

THESIS FOR THE DEGREE OF LICENTIATE

On algebraic function fields and their associated L -functions

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Abstract

We study counting functions of field extensions of the rational function field $\mathbb{F}_q(T)$. Moreover, we study the distribution of the low-lying zeros of certain L -functions associated to these extensions.

Our first results concern S_3 -cubic extensions of $\mathbb{F}_q(T)$, ordered by discriminant, with q coprime to 2 and 3. We derive an asymptotic formula, with an error term of order $\ll_{\epsilon} X^{2/3+\epsilon}$, matching the current best result over number fields due to Bhargava, Taniguchi and Thorne. We also obtain an asymptotic formula for a refined counting function where one specifies the splitting type of finitely many primes. In addition to obtaining an upper bound for the error term, we also obtain a lower bound by studying the one-level density of certain Artin L -functions associated with these fields. This generalises conditional results over \mathbb{Q} obtained by Cho, Fiorilli, Lee and Södergren.

Next, we study a certain family of Artin L -functions associated with D_4 -quartic extensions of $\mathbb{F}_q(T)$. We prove for large q coprime to 2 that, when ordered by conductor, at least 77% of these L -functions are non-vanishing at the central point $s = 1/2$, improving results over \mathbb{Q} by Durlanık. We also obtain an asymptotic formula for the counting function of these fields with a power-saving error term, generalising results due to Friedrichsen.

Keywords: field discriminant, conductor, function field, low-lying zeros, one-level density, prime splitting, cubic field, D_4 -quartic field

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List of publications

This thesis consists of an extended summary and the following appended manuscripts.

Paper I: V. Ahlquist, “On the counting function of cubic function field”, Preprint, 2025, arXiv:2504.12160.

Paper II: V. Ahlquist, “Non-vanishing of Artin L -functions associated with D_4 -quartic function fields ordered by conductor”, (Manuscript).

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Chapter 1

Introduction

1.1 A classical counting problem

Arithmetic statistics is the study of distributional properties of arithmetic objects. If these objects have a notion of size, then possibly the most basic distributional property that one can study is the number of objects of size bounded by x , as x grows large. Before introducing more intricate objects in Chapters 2 and 3, we begin by considering one of the simplest arithmetic objects of interest, namely the set of integers \mathbb{Z} . Studying the number of integers of bounded size is essentially trivial. Indeed, restricting to positive integers for simplicity, we may estimate for $0 < x \in \mathbb{R}$, that

$$\#\{n \in \mathbb{Z}_+ : n \leq x\} = x + f(x), \quad (1.1)$$

where $|f(x)| < 1$ for all x .

More interesting questions may be asked about the integers if one considers additional algebraic structure. Specifically, \mathbb{Z} is naturally endowed with an addition $+$ and a multiplication \cdot . By multiplying nonzero integers with each other, one obtains a new integer of larger (or possibly equal) size. Similarly, one may divide an integer with one of its divisors and obtain a smaller integer. If one is not allowed

to divide an integer by itself, or 1, then assuming that one starts at an integer larger than 2, this process terminates at a prime. Here, a prime number is a positive integer that has precisely 2 positive divisors. Examples of prime numbers are 2, 11, and 23. The Fundamental Theorem of Arithmetic sheds light on the multiplicative structure of the integers and asserts that every positive integer can be written uniquely as a product of powers of distinct primes, up to ordering.

Having realised that primes are an important class of integers, one may ask how many primes there are. Let us define the counting function

$$\pi(x) = \#\{p \leq x : p \text{ is prime}\}.$$

The following theorem concerning the growth of $\pi(x)$ is essentially due to Euclid, and was proven more than 2000 years ago.

Theorem 1.1.1 (Euclid). *There are an infinite number of primes, i.e. $\pi(x) \rightarrow \infty$ as $x \rightarrow \infty$.*

The question of obtaining more precise asymptotics for the function $\pi(x)$ was partially resolved in the 19th century. Chebyshev [5] proved that up to constants, $\pi(x)$ grows like $x/\log x$. More precisely, he proved the following theorem.

Theorem 1.1.2. *There are positive constants C_1 and C_2 such that*

$$C_1 \leq \liminf_{x \rightarrow \infty} \frac{\pi(x)}{x/\log x} \leq \limsup_{x \rightarrow \infty} \frac{\pi(x)}{x/\log x} \leq C_2.$$

At the end of the 19th century, Hadamard [16] and de la Vallée Poussin [10] independently proved that one may in fact take $C_1 = C_2 = 1$ in the above theorem, establishing what is known as the Prime Number Theorem. Compared to $x/\log x$, a better estimate for $\pi(x)$ is the logarithmic integral

$$\text{Li}(x) = \int_2^x \frac{1}{\log u} du.$$

In fact, de la Vallée Poussin [11] later proved a sharper version of the Prime Number Theorem, which includes an upper bound for the error term when one approximates $\pi(x)$ with $\text{Li}(x)$.

Theorem 1.1.3 (The Prime Number Theorem). *There is a constant $c > 0$ so that*

$$\pi(x) = \text{Li}(x) + \mathcal{O}\left(xe^{-c\sqrt{\log x}}\right).$$

1.2 The Riemann zeta function

The size of the error term in the Prime Number Theorem is intimately connected with the non-trivial zeros of the Riemann zeta function $\zeta(s)$, studied by Riemann in his celebrated memoir [25]. Specifically, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for a complex variable s . This series is absolutely convergent when $\Re(s) > 0$. However, one may meromorphically continue $\zeta(s)$ to a function with a single simple pole, at $s = 1$. Using the fundamental theorem of arithmetic, we may write the sum defining $\zeta(s)$ as an infinite so-called Euler product over primes p :

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

In particular, this implies that $\zeta(s)$ is nonzero for $\Re(s) > 1$. We remark that de la Vallée Poussin proves Theorem 1.1.3 by establishing a zero-free region of $\zeta(s)$, showing that the Riemann zeta function is non-zero in a region (which is shrinking in the vertical direction) extending to the left of the line $\Re(s) = 1$.

The Riemann zeta function obeys a functional equation, relating values at s to values at $1 - s$. To state this more precisely, we require the gamma function $\Gamma(s)$, defined by

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} du,$$

for $\Re(s) > 1$, and analytically continued through the relation $\Gamma(s+1) = s\Gamma(s)$. Then, if we let

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

we have the functional equation

$$\xi(s) = \xi(1 - s).$$

From this relation, one sees that $\zeta(s)$ has so-called trivial zeros at every negative even integer, coming from poles of the gamma function. Moreover, one realises that the non-existence of zeros with $\Re(s) > 1$ implies the non-existence of non-trivial zeros with $\Re(s) < 0$. In fact, one can prove that there are no zeros on the lines $\Re(s) = 1$ and $\Re(s) = 0$, which is essentially equivalent to the version of the Prime Number Theorem without error term.

The vertical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$ is known as the critical strip, and it is here that all the non-trivial zeros of $\zeta(s)$ lie. The celebrated Riemann Hypothesis, stated by Riemann in his memoir, asserts that all non-trivial zeros lie on the so-called critical line.

Conjecture 1.2.1 (The Riemann Hypothesis). *All non-trivial zeros of $\zeta(s)$ lie on the line $\{s \in \mathbb{C} : \Re(s) = 1/2\}$.*

A proof of this conjecture has eluded number theorists for more than 150 years. Assuming its validity, the error term in Theorem 1.1.3 could be improved to $x^{1/2} \log x$, a result due to von Koch [33].

In this thesis, we study certain L -functions, which are generalisations of the Riemann zeta function, associated with analogues of \mathbb{Z} in more algebraic settings. In particular, we will be interested in their so-called low-lying zeros, which are zeros with a small imaginary part. The next two chapters are dedicated to first introducing the arithmetic objects, known as function fields, that these L -functions are associated with, and second providing an introduction to the theory of L -functions in general, and their low-lying zeros in particular.

Chapter 2

Function fields

We now provide the setup required to define and understand the function fields that we are interested in. The material in this chapter is standard and mostly taken from [22], [24], [26], and [32].

2.1 The rational function field

Let q be a power of a prime. Recall that there is a unique finite field, up to isomorphism, of order q , denoted \mathbb{F}_q . Moreover, its multiplicative group \mathbb{F}_q^* of nonzero elements is cyclic of order $q - 1$. The rational function field is the field $\mathbb{F}_q(T)$, consisting of quotients of polynomials over \mathbb{F}_q in the indeterminate T . This field will serve as an analogue of the rational numbers \mathbb{Q} .

Having found an analogue of \mathbb{Q} , one is left with the question of finding a suitable subring of $\mathbb{F}_q(T)$ that could serve as an analogue of \mathbb{Z} . In a certain sense, there is no canonical choice of such a ring, as we shall see later in Section 2.3.1. However, a non-canonical, but common choice is the ring $R := \mathbb{F}_q[T]$, consisting of the polynomials in T over \mathbb{F}_q . Irreducible polynomials in this ring then provide a natural analogue for primes in \mathbb{Z} .

The size of a polynomial can be defined through its degree, giving rise

to an absolute value

$$|f| = q^{\deg(f)}.$$

When we study integers, we often restrict to positive integers for convenience. This means that we essentially consider \mathbb{Z}^* up to multiplication with a unit belonging to the set $\{1, -1\}$. The group of units in $\mathbb{F}_q[T]$ is \mathbb{F}_q^* , so that an analogue of restricting to positive integers is restricting to monic polynomials. We denote the set of all monic polynomials by M , and then, similarly to how we estimated the number of integers of bounded size, we may estimate

$$\#\{f \in M : |f| = q^d\} = q^d,$$

for integers $d \geq 0$. Note that compared to (1.1), we have an equality instead of an inequality in the definition of our counting function. This is common practice when studying counting problems over function fields, and produces results that more closely resemble analogous results over \mathbb{Q} .

Similar to counting problems over \mathbb{Q} , one may study counting problems over $\mathbb{F}_q(T)$ using L -functions. One of the main advantages of working with function fields is that the L -functions often have a particularly simple form. We shall return to this subject in the next chapter when we study L -functions in more detail.

The ring $\mathbb{F}_q[T]$ is quite similar to \mathbb{Z} , and in Paper I, our approach is based on the idea that $\mathbb{F}_q[T]$ is a natural analogue of \mathbb{Z} . There are, however, some arithmetic differences between the two rings. One such difference is the fact that, compared to \mathbb{Z} , $\mathbb{F}_q[T]$ does not contain every so-called finite prime. In Paper II, we take a more global approach, not requiring us to distinguish any particular subring of $\mathbb{F}_q(T)$. We will return to this discussion in Section 2.3.1, where we define the notion of a prime, as well as that of a finite prime.

2.2 Fields and their extensions

To study function fields other than $\mathbb{F}_q(T)$, we require notions from more abstract field theory. We begin with the concept of a field extension.

Let K be a field. Then, a field extension of K is another field L such that L contains K as a subfield. We sometimes write L/K to specify that L is a field extension of K . The extension L/K is called algebraic if every element of L satisfies a polynomial equation with coefficients in K . These are the only types of extensions that we shall concern ourselves with in this thesis. Given an element x in an algebraic extension L of K , we may consider the set of polynomials $f \in K[X]$ such that $f(x) = 0$ and one sees that there is a unique such monic polynomial of minimal degree. This polynomial is known as the minimal polynomial of x over K .

The extension L/K is called finite if L is of finite dimension, when viewed as a vector space over K , and we write $[L : K]$ for this dimension. One realises by taking powers of generating elements that an extension being finite immediately implies algebraicity. In fact, a finite extension can be characterised as an extension generated by a finite number of algebraic elements, as a K -algebra. Another important notion is separability. We say that an element x in L/K is separable if its minimal polynomial over K does not have multiple roots, in any extension of K . The extension L/K is separable if every element of L is separable over K .

From the definition, one sees that $x \in L$ is algebraic over K if and only if $K(x)$, the smallest field extension of K containing x , is finite over K . In particular, this implies that the property of being an algebraic extension is transitive. Given a field K , there is a maximal algebraic extension \overline{K} of K , which admits no nontrivial algebraic extensions. The field \overline{K} is called the algebraic closure of K , and it is unique up to isomorphism.

We will be interested in maps between field extensions. In particular,

we define the group of K -automorphisms of L by

$$\text{Aut}(L/K) = \{\sigma : L \rightarrow L : \sigma|_K = \text{id}_K, \sigma \text{ is a field automorphism}\}.$$

One can show that every $\sigma \in \text{Aut}(L/K)$ can be extended to an automorphism in $\text{Aut}(\overline{K}/K)$. Similarly, by restriction, every $\tau : \overline{K} \rightarrow \overline{K}$, fixing K , restricts to a map $\sigma : L \rightarrow \overline{K}$. An important property of elements in $\text{Aut}(\overline{K}/K)$ is that they permute the roots of polynomials with coefficients in K , which follows from the relation $\sigma(f(x)) = f(\sigma(x))$. We say that L/K is normal if the restriction of every $\sigma \in \text{Aut}(\overline{K}/K)$ satisfies $\sigma(L) = L$. This is equivalent to requiring that the minimal polynomial over K of every element in L splits completely in L .

A special class of extensions which are particularly important are Galois extensions. An extension L/K is Galois if it is finite, separable and normal. A few equivalent characterisations are given in the following theorem [22, Theorem 3.10].

Theorem 2.2.1. *For an extension L/K , the following are equivalent.*

1. L is the splitting field of a polynomial $f \in K[X]$, i.e. $L = K(x_1, \dots, x_n)$, where the x_i are the roots of f .
2. L is finite over K , and $K = L^{\text{Aut}(L/K)} := \{x \in L : \sigma(x) = x, \text{ for every } \sigma \in \text{Aut}(L/K)\}$.
3. L is a Galois extension of K .

When L/K is Galois, we write $\text{Gal}(L/K)$ for $\text{Aut}(L/K)$, and call this group the Galois group of L/K .

The main theorem on Galois extensions is the Fundamental Theorem of Galois Theory, see [22, Theorem 3.17].

Theorem 2.2.2. *Let L/K be Galois. Then, the map $H \mapsto L^H := \{x \in L : \sigma(x) = x, \text{ for every } \sigma \in H\}$ is a bijection between the collection of subgroups of $\text{Gal}(L/K)$ and fields E such that $K \subseteq E \subseteq L$. Its inverse is $E \mapsto \text{Gal}(L/E)$. Moreover,*

1. $G \supseteq H \implies L^G \subseteq L^H$,

2. $(G : H) = [L^H : L^G]$,
3. H is normal in $\text{Gal}(L/K) \iff L^H$ is normal, and thus Galois, over K .

In Paper II, this theorem is used to construct the subfield lattice of the Galois closure of a D_4 -quartic field from the subgroup lattice of D_4 .

2.3 The arithmetic of function fields

We now describe the arithmetic of global fields, with a focus towards function fields. A global field is simply a finite extension of \mathbb{Q} , i.e. a number field, or a finite extension of $\mathbb{F}_q(T)$, i.e. a function field.

2.3.1 Primes

We begin by defining the notion of a prime. To motivate the definition, we first consider the situation over \mathbb{Z} .

Let $p \in \mathbb{Z}$ be a prime. Then, given any integer n , we define the valuation function ν_p by letting $\nu_p(n)$ denote the largest exponent e such that $p^e \mid n$, and we let $\nu_p(0) = \infty$. As p is prime, $\nu_p(nm) = \nu_p(n) + \nu_p(m)$ for any integers n, m . Furthermore, we have the inequality $\nu_p(n + m) \geq \min\{\nu_p(n), \nu_p(m)\}$. We may define a corresponding absolute value, $|\cdot|_p$, the p -adic absolute value, by $|n|_p = 2^{-\nu_p(n)}$. In fact, we can extend to all of \mathbb{Q} multiplicatively. The choice of the base 2 in the definition of $|\cdot|_p$ is rather arbitrary, any number larger than 1 works, and p itself is a common choice. We say that two absolute values $|\cdot|$ and $|\cdot|'$ are equivalent if $|\cdot| = (|\cdot|')^c$, for some $c > 0$.

We see that we may associate an equivalence class of absolute values to each prime. These absolute values are known as non-archimedean as they satisfy the inequality $|x + y| \leq \max\{|x|, |y|\}$, which is a sharpening of the usual triangle inequality. Now, on \mathbb{Q} , we also have the usual absolute value. This absolute value is called archimedean, as it is not non-archimedean, and its corresponding valuation is $-\log$

composed with the usual absolute value. As we can associate absolute values to primes, one might be led to think that we can associate primes to absolute values. However, there is no prime number in \mathbb{Z} associated with the usual absolute value $|\cdot|$. Instead, we call this an infinite prime, or an infinite place.

In general, a valuation $\nu : K^* \rightarrow \mathbb{R}$, on a field K , is a function satisfying

1. $\nu(xy) = \nu(x) + \nu(y)$, for all $x, y \in K^*$,
2. $\nu(x + y) \geq C \min\{\nu(x), \nu(y)\}$, for some $C \geq 1$.

We also let $\nu(0) = \infty$. After scaling, one sees that it suffices to consider the cases $C = 1$ and $C = 2$, say. As above, from a valuation ν , we obtain an absolute value $2^{-\nu}$. A famous theorem of Ostrowski asserts that on \mathbb{Q} , the only valuations are either $-\log|x|$ with $|\cdot|$ the usual absolute value, or ν_p for some prime p , up to equivalence of the corresponding absolute values. The valuation is called non-archimedean if the above holds with $C = 1$.

When K is a global field, we define a prime in K to be a nontrivial valuation ν , satisfying the above. We have the following result for the primes in $\mathbb{F}_q(T)$ [32, Theorem 1.3.2].

Theorem 2.3.1. *The primes in $\mathbb{F}_q(T)$ are of the form ν_P , for irreducible polynomials P in $\mathbb{F}_q[T]$, together with the valuation v_∞ , defined by $v_\infty(f/g) = \deg(g) - \deg(f)$.*

Remark. All of these valuations are non-archimedean. In particular, $\mathbb{F}_q(T)$ does not admit an infinite prime. Moreover, these are all discrete valuations, meaning that they are non-archimedean, and that their image is \mathbb{Z} .

Remark. The finite prime v_∞ is often called the prime at infinity. The associated prime ideal is denoted P_∞ .

To a discrete valuation ν , we may associate its discrete valuation ring $\mathcal{O}_\nu = \{x \in K : \nu(x) \geq 0\}$. This is a local ring with a unique prime ideal $\{x \in K : \nu(x) \geq 1\}$. We sometimes call this prime ideal a prime

of K . Given a finite set S of valuations, we may define $\mathcal{O}_S = \{x \in K : \nu(x) \geq 0 \text{ for all } \nu \notin S\}$. One checks for $K = \mathbb{F}_q(T)$ that if $S = v_\infty$, then $\mathcal{O}_S = \mathbb{F}_q[T]$, however $\mathcal{O}_\emptyset = \mathbb{F}_q$, and it is in this sense, that $\mathbb{F}_q(T)$ does not have a suitable subring containing all finite primes.

Finally, given a prime \mathcal{P} in a field K , we may form the completion of K with respect to the corresponding absolute value $|\cdot|_{\mathcal{P}}$. We then obtain a complete field, containing as a subring the completion of $\mathcal{O}_{\mathcal{P}}$. Moreover, if $S = \{\nu_{\mathcal{P}}\}$, then \mathcal{O}_S forms a "lattice" (albeit \mathcal{O}_S is not discrete). In particular, when $K = \mathbb{F}_q(T)$, and $S = \{\nu_\infty\}$, then one may study the inclusion $\mathcal{O}_S = \mathbb{F}_q[T] \subseteq K_\infty$, where K_∞ is the completion of K at ν_∞ . We apply this perspective in Paper I to study $\mathbb{F}_q[T]$ -integral points of a certain lattice using analytic methods, analogous to working over \mathbb{R} when counting integral points in a \mathbb{Z} -lattice.

2.3.2 Primes and extensions

We now study how primes behave under extensions. Let L/K be a finite, separable extension of function fields. Every prime ν on L clearly restricts to a prime on K , and we say that ν lies over $\nu|_K$. Denoting the ideal that ν corresponds to by \mathfrak{P} , and the ideal that $\nu|_K$ corresponds to by \mathcal{P} , we write $\mathfrak{P} | \mathcal{P}$.

In the other direction, it turns out that given a prime ν in K , it admits a factorisation

$$\nu_{\mathcal{P}} = e_1 \nu_{\mathfrak{P}_1}|_K + \dots + e_r \nu_{\mathfrak{P}_r}|_K, \quad (2.1)$$

where the $\nu_{\mathfrak{P}_i}$ are primes in L . The integer $e_i =: e(\mathfrak{P}_i/\mathcal{P})$ is called the ramification degree of \mathfrak{P}_i over \mathcal{P} .

When $\mathfrak{P} | \mathcal{P}$, the quotient $\mathcal{O}_{\mathfrak{P}}/\mathfrak{P}$ is an \mathbb{F}_q -vector space over $\mathcal{O}_{\mathcal{P}}/\mathcal{P}$, and we define the inertial degree $f(\mathfrak{P}/\mathcal{P})$ as the corresponding index

$$f(\mathfrak{P}/\mathcal{P}) = [\mathcal{O}_{\mathfrak{P}}/\mathfrak{P} : \mathcal{O}_{\mathcal{P}}/\mathcal{P}].$$

The decomposition (2.1) together with the constants $f(\mathfrak{P}/\mathcal{P})$ determine the so-called splitting type of \mathcal{P} in L .

We have the following equality, relating the above constants to the degree of the extension L/K :

$$[L : K] = \sum_{i=1}^r e(\mathfrak{P}_i/\mathcal{P})f(\mathfrak{P}_i/\mathcal{P}),$$

see [32, Corollary 1.4.3]. Moreover, if L/K is Galois, then the numbers $e(\mathfrak{P}_i/\mathcal{P})$, and $f(\mathfrak{P}_i/\mathcal{P})$ are independent of i , so that we may simply write $e(\mathcal{P})$, $f(\mathcal{P})$. Writing $r =: r(\mathcal{P})$, we then have that

$$[L : K] = r(\mathcal{P})e(\mathcal{P})f(\mathcal{P}).$$

In the case that $r = [L : K]$, we say that \mathcal{P} is totally split. If some $e_i > 1$, we say that \mathcal{P} is ramified, and it is totally ramified if $e_1 = [L : K]$ (in which case $r = 1$). Finally, \mathcal{P} is inert if $r = 1$ and $f(\mathfrak{P}_1/\mathcal{P}) = [L : K]$.

2.3.3 Divisors and divisor classes

Let K be a function field. The group of divisors D_K of K is the free abelian group generated by the primes. We use multiplicative notation for this group to emphasise the similarities to the ideal group. A general element in D_K is then a product of integer powers of finitely many distinct primes.

The degree of a prime is the degree $[\mathcal{O}_{\mathcal{P}}/\mathcal{P} : \mathbb{F}_q]$. For prime polynomials $P \in \mathbb{F}_q[T]$, this agrees with the usual degree. We define the degree map on divisors by

$$D = \prod_{\mathcal{P}} \mathcal{P}^{\delta_{\mathcal{P}}} \mapsto \sum_{\mathcal{P}} \delta_{\mathcal{P}} \deg_K(\mathcal{P}).$$

We now turn our attention to divisor classes. We have a map $K^* \rightarrow D_K$ given by mapping an element x to the principal divisor

$$(x) = \prod_{\mathcal{P}} \mathcal{P}^{\nu_{\mathcal{P}}(x)}.$$

The kernel is known as the constant field of K . Note that the above map is well-defined as $\nu_{\mathcal{P}}(x)$ is zero for all but finitely many \mathcal{P} . We now

define the group of divisor classes $\text{Cl}(K)$ as the quotient of D_K with the group of principal divisors. This group will play a role in Paper II, when we study quadratic extensions of a quadratic extension of $\mathbb{F}_q(T)$.

In a general global field, we have a "product formula"

$$\sum_{\mathcal{P}} \nu_{\mathcal{P}}(x) \deg_K(\mathcal{P}) = 0.$$

In particular, this shows that Cl_K is never trivial when K is a function field. Moreover, this proves that the degree map in fact induces a map on divisor classes. In contrast to the situation for number fields, $\text{Cl}(K)$ is never finite. However, the group $\text{Cl}^0(K)$ of degree zero-divisors is finite, and the cardinality of this group is the class number h_K of K .

Given an extension L/K of function fields, we may define a norm map $N_{L/K} : D_L \rightarrow D_K$ induced by

$$\mathfrak{P} \mapsto f(\mathfrak{P}/\mathcal{P})\mathcal{P},$$

where $\mathfrak{P} | \mathcal{P}$. This map will be used for defining the discriminant of a function field in the next section. We also define a map

$$|\cdot| : D_L \rightarrow \mathbb{Q},$$

induced by $\mathfrak{P} \mapsto |\mathfrak{P}| = \#(\mathcal{O}_{\mathfrak{P}}/\mathfrak{P})$, and we often refer to this map as an absolute value. We remark, however, that it is evidently not an absolute value in the sense of the previous sections.

2.3.4 The discriminant

An important algebraic invariant of a field extension L/K is the discriminant. We will only need to consider the case of a separable function field extension. To define the discriminant, we first define the different divisor.

Let \mathcal{P} be a prime in K , and consider the corresponding local ring $\mathcal{O}_{\mathcal{P}}$. We may consider its integral closure $R_{\mathcal{P}}$ inside L , i.e. $R_{\mathcal{P}} = \{x \in L : f(x) = 0, \text{ for some monic } f \in \mathcal{O}_{\mathcal{P}}[x]\}$. As $\mathcal{O}_{\mathcal{P}}$ is a PID, this becomes

a finite, free $\mathcal{O}_{\mathcal{P}}$ -module, see e.g. [24, Proposition I.8.1], so that we may find a basis $\{x_1, \dots, x_n\}$.

We are now almost ready to define the inverse different. First, if $\beta \in L$, then there is a multiplication map $L \rightarrow L$ given by $x \mapsto \beta x$, and we denote its trace by $\text{Tr}(\beta)$. Now, we define the (semilocal) inverse different by $C_{R_{\mathcal{P}}/\mathcal{O}_{\mathcal{P}}} = \{x \in L : \text{Tr}(xb) \in \mathcal{O}_{\mathcal{P}} \text{ for all } b \in R_{\mathcal{P}}\}$. This becomes a fractional ideal, and its ideal inverse $\mathcal{D}_{R_{\mathcal{P}}/\mathcal{P}}$ is called the semilocal different at \mathcal{P} .

The prime ideals inside $R_{\mathcal{P}}$ are exactly the primes lying over \mathcal{P} . Hence, $\mathcal{D}_{R_{\mathcal{P}}/\mathcal{P}}$ has the form

$$\prod_{\mathfrak{P}|\mathcal{P}} \mathfrak{P}^{\delta(\mathfrak{P})},$$

for some $\delta(\mathfrak{P}) \geq 0$. We now define the global different divisor by setting

$$\mathcal{D}_{L/K} = \prod_{\mathfrak{P}} \mathfrak{P}^{\delta(\mathfrak{P})},$$

and one shows that all but finitely many $\delta(\mathfrak{P})$ are zero. Finally, we define the (relative) discriminant $\text{Disc}(L/K)$ as the norm $N_{L/K}(\mathcal{D}_{L/K})$. Moreover, the absolute discriminant $\text{Disc}(L)$ of a field L is $\text{Disc}(L/\mathbb{F}_q(T))$.

The different is closely related to the ramification in L/K . We say that $\mathfrak{P} | \mathcal{P}$ is tamely ramified in L , if its ramification degree $e(\mathfrak{P}/\mathcal{P})$ is coprime to the characteristic of the residue field $\mathcal{O}_{\mathcal{P}}/\mathcal{P}$. In this case, we have that $\delta(\mathfrak{P}) = e(\mathfrak{P}/\mathcal{P}) - 1$, so that the ramification degrees can be read from the different. As the residue field is a finite extension of \mathbb{F}_q , the characteristic is the prime p dividing q . In particular, for certain q , one may assume that all primes are tamely ramified. Indeed, in Paper I, we consider extensions of degree 3, and require that q be coprime to 2, 3. Moreover, in Paper II, we study quadratic extensions of quadratic extensions and require that q be coprime to 2.

In Paper I, we make use of a semilocal description of the discriminant. Specifically, with $K = \mathbb{F}_q(T)$ and $S = \{\nu_{\infty}\}$, the ring $\mathcal{O}_S = \mathbb{F}_q[T]$ is a PID. Let R_S denote the integral closure of $\mathbb{F}_q[T]$ in L , and let

$\{x_1, \dots, x_n\}$ be a basis. From the situation over number fields, one is led to define the S -semilocal discriminant as

$$\text{Disc}(R_S/\mathbb{F}_q[T]) = (\det(\text{Tr}(x_i x_j))_{1 \leq i, j \leq n}),$$

and this agrees with

$$\prod_{\mathcal{P} \notin S} N_{L/K}(\mathcal{D}_{R_{\mathcal{P}}/\mathcal{P}}),$$

see [26, Proposition 7.9, Lemma 7.10] and the preceding discussion.

2.4 Counting field extensions

Given some real number X , a classical result due to Hermite [24, Theorem III.2.16] asserts that there are only finitely many number fields with absolute discriminant of size bounded by X , up to isomorphism. For function fields, the situation is more delicate. For example, there are infinitely many unramified extensions of $\mathbb{F}_q(T)$, namely the extensions $\mathbb{F}_{q^m}(T)$ for every m . These extensions are all non-geometric, in the sense that they enlarge the field of constants. It turns out that if one restricts to the setting of separable, geometric extensions, then Hermite's theorem holds also for function fields, see e.g. [27].

Knowing that there are finitely many extensions of bounded absolute discriminant, one is led to the question of more precise asymptotics for the number of fields. For example, one may be interested in also fixing the degree n of the extension. A landmark result for counting extensions with fixed degree is due to Ellenberg and Venkatesh [13], proving upper bounds for the number of such number field extensions with discriminant bounded by X .

One may restrict the counting function even further by e.g. specifying the Galois group of the Galois closure of the extension. For general G and n , and a base field K , Malle's conjecture predicts that the counting function of degree n extensions of K with Galois group G grows asymptotically like

$$X^{1/a(G)} (\log X)^{b(K,G)-1},$$

for some $a(G)$ and $b(K, G)$ given in terms of data from G and K , see [21, Conjecture 1.1]. This conjecture is known to hold for several different K, G and n , however in general the conjecture is false, see e.g. [19] for a counterexample.

Türkelli proposed a modification to the constant $b(K, G)$ [31], however the conjecture with this modification is also known to not hold over \mathbb{Q} , see [34]. We remark that over $K = \mathbb{F}_q(T)$, Türkelli's conjecture is known to be essentially true if one restricts to geometric extensions, see the recent work of Landesman and Levy [20, Theorem 10.1.10].

We now briefly describe some of the work that has been done on studying the counting function for specific choices of G and K . This description is by no means meant to be exhaustive. The simplest nontrivial case is when $n = 3$ and with Galois group G equal to either C_3 or S_3 . The main term in the counting function for the abelian group $G = C_3$ was first found by Cohn [7] over \mathbb{Q} . When G is a general abelian group and $K = \mathbb{Q}$, the main term was found by Mäki [23], and the main term in the case of a general global K was later found by Wright [36].

The non-abelian and non-Galois case $G = S_3$ was first studied by Davenport and Heilbronn [9] over \mathbb{Q} , and by Datskovsky and Wright [8] over a global field. Their results were later improved by several different authors. This is described in more detail in Paper I, where we study cubic S_3 -extensions of $\mathbb{F}_q(T)$. The main term for $G = S_n$, with $n \geq 4$ has been found for $n = 4$ or 5 by Bhargava [1], [2], for $K = \mathbb{Q}$, and this was later extended to a general (global) base field by Bhargava, Shankar and Wang [3].

Many other choices of G and K have been studied, but for the sake of brevity we finish our overview here. Before ending this section, we remark that field extensions may be studied using invariants other than the discriminant. For example, in Paper II, we study D_4 -quartic extensions when ordered by conductor, an invariant that in this case is closely related to the discriminant, see Section 4.2.

Chapter 3

L -functions

Having introduced the general function field setting, we are now ready to define their associated L -functions. Throughout, we will let K denote a general function field. The theory from all but the last section is taken from [24] and [26].

3.1 Dedekind zeta functions

Recall the group of divisors D_K . For $D \in D_K$, we write $D \geq 0$ if no exponent in the prime decomposition of D is negative, and we say that D is an effective divisor. We may then define the Dedekind zeta function

$$\zeta_K(s) = \sum_{D \geq 0} \frac{1}{|D|^s} = \prod_{\mathcal{P}} (1 - |\mathcal{P}|^{-s})^{-1}, \quad (3.1)$$

where the equality follows from unique factorisation. Both the product and sum above are absolutely convergent when $\Re(s) > 1$.

As an example, we may consider the rational function field $K = \mathbb{F}_q(T)$. Recall that the set of primes in $\mathbb{F}_q(T)$ consists of the prime at infinity P_∞ , together with primes corresponding to irreducible polynomials in

$\mathbb{F}_q[T]$. One shows that $\deg P_\infty = 1$, so that $\zeta_{\mathbb{F}_q(T)}(s)$ equals

$$(1 - q^{-s})^{-1} \prod_{P \neq P_\infty} (1 - |P|^{-s})^{-1} = \frac{1}{(1 - q^{-s})} \sum_{k=0}^{\infty} \left(\sum_{\substack{\text{monic } f \in \mathbb{F}_q[T] \\ \deg f = k}} 1 \right) q^{-ks}.$$

Using that there are precisely q^k monic polynomials of degree k , we find that

$$\zeta_{\mathbb{F}_q(T)}(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})}. \quad (3.2)$$

In particular, compared to the situation over \mathbb{Q} , the Riemann Hypothesis is trivial to prove for the special case $K = \mathbb{F}_q(T)$, as $\zeta_{\mathbb{F}_q(T)}$ does not have any zeros.

By way of e.g. the Riemann-Roch theorem, see [26, Theorem 5.4], which can be viewed as a function field substitute for the classical Poisson summation formula, one proves for a general K that $\zeta_K(s)$ can be analytically continued to a meromorphic function on \mathbb{C} , with a simple pole at $s = 1$. In fact, the Dedekind zeta function satisfies the functional equation

$$\xi_K(s) = \xi_K(1 - s),$$

where

$$\xi_K(s) = q^{(g_K - 1)s} \zeta_K(s). \quad (3.3)$$

Here, $g_K \in \mathbb{Z}_{\geq 0}$ is the so-called genus of K .

We will not provide a general definition for g_K , instead, we note that it is closely related to the discriminant. First, for the rational function field, one may show that $g_{\mathbb{F}_q(T)} = 0$. Moreover, if L/K is a separable extension, which is also geometric, i.e. an extension in which the constant field is not enlarged, then one has the equality

$$2g_L - 2 = [L : K](2g_K - 2) + \deg_K(\text{Disc}(L/K)), \quad (3.4)$$

known as the Riemann–Hurwitz theorem [26, Theorem 7.16]. Using this, we can determine the genus for any separable, geometric extension of $\mathbb{F}_q(T)$.

If one makes the change of variables $u = q^{-s}$ and writes, with some abuse of notation, $\zeta_K(u)$ for the result, then it turns out that $\zeta_K(u)$ becomes a rational function in u . In fact, we have that

$$\zeta_K(u) = \frac{P_K(u)}{(1 - qu)(1 - u)},$$

for a polynomial P_K of degree $2g_K$, cf. (3.2) where the genus is 0. Moreover, the Riemann Hypothesis is a theorem over function fields, originally proven by Weil [35], asserting that all zeros of $\zeta_K(s)$ have real part equal to $1/2$. Equivalently, all roots of P_K have absolute value $q^{-1/2}$.

Just as over \mathbb{Q} , the Riemann Hypothesis for $\zeta_K(u)$ implies strong bounds for the error term when counting primes. Specifically, we have that [26, Theorem 5.12]

$$\#\{\mathcal{P} : \deg_K \mathcal{P} = n\} = \frac{q^n}{n} + \mathcal{O}_K\left(\frac{q^{n/2}}{n}\right),$$

cf. the comments after Conjecture 1.2.1.

3.2 Hecke L -functions

Before introducing Artin L -functions, which play a major role in both Papers I and II, we define Hecke L -functions. These are, in a certain sense, more general than Artin L -functions, and they are used in Paper II for estimating sums connected to the class group $\text{Cl}(K)$. Furthermore, the functional equation for Hecke L -function can be used to deduce the functional equation of Artin L -functions. The presentation below follows [26, Chapter 9]. We will only need to consider Hecke characters of finite order for our purposes.

Let \mathcal{F} be an effective divisor and consider the group $\mathcal{D}(\mathcal{F})$ of divisors D coprime to \mathcal{F} , in the sense that the power of any divisors of \mathcal{F} in the product decomposition of D is zero. Moreover, let $H^{\mathcal{F}}$ denote the set of principal divisors, generated by elements x congruent to 1 modulo

\mathcal{F} , i.e. $\nu_{\mathcal{P}}(x-1) \geq 1$ for all primes $\mathcal{P} \mid \mathcal{F}$. The ray class group modulo \mathcal{F} , $\text{Cl}_{\mathcal{F}}(K)$, is then the quotient $D(\mathcal{F})/H^{\mathcal{F}}$.

Once again, $\text{Cl}_{\mathcal{F}}(K)$ is not a finite group, however, the subgroup of divisor classes of degree zero $\text{Cl}_{\mathcal{F}}^0(K)$ is finite. A Hecke character on $\text{Cl}_{\mathcal{F}}(K)$ is a map

$$\chi : \text{Cl}_{\mathcal{F}}(K) \rightarrow S^1$$

of finite order. When $\mathcal{F} \mid \mathcal{F}'$, we have a reduction map

$$\text{Cl}_{\mathcal{F}'}(K) \rightarrow \text{Cl}_{\mathcal{F}}(K),$$

which thus induces a map on the character groups

$$\widehat{\text{Cl}}_{\mathcal{F}}(K) \rightarrow \widehat{\text{Cl}}_{\mathcal{F}'}(K),$$

so that a Hecke character χ modulo \mathcal{F} induces a character modulo any multiple of \mathcal{F} . We say that a character on $\text{Cl}_{\mathcal{F}}(K)$ is primitive modulo \mathcal{F} if it is not induced from a character modulo a proper divisor of \mathcal{F} .

Having defined (finite order) Hecke characters, we are now in a position to define their L -functions. When χ is a Hecke character modulo \mathcal{F} , we let

$$L(s, \chi) = \prod_{\mathcal{P} \nmid \mathcal{F}} (1 - \chi(\mathcal{P})|\mathcal{P}|^{-s})^{-1}.$$

It turns out that for primitive χ , this becomes a polynomial in $u = q^{-s}$. Indeed, we have [26, Theorem 9.24A]:

Theorem 3.2.1. *Let χ be a primitive Hecke character on $\text{Cl}_{\mathcal{F}}(K)$, and suppose that χ is not trivial on $\text{Cl}_{\mathcal{F}}^0(K)$. Then $L(s, \chi)$ is a polynomial in $u = q^{-s}$ of degree $2g_K - 2 + \deg_K \mathcal{F}$. Moreover, the completed function*

$$\Lambda(s, \chi) := q^{(g_K-1)s} |\mathcal{F}|^{s/2} L(s, \chi)$$

satisfies the functional equation

$$\Lambda(s, \chi) = \epsilon(\chi) \Lambda(1-s, \bar{\chi}),$$

with $|\epsilon(\chi)| = 1$.

Having seen the rationality of $L(s, \chi)$, one is left with the question of the location of its zeros. It turns out that the Riemann Hypothesis holds for these L -functions as well, and in fact this follows from the validity of the Riemann Hypothesis for Artin L -functions, together with input from class field theory.

3.3 Artin L -functions

Let L/K be a finite geometric Galois extension of function fields, and let $\rho : \text{Gal}(L/K) \rightarrow \text{GL}(V)$ be a finite-dimensional (complex) representation of the Galois group $\text{Gal}(L/K)$. We wish to define the Artin L -function attached to this representation, cf. [24, Section VII.10] [26, Chapter 9], but for this, we need some additional definitions from algebraic number theory.

Now, let $\mathfrak{P} \mid \mathcal{P}$ be primes in L and K respectively. Any element $\sigma \in \text{Gal}(L/K)$ permutes the primes lying over \mathcal{P} , so we may define the decomposition group $G_{\mathfrak{P}}$ by

$$G_{\mathfrak{P}} = \{\sigma \in \text{Gal}(L/K) : \sigma(\mathfrak{P}) = \mathfrak{P}\}.$$

By reduction modulo \mathfrak{P} , we obtain an induced map

$$G_{\mathfrak{P}} \rightarrow \text{Gal}(\kappa_{\mathfrak{P}}/\kappa_{\mathcal{P}}),$$

where $\kappa_{\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}}/\mathfrak{P}$ and $\kappa_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}/\mathcal{P}$. This map turns out to be surjective, and we denote its kernel by $I_{\mathfrak{P}}$, so that

$$G_{\mathfrak{P}}/I_{\mathfrak{P}} \simeq \text{Gal}(\kappa_{\mathfrak{P}}/\kappa_{\mathcal{P}}).$$

The extension $\kappa_{\mathfrak{P}}/\kappa_{\mathcal{P}}$ is a finite extension of finite fields, whence its Galois group is generated by the Frobenius element $x \mapsto x^{q'}$, with $q' = q^{\deg_K \mathcal{P}}$. The isomorphism above associates this Frobenius element to an element $\varphi_{\mathfrak{P}} \in \text{Gal}(L/K)/I_{\mathfrak{P}}$. If we let $V^{I_{\mathfrak{P}}}$ denote the subspace of V fixed pointwise by all elements of $I_{\mathfrak{P}}$, then $\varphi_{\mathfrak{P}}$ acts on $V^{I_{\mathfrak{P}}}$.

We can now define the Artin L -function

$$L(s, \rho) = \prod_{\mathcal{P}} \det(I - \varphi_{\mathfrak{P}} |\mathcal{P}|^{-s}; V^{I_{\mathfrak{P}}})^{-1},$$

where the local factor is the inverse of the characteristic polynomial, evaluated at $|\mathcal{P}|^{-s}$. The fact that the Galois group acts transitively on primes lying above \mathcal{P} and that the characteristic polynomial is invariant under conjugation, shows that the above does not depend on the choice of $\mathfrak{P} | \mathcal{P}$.

When ρ is the trivial representation, this definition reduces to (3.1) and we simply recover the Dedekind zeta function of K . The L -function of a sum of two representations simply becomes the product of the two L -functions associated with each summand. Recall that the regular representation contains each irreducible representation ρ of $\text{Gal}(L/K)$ with multiplicity $\chi_{\rho}(1)$, where χ_{ρ} is the character of ρ . Hence, we find that [24, Corollary VII. 10.5]

$$L(s, \rho_{\text{reg}}) = \zeta_K(s) \prod_{\rho \neq 1} L(s, \rho)^{\chi_{\rho}(1)}, \quad (3.5)$$

where the product ranges over nontrivial, irreducible representations of $\text{Gal}(L/K)$. Moreover, using the fact that the regular representation is induced from the trivial representation of the trivial subgroup, one can in fact prove that the left-hand side above is $\zeta_L(s)$. We thus see that for a Galois extension L/K the Dedekind zeta function $\zeta_L(s)$ factors into a product of Artin L -functions:

$$\zeta_L(s) = \zeta_K(s) \prod_{\rho \neq 1} L(s, \rho)^{\chi_{\rho}(1)}. \quad (3.6)$$

We have not yet mentioned the functional equation or meromorphic continuation of an Artin L -function. This follows, in fact, from the analogous properties for Hecke L -functions, but we will not go into details here. The main idea is that by using class field theory, one may associate a Hecke character to any one-dimensional representation of an abelian extension, see e.g. [26, p. 141].

Over number fields, the analyticity of Artin L -functions is not known in general. However, for Artin L -functions, Weil proved that for a non-trivial irreducible representation, the corresponding Artin L -function is a polynomial, see e.g. [26, Theorem 9.16B]. The Riemann Hypothesis for such L -functions then follows from the product formula (3.6), and the Riemann Hypothesis for Dedekind zeta functions, see [26, p.130].

Before ending this section, we briefly describe an important invariant associated to an Artin L -function, namely its conductor. This is related to the Artin conductor of the associated representation, but we will not go into the details of this relationship. Given a representation ρ , the completed L -function has the form

$$\xi(s, \rho) = c(\rho)^{s/2} L(s, \rho),$$

for some $c(\rho)$, and this function is invariant under the map $s \mapsto 1 - s$. In Papers I and II, we are interested in Artin L -functions of the form

$$\frac{\zeta_L(s)}{\zeta_K(s)},$$

for some L/K . In this case, the value of $c(\rho)$ can be deduced from the functional equations of the Dedekind zeta functions ζ_L and ζ_K , and we find that

$$c(\rho) = q^{(2g_L - 2) - (2g_K - 2)} = q^{2g_L - 2g_K}.$$

In both Papers I and II, we will be interested in how a prime \mathcal{P} in some field K , splits in an extension L . When L/K is Galois, this can be determined by studying the cycle structure of the Frobenius element $\varphi_{\mathfrak{p}}$ in $\text{Gal}(L/K)$. The case of a general geometric, separable extension L/K can be inferred from the Galois case by considering the Galois closure of L .

We end this section by mentioning a result which allows us to study the number of primes in K , splitting in a certain way in L , namely Chebotarev's Density Theorem. One obtains this result as an application of the Riemann Hypothesis for Artin L -functions, and the orthogonality relations, see [26, Theorem 9.13B].

Theorem 3.3.1 (Chebotarev's Density Theorem). *Let L/K be a geometric Galois extension of function fields, and let $G = \text{Gal}(L/K)$, with $C \subseteq G$ a conjugacy class. Letting S'_K denote the set of primes in K , which are unramified in L , we have*

$$\#\{\mathcal{P} \in S'_K : \deg_K \mathcal{P} = n, \varphi_{\mathcal{P}} \in C\} = \frac{\#C}{\#G} \frac{q^n}{n} + \mathcal{O}_{G,K} \left(\frac{q^{n/2}}{n} \right).$$

3.4 The one-level density

We have seen that the Riemann Hypothesis holds true for a wide class of L -functions defined over function fields. Knowing this, we may ask ourselves how the zeros distribute themselves along the line $\Re(s) = 1/2$, or equivalently on the circle $|u| = q^{-1/2}$. In particular, one can study the zeros lying close to the point $1/2$. These zeros are known as the low-lying zeros and can be studied using the so-called one-level density.

Let $\mathcal{F}(X)$ be a family of L -functions, ordered by X . For a Schwartz function ϕ , the one-level density is then defined as the average

$$\frac{1}{\#\mathcal{F}(X)} \sum_{L(s,f) \in \mathcal{F}(X)} \sum_{\rho_f} \phi \left(\frac{\gamma_f \log c_X}{2\pi} \right), \quad (3.7)$$

where f is an index, and $\rho_f = 1/2 + i\gamma_f$ ranges over the nontrivial zeros of $L(s, f)$. The parameter c_X should satisfy $c_X \asymp c_f$, where c_f is the conductor of $L(s, f)$.

Aside from being Schwartz, we require the Fourier transform $\widehat{\phi}$ of ϕ to have compact support, contained in $[-\sigma, \sigma]$, for some $\sigma > 0$. With this assumption, we can analytically continue ϕ to an entire function on \mathbb{C} through the inverse Fourier transform. Then, even without assuming the Riemann Hypothesis, one may still study the one-level density, but in this case it is possible that $\gamma_f \notin \mathbb{R}$.

Katz and Sarnak conjectured [18] that for a natural family \mathcal{F} (see e.g. [29] for one definition of what makes a family natural) as $X \rightarrow \infty$, the

one-level density (3.7) converges to an integral

$$\int_{-\infty}^{\infty} \phi(x) W_G(x) dx,$$

where $W_G(X)$ equals one of five density functions, depending on the so-called symmetry type G of the family \mathcal{F} . This is known as the Katz–Sarnak Conjecture. The symmetry type is either unitary, unitary symplectic, or one of three orthogonal symmetry types. The names of the symmetry types come from Random matrix theory, where one can prove a similar limiting behaviour for eigenvalues of random matrices lying close to the point 1.

Of particular interest to us is the density

$$W_{Sp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}.$$

Indeed, in Papers I and II, we study the one-level density associated to two families of Artin L -functions, and in both of these families, the symmetry type is symplectic. Note that $W_{Sp}(0) = 0$, so that one does not expect many zeros of the L -functions in $\mathcal{F}(X)$ to lie at, or very close to, the point $s = 1/2$.

Much work has been done on verifying the Katz–Sarnak Conjecture for various families of L -functions. See e.g. [17], [38], [28], [30], and [37] for families of modular form L -functions, elliptic curve L -functions, Hecke L -functions and Artin L -functions.

Chapter 4

Summary of appended papers

4.1 Paper I

In this paper, we study the counting function $\mathcal{F}(X)$ of non-Galois cubic function field extensions of $\mathbb{F}_q(T)$, ordered by discriminant X . In particular, we find the main and secondary terms of this counting function, together with an error term of the same quality as the current best result, due to Bhargava, Taniguchi and Thorne [4], over \mathbb{Q} .

We also study a refined counting function, where one specifies the splitting behaviour of finitely many rational primes. In particular, if S is a splitting type, and $P \in \mathbb{F}_q[T]$ is a rational prime, then we prove that the number of cubic non-Galois extensions of $\mathbb{F}_q(T)$, where P splits according to S , equals

$$C_{1,P,S}X + C_{2,P,S}X^{5/6} + \mathcal{O}_\epsilon(X^{2/3+\epsilon}|P|^{2/3}), \quad (4.1)$$

for certain coefficients $C_{i,P,S}$. For unramified S , the coefficient $C_{1,P,S}$ is the same as the coefficient of the main term from Chebotarev's Density Theorem.

By computing the one-level density of a family of Artin L -functions associated with these cubic extensions, we also prove a lower bound for the error term in (4.1). Specifically, we prove that if (4.1) holds

with an error term

$$\mathcal{O}_\epsilon(X^{\theta+\epsilon}|P|^\omega),$$

then one must in fact have that $\theta + \omega \geq 1/2$. This generalises an analogous conditional result of Cho, Fiorilli, Lee and Södergren [6] over \mathbb{Q} .

4.2 Paper II

In this paper, we study the family of Artin L -functions ζ_L/ζ_K associated with the unique irreducible two-dimensional representation of a D_4 extension $L/\mathbb{F}_q(T)$, with K the quadratic subfield of L . In particular, by computing the one-level density of this family, we prove for large enough q , that at least 77% of these L -functions, when ordered by conductor, are non-vanishing at the point $s = 1/2$, improving results due to Durlanik [12] over \mathbb{Q} .

More specifically, these results are obtained by verifying the Katz–Sarnak Conjecture for certain subfamilies of our family, where we control the size of the discriminant of the quadratic subfield K . In particular, we are able to vary the allowed support σ between the different subfamilies, which allows us to isolate the contribution to the one-level density of the L -functions which are particularly difficult to handle.

Finally, we remark that the methods we use are also applicable to the study of the one-level density of the family of quadratic extensions of a general function field K . We use this observation to establish a rate of non-vanishing larger than 94% for such a family, with a fixed K . This generalises results over number fields due to Gao and Zhao [14], [15], who studied this one-level density for finitely many choices of K defined over \mathbb{Q} .

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