d\omega \otimes \xi + (-1)^{\deg d\omega} d\omega \otimes \xi \tag{4.17}$$

and since $\deg d\omega = \deg \omega + 1$ if $d\omega \neq 0$, we see that $D\gamma = 0$. This implies that $D\eta = D\varepsilon \oplus D\gamma = D\varepsilon$. Therefore, the differentials

$$a_0^{-1} D\eta \wedge a_n = b_0^{-1} D\varepsilon \wedge b_n \tag{4.18}$$

representing the residue with respect to H and F, respectively, coincide, whence $\operatorname{Res}_M^H = \operatorname{Res}_M^F$. This implies $\operatorname{Res}_M^E = \operatorname{Res}_M^F$, as required. ∎

4.3 Residues and intersection multiplicities

In this section we prove the main result of the thesis, namely that the residue of $\zeta \in \mathscr{O}$ with respect to M satisfies the following residue formula

$$\operatorname{Res}_M[\zeta] = (\dim_{\mathbb{C}} M)\, \zeta(0) \tag{4.19}$$

so that in particular we obtain the intersection multiplicity in the case when M is a cyclic module $\mathscr{O}/J$. The idea is to disassemble the module M using a so-called *composition series*. One then shows that the residue behaves like the dimension with respect to short exact sequences. Finally, by induction we need only the calculation

$$\operatorname{Res}_{\mathfrak{m}}[\zeta] = \zeta(0) \tag{4.20}$$

for the residue of ζ with respect to $\mathscr{O}/\mathfrak{m}$. This calculation is just the Cauchy integral formula at the origin applied to the function ζ.

Lemma 4.3.1. *An Artinian $\mathscr{O}$-module M with $\dim_{\mathbb{C}} M = \ell$ has a composition series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\ell-1} \subseteq M_\ell = M$$

with $M_j/M_{j-1} \cong \mathscr{O}/\mathfrak{m}$ for $j = 1, \ldots, \ell$.

Proof. We construct a composition series as follows. Let $\{v_1, \ldots, v_\ell\}$ be a basis for M as a complex vector space. Let $\{v_{i_1}, \ldots, v_{i_k}\}$ be a minimal subset generating M as an $\mathscr{O}$-module. Take $M_{\ell-1}$ to be the module generated by $\{v_1 \ldots, v_\ell\} \setminus \{v_{i_k}\}$. We claim that $M_{\ell-1}$ is a maximal proper submodule. Note first that $M_{\ell-1} \subsetneq M$ as we have excluded a generator, so in particular we see that $\dim_{\mathbb{C}} M_{\ell-1} = \ell - 1$. Now, any chain $L \subsetneq N \subseteq M$ satisfies $\dim_{\mathbb{C}} L < \dim_{\mathbb{C}} N \leqslant \ell$ and hence $M_{\ell-1}$ can not be properly contained in a proper submodule of M as it is maximal as a subspace. The result follows by induction. ■

Theorem 4.3.2. *For a short exact sequence of Artinian $\mathscr{O}$-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

the residues are related by $\operatorname{Res}_M = \operatorname{Res}_L + \operatorname{Res}_N$.

We provide a proof based on the construction of a resolution of M from resolutions of L and N.

Lemma 4.3.3 (Horseshoe Lemma)**.** *A diagram*

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & E_1 \arrow[r, "\varepsilon_1"] & E_0 \arrow[r, "\varepsilon_0"] & L \arrow[r] \arrow[d] & 0 \\
 & & & M \arrow[d] & \\
\cdots \arrow[r] & G_1 \arrow[r, "\gamma_1"] & G_0 \arrow[r, "\gamma_0"] & N \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}
$$

where the column is exact and where the rows are projective resolutions, can be completed to a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & E_1 \arrow[r, "\varepsilon_1"] \arrow[d] & E_0 \arrow[r, "\varepsilon_0"] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F_1 \arrow[r, "\varphi_1"] \arrow[d] & F_0 \arrow[r, "\varphi_0"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & G_1 \arrow[r, "\gamma_1"] \arrow[d] & G_0 \arrow[r, "\gamma_0"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

with exact columns and where the middle row is also a projective resolution. Moreover, $F_0 = E_0 \oplus G_0$ $\varphi_0 = \varepsilon_0 \oplus \gamma_0'$, for some lifting $\gamma_0' : G_0 \to M$ of γ_0. At each level $\ell \geqslant 1$, we get a similar splitting $F_\ell = E_\ell \oplus G_\ell$, and φ_ℓ is given as a matrix

$$\varphi_\ell = \begin{pmatrix} \varepsilon_\ell & \lambda_\ell \\ 0 & \gamma_\ell \end{pmatrix}.$$

The lemma is proven by successively lifting γ and applying the Snake lemma.

Proof of Theorem 4.3.2. A priori, the residues are already defined using some free resolutions of our modules, but by invoking Lemma 4.2.8 we may choose any resolutions we want. In particular, the Horseshoe lemma gives a free resolution (F, φ) of M from free resolutions (E, ε) and (G, γ) of L and N respectively. Moreover, this free resolution fits into a short exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

Suppose also that E, G are hermitian resolutions and note that since $F_k = E_k \oplus G_k$ we get that as a bundle $F = E \oplus G$ and hence we may equip F with the direct sum connection $D = D_E \oplus D_G$.
There is a zero-dimensional complete intersection $I = (f) \subseteq \operatorname{ann} M \subseteq \operatorname{ann} L \cap \operatorname{ann} N$ by Proposition 4.2.1. Let K be the Koszul complex associated to f. Let $a_0 : K_0^{\oplus s} \overset{\cong}{\to} E_0$ and $c_0 : K_0^{\oplus t} \overset{\cong}{\to} G_0$ be identifications. By Proposition 4.1.4 there exists morphisms a and c of complexes extending a_0 and c_0 respectively. Since the sequence of resolutions is free, it splits at each level. We therefore get an identification $b_0 : K_0^{\oplus r} \to F_0$ by defining $b_0 = a_0 \oplus c_0$. Note also that the Koszul complex $(K^{\oplus r}, \psi)$ splits as $(K^{\oplus s} \oplus K^{\oplus t}, \psi' \oplus \psi'')$. We show how to extend b_0 to a chain morphism $b : K^{\oplus r} \to F$ that is easy to work with by studying the commutation relations $\varphi b = b\psi = b(\psi' \oplus \psi'')$. Since the complex splits at each level we see that b_ℓ may be represented as a 2×2 matrix of homomorphisms

$$b_\ell = \begin{pmatrix} d_\ell & g_\ell \\ h_\ell & e_\ell \end{pmatrix}.$$

Furthermore, since we defined $b_0 = a_0 \oplus c_0$, we have $d_0 = a_0$, $e_0 = c_0$ and $g_0 = h_0 = 0$. We begin by constructing b_1 by using the relation $\varphi_1 b_1 = b_0 \psi_1$.
Recall that $\varphi_0 = \varepsilon_0 \oplus \gamma_0'$ for some lifting $\gamma_0' : G_0 \to M$ of γ_0, and that generally φ_ℓ can be described by a matrix

$$\varphi_\ell = \begin{pmatrix} \varepsilon_\ell & \lambda_\ell \\ 0 & \gamma_\ell \end{pmatrix}$$

for $\ell \geqslant 1$. The equation $\varphi_1 b_1 = b_0 \psi_1$ translates to

$$\begin{pmatrix} \varepsilon_1 d_1 + \lambda_1 h_1 & \varepsilon_1 g_1 + \lambda_1 e_1 \\ \gamma_1 h_1 & \gamma_1 e_1 \end{pmatrix} = \begin{pmatrix} a_0 \psi_1' & 0 \\ 0 & c_0 \psi_1'' \end{pmatrix}. \tag{4.21}$$

Let $h_1 = 0$, then we may choose $d_1 = a_1$ and $e_1 = c_1$ since a and c are chain maps. It remains therefore to show that we can define g_1 such that $\varepsilon_1 g_1 + \lambda_1 c_1 = 0$. Since $\varphi_0 \varphi_1 = 0$, we see that $\varepsilon_0 \lambda_1 = 0$. In particular

$$-\varepsilon_0 \lambda_1 c_1 = 0 \tag{4.22}$$

i.e. $\operatorname{im} \lambda_1 c_1 \subseteq \ker \varepsilon_0 = \operatorname{im} \varepsilon_1$. Hence, if $x \in K_1''$, we may define $g_1(x)$ to be any

$y \in E_1$ such that $\varepsilon_1(y) = -\lambda_1 c_1(x)$. Then, by construction $\varepsilon_1 g_1(x) + \lambda_1 c_1(x) = 0$. Inductively, we do a similar diagram chase to find an upper triangular matrix

$$b_{\ell+2} = \begin{pmatrix} a_{\ell+2} & g_{\ell+2} \\ 0 & c_{\ell+2} \end{pmatrix}.$$

Indeed, $b_{\ell+2}$ can be chosen like this if we have b_ℓ and $b_{\ell+1}$ of the same form, since we may find $g_{\ell+2}$ by moving around the terms in the commutation relation using the property

$$\varepsilon\lambda + \lambda\gamma = 0.$$

Since φ is upper triangular and D is the direct sum connection, the differential form $D\varphi$ has a matrix representation

$$D\varphi = \begin{pmatrix} D\varepsilon & \omega \\ 0 & D\gamma \end{pmatrix}$$

where ω is some form that is not of interest for our calculations. Therefore, we see that

$$\begin{aligned} \operatorname{tr}\left(b_0^{-1} D\varphi \wedge b_n\right) &= \operatorname{tr}\begin{pmatrix} a_0^{-1} D\varepsilon \wedge a_n & \omega' \\ 0 & c_0^{-1} D\gamma \wedge c_n \end{pmatrix} \\ &= \operatorname{tr}\left(a_0^{-1} D\varepsilon \wedge a_n\right) + \operatorname{tr}\left(c_0^{-1} D\gamma \wedge c_n\right) \end{aligned}$$

whence the statement $\operatorname{Res}_M = \operatorname{Res}_L + \operatorname{Res}_N$ follows. ■

Theorem 4.3.4. *The residue satisfies* $\operatorname{Res}_M[\xi] = (\dim_{\mathbb{C}} M)\, \xi(0)$. *Writing this out, we get an integral formula*

$$\int \bar{\partial}\chi_\varepsilon \wedge \xi \operatorname{tr}(a_0^{-1} D\varphi \wedge a_n) \wedge u_n^f = (2\pi i)^n n! (\dim_{\mathbb{C}} M) \xi(0) \tag{4.23}$$

generalising the argument principle.

In the particular case when $M = \mathscr{O}/(f_1, \ldots, f_N)$ we recall that the algebraic multiplicity of the vanishing at the origin for f was defined precisely as $\mathbf{D}(f) = \dim_{\mathbb{C}} M$ and (4.23) gives a relation between the residue integral to this multiplicity.

Proof. Applying Proposition 3.2.8 and Corollary 3.2.9 we get

$$\operatorname{Res}_{\mathfrak{m}}[\xi] = \frac{1}{n!} \operatorname{Res}_z[n! \xi dz] = \xi(0)$$

where the last equality follows from Cauchy's integral formula.

The result now follows from the existence of a composition series of M. Indeed, for a composition series, the inclusions induce short exact sequences

$$0 \longrightarrow M_k \longrightarrow M_{k+1} \longrightarrow \mathscr{O}/\mathfrak{m} \longrightarrow 0 \tag{4.24}$$

and hence we may apply Theorem 4.3.2 to see that $\mathrm{Res}_{M_{k+1}} = \mathrm{Res}_{M_k} + \mathrm{Res}_{\mathfrak{m}}$. Induction together with the fact that $\mathrm{Res}_{M_1} = \mathrm{Res}_{\mathfrak{m}}$ concludes the proof, since the series has length $\dim_{\mathbb{C}} M$. ■

Example 4.3.5. We give an example calculating the residue for a non-complete intersection in $\mathscr{O} = {}_2\mathscr{O}_0$. Consider the ideal $J = (z_1^2, z_1 z_2, z_2^2) \subseteq \mathscr{O}$ and note that J is zero-dimensional but not a complete intersection. Indeed, note that none of the generators are redundant, and hence J is minimally generated by 3 elements, which is seen e.g. by a corollary of Nakayama's lemma that any two minimally generating sets of a module over a local ring have the same cardinality. But since $\mathscr{O}$ is 2-dimensional the maximal length of a regular sequence is 2. Therefore, J is not a complete intersection and we must use Definition 4.2.2 to define a residue for $\mathscr{O}/J$. We do this and demonstrate the residue formula (4.23).
Now, choose the complete intersection $I = (z_1^2, z_2^2) \subseteq J$. The cyclic module $\mathscr{O}/J$ has a (minimal) free resolution

$$0 \longrightarrow \mathscr{O}^2 \xrightarrow{\varphi_2} \mathscr{O}^3 \xrightarrow{\varphi_1} \mathscr{O} \longrightarrow \mathscr{O}/J \longrightarrow 0 \tag{4.25}$$

where

$$\varphi_1 = \begin{pmatrix} z_1^2 & z_1 z_2 & z_2^2 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} -z_2 & 0 \\ z_1 & -z_2 \\ 0 & z_1 \end{pmatrix}.$$

Let (K, ψ) be the Koszul complex

$$0 \longrightarrow \mathscr{O} \xrightarrow{\begin{pmatrix} z_2^2 \\ -z_1^2 \end{pmatrix}} \mathscr{O}^{\oplus 2} \xrightarrow{\begin{pmatrix} z_1{}^2 & z_2{}^2 \end{pmatrix}} \mathscr{O} \longrightarrow \mathscr{O}/I \longrightarrow 0 \tag{4.26}$$

associated to the regular sequence (z_1^2, z_2^2). We find the morphism a by solving the commutation relations. Note that a_0 is the identity and a_{-1} is the natural surjection $\mathscr{O}/I \to \mathscr{O}/J$. We claim that one can choose a_1 and a_2 to be

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}.$$

We now calculate the differential $D\varphi \wedge a_2$ (where the sign comes from the super-structure in matrix notation, see [LW18b, section 2.5])

$$\begin{aligned}
D\varphi_1 D\varphi_2 a_2 &= (-1)^{1\cdot 1} \begin{pmatrix} d(z_1^2) & d(z_1 z_2) & d(z_2^2) \end{pmatrix} \begin{pmatrix} -dz_2 & 0 \\ dz_1 & -dz_2 \\ 0 & dz_1 \end{pmatrix} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} \\
&= (-1) \begin{pmatrix} -d(z_1^2) \wedge dz_2 + d(z_1 z_2) \wedge dz_1 & -d(z_1 z_2) + d(z_2^2) \wedge dz_1 \end{pmatrix} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} \\
&= 3 \cdot 2 z_1 z_2 dz_1 \wedge dz_2.
\end{aligned}$$

Finally, we compute the residue

$$\begin{aligned}
\mathrm{Res}_M[\xi] = \frac{1}{2} \mathrm{Res}_{(z_1^2\ z_2^2)}[3 \cdot 2 z_1 z_2 \xi dz_1 \wedge dz_2] &= \frac{3}{(2\pi i)^2} \int_\Gamma \frac{z_1 z_2 \xi dz_1 dz_2}{z_1^2 z_2^2} \\
&= \frac{3}{(2\pi i)^2} \int_\Gamma \frac{\xi dz_1 dz_2}{z_1 z_2} = 3 \cdot \xi(0)
\end{aligned}$$

which is precisely what we expected, since $\dim_{\mathbb{C}} \mathscr{O}/J = 3$ as it has as a basis the functions $1, z_1, z_2$ modulo J. ■

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