GÖTEBORGS UNIVERSITET

# What's the Name of the Game?

## Dialogue Game Semantics for Intuitionistic Modal Logic with Strict Implication


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In this thesis we will develop dialogue game semantics for intuitionistic modal logic with strict implication. We begin by introducing Kripke semantics and proof systems for intuitionistic modal logic. Afterwards, we define a dialogue semantics for classical and intuitionistic propositional logic similar to the one found in [1] and a dialogue semantics for modal logic based on the work in [11]. We then discuss a suitable way to include strict implication in this framework. However, when adding strict implication to the existing dialogue semantics for intuitionistic modal logic, the interdefinability of strict implication and the box operator is valid, even though it does not hold in intuitionistic modal logic. We suggest two ways of fixing this issue. First, we adapt the dialogue semantics for intuitionistic modal logic to mirror the approach that is taken for the Kripke semantics. This involves adapting the dialogue semantics for classical modal logic into dialogue semantics for intuitionistic propositional logic and then creating a Bi-Modal dialogue semantics for intuitionistic modal logic. We then add strict implication to this system and show that the resulting semantics is sound and complete with respect to intuitionistic modal logic. The second approach that we explore is to restrict the way players may use previous information in our dialogue games. We then show that these two approaches are equivalent. Finally we discuss the benefits of both systems and discuss directions for further research.

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*All games are meaningless if you do not know the rules.*

B.K.S. IYENGAR

# CONTENTS

# 1 Introduction

In this thesis we will introduce dialogue game semantics for intuitionistic modal logic with strict implication. We will give a brief introduction to each of the three components of this topic before we motivate their combination.

## 1.1 Strict Implication

Lewis [8] originally introduced strict implication because he believed that the traditional material implication did not accurately capture the meaning that implication has in our natural language. According to [3], he believed that an implication being true should not solely be dependant on the truth values of the antecedent and the conclusion. Instead, it should be a function on the meaning of a proposition which determines whether it implies the consequent or not. The statement "A implies B" conveys that the proposition $B$ can somehow be deduced from the information in proposition $A$. Lewis defined strict implication $\prec$ to express the relation that a valid deduction of the consequent is possible from the conclusion. Specifically, the satisfaction clause for $\prec$ is

$$w \Vdash A \prec B \text{ iff for every } v \text{ with } wRv \text{ if } v \Vdash A \text{ then } v \Vdash B.$$

The systems that Lewis created to give an account of strict implication are nowadays used as the basis for modal logic. However, strict implication has mostly been replaced by the unary modal operators $\Box$ and $\Diamond$. Lewis had also introduced the $\Box$ operator as a defined operator in his systems, through

$$\Box A = \top \prec A.$$

In fact, $\prec$ and $\Box$ are interdefinable through

$$A \prec B = \Box(A \to B).$$

While Lewis' systems had various problems, see also [9], which kept them from having the impact he probably hoped, modal logic has since gone on to be very successful.

Even though strict implication is no longer commonly used in modal logic, the desire to have a relation between two propositions which conveys that the antecedent is relevant to proving the consequent can still be found in logic. These include for example linear logic, relevance logic and the logic of bunched implication [9]. Additionally, as we will discuss in more detail in the next section, strict implication behaves differently in intuitionistic logic, which is motivation enough to study it further.

## 1.2 INTUITIONISTIC MODAL LOGIC

Intuitionism was founded by Brouwer who was motivated by the idea that the meaning of logical statements should come from the act of proving them. Heyting later formalised this sentiment into intuitionistic logic, which has been a very successful area of logic ever since. As [15] points out, it is mathematically natural to want to combine intuitionistic logic with the even more successful modal logic. Other motivations to study intuitionistic modal logic include computer science applications and philosophical ideas. Some early developments of intuitionistic modal logic include for example [14], who provided an embedding of intuitionistic modal logic into bi-modal classical logic. A more in depth survey of the work that has been done in this area can be found in [15].

Many of the sources which discuss intuitionistic modal logic with strict implication are actually from provability and preservativity logic, see for example [6], [7] and [18]. This is also the context in which it was first discussed that $\prec$ is more expressive than $\Box$ in intuitionistic logic. While $\Box$ is still definable by

$$\Box A \leftrightarrow (\top \prec A),$$

the equivalence

$$(A \prec\!\!\!\!\!\rightarrow B) \leftrightarrow \Box(A \rightarrow B)$$

no longer holds. While we will not go further into provability and preservativity logics here, they clearly provide a motivation to study intuitionistic logic with strict implication further.

## 1.3 Dialogue Game Semantics

Dialogue semantics were first introduced by Paul Lorenzen in his paper Logik und Agon [10]. He motivated this idea by stating that logic originally became a field of interest in Ancient Greece as a tool to protect oneself from rhetoric tricks that others would use in an argument. In contrast to this, modern logic at the time of Lorenzen had become a “single player game” in which logicians attempt to uncover truths using logical rules as tools. Lorenzen proposed to bring back the argumentative origins of logic by introducing a semantics in which a proposition is valid iff a player can always convince an opponent that this proposition is true in a dialogue which follows certain rules.

The idea is to associate each logical statement with a finite two-player zero-sum game between a proponent who attempts to defend the thesis and an opponent who attacks it. Each move of a player consists either of them attacking a previous statement or defending against a previous attack. The validity of the statement is not decided by which player wins an individual play-through of the game but by which of the players has a winning strategy for the corresponding dialogue game. Since these dialogue games are set up to be finite two-player zero-sum games, we know that one of the players must have a winning strategy. If it is the proponent then the statement is valid, if it is the opponent then it is invalid.

## 1.4 Outline

By now dialogue semantics have been introduced for multiple different logics, including classical and modal logic, but the original dialogues that Lorenzen proposed were designed to model intuitionistic propositional logic. The idea that a proposition should be valid if one can convince an opponent to accept it is very similar to the intuitionistic principle that a statement is true if we can give a correct proof for it. Because of this, it is natural for us to consider this semantics when studying other types of intuitionistic logics as well. As we have already touched upon, intuitionistic modal logic with strict implication is a logic that finds applications in multiple areas so there is a motivation to study it further. As there are interesting results when combining between both strict implication and intuitionistic modal logic and dialogue game semantics and intuitionistic modal logic, it seems natural to study the combination of these three components as well. As we will see in the course of this thesis, combining these elements is not necessarily as trivial as we may first hope, which makes the systems we introduce an interesting result by themselves as well.

We will begin by introducing existing semantics and proof systems for intuitionistic modal logic, which we will use to show the validity of our dialogue systems in the later chapters. In the third chapter we will introduce some of the previous approaches to dialogue game semantics that can be found in the literature in more detail. We will present a dialogue semantics for propositional logic based on [1] and a dialogue semantics for modal logic based on [11]. We will also give the natural way to include strict implication in these systems. While this has, as far as we can tell, not been done before, it follows trivially from the satisfaction clause of strict implication. Once we have introduced these basics, we show that combining all these elements in the trivial way does not yield a valid dialogue game semantics for intuitionistic modal logic with strict implication. In the fourth chapter we introduce two alternative dialogue semantics for intuitionistic modal logic with strict implication which are both original contributions by this thesis. The first approach will be to translate the Kripke semantics that we have introduced in chapter 2 into a bi-modal dialogue semantics. This approach is novel only in the sense that it has not yet been formulated within dialogue game semantics, the technique of translating intuitionistic modal logic into bi-modal logic is well known and is explained for Kripke semantics in more detail in section 2.1. We show that this dialogue game semantics is sound and complete with respect to intuitionistic modal logic. The second approach will be to adapt a dialogue game for classical modal logic by restricting the way the players are allowed to use the information that is generated throughout the game. This approach requires some novel ideas on how to change the structure of the dialogue games we have introduced before. We will show that this dialogue game semantics is equivalent to the bi-modal one. Finally, in the conclusion we briefly discuss the benefits of each approach and highlight some further research that can be done.

Throughout the thesis we will be using $p,q$ or $p_1,p_2,\ldots$ to denote propositional variables and $A,B,C$ to denote formulas. We take $\mathcal{L}$ to be the language of (intuitionistic) propositional logic, with the connectives $\bot,\wedge,\vee,\to$. Additionally, we use $\mathcal{L}_\Box = \mathcal{L} \cup \{\Box\}$ and $\mathcal{L}_{\multimap} = \mathcal{L}_\Box \cup \{\multimap\}$. We define $\top = \bot \to \bot$.

# 2 Systems for Intuitionistic Modal Logic

In this chapter we present Kripke semantics, a sequent calculus and a Hilbert calculus for intuitionistic modal logic with strict implication. These are not necessarily the only systems that have been discussed in the literature, for example a natural deduction system for intuitionistic modal logic which can be expanded to include strict implication can be found in [15]. However, in this thesis we will only use the following systems alongside the dialogue game semantics that we will introduce in the next chapter.

## 2.1 Kripke Semantics

Ever since Gödel provided a translation of intuitionistic propositional logic into the modal logic S4 in a way that can preserve Kripke completeness, decidability and the finite model property [16], this connection has been used to study intuitionistic logic further. When Kripke semantics for classical modal logic were introduced, it was therefore natural that these could also be used to model intuitionistic propositional logic. Kripke models for classical modal logic are of the form $(W,R,V)$, where $W$ is a set of worlds, $R$ is a relation between these worlds and $V$ is a valuation function which assigns each propositional variable to a subset of $W$. In order to adapt these for intuitionistic propositional logic, we use Kripke models of the form $(W,\leq,V)$ where $\leq$ is a partial order on $W$ and $V$ is a valuation that assigns each propositional variable to a subset of $W$ so that monotonicity holds. By monotonicity we mean that for every propositional variable $p$, whenever $w \in V(p)$ and $w \leq v$ for $w,v \in W$ it must also hold that $v \in V(p)$. For an example, see figure 2.1. On an intuitive level, we can interpret each world $w \in W$ as a state of knowledge and $\leq$ as a relation on these worlds that respects the ordering by information content. When moving through such a Kripke model we may therefore gain new information, but never lose or refute information that we already had.

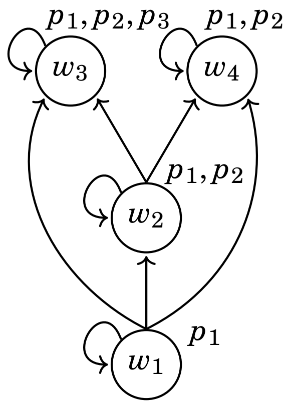


Figure 2.1: Kripke model for intuitionistic logic

The satisfaction clauses for $\bot, \wedge$ and $\vee$ are the same as classical logic and therefore defined locally in each world. In particular, each world locally uses classical reasoning. The satisfaction clause for $\rightarrow$ we use is

$$w \Vdash A \rightarrow B \text{ iff for all } v \text{ with } w \leq v \text{ if } v \Vdash A \text{ then } v \Vdash B.$$

This means that the truth value of a formula $w : A \rightarrow B$ is not determined locally in one world but rather in the upwards-closure of $w$ in $W$ with respect to $\leq$. As usual in intuitionistic logic, we define $\neg A$ as $A \rightarrow \bot$. We can easily see that if monotonicity holds for all propositional variables $p$, it also holds for all propositional formulas $A$.

When considering intuitionistic modal logic, it is natural to want the relation between intuitionistic propositional and modal logic to be similar to the one between classical propositional and modal logic. In order to model classical modal logic we use Kripke models in which possible worlds are accessible through an accessibility relation $R$. The accessibility relation is used to interpret the modalities while the other connectives are interpreted locally in each world.

It therefore seems obvious that intuitionistic modal logic is closely related to bi-modal classical logic. In fact, this embedding has already been formulated by

[14] in the late 1970s, quite some time before more general advances were made in the study of poly-modal logics. A more in depth introduction to such an embedding and an explicit translation can be found in [16].

The natural choice to model intuitionistic model logic is therefore a bi-relational Kripke model of the form $(W, \leq, R, V)$ where $\leq$ is used to interpret $\rightarrow$ as before and $R$ is used to interpret the modal operators.

While the satisfaction of $\rightarrow$ is not determined locally, it is independent of the modal relation $R$, which is sufficient in order to mirror the approach taken with classical modal Kripke models. The modalities are interpreted by

$$w \Vdash \Box A \text{ iff for all } v \text{ with } wRv \text{ it holds that } v \Vdash A$$

and

$$w \Vdash \Diamond A \text{ iff there is a } v \text{ with } wRv \text{ so that } v \Vdash A.$$

While these bi-relational Kripke models seem natural for intuitionistic modal logic, there are still some difficulties that must be taken care of in order to ensure that they behave in the intended way.

At this point we note that, unlike in classical modal logic, $\Diamond$ and $\Box$ are not dual in intuitionistic modal logic. This is analogous to what we observe with $\vee$ and $\wedge$ in propositional logic and $\exists$ and $\forall$ in first-order logic. This stems from negation not being a primitive connective in intuitionistic logic but rather being defined as $\neg A = A \rightarrow \bot$. For simplicity, from now on we will focus on intuitionistic modal logic which contains only the $\Box$ operator, without adding the $\Diamond$ operator to our language as well. We will still simply refer to this as intuitionistic modal logic for better readability.

As we have mentioned, an important property of intuitionistic propositional logic is monotonicity. We now want to make sure that this is maintained in our modal logic as well. Specifically, we want $\Box A \rightarrow (\top \rightarrow \Box A)$ to be valid in every model. The difficulty of this becomes clear when looking at the example in figure 2.2. In this model $M$, we know that $M, v \Vdash p$ and $M, w \Vdash \Box p$. By monotonicity it should also hold that $M, w' \Vdash \Box p$ since $w \leq w'$. However, this would require us to somehow ensure that $M, v' \Vdash p$.

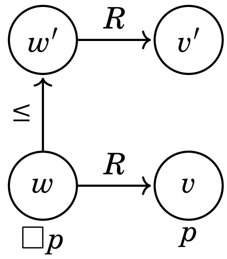


Figure 2.2: Problems with Monotonicity of □

In [15] two ways to fix this issue are presented. There one can also find a more detailed overview of the literature in which they appear. The first option is to change the satisfaction clauses of the modality in order to include monotonicity. This would make the satisfaction clause for □:

$$w \Vdash \Box A \text{ iff for all } w' \text{ with } w \leq w', \text{ for all } v' \text{ with } w'Rv' \text{ it holds that } v' \Vdash A.$$

In the example shown in figure 2.2, $w \Vdash \Box p$ could therefore only hold if $M, v' \Vdash p$, just as we previously required.

The second option is to place restrictions on the frames our models are based on, specifically on how the two relations $\leq$ and $R$ interact with one another. In [2] it is proven that for intuitionistic modal logic with only the □ operator, monotonicity of modal formulas in a model is equivalent to the frame of the model satisfying

$$\Box\text{-p: if } w \leq w'Rv' \text{ then there exists some } v \text{ so that } wRv \leq v'.$$

This is visualised in figure 2.3. The name □-p is used because it ensures persistence, another word for monotonicity, of □ formulas.

In our example from figure 2.2 this would mean that there is a world $u$ in our model so that $wRu \leq v'$. From $w \Vdash \Box p$ it follows that $u \Vdash p$ and therefore also $v' \Vdash p$. Since this holds for any $v'$ with $w'Rv'$, it also holds that $w' \Vdash \Box p$.

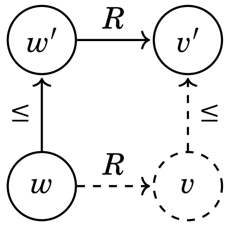


Figure 2.3: □-p, dashed lines indicate forced existence

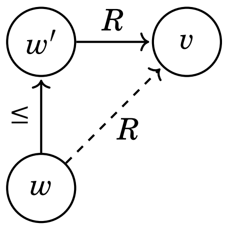


Figure 2.4: ⥽-p, dashed lines indicate forced existence

While □-p is the least restrictive option, there are also other frame properties which are used in the literature. For example, as visualised in figure 2.4, in [5] the models are required to satisfy

$$\text{⥽-p: if } w \leq w'Rv \text{ then } wRv.$$

An even stronger restriction which is used for example in [17] and illustrated in figure 2.5 is

$$\text{mix: if } w \leq w'Rv \leq v' \text{ then } wRv'.$$

Even though mix is more restrictive than □-p, [9] states that when working in an intuitionistic modal logic without ◊, it is “mostly harmless”. In fact, in a model in which □-p holds, mix holds if we simply additionally require the model to satisfy

$$\text{Brilliancy: if } wRv \leq v' \text{ then } wRv'.$$

This can be seen in figure 2.6. We can read this restriction to mean that for any world $w$, the set of its $R$ successors is upwards closed with respect to $\leq$. The

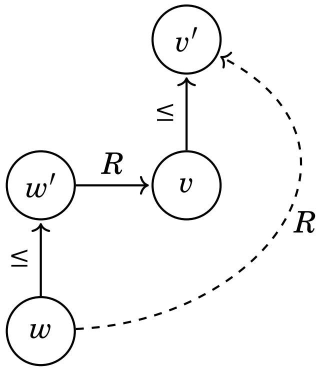


Figure 2.5: Mix, dashed lines indicate forced existence

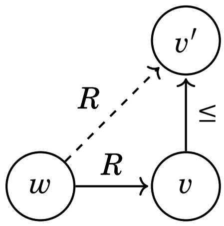


Figure 2.6: Brilliancy, dashed lines indicate forced existence

name brilliancy is attributed to Iemhoff [6], other sources such as [2] also call this property strongly condensed.

**2.1 Lemma.** Both ⊰-p and mix imply □-p.

*Proof.* This is clear due to the fact that $v \leq v$ for every world $v$. □

Both changing the satisfaction clause and placing restrictions on the frames has its advantages. For example, one could argue that changing the satisfaction clause of the □ operator ensures a more intuitionistic reading of the modality. This would also let us study intuitionistic modal formulas in arbitrary models. On the other hand, we may want to keep the usual satisfaction clauses in order to stay closer to classical modal logic. Either way, [15] states that since we only add the □ modality, both approaches induce the same intuitionistic modal logic. In the following we will be using the Kripke semantics that use the satisfaction condition as we know it from classical modal logic and requires each frame to satisfy □-p. We will refer to this semantics as *K-IntMod*.

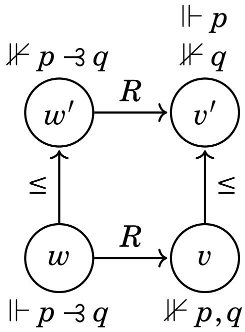


Figure 2.7: Problems with Monotonicity of $⥽$

### 2.1.1 STRICT IMPLICATION

In this section we briefly examine how adding strict implication to our language affects the Kripke models we have just introduced. We will refer to to this Kripke semantics for $\mathcal{L}_{⥽}$ as *K-IntModSI*.

As said in section 1.1, strict implication is defined on $R$ by

$$w \Vdash A ⥽ B \text{ iff for all } v \text{ with } wRv \text{ if } v \Vdash A \text{ then } v \Vdash B.$$

This is symmetric to the intuitionistic satisfaction clause for $\to$, but defined on the modal accessibility relation $R$ rather than the intuitionistic relation $\leq$. If we add strict implication to our intuitionistic modal logic, $\Box$-p alone is no longer enough to ensure monotonicity of all formulas. We can see an example of this in figure 2.7. While this frame satisfies $\Box$-p, that is not enough to ensure that $w' \Vdash p ⥽ q$.

In [18] it is proven that persistence of $⥽$ formulas in a model is equivalent to the frame satisfying $⥽$-p. As we have already discussed, $⥽$-p also ensures monotonicity of $\Box$ formulas, therefore we include $⥽$-p in *K-IntModSI* and omit $\Box$-p.

Additionally, we remark that unlike in classical modal logic, $⥽$ is not interdefinable with $\Box$. In particular, $A ⥽ B \leftrightarrow \Box(A \to B)$ does not hold in every model. This is due to the fact that the intuitionistic implication is not satisfied locally but dependant on the worlds accessible by the $\leq$ relation. While $\Box(A \to B)$ still implies $A ⥽ B$ the inverse does not hold anymore. This means that $⥽$ is more expressive than $\Box$ in this logic, since $\Box$ is a definable connective, as $\Box A \leftrightarrow (\top ⥽ A)$ still holds.

**2.2 Lemma.** *K-IntModSI* $\vDash \Box(A \to B) \to A ⥽ B$.

*Proof.* This claim holds if every world $w$ that satisfies $\Box(A \to B)$ also satisfies $A \dashv\!\!\!\! \exists\, B$. Let us assume $w \Vdash \Box(A \to B)$. In order to show that $w \Vdash A \dashv\!\!\!\! \exists\, B$, we need to show that for every $v$ with $wRv$ and $v \Vdash A$ it follows that $v \Vdash B$. If there is no such $v$ then this is vacuously true. From $w \Vdash \Box(A \to B)$ it follows that $v \Vdash A \to B$. Since $v \leq v$ and $v \Vdash A$, this means that $v \Vdash B$. Therefore $w \Vdash A \dashv\!\!\!\! \exists\, B$ and $K-IntModSI \vDash \Box(A \to B) \to A \dashv\!\!\!\! \exists\, B$. □

**2.3 Lemma.** *K-IntModSI* $\nvDash (A \dashv\!\!\!\! \exists\, B) \to \Box(A \to B)$.

*Proof.* We can easily prove this by providing a counterexample. In figure 2.8 we can see that $w \Vdash p \dashv\!\!\!\! \exists\, q$ but $w \nVdash \Box(p \to q)$. □

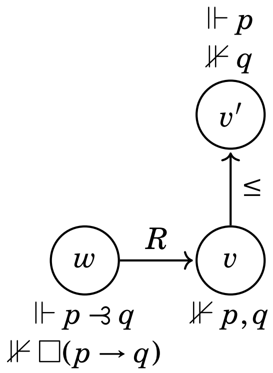


Figure 2.8: $p \dashv\!\!\!\! \exists\, q$ does not imply $\Box(p \to q)$

However, it is possible to restore the interdefinability of $\Box$ and $\dashv\!\!\!\! \exists$ by placing additional restrictions on the frames of our models. In fact, we show that a frame models $(A \dashv\!\!\!\! \exists\, B) \to \Box(A \to B)$ iff the frame is brilliant.

**2.4 Lemma.** Every brilliant frame models $(A \dashv\!\!\!\! \exists\, B) \to \Box(A \to B)$.

*Proof.* We assume that $w \Vdash A \dashv\!\!\!\! \exists\, B$. If there is no world $v$ with $wRv$, then $w \Vdash \Box(A \to B)$ vacuously holds.

If there is a world $v$ so that $wRv$ then we need to show that $v \Vdash A \to B$. If there is a $v'$ so that $v \leq v'$ then by brilliancy it must also hold that $wRv'$. Therefore, if $v' \Vdash A$ we know that $v' \Vdash B$ because $w \Vdash A \dashv\!\!\!\! \exists\, B$. Since this holds for an arbitrary $v'$ and $v$ it follows that $v \Vdash A \to B$ and $w \Vdash \Box(A \to B)$. □

**2.5 Lemma.** If a frame $\mathcal{F}$ models $(A \multimap B) \to \Box(A \to B)$ then $\mathcal{F}$ is brilliant.

*Proof.* We assume there is a frame $\mathcal{F}$ which is not brilliant, which means there must exist worlds $w, v, v'$ so that $wRv \leq v'$ but not $wRv'$. We can now easily define a valuation just like in figure 2.8 so that $w \Vdash (p \multimap q)$ but $w \nVdash \Box(p \to q)$. Therefore, if $\mathcal{F}$ is not brilliant, then $\mathcal{F} \nvDash (p \multimap q) \to \Box(p \to q)$ and thereby also $\mathcal{F} \nvDash (A \multimap B) \to \Box(A \to B)$. □

## 2.2 SEQUENT CALCULUS

We will now give a sequent calculus for intuitionistic modal logic which we will refer to as *S-IntMod* in the following. This sequent calculus is taken from [15].

$$\frac{}{\mathcal{G};\Gamma,x:A \Rightarrow x:A}\,Ass \qquad \frac{}{\mathcal{G};\Gamma x:\bot \Rightarrow z:A}\,\bot L$$

$$\frac{\mathcal{G};\Gamma,x:A,x:B \Rightarrow z:C}{\mathcal{G};\Gamma x:A\wedge B \Rightarrow z:C}\,\wedge L \qquad \frac{\mathcal{G};\Gamma \Rightarrow x:A \quad \mathcal{G};\Gamma \Rightarrow x:B}{\mathcal{G};\Gamma \Rightarrow x:A\wedge B}\,\wedge R$$

$$\frac{\mathcal{G};\Gamma,x:A \Rightarrow z:C \quad \mathcal{G};\Gamma,x:B \Rightarrow z:C}{\mathcal{G};\Gamma,x:A\vee B \Rightarrow z:C}\,\vee L \qquad \frac{\mathcal{G};\Gamma \Rightarrow x:A_i}{\mathcal{G};\Gamma \Rightarrow x:A_1\vee A_2}\,\vee R_i$$

$$\frac{\mathcal{G};\Gamma,x:B \Rightarrow z:C \quad \mathcal{G};\Gamma \Rightarrow x:A}{\mathcal{G};\Gamma,x:A\to B \Rightarrow z:C}\,\to L \qquad \frac{\mathcal{G};\Gamma,x:A \Rightarrow x:B}{\mathcal{G};\Gamma \Rightarrow x:A\to B}\,\to R$$

$$\frac{\mathcal{G};\Gamma,y:A \Rightarrow z:B}{\mathcal{G},xRy;\Gamma,x:\Box A \Rightarrow z:B}\,\Box L \qquad \frac{\mathcal{G},xRy;\Gamma \Rightarrow y:A}{\mathcal{G};\Gamma \Rightarrow x:\Box A}\,\Box R \text{ [2.a]}$$

Table 2.1: Sequent Calculus for Intuitionistic Modal Logic

Intuitively we read $x$ as a world in a modal model and $x:A$ as the assertion that $A$ holds in the world $x$. $\Gamma$ is a finite set of formulas which are each labelled with a world $x$. $\mathcal{G}$ is a finite set of assumptions of the form $xRy$ which we can interpret as a finite graph.

In [15] it is proven that *S-IntMod* is sound and complete with respect to *K-IntMod*. This means that a sequent $\mathcal{G};\Gamma \Rightarrow x:A$ is derivable in *S-IntMod* if and only if every

[2.a] Restriction on $\Box R$: y must not occur in $\mathcal{G};\Gamma \Rightarrow x:\Box A$.

Kripke model $\mathcal{M}$ which has a finite sub-graph that is equal to $\mathcal{G}$ and satisfies all labelled formulas in $\Gamma$ also satisfies $x : A$.

## 2.3 HILBERT CALCULUS

We present a Hilbert-style axiomatisation of *K-IntModSI* which is taken from [9]. We will refer to this as *H-IntModSI* in the following. Proofs of soundness and completeness of this axiomatisation can be found in [7].

**2.6 Definition.** *H-IntModSI* is given by the axioms

- The axioms of intuitionistic propositional logic
- $(A \prec\!\!\!\!\!\!- B) \to ((B \prec\!\!\!\!\!\!- C) \to (A \prec\!\!\!\!\!\!- C))$
- $(A \prec\!\!\!\!\!\!- B) \to ((A \prec\!\!\!\!\!\!- C) \to (A \prec\!\!\!\!\!\!- (B \wedge C)))$
- $(A \prec\!\!\!\!\!\!- C) \to ((B \prec\!\!\!\!\!\!- C) \to ((A \vee B) \prec\!\!\!\!\!\!- C))$

and the inference rules

- Modus Ponens:
  $\vdash A \to B$ and $\vdash A$ implies $\vdash B$
- $N_a$:
  $\vdash A \to B$ implies $\vdash A \prec\!\!\!\!\!\!- B$

$N_a$ is the $\prec\!\!\!\!\!\!-$ variant of the standard necessitation rule Nec which states that $\vdash A$ implies $\vdash \Box A$. Since we have already discussed that $\prec\!\!\!\!\!\!-$ is more expressive than $\Box$ in intuitionistic modal logic, it is clear that $N_a$ implies Nec.

# 3 Dialogue Games

In this chapter we will introduce the basic concepts of dialogue game semantics that we need in the next chapter. We begin by introducing dialogue games for propositional logic, then expand these to dialogue games for modal logic and finally discuss how strict implication can be included. At the end we combine these elements in the trivial way and discuss why this isn't a valid system. Based on our observation we propose two different dialogue game semantics which we will introduce in the next chapter.

## 3.1 Propositional Logic

In this section we will define a dialogue game semantics for propositional logic. Our games are based on the ones defined in [1] with some minor, mostly cosmetic differences.

Just like all other games, dialogues are defined by rules. We differentiate between particle rules and structural rules. The particle rule specify which moves are possible, while the structural rules determine the context of the game. If we compare this to chess for example, the particle rules would tell us how each piece is allowed to move across the board and the structural rules would tell us how big the board is, what the starting positions are, what the winning condition is, etc.

In the case of our dialogue games, the particle rules specify the local properties of the dialogue by stating how each formula can be attacked and how these attacks can be defended against. The structural rules on the other hand, determine the notion of proof in a given dialogue system and ensure that a dialogue is a valid argument for a thesis.

From a logical perspective, we can compare these two different types of rules to the way we determine the meaning of a pronoun in a sentence. Even if we know the definition of a pronoun by itself, it is only within the context of a sentence that we know exactly what the pronoun is referring to. The particle rules give us the definition, or rather meaning, of each connective but the structural rules determine the context of the dialogue and therefore the logic we are working in.

When changing an existing dialogue game system to adapt it to a different logic we can therefore choose between changing the particle rules, the structural rules or both. In some cases it may even be possible to reach different logics in different ways. We will go more into this later on.

Before we introduce these rules however, we must introduce some basic notation. The two players of the game are the proponent ($P$) and the opponent ($O$). We will use the variables $X$ and $Y$ to range over $\{O,P\}$ with the assumption that $X \neq Y$. In addition to the usual connectives $\bot, \wedge, \vee, \rightarrow, \neg$ we also use the attack symbols $?_L, ?_R$ and $?_\vee$.

Each move is signified by a dialogial expression. We define these to be either a formula of our language or one of the attack symbols.

While it is possible to let a dialogue begin with a set of formulas that $O$ initially grants as done in [1], we will be focusing on dialogues that are equivalent to validity and therefore need no starting assumptions.

### 3.1.1 Structural Rules

**Rule 0** (Starting rule)**.** All expressions in a dialogue game are numbered, starting with the thesis at 0. The thesis is stated by player $P$. Players $O$ and $P$ take turns making moves, so the odd numbered expressions are stated by $O$ and the even numbered expressions are stated by $P$. Every move after the thesis is a response to a previous expression by the other player and must adhere to both the particle and the structural rules.

**Rule 1** (Winning rule)**.** Player $X$ wins if it is player $Y$s turn but $Y$ has no more possible moves.

We now need to regulate which moves players are allowed to make during their turn. The following rule is also how we differentiate between dialogues for classical and intuitionistic logic.

**Rule 2c** (Classical move rule)**.** During their turn, player $X$ can either attack any assertion by $Y$ or defend against any attack by $Y$, including those which have already been defended against.

**Rule 2i** (Intuitionistic move rule)**.** During their turn, player $X$ can either attack any assertion by player $Y$ or defend themselves against the last open attack by $Y$.

We will look at an example of how the differences between these two versions of rule 2 affect the dialogues once we have given all the rules.

So far, all of our rules have been equal for both players. However, proving validity of a formula is not necessarily symmetric to proving invalidity. In order to show that a formula is not valid, we only need to supply one counterexample. To prove validity however, we need to show that the formula holds in every case. In propositional logic this means that the formula holds for any valuation of the propositional variables. If we want a winning strategy for $P$ to be equivalent to validity then we must make sure that $P$'s strategy is independent of the valuations of the propositional variables. We can achieve this by restricting $P$'s ability to assert a propositional variable $p$. Intuitively, we view the propositional variables that $O$ asserts as the ones which are valued as true and the others as false. As we know from game theory, vaguely speaking, $P$ has a winning strategy for a game if they have a winning strategy for every move that $O$ can make. Therefore $P$ having a winning strategy means they have a winning strategy for any valuation that $O$ may choose.

**Rule 3** (Atomic formula introduction rule)**.** Player $P$ is not allowed to assert atomic formulas unless they have previously been asserted by player $O$.

We want our games to be finite, but, according to our current rules it is still possible for each of the players to repeat an attack infinitely many times in order to delay the game. We will prevent this by forbidding players to repeat their moves.

**3.1 Definition.** We call a move a strict repetition if the player has previously already played the same move in response to the same expression.

**Rule 4** (No delaying rule)**.** Players may not play strict repetitions.

When playing with the intuitionistic rule set, the rule 2i is an even stronger restriction for defensive moves since an attack that has already been defended may not be defended again, even if it is a different move.

As mentioned, we are not necessarily interested in a single play-through of a dialogue game but rather in winning strategies for player $P$. In order to make our work with these easier, we introduce one more structural rule.

**Rule E** (Opponent move restriction rule)**.** The opponent may only respond to the last move that player $P$ has made.

This rule may seem controversial at first. We will only give a brief intuition here as to why this rule is admissible, as a full proof would go beyond the scope of this thesis. In general, throughout a play-through of a dialogue game the players each assert formulas which they take to be true. As we will see in the particle rules later on, these formulas will all be subformulas of the thesis. The game ends when either player is forced to attack a propositional variable or $P$ is unable to assert a propositional variable. Player $P$'s ability to respond to a move by $O$ is restricted by rule 3. Therefore $P$ will occasionally need to respond to other moves, in order to gather additional information. Player $O$ faces no such restrictions. If $O$ has a move which is a response to a move $m$ by $P$ and a part of a winning strategy for $O$, then $O$ can always make this move immediately after $P$ has made the move $m$. For this reason we can theoretically find a winning strategy for $O$ which follows rule E whenever we have any winning strategy for $O$. A formal proof of this can be very technical, see for example [4] for a proof that rule E is admissible in dialogue games for first order logic. In the literature authors often distinguish between D and E dialogues to show whether they use rule E or not.

In table 3.1 we have listed a brief overview of these rules. Rules 0, 1, 2c, 3, 4 and E are needed for classical propositional logic. In order to obtain dialogue games for intuitionistic propositional logic we add rule 2i.

### 3.1.2 Particle Rules

While the structural rules determine the general structure of our dialogue games, the particle rules specify which individual moves are possible. This is done by

| # | Description |
|---|---|
| 0 | The thesis is stated by $P$, the players alternate |
| 1 | Players win if the other player has no more possible moves |
| 2c | Players can react to any previous move by the opponent |
| 3 | Only $O$ can assert new atomic formulas |
| 4 | Strict repetitions are prohibited |
| E | $O$ can only react to the previous move |
| 2i | A player can only defend against the newest open attack |

Table 3.1: Standard rules for classical propositional logic
+ rule for intuitionistic propositional logic

specifying how each kind of dialogical expression can be attacked and how these attacks can be defended against. The particle rules for propositional logic can be found in table 3.2. Since $?_L$, $?_R$ and $?_\vee$ are not assertions but only attacks, they cannot be attacked themselves.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\bot$ | cannot be asserted | | |
| $p$ | $p$ | cannot be attacked | |
| $\wedge$ | $A \wedge B$ | $?_L$<br>$?_R$ | $A$<br>$B$ |
| $\vee$ | $A \vee B$ | $?_\vee$ | $A$ or $B$ |
| $\rightarrow$ | $A \rightarrow B$ | $A$ | $B$ |
| $\neg$ | $\neg A$ | $A$ | cannot be defended |

Table 3.2: Particle rules for propositional logic

When making sense of these rules intuitively, it is important to remember how exactly we want to use the dialogue games. As described in section 1.3 we want a formula to be valid iff the proponent has a winning strategy for the corresponding dialogue game. The player attacking a formula $A \wedge B$ can choose which side of the conjunction they want to challenge, forcing the player who stated $A \wedge B$ to have a winning strategy for either side of the conjunction. This ensures that $A \wedge B$ is valid iff both $A$ and $B$ are valid. The particle rules for disjunction can also be explained in this way.

The rule for implication works a little differently. As we know, a statement $A \rightarrow B$ is only invalid if it is possible for $A$ to hold while $B$ does not. For this reason, a player attacking $A \to B$ must in turn assert $A$. This gives the other player a choice between launching a counterattack against the antecedent or defending the consequent.

We can also see that the particle rules for $\to$ and $\neg$ are compatible with both the classical and the intuitionistic definition of these connectives. In intuitionistic logic negation is defined by $\neg A := A \to \bot$. This means that, as an implication, an assertion $\neg A$ can be attacked by the other player asserting $A$, but the only defence against this is to assert falsity, which isn't possible, just as the negation rule states.

Note that it does not count as a strict repetition when an expression $A \wedge B$ is attacked with both $?_L$ and $?_R$. The same holds for a defence against $?_\vee$.

**3.2 Definition.** We will refer to the dialogue game semantics with the classical rule set as *D-CProp* and the intuitionistic rule set as *D-IntProp*.

### 3.1.3 Winning Strategies

Now that we have defined how the dialogue games work, we need a way to represent them which will help us study winning strategies for $P$.

**3.3 Definition.** We say that the dialogue game which begins with $P$ stating the thesis $A$ is the dialogue game associated with $A$.

**3.4 Definition.** A dialogue tree for $A$ is a rooted, directed tree in which each node is a state in a dialogue game. Every branch corresponds to a play-through of a dialogue game with the thesis $A$ which obeys the particle and structural rules. This is simply a representation of the dialogue game associated with $A$ in extensive form.

**3.5 Definition.** The depth of a dialogue tree is the length of its longest branch.

**3.6 Definition.** We call a node an $O$- or a $P$-node if it is a state in which it is $O$ or $P$s turn to move respectively.

**3.7 Definition.** A winning strategy for player $P$ is a finite sub-tree of a dialogue tree in which every leaf is an $O$-node at which the winning condition for $P$ holds. Every $P$-node only has one child whereas every $O$-node has a child for each possible $O$-move.

Essentially this is a sub-tree of a dialogue tree in which we can see which moves $P$ must make in order to win, regardless of what $O$ chooses.

We can see an example of a dialogue tree in figure 3.1 with a subtree which is a winning strategy marked in red. This is the dialogue tree for the intuitionistic dialogue game associated with $((p_1 \vee p_2) \to q) \to ((p_1 \wedge p_2) \to q)$. It is not important to understand this example in detail yet, we will introduce a notation style for these trees that make them easier to read in the next section. Note that in the dialogue tree, not every leaf-node satisfies the winning condition for $P$, but we can still find a sub-tree that is a winning strategy for $P$.

An easy way to show that $P$ does not have a winning strategy is to show that every leaf-node of the dialogue tree satisfies the winning condition for $O$, since in this case there is clearly no sub-tree in which all leaf-nodes satisfy the winning condition for $P$.

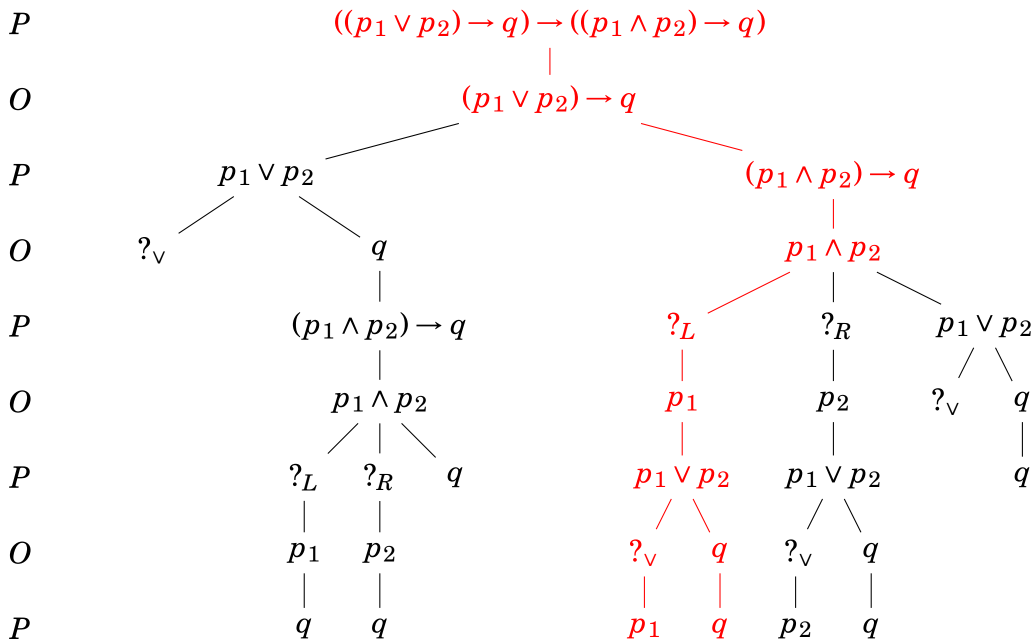


Figure 3.1: Dialogue tree with a winning strategy for $P$ marked in red

**3.8 Definition.** The active formula of a round of a dialogue game is either

(i) the last formula asserted by $P$ that $O$ is required to attack or

(ii) the last formula that $O$ has attacked.

**3.9 Remark.** Note that if $P$ asserts a formula $A$ in order to attack a formula $A \to B$ then $O$ is not required to attack since they could also defend the implication by asserting $B$.

**3.10 Definition.** We associate each turn of a dialogue game with a dialogue sequent $\Pi \vdash A$ where $\Pi$ is the set of formulas that have been granted by $O$ so far and $A$ is the active formula.

**3.11 Theorem.** A formula is valid in classical propositional logic iff $P$ has a winning strategy for the corresponding *D-CProp* dialogue game.

A formula is valid in intuitionistic propositional logic iff $P$ has a winning strategy for the corresponding *D-IntProp* dialogue game.

A detailed proof of this is given in [1].

### 3.1.4 Examples

As with most games, it is easiest to understand how dialogue games are played by looking at a few examples. We will also use this opportunity to introduce a presentation of the dialogues that is easier to read. In general, we will focus on presenting a winning strategy for $P$ if this is possible. If this is not possible, we give the entire dialogue tree. In order to save space, we omit the edges of the tree when a node only has one child. For each move we also indicate whether it is an attack or a defence and which previous move it refers to. In dialogue 3.1 we can see the same winning strategy that we also found in figure 3.1 in this new notation style.

In dialogue 3.2 we can see how the difference between rules 2c and 2i affects the games concretely. When playing with rule 2i player $P$ has no winning strategy. This is because move number IV is a defence against an attack which has already been defended and therefore not a legal move in an intuitionistic dialogue. However, $P$ has no alternative strategy since they are not able to assert $p$ before $O$ has done so because of rule 3. This means dialogue 3.2 above the dashed line shows the complete dialogue tree for this game and every leaf-node satisfies the winning condition for $O$. Therefore player $P$ has no winning strategy for the game $p \vee \neg p$,

| # | Player | Expression | Description | Dialogue Sequent |
|---|---|---|---|---|
| T | $P$ | $F$ | Thesis | $\vdash F$ |
| I | $O$ | $(p_1 \vee p_2) \to q$ | Attack T | $(p_1 \vee p_2) \to q \vdash F$ |
| II | $P$ | $(p_1 \wedge p_2) \to q$ | Defence I | $(p_1 \vee p_2) \to q \vdash (p_1 \wedge p_2) \to q$ |
| III | $O$ | $p_1 \wedge p_2$ | Attack II | $p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash (p_1 \wedge p_2) \to q$ |
| IV | $P$ | $?_L$ | Attack III | $p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash (p_1 \wedge p_2) \to q$ |
| V | $O$ | $p_1$ | Defence IV | $p_1, p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash (p_1 \wedge p_2) \to q$ |
| VI | $P$ | $p_1 \vee p_2$ | Attack I | $p_1, p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash (p_1 \wedge p_2) \to q$ |

| | | | | |
|---|---|---|---|---|
| VII | $O$ | $?_\vee$ | Attack VI | $p_1, p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash p_1 \vee p_2$ |
| VIII | $P$ | $p_1$ | Defence VII | $p_1, p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash p_1$ |

| | | |
|---|---|---|
| $q$ | Defence VI | $q, p_1, p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash (p_1 \wedge p_2) \to q$ |
| $q$ | Defence III | $q, p_1, p_1 \wedge p_2, (p_1 \vee p_2) \to q \vdash q$ |

Dialogue 3.1: $F = ((p_1 \vee p_2) \to q) \to ((p_1 \wedge p_2) \to q)$

| # | Player | Expression | Description | Dialogue Sequent |
|---|---|---|---|---|
| T | $P$ | $\neg p \vee p$ | Thesis | $\vdash \neg p \vee p$ |
| I | $O$ | $?_\vee$ | Attack T | $\vdash \neg p \vee p$ |
| II | $P$ | $\neg p$ | Defence I | $\vdash \neg p$ |
| III | $O$ | $p$ | Attack II | $p \vdash \neg p$ |
| IV | $P$ | $p$ | Defence I | $p \vdash p$ |

Dialogue 3.2: $\neg p \vee p$, with and without 2i

which makes sense since we know that the law of excluded middle is not valid in intuitionistic logic.

In classical logic on the other hand, the law of excluded middle is valid. When playing with rule 2c player $P$ is able to defend the disjunction again, using the $p$ that $O$ was forced to assert, so dialogue 3.2 is clearly a winning strategy for $P$.

## 3.2 Modal Logic

In this section we will expand our dialogue games to modal logic, which means we extend our language with the modal operators $\Box$ and $\Diamond$. The game semantics presented here are mostly based on the modal dialogues presented in [11].

Since modalities by their very nature make the context of the formula explicit, we need a way to keep track of the context the players are playing in. For this purpose we identify each dialogical expression with a world $w$, just as in a Kripke model. In this thesis we will use integers to denote these worlds. The world in which the dialogue game begins will be 1, each new world after that will be denoted by a new number. We also introduce labels, which will keep track of the relations between the worlds. Each label consists of a finite string of integers separated by dots, $x = x_0.x_1 \dots x_n$. The set of all labels is semi-ordered by the succession relation. If $v$ denotes a world, the labels $\vec{x}.v.w_i.\vec{y}$ denote that $w_i$ is a successor of $v$.

We update our definition of the dialogue sequent to include the label information.

**3.12 Definition.** We associate each round of a dialogue game with a dialogue sequent $\mathcal{G};\Pi \vdash A$ where $\Pi$ is the set of formulas that have been granted by O so far and A is the active formula. $\mathcal{G}$ is the set of all statements $wRv$ for which a label of the form $\vec{x}.w.v.\vec{y}$ has been introduced.

For readability we will denote the labels separately instead of including them in the dialogue sequent when presenting a dialogue tree.

We also introduce two new attack symbols $?_\Diamond$ and $?_v$, where $v$ is a world, to our language.

### 3.2.1 PARTICLE RULES

We can now formulate the particle rules for modal operators. Since the particle rules for all other connectives are independent of the modal context, they are the same as in table 3.2, except with a world $v$ added to each expression.

Intuitively the rules for $\Box$ and $\Diamond$ can be explained similarly to the particle rules for $\wedge$ and $\vee$. If a player $X$ asserts $v : \Box A$, they should be able to defend it at every world $w$ with $vRw$. By letting the attacker choose the world we ensure that player $X$ needs to have a strategy for every such world in order to have a winning strategy for the statement $\Box A$. If a player $X$ asserts $v : \Diamond A$ however, they only need to be able to choose one world in which $A$ holds.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\Box$ | $v : \Box A$ | $v : ?_w$ for any $w$ with $vRw$ | $w : A$ |
| $\Diamond$ | $v : \Diamond A$ | $v : ?_\Diamond$ | $w :\ A$ for any $w$ with $vRw$ |

Table 3.3: Particle rules for modal logic

If a player attacks an assertion $v : \Box A$ or defends against an attack $?_\Diamond$ with a world $w$ which has not been used in the dialogue game so far, this introduces the label $\vec{x}.v.w$ to the dialogue game, where $\vec{x}.v$ is any label of this form that has already been introduced within the dialogue game. If a label of the form $\vec{x}.v.w$ has already been introduced we say that the player uses the label.

If a world $w$ has already been used in the dialogue game it cannot be used to introduce a new label unless the structural rules specifically allow it.

### 3.2.2 Structural Rules

Since we now have multiple worlds which can each make different propositional variables true, we need to adapt our rule for atomic formula introduction. All other rules from the dialogues for propositional logic remain unchanged.

**Rule 3m** (Atomic formula introduction rule)**.** Player $P$ is not allowed to assert atomic formulas at a given world unless they have previously been asserted by player $O$ at that world. An atomic formula cannot be attacked by either player. Neither player can assert $\bot$ at any world.

We also note that moves which differ only in the world that they are played in are considered different moves and are not a strict repetition.

We now need to add some new structural rules to specify how the players may introduce and use the labels what we have added into the game.

**Rule 5.** All expressions in a dialogue game are labelled with a world $v$, starting with the thesis at 1. Player $O$ may introduce a label anytime the particle rules require them to use a label. Player $P$ on the other hand cannot introduce a label unless it is specifically allowed in the version of rule 6 that is currently being used.

In order to ensure that player $O$ cannot stall the game indefinitely by simply creating infinitely many lables, we would need to restrict how often they can attack an assertion of the form $\Box A$ or defend against an attack $?_\Diamond$. However, rule E ensures that $O$ can only react to each of these statements once, therefore a further restriction is not necessary. Such a restriction is also not necessary for player $P$ because in all cases relevant to us, rule 5 prevents $P$ from creating labels at all.

The last rule varies depending on the properties of the accessibility relation. One of the great benefits of modal logic is how easy it is to specify certain properties that we want the accessibility relation of a model to have. It makes sense to also try to include this flexibility in our dialogue games. Here, we will give the rule for the modal logics $K$ and $S4$.

**Rule 6K.** When playing in a world with the label $v$, if the rules allow it, Player $P$ may use any label of the form $v.i$ that has already been introduced by player $O$.

**Rule 6S4.** When playing in a world $v$, both players may choose the world $v$ when a particle rule requires them to use a label. If a label of the form $\vec{x_1}.v.\vec{x_2}.w.\vec{x_3}$ has been introduced, both players may introduce a label of the form $\vec{x_1}.v.w.\vec{x_3}$ whenever the particles require them to use a label.

| # | Description |
|---|---|
| 0 | The thesis is stated by $P$, the players alternate |
| 1 | Players win if the other player has no more possible moves |
| 2c/ 2i | Players can respond to any previous/ only the last attack by the opponent |
| 3m | Only $O$ can assert new atomic formulas |
| 4 | Strict repetitions are prohibited |
| E | $O$ can only react to the previous move |
| 5 | Only $O$ can introduce new labels |
| 6K | Player $P$ may only use existing labels |

Table 3.4: Rules for propositional logic
+ rules for modal logic

**3.13 Proposition.** A formula is valid in classical or intuitionistic modal logic iff $P$ has a winning strategy for the corresponding classical or intuitionistic modal dialogue game.

A proof of this is given for example in [11], where several possible frame conditions are also discussed.

### 3.2.3 EXAMPLES

We will look at two examples of modal dialogue games in order to better understand how the labels are used.

Dialogue 3.3 is a winning strategy for player $P$ for the game associated with $\Diamond\neg p \to \neg\Box p$. Clearly $O$ has no other possibles choices they can make and cannot respond to $P$'s last move.

Dialogue 3.4 shows the game associated with $\neg\Box p \to \Diamond\neg p$. We can see that when using the classical move rule 2c $P$ has a winning strategy but not when using the intuitionistic move rule 2i. This makes sense as $\Box$ and $\Diamond$ are dual in classical modal logic but not in intuitionistic logic.

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : \Diamond\neg p \to \neg\Box p$ | 1 | Thesis |
| I | $O$ | $1 : \Diamond\neg p$ | 1 | Attack on T |
| II | $P$ | $1 : \neg\Box p$ | 1 | Defence I |
| III | $O$ | $1 : \Box p$ | 1 | Attack II |
| IV | $P$ | $1 : ?_\Diamond$ | 1 | Attack I |
| V | $O$ | $2 : \neg p$ | 1.2 | Defence IV |
| VI | $P$ | $1 : ?_2$ | 1.2 | Attack III |
| VII | $O$ | $2 : p$ | 1.2 | Defence VI |
| VIII | $P$ | $2 : p$ | 1.2 | Attack V |

Dialogue 3.3: Example Dialogue Game

| # | Player | Expression | Label | Description | Dialogue Sequent |
|---|---|---|---|---|---|
| T | $P$ | $1 : \neg\Box p \to \Diamond\neg p$ | 1 | Thesis | $\vdash \neg\Box p \to \Diamond\neg p$ |
| I | $O$ | $1 : \neg\Box p$ | 1 | Attack T | $\neg\Box p \vdash \neg\Box p \to \Diamond\neg p$ |
| II | $P$ | $1 : \Diamond\neg p$ | 1 | Defence I | $\neg\Box p \vdash \Diamond\neg p$ |
| III | $O$ | $1 : ?_\Diamond$ | 1 | Attack II | $\neg\Box p \vdash \Diamond\neg p$ |
| IV | $P$ | $1 : \Box p$ | 1 | Attack I | $\neg\Box p \vdash \Box p$ |
| V | $O$ | $1 : ?_2$ | 1.2 | Attack IV | $\neg\Box p \vdash \Box p$ |
| VI | $P$ | $2 : \neg p$ | 1.2 | Defence III | $\neg\Box p \vdash \neg p$ |
| VII | $O$ | $2 : p$ | 1.2 | Attack VI | $p, \neg\Box p \vdash \neg p$ |
| VIII | $P$ | $2 : p$ | 1.2 | Defence V | $p, \neg\Box p \vdash p$ |

Dialogue 3.4: Duality of Box and Diamond in intuitionistic logic vs classical logic

## 3.3 Strict Implication

We now want to think about how to add strict implication to a dialogue game. We remember that strict implication is defined as

$$w \Vdash A \prec B \text{ iff for all } v \text{ with } wRv \text{ if } v \Vdash A \text{ then } v \Vdash B.$$

We have already remarked the similarity to the satisfaction clause of the intuitionistic implication. It therefore makes sense that the particle rules for strict implication will be similar to the ones we have introduced for $\rightarrow$. The difference is of course that strict implication is not satisfied locally but depends on the worlds accessible by $R$. Since we want to ensure that if $v \Vdash A$ then $v \Vdash B$ holds at every accessible world, we must give the attacker the choice of which world should be regarded for this. This is similar to the particle rules for $\Box$ that we have introduced. We therefore feel justified in suggesting the particle rules for $\prec$ that can be found in table 3.5.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\prec$ | $w : A \prec B$ | $v : A$ for any $v$ with $wRv$ | $v : B$ |

Table 3.5: Strict Implication Particle Rules

We will look at a few example dialogues using these rules in the next section.

## 3.4 Intuitionistic Modal Logic with Strict Implication

If we define a dialogue semantics using the particle rules we have introduced for $\mathcal{L}_{\prec}$ and the structural rules for modal logic with the intuitionistic move rule 2i we should get a dialogue game for intuitionistic modal logic with strict implication. We call this semantics *D-IntModSI*.

One difference between *D-IntModSI* and the possible worlds semantics that we discussed in chapter 2.1 is that we have no representation of the intuitionistic relation $\leq$. In the possible world semantics, we needed to add the $\Box$-p restriction to our models to ensure that monotonicity of $\Box$ statements still holds. Specifically we encountered this problem when looking at $\Box p \to (\top \to \Box p)$, where $\top = \neg\bot = \bot \to \bot$. We take a look at this in example 3.5. This is clearly a winning strategy for $P$, which means that monotonicity of $\Box$ formulas is given in our *D−IntModSI*.

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : \Box p \to (\top \to \Box p)$ | 1 | Thesis |
| I | $O$ | $1 : \Box p$ | 1 | Attack T |
| II | $P$ | $1 : \top \to \Box p$ | 1 | Defence I |
| III | $O$ | $1 : \top$ | 1 | Attack II |
| IV | $P$ | $1 : \Box p$ | 1 | Defence III |
| V | $O$ | $1 : ?_2$ | 1.2 | Attack IV |
| VI | $P$ | $1 : ?_2$ | 1.2 | Attack I |
| VII | $O$ | $2 : p$ | 1.2 | Defence VI |
| VIII | $P$ | $2 : p$ | 1.2 | Defence V |

Dialogue 3.5: Monotonicity of $\Box$

When adding strict implication to our possible world semantics we saw that $\Box(p \to q) \to (p \prec q)$ holds, while the inverse is only true when additional restrictions are placed on the accessibility relations. However, in *D-IntModSI* both directions are valid.

**3.14 Proposition.** *D-IntModSI* $\vDash \Box(p \to q) \to (p \prec q)$.

*Proof.* We show this by providing a winning strategy for $P$ for the dialogue game associated with $\Box(p \to q) \to (p \prec q)$.

We can easily see that dialogue 3.6 is a winning strategy for $P$. Every $O$-node has all possible children, in this case $O$ always only has one possible move, and the leaf-node satisfies the winning condition for $P$. Therefore *D-IntModSI* $\vDash \Box(p \to q) \to (p \prec q)$. $\Box$

**3.15 Proposition.** *D-IntModSI* $\vDash (p \prec q) \to \Box(p \to q)$

*Proof.* Just as in the previous proof, we provide a winning strategy for $P$ for the dialogue game associated with $(p \prec q) \to \Box(p \to q)$.

We can easily see that dialogue 3.7 is a winning strategy for $P$ as every $O$-node has all possible child-nodes and the winning condition for $P$ holds in the leaf-node. Therefore it follows that *D-IntModSI* $\vDash (p \prec q) \to \Box(p \to q)$. $\Box$

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : \Box(p \to q) \to (p \dashv q)$ | 1 | Thesis |
| I | $O$ | $1 : \Box(p \to q)$ | 1 | Attack T |
| II | $P$ | $1 : p \dashv q$ | 1 | Defence I |
| III | $O$ | $2 : p$ | 1.2 | Attack II |
| IV | $P$ | $1 : ?_2$ | 1.2 | Attack I |
| V | $O$ | $2 : p \to q$ | 1.2 | Defence IV |
| VI | $P$ | $2 : p$ | 1.2 | Attack V |
| VII | $O$ | $2 : q$ | 1.2 | Defence VI |
| VIII | $P$ | $2 : q$ | 1.2 | Defence III |

Dialogue 3.6: $\Box(p \to q) \to (p \dashv q)$

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : (p \dashv q) \to \Box(p \to q)$ | 1 | Thesis |
| I | $O$ | $1 : p \dashv q$ | 1 | Attack T |
| II | $P$ | $1 : \Box(p \to q)$ | 1 | Defence I |
| III | $O$ | $1 : ?_2$ | 1.2 | Attack II |
| IV | $P$ | $2 : p \to q$ | 1.2 | Defence III |
| V | $O$ | $2 : p$ | 1.2 | Attack IV |
| VI | $P$ | $2 : p$ | 1.2 | Attack I |
| VII | $O$ | $2 : q$ | 1.2 | Defence VI |
| VIII | $P$ | $2 : q$ | 1.2 | Defence V |

Dialogue 3.7: $(p \dashv q) \to \Box(p \to q)$

From propositions 3.14 and 3.15 it follows that $\dashv$ and $\Box$ are interdefinable in *D-IntModSI*. From lemmas 2.4 and 2.5 we know that the interdefinabilty of $\Box$ and $\dashv$ is equivalent to the frame of our models having the brilliancy property. Apparently we have implicitly added this property to our semantics. This makes sense, as we are not working with the intuitionistic relation $\leq$, therefore we have no need to regulate how the two different accessibility relations harmonise.

Since our goal however is to define dialogue semantics that are sound and complete with respect to intuitionistic modal logic with strict implication, we do not want $\Box$ and $\dashv$ to be interdefinable. We must therefore adapt our current dialogues, which we will attempt in two different ways.

Firstly we will try to mirror the approach we used to get possible world semantics for intuitionistic modal logic. This means we will begin by creating a dialogue game for intuitionistic logic which treats the intuitionistic $\leq$ relation like a modal accessibility relation which is reflexive, transitive and anti-symmetric. We will then use the bi-modal approach to define dialogue games for intuitionistic modal logic.

Our second approach will define a dialogue semantics with the same particle rules as we have used in this chapter. Instead of modelling the intuitionistic relation $\leq$, we will restrict the way that the players can use previous information in the course of the game.

# 4 | Games for Strict Implication

## 4.1 | BiModal Approach

In this section we will introduce a dialogue game semantics which is designed to mirror the connection between intuitionistic modal logic and bi-modal logic in the same way as the Kripke semantics we have introduced in section 2.1. We begin by adapting the dialogue games we have introduced for modal logic in section 3.2 so that they can be used to model intuitionistic propositional logic, we will call this semantics *UniMod*. We then add the $\Box$ operator and introduce the bi-modal dialogue game semantics *BiMod*. Here we also show that the semantics we have developed is sound and complete. After this we add strict implication and show that this dialogue game semantics, which we call *BiModSI*, is indeed a suitable candidate since it models strict implication in the intended way.

### 4.1.1 | Modelling Intuitionistic Logic as Modal Logic

Since we are using the dialogue games for modal logic as a starting point, we will again be working with labelled expressions. The accessibility relation we are considering is the intuitionistic relation $\leq$, so we will write $x.y$ to signify $x \leq y$.

In contrast to the particle rules for *D-IntProp*, we define the rules for implication to correspond to the interpretation in Kripke models. The particle rules can be found in table 4.1. We want $w \Vdash A \to B$ to hold if for all $w'$ with $w \leq w'$, if $w' \Vdash A$ then also $w' \Vdash B$. We achieve this by changing the attack of an expression $w : A \to B$ to $w' : A$ for $w \leq w'$. The particle rule for negation isn't strictly necessary since negation is defined as $\neg A := A \to \bot$ but for simplicity we also give it here.

The structural rules have mostly already been introduced but we collect them here again as a reminder.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\bot$ | cannot be asserted | | |
| $p$ | $w : p$ | cannot be attacked | |
| $\wedge$ | $w : A \wedge B$ | $w : ?_L$<br>$w : ?_R$ | $w : A$<br>$w : B$ |
| $\vee$ | $w : A \vee B$ | $w : ?_\vee$ | $w : A$ or $w : B$ |
| $\rightarrow$ | $w : A \rightarrow B$ | $w' : A$ for any $w'$ with $w \leq w'$ | $w' : B$ |
| $\neg$ | $w : \neg A$ | $w' : A$ for any $w'$ with $w \leq w'$ | cannot be defended |

Table 4.1: Particle Rules for intuitionistic logic

**Rule 0** (Starting rule)**.** All dialogical expressions in a dialogue game are numbered, starting with the thesis at 0. The thesis is stated by player $P$. Players $O$ and $P$ take turns, so the odd numbered moves are made by $O$ and the even numbered moves are made by $P$. Every move after the thesis is a reaction to a previous expression by the other player and must adhere to both the particle and the structural rules.

**Rule 1** (Winning rule)**.** Player $X$ wins if it is player $Y$s turn but $Y$ has no more possible moves.

Since we are modelling intuitionistic logic by using the $\leq$ relation to interpret implication just as in the Kripke semantics, we will be using the classical move rule in this dialogue semantics. This corresponds to using classical reasoning within each world of a Kripke models.

**Rule 2c** (Classical move rule)**.** During their turn, player $X$ can either attack any assertion by $Y$ or defend against any attack by $Y$, including those which have already been defended against.

**Rule 3m** (Atomic formula introduction rule)**.** Player $P$ is not allowed to assert atomic formulas in a given context unless they have previously been asserted by player $O$ in that context.

**Rule 4** (No delaying rule)**.** Players may not preform strict repetitions of moves.

**Rule E** (Opponent move restriction rule)**.** The opponent may only react to the last move that player $P$ has made.

**Rule 5** (Label creation rule)**.** All expressions in a dialogue game are labelled, starting with the thesis at 1. Player $O$ may introduce a label anytime the particle rules allow it. Player $P$ on the other hand cannot introduce a label unless it is specifically allowed in the version of rule 6 that is currently being used.

Since $\leq$ is a pre-order, we will use the corresponding structural rule.

**Rule 6≤** (Pre-order rule)**.** When playing in the label $w$, both players may choose the world $w$ when a particle rule requires them to use a label $w.i$. They may also choose any world $w$ if a label of the form $v.\vec{x}$ has already been introduced.

Finally, we also need to add a rule that ensures that monotonicity of propositional variables holds.

**Rule 7** (Monotonicity)**.** Any atomic formula that has been asserted at a world $v$ can also be used at a world $w$ if a label of the form $\vec{x}.v.w$ can be used.

**4.1 Remark.** If we assume that both $O$ and $P$ are playing an optimal strategy then $O$ will always choose to introduce a new label while $P$ will always choose to play the current label. This is because $O$ does not need any previous information in order to make their move. $P$ on the other hand relies on $O$ to state the necessary propositional variables in a given world. Due to rule 7 it is $P$s best interest to stay in the same world, since they can use all information they obtain there in the successor worlds as well.

We can see how rule 7 is used in dialogue 4.1. In move IV, player $P$ can assert $p$ at the world 3 because $O$ has previously asserted it at the world 2 and introduced the label 1.2.3.

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : p \to (\top \to p)$ | 1 | Thesis |
| I | $O$ | $2 : p$ | 1.2 | Attack T |
| II | $P$ | $2 : (\top \to p)$ | 1.2 | Defence I |
| III | $O$ | $3 : \top$ | 1.2.3 | Attack II |
| IV | $P$ | $3 : p$ | 1.2.3 | Defence III |

Dialogue 4.1: $p \to (\top \to p)$

To justify our choice to use the classical move rule instead of the intuitionistic one we show that this dialogue semantics does not model classical logic.

| # | Description |
|---|---|
| 0 | The thesis is stated by $P$, the players alternate |
| 1 | Players win if the other player has no more possible moves |
| 2c | Players can react to react to any previous move by the opponent |
| 3m | Only $O$ can assert new atomic formulas at a label |
| 4 | Strict repetitions are prohibited |
| E | $O$ can only react to the previous move |
| 5 | Only $O$ may introduce new labels |
| 6≤ | Reflexivity and transitivity of labels can be used |
| 7 | Atomic formulas can be used in all following labels |

Table 4.2: Standard rules for classical modal logic
+ rules to adapt it to intuitionistic propositional logic

**4.2 Proposition.** *UniMod* $\nvDash \neg p \vee p$.

*Proof.* In order to prove this, we need to show that there is no winning strategy for $P$. We do this by looking at the complete dialogue tree as seen in dialogue 4.2 to see that there is no sub-tree which is a winning strategy. In this case, the dialogue tree does not branch because neither $O$ nor $P$ have any choices of which move to make. The only leaf-node satisfies the winning condition for $O$, therefore it is clearly impossible to find a sub-tree with a leaf-node which satisfies the winning condition for $P$. □

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : \neg p \vee p$ | 1 | Thesis |
| I | $O$ | $1 : ?_{\vee}$ | 1 | Attack on T |
| II | $P$ | $1 : \neg p$ | 1 | Defence against I |
| III | $O$ | $2 : p$ | 1.2 | Attack on II |

Dialogue 4.2: $\neg p \vee p$

We also want to make sure that even though we have only enforced the monotonicity of propositional variables in rule 7, it holds for every formula $A$. This property is equivalent to the validity of $A \to (\top \to A)$ for every formula $A$.

**4.3 Proposition** (Monotonicity). *UniMod* $\vDash A \to (\top \to A)$ for every formula $A$.

*Proof.* In dialogue 4.3 we can see the beginning of $P$s winning strategy for the dialogue game associated with $A \to (\top \to A)$. If $A$ consists only of a propositional variable $p$, then this is a complete winning strategy for $P$. Otherwise, move V must be that $O$ attacks the expression $3:A$.

$P$ can now copy any move that $O$ makes in order to attack move I. This is obvious if $A$ is a conjunction or a disjunction. If $A = B \to C$ then $O$ attacks $3 : B \to C$ by stating $4:B$. $P$ can copy this attack due to the transitivity of $\leq$, which is captured in rule 6≤. Since $O$ must defend against this move, $P$ can copy this defence in order to defend against move V. In particular, this means that there cannot be a leaf-node of the dialogue tree in which the win-condition for $O$ is satisfied.

Using this strategy, $P$ will always have another possible move after $O$ makes a move. Therefore, $P$ has a winning strategy for the game associated with $A \to (\top \to A)$. □

| # | Player | Expression | Label | Description | Dialogue Sequent |
|---|---|---|---|---|---|
| T | $P$ | $1 : A \to (\top \to A)$ | 1 | Thesis | $\vdash 1 : A \to (\top \to A)$ |
| I | $O$ | $2 : A$ | 1.2 | Attack T | $2 : A \vdash 1 : A \to (\top \to A)$ |
| II | $P$ | $2 : \top \to A$ | 1.2 | Defence I | $2 : A \vdash 2 : \top \to A$ |
| III | $O$ | $3 : \top$ | 1.2.3 | Attack II | $2 : A, 3 : \top \vdash 2 : \top \to A$ |
| IV | $P$ | $3 : A$ | 1.2.3 | Defence III | $2 : A, 3 : \top \vdash 3 : A$ |

Dialogue 4.3: Monotonicity

We can use the argumentation in this proof to show that if $P$ has a winning strategy for the dialogue game associated with $A$ then $P$ also has a winning strategy for the dialogue game associated with $A[B/p]$.

**4.4 Proposition.** If $P$ has a winning strategy for the dialogue game associated with $A$ then $P$ also has a winning strategy for the dialogue game associated with $A[B/p]$.

*Proof.* Since $P$ is only able to state assert $p$ in a context after $O$ has done so, $P$ has a strategy for the game associated with $A[B/p]$ in which they only assert $B$ after $O$ has done so. If $O$ attacks $B$, $P$ can copy this move, resulting in a winning strategy just like in the proof of proposition 4.3. □

We will not prove soundness and completeness of this system since it is only a first step towards the semantics we are working towards in this section. However, the soundness and completeness proof from the next section can easily be adapted to work here as well.

### 4.1.2 INTUITIONISTIC MODAL LOGIC AS A BI-MODAL DIALOGUE GAME

We now want to model intuitionistic modal logic as a bi-modal dialogue game. Since we now have two separate relations, we use $w.w'$ to signify that $w \leq w'$ and $wRv$ to signify that $v$ is an $R$ successor of $w$.

Just as when adapting the dialogues for propositional logic for modal logic, the previous particle rules remain unchanged and we add the rule for the $\Box$ operator which can be found in table 4.3.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\Box$ | $w : \Box A$ | $w : ?_v$ for any $v$ with $wRv$ | $v : A$ |

Table 4.3: Particle Rules for $\Box$ operator

We also need to add a new structural rule to regulate how the players can use the new accessibility relation.

**Rule 6R-K.** When playing in a world with the label $w$, if the rules allow it, both players may choose any label of the form $wRx$ that has already been introduced by player $O$.

Since we are basing this version of dialogues on the bi-modal Kripke semantics for intuitionistic modal logic, it makes sense that we encounter similar difficulties. In particular, we remember that in section 2.1 we were faced with the problem of maintaining monotonicity for boxed formulas in situations as pictured in figure 4.1.

We can examine a similar situation in the game associated with $\Box p \to (\top \to \Box p)$ as shown in dialogue 4.4. Using only the rules we have introduced so far, moves T to V form the complete dialogue tree for this game. Since the tree does not branch

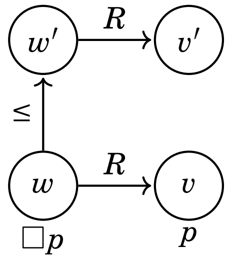


Figure 4.1: Monotonicity of Box

and the only leaf-node satisfies the winning condition for $O$, $P$ does not have a winning strategy for this game.

| # | Player | Expression | Label | Description | Dialogue Sequent |
|---|---|---|---|---|---|
| T | $P$ | $1 : F$ | 1 | Thesis | $\vdash 1 : F$ |
| I | $O$ | $2 : \Box p$ | 1.2 | Attack T | $2 : \Box p \vdash 1 : F$ |
| II | $P$ | $2 : \top \to \Box p$ | 1.2 | Defence I | $2 : \Box p \vdash 2 : \top \to \Box p$ |
| III | $O$ | $3 : \top$ | 1.2.3 | Attack II | $2 : \Box p, 3 : \top \vdash 2 : \top \to \Box p$ |
| IV | $P$ | $3 : \Box p$ | 1.2.3 | Defence III | $2 : \Box p, 3 : \top \vdash 3 : \Box p$ |
| V | $O$ | $3 : ?_4$ | $1.2.3R4$ | Attack IV | $2 : \Box p, 3 : \top \vdash 3 : \Box p$ |
| VI | $P$ | $2 : ?_5$ | $1.2R5.4$ | Attack I | $2 : \Box p, 3 : \top \vdash 3 : \Box p$ |
| VII | $O$ | $5 : p$ | $1.2R5.4$ | Defence VI | $2 : \Box p, 3 : \top, 5 : p \vdash 3 : \Box p$ |
| VIII | $P$ | $4 : p$ | $1.2R5.4$ | Defence V | $2 : \Box p, 3 : \top, 5 : p \vdash 4 : p$ |

Dialogue 4.4: $F = \Box p \to (\top \to \Box p)$ without and with rule □-p

In order to regain the monotonicity property we need to add a structural rule which corresponds to the □-p restriction for Kripke semantics.

**Rule □-p.** If a label of the form $x \cdot w.w'Rv \cdot y$, where $\cdot \in \{., R\}$ and $x$ and $y$ labels, has been introduced, then both players can use a label of the form $x \cdot wRv'.v \cdot y$ where $v'$ is a new world.

If we revisit dialogue 4.4 with this additional rule, we can see that $P$ is now able to win as seen below the dashed line. Because of rule □-p, $P$ can use the fact that $O$ has introduced label $1.2.3R4$ to introduce a label $1.2R5.4$. A summary of the structural rules can be found in table 4.4.

We can now also extend the proof from proposition 4.3 to show that $BiMod \vDash A \to (\top \to A)$.

| # | Description |
|---|---|
| 0 | The thesis is stated by $P$, the players alternate |
| 1 | Players win if the other player has no more possible moves |
| 2c | Players can react to any previous move by the opponent |
| 3m | Only $O$ can assert new atomic formulas at a label |
| 4 | Strict repetitions are prohibited |
| E | $O$ can only react to the previous move |
| 5 | Only $O$ may introduce new labels |
| 6≤ | Reflexivity and transitivity of labels can be used |
| 7 | Atomic formulas can be used in all following labels |
| $6R-K$ | $P$ may only use labels $wRv$ that $O$ has previously introduced |
| $\Box$-p | A label of the form $x \cdot w.w'Rv \cdot y$ implies a label of the form $x \cdot wRv.v' \cdot y$ |

Table 4.4: Rules for *UniMod*
+ rules for *BiMod*

**4.5 Proposition** (Monotonicity). *BiMod* $\vDash A \rightarrow (\top \rightarrow A)$ for every formula $A$.

*Proof.* Clearly the proof from proposition 4.3 still works for non-modal formulas. In case $A = \Box B$, the winning strategy can easily be found by replacing $p$ with $B$ in dialogue 4.4 and then continuing with the copy-cat strategy introduced in proposition 4.4. □

As before, we can generalise this result to say that if $P$ has a winning strategy for the dialogue game associated with $A$ then $P$ also has a winning strategy for the dialogue game associated with $A[B/p]$.

**4.6 Remark.** Player $O$ wins a game if $P$ has no more possible moves to make. Since we are using the classical move rule, $P$ is able to respond to any previous move by $O$ unless this would be a strict repetition. This means that we cannot necessarily determine what $P$s next move will be just by knowing the dialogue sequent. However, we know that in order for $O$ to have a winning strategy, at the end of the game, for every move that $O$ has made, $P$ either is not able to respond or has already done so. In order to show that $O$ has a winning strategy it is therefore sufficient to look at $P$s response to $O$s move, without specifying at what point in the game it may occur.

Because $P$ can respond to any move, we will update our definition of the dialogue sequent.

**4.7 Definition.** We associate each round of a dialogue game with a dialogue sequent $\mathcal{G};\Pi \vdash \Delta, A$ where $\mathcal{G}$ is the set of all statements $wRv$ for which a label of the form $\vec{x}.w.v.\vec{y}$ has been introduced, $\Pi$ is the set of formulas that have been asserted by $O$, $\Delta$ is the set of all formulas that have been asserted by $P$ and $A$ is the active formula.

We will now show that the semantics we have developed in this section are sound and complete with respect to intuitionistic modal logic.

**4.8 Notation.** For a set $\mathcal{G}$ of assumptions of the form $wRv$ and $w \leq w'$ we write $M \vDash \mathcal{G}$ to say that the frame of a Kripke model $M$ satisfies these relations. This means that if we read $\mathcal{G}$ as a graph, the frame of $M$ has a sub-graph which is equivalent to $\mathcal{G}$, provided we rename the worlds of $M$ appropriately. We write $\mathcal{G}, \Pi \vDash \Delta$ to mean that for every model $M$ with $M \vDash \mathcal{G}$ and $M \vDash \Pi$ it also holds that $M \vDash \Delta$.

**4.9 Proposition.** If $P$ has a winning strategy for the *BiMod* dialogue game associated with a formula $F$ then $F$ is valid in K-IntMod.

*Proof.* We do this proof by contraposition, showing that if a formula $F$ is not valid in *K-IntMod*, then $O$ has a winning strategy for the *BiMod* dialogue game associated with $F$. We will prove this by showing that if $\mathcal{G};\Pi \vdash \Delta, A$ is the dialogue sequent at an O-node and $\mathcal{G};\Pi \nvDash \Delta, A$, then there is a possible move for $O$ which is a part of a winning strategy. $\mathcal{G}$, $\Pi$ and $\Delta$ may all be empty, so this will also prove that if $F$ is the thesis of a dialogue game and $\nvDash A$, then $O$ has a winning strategy for this dialogue game. More specifically, we show that $O$ has a strategy so that no $P$ strategy is a winning strategy.

Because $\mathcal{G};\Pi \nvDash \Delta, A$, there must be a model $M$ so that $M \vDash \mathcal{G}$, $M \vDash \Pi$ but $M \nvDash \Delta, A$. $O$ will only assert formulas that hold in this model $M$. Each attack of an asserted formula has one or two formulas which can be stated as a defence against this attack. $O$ will only attack formulas in such a way that all possible defence formulas do not hold in the model $M$. We claim that this describes a winning strategy for $O$. We prove by induction that for every dialogue tree and every $P$-node $x$ in the tree which has a dialogue sequent $\mathcal{G};\Pi \vdash \Delta, A$ such that the generating sub-tree is of depth $< d$ then if $\mathcal{G};\Pi \nvDash \Delta, A$ then there is no winning strategy for P.

If $d = 0$, this means player $P$ is unable to make a move. Clearly in this case $O$ has a winning strategy, whatever the dialogue sequent is.

We now assume that the assumption holds for $d < n$. This means that for every tree of depth $d < n$, if the dialogue sequent at the root of the tree is $\mathcal{G};\Pi \vdash \Delta, A$ and $\mathcal{G};\Pi \nvDash \Delta, A$ then there is no sub-tree which is a winning strategy for $P$.

We now look at a dialogue tree of depth $n$. Player $O$ can only react to the previous move by $P$, so we will show that for any move that $P$ makes, we can find a move $m$ with which $O$ can respond that is a part of the winning strategy. By remark 4.6, we can show that $m$ is a part of a winning $O$ strategy by showing that $O$ has a winning strategy after $P$ responds to $m$. If $P$ has no possible response to $m$, this is trivial.

Before we go into the case distinction however, we differentiate between three different scenarios in which $P$ can assert $w : A$.

(i) $w : A$ is the thesis

(ii) $w : A$ is the defence of a formula that $O$ has attacked

(iii) $w : A$ is the attack of a formula $v : A \to B$ that $O$ has stated

In case (i), there is no formula in $\Delta$. Clearly if $\mathcal{G}, \Pi \nvDash \Delta, w : A$ then $\mathcal{G}, \Pi \nvDash w : A$. In case (ii), $O$ has attacked a formula $u : B$ that $P$ has stated. By our strategy, this means $M \nvDash w : A$ and therefore that $\mathcal{G}, \Pi \nvDash w : A$. In case (iii) it must not necessarily hold that $\mathcal{G}, \Pi \nvDash w : A$, however we shall mention this case separately later on. We can therefore assume that if $P$ asserts $w : A$, we know that $M \nvDash w : A$.

1. $P$ asserts $w : A \wedge B$
   We therefore know that the dialogue sequent is of the form $\mathcal{G};\Pi \vdash \Delta, w : A \wedge B$ and $\mathcal{G};\Pi \nvDash \Delta, w : A \wedge B$. This means that there is a model $M$ and a world $w$ so that $M \vDash \mathcal{G}$, $M \vDash \Pi$ and $M, w \nVdash A \wedge B$. Clearly this means that $M, w \nVdash A$ or $M, w \nVdash B$. Based on this $O$ can play $?_L$ or $?_R$ respectively. Without loss of generality we assume $M, w \nVdash A$. Independently of when player $P$ responds to this attack, they will have to assert $A$, making the dialogue sequent $\mathcal{G}';\Pi' \vdash \Delta', w : A \wedge B, w : A$. Due to $O$s strategy we know that $M \vDash \mathcal{G}'$ and $M \vDash \Pi'$ but also that $M \nvDash A$. The induction hypothesis now tells us that $O$ still has a winning strategy.

2. $P$ asserts $w : A \vee B$
   We therefore know that the dialogue sequent is of the form $\mathcal{G};\Pi \vdash \Delta, w : A \vee B$ and $\mathcal{G};\Pi \not\models w : A \vee B$. This means that there is a model $M$ and a world $w$ so that $M \models \mathcal{G}$, $M \models \Pi$ and $M, w \not\Vdash A \vee B$. In particular, $M, w \not\Vdash A$ and $M, w \not\Vdash B$. If player $O$ attacks the disjunction with $?_{\vee}$, player $P$ may at some point defend this attack by asserting either $w : A$ or $w : B$, changing the dialogue sequent to $\mathcal{G}';\Pi' \vdash \Delta, w : A \vee B, w : A$ or $\mathcal{G}';\Pi' \vdash \Delta, w : A \vee B, w : B$. Clearly neither $w : A$ nor $w : B$ hold in $M$, so by our induction hypothesis $O$ has a winning strategy.

3. $P$ asserts $w : A \rightarrow B$
   We therefore know that the dialogue sequent is of the form $\mathcal{G};\Pi \vdash \Delta, w : A \rightarrow B$. This means there is a model $M$ and a world $w$ so that $M \models \mathcal{G}$, $M \models \Pi$ and $M, w \not\Vdash A \rightarrow B$. Specifically, there is a world $w'$ with $w \leq w'$, $M, w' \Vdash A$ and $M, w' \not\Vdash B$. $O$ may attack $w : A \rightarrow B$ by creating or using the label $w'$ and asserting $w' : A$. Player $P$ can either attack $w' : A$ or defend against the attack by stating $w' : B$.
   (i) If $P$ attacks $w' : A$, player $O$ has a winning strategy based on cases number 6 to 9.
   (ii) If $P$ asserts $w' : B$, the dialogue sequent is $\mathcal{G}';\Pi' \vdash \Delta', w' : B$. We know that $M \models \mathcal{G}'$, $M \models \Pi'$ but $M, w' \not\Vdash B$. Therefore $O$ has a winning strategy by induction hypothesis.

4. $P$ asserts $w : \Box A$
   We therefore know that the dialogue sequent is of the form $\mathcal{G};\Pi \vdash \Delta, w : \Box A$. This means there is a model $M$ and a world $w$ so that $M \models \mathcal{G}$, $M \models \Pi$ and $M, w \not\models \Box A$. Specifically, there is a world $v$ with $wRv$ and $M, v \not\models A$. $O$ may attack $w : \Box A$ by creating or using the label $v$ and playing $?_v$. When $P$ defends this move they must assert $v : A$. Since $v : A$ is not satisfied in $M$, $O$ has a winning strategy by induction hypothesis.

5. $P$ asserts $w : p$
   $P$ can assert $w : p$ in two situations.
   (i) As a defence against an attack by $O$

(ii) As an attack on an assertion $u : p \to B$

We also know that $P$ can only assert a propositional variable if $O$ has previously already done so. Since $O$ only asserts statements that hold in our model $M$, $w : p$ must hold there too. However, we have also stated that $O$ only attacks formulas which do not hold in the model. Therefore it cannot be that $P$ has asserted $w : p$ in situation $(i)$. In situation $(ii)$, $O$ can simply assert $w : B$, just as in case number 3.

6. P plays $w : ?_L$ or $w : ?_R$
   We will only go through the case for $w : ?_L$. This means that $O$ has previously asserted $w : A \wedge B$ and therefore the dialogue sequent is of the form $\mathcal{G}; \Pi, w : A \wedge B \vdash \Delta$. We know that $\mathcal{G}; \Pi, w : A \wedge B \not\models \Delta$. This means there is a model $M$ so that $M \models \mathcal{G}, \Pi$, $M, w \Vdash A \wedge B$ and $M \not\models \Delta$. In particular this means that $M, w \Vdash A$ and $M, w \Vdash B$. $O$ must defend against the attack by stating $w : A$. Player $O$ can always assert $w : A$, unless it is $w : \bot$. However, $A$ cannot be $w : \bot$ because otherwise from $M, w \Vdash \bot \wedge B$ it would follow that $M \models \Delta$. $P$ is allowed to attack $w : A$ at some later point, where the dialogue sequent would be $\mathcal{G}'; \Pi', w : A \wedge B, w : A \vdash \Delta'$. It clearly also holds that $\mathcal{G}'; \Pi', w : A \wedge B, w : A \not\models \Delta$ based on our assumption, therefore $O$ has a winning strategy by induction hypothesis.

7. $P$ plays $w : ?_\vee$
   This means that $O$ has previously asserted $w : A \vee B$ and therefore the dialogue sequent is of the form $\mathcal{G}; \Pi, w : A \vee B \vdash \Delta$. We know that $\mathcal{G}; \Pi, w : A \vee B \not\models \Delta$. In particular this means that either $M, w \Vdash A$ or $M, w \Vdash B$. $O$ must defend against the attack by stating $w : A$ or $w : B$, they can choose the one which holds in $M$, without loss of generality we assume it is $w : A$. $P$ is allowed to attack $A$ at some later point, where the dialogue sequent would be $\mathcal{G}'; \Pi', w : A \vee B, w : A \vdash \Delta'$. It clearly also holds that $\mathcal{G}; \Pi', w : A \vee B, w : A \not\models \Delta'$ based on our assumption, therefore $O$ has a winning strategy by induction hypothesis.

8. $P$ attacks $w : A \to B$ by asserting $w' : A$ at a world $w'$ with $w \leq w'$
   This means that $O$ has previously asserted $w : A \to B$ and introduced the

label $w \leq w'$ and therefore the dialogue sequent is of the form $\mathcal{G}, w \leq w'; \Pi, w : A \to B \vdash \Delta$. We know that $\mathcal{G}; \Pi, w : A \to B \nvDash \Delta$. This means that there is a model so that $M \vDash \mathcal{G}, w \leq w'$, $M \vDash \Pi, w : A \to B$ and $M, w \nVdash \Delta$. Specifically, either $M, w' \nVdash A$ or $M, w' \Vdash B$. We distinguish between these cases.

(i) $M, w' \nVdash A$
$O$ chooses to attack $w' : A$. The case that $A = p$ has already been discussed in case number 3. Depending on the formula $A$, this is handled in cases number 1 to 5.

(ii) $M, w' \Vdash B$
$O$ chooses to defend against the attack by asserting $w' : B$. $P$ is allowed to attack $w' : B$ at some later point, where the dialogue sequent would be $\mathcal{G}', w \leq w'; \Pi', w : A \to B, w' : B \vdash \Delta', w' : A$. It clearly also holds that $\mathcal{G}', w \leq w'; \Pi', w : A \to B, w' : B \nvDash \Delta', w' : A$ based on our assumption, therefore $O$ has a winning strategy by induction hypothesis.

9. $P$ plays $w : ?_v$
This means that $O$ has previously asserted $w : \Box A$ and introduced the label $wRv$ and therefore the dialogue sequent is of the form $\mathcal{G}, wRv; \Pi, w : \Box A \vdash \Delta$. We know that $\mathcal{G}, wRv; \Pi, w : \Box A \nvDash \Delta$. This means that there is a model $M$ so that $M \vDash \mathcal{G}, wRv$, $M \vDash \Pi, w : \Box A$ and $M, w \nvDash \Delta$. Specifically, for all worlds $u$ with $wRu$ it holds that $M, u \vDash A$. $O$ can now defend against this attack by stating $v : A$. $P$ is allowed to attack $v : A$ at some later point, where the dialogue sequent would be $\mathcal{G}', wRv; \Pi', w : \Box A, v : A \vdash \Delta$. It clearly also holds that $\mathcal{G}', wRv; \Pi', w : \Box A, v : A \nvDash \Delta$ based on our assumption, therefore $O$ has a winning strategy by induction hypothesis.

$\Box$

**4.10 Proposition.** If $A$ is valid in *K-IntMod* then $P$ has a winning strategy for the *BiMod* game associated with $A$.

*Proof.* We know that $A$ is valid in *K-IntMod* iff there is a *S-IntMod* derivation of $A$. We will show that if there is a *S-IntMod* derivation $\delta$ of a sequent $\mathcal{G}; \Gamma \Rightarrow x : A$, then $P$ has a winning strategy for the $O$-node with the dialogue sequent $\mathcal{G}; \Gamma \vdash x : A, \Delta$.

We prove this by induction on the depth $d$ of the derivation $\delta$.

While we are using the classical move rule and our dialogue sequents may have more than one formula on the right hand side, we can assume that the formulas on the right hand side of the sequent is the active formula.

If $d = 1$ then the derivation consists only of an axiom.

1. $\dfrac{}{\mathcal{G};\Gamma, x:A \Rightarrow x:A}\ Ass$
   Then the dialogue sequent is $\mathcal{G};\Gamma, x:A \vdash x:A, \Delta$ and $O$ must attack the formula $x:A$. If $A = p$ then this is not possible and the winning condition for $P$ is fulfilled. Otherwise, $P$ can copy this move to attack $x:A$ and use the copying strategy we have encountered several times before in order to have a winning strategy.

2. $\dfrac{}{\mathcal{G};\Gamma x:\bot \Rightarrow z:A}\ \bot L$
   $O$ is not allowed to state $x:\bot$ and it is therefore not possible to reach this exact game state. However, we will show that $P$ in this case has a winning strategy for $\mathcal{G},\Gamma \vdash z:A$. If the dialogue sequent were $\mathcal{G},\Gamma, x:\bot \vdash z:A, \Delta$, then $P$ would have asserted $z:A$ as a response to a previous move by $O$ since there is no response to $x:\bot$. Clearly $P$ would have been able to assert $z:A$ in the previous turn already since no additional information was gained by $O$ asserting $x:\bot$. If we switch the order of these two moves by $P$ then the dialogue sequent at the last O-node is $\mathcal{G},\Gamma \vdash z:A$. Since $O$ clearly has no possible response to the now last move by $P$, the winning condition for $P$ is fulfilled. Therefore $P$ has a winning strategy for $\mathcal{G},\Gamma \vdash z:A$. Using this argumentation we from now on assume that all dialogue sequents we look at are possible game states.

We now assume the claim holds for derivations of depth $d < n$ and look at a derivation of depth $n$. We differentiate by the last rule that is used in the derivation.

1. $\dfrac{\mathcal{G};\Gamma, x:A, x:B \Rightarrow z:C}{\mathcal{G};\Gamma x:A \wedge B \Rightarrow z:C}\ \wedge L$
   The dialogue sequent is therefore $\mathcal{G};\Gamma, x:A \wedge B \vdash z:C, \Delta$ and $O$ must attack the formula $z:C$. $P$ can now attack $x:A \wedge B$ with both $x:?_L$ and $x:?_R$

which $O$ must both defend. After this the dialogue sequent is $\mathcal{G};\Gamma,x:A\wedge B,x:A,x:B\vdash z:C,\Delta$. Since we know there is a derivation of the sequent $\mathcal{G};\Gamma,x:A,x:B\Rightarrow z:C$, by induction hypothesis $P$ has a winning strategy.

2. $\dfrac{\mathcal{G};\Gamma\Rightarrow x:A \quad \mathcal{G};\Gamma\Rightarrow x:B}{\mathcal{G};\Gamma\Rightarrow x:A\wedge B}\wedge R$

The dialogue sequent is $\mathcal{G};\Gamma\vdash x:A\wedge B,\Delta$ and $O$ must attack $x:A\wedge B$, without loss of generality we assume $O$ plays $x:?_L$. $P$ can respond by asserting $x:A$, making the dialogue sequent $\mathcal{G};\Gamma\Rightarrow x:A,\Delta$. Since we know there is a derivation of $\mathcal{G};\Gamma\vdash x:A$, there is a winning strategy for $P$ by induction hypothesis. We know that $P$ can assert $x:A$ because there is a winning strategy for the dialogue sequent $\mathcal{G};\Gamma\vdash x:A$, which means it is a possible game state. From now on we will not mention this explicitly every time.

3. $\dfrac{\mathcal{G};\Gamma,x:A\Rightarrow z:C \quad \mathcal{G};\Gamma,x:B\Rightarrow z:C}{\mathcal{G};\Gamma,x:A\vee B\Rightarrow z:C}\vee L$

The dialogue sequent is $\mathcal{G};\Gamma,x:A\vee B\vdash z:C,\Delta$ and $O$ must attack $z:C$. $P$ can now attack $x:A\vee B$ by playing $x:?_\vee$. Without loss of generality we assume $O$ defends by asserting $x:A$. After this the dialogue sequent is $\mathcal{G};\Gamma,x:A\vee B,x:A\vdash z:C,\Delta$. Since we know there is a derivation of $\mathcal{G};\Gamma,x:A\Rightarrow z:C$, $P$ has a winning strategy by induction hypothesis.

4. $\dfrac{\mathcal{G};\Gamma\Rightarrow x:A_i}{\mathcal{G};\Gamma\Rightarrow x:A_1\vee A_2}\vee R_i$

The dialogue sequent is $\mathcal{G};\Gamma\Rightarrow x:A_1\vee A_2$ and $O$ must attack $x:A_1\vee A_2$ by playing $?_\vee$. We know there is a derivation of either $\mathcal{G};\Gamma\Rightarrow x:A_1$ or $\mathcal{G};\Gamma\Rightarrow x:A_2$ so $P$ plays $x:A_1$ or $x:A_2$ respectively and the dialogue sequent is now $\mathcal{G};\Gamma\Rightarrow x:A_i,x:A_1\vee A_2,\Delta$. Since we know there is a derivation of $\mathcal{G};\Gamma\Rightarrow x:A_i$, $P$ has a winning strategy by induction hypothesis.

5. $\dfrac{\mathcal{G};\Gamma,x:B\Rightarrow z:C \quad \mathcal{G};\Gamma\Rightarrow x:A}{\mathcal{G};\Gamma,x:A\to B\Rightarrow z:C}\to L$

The dialogue sequent is $\mathcal{G};\Gamma,x:A\to B\vdash z:C,\Delta$ and $O$ must attack $z:C$. $P$ can now attack $x:A\to B$ by asserting $x:A$. $O$ now has two options of which move to make.

(i) If $O$ attacks $x:A$ then the dialogue sequent is now $\mathcal{G};\Gamma,x:A\to B\vdash x:A,\Delta$. Since we know there is a derivation of $\mathcal{G};\Gamma\Rightarrow x:A$, $P$ has a winning strategy by induction hypothesis.

(ii) If $O$ asserts $x:B$ then the dialogue sequent is $\mathcal{G};\Gamma,x:A\to B,x:B\vdash z:C,x:A,\Delta$. Since we know there is a derivation of $\mathcal{G};\Gamma,x:B\Rightarrow z:C$, $P$ has a winning strategy by induction hypothesis.

6. $\dfrac{\mathcal{G};\Gamma,x:A\Rightarrow x:B}{\mathcal{G};\Gamma\Rightarrow x:A\to B}\to R$

The dialogue sequent is $\mathcal{G};\Gamma\vdash x:A\to B,\Delta$ and $O$ must attack $x:A\to B$ by asserting $x':A$ and $x\leq x'$. Since there is a derivation of $\mathcal{G};\Gamma,x:A\Rightarrow x:B$, we know that $P$ has a winning strategy for the dialogue sequent $\mathcal{G};\Gamma,x:A\vdash x:B,\Delta$. In particular this means that it is possible for $P$ to defend this attack by asserting $x':B$, even if it may be necessary for $P$ to make some other moves first in order to gain more information. By induction hypothesis, $P$ has a winning strategy.

7. $\dfrac{\mathcal{G};\Gamma,y:A\Rightarrow z:B}{\mathcal{G},xRy;\Gamma,x:\Box A\Rightarrow z:B}\Box L$

The dialogue sequent is $\mathcal{G},xRy;\Gamma,x:\Box A\vdash z:B,\Delta$ and $O$ must attack $z:B$. $P$ can now attack $x:\Box A$ by playing $?_y$. $O$ defends against this attack by asserting $y:A$ making the dialogue sequent $\mathcal{G}xRy;\Gamma,x:\Box A,y:A\vdash z:B,\Delta$. Since we know there is a derivation of $\mathcal{G};\Gamma,y:A\Rightarrow z:B$, $P$ has a winning strategy by induction hypothesis.

8. $\dfrac{\mathcal{G},xRy;\Gamma\Rightarrow y:A}{\mathcal{G};\Gamma\Rightarrow x:\Box A}\Box R$

The dialogue sequent is $\mathcal{G};\Gamma\vdash x:\Box A,\Delta$ and $O$ must attack $x:\Box A$ by playing $x:?_y$ and asserting $xRy$. $P$ can respond to this by stating $y:A$, making the dialogue sequent $\mathcal{G},xRy;\Gamma\vdash y:A,\Delta$. Since there is a derivation of $\mathcal{G},xRy;\Gamma\Rightarrow y:A$, $P$ has a winning strategy by induction hypothesis.

We have thereby proven our claim that if there is a *S-IntMod* derivation $\delta$ of a sequent $\mathcal{G};\Gamma\Rightarrow x:A$, then $P$ has a winning strategy for the $O$-node with the dialogue sequent $\mathcal{G};\Gamma\vdash x:A,\Delta$. In particular this means that if there is a derivation of a sequent $\Rightarrow x : A$, then $P$ has a winning strategy for the *BiMod* dialogue game associated with $A$. □

**4.11 Theorem.** *BiMod* dialogue semantics are sound and complete with respect to *IntMod*.

*Proof.* This follows directly from propositions 4.9 and 4.10 □

This makes *BiMod* dialogue semantics a suitable alternative to the dialogue games for intuitionistic modal logic we introduced in section 3.2.

### 4.1.3 ADDING STRICT IMPLICATION

In this section we will now show that the *BiMod* dialogue semantics we have established can be extended to model *IntModSI*. We do this by adding the particle rules for $\multimap$ that we have introduced in section 3.3 which can be found again in table 4.5.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\multimap$ | $w : A \multimap B$ | $v : A$ for any $v$ with $wRv$ | $v : B$ |

Table 4.5: Particle rules for strict implication

We now also need to ensure that monotonicity of $\multimap$ formulas is guaranteed. This is achieved by adding a structural rule with corresponds to $\multimap$-p from 2.1.

**Rule $\multimap$-p.** If a label of the form $x \cdot w.w'Rv \cdot y$, where $\cdot \in \{.,R\}$ and $x$ and $y$ are labels, has been introduced, then both players can use a label of the form $x \cdot wRv \cdot y$.

The complete list of structural rules for *BiModSI* can be found in table 4.6.

We now show that monotonicity of strict implication formulas holds in this dialogue semantics.

**4.12 Proposition** (Monotonicity). *BiModSI* $\vDash A \to (\top \to A)$ for every $\mathcal{L}_{\multimap}$ formula $A$.

*Proof.* Clearly it is enough to prove that *BiModSI* $\vDash (A \multimap B) \to (\top \to (A \multimap B))$ since all other cases have already been proven in propositions 4.3 and 4.5.

We prove this by giving the beginning of a winning strategy for $P$ in dialogue 4.5. From move VI onwards player $P$ can simply copy any move that $O$ makes. □

| # | Description |
|---|---|
| 0 | The thesis is stated by $P$, the players alternate |
| 1 | Players win if the other player has no more possible moves |
| 2c | Players can react to react to any previous move by the opponent |
| 3m | Only $O$ can assert new atomic formulas at a label |
| 4 | Strict repetitions are prohibited |
| E | $O$ can only react to the previous move |
| 5 | Only $O$ may introduce new labels |
| 6≤ | Reflexivity and transitivity of labels can be used |
| 7 | Atomic formulas can be used in all following labels |
| $6R-K$ | $P$ may only use labels $wRv$ that $O$ has previously introduced |
| $\Box$-p | A label of the form $x \cdot w.w'Rv \cdot y$ implies a label of the form $x \cdot wRv.v' \cdot y$ |
| $\prec$-p | A label of the form $x \cdot w.w'Rv \cdot y$ implies a label of the form $x \cdot wRv \cdot y$ |

Table 4.6: Rules for *BiMod*
+ rule for *BiModSI*

| # | Player | Expression | Label | Description | Dialogue Sequent |
|---|---|---|---|---|---|
| T | $P$ | $1:F$ | 1 | Thesis | $\vdash 1:F$ |
| I | $O$ | $2:A \prec B$ | 1.2 | Attack T | $2:\Box p \vdash 1:F$ |
| II | $P$ | $2:\top \to (A \prec B)$ | 1.2 | Defence I | $2:\Box p \vdash 2:\top \to (A \prec B)$ |
| III | $O$ | $3:\top$ | 1.2.3 | Attack II | $2:\Box p, 3:\top \vdash 2:\top \to (A \prec B)$ |
| IV | $P$ | $3:A \prec B$ | 1.2.3 | Defence III | $2:\Box p, 3:\top \vdash 3:A \prec B$ |
| V | $O$ | $4:A$ | $1.2.3R4$ | Attack IV | $2:\Box p, 3:\top, 4:A \vdash 3:A \prec B$ |
| VI | $P$ | $4:A$ | $1.2R4$ | Attack I | $2:\Box p, 3:\top, 4:A \vdash 4:A$ |

Dialogue 4.5: Monotonicity of $\prec$, $F = (A \prec B) \to (\top \to (A \prec B))$

We now show that in the semantics *BiModSI*, the interdefinability of $\Box$ and $\prec$ does not hold, just as we intended.

**4.13 Proposition.** *BiModSI* $\vDash \Box(p \to q) \to (p \prec q)$.

*Proof.* We provide a winning strategy for $P$ for the associated game in dialogue 4.6. $\Box$

**4.14 Proposition.** *BiModSI* $\nvDash (p \prec q) \to \Box(p \to q)$

*Proof.* We can easily see that the complete dialogue tree from dialogue 4.7 does not contain a sub-tree that is a winning strategy for $P$. $\Box$

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : \Box(p \to q) \to (p \prec q)$ | 1 | Thesis |
| I | $O$ | $2 : \Box(p \to q)$ | 1.2 | Attack T |
| II | $P$ | $2 : p \prec q$ | 1.2 | Defence I |
| III | $O$ | $3 : p$ | $1.2R3$ | Attack II |
| IV | $P$ | $2 : ?_3$ | $1.2R3$ | Attack I |
| V | $O$ | $3 : p \to q$ | $1.2R3$ | Defence IV |
| VI | $P$ | $3 : p$ | $1.2R3$ | Attack V |
| VII | $O$ | $3 : q$ | $1.2R3$ | Defence VI |
| VIII | $P$ | $3 : q$ | $1.2R3$ | Defence III |

Dialogue 4.6: $\Box(p \to q) \to (p \prec q)$

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : (p \prec q) \to \Box(p \to q)$ | 1 | Thesis |
| I | $O$ | $2 : p \prec q$ | 1.2 | Attack T |
| II | $P$ | $2 : \Box(p \to q)$ | 1.2 | Defence I |
| III | $O$ | $2 : ?_3$ | $1.2R3$ | Attack II |
| IV | $P$ | $3 : p \to q$ | $1.2R3$ | Defence III |
| V | $O$ | $4 : p$ | $1.2R3.4$ | Attack IV |

Dialogue 4.7: $(p \prec q) \to \Box(p \to q)$

We will now also show that *BiModSI* is sound and complete with respect to intuitionistic modal logic with strict implication.

**4.15 Proposition.** If a formula $A$ is not valid in intuitionistic modal logic, then $O$ has a winning strategy for the *BiModSI* dialogue game associated with $A$.

*Proof.* We simply extend the proof from proposition 4.9 with the cases concerning $\prec$.

1. $P$ asserts $w : A \prec B$
   We therefore know that the dialogue sequent is of the form $\mathcal{G};\Pi \vdash \Delta, w : A \prec B$. This means there is a model $M$ and a world $w$ so that $M \vDash \mathcal{G}$, $M \vDash \Pi$ and $M,w \nVdash A \prec B$. Specifically, there is a world $v$ with $wRv$, $M,v \Vdash A$ and $M,v \nVdash B$. $O$ may attack $w : A \prec B$ by creating or using the label $v$ and asserting $v : A$. $P$ can either attack $v : A$ or defend by stating $v : B$. If $P$ attacks $v : A$, player $O$ has a winning strategy based on the other cases

and the induction hypothesis. If $P$ asserts $v : B$, the dialogue sequent is $\mathcal{G}'; \Pi' \vdash \Delta, w : A \prec\!\!\!\!\prec B, v : B$. Since $O$ only asserts formulas which hold in the model $M$, we know that $M \vDash \mathcal{G}'$, $M \vDash \Pi'$ but $M, w' \nVdash v : B$. Therefore $O$ has a winning strategy by induction hypothesis.

2. $P$ attacks $w : A \prec\!\!\!\!\prec B$ by asserting $v : A$
   This means that $O$ has previously asserted $w : A \prec\!\!\!\!\prec B$ and $wRv$, therefore the dialogue sequent is of the form $\mathcal{G}, wRv; \Pi, w : A \prec\!\!\!\!\prec B \vdash \Delta, x : F$. We know that $\mathcal{G}, wRv; \Pi, w : A \prec\!\!\!\!\prec B \nvDash \Delta, x : F$. This means that there is a model so that $M \vDash \mathcal{G}, wRv$, $M \vDash \Pi, A \prec\!\!\!\!\prec B$ and $M \nvDash \Delta, x : F$. Specifically, either $M, v \nVdash A$ or $M, v \Vdash B$. We distinguish between these cases.

   (i) $M, v \nVdash A$
   $O$ chooses to attack $v : A$. The case that $A = p$ is not possible, just as the case 3 in the proof of proposition 4.9. Depending on the formula $A$, this is handled in the other cases of this proof.

   (ii) $M, v \Vdash B$
   $O$ chooses to defend against the attack by asserting $v : B$. $P$ is allowed to attack $v : B$ at some later point, where the dialogue sequent would be $\mathcal{G}', wRv; \Pi', w : A \prec\!\!\!\!\prec B, v : B \vdash \Delta, x' : F'$. It clearly also holds that $\mathcal{G}', wRv; \Pi', w : A \prec\!\!\!\!\prec B, v : B \nvDash \Delta, x : F$ based on our assumption, therefore $O$ has a winning strategy by induction hypothesis.

□

Since we have not found a suitable sequent calculus for *IntModSI* in the literature, we will not extend the proof of proposition 4.10. However, we have given a Hilbert-style axiomatisation of *IntModSI* in section 2.3. We will show that if there is a Hilbert-style deduction of a formula $A$, then $P$ has a winning strategy for the associated *BiModSI* game. We do this by showing that all axioms of $H\text{-}IntModSI$ and the inference rules are valid in *BiModSI*.

There are two main reasons why we would prefer to use a sequent calculus instead. Firstly, dialogue game semantics are, by their very nature, very closely related to sequent calculus. They are considered to be "proof-theoretic semantics" since a winning strategy for player $P$ can easily be transformed into a proof of the thesis, just as we have seen in proposition 4.10. Therefore it is desirable to use and highlight this connection by showing that every sequent calculus derivation can be translated into a winning strategy for $P$. Secondly, in the proof of 4.17 we assume that $O$ has a winning strategy for a dialogue which does not utilise rule E if and only if $O$ has a winning strategy for a dialogue which does utilise rule E. However, we do not formally prove this claim. While we have given an intuition in section 3.1.1, this does mean there is a slight formal gap in our proof. By using the sequent calculus where it is possible we are able to ensure the formal completeness of this proof at least. Nevertheless, we present the proof for the Hilbert-style calculus in order to to provide an assurance that the dialogue semantics we have introduced are sound and complete.

**4.16 Proposition.** $P$ has a winning strategy for all axioms of $H\text{-}IntModSI$.

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : (p_1 \prec p_2) \rightarrow ((p_2 \prec p_3) \rightarrow (p_1 \prec p_3))$ | 1 | Thesis |
| I | $O$ | $2 : p_1 \prec p_2$ | 1.2 | Attack T |
| II | $P$ | $2 : (p_2 \prec p_3) \rightarrow (p_1 \prec p_3)$ | 1.2 | Defence I |
| III | $O$ | $3 : p_2 \prec p_3$ | 1.2.3 | Attack II |
| IV | $P$ | $3 : p_1 \prec p_3$ | 1.2.3 | Defence III |
| V | $O$ | $4 : p_1$ | 1.2.3$R$4 | Attack IV |
| VI | $P$ | $4 : p_1$ | 1.2$R$4 | Attack I |
| VII | $O$ | $4 : p_2$ | 1.2$R$4 | Defence VI |
| VII | $P$ | $4 : p_2$ | 1.2.3$R$4 | Attack III |
| VIII | $O$ | $4 : p_3$ | 1.2.3$R$4 | Defence VII |
| IX | $P$ | $4 : p_3$ | 1.2.3$R$4 | Defence V |

Dialogue 4.8: $(p_1 \prec p_2) \rightarrow ((p_2 \prec p_3) \rightarrow (p_1 \prec p_3))$

*Proof.* Since we have already shown that *BiMod* is sound and complete with respect to intuitionistic modal logic, we will only go through the axioms that contain strict implication. Specifically we will show that

1. *BiModSI* $\vDash (p_1 \prec p_2) \rightarrow ((p_2 \prec p_3) \rightarrow (p_1 \prec p_3))$
2. *BiModSI* $\vDash (p_1 \prec p_2) \rightarrow ((p_1 \prec p_3) \rightarrow (p_1 \prec (p_2 \wedge p_3)))$
3. *BiModSI* $\vDash (p_1 \prec p_3) \rightarrow ((p_2 \prec p_3) \rightarrow ((p_1 \vee p_2) \prec p_3))$

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : F$ | 1 | Thesis |
| I | $O$ | $2 : p_1 \strictif p_2$ | 1.2 | Attack T |
| II | $P$ | $2 : (p_1 \strictif p_3) \rightarrow (p_1 \strictif (p_2 \wedge p_3))$ | 1.2 | Defence I |
| III | $O$ | $3 : p_1 \strictif p_3$ | 1.2.3 | Attack II |
| IV | $P$ | $3 : p_1 \strictif (p_2 \wedge p_3)$ | 1.2.3 | Defence III |
| V | $O$ | $4 : p_1$ | 1.2.3$R$4 | Attack IV |
| VI | $P$ | $4 : p_2 \wedge p_3$ | 1.2.3$R$4 | Defence V |

| # | Player | Expression | Label | Description | Expression | Label | Description |
|---|---|---|---|---|---|---|---|
| VII | $O$ | $4 : ?_L$ | 1.2.3$R$4 | Defence VI | $4 : ?_R$ | 1.2.3$R$4 | Defence VI |
| VIII | $P$ | $4 : p_1$ | 1.2$R$4 | Attack I | $4 : p_1$ | 1.2.3$R$4 | Attack III |
| IX | $O$ | $4 : p_2$ | 1.2$R$4 | Defence VIII | $4 : p_3$ | 1.2.3$R$4 | Defence VIII |
| X | $P$ | $4 : p_2$ | 1.2.3$R$4 | Defence VI | $4 : p_3$ | 1.2.3$R$4 | Defence VI |

Dialogue 4.9: $F = (p_1 \strictif p_2) \rightarrow ((p_1 \strictif p_3) \rightarrow (p_1 \strictif (p_2 \wedge p_3)))$

For each axiom we will give a winning strategy for $P$.

1. A winning strategy for $P$ for the *BiModSI* game associated with $(p_1 \strictif p_2) \rightarrow ((p_2 \strictif p_3) \rightarrow (p_1 \strictif p_3))$ can be found in dialogue 4.8.

2. A winning strategy for $P$ for the *BiModSI* game associated with $(p_1 \strictif p_2) \rightarrow ((p_1 \strictif p_3) \rightarrow (p_1 \strictif (p_2 \wedge p_3)))$ can be found in dialogue 4.9.

3. A winning strategy for $P$ for the *BiModSI* game associated with $(p_1 \strictif p_3) \rightarrow ((p_2 \strictif p_3) \rightarrow ((p_1 \vee p_2) \strictif p_3))$ can be found in dialogue 4.10.

Since we have found a winning strategy for $P$ for every axiom of *HIntModSI*, they are all valid in *BiModSI*. □

**4.17 Proposition.** If $P$ has a winning strategy for the *BiModSI* games associated with $A \rightarrow B$ and $A$ then $P$ also has a winning strategy for the *BiModSI* game associated with $B$.

*Proof.* We prove this by showing that if $P$ does not have a winning strategy for $B$ then $P$ either does not have a winning strategy for $A \rightarrow B$ or for $A$.

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : F$ | 1 | Thesis |
| I | $O$ | $2 : p_1 -\!< p_3$ | 1.2 | Attack T |
| II | $P$ | $2 : (p_2 -\!< p_3) \to ((p_1 \vee p_2) -\!< p_3)$ | 1.2 | Defence I |
| III | $O$ | $3 : p_2 -\!< p_3$ | 1.2.3 | Attack II |
| IV | $P$ | $3 : (p_1 \vee p_2) -\!< p_3$ | 1.2.3 | Defence III |
| V | $O$ | $4 : p_1 \vee p_2$ | 1.2.3$R$4 | Attack IV |
| VI | $P$ | $4 : ?_\vee$ | 1.2.3$R$4 | Attack V |

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| VII | $O$ | $4 : p_1$ | 1.2.3$R$4 | Defence VI |
| VIII | $P$ | $4 : p_1$ | 1.2$R$4 | Attack I |
| IX | $O$ | $4 : p_3$ | 1.2$R$4 | Defence VIII |
| X | $P$ | $4 : p_3$ | 1.2.3$R$4 | Defence V |

| Expression | Label | Description |
|---|---|---|
| $4 : p_2$ | 1.2.3$R$4 | Defence VI |
| $4 : p_2$ | 1.2.3$R$4 | Attack I |
| $4 : p_3$ | 1.2.3$R$4 | Defence VIII |
| $4 : p_3$ | 1.2.3$R$4 | Defence V |

Dialogue 4.10: $F = (p_1 -\!< p_3) \to ((p_2 -\!< p_3) \to ((p_1 \vee p_2) -\!< p_3))$

| # | Player | Expression | Label | Description |
|---|---|---|---|---|
| T | $P$ | $1 : F$ | 1 | Thesis |
| I | $O$ | $2 : p_1 -\!< p_2$ | 1.2 | Attack T |
| II | $P$ | $2 : (p_1 -\!< p_3) \to (p_1 -\!< (p_2 \wedge p_3))$ | 1.2 | Defence I |
| III | $O$ | $3 : p_1 -\!< p_3$ | 1.2.3 | Attack II |
| IV | $P$ | $3 : p_1 -\!< (p_2 \wedge p_3)$ | 1.2.3 | Defence III |
| V | $O$ | $4 : p_1$ | 1.2.3$R$4 | Attack IV |
| VI | $P$ | $4 : p_2 \wedge p_3$ | 1.2.3$R$4 | Defence V |

Dialogue 4.11: $F = (p_1 -\!< p_2) \to ((p_1 -\!< p_3) \to (p_1 -\!< (p_2 \wedge p_3)))$

We know that the *BiModSI* game associated with $A \to B$ begins with $O$ attacking the thesis by creating the label 1.2 and asserting $2 : A$. We assume that $P$ has a winning strategy $S_1$ for the game associated with $A$. If we disregard rule E, $O$ can copy strategy $S_1$ and therefore also has a winning strategy for asserting $2 : A$. In particular however, this is a winning strategy for $O$ in which $O$ does not state any propositional variables $p$. This is because, since it was originally $P$s strategy, $S_1$ only contains assertions of propositional variables after the other player has asserted them. Since $P$ is not able to assert propositional variables first, the strategy does not contain any assertions of propositional variables by either player. This means that whenever $P$ attacks $2 : A$ or responds to moves following from this, $O$ will have a response which is a part of a winning strategy and does not force $O$ to assert any propositional variables.

Therefore, $P$ cannot win the game $A \to B$ without defending $O$s original attack and stating $2:B$.

If we assume that $P$ does not have a winning strategy for the game associated with $B$ then, because this is a finite two player zero-sum game, $O$ must have a winning strategy $S_2$ for this game. Since $O$ is able to play $S_1$ without giving $P$ the possibility of asserting any additional propositional variables, we can combine $S_1$ and $S_2$ to obtain $S_3$, a winning strategy for $O$ for the *BiModSI* game associated with $A \to B$. Of course the way we have constructed this strategy means it may not necessarily follow rule E. However, as we described briefly when introducing this rule, it is possible to translate every winning strategy for $O$ into a winning strategy for $O$ that obeys rule E.

However, this contradicts the fact that $P$ has a winning strategy for the game associated with $A \to B$. Therefore $P$ must have a winning strategy for the *BiModSI* game associated with $B$. □

**4.18 Proposition.** If $P$ has a winning strategy for the *BiModSI* game associated with $A \to B$ then $P$ has a winning strategy for the *BiModSI* game associated with $A \multimap B$.

*Proof.* We assume $P$ has a winning strategy for $A \to B$. We know that this game begins with $O$ attacking the thesis by creating the label 1.2 and asserting $2:A$. After this, $P$ follows their winning strategy, which we shall call $S$. The game associated with $A \multimap B$ begins with $O$ attacking the thesis by creating the label $1R2$ and asserting $2:A$. Clearly $P$ has the same amount of information about the world 2 in both games, even though the labels differ slightly. $P$ can simply copy the winning strategy $S$ in this game and therefore also has a winning strategy for $A \multimap B$. □

**4.19 Theorem.** If there is a *H-IntModSI* deduction of $A$ then $P$ has a winning strategy for the *BiModSI* game associated with $A$.

*Proof.* Directly follows from propositions 4.4, 4.16, 4.17 and 4.18. □

## 4.2 SUBGAME APPROACH

In this section we will define a game semantics for intuitionistic modal logic which uses the same particle rules that we also used for modal logic without strict implication.

Our main goal is to ensure that the interdefinability of $\Box$ and $⥽$ does not hold, specifically we do not want $(p ⥽ q) \to \Box(p \to q)$ to be valid. If we think back to dialogue 3.7, the reason $P$ had a winning strategy was because they were able to use the information that $O$ asserted while attacking an implication. However, in the counterexample we constructed in the proof of lemma 2.3, the antecedent of the implication was not true in $v$ itself but in a world $v'$ with $v \leq v'$.

Intuisionistically, we therefore want to interpret $O$ attacking the implication $v : A \to B$ by asserting $v : A$ as $O$ stating that there is some world in the cone above $v$ where $A$ holds. Player $P$ can use this information but they do not know in which of the world accessible from $v$ they are currently playing in. Most importantly, $P$ should not be able to use the assertion of $A$ in $v$, since that is not necessarily where $O$ has asserted it. In our *BiModSI* dialogues we have modelled this directly using the intuitionistic accessibility relation. In the dialogue semantics we define in this section, we will instead represent this as a restriction of information usage by introducing subgames to the structure of our dialogue games. Each subgame will represent a dialogue about a cone above some world $v$ and the players will not be able to use the information they gained outside of it. We will call this semantics *Sub*.

We begin by giving the particle rules we will be using for these dialogues. As mentioned above, these are the same rules that we have introduced in chapter 3. They can be found again in table 4.7.

The structural rules we will be using also begin with the standard rules we have introduced for modal dialogue games. We only repeat them here briefly in the form of table 4.8, the detailed versions can be found in section 3.2. Since we are modelling the intuitionistic aspect of the logic by introducing subgames, we will be using the classical move rule.

| | Assertion | Attack | Defence |
|---|---|---|---|
| $\bot$ | cannot be asserted | | |
| $p$ | $w:p$ | cannot be attacked | |
| $\wedge$ | $w:A\wedge B$ | $w:?_L$<br>$w:?_R$ | $w:A$<br>$w:B$ |
| $\vee$ | $w:A\vee B$ | $w:?_\vee$ | $w:A$ or $w:B$ |
| $\rightarrow$ | $w:A\rightarrow B$ | $w:A$ | $w:B$ |
| $\neg$ | $w:\neg A$ | $w:A$ | cannot be defended |
| $\Box$ | $w:\Box A$ | $w:?_v$ for any $v$ with $wRv$ | $v:A$ |
| $\prec$ | $w:A\prec B$ | $v:A$ for any $v$ with $wRv$ | $v:B$ |

Table 4.7: Particle Rules for Sub

We will now introduce the structural rules which regulate the way subgames work in these dialogues, but first we need to define what we mean by subgames in this setting.

**4.20 Definition.** We will call the dialogue game that begins with $P$ stating the thesis in move 0 the main game. A subgame is a game which can be created within a parent game when a formula of the form $A\rightarrow B$ is attacked. All formulas that are asserted by $O$ in the parent game are treated as assumptions in the subgame. All structural and particle rules that hold in the main game also hold in every subgame.

**4.21 Remark.** It is important to note that the way we define subgame here is different to the usual definition of a subgame in game theory. In game theory a subgame is taken to mean the game that is given by a sub-tree of the game in extensive form. This will not necessarily be the case here.

Rule 8 will specify how we denote subgames in our dialogue trees. Just as only $O$ is allowed to introduce new labels, only $O$ is allowed to create a new subgame. $P$ can of course still attack an implication but they must either stay in the same game or use a subgame that $O$ has previously created.

**Rule 8** (Subgame creation)**.** $O$ may, but is not forced to, begin a new subgame whenever they attack a formula of the form $v:A\rightarrow B$ by asserting $v:A$. If they choose to open such a subgame, we denote this by adding a, b, c, etc. next to the number of the move.

| # | Description |
|---|---|
| 0 | The thesis is stated by $P$, the players alternate |
| 1 | Players win if the other player has no more possible moves |
| 2c | Players can react to react to any previous move by the opponent |
| 3m | Only $O$ can assert new atomic formulas at a label |
| 4 | Strict repetitions are prohibited |
| E | $O$ can only react to the previous move |
| 5 | Only $O$ may introduce new labels |
| 6$R$ | Only previously introduced labels may be used |
| 8 | $O$ may create a subgame when attacking an implication |
| 9 | All formulas asserted in the parent-game may be used the subgame |
| 10 | Any label introduced in a subgame can be used in all parent-games |
| 11 | $P$ wins the main game if they win any subgame |

Table 4.8: Standard rules for classical modal logic
+ rules specific to Sub

A reponse to a move within a subgame is also within that subgame.

**4.22 Remark.** If we assume that both $O$ and $P$ are playing an optimal strategy then $O$ will always choose to start a new subgame while $P$ will always choose to stay in the game that is currently being played. This is analogous to remark 4.1.

**4.23 Remark.** The only way to exit a subgame is to respond to a move which is outside of that subgame. As a result of rule E, $O$ is not able to exit a subgame.

The difference between a main game and a subgame is that a subgame may have assumptions. More specifically, every formula that $O$ has asserted in a parent game is taken to be an assumption for the subgame. This means that if $O$ has asserted $w : p$ in the parent game, $P$ may also assert $w : p$ in the subgame. This is the structural rule that corresponds to rule 7 in our BiMod dialogues, ensuring monotonicity of propositional variables. However, this rule also means that $P$ may attack any formula that $O$ has stated in the parent game within the subgame. Note that we don't only mean formulas that were asserted in the parent game before the subgame was created, but at any point in the game.

**Rule 9** (Subgame Assumptions)**.** All formulas that have been asserted in the parent game can also be used or responded to in a subgame. Labels that have been introduced in the parent game cannot be used in a subgame.

Rule 9 clearly does not affect $O$ as they don't need previous information in order to be allowed to make any move. Additionally, because of rule E, player $O$ is unable to use any information except from the previous move of $P$.

The second part of rule 9 specifically states that we do not assume our models to have the brilliancy property. If we want to add brilliancy to our dialogues, we simply need to change rule 9 to state that labels that are introduced in the parent game can also be used in the subgame.

**Rule 10** (Subgame Information Export)**.** Any label that has been introduced within a subgame can also be used in the parent game. Formulas that have been asserted in a subgame cannot be used in the parent game.

Rule 10 is necessary in order to ensure monotonicity of $\Box$ as we can see in dialogue 4.12. Without rule 10, the game ends after move V because $P$ is unable to respond to any of the open moves by $O$. With Rule 10, $P$ can use the label from the subgame ab in the parent game a.

Rule 10 also ensures monotonicity of $\prec\!\!\!\!\!\!\prec$ formulas as we can see in dialogue 4.13. Essentially, rule 10 corresponds to $\prec\!\!\!\!\!\!\prec$-p from the *BiModSI* dialogues.

| # | | Player | Expression | Label | Description | Dialogue Sequent |
|---|---|---|---|---|---|---|
| T | | $P$ | $1:\Box p \to (\top \to \Box p)$ | 1 | Thesis | $\vdash 1:\varphi$ |
| I | a | $O$ | $1:\Box p$ | 1 | Attack T | $1:\Box p \vdash 1:\varphi$ |
| II | a | $P$ | $1:\top \to \Box p$ | 1 | Defence I | $1:\Box p \vdash 1:\top \to \Box p$ |
| III | ab | $O$ | $1:\top$ | 1 | Attack II | $1:\Box p, 1:\top \vdash 1:\top \to \Box p$ |
| IV | ab | $P$ | $1:\Box p$ | 1 | Defence III | $1:\Box p, 1:\top \vdash 1:\Box p$ |
| V | ab | $O$ | $1:\ ?_2$ | $1R2$ | Attack IV | $1:\Box p, 1:\top \vdash 1:\Box p$ |
| VI | a | $P$ | $1:\ ?_2$ | $1R2$ | Attack I | $1:\Box p, 1:\top \vdash 1:\Box p$ |
| VII | a | $O$ | $2:p$ | $1R2$ | Defence VI | $1:\Box p, 1:\top, 2:p \vdash 1:\Box p$ |
| VIII | a | $P$ | $2:p$ | $1R2$ | Defence V | $1:\Box p, 1:\top, 2:p \vdash 2:p$ |

Dialogue 4.12: Monotonicity of $\Box$ without and with rule 10

**4.24 Proposition.** If $P$ has a winning strategy for the *Sub* dialogue game associated with $A$, then $P$ also has a winning strategy for the dialogue game associated with $A[B/p]$

*Proof.* $P$ is only able to assert an atomic formula $w:p$ in a game x if $O$ has previously asserted $w:p$ in either x or a parent game of x. Therefore if we replace

| # | Player | Expression | Label | Description | Dialogue Sequent |
|---|---|---|---|---|---|
| T | $P$ | $1: F$ | 1 | Thesis | $\vdash 1: F$ |
| I a | $O$ | $1: p \prec q$ | 1 | Attack T | $1: \Box p \vdash 1: F$ |
| II a | $P$ | $1: \top \to (p \prec q)$ | 1 | Defence I | $1: \Box p \vdash 1: \top \to \Box p$ |
| III ab | $O$ | $1: \top$ | 1 | Attack II | $1: \Box p, 1: \top \vdash 1: \top \to \Box p$ |
| IV ab | $P$ | $1: p \prec q$ | 1 | Defence III | $1: \Box p, 1: \top \vdash 1: \Box p$ |
| V ab | $O$ | $2: p$ | $1R2$ | Attack IV | $1: \Box p, 1: \top \vdash 1: \Box p$ |
| VI ab | $P$ | $2: p$ | $1R2$ | Attack I | $1: \Box p, 1: \top \vdash 1: \Box p$ |
| VII ab | $O$ | $2: q$ | $1R2$ | Defence VI | $1: \Box p, 1: \top, 2: p \vdash 1: \Box p$ |
| VIII ab | $P$ | $2: q$ | $1R2$ | Defence V | $1: \Box p, 1: \top, 2: p \vdash 2: q$ |

Dialogue 4.13: Monotonicity of $\prec$, $F = (p \prec q) \to (\top \to (p \prec q))$
dashed line represents game without rule 10

$p$ with any formula $B$, then $P$ has a strategy for the game $A[B/p]$ in which $P$ only states $w : B$ after $O$ has already done so. If $O$ attacks $w : B$ in a game x, $P$ can copy this move, resulting in a winning strategy. □

**4.25 Proposition.** If $A$ is not valid then $O$ has a winning strategy for the Sub game associated with $A$.

*Proof.* In order to prove this we will simply show that we can translate $O$s winning *BiModSI* strategy from the proof of proposition 4.9 into a winning strategy for the corresponding *Sub* game. Each subgame $x$ is created when attacking an assertion $w : A \to B$ and will represent a world $w'$ with $w \leq w'$ in the associated *BiModSI* game. In particular, $O$ will only assert formulas $C$ in the subgame $x$ if they would assert $w' : C$ in the *BiModSI* game.

We will only go through the non-trivial cases here.

1. $P$ asserts $w : A \to B$
   In *BiMod* $O$s strategy is to introduce a label $w \leq w'$ and assert $w' : A$. In *Sub* this is not possible. Instead, $O$ will create a subgame $x$ and assert $w : A$ there. This subgame represents the world $w'$ with $w \leq w$, $M, w' \Vdash A$ and $M, w' \nVdash B$. P can now either attack $A$ in the subgame $x$ or defend the implication by asserting $w : B$ in the subgame $x$.
   (i) If $P$ attacks $w : A$, then $O$ has a winning strategy because of the induction hypothesis.

(ii) If $P$ asserts $w : B$ then $O$ has a winning strategy by induction hypothesis since $M, w' \nVdash B$.

2. $P$ attacks $w : A \rightarrow B$ by asserting $w : A$ in the same game or in a subgame $x$ that $O$ has previously created. This means that $O$ has previously asserted $w : A \rightarrow B$ and created the subgame $x$. In particular this means that in the model $M$, $w \Vdash A \rightarrow B$. The subgame $x$ represent a world $w'$ with $w \leq w'$.

   (i) $M, w' \nVdash A$
   In this case $O$ chooses to attack $w : A$. This is a winning strategy based on the other cases of this induction.

   (ii) $M, w' \Vdash B$
   In this case $O$ chooses to defend against the attack by asserting $w : B$ in the subgame $x$. By induction hypothesis this is a winning strategy for $O$.

Therefore, if $A$ is not valid in intuitionistic modal logic with strict implication, then $O$ has a winning strategy in *BiModSI* which can be translated into a winning strategy for $O$ in *Sub*. □

**4.26 Proposition.** If $A$ is valid in *K-IntMod* then $P$ has a winning strategy for the *Sub* game associated with $A$.

*Proof.* Just as in the previous proof, we will translate the winning strategy for $P$ from the proof of proposition 4.10 into a winning strategy for Sub. We will again only give the non-trivial cases.

1. $\dfrac{\mathcal{G}; \Gamma, w : B \Rightarrow z : C \quad \mathcal{G}; \Gamma \Rightarrow x : A}{\mathcal{G}; \Gamma, w : A \rightarrow B \Rightarrow z : C} \rightarrow L$
   When $P$ attacks the assertion $w : A \rightarrow B$ by stating $w : A$, they do not begin a new subgame, therefore the BiMod strategy can be replicated in Sub.

2. $\dfrac{\mathcal{G}; \Gamma, x : A \Rightarrow x : B}{\mathcal{G}; \Gamma \Rightarrow x : A \rightarrow B} \rightarrow R$
   When $O$ attacks $w : A \rightarrow B$ they create a new subgame $x$ and assert $w : A$ in this subgame. This subgame represents a world $w'$ in $M$ with $w \leq w'$. The information that $P$ has available in the subgame and the information that $P$

has at $w'$ in the *BiModSI* game are equivalent. Therefore any move that is a part of $P$s winning strategy in *BiModSI* can also be made in *Sub*.

Therefore, if $A$ is valid in *K-IntMod* then we can translate $P$s winning strategy for the associated *BiModSI* game into a winning strategy for the associated *Sub* game.

□

**4.27 Theorem.** The dialogue semantics *Sub* is sound and complete with respect to intuitionistic modal logic with strict implication.

*Proof.* This follows directly from propositions 4.25 and 4.26. □

# 5 Conclusion

In this thesis we have shown that standard dialogue semantics cannot accurately model intuitionistic modal logic with strict implication. After discussing why this problem occurs, we suggest two possible alternative dialogue semantics which can capture intuitionistic modal logic with strict implication.

The first option we introduced are *BiModSI* dialogues. This approach is inspired by Kripke models for intuitionistic modal logic. We started by adapting the dialogue semantics we have introduced for classical modal logic to model intuitionistic propositional logic. We then added a second accessibility relation to model the $\Box$ operator and $\dashv$. In order to ensure that monotonicity of all formulas of our language is given, we added a structural rule that corresponds to the $\dashv$-p restriction on Kripke models. We showed that $O$ has a winning strategy for a *BiMod* game if the thesis is not valid in Kripke models for intuitionistic modal logic and $P$ has a winning strategy if there is a sequent calculus deduction of the thesis. For strict implication we instead showed that any formula for which there is a Hilbert-style deduction, there is also a winning strategy $P$.

The second option we introduced are *Sub* dialogues. These dialogue games function very similarly to *BiModSI* games but use the same particle rules as the standard dialogue games we introduced in the beginning. Instead of having an intuitionistic accessibility relation, we restricted the way that labels and formulas can be used in the course of the game. We then showed that winning strategies for *BiModSI* dialogue games can be translated into winning strategies for *Sub* dialogue games meaning that the two dialogue systems are equivalent.

Both options have their merits. On the one hand, the *BiModSI* games closely mirror the Kripke semantics and they mostly use structural rules which we are already familiar with from dialogues for classical modal logic which make them easy to understand. On the other hand, *Sub* games give a dialogue semantics which is closer to the one for classical modal logic in some ways. Since *Sub* games use the same particle rules as games for classical modal logic, the difference is only structural, just like between the standard dialogue semantics for intuitionistic and classical propositional logic.

While both approaches are equivalent for the logic we focused on in this thesis, we have not considered which effect it would have to change our logic. Since we are working with modal logic, it is natural to want to be able to include different frame conditions like reflexivity, transitivity, etc.. While we mentioned that these frame conditions can be introduced by using variations of rule 6K, we have not studied whether using these different rules also works in *BiModSI* and *Sub*. Another natural next step is to attempt to include $\Diamond$ in our logic as well. When using Kripke models, adding a $\Diamond$ operator to our language causes the need for additional frame conditions to ensure monotonicity of all formulas. While we are confident that our dialogue games can be expanded to model $\Diamond$ as well, this is outside the scope of this thesis.

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