Thesis for the Degree of Doctor of Philosophy

Hyperuniformity and Hyperfluctuations for Random Measures on Euclidean and Non-Euclidean Spaces

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Abstract

In this thesis we study fluctuations of generic random point configurations in Euclidean and symmetric curved geometries. Mathematically, such configurations are interpreted as isometrically invariant point processes, and fluctuations are recorded by the variance of the number of points in a centered ball, the *number variance*. Hyperuniformity and hyperfluctuation of such configurations in the sense of Stillinger-Torquato is characterized in terms of large-scale asymptotics of the number variance in relation to that of an ideal gas, and equivalently by small-scale asymptotics of the *Bartlett spectral measure* in the diffraction picture.

Appended to the Thesis are three papers:

In Paper I we provide lower asymptotic bounds for number variances of isometrically invariant random measures in Euclidean and hyperbolic spaces, generalizing a result by Beck. In particular, we find that geometric hyperuniformity fails for every isometrically invariant random measure on hyperbolic space. In contrast to this, we define a notion of spectral hyperuniformity which is satisfied by certain invariant random lattice configurations.

In Paper II we establish similar lower asymptotic bounds for number variances of automorphism invariant point processes in regular trees. The main result is that these lower bounds are not uniform for the invariant random lattice configurations defined by the fundamental groups of complete regular graphs and the Petersen graph. We also provide a criterion for when these lower bounds are uniform in terms of certain rational peaks appearing in the diffraction picture.

In Paper III we prove the existence and uniqueness of Bartlett spectral measures for invariant random measures on a large class of non-compact commutative spaces, which includes those in Papers I and II. For higher rank symmetric spaces governed by simple Lie groups, we prove that there is a power strictly less than 2 of the volume of balls that asymptotically bounds the number variance of any invariant random measure from above. Moreover, we derive Bartlett spectral measures for invariant determinantal point processes on commutative spaces and define a notion of *heat kernel hyperuniformity* on Euclidean and hyperbolic spaces that is equivalent to spectral hyperuniformity.

Keywords: Point processes, number variance, hyperuniformity, spectral measures, spherical harmonic analysis.

Sammanfattning

I denna avhandling studerar vi fluktuationer av generiska slumpmässiga punktkonfigurationer i Euklidiska och symmetriskt krökta geometrier. Matematiskt tolkas sådana konfigurationer som isometriskt invarianta punktprocesser, och fluktuationer bestäms av variansen av antalet punkter i en centrerad boll, *talvariansen*. Hyperuniformitet och hyperfluktuation av sådana konfigurationer enligt Stillinger-Torquato är karakteriserade i termer av storskalig asymptotik av talvariansen relativt talvariansen av en ideal gas, och ekvivalent genom småskalig asymptotik av *Bartlett-spektralmåttet* i diffraktionsbilden.

Till avhandlingen hör tre artiklar:

I Artikel I finner vi nedre asymptotiska begränsningar för talvarianser av isometriskt invarianta slumpmått i Euklidiska och hyperboliska rum, vilket generaliserar ett resultat av Beck. Särskilt finner vi att geometrisk hyperuniformitet inte gäller för något isometriskt invariant slumpmått på hyperboliska rum. I kontrast till detta definierar vi spektral hyperuniformitet som uppfylls av vissa invarianta slumpmässiga gitterkonfigurationer.

I Artikel II finner vi liknande nedre asymptotiska begränsningar för talvariansen av automorfi-invarianta punktprocesser i reguljära träd. Huvudresultatet är att dessa nedre begränsningar inte är likformiga för invarianta slumpmässiga gitterkonfigurationer tillhörande fundamentalgrupper av fullständiga reguljära grafer och Petersen-grafen. Vi ger också ett kriterium för när dessa nedre begränsningar är likformiga i termer av uppträdandet av vissa rationella toppar i diffraktionsbilden.

I Artikel III bevisar vi existens och entydighet av Bartlett-spektralmått för invarianta slumpmått på en stor klass av icke-kompakta kommutativa rum, vilket inkluderar de som behandlas i Artikel I och II. För symmetriska rum av högre rang som kommer från enkla Liegrupper bevisar vi att det finns en potens strikt mindre än 2 av volymen av bollar som asymptotiskt begränsar alla talvarianser av invarianta slumpmått från ovan. Vi härleder också Bartlett-spektralmått för invarianta determinantprocesser på kommutativa rum och definierar *värmekärne-hyperuniformitet* på Euklidiska och hyperboliska rum som är ekvivalent med spektral hyperuniformitet.

Nyckelord: Punktprocesser, talvarians, hyperuniformitet, spektralmått, sfärisk harmonisk analys.

List of appended papers

The following three papers are appended to the Thesis:

I. Michael Björklund, Mattias Byléhn,
Hyperuniformity of random measures on Euclidean and hyperbolic spaces
Preprint, submitted.

II. Mattias Byléhn,
Hyperuniformity in regular trees
Preprint, submitted.

III. Michael Björklund, Mattias Byléhn,
Hyperuniformity and hyperfluctuations of random measures in commutative spaces
Preprint, submitted.

The contributions made by the author to the respective papers are:

I. I worked out most of the proofs for general dimensions, guided by ideas and proofs of the first author. I also wrote the paper.

II. The main problem was posed by the supervisor. I proved all results and wrote the paper.

III. I wrote the paper and worked out mathematical details in proofs and derivations, following the main ideas, proofs and sketches provided by the first author.

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Mattias Byléhn
Göteborg, May 2025

Contents

Preface

Motivation and setting

In this Thesis we investigate notions of fluctuations of generic random point configurations in Euclidean flat geometries as well as certain curved geometries that are globally determined by their isometries. The fundamental philosophy that motivates the investigation of fluctuations comes from statistical physics, in which fluctuations are in many physical contexts closely related to notions of order in a physical system. With respect to density, the suppression of large-scale fluctuations of a random point configuration is related to geometric long-range order via the notion of *hyperuniformity*, introduced by Stillinger-Torquato in [23]. A random point configuration in Euclidean space is said to be hyperuniform if the variance of the number of points in a centered ball is asymptotically suppressed by the volume of the ball in the infinite radial limit. Equivalently, hyperuniformity of such a configuration is characterized by the suppression of small spatial frequencies in the Fraunhofer diffraction picture of the configuration. Also, in the spectral picture, an even more rigid notion of long-range order is governed by *stealth* of a point process, introduced by Batten-Stillinger-Torquato in [5]. A generic random point configuration is *stealthy* if all sufficiently small frequencies completely vanish in the Fraunhofer diffraction picture.

Mathematically, generic random point sets are realized as random atomic measures that are invariant under a large group of isometries, i.e. distance preserving transformations. In this Thesis we deal with invariant point processes and, more generally, invariant random measures. In d-dimensional Euclidean space $\mathbb{R}^d$, hyperuniformity of a point process is detected geometrically by large radial asymptotics of the *number variance*, and spectrally by small scale asymptotics of the *structure factor/diffraction measure/Bartlett spectral measure* of the point process. More precisely, if $\mathcal{P}$ is an invariant point process then it is hyperuniform if the random variables $|\mathcal{P} \cap B_r(0)| = |\{p \in \mathcal{P} \mid \|p\| \leq r\}|$, recording the random number of points of $\mathcal{P}$ in the ball $B_r(0)$ of radius $r > 0$ centered at the origin, satisfy

$$\lim_{r \to +\infty} \frac{\mathrm{Var}(|\mathcal{P} \cap B_r(0)|)}{\mathrm{Vol}_{\mathbb{R}^d}(B_r(0))} = 0\,.$$

If this limit is instead infinite, we say that $\mathcal{P}$ is hyperfluctuating. The spectral formulation of hyperuniformity states that the so called Bartlett spectral measure $\sigma_{\mathcal{P}}$ of $\mathcal{P}$, which is positive measure on $\mathbb{R}^d$, satisfies the following small-scale asymptotics in the frequency domain:

$$\lim_{\varepsilon\to 0^+} \frac{\sigma_{\mathcal{P}}(B_\varepsilon(0))}{\mathrm{Vol}_{\mathbb{R}^d}(B_\varepsilon(0))} = 0\,.$$

The addition of $\mathcal{P}$ being stealthy requires that $\sigma_{\mathcal{P}}(B_{\varepsilon_o}(0)) = 0$ for some $\varepsilon_o > 0$.

Structure

The purpose of the Introduction to this Thesis is to provide the reader some mathematical rigor regarding preliminary material that is not covered in detail in the appended papers, and to give an overview of the theory and tools used in the papers. Most importantly, we make an effort to highlight *examples*, specifically of invariant point processes.

The Introduction is structured as follows: In Chapter 1 we define spaces of locally finite Borel measures on proper separable metric spaces, keeping Euclidean spaces in mind. Linear statistics are defined next, and using these we generate sigma algebras on spaces of measures, which allows us to then define what we mean by a point process and by a random measure on a proper separable metric space. We also define moment measures and n-point correlation measures of random measures and exemplify the theory by surveying Poisson point processes, determinantal and permanental point processes.

In Chapter 2 we restrict our attention to random measures on Euclidean spaces and start out by defining stationarity and isotropy, the two central notions of invariance of random measures. We then derive reduced moment measures of stationary random measures, in particular the autocorrelation measure that records the variance of linear statistics and compute it for some examples. The examples we consider are Poisson, determinantal/permenental point processes, invariant random translates of a lattice and i.i.d. perturbations of such, as well as invariant random quasicrystals/model sets. In the subsequent Section we survey some general results on asymptotics of number variances of stationary random measures. In order to define the spectral analogue of the autocorrelation measure and number variance we introduce Fourier transforms and arrive at a central Theorem in this Thesis due to Bochner, generalized by Schwartz, stating that positive-definite distributions are distributional Fourier transforms of positive tempered locally finite Borel measures. In particular, this holds for the autocorrelation measure of a stationary random measure, and we define the associated *Bartlett spectral measure* as its distributional Fourier transform. Using this we recall the definitions of spectral hyperuniformity and stealth of stationary random measures, and check these properties for the previously mentioned examples.

In Chapter 3 we introduce the general setup of the homogeneous spaces that we consider and invariant random measures on them. Invariant Poisson and determinantal/permanental point processes readily generalize to these spaces. In order to give a better sense of how geometric and spectral fluctuations of invariant random measures behave on non-Euclidean spaces, we restrict our attention to real hyperbolic spaces and regular trees as the central examples. These belong to a wider class of *commutative spaces* on which there is a well-behaved notion of Fourier transform available for radial functions, and we cover some general properties of this transform. Finally, we prove that invariant determinantal point processes associated with orthogonal projections are not spectrally hyperuniform in the sense of Papers I and II.

"There was math, I got confused."

- Ted Mosby

1. Point processes and random measures

1.1 Point processes: some intuition

From a probabilistic point of view, a point process in a measurable space X can be described as a countable collection

$$\xi = (\xi_n \,|\, n \in \mathbb{N})$$

of $(X \cup \{\varnothing\})$-valued random variables, thought of as collectively describing an (in)finite random point set in X, where the variables taking the value $\varnothing$ are thought of as not being observed. On the real line $X = \mathbb{R}$ we can think of ξ as describing the time stamps of a series of (random) instantaneous events, for example the decay of atoms in a radioactive material, the times each new hour starts or the clicks on the screens of the coffee machines in the lunchroom during the day. In 2- and 3-dimensional spaces $X = \mathbb{R}^2, \mathbb{R}^3$ we can for example think of ξ as the random configuration of atoms in a (quasi)crystalline material, or the snapshot of a gas in thermal equilibrium, say the air molecules bouncing around in my office. The three examples on the real line $\mathbb{R}$ mentioned are in a sense central to this Thesis in that they exhibit the following three characteristic properties. The decay of atoms in an (idealized) radioactive sample occur at "purely random" (Poissonian) times, the start of each new hour is deterministic and regular (periodic) and the times that people at the department get coffee are random but tend to cluster, especially around 10am and 3pm here. A related fundamental question is how to mathematically observe the occurrence of these three phenomena, and if we can do so using some type of index. The idea is that fluctuations/variances of point counts in large bounded domains encode a lot of this information.

Mathematically, it will be fruitful to interpret a point process ξ as a random atomic measure

$$\mu_\xi = \sum_{n=1}^{\infty} \delta_{\xi_n}$$

where δ_{ξ_n} denotes the Dirac delta measure at the random point ξ_n and $\delta_\varnothing = 0$ by convention. The notion of a point process can then also be generalized to include weights

$$\mu_{\xi,w} = \sum_{n=1}^{\infty} w(\xi_n)\delta_{\xi_n} \,,$$

where $w : X \to \mathbb{R}_{\geq 0}$ is a non-negative measurable function on X. If we for example have a sample of some material consisting of different families of particles, introducing a weight as above can be useful to distinguish the different families of particles mathematically when modeling the sample. In the next Section we will start to formalize what we mean by a point process and random measure and on what spaces they will live on.

1.2 Spaces of positive locally finite measures

Let (X, d) be a proper separable metric space, in other words a metric space with a countable dense subset and in which closed balls $B_r(x) = \{y \in X : d(x, y) \leq r\}$ are compact for all $x \in X$ and all $r \geq 0$. For now we emphasize the focus on d-dimensional Euclidean space $X = \mathbb{R}^d$ with the standard Euclidean metric

$$d_{\mathbb{R}^d}(x, y) = \|x - y\| = \Big(\sum_{j=1}^{d}(x_j - y_j)^2\Big)^{1/2}.$$

For our purposes, a proper separable metric space X is naturally endowed with the metric topology and the associated Borel σ-algebra $\mathscr{B}_X$.

For the precise mathematical definition of random measures and point processes, we consider the set $\mathscr{M}_+(X)$ of locally finite positive Borel measures on X, in other words non-negative measures p on the Borel σ-algebra $\mathscr{B}_X$ satisfying $p(B) < +\infty$ for all bounded Borel sets $B \subset X$. A *random measure* will be an $\mathscr{M}_+(X)$-valued random variable, which requires us to define a suitable σ-algebra on $\mathscr{M}_+(X)$. But first, to give some context we note that an atomic positive Borel measure of the form

$$\delta_{(P,w)} = \sum_{x \in P} w(x)\delta_x$$

for some countable subset $P \subset X$ and some non-negative measurable function $w : X \to \mathbb{R}_{\geq 0}$ is locally finite if and only if

$$\sum_{x \in P \cap B} w(x) < +\infty$$

for every bounded Borel subset $B \subset X$. With this in mind we denote the subset of *pure point measures* in $\mathscr{M}_+(X)$ by

$$\mathscr{M}_+^{\text{p.p.}}(X) = \Big\{\delta_{(P,w)} = \sum_{x \in P} w(x)\delta_x \,\Big|\, \sum_{x \in P \cap B} w(x) < +\infty \quad \forall \, \text{bounded } \; B \in \mathscr{B}_X\Big\},$$

where we implicitly have assumed that P, w are as previously described. Note that if $w(x) \geq w_o > 0$ for all $x \in X$, then $\delta_{(P,w)} \in \mathscr{M}_+^{\text{p.p.}}(X)$ if and only if $P \subset X$ is locally finite in the sense that $|P \cap B| < +\infty$ for all bounded Borel $B \subset X$. Before moving on, we also define the subset of *simple pure point measures* in $\mathscr{M}_+(X)$ to be

$$\mathscr{M}_+^{\text{s.p.p.}}(X) = \Big\{\delta_P = \sum_{x \in P} \delta_x \,\Big|\, P \subset X \text{ locally finite}\Big\}.$$

To give explicit examples we can consider the following pure point measures on the real line,

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \delta_{\frac{1}{n}} \notin \mathscr{M}_+(\mathbb{R})\,, \qquad \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-2}\delta_{\frac{1}{n}} \in \mathscr{M}_+^{\text{p.p.}}(\mathbb{R})\,, \quad \sum_{n\in\mathbb{Z}} \delta_n \in \mathscr{M}_+^{\text{s.p.p.}}(\mathbb{R})\,.$$

Remark 1.2.1 (Radon measures)**.** From Theorem 17.10 in [18] one can deduce that if $p \in \mathscr{M}_+(X)$ then it is inner and outer regular in the sense that

$$p(B) = \sup\Big\{p(C)\,|\,C \subset B \text{ is compact in } X\Big\} = \inf\Big\{p(U)\,|\,U \supset B \text{ is open in } X\Big\}\,.$$

Such measures p are the *Radon measures* on X, so $\mathscr{M}_+(X)$ is the space of positive Radon measures on X. We will stick to "positive locally finite Borel" in the Introduction, but note that there are slight variations of terminology in the appended Papers.

1.3 Linear statistics

The set $\mathscr{M}_+(X)$ of locally finite Borel measures carries a large family of canonical functions that separate points and naturally define a σ-algebra on $\mathscr{M}_+(X)$. These functions are known as *linear statistics*, defined as follows. Denote by $\mathscr{L}_c^\infty(X)$ the vector space of complex-valued bounded measurable functions with compact support. For each $f \in \mathscr{L}_c^\infty(X)$, the associated linear statistic is the function $\mathbb{S}f : \mathscr{M}_+(X) \to \mathbb{C}$ defined by

$$\mathbb{S}f(p) = \int_X f(x)dp(x)\,, \quad p \in \mathscr{M}_+(X)\,.$$

The term "linear" in linear statistic comes from the observation that $\mathscr{M}_+(X)$ is an additive convex cone, namely that if $p_1, p_2 \in \mathscr{M}_+(X)$ and $t_1, t_2 \geq 0$ then $t_1p_1 + t_2p_2 \in \mathscr{M}_+(X)$, and that the function $\mathbb{S}f$ satisfies the linearity condition

$$\mathbb{S}f(t_1p_1 + t_2p_2) = t_1\mathbb{S}f(p_1) + t_2\mathbb{S}f(p_2)\,.$$

As an example, if $B \subset X$ is a bounded Borel set then the indicator function

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

is an element of $\mathscr{L}_c^\infty(X)$ and $\mathbb{S}\chi_B(p) = p(B)$. In particular, if p is an atomic measure of the form

$$p = \sum_{n=1}^\infty \delta_{x_n}$$

then $\mathbb{S}\chi_B(p) = |\{n \in \mathbb{N}\,|\,x_n \in B\}|$ is simply the count of points x_n in B.

We endow the set $\mathscr{M}_+(X)$ of locally finite positive Borel measures with the smallest σ-algebra $\mathscr{B}_{\mathscr{M}_+}$ such that each linear statistic $\mathbb{S}f$ with $f \in \mathscr{L}_c^\infty(X)$ is measurable. In other words, $\mathscr{B}_{\mathscr{M}_+}$ is the σ-algebra generated by the inverse images

$$(\mathbb{S}f)^{-1}(D) = \Big\{p \in \mathscr{M}_+(X)\,\Big|\, \int_X f(x)dp(x) \in D\Big\}$$

for measurable subsets $D \subset \mathbb{C}$ and functions $f \in \mathscr{L}^\infty_c(X)$. Similarly, we let $\mathscr{B}_{\mathscr{M}_+^{\text{p.p.}}}$ and $\mathscr{B}_{\mathscr{M}_+^{\text{s.p.p.}}}$ denote the subspace σ-algebras on $\mathscr{M}_+^{\text{p.p.}}(X)$ and $\mathscr{M}_+^{\text{s.p.p.}}(X)$.

Remark 1.3.1 (Weak*-topology on $\mathscr{M}_+(X)$)**.** As a subspace of the topological dual $C_c(X)^*$, the space $\mathscr{M}_+(X)$ of positive locally finite Borel measures can be endowed with the subspace weak*-topology, sometimes called the *weak* or *vague* topology, in which convergence of a sequence $p_n \to p$ is defined by

$$\int_X f(x)dp_n(x) \longrightarrow \int_X f(x)dp(x)\,, \quad \forall f \in C_c(X)$$

as $n \to +\infty$. In other words, it is the smallest topology for which linear statistics $\mathbb{S}f$ with $f \in C_c(X)$ are continuous. By Theorem 1.5 in [17], this topology on $\mathscr{M}_+(X)$ is Polish. Since pointwise limits of continuous functions are Borel measurable, the linear statistics $\mathbb{S}f$ with $f \in \mathscr{L}^\infty_c(X)$ are measurable with respect to the weak*-Borel σ-algebra. In fact, one can show that the the weak*-Borel σ-algebra coincides with the σ-algebra $\mathscr{B}_{\mathscr{M}_+}$, see for example [17, Lemma 4.7]. In particular, $\mathscr{M}_+^{\text{s.p.p.}}(X)$ is weak*-closed and hence measurable.

1.4 Random measures

We now have a measurable space $(\mathscr{M}_+(X), \mathscr{B}_{\mathscr{M}_+})$ of locally finite positive Borel measures on a proper separable metric space X, which allows us to define point processes and random measures.

Definition 1.4.1 (Random measures and point processes)**.** Let $(\Omega, \mathscr{B}, \mathbb{P})$ be a probability space.

1. A *random (locally finite) measure on* X is a measurable map $\mu : \Omega \to \mathscr{M}_+(X)$.
2. A *(locally finite) point process on* X is a measurable map $\mu : \Omega \to \mathscr{M}_+^{\text{p.p.}}(X)$.
3. A *(locally finite) simple point process on* X is a measurable map $\mu : \Omega \to \mathscr{M}_+^{\text{s.p.p.}}(X)$.

Since the inclusions $\mathscr{M}_+^{\text{s.p.p.}}(X) \hookrightarrow \mathscr{M}_+^{\text{p.p.}}(X) \hookrightarrow \mathscr{M}_+(X)$ are measurable by definition we see that a simple point process defines a point process, which in turn defines a random measure on X. The *law* of a random measure μ is the push-forward probability measure $\mathbb{P}_\mu \in \text{Prob}(\mathscr{M}_+(X))$ given by $\mathbb{P}_\mu(E) = \mathbb{P}(\mu^{-1}(E))$ for Borel sets $E \in \mathscr{B}_{\mathscr{M}_+}$.

Remark 1.4.2 (Notational conventions regarding random measures)**.** In this Thesis we typically consider random measures μ one at a time, so we will frequently identify a random measure μ with its law $\mathbb{P}_\mu$ and write $\mu \in \text{Prob}(\mathscr{M}_+(X))$. If $\mu \in \text{Prob}(\mathscr{M}_+(X))$ and $\psi : \mathscr{M}_+(X) \to \mathbb{C}$ is μ-integrable, then the *μ-expectation* of ψ is

$$\mathbb{E}_\mu(\psi) = \int_{\mathscr{M}_+(X)} \psi(p)d\mu(p)\,.$$

Remark 1.4.3 (Integer-valued random measures are point processes)**.** As shown in [19, Cor. 6.5], if a random measure $\mu : \Omega \to \mathscr{M}_+(X)$ is almost surely an integer-valued measure, then there are X-valued random variables $\xi_1, \xi_2, \ldots$ and an $\mathbb{N}_0 \cup \{+\infty\}$-valued random variable N on Ω such that

$$\mu = \sum_{n=1}^{N} \delta_{\xi_n} \quad \text{a.s.}$$

If μ is in addition almost surely $\{0,1\}$-valued, then it is equivalent up to a null set in Ω to a simple point process on X. We note that point processes are often defined to be integer-valued random measures, and that our definition as $\mathscr{M}_+^{\text{p.p.}}(X)$-valued random measures are commonly referred to as *weighted point processes*.

1.4.1 Moment measures

Given $r \geq 1$, a random measure $\mu \in \operatorname{Prob}(\mathscr{M}_+(X))$ on a proper separable metric space (X,d) is said to be *locally r-integrable* if

$$\int_{\mathscr{M}_+(X)} p(B)^r d\mu(p) < +\infty$$

for every bounded Borel set $B \subset X$. Note that this is equivalent to $\mathbb{E}_\mu(|\mathbb{S}f|^r) < +\infty$ for every $f \in \mathscr{L}_c^\infty(X)$, and that every locally r-integrable random measure is locally r'-integrable for every $r' \leq r$. A locally 1-integrable random measure will be called *locally integrable*, and a locally 2-integrable random measure will be called *locally square-integrable*. In the locally square-integrable case, we will pay special attention to the *variance* of linear statistics,

$$\operatorname{Var}_\mu(\mathbb{S}f) = \mathbb{E}_\mu(|\mathbb{S}f - \mathbb{E}_\mu(\mathbb{S}f)|^2) = \int_{\mathscr{M}_+(X)} \Big| \int_{\mathscr{M}_+(X)} \int_X f(x) d(p-q)(x) d\mu(q) \Big|^2 d\mu(p)\,.$$

We emphasize already the central quantity of this Thesis in the following definition.

Definition 1.4.4 (Number variance)**.** Let μ be a locally square-integrable random measure on a proper separable pointed metric space (X,d,x_o). The *number variance* of μ is the function $\operatorname{NV}_\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$\operatorname{NV}_\mu(r) = \operatorname{Var}_\mu(\mathbb{S}\chi_{B_r(x_o)})\,.$$

If $n \in \mathbb{N}_0$ and μ is locally n-integrable, then by the Riesz representation Theorem [13, Thm 7.44, p.239] there is a unique positive locally finite Borel measure $\eta_\mu^{(n)}$ on X^n satisfying

$$\int_{X^n} F\, d\eta_\mu^{(n)} = \int_{\mathscr{M}_+(X)} \Big(\int_{X^n} F\, dp^{\otimes n} \Big) d\mu(p)\,, \quad \forall\, F \in \mathscr{L}_c^\infty(X^n)\,,$$

where $p^{\otimes n}$ is the product measure of p with itself n times. The measure $\eta_\mu^{(n)}$ is called the *n'th moment measure* of μ. Note that if $F = f^{\otimes n}$ for some $f \in \mathscr{L}_c^\infty(X)$, then we recover the n'th moment of the linear statistic $\mathbb{S}f$ with respect to μ,

$$\int_{X^n} f^{\otimes n} d\eta_\mu^{(n)} = \mathbb{E}_\mu(|\mathbb{S}f|^n)\,.$$

In particular, for locally square-integrable random measures μ, the variance of linear statistics can be written as

$$\mathrm{Var}_\mu(\mathbb{S}f) = \eta_\mu^{(2)}(f \otimes \overline{f}) - |\eta_\mu^{(1)}(f)|^2 .$$

In some situations, for example when dealing with determinantal/permanental point processes, one would also like to consider the *n-point correlation measure* $\rho_\mu^{(n)}$ on X^n, defined as the restriction of the n'th moment measure $\eta_\mu^{(n)}$ to the n-point configuration space

$$X^{(n)} = \left\{ (x_1, \dots, x_n) \in X^n \,\middle|\, x_i \neq x_j \ \forall\, 1 \leq i \neq j \leq n \right\} .$$

In particular, if $f_1, \dots, f_n \in \mathscr{L}_c^\infty(X)$ are functions with disjoint supports, then

$$\int_{X^{(n)}} (f_1 \otimes \cdots \otimes f_n) d\rho_\mu^{(n)} = \mathbb{E}_\mu(\mathbb{S}f_1 \dots \mathbb{S}f_n) .$$

For a random measure μ on X that is locally n-integrable for every $n \in \mathbb{N}$, it is not always the case that it is uniquely determined by its moment measures $\eta_\mu^{(n)}$ or n-point correlation measures $\rho_\mu^{(n)}$. However, there are many interesting examples where this holds.

1.4.2 Examples

It is time for us to provide some examples.

Poisson point processes: Let $\nu \in \mathscr{M}_+(X)$ be a positive locally finite Borel measure on X. The *Poisson point process on X with intensity measure ν* is the unique random measure $\mathrm{Poi}(\nu) \in \mathrm{Prob}(\mathscr{M}_+(X))$ satisfying

1. for every bounded Borel set $B \subset X$, the linear statistic $\mathbb{S}\chi_B$ is Poisson distributed with intensity $\nu(B)$ with respect to $\mathrm{Poi}(\nu)$,
2. if $B_1, \dots, B_n \subset X$ are *disjoint* bounded Borel sets, then the linear statistics $\mathbb{S}\chi_{B_1}, \dots, \mathbb{S}\chi_{B_n}$ are independent with respect to $\mathrm{Poi}(\nu)$.

In particular, if ν does not admit any atoms then one shows that $\mathrm{Poi}(\nu)$ is a simple point process. To gain some intuition we briefly recall the construction of such a process: Partition X into subsets $A_1, A_2, \dots$ of unit ν-measure and let $\nu_i = \nu|_{A_i}$. Independently for each $i \in \mathbb{N}$, suppose that $\xi_1^{(i)}, \xi_2^{(i)}, \dots$ are independent ν_i-distributed random variables and let N_i be a Poisson-distributed random variable with intensity 1. Then the random measure $\mu : \Omega \to \mathscr{M}_+(X)$ defined by

$$\mu = \sum_{i=1}^{\infty} \sum_{n=1}^{N_i} \delta_{\xi_n^{(i)}}$$

can be shown to satisfy (1) and (2) above, see [19, Section 3.2]. Moreover, a proof of uniqueness can be found in [19, Prop. 2.10].

The n-point correlation measures of $\mathrm{Poi}(\nu)$ can now readily be computed using (1) and (2) above to be

$$\int_{X^{(n)}} (f_1 \otimes \cdots \otimes f_n) d\rho_{\mathrm{Poi}(\nu)}^{(n)} = \mathbb{E}_{\mathrm{Poi}(\nu)}(\mathbb{S}f_1 \dots \mathbb{S}f_n)$$

$$= \mathbb{E}_{\mathrm{Poi}(\nu)}(\mathbb{S}f_1)\dots\mathbb{E}_{\mathrm{Poi}(\nu)}(\mathbb{S}f_n) = \int_{X^{(n)}} (f_1 \otimes \cdots \otimes f_n) d\nu^{\otimes n} ,$$

for functions $f_1, \dots, f_n \in \mathscr{L}_c^\infty(X)$ with pairwise disjoint supports, so the n-point correlation measure is $\rho^{(n)}_{\mathrm{Poi}(\nu)} = \nu^{\otimes n}|_{X^{(n)}}$.

Determinantal point processes: Let $\nu \in \mathscr{M}_+(X)$ be a positive locally finite Borel measure and $L : X \times X \to \mathbb{C}$ a continuous Hermitian positive-definite kernel, meaning that $L(x_1, x_2) = \overline{L(x_2, x_1)}$ for all $x_1, x_2 \in X$ and

$$\int_X \int_X f(x_1)\overline{f(x_2)} L(x_1, x_2) d\nu(x_2) d\nu(x_1) \geq 0 , \quad \forall f \in \mathscr{L}_c^\infty(X) .$$

Moreover, assume that L is of *local trace class* in the sense that

$$\int_B L(x, x) d\nu(x) < +\infty$$

for all bounded Borel $B \subset X$. The *determinantal point process* associated with L is a simple point process μ_L on X, uniquely determined by the n-point correlation measures

$$d\rho^{(n)}_{\mu_L}(x_1, \dots, x_n) = \det(L(x_i, x_j))^n_{i,j=1} d\nu^{\otimes n}(x_1, \dots, x_n) .$$

Similarly, the *permanental point process* associated with L is a simple point process μ_L on X, uniquely determined by the n-point correlation measures

$$d\rho^{(n)}_{\mu_L}(x_1, \dots, x_n) = \mathrm{per}(L(x_i, x_j))^n_{i,j=1} d\nu^{\otimes n}(x_1, \dots, x_n) ,$$

where the permanent is of a complex matrix $(a_{ij})^n_{i,j=1}$ is

$$\mathrm{per}(a_{ij})^n_{i,j=1} = \sum_{\sigma \in \mathrm{Sym}(n)} \prod_{i=1}^n a_{i,\sigma(i)} .$$

For the existence of determinantal and permanental point processes we refer to [2, Section 5.2.4 + Thm 5.2.17]. For the Poisson point process $\mathrm{Poi}(\nu)$, one sees from its n-point correlation measures that it defines a determinantal/permanental point process with kernel

$$L(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{else} \end{cases} .$$

We will get back to more examples of determinantal/permanental point processes in the coming Sections.

2. Fluctuations in Euclidean space

2.1 Stationary and isotropic random measures

In this Subsection we will restrict our attention to random measures and point processes on Euclidean spaces that are *stationary*, which roughly means that the random measure is statistically generic across every location in the space, and also random measures that in addition to this are *isotropic*, in other words that look statistically the same in every radial direction.

Let $d \in \mathbb{N}$ and consider d-dimensional Euclidean space $\mathbb{R}^d$ with the Euclidean metric

$$d_{\mathbb{R}^d}(x,y) = \|x-y\| = \Big(\sum_{j=1}^{d}(x_j-y_j)^2\Big)^{1/2}.$$

This metric has the important property of being preserved by translations and orthogonal transformations (including rotations) in the sense that for every $x, y \in \mathbb{R}^d$,

$$d_{\mathbb{R}^d}(kx+t, ky+t) = d_{\mathbb{R}^d}(x,y)$$

for any translation $t \in \mathbb{R}^d$ and any orthogonal transformation $k \in \mathrm{O}(d)$. In fact, one can verify that every isometry of $(\mathbb{R}^d, \|\cdot\|)$ is of the form $x \mapsto kx + t$ for some $t \in \mathbb{R}^d$ and $k \in \mathrm{O}(d)$ and the set of all these isometries form the *Euclidean motion group* $G_d = \mathrm{O}(d) \ltimes \mathbb{R}^d$, which consists of pairs $(k,t) \in \mathrm{O}(d) \times \mathbb{R}^d$ with the group law

$$(k_1,t_1)(k_2,t_2) = (k_1k_2, k_1t_2 + t_1).$$

Viewing this as a continuous group action $G_d \curvearrowright \mathbb{R}^d$ with stabilizer $K_d = \mathrm{O}(d) \ltimes \{0\}$ of the origin, we can think of Euclidean space as the quotient space G_d/K_d. The benefit from this point of view is that this action of G_d lifts to an action $G_d \curvearrowright \mathscr{M}_+(\mathbb{R}^d)$ on the cone of positive locally finite Borel measures by considering push-forwards,

$$(k,t)_*p(B) = p((k^{-1}, -k^{-1}t).B), \quad (k,t) \in G_d, \ B \subset X \text{ bounded Borel.}$$

In turn, this lifts to an action on the set $\mathrm{Prob}(\mathscr{M}_+(\mathbb{R}^d))$ of random measures on $\mathbb{R}^d$ by an additional push-forward lift,

$$(k,t).\mu(E) = \mu((k^{-1}, -k^{-1}t)_*E), \qquad (k,t) \in G_d, \ E \in \mathscr{B}_{\mathscr{M}_+}.$$

We also note that this action preserves point processes and simple point processes. In this Subsection we will be considering random measures that are invariant under a large subgroup of isometries in G_d.

Definition 2.1.1. A random measure μ on $\mathbb{R}^d$ is

- *stationary* if it is invariant under translations, in other words $t.\mu = \mu$ for all $t \in \mathbb{R}^d$.
- *isotropic* if it is invariant under orthogonal transformations, in other words $k.\mu = \mu$ for all $k \in \mathrm{O}(d)$

Remark 2.1.2. In light of the larger generality of random measures on homogeneous spaces, we will refer to stationary (and possibly isotropic) random measures μ on $\mathbb{R}^d$ as *invariant* random measures. We note however that the term "stationary" is the more commonly used term in the literature.

2.1.1 Reduced moment measures

For an invariant locally n-integrable random measure μ on $\mathbb{R}^d$, the moment measure $\eta_\mu^{(n)}$ on $(\mathbb{R}^d)^n$ is invariant under diagonal $\mathbb{R}^d$-translations in the sense that

$$\int_{(\mathbb{R}^d)^n} F(x_1 + t, \ldots, x_n + t) d\eta_\mu^{(n)}(x_1, \ldots, x_n) = \int_{(\mathbb{R}^d)^n} F(x_1, \ldots, x_n) d\eta_\mu^{(n)}(x_1, \ldots, x_n)$$

for all $F \in \mathscr{L}_c^\infty((\mathbb{R}^d)^n)$ and all translations $t \in \mathbb{R}^d$. Thus the measure $\eta_\mu^{(n)}$ on $(\mathbb{R}^d)^n$ should descend down to a measure $\overline{\eta}_\mu^{(n)}$ on $(\mathbb{R}^d)^{n-1}$. To make this explicit we consider for every $n \in \mathbb{N}$ the diagonal averaging transformation $\mathrm{Av}_n : \mathscr{L}_c^\infty((\mathbb{R}^d)^n) \to \mathscr{L}_c^\infty((\mathbb{R}^d)^{n-1})$,

$$\mathrm{Av}_n(F)(x_1, \ldots, x_{n-1}) = \int_{\mathbb{R}^d} F(x_1 + t, \ldots, x_{n-1} + t, t) dt \,.$$

We claim that the map Av_n is surjective.

Lemma 2.1.3. *Let $f \in \mathscr{L}_c^\infty((\mathbb{R}^d)^{n-1})$ and let $b \in \mathscr{L}_c^\infty(\mathbb{R}^d)$ be a positive function with $\int_{\mathbb{R}^d} b(x)dx = 1$. Then the function*

$$F_f(x_1, \ldots, x_n) = b(x_n) f(x_1 - x_n, \ldots, x_{n-1} - x_n)$$

belongs to $\mathscr{L}_c^\infty((\mathbb{R}^d)^n)$ and satisfies $\mathrm{Av}_n(F_f) = f$.

Proof. It is not hard to see that $F_f \in \mathscr{L}_c^\infty((\mathbb{R}^d)^n)$, and the function $\mathrm{Av}_n(F_f)$ is

$$\begin{aligned} \mathrm{Av}_n(F_f)(x_1, \ldots, x_n) &= \int_{\mathbb{R}^d} F(x_1 + t, \ldots, x_{n-1} + t, t) dt \\ &= f(x_1, \ldots, x_{n-1}) \int_{\mathbb{R}^d} b(t) dt = f(x_1, \ldots, x_{n-1}) \,. \end{aligned}$$

□

Definition 2.1.4. Let μ be an invariant locally n-integrable random measure on $\mathbb{R}^d$. The *n'th reduced moment measure* $\overline{\eta}_\mu^{(n)}$ of μ is the positive Radon measure on $(\mathbb{R}^d)^{n-1}$ satisfying

$$\overline{\eta}_\mu^{(n)}(f) = \int_{X^n} b(x_n) f(x_1 - x_n, \ldots, x_{n-1} - x_n) d\eta_\mu^{(n)}(x_1, \ldots, x_n)$$

for a positive function $b \in \mathscr{L}_c^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} b(x)dx = 1$.

Implicitly in this definition is the independence of the bump function $b \in \mathscr{L}_c^\infty(\mathbb{R}^d)$ with unit integral. Special attention will be paid to the first and second reduced moment measures $\overline{\eta}_\mu^{(1)}, \overline{\eta}_\mu^{(2)}$ when μ is an invariant locally square-integrable random measure. The first reduced moment measure is a positive scalar $i_\mu = \overline{\eta}_\mu^{(1)} > 0$, the *intensity* of μ, and it satisfies

$$\mathbb{E}_\mu(\mathbb{S}f) = \eta_\mu^{(1)}(f) = i_\mu \int_{\mathbb{R}^d} f(x)dx\,, \quad f \in \mathscr{L}_c^\infty(\mathbb{R}^d)\,.$$

For the second reduced moment measure, we first note that

$$\mathrm{Av}_2(f_1 \otimes \overline{f_2})(x) = \int_{\mathbb{R}^d} f_1(x+t)\overline{f_2(t)}dt = \int_{\mathbb{R}^d} \overline{f_2(-(x-t))} f_1(t)dt = (f_2^* * f_1)(x)\,,$$

where $f^*(x) = \overline{f(-x)}$ and $*$ denotes convolution. In particular, the reduced second moment measure $\overline{\eta}_\mu^{(2)}$ on $\mathbb{R}^d$ satisfies

$$\mathbb{E}_\mu(\mathbb{S}f_1 \overline{\mathbb{S}f_2}) = \eta_\mu^{(2)}(f_1 \otimes \overline{f_2}) = \overline{\eta}_\mu^{(2)}(f_2^* * f_1)\,.$$

Using this we can write the variance of a linear statistic as

$$\begin{aligned}\mathrm{Var}_\mu(\mathbb{S}f) = \mathbb{E}_\mu(|\mathbb{S}f|^2) - |\mathbb{E}_\mu(\mathbb{S}f)|^2 &= \int_{\mathbb{R}^d} (f^* * f)(x) d\overline{\eta}_\mu^{(2)}(x) - i_\mu^2 \Big| \int_{\mathbb{R}^d} f(x)dx \Big|^2 \\ &= \int_{\mathbb{R}^d} (f^* * f)(x) d\overline{\eta}_\mu^{(2)}(x) - i_\mu^2 \int_{\mathbb{R}^d} (f^* * f)(x)dx\,,\end{aligned}$$

which leads us to the following definition.

Definition 2.1.5. Let μ be an invariant locally square-integrable random measure on $\mathbb{R}^d$. The *autocorrelation measure* η_μ of μ is the signed Radon measure on $\mathbb{R}^d$ satisfying

$$\mathrm{Var}_\mu(\mathbb{S}f) = \eta_\mu(f^* * f)\,.$$

In terms of the first and second reduced moment measures, $d\eta_\mu(x) = d\overline{\eta}_\mu^{(2)}(x) - i_\mu^2 dx$. In particular, the number variance of an invariant locally square-integable random measure μ on $\mathbb{R}^d$ can be written as

$$\mathrm{NV}_\mu(r) = \eta_\mu(\chi_{B_r(0)} * \chi_{B_r(0)}) = \int_{\mathbb{R}^d} \mathrm{Vol}_{\mathbb{R}^d}(B_r(x) \cap B_r(0)) d\eta_\mu(x)\,,$$

where $\mathrm{Vol}_{\mathbb{R}^d}$ denotes the Lebesgue volume.

2.1.2 Examples

In order to get a sense of what the number variance of an invariant random measure on $\mathbb{R}^d$ can be, we provide some examples of different natures.

Homogeneous Poisson point processes: A Poisson point process $\mathrm{Poi}(\nu)$ on $\mathbb{R}^d$ with intensity measure $\nu \in \mathscr{M}_+(\mathbb{R}^d)$ is invariant if and only if $d\nu(x) = i \cdot dx$ for some $i > 0$, and a Poisson point process of the form $\mathrm{Poi}(i) := \mathrm{Poi}(i \cdot dx)$ is called *homogeneous* with intensity i. Such a simple point process is both stationary and isotropic since the Lebesgue measure dx is. The autocorrelation of a homogeneous Poisson point process is $\eta_{\mathrm{Poi}(i)} = i \cdot \delta_0$ since

$$\eta_{\mathrm{Poi}(i)}(f^* * f) = \mathrm{Var}_{\mathrm{Poi}(i)}(\mathbb{S}f) = i \int_{\mathbb{R}^d} |f(x)|^2 dx = i(f^* * f)(0) = i\delta_0(f^* * f)$$

for all $f \in \mathscr{L}_c^\infty(\mathbb{R}^d)$.

Determinantal/permanental point processes with equivariant kernel: Fix the background measure $d\nu(x) = dx$ and let $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a continuous positive-definite Hermitian locally trace class kernel as in end of the last chapter. We assume that L satisfies the equivariance property

$$L(x_1 + t, x_2 + t) = c(t, x_1)\overline{c(t, x_2)}L(x_1, x_2)$$

for every $x_1, x_2, t \in \mathbb{R}^d$, where $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is some measurable map with $|c(t, x)| = 1$ for all $t, x \in \mathbb{R}^d$. For such kernels L, the associated determinantal and permanental point processes $\mu_L^{\mathrm{det}}, \mu_L^{\mathrm{per}}$ are translation invariant since their n-point correlation measures are translation invariant. For example, in the permanental case we have that

$$\begin{aligned}\mathrm{per}(L(x_i + t, x_j + t))_{i,j=1}^n &= \mathrm{per}(c(t, x_i)\overline{c(t, x_j)}L(x_i, x_j))_{i,j=1}^n \\ &= |c(t, x_1) \dots c(t, x_n)|^2 \mathrm{per}(L(x_i, x_j))_{i,j=1}^n = \mathrm{per}(L(x_i, x_j))_{i,j=1}^n\end{aligned}$$

and similarly for the determinantal case. Introducing the function $\kappa_L(x) = |L(x, 0)|^2$, the variance of linear statistics $\mathbb{S}f$ with $f \in \mathscr{L}_c^\infty(\mathbb{R}^d)$ is computed in Section 6 of Paper III to be

$$\begin{aligned}\mathrm{Var}_{\mu_L}(\mathbb{S}f) &= L(0,0) \int_{\mathbb{R}^d} |f(x)|^2 dx \pm \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x_1)\overline{f(x_2)}\kappa_L(x_1 - x_2)dx_1 dx_2 \\ &= L(0,0)(f^* * f)(0) \pm \int_{\mathbb{R}^d} (f^* * f)(x)\kappa_L(x)dx\end{aligned}$$

with $-$ for $\mu_L = \mu_L^{\mathrm{det}}$ and $+$ for $\mu_L = \mu_L^{\mathrm{per}}$. In particular, the autocorrelation measure for the respective point process is

$$\eta_{\mu_L}(f) = L(0,0)f(0) \pm \int_{\mathbb{R}^d} f(x)|L(x,0)|^2 dx\,.$$

Explicit examples of kernels for invariant determinantal and permanental point processes on $\mathbb{R}^d$ include the Linnik-type kernels on the real line of De Coninck-Dunlop-Huillet in [12, Section 5], given by

$$L_{\alpha,\zeta}(t_1, t_2) = (1 + |t_1 - t_2|^\alpha)^{-\zeta/2}\,, \quad 0 < \alpha \leq 2\,,\ \zeta > 0$$

and kernels governing the determinantal *infinite polyanalytic ensembles of pure type* in the plane of Abreu-Pereira-Romero-Torquato in [1],

$$L_{\lambda,n}(z_1, z_2) = \mathcal{L}_n(\tfrac{1}{2}\lambda|z_1 - z_2|^2)\mathrm{e}^{-\frac{1}{4}\lambda(|z_1|^2 + |z_2|^2 - 2z_1\overline{z}_2)}\,, \quad \lambda > 0\,,\ n \in \mathbb{N}_0\,.$$

Here, $\mathcal{L}_n(t) = (n!)^{-1}\mathrm{e}^t \frac{d^n}{dt^n}(t^n \mathrm{e}^{-t})$ are Laguerre polynomials. We also note that if $n = 0$

and $\lambda = 2\pi$ we retrieve the *infinite Ginibre ensemble* in the determinantal case,

$$L_{2\pi,0}(z_1,z_2) = \mathrm{e}^{\pi z_1 \overline{z}_2 - \frac{\pi}{2}(|z_1|^2+|z_2|^2)}\,.$$

Invariant random lattices: Let $g \in \mathrm{GL}_d(\mathbb{R})$ be an invertible linear map on $\mathbb{R}^d$ and consider the lattice $\Gamma = g\mathbb{Z}^d$ with the fundamental domain $\mathcal{F}_\Gamma = g([0,1)^d) \subset \mathbb{R}^d$. Recall that the *covolume* of Γ is

$$\mathrm{covol}(\Gamma) = \mathrm{Vol}_{\mathbb{R}^d}(\mathcal{F}_\Gamma) = |\det(g)|\,.$$

If we consider the coset space $\Gamma\backslash\mathbb{R}^d$ with the probability measure $m_{\Gamma\backslash\mathbb{R}^d}$ coming from the normalized Lebesgue measure $\mathrm{covol}(\Gamma)^{-1}dx|_{\mathcal{F}_\Gamma}$ on $\mathcal{F}_\Gamma$, then the *invariant random* Γ*-lattice* in $\mathbb{R}^d$ is the simple point process μ_Γ on $\mathbb{R}^d$ coming from the map

$$\Gamma\backslash\mathbb{R}^d \ni \Gamma + x \longmapsto \delta_{\Gamma+x} = \sum_{\gamma\in\Gamma} \delta_{\gamma+x} \in \mathscr{M}_+^{\text{s.p.p.}}(\mathbb{R}^d)\,.$$

Since the Lebesgue probability measure on $\Gamma\backslash\mathbb{R}^d$ is invariant under the canonical $\mathbb{R}^d$-action, then μ_Γ is $\mathbb{R}^d$-invariant. Moreover, using that the lattice Γ is uniformly discrete, linear statistics $\mathbb{S}f$ with $f \in \mathscr{L}_c^\infty(\mathbb{R}^d)$ are bounded on the support of the measure μ_Γ, and hence μ_Γ is locally n-integrable for every $n \in \mathbb{N}$. The expectation of a linear statistic with respect to μ_Γ is

$$\mathbb{E}_{\mu_\Gamma}(\mathbb{S}f) = \int_{\Gamma\backslash\mathbb{R}^d} \sum_{\gamma\in\Gamma} f(\gamma+x)\,dm_{\Gamma\backslash\mathbb{R}^d}(\Gamma+x) = \frac{1}{\mathrm{covol}(\Gamma)}\int_{\mathbb{R}^d} f(x)dx\,,$$

and the second moment of linear statistics can be computed as

$$\begin{aligned}
\mathbb{E}_{\mu_\Gamma}(|\mathbb{S}f|^2) &= \int_{\Gamma\backslash\mathbb{R}^d} \Big|\sum_{\gamma\in\Gamma} f(\gamma+x)\Big|^2 dm_{\Gamma\backslash\mathbb{R}^d}(\Gamma+x) \\
&= \frac{1}{\mathrm{covol}(\Gamma)} \sum_{\gamma'\in\Gamma} \int_{\mathcal{F}_\Gamma+\gamma'} \Big(\sum_{\gamma\in\Gamma} f(\gamma+x)\Big)\overline{f(x)}dx \\
&= \frac{1}{\mathrm{covol}(\Gamma)} \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^d} f(\gamma+x)\overline{f(x)}dx = \frac{1}{\mathrm{covol}(\Gamma)} \sum_{\gamma\in\Gamma} (f^* * f)(\gamma)\,.
\end{aligned}$$

Thus the autocorrelation measure $\eta_\Gamma := \eta_{\mu_\Gamma}$ is given by

$$\eta_\Gamma(f) = \frac{1}{\mathrm{covol}(\Gamma)} \sum_{\gamma\in\Gamma} f(\gamma) - \frac{1}{\mathrm{covol}(\Gamma)^2}\int_{\mathbb{R}^d} f(x)dx\,.$$

Invariant random quasicrystals: Let $d, d' \geq 1$ and let $\Gamma < \mathbb{R}^d \times \mathbb{R}^{d'}$ be a lattice projecting injectively into $\mathbb{R}^d$ and densely to $\mathbb{R}^{d'}$. For a bounded subset $W \subset \mathbb{R}^{d'}$ with dense interior, the *window*, we define the *model set*

$$\mathcal{P}_{d,d'}(\Gamma, W) = \Big\{\gamma \in \mathbb{R}^d \,\Big|\, \exists\, \gamma' \in W : (\gamma,\gamma') \in \Gamma\Big\}\,.$$

Such model sets can be shown to be uniformly discrete using the uniform discreteness of the lattice Γ and pre-compactness of the window W.

Given a model set $\mathcal{P}_o = \mathcal{P}_{d,d'}(\Gamma, W)$ in $\mathbb{R}^d$, we consider the coset space $\Gamma\backslash(\mathbb{R}^d \times \mathbb{R}^{d'})$ with the Lebesgue probability measure and note that the $\mathbb{R}^d$-action $(x, x') \mapsto (x + t, x')$ is ergodic and therefore has dense orbits. In fact, the orbits are embeddings of $\mathbb{R}^d$ in this torus and with this in mind we get an invariant simple point process $\mu_{\mathcal{P}_o}$ on $\mathbb{R}^d$ from the map

$$\Gamma\backslash(\mathbb{R}^d \times \mathbb{R}^{d'}) \ni \Gamma + (x, x') \longmapsto \delta_{\mathcal{P}_o + x} = \sum_{\gamma \in \mathcal{P}_o} \delta_{\gamma + x} \in \mathscr{M}_+^{\text{s.p.p.}}(\mathbb{R}^d)\,.$$

We refer to $\mu_{\mathcal{P}_o}$ as an *invariant random quasicrystal* or *invariant random model set*. One can write the measures $\delta_{\mathcal{P}_o+x}$ in terms of Γ as

$$\delta_{\mathcal{P}_o+x} = \sum_{(\gamma,\gamma') \in \Gamma} \chi_W(\gamma')\delta_{\gamma+x}\,,$$

so from the computations for invariant random lattices we see that the autocorrelation measure $\eta_{\mathcal{P}_o} := \eta_{\mu_{\mathcal{P}_o}}$ is

$$\eta_{\mathcal{P}_o}(f) = \frac{1}{\operatorname{covol}(\Gamma)} \sum_{(\gamma,\gamma') \in \Gamma} (\chi_W^* * \chi_W)(\gamma') f(\gamma) - \frac{\operatorname{Vol}_{\mathbb{R}^{d'}}(W)^2}{\operatorname{covol}(\Gamma)^2} \int_{\mathbb{R}^d} f(x)\,dx\,, \quad f \in \mathscr{L}_c^\infty(\mathbb{R}^d)\,.$$

I.i.d. perturbed invariant random lattices: Let $\Gamma < \mathbb{R}^d$ be a lattice and $\nu \in \operatorname{Prob}(\mathbb{R}^d)$ a probability measure that is absolutely continuous with respect to the Lebesgue measure. Consider the action of $\mathbb{R}^d \times \Gamma$ on the product space $\mathbb{R}^d \times (\mathbb{R}^d)^\Gamma$ given by

$$(t, \gamma).(x, (z_{\gamma'})_{\gamma' \in \Gamma}) = (x + t, (z_{\gamma+\gamma'})_{\gamma' \in \Gamma})\,.$$

We will write $z = (z_\gamma)_{\gamma \in \Gamma}$ for elements of $(\mathbb{R}^d)^\Gamma$. The space $\Gamma\backslash\mathbb{R}^d \times (\mathbb{R}^d)^\Gamma$ can then be equipped with a $(\mathbb{R}^d \times \Gamma)$-invariant probability measure $m_{\Gamma\backslash\mathbb{R}^d} \otimes \nu^{\otimes\Gamma}$ and we define the *i.i.d. ν-perturbed invariant random Γ-lattice* to be the invariant point process $\mu_{\Gamma,\nu}$ on $\mathbb{R}^d$ obtained via the map

$$\Gamma\backslash\mathbb{R}^d \times (\mathbb{R}^d)^\Gamma \ni (\Gamma + x, z) \longmapsto \delta_{(\Gamma+x,z)} = \sum_{\gamma \in \Gamma} \delta_{\gamma+x+z_\gamma} \in \mathscr{M}_+^{\text{p.p.}}(\mathbb{R}^d)\,.$$

Since the perturbation law ν is absolutely continuous with respect to the Lebesgue measure, one can show that $\mu_{\Gamma,\nu}$ is up to null sets a simple point process. In Section 2 of Paper I we compute the autocorrelation measure of $\mu_{\Gamma,\nu}$ to be

$$\eta_{\Gamma,\nu} = \check{\nu} * \nu * \eta_\Gamma + \tfrac{1}{\operatorname{covol}(\Gamma)}(\delta_0 - \check{\nu} * \nu)\,,$$

where η_Γ is the autocorrelation measure of the invariant random Γ-lattice and $\check{\nu}(B) = \nu(-B)$ for Borel sets $B \subset X$. Here, the convolution of a finite measure ν and a positive locally finite Borel measure η is given by

$$(\nu * \eta)(B) = \int_{\mathbb{R}^d} \eta(B - x)\,d\nu(x)\,, \quad B \subset X \text{ bounded Borel.}$$

2.2 Geometric fluctuations

2.2.1 Upper and lower bounds on number variances

Having some examples in mind, we are in a position of stating some results regarding number variances of invariant random measures on Euclidean spaces $\mathbb{R}^d$. The first is an asymptotic upper bound on the number variance that can be viewed as a consequence of the mean ergodic Theorem.

Proposition 2.2.1. *Let μ be an invariant locally square-integrable random measure on $\mathbb{R}^d$. Then*

$$\lim_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{\mathbb{R}^d}(B_r(0))^2} = 0\,.$$

We save the proof of this result for Section 2.5. Achieving a universal lower bound is a much more subtle result due to Beck in [6, p.3-4]. There it is shown for any infinite locally finite set $\mathcal{P} \subset \mathbb{R}^d$ and any compact convex $C \subset \mathbb{R}^d$ with non-empty interior that

$$\limsup_{r\to+\infty} \frac{1}{(2r)^d} \int_{[-r,r)^d} \int_0^1 \int_{\mathrm{O}(d)} \Big| |\mathcal{P} \cap (skC + x)| - \mathrm{Vol}_{\mathbb{R}^d}(C) \Big|^2 dkdsdx \gg \mathrm{Vol}_{d-1}(\partial C)\,,$$

where dk denotes the Haar probability measure on $\mathrm{O}(d)$. In Paper I we extend this result to random measures. The result is the following.

Theorem 2.2.2 (Beck's Theorem)**.** *Let μ be an invariant locally square-integrable random measure on $\mathbb{R}^d$. Then*

$$\limsup_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} > 0\,.$$

Moreover, we provide point processes in dimensions $d = 1, 5, 9, 13, \dots$ for which this lower threshold is not uniformly bounded from below. More precisely, we show for invariant random lattices in $\mathbb{R}^5$ that the lower limit of the quantity in Beck's Theorem vanishes. The complete statement for random lattices in $\mathbb{R}^d$ for general $d \geq 1$ is the following, which we prove in Section 2.6

Proposition 2.2.3. *Let $\Gamma < \mathbb{R}^d$ be a lattice and consider the random lattice orbit μ_Γ in $\mathbb{R}^d$. Then*

$$\liminf_{r\to+\infty} \frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} = 0\,.$$

if and only if $d \equiv 1 \bmod 4$.

2.2.2 Geometric hyperuniformity

We have finally arrived at the first definition of hyperuniformity and hyperfluctuations of an invariant/stationary point process in $\mathbb{R}^d$.

Definition 2.2.4. An invariant locally square-integrable random measure μ on $\mathbb{R}^d$ is *geometrically hyperuniform* if

$$\limsup_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{\mathbb{R}^d}(B_r(0))} = 0\,.$$

Moreover, if this upper limit is positive and finite we say that μ is *Poissonian*, and if it is infinite we say that μ is *hyperfluctuating*.

We highlight two interpretations of this property:

1. As mentioned, geometric hyperuniformity of a random measure can be interpreted as the asymptotic suppression of mass fluctuations compared to the volume of the observation domain in the radial limit.

2. Alternatively, the volume $\mathrm{Vol}_{\mathbb{R}^d}(B_r(0))$ can be interpreted as the number variance of the unit intensity homogeneous Poisson point process, so that geometric hyperuniformity corresponds to "sub-Poissonian" mass fluctuations in the radial limit.

The following classes of invariant point processes in $\mathbb{R}^d$ are geometrically hyperuniform:

- Invariant determinantal point processes associated with locally trace class projections in $L^2(\mathbb{R}^d)$,
- invariant random lattices,
- invariant random quasicrystals/model sets associated with arithmetic lattices $\Gamma < \mathbb{R}^d \times \mathbb{R}^{d'}$ [10],
- i.i.d. perturbed invariant random lattices.

Moreover, homogeneous Poisson point processes are by definition Poissonian and invariant permanental point processes are Poissonian or hyperfluctuating. On the real line there are parameters α, ζ such that the invariant permanental point process with the Linnik-type kernel $L_{\alpha,\zeta}(t_1, t_2) = (1+|t_1-t_2|^\alpha)^{-\zeta/2}$ is hyperfluctuating, see [12, Section 5]. We provide more details in Section 2.4.3.

2.3 Some Fourier analysis

2.3.1 The Fourier transform

Given a complex-valued Lebesgue-integrable function $f \in \mathscr{L}^1(\mathbb{R}^d)$, its Fourier transform is the continuous function

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-i\langle x,\xi\rangle}\,dx$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^d$. The Riemann-Lebesgue lemma states that $\widehat{f}$ vanishes at infinity, so $\widehat{f} \in C_0(\mathbb{R}^d)$. Moreover, standard properties of the Fourier transform is that it takes the convolution of two Lebesgue-integrable functions to the product of their respective Fourier transforms, and the $*$-involution to complex conjugation:

$$(\widehat{f_1 * f_2})(\xi) = \widehat{f_1}(\xi)\widehat{f_2}(\xi) \quad \text{and} \quad \widehat{f^*}(\xi) = \overline{\widehat{f}(\xi)}\,.$$

If the Fourier transform $\widehat{f} \in C_0(\mathbb{R}^d)$ is itself integrable, then the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) \mathrm{e}^{i\langle x,\xi\rangle} \frac{d\xi}{(2\pi)^d}$$

holds for Lebesgue almost every $x \in \mathbb{R}^d$. In particular, for $f \in \mathscr{L}^1(\mathbb{R}^d) \cap \mathscr{L}^2(\mathbb{R}^d)$ the *Plancherel formula* holds,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = (f^* * f)(0) = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \frac{d\xi}{(2\pi)^d} .$$

We will frequently consider Fourier transforms of *radial* functions, meaning $f \in \mathscr{L}^1(\mathbb{R}^d)$ such that $f(kx) = f(x)$ for all orthogonal transformations $k \in \mathrm{O}(d)$ and all $x \in \mathbb{R}^d$. Equivalently, there is a function $f_o : \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that $f(x) = f_o(\|x\|)$ for all $x \in \mathbb{R}^d$. By introducing rotational averaging over the unit sphere $S^{d-1} \subset \mathbb{R}^d$, the Fourier transform of a radial function $f \in \mathscr{L}^1(\mathbb{R}^d)$ can be written as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\omega_{\|\xi\|}(x) dx ,$$

where

$$\omega_\lambda(x) = \int_{S^{d-1}} \mathrm{e}^{-i\lambda\langle x,u\rangle} dm_{S^{d-1}}(u)$$

for $\lambda \geq 0$, and $m_{S^{d-1}}$ is the $\mathrm{O}(d)$-invariant probability surface measure on S^{d-1}. Introducing spherical coordinates on S^{d-1}, one can rewrite the functions ω_λ with $\lambda \geq 0$ in terms of Bessel functions of the first kind,

$$\omega_\lambda(x) = \frac{2^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{(\lambda\|x\|)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(\lambda\|x\|) ,$$

where the Bessel function is

$$J_\alpha(z) = \frac{2^{1-\alpha} z^\alpha}{\pi^{1/2}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-s^2)^{\alpha-\frac{1}{2}} \cos(zs) ds , \quad z, \alpha \in \mathbb{C} , \ \mathrm{Re}(\alpha) > -\tfrac{1}{2} .$$

We mention briefly the asymptotic behaviour of this function for small and large real argument, which will be useful for later. As $s \to 0^+$ then

$$J_\alpha(s) \sim \frac{s^\alpha}{2^\alpha \Gamma(\alpha+1)} \tag{2.3.1}$$

and for $s \geq 1$, there is a constant $C_\alpha > 0$ such that

$$\left| J_\alpha(s) - \sqrt{\tfrac{2}{\pi s}} \sin(s - \tfrac{2\alpha-1}{4}\pi) \right| \leq C_\alpha s^{-3/2} , \tag{2.3.2}$$

see [26, p. 199] or Lemma 3.2 in Paper I. Writing the Fourier transform of a radial function $f \in \mathscr{L}^1(\mathbb{R}^d)$ in polar coordinates, we now get that

$$\widehat{f}(\xi) = \frac{2^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\|\xi\|^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} f(x) J_{\frac{d-2}{2}}(\|\xi\|\|x\|) \frac{dx}{\|x\|^{\frac{d-2}{2}}} = \frac{(2\pi)^{\frac{d}{2}}}{\|\xi\|^{\frac{d-2}{2}}} \int_0^\infty f_o(s) J_{\frac{d-2}{2}}(\|\xi\| s) s^{\frac{d}{2}} ds$$

where $f_o : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is such that $f(x) = f_o(\|x\|)$ for all $x \in \mathbb{R}^d$.

Example 2.3.1 (The Fourier transform of indicators on balls)**.** Consider the indicator function $\chi_{B_r(0)}$ on the Euclidean ball centered at the origin of radius $r > 0$. It is radial and so its Fourier transform is

$$\begin{aligned}\widehat{\chi}_{B_r(0)}(\xi) &= \frac{2^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\|\xi\|^{\frac{d-2}{2}}} \int_{B_r(0)} J_{\frac{d-2}{2}}(\|\xi\|\|x\|) \frac{dx}{\|x\|^{\frac{d-2}{2}}} \\ &= \frac{(2\pi)^{\frac{d}{2}}}{\|\xi\|^{\frac{d-2}{2}}} \int_0^r J_{\frac{d-2}{2}}(\|\xi\|s) s^{\frac{d}{2}} ds = \frac{(2\pi)^{\frac{d}{2}}}{\|\xi\|^d} \int_0^{\|\xi\|r} J_{\frac{d-2}{2}}(t) t^{\frac{d}{2}} dt\,.\end{aligned}$$

Using the integral formula

$$\int_0^r J_{\frac{d-2}{2}}(t) t^{\frac{d}{2}} dt = J_{\frac{d}{2}}(r) r^{\frac{d}{2}}\,,$$

which can be found for example in [26, p. 132], the latter integral can be computed as

$$\widehat{\chi}_{B_r(0)}(\xi) = \Big(\frac{2\pi r}{\|\xi\|}\Big)^{\frac{d}{2}} J_{\frac{d}{2}}(\|\xi\|r)\,.$$

To handle the Fourier transform of measures and more generally distributions, we need to understand the behavior of the Fourier transform of test functions. Let $C_c^\infty(\mathbb{R}^d)$ denote the space of compactly supported smooth functions on $\mathbb{R}^d$. The Fourier transform of such functions is characterized by Schwartz' version of the Paley-Wiener Theorem.

Theorem 2.3.1 (Paley-Wiener, Schwartz)**.** *A holomorphic function $F : \mathbb{C}^d \to \mathbb{C}$ is the Fourier transform of a function $f \in C_c^\infty(\mathbb{R}^d)$ if and only if there is constant $b > 0$ and for every $N \in \mathbb{N}$ a constant $C_N > 0$ such that*

$$|F(z)| \leq C_N(1 + \|z\|)^{-N} \mathrm{e}^{b\|\mathrm{Im}(z)\|}\,, \quad \forall\, z \in \mathbb{C}^d\,.$$

We denote by $\mathrm{PW}(\mathbb{C}^d)$ the space of holomorphic functions satisfying this property, and the Fourier transform defines an isomorphism between $C_c^\infty(\mathbb{R}^d)$ and $\mathrm{PW}(\mathbb{C}^d)$. Dualizing this isomorphism, we can define the Fourier transform of a distribution $\xi \in C_c^\infty(\mathbb{R}^d)^*$ to be the continuous linear functional $\widehat{\xi}$ on $\mathrm{PW}(\mathbb{C}^d)$ satisfying

$$\widehat{\xi}(\widehat{f}) = \xi(f)$$

for all $f \in C_c^\infty(\mathbb{R}^d)$. It is however not an easy task to describe the dual space of $\mathrm{PW}(\mathbb{C}^d)$, so the usefulness of the distributional Fourier transform is lacking when only talking about test functions in $C_c^\infty(\mathbb{R}^d)$. However, if we extend the space of test functions to the *Schwartz space* $\mathscr{S}(\mathbb{R}^d)$ of smooth functions whose derivatives all have sub-polynomial decay, then as a consequence of Schwartz' Paley-Wiener Theorem one can show that the Fourier transform restricts to an isomorphism $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$. The strong dual space $\mathscr{S}'(\mathbb{R}^d)$ consists of *tempered* distributions, which for measures is characterized by having at most polynomial growth, and the distributional Fourier transform of tempered distributions are tempered and vice versa.

2.3.2 Bochner's Theorem

Let $\sigma \in \mathscr{M}_+(\mathbb{R}^d)$ be a positive locally finite *tempered* measure. By the Paley-Wiener Theorem, the distributional Fourier transform $\widehat{\sigma}$ of σ is a tempered distribution, and it satisfies

$$\widehat{\sigma}(f^* * f) = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\sigma(\xi) \geq 0$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$. We say that such a distribution is *positive-definite*. It turns out that every positive-definite distribution is of the form $\widehat{\sigma}$ for some positive measure σ as described.

Theorem 2.3.2 (Bochner, Schwartz). *A distribution η on $\mathbb{R}^d$ is positive-definite if and only if there is a positive locally finite tempered measure σ_η on $\mathbb{R}^d$ such that*

$$\eta(f) = \int_{\mathbb{R}^d} \widehat{f}(\xi) d\sigma_\eta(\xi)$$

for every function $f \in C_c^\infty(\mathbb{R}^d)$. In particular, η is a tempered distribution and the identity extends to Schwartz functions $f \in \mathscr{S}(\mathbb{R}^d)$.

The original Theorem due to Bochner is that every positive-definite continuous function $\eta : \mathbb{R}^d \to \mathbb{C}$ is the Fourier transform of a *finite* positive Borel measure σ_η on $\mathbb{R}^d$ in the sense that

$$\eta(x) = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle x,\xi\rangle} d\sigma_\eta(\xi)$$

for all $x \in \mathbb{R}^d$. In particular, η is bounded and achieves its maximum at $x = 0$. Moreover, each finite positive Borel measure on $\mathbb{R}^d$ gives rise to a continuous positive-definite function on $\mathbb{R}^d$, so Bochner's result is an equivalence.

2.4 Spectral fluctuations

2.4.1 Bartlett spectral measures

Let μ be an invariant/stationary locally square-integrable random measure on $\mathbb{R}^d$. Then the autocorrelation measure η_μ of μ on $\mathbb{R}^d$ is positive-definite, since

$$\eta_\mu(f^* * f) = \mathrm{Var}_\mu(\mathbb{S}f) \geq 0$$

for every $f \in \mathscr{L}_c^\infty(\mathbb{R}^d)$. The Bochner-Schwartz Theorem then tells us that there is a positive locally finite tempered measure σ_μ on $\mathbb{R}^d$ such that

$$\mathrm{Var}_\mu(\mathbb{S}f) = \eta_\mu(f^* * f) = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\sigma_\mu(\xi)\,, \quad f \in \mathscr{S}(\mathbb{R}^d)\,.$$

In fact, one can show that this equality extends all $f \in \mathscr{L}_c^\infty(\mathbb{R}^d)$, see Lemma 4.3 in Paper I. The measure σ_μ is the *Bartlett spectral measure* of μ introduced by Bartlett in [4], also called the *diffraction measure* of μ. This measure can be shown to always assign zero mass to $\xi = 0$,

$$\sigma_\mu(\{0\}) = 0\,,$$

see [10, Prop. 2.5 + Lemma 2.6], and it is moreover *translation bounded* in the sense that

$$\sup_{t \in \mathbb{R}^d} \sigma_\mu(B + t) < +\infty \tag{2.4.1}$$

for all bounded Borel $B \subset \mathbb{R}^d$, see [7, Prop. 4.9, p.25]. In particular, the number variance of an invariant locally square-integrable random measure can be expressed as

$$\mathrm{NV}_\mu(r) = (2\pi r)^d \int_{\mathbb{R}^d} J_{\frac{d}{2}}(r\|\xi\|)^2 \frac{d\sigma_\mu(\xi)}{\|\xi\|^d}\,.$$

Whenever σ_μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, one can define the *structure factor* of μ to be the density

$$\mathcal{S}_\mu(\xi) = \frac{d\sigma_\mu(\xi)}{d\xi}\,.$$

The existence of this Radon-Nikodym derivative is heuristically related to "local disorder" of the random measure, and we will see that it exists for invariant determinantal/permanental point processes and i.i.d. perturbed invariant random lattices.

2.4.2 Spectral hyperuniformity and stealth

With Bartlett spectral measures defined for invariant random measures on $\mathbb{R}^d$, we can characterize geometric hyperuniformity in terms of sufficiently rapid decay of the Bartlett spectral measure around the origin, a Fourier duality for the number variance for point processes. The following result can be found in [10, Prop. 3.3].

Proposition 2.4.1 (Björklund-Hartnick)**.** *An invariant random measure μ on $\mathbb{R}^d$ is geometrically hyperuniform if and only if*

$$\lim_{\varepsilon \to 0^+} \frac{\sigma_\mu(B_\varepsilon(0))}{\mathrm{Vol}_{\mathbb{R}^d}(B_\varepsilon(0))} = 0\,.$$

We refer to this property as *spectral hyperuniformity* of μ. If the Bartlett spectral measure is absolutely continuous with respect to the Lebesgue measure, $d\sigma_\mu(\xi) = \mathcal{S}_\mu(\xi)d\xi$, then spectral hyperuniformity of μ is equivalent to

$$\lim_{\xi \to 0^+} \mathcal{S}_\mu(\xi) = 0\,.$$

With this characterization of hyperuniformity, we emphasize a particularly rigid class of hyperuniform random measures named *stealthy* random measures. Such random measures where introduced by Batten-Stillinger-Torquato in [5], and elaborated upon in [24].

Definition 2.4.2 (Stealth)**.** An invariant locally square-integrable random measure μ on $\mathbb{R}^d$ is *stealthy* if there is an $\varepsilon_o > 0$ such that $\sigma_\mu(B_{\varepsilon_o}(0)) = 0$.

Clearly stealthy random measures are spectrally hyperuniform. Next, we return to the examples from earlier.

2.4.3 Examples

Let us return to the examples from Section 2.1.2, compute Bartlett spectral measures and verify spectral hyperuniformity for them.

Homogeneous Poisson point processes: For Poi(i) where $i > 0$ is the intensity, the autocorrelation measure is

$$\eta_\mu(f) = if(0) = i\int_{\mathbb{R}^d} \widehat{f}(\xi)\frac{d\xi}{(2\pi)^d}\ , \quad f \in \mathscr{S}(\mathbb{R}^d)$$

by the Plancherel formula. Hence the Bartlett spectral measure is $d\sigma_{\mathrm{Poi}(i)}(\xi) = i(2\pi)^{-d}d\xi$. The Poisson point process is not geomtrically hyperunifrom and hence not spectrally hyperuniform. Alternatively, one can easily verify directly that

$$\lim_{\varepsilon\to 0^+} \frac{\sigma_{\mathrm{Poi}(i)}(B_\varepsilon(0))}{\mathrm{Vol}_{\mathbb{R}^d}(B_\varepsilon(0))} = \frac{i}{(2\pi)^d} > 0\,.$$

Determinantal/permanental point processes with equivariant kernel: Let $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a kernel as in Section 2.1.2 and assume that the function $\kappa_L(x) = |L(x,0)|^2$ is square-integrable on $\mathbb{R}^d$, meaning

$$\int_{\mathbb{R}^d} |L(x,0)|^4 dx < +\infty\,.$$

Then we can use the Plancherel formula to write the autocorrelation measure of the associated determinantal/permanental point process μ_L as

$$\eta_{\mu_L}(f) = L(0,0)f(0) \pm \int_{\mathbb{R}^d} f(x)\kappa_L(x)dx = \int_{\mathbb{R}^d} \widehat{f}(\xi)(L(0,0) \pm \widehat{\kappa}_L(\xi))\frac{d\xi}{(2\pi)^d}\,,$$

so that the Bartlett spectral measure is

$$d\sigma_{\mu_L}(\xi) = (2\pi)^{-d}(L(0,0) \pm \widehat{\kappa}_L(\xi))d\xi\,.$$

In the determinantal case, spectral hyperuniformity of μ_L^{det} is equivalent to $\widehat{\kappa}_L(0) = L(0,0)$ since

$$\lim_{\varepsilon\to 0^+} \frac{\sigma_\mu(B_\varepsilon(0))}{\mathrm{Vol}_{\mathbb{R}^d}(B_\varepsilon(0))} = (2\pi)^{-d}(L(0,0) - \widehat{\kappa}_L(0))\,.$$

Moreover, $L(0,0) > 0$ since L is positive-definite and $\widehat{\kappa}_L(0) = \int_{\mathbb{R}^d} |L(x,0)|^2 dx > 0$, so permanental point processes are never spectrally hyperuniform.

We also note that when the integral operator

$$Lf(x) = \int_{\mathbb{R}^d} L(x,y)f(y)dy$$

defines an orthogonal projection onto a closed subspace of $L^2(\mathbb{R}^d)$, then $L(0,0) = 1$ and

L is *reproducing* in the sense that

$$L(x_1, x_2) = \int_{\mathbb{R}^d} L(x_1, y) L(y, x_2) dy$$

for all $x_1, x_2 \in \mathbb{R}^d$. In particular,

$$\widehat{\kappa}_L(0) = \int_{\mathbb{R}^d} |L(x,0)|^2 dx = L(0,0) = 1\,,$$

and we conclude that for such kernels L, the associated determinantal point point process $\mu_L^{\det}$ is spectrally hyperuniform. As an example, if we consider the kernel for the infinite Ginibre ensemble in the complex plane

$$L_{2\pi,0}(z_1, z_2) = \mathrm{e}^{\pi z_1 \overline{z}_2 - \frac{\pi}{2}(|z_1|^2 + |z_2|^2)}\,,$$

then $L(0,0) = 1$ and $\kappa_{2\pi,0}(z) := |L_{2\pi,0}(z,0)|^2 = \mathrm{e}^{-\frac{\pi}{2}|z|^2}$ satisfies

$$\widehat{\kappa}_{2\pi,0}(0) = \int_{\mathbb{C}} \mathrm{e}^{-\frac{\pi}{2}|z|^2} dA(z) = 1\,,$$

where A is the Lebesgue measure, so the infinite Ginibre ensemble is spectrally hyperuniform.

Invariant random lattices: Given a lattice $\Gamma < \mathbb{R}^d$, the *Poisson summation formula* states that

$$\sum_{\gamma \in \Gamma} f(\gamma) = \frac{1}{\operatorname{covol}(\Gamma)} \sum_{\xi \in \Gamma^\perp} \widehat{f}(\xi)\,, \quad f \in \mathscr{S}(\mathbb{R}^d)$$

where $\Gamma^\perp = \{\xi \in \mathbb{R}^d \,|\, \langle \gamma, \xi \rangle \in \mathbb{Z}\ \forall\, \gamma \in \Gamma\}$ is the *dual lattice* of Γ. For the invariant random lattice μ_Γ we can write the autocorrelation measure η_Γ using the Poisson summation formula as

$$\begin{aligned}
\eta_\Gamma(f) &= \frac{1}{\operatorname{covol}(\Gamma)} \sum_{\gamma \in \Gamma} f(\gamma) - \frac{1}{\operatorname{covol}(\Gamma)^2} \int_{\mathbb{R}^d} f(x) dx \\
&= \frac{1}{\operatorname{covol}(\Gamma)^2} \sum_{\xi \in \Gamma^\perp \setminus \{0\}} \widehat{f}(\xi)\,, \quad f \in \mathscr{S}(\mathbb{R}^d)\,,
\end{aligned}$$

so that the Bartlett spectral measure $\sigma_\Gamma := \sigma_{\mu_\Gamma}$ is realized as

$$\sigma_\Gamma = \frac{1}{\operatorname{covol}(\Gamma)^2} \sum_{\xi \in \Gamma^\perp \setminus \{0\}} \delta_\xi\,.$$

In particular, the invariant random lattice orbit μ_Γ is stealthy since $\Gamma^\perp$ is uniformly discrete.

Invariant random quasicrystals: For the invariant random quasicrystal $\mu_{\mathcal{P}_o}$ coming from a model set $\mathcal{P}_o = \mathcal{P}_{d,d'}(\Gamma, W)$ in $\mathbb{R}^d$, the Poisson summation formula allows us to write the autocorrelation measure as

$$\begin{aligned}\eta_{\mathcal{P}_o}(f) &= \frac{1}{\operatorname{covol}(\Gamma)} \sum_{(\gamma,\gamma')\in\Gamma} (\chi_W^* * \chi_W)(\gamma') f(\gamma) - \frac{\operatorname{Vol}_{\mathbb{R}^{d'}}(W)^2}{\operatorname{covol}(\Gamma)^2} \int_{\mathbb{R}^d} f(x)dx \\ &= \frac{1}{\operatorname{covol}(\Gamma)^2} \sum_{(\xi,\xi')\in\Gamma^\perp\setminus\{(0,0)\}} |\widehat{\chi}_W(\xi')|^2 \widehat{f}(\xi)\,, \quad f \in \mathscr{S}(\mathbb{R}^d)\,.\end{aligned}$$

Thus the Bartlett spectral measure is

$$\sigma_{\mathcal{P}_o} = \frac{1}{\operatorname{covol}(\Gamma)^2} \sum_{(\xi,\xi')\in\Gamma^\perp\setminus\{(0,0)\}} |\widehat{\chi}_W(\xi')|^2 \delta_\xi$$

Note that $\widehat{\chi}_W$ has non-compact support by the Paley-Wiener Theorem, so the Bartlett spectral measure $\sigma_{\mathcal{P}_o}$ is typically supported on a dense subset of $\mathbb{R}^d$. Most invariant random quasicrystals are spectrally hyperuniform, in particular those coming from arithmetic lattices $\Gamma < \mathbb{R}^d \times \mathbb{R}^{d'}$, and there are non-spectrally hyperuniform examples in every dimension d, see [10, Section 5 + 6].

Independently identically distributed perturbed invariant random lattices: The Fourier transform of the perturbation law $\nu \in \operatorname{Prob}(\mathbb{R}^d)$ is

$$\widehat{\nu}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-i\langle x,\xi\rangle} d\nu(x)\,,$$

and with this one computes the Bartlett spectral measure of the i.i.d. perturbed invariant random lattice $\mu_{\Gamma,\nu}$ is

$$d\sigma_{\Gamma,\nu}(\xi) = |\widehat{\nu}(\xi)|^2 d\sigma_\Gamma(\xi) + \frac{1-|\widehat{\nu}(\xi)|^2}{\operatorname{covol}(\Gamma)} \frac{d\xi}{(2\pi)^d}\,,$$

where σ_Γ is the Bartlett spectral measure of the invariant random Γ-lattice μ_Γ. Since the invariant random lattice μ_Γ is stealthy and $\widehat{\nu}(0) = \nu(\mathbb{R}^d) = 1$, then

$$\lim_{\varepsilon\to 0^+} \frac{\sigma_{\Gamma,\nu}(B_\varepsilon(0))}{\operatorname{Vol}_{\mathbb{R}^d}(B_\varepsilon(0))} = \frac{1-|\widehat{\nu}(0)|^2}{\operatorname{covol}(\Gamma)(2\pi)^d} = 0\,,$$

so that $\mu_{\Gamma,\nu}$ is spectrally hyperuniform.

2.5 Proof of Proposition 2.2.1

Let μ be an invariant locally square-integrable random measure on $\mathbb{R}^d$. By definition of the Bartlett spectral measure σ_μ we are able to write the number variance of μ divided by the volume squared as

$$\frac{\operatorname{NV}_\mu(r)}{\operatorname{Vol}_{\mathbb{R}^d}(B_r(0))^2} = \int_{\mathbb{R}^d} \left| \frac{\widehat{\chi}_{B_r(0)}(\xi)}{\operatorname{Vol}_{\mathbb{R}^d}(B_r(0))} \right|^2 d\sigma_\mu(\xi)\,.$$

Using the computation in Example 2.3.1, we consider the sequence of positive continuous functions

$$\psi_r(\xi) = \frac{\widehat{\chi}_{B_r(0)}(\xi)}{\mathrm{Vol}_{\mathbb{R}^d}(B_r(0))} = \frac{2^d\Gamma(\frac{d}{2}+1)^2}{(r\|\xi\|)^d} J_{\frac{d}{2}}(r\|\xi\|)^2$$

and we claim that ψ_r converges pointwise to $\chi_{\{0\}}$ as $r \to +\infty$. To see this, first note that $\psi_r(\xi)$ is continuous in both $\xi \in \mathbb{R}^d$ and $r > 0$ with $\psi_r(0) = 1$, and with the large argument asymptotic expansion from Equation 2.3.2 we see that if $r\|\xi\| \geq 1$, then

$$\psi_r(\xi) \leq \frac{2^d\Gamma(\frac{d}{2}+1)^2}{(r\|\xi\|)^{d+1}} \Big(\sqrt{\frac{2}{\pi}} \sin(r\|\xi\| - \tfrac{d-1}{4}\pi) + C_d(r\|\xi\|)^{-1/2}\Big)^2 \leq \frac{C_d'}{(r\|\xi\|)^{d+1}}$$

for some $C_d' > 0$ and each $\xi \neq 0$ in $\mathbb{R}^d$. In particular, $\psi_r \to \chi_{\{0\}}$ pointwise as desired. This estimate together with the knowledge that $\psi_r(0) = 1$ for all $r > 0$ allows for a refined upper bound of the form

$$\psi_r(\xi) \leq \frac{C_d''}{(1 + r\|\xi\|)^{d+1}}$$

for some $C_d'' > 0$. Next we claim that

$$\int_{\mathbb{R}^d} \frac{d\sigma_\mu(\xi)}{(1+\|\xi\|)^{d+1}} < +\infty\,, \tag{2.5.1}$$

from which it follows by dominated convergence that

$$\lim_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{\mathbb{R}^d}(B_r(0))^2} = \lim_{r\to+\infty} \int_{\mathbb{R}^d} \psi_r(\xi) d\sigma_\mu(\xi) = 0\,.$$

To verify Equation 2.5.1 we consider the upper bound

$$\int_{\mathbb{R}^d} \frac{d\sigma_\mu(\xi)}{(1+\|\xi\|)^{d+1}} \leq \sum_{n=0}^{\infty} \frac{\sigma_\mu(B_{n+1}(0)\backslash B_n(0))}{(1+n)^{d+1}}\,.$$

By a standard covering argument there is for every $n \geq 0$ a finite set $F_{n+1} \subset \mathbb{R}^d$ with $|F_{n+1}| \leq M(n+1)^{d-1}$ for some $M > 0$ such that

$$B_{n+1}(0)\backslash B_n(0) \subset \bigcup_{\xi \in F_{n+1}} B_1(\xi)\,.$$

Finally, we use that σ_μ is translation bounded as in Equation 2.4.1 to find that

$$\sigma_\mu(B_{n+1}(0)\backslash B_n(0)) \leq |F_{n+1}| \sup_{\xi\in\mathbb{R}^d} \sigma_\mu(B_1(\xi)) \leq M(n+1)^{d-1} \sup_{\xi\in\mathbb{R}^d} \sigma_\mu(B_1(\xi))\,,$$

is finite, so

$$\int_{\mathbb{R}^d} \frac{d\sigma_\mu(\xi)}{(1+\|\xi\|)^{d+1}} \leq M \sup_{\xi\in\mathbb{R}^d} \sigma_\mu(B_1(\xi)) \sum_{n=0}^{\infty} \frac{1}{(1+n)^2} < +\infty\,.$$

□

2.6 Proof of Proposition 2.2.3

The quantity of interest is

$$\frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} = \frac{2^{d-1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2}\, r \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{J_{d/2}(r\|\xi\|)^2}{\|\xi\|^d}\,.$$

For the proof we need uniform convergence of this series, which will be ensured by the following Lemma.

Lemma 2.6.1. *The Epstein zeta series*

$$E_{\Gamma^\perp}(s) = \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{1}{\|\xi\|^s}$$

is absolutely convergent whenever $\mathrm{Re}(s) > d$*.*

The proof is essentially the same as Lemma 5.5 in Paper I. Using the asymptotic expansion of Bessel functions from Equation 2.3.2, we can bound the square of a Bessel function from above and below by

$$\frac{2}{\pi r\|\xi\|}\sin^2(r\|\xi\| - \tfrac{d-1}{4}\pi) - \frac{C_d}{r^2\|\xi\|^2} \le J_{\frac{d}{2}}(r\|\xi\|)^2 \le \frac{2}{\pi r\|\xi\|}\sin^2(r\|\xi\| - \tfrac{d-1}{4}\pi) + \frac{C_d}{r^2\|\xi\|^2}$$

for some positive constant $C_d > 0$.

First case: Assume that $d \equiv 1 \bmod 4$. Then the upper bound on the square of the Bessel function above yields

$$\frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} \le \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{1}{\|\xi\|^{d+1}}\Big(\sin^2(r\|\xi\|) + \frac{\pi C_d}{2r\|\xi\|}\Big)\,.$$

By Lemma 2.6.1, this upper bound is absolutely convergent so taking the lower limit as $r\to+\infty$ yields

$$\liminf_{r\to+\infty}\frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} \le \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2}\liminf_{r\to+\infty} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{\sin^2(r\|\xi\|)}{\|\xi\|^{d+1}}\,.$$

It suffices to find for every $\varepsilon > 0$ a sequence $r_j \to +\infty$ such that

$$\lim_{j\to+\infty} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{\sin^2(r_j\|\xi\|)}{\|\xi\|^{d+1}} < \varepsilon\,.$$

To do this we use Lemma 2.6.1 and take finitely many $\xi_1,\ldots,\xi_N \in \Gamma^\perp\setminus\{0\}$ for some $N = N(\varepsilon)\in\mathbb{N}$ such that

$$\sum_{\xi\in\Gamma^\perp\setminus\{0,\xi_1,\ldots,\xi_N\}} \frac{1}{\|\xi\|^{d+1}} < \varepsilon\,.$$

Then it suffices to pick the sequence $r_j\to+\infty$ such that $\sin^2(r_j\|\xi_n\|)\to 0$ as $j\to+\infty$ for every $n = 1, \dots, N$. In light of this we let $\alpha = (\|\xi_1\|, \dots, \|\xi_N\|) + 2\pi\mathbb{Z}^N \in \mathbb{R}^N/2\pi\mathbb{Z}^N$ in the N-torus and take a sequence $R_k \to +\infty$ such that the sequence

$$R_k.\alpha = (R_k\|\xi_1\|, \dots, R_k\|\xi_N\|) + 2\pi\mathbb{Z}^N$$

converges in $\mathbb{R}^N/2\pi\mathbb{Z}^N$ (which can be done due to compactness). Then $(R_k.\alpha)_k$ is Cauchy, so

$$(R_k - R_\ell).\alpha = R_k.\alpha - R_\ell.\alpha \longrightarrow 0 + 2\pi\mathbb{Z}^N$$

as $j, k \to +\infty$. In particular, we can take two subsequences $k_j, \ell_j \to +\infty$ such that $R_{k_j} - R_{\ell_j} \geq j$ for every $j \in \mathbb{N}$, so that $r_j := R_{k_j} - R_{\ell_j} \to +\infty$ as $j \to +\infty$ and

$$r_j.\alpha \to 0 + 2\pi\mathbb{Z}^N$$

in $\mathbb{R}^N/2\pi\mathbb{Z}^N$. Gathering everything, we have the upper bound

$$\begin{aligned}\liminf_{r\to+\infty} \frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} &\leq \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2} \lim_{j\to+\infty} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{\sin^2(r_j\|\xi\|)}{\|\xi\|^{d+1}} \\ &< \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2}\Big(\sum_{n=1}^N \frac{\lim_{j\to+\infty}\sin^2(r_j\|\xi_n\|)}{\|\xi_n\|^{d+1}} + \varepsilon\Big) = \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2}\varepsilon\,,\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary we are done.

Second case: Assume that $d \not\equiv 1 \bmod 4$. Similarly to the upper bound on the number variance in the first case, we have the lower bound

$$\frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} \geq \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{1}{\|\xi\|^{d+1}}\Big(\sin^2(r\|\xi\| - \tfrac{d-1}{4}) - \frac{\pi C_d}{2r\|\xi\|}\Big)\,.$$

and taking lower limits as $r \to +\infty$ yields

$$\liminf_{r\to+\infty} \frac{\mathrm{NV}_{\mu_\Gamma}(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} \geq \frac{2^d\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\mathrm{covol}(\Gamma)^2} \liminf_{r\to+\infty} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{\sin^2(r\|\xi\| - \frac{d-1}{4}\pi)}{\|\xi\|^{d+1}}\,.$$

To see that the latter lower limit is positive when d is not congruent to 1 modulo 4, assume by way of contradiction that

$$\liminf_{r\to+\infty} \sum_{\xi\in\Gamma^\perp\setminus\{0\}} \frac{\sin^2(r\|\xi\| - \frac{d-1}{4}\pi)}{\|\xi\|^{d+1}} = 0\,.$$

Then there must a sequence $r_j \to +\infty$ such that $\sin^2(r_j\|\xi\| - \frac{d-1}{4}\pi) \to 0$ as $j \to +\infty$ for every $\xi \in \Gamma^\perp\setminus\{0\}$. On the other hand, if we consider $2\xi \in \Gamma^\perp\setminus\{0\}$ then we must have

$$0 = \lim_{j\to+\infty} \sin^2(2r_j\|\xi\| - \tfrac{d-1}{4}\pi) = \sin^2(\tfrac{d-1}{4}\pi)\,,$$

which forces $d \equiv 1 \bmod 4$, a contradiction. $\square$

3. Fluctuations in non-Euclidean spaces

We are similarly to the Euclidean setting interested in random measures that are statistically independent of location in the underlying geometric space, and therefore we first restrict our attention to proper separable metric spaces for which one can transport an arbitrary pair of points to one another via some distance-preserving map. These are the *homogeneous* proper separable metric spaces, and we will later restrict our attention to hyperbolic spaces and regular trees.

3.1 Homogeneous spaces

Let (X, d) be a proper separable metric space and fix a reference point $x_o \in X$. The set $\mathrm{Isom}(X, d)$ of isometries of (X, d) forms a group under composition, and endowed with the topology of pointwise convergence, $\mathrm{Isom}(X, d)$ defines a locally compact second countable Hausdorff (lcsc) topological group, see for example [11, Lemma 5.B.4, p. 143]. We will assume that there is a closed subgroup $G < \mathrm{Isom}(X, d)$ acting transitively on X, meaning that for every $x \in X$ there is an isometry $g \in G$ such that $g.x_o = x$. Denoting by $K = \{g \in G \,|\, g.x_o = x_o\}$ the stabilizer subgroup of $x_o \in X$, we find that the orbit map $G \to X$, $g \mapsto g.x_o$ descends to a homeomorphism $G/K \to X$. We say that X is *G-homogeneous*, or *homogeneous* for short. This includes the linear spaces $\mathbb{R}^d$, which are homogeneous with respect to any transitive subgroup of the Euclidean motions $\mathrm{O}(d) \ltimes \mathbb{R}^d$.

Every such lcsc group G as described admits left- and right-G-invariant positive locally finite Borel measures $m_G^{(\ell)}, m_G^{(r)}$, the left- and right-invariant *Haar measures* for G. These measures are the unique left- and right-invariant measures on G up to scaling by a positive constant. In the appended papers we always assume G to be *unimodular*, meaning that every left-invariant Haar measure is right-invariant and vice versa. Moreover, the stabilizer subgroup $K < G$ is always assumed to be *compact*. If we fix a Haar measure m_G on G then we endow the proper separable homogeneous metric space $X = G/K$ with the G-invariant measure

$$m_X(B) = \int_G \chi_B(g.x_o)dm_G(g)\,, \quad B \subset X \text{ bounded Borel.}$$

Since the orbit map $G \to X$ is proper, m_X defines a positive locally finite Borel measure on X, and it is moreover unique up scaling since the m_G is. In the case of $\mathbb{R}^d$, the positive invariant measures are positive multiples of the Lebesgue measure.

3.2 Invariant random measures

If (X, d, x_o) is a proper separable pointed G-homogeneous metric space with compact stabilizer $K < G$ as in the previous Subsection, the left action of G on X lifts to a continuous action of G on $\mathscr{M}_+(X)$ given by push-forwards $g_*p(B) = p(g^{-1}.B)$ for $g \in G$ and bounded Borel sets $B \subset X$. In turn, this action lifts to an action on random measures $\mu \in \text{Prob}(\mathscr{M}_+(X))$ by an additional push-forward operation, $g.\mu(E) = \mu(g_*^{-1}E)$ for $E \in \mathscr{B}_{\mathscr{M}_+}$, and similarly for (simple) point processes. Stationarity and isotropy in the Euclidean setting now generalizes to G-invariance of random measures on X.

Definition 3.2.1. A random measure μ on a proper separable G-homogeneous metric space X is *invariant* if $g.\mu = \mu$ for all $g \in G$.

An immediate consequence is that if μ is an invariant locally n-invariant random measure on a proper separable G-homogeneous metric space X as described, then the moment measure $\eta_\mu^{(n)}$ and n-point correlation measure $\rho_\mu^{(n)}$ are both diagonally G-invariant on X^n. In particular, the μ-expectation of linear statistics determines a positive G-invariant linear functional

$$\mathscr{L}_c^\infty(X) \ni f \longmapsto \mathbb{E}_\mu(\mathbb{S}f) \in \mathbb{C}\,,$$

so by uniqueness of the Haar measure on G there is an *intensity* $i_\mu > 0$ satisfying

$$\mathbb{E}_\mu(\mathbb{S}f) = i_\mu \int_X f(x)dm_X(x)\,.$$

Examples of invariant random measures in the general homogeneous setting that we have described can be found in Section 2 of Paper I, where Poisson point processes, invariant random lattice orbits and i.i.d. perturbed analogues are introduced, as well as Section 6 of Paper III, where determinantal point processes are considered. For the sake of completeness, we briefly mention Poisson, determinantal and permanental point processes here. For a more detailed exposition on invariant random measures on homogeneous spaces, their moment measures, Palm measures and reduced moment measures, we refer to [20].

3.3 Examples

Homogeneous Poisson point processes: A *homogeneous* Poisson point process is a Poisson point process $\text{Poi}(m_X)$ for which the intensity measure is an invariant positive locally finite measure m_X on X. Such a point process is invariant since m_X is invariant. The variance of linear statistics is

$$\text{Var}_{\text{Poi}(m_X)}(\mathbb{S}f) = \int_X |f(x)|^2 dm_X(x)\,,$$

and we point out that the number variance of $\text{Poi}(m_X)$ is

$$\text{NV}_{\text{Poi}(m_X)}(r) = m_X(B_r(x_o))\,.$$

Equivariant determinantal and permanental point processes: Fix a positive G-invariant background measure $\nu = m_X$ and let $L : X \times X \to \mathbb{C}$ be a continuous postive-definite Hermitian locally trace class kernel satisfying the equivariance property

$$L(g.x_1, g.x_2) = c(g,x_1)\overline{c(g,x_2)}L(x_1,x_2)$$

for every $x_1, x_2 \in X$, where $c : G \times X \to \mathbb{C}$ is some measurable map with $|c(g,x)| = 1$. Then the associated determinantal and permanental point processes $\mu_L^{\mathrm{det}}, \mu_L^{\mathrm{per}}$ are G-invariant since the n-point correlation densities satisfy

$$\begin{aligned}\mathrm{per}(L(g.x_i, g.x_j))_{i,j=1}^n &= \mathrm{per}(c(g,x_i)\overline{c(g,x_j)}L(x_i,x_j))_{i,j=1}^n \\ &= |c(g,x_1)\dots c(g,x_n)|^2 \mathrm{per}(L(x_i,x_j))_{i,j=1}^n = \mathrm{per}(L(x_i,x_j))_{i,j=1}^n\end{aligned}$$

and similarly for the determinant. Following the computations of Section 6 in Paper III, we can show that the variance of linear statistics $\mathbb{S}f$ with $f \in \mathscr{L}_c^\infty(X)$ is

$$\mathrm{Var}_{\mu_L}(\mathbb{S}f) = L(x_o,x_o)\int_X f(x)dm_X(x) \pm \int_{X^2} f(x_1)\overline{f(x_2)}|L(x_1,x_2)|^2 dm_X^{\otimes 2}(x_1,x_2)$$

with $-$ for $\mu_L = \mu_L^{\mathrm{det}}$ and $+$ for $\mu_L = \mu_L^{\mathrm{per}}$.

3.4 Polar coordinates on hyperbolic spaces and trees

In order to define Bartlett spectral measures, we need a notion of Fourier transform for "radial" functions on the reference homogeneous space, and this requires some type of polar coordinate system on the homogeneous space. We will from now on focus on real hyperbolic spaces and regular trees, and we start by introducing polar coordinates on these spaces.

3.4.1 Real hyperbolic spaces

Consider $\mathbb{R}^{1+d}$ with $d \geq 2$ and the bilinear Lorentz form

$$[x,y] = x_0y_0 - x_1y_1 - \dots - x_dy_d$$

for $x, y \in \mathbb{R}^{1+d}$. The group of linear transformations preserving the form $[\cdot,\cdot]$ is by definition the indefinite orthogonal group $\mathrm{O}(1,d)$, and if we restrict our attention to the one-sheeted hyperboloid

$$\mathbb{H}^d = \left\{ x \in \mathbb{R}^{1+d} \,\middle|\, [x,x] = 1\,,\ x_0 > 0 \right\}$$

with the proper metric $d(x,y) = \mathrm{arccosh}([x,y])$, then the group of orientation-preserving isometries of $(\mathbb{H}^d, d)$ is the connected subgroup $G_d = \mathrm{SO}^\circ(1,d) < \mathrm{O}(1,d)$ of linear transformations $g \in \mathrm{O}(1,d)$ with $\det(g) = 1$ and $(gx)_0 > 0$ whenever $x_0 > 0$. Moreover, the action of G_d on $\mathbb{H}^d$ is transitive with compact stabilizer subgroup

$$K_d = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & k \end{array}\right) : k \in \mathrm{SO}(d) \right\} < G_d\,.$$

of the reference point $x_o = (1, 0, \dots, 0) \in \mathbb{H}^d$. The proper separable homogeneous metric space $(\mathbb{H}^d, d)$ is *d-dimensional real hyperbolic space.* We briefly mention that when $d = 2$ we can identify $\mathbb{H}^2$ with $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ from the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper-half plane

$\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ by Möbius transformations, and also with $\mathrm{SU}(1,1)/\mathrm{U}(1)$ from the action of $\mathrm{SU}(1,1)$ on the open unit disk $\mathbb{D} \subset \mathbb{C}$ by Möbius transformations. When $d = 3$ we can identify $\mathbb{H}^3$ with $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ from a (left) action of $\mathrm{SL}_2(\mathbb{C})$ on the upper-half space $\mathbb{C} \times j\mathbb{R}_{>0}$ in the quarternions by fractional transformations (with respect to right division).

The polar coordinates on $\mathbb{H}^d$ are obtained from first observing that the 1-parameter semigroup

$$A_d^+ = \left\{ a_t = \left(\begin{array}{c|c|c} \cosh(t) & 0 & \sinh(t) \\ \hline 0 & \mathrm{Id}_{\mathbb{R}^{d-1}} & 0 \\ \hline \sinh(t) & 0 & \cosh(t) \end{array} \right) \,\middle|\, t \in \mathbb{R}_{\geq 0} \right\} \subset G_d$$

with $a_t.o = (\cosh(t), 0, \ldots, 0, \sinh(t))$ satisfies $d(a_t.x_o, x_o) = t$ for each $t \geq 0$. Thus $\gamma_o = A_d^+.x_o$ is a one-sided geodesic in $\mathbb{H}^d$, and the level sets

$$L_t = \left\{ x \in \mathbb{H}^d \,\middle|\, d(x, x_o) = t \right\} = \left\{ (\cosh(t), \sinh(t)u) \in \mathbb{H}^d \,|\, u \in S^{d-1} \right\}$$

are precisely intersections of spheres in $\mathbb{R}^{1+d}$ with $\mathbb{H}^d$. This means that every $x \in \mathbb{H}^d$ can be written as $x = (\cosh(t), \sinh(t)u)$, where $t = d(x, o)$ and $u = x'/\|x'\|$ for $x' = (x_1, \ldots, x_d)$. Moreover, the stabilizer subgroup K_d acts transitively on L_t for each $t \geq 0$, so for every $x = (\cosh(t), \sinh(t)u) \in \mathbb{H}^d$ can be written as $x = k_u a_t.o$ where $k_u \in K_d$ is any element corresponding to some special orthogonal matrix that maps $(0, \ldots, 0, 1) \in S^{d-1}$ to $u \in S^{d-1}$. In total, we have described the *Cartan decomposition*

$$G_d = K_d A_d^+ K_d \,.$$

Note that the Cartan decomposition induces a bijection $K_d \backslash G_d / K_d \to \mathbb{R}_{\geq 0}$ by sending a double coset $K_d g K_d$ to the distance $d(g.o, o)$. In particular, since $d(g^{-1}.o, o) = d(g.o, o)$ this means that

$$K_d g^{-1} K_d = K_d g K_d$$

for every $g \in G_d$.

Lastly, from the polar decomposition we get a G_d-invariant positive locally finite Borel measure on $\mathbb{H}^d$,

$$\int_{\mathbb{H}^d} f(x) dm_{\mathbb{H}^d}(x) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty \int_{S^{d-1}} f(\cosh(t), \sinh(t)u) dm_{S^{d-1}}(u) \sinh(t)^{d-1} dt$$

for $f \in \mathscr{L}_c^\infty(\mathbb{H}^d)$, where $m_{S^{d-1}}$ is the $\mathrm{SO}(d)$-invariant probability surface measure on S^{d-1}. Note that with respect to this measure, the volume of metric balls behaves exponentially in the large radial limit,

$$m_{\mathbb{H}^d}(B_r(x_o)) \asymp \mathrm{e}^{(d-1)r} \,,$$

which reflects the non-amenability of the orientation-preserving isometry group $\mathrm{SO}^\circ(1, d)$.

3.4.2 Regular trees

Let $v \geq 2$ be an integer and $\mathbb{T}_v$ the unique countably infinite cycle-free simple connected graph in which every vertex has v neighbors. The graph $\mathbb{T}_v$ is called the *v-regular tree* and v is referred to as the *valence* of the tree. We will frequently identify the graph $\mathbb{T}_v$ with its vertex set. A natural metric on $\mathbb{T}_v$ is the graph metric $d(x_1, x_2)$ given by the minimal length of paths between two vertices $x_1, x_2 \in \mathbb{T}_v$, and the group $G_v = \mathrm{Aut}(\mathbb{T}_v)$ of graph automorphisms of $\mathbb{T}_v$ coincides with the isometry group of $(\mathbb{T}_v, d)$. Moreover, given a root $x_o \in \mathbb{T}_v$ and any $x \in \mathbb{T}_v$ one can inductively construct a graph automorphism $g \in G_v$ such that $g.x_o = x$, so that $(\mathbb{T}_v, d)$ becomes a proper separable homogeneous metric space under the action of G_v with stabilizer $K_v < G_v$ of the root $x_o \in \mathbb{T}_v$. Using that each "sphere"

$$\partial B_n(x_o) = \Big\{ x \in \mathbb{T}_v \,|\, d(x, x_o) = n \Big\}\,, \quad n \in \mathbb{N}_0$$

is finite and that G_v acts isometrically on $\mathbb{T}_v$, one can show that the stabilizer $K_v < G_v$ defines a profinite subgroup and is therefore compact. One can moreover show that K_v is open in G_v, so that G_v is locally profinite.

Remark 3.4.1 (Regular trees as moduli spaces of rank 2 $\mathbb{Z}_p$-modules)**.** We mention that when the valence is $v = p+1$ for some prime $p \in \mathbb{N}$ there is a purely algebraic description of $\mathbb{T}_v$. In the 2-dimensional p-adic vector space $\mathbb{Q}_p^2$, two $\mathbb{Z}_p$-modules $\Lambda_1, \Lambda_2 < \mathbb{Q}_p^2$ are said to be *homothetic* if there is a $\lambda \in \mathbb{Q}_p \backslash \{0\}$ such that $\Lambda_1 = \lambda\Lambda_2$, and this defines an equivalence relation on the set of rank 2 $\mathbb{Z}_p$-modules. If we denote by $\mathscr{L}_p$ the set of equivalence classes of rank 2 $\mathbb{Z}_p$-modules in $\mathbb{Q}_p^2$ modulo homothety and introduce the reflexive symmetric relation given by

$$[\Lambda_1] \sim [\Lambda_2] \quad \Longleftrightarrow \quad p\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$$

for some representatives Λ_1, Λ_2, then one readily shows that $(\mathscr{L}_p, \sim)$ defines a $(p+1)$-regular tree. Moreover, the projective general linear group

$$\mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \mathrm{id}\,, \quad \mathbb{Q}_p^\times = \{\lambda \in \mathbb{Q}_p \,|\, |\lambda|_p = 1\}$$

acts transitively on $\mathscr{L}_p$ with maximal compact stabilizer $\mathrm{PGL}_2(\mathbb{Z}_p)$, so the $(p+1)$-regular tree $\mathbb{T}_{p+1}$ can be realized as the homogeneous space $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$. In this sense, regular trees can also be thought of as discrete analogues of the hyperbolic plane $\mathbb{H}^2 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$.

Similarly to real hyperbolic spaces, we obtain polar coordinates on the tree $\mathbb{T}_v$ by considering a one-sided "geodesic" $\gamma_o = (x_o, x_1, x_2, \dots)$ of adjacent vertices starting at the root $x_o \in \mathbb{T}_v$, which lifts to a 1-parameter semigroup $A_v^+ \subset G_v$ generated by the unique graph automorphism $s_o \in G_v$ sending x_n to x_{n+1} for all n. Since K_v acts transitively on each "sphere" $\partial B_n(x_o)$ we can write each $x \in \mathbb{T}_v$ as

$$x = k_x s_o^n.x_o\,,$$

where $n = d(x, x_o)$ and $k_x.x_n = x$ for all n. Similarly to the real hyperbolic case, we have described the *Cartan decomposition*

$$G_v = K_v A_v^+ K_v$$

and the map $K_v \backslash G_v / K_v \to \mathbb{N}_0$ sending a double coset $K_v g K_v$ to the distance $d(g.x_o, x_o)$ is a bijection. In particular, $K_v g^{-1} K_v = K_v g K_v$ for all $g \in G_v$.

Regarding invariant measures, it is clear to see that the counting measure

$$\sum_{x \in \mathbb{T}_v} f(x) = \sum_{n=0}^{\infty} \int_{K_v} f(k s_o^n.x_o) dm_{K_v}(k) |\partial B_n(x_o)|$$

where m_{K_v} is the Haar probability on K_v, is a positive G_v-invariant measure on $\mathbb{T}_v$. The group $G_v = \mathrm{Aut}(\mathbb{T}_v)$ is non-amenable, and the volume of metric balls in the v-regular tree grows exponentially in the radial limit. Explictly, we have the estimate

$$|B_n(x_o)| \asymp (v-1)^n \,.$$

3.4.3 The Gelfand property

We want to define an analogue of the Fourier transform for "radial" functions on $\mathbb{H}^d$ and $\mathbb{T}_v$, specifically one for indicator functions $\chi_{B_r(x_o)}$, and the transform should take the group convolution of two "radial" functions on $G_d = \mathrm{SO}^\circ(1,d)$, $G_v = \mathrm{Aut}(\mathbb{T}_v)$ to the multiplication of their transforms. In the Euclidean case this works because the space $\mathscr{L}_c^\infty(\mathbb{R}^d)$ of bounded measurable functions with compact support forms a commutative algebra under convolution. For a non-abelian lcsc group G, the algebra $\mathscr{L}_c^\infty(G)$ is *never* commutative with respect to the group convolution

$$(\varphi_1 * \varphi_2)(g) = \int_G \varphi_1(h)\varphi_2(h^{-1}g) dm_G(h) \,, \quad \varphi_1, \varphi_2 \in \mathscr{L}_c^\infty(G) \,,$$

but when restricting to "radial" functions then one can find commutative subalgebras.

Let $G = G_d$ or G_v and $K = K_d$ or K_v respectively and $X = G/K$ with basepoint $x_o = eK$. A function $f : X \to \mathbb{C}$ is *radial* if it is invariant under K, meaning $f(k.x) = f(x)$ for all $k \in K$, $x \in X$. Every radial function f on X can equivalently be considered as a bi-K-invariant function $\varphi_f : G \to \mathbb{C}$ by setting $\varphi_f(g) = f(g.x_o)$, and we denote by $\mathscr{L}_c^\infty(G,K) \subset \mathscr{L}_c^\infty(G)$ the subalgebra of bi-K-invariant functions. The double coset symmetry $Kg^{-1}K = KgK$ for all $g \in G$ that we obtained from the Cartan decompositions of G_d, G_v will now be used to prove the *Gelfand property*, defined as follows.

Definition 3.4.2. Let G be a lcsc group and $K < G$ a compact subgroup. The pair (G,K) is *Gelfand* if the algebra $\mathscr{L}_c^\infty(G,K)$ is commutative under group convolution, and we say that the homogeneous space $X = G/K$ is *commutative*.

Lemma 3.4.3. *Let G be lcsc and $K < G$ compact. If $Kg^{-1}K = KgK$ for all $g \in G$ then (G,K) is Gelfand.*

In particular, the hyperbolic pairs (G_d, K_d) and regular tree pairs (G_v, K_v) are Gelfand. We note however that this double coset criterion is not guaranteed for general Gelfand pairs.

Proof of Lemma 3.4.3. By assumption, every $\varphi \in \mathscr{L}_c^\infty(G,K)$ satisfies $\varphi(g^{-1}) = \varphi(g)$ for all $g \in G$. Thus

$$\begin{aligned}(\varphi_1 * \varphi_2)(g) &= \int_G \varphi_1(h)\varphi_2(h^{-1}g)dm_G(h) = \int_G \varphi_1(h^{-1})\varphi_2(g^{-1}h)dm_G(h) \\ &= \int_G \varphi_1(h^{-1}g)\varphi_2(h)dm_G(h) = (\varphi_2 * \varphi_1)(g)\end{aligned}$$

for all $g \in G$ and every $\varphi_1, \varphi_2 \in \mathscr{L}_c^\infty(G, K)$. □

3.5 Spherical functions and spherical transforms

Every lcsc Gelfand pair (G, K) with $K < G$ compact admits a special class of so called *spherical functions* on G that are in a sense analogues of complex exponentials $\mathrm{e}^{-i\langle\cdot,\xi\rangle}$ for $\xi \in \mathbb{C}^d$ in the Euclidean case. A non-zero continuous function $\omega : G \to \mathbb{C}$ is K-*spherical* or *spherical* for short if it is bi-K-invariant and satisfies $\omega(e) = 1$ as well as the functional equation

$$\omega(g_1)\omega(g_2) = \int_K \omega(g_1 k g_2) dm_K(k)\,, \quad \forall\, g_1, g_2 \in G\,,$$

where m_K denotes the Haar probability measure on K. We denote by $\mathcal{S}(G, K)$ the collection of all K-spherical functions on G. We have actually already seen spherical functions coming from a non-abelian Gelfand pair in the previous Chapter. If $G = \mathrm{O}(d) \ltimes \mathbb{R}^d$ is the Euclidean motion group and $K = \mathrm{O}(d) \ltimes \{0\}$ is the stabilizer of the origin in $\mathbb{R}^d$, then the K-spherical functions are

$$\omega_\lambda(x) = \int_{S^{d-1}} \mathrm{e}^{-i\lambda\langle x,u\rangle} dm_{S^{d-1}}(u) = \frac{2^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{(\lambda\|x\|)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(\lambda\|x\|)\,, \quad x \in \mathbb{R}^d,\, \lambda \in \mathbb{C}\,.$$

For real hyperbolic spaces $\mathbb{H}^d$ the spherical functions are given in polar coordinates by

$$\omega_\lambda(a_t) = \frac{2^{\frac{d-1}{2}}\Gamma(\frac{d}{2})}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})} \sinh(t)^{2-d} \int_0^t (\cosh(t) - \cosh(s))^{\frac{d-3}{2}} \cos(\lambda s) ds\,, \quad t \geq 0, \lambda \in \mathbb{C}$$

and for regular trees $\mathbb{T}_v$ they are

$$\omega_\lambda(s_o^n) = \frac{\sin((n+1)\lambda\log(q)) - q^{-1}\sin((n-1)\lambda\log(q))}{(1+q^{-1})\sin(\lambda\log(q))}\, q^{-n/2}\,, \quad n \in \mathbb{N}_0,\, \lambda \in \mathbb{C}\,,$$

where $q = v - 1$.

Of particular importance to us are the *positive-definite* spherical functions $\omega \in \mathcal{S}(G, K)$, satisfying

$$\int_G (\varphi^* * \varphi)(g)\omega(g) dm_G(g) \geq 0$$

for all $\varphi \in \mathscr{L}_c^\infty(G)$. If we denote the collection of such by $\mathcal{S}^+ = \mathcal{S}^+(G, K) \subset \mathcal{S}(G, K)$, then one can show that the *spherical transform* of functions $\varphi \in \mathscr{L}_c^\infty(G,K)$ defined by

$$\widehat{\varphi}(\omega) = \int_G \varphi(g)\omega(g^{-1})dm_G(g)\,, \quad \omega \in \mathcal{S}^+$$

satisfies the desired properties

$$(\widehat{\varphi_1 * \varphi_2})(\omega) = \widehat{\varphi}_1(\omega)\widehat{\varphi}_2(\omega) \quad \text{and} \quad \widehat{\varphi^*}(\omega) = \overline{\widehat{\varphi}(\omega)}$$

for all $\omega \in \mathcal{S}^+$. For real hyperbolic spaces $\mathbb{H}^d$, the set of positive-definite spherical functions is

$$\mathcal{S}^+(G_d,K_d) = \left\{\omega_\lambda \in \mathcal{S}(G_d,K_d) \,\middle|\, \lambda \in i(0,\tfrac{d-1}{2}] \cup [0,+\infty)\right\}$$

with $\lambda = i\frac{d-1}{2}$ corresponding to the constant spherical function. For regular trees $\mathbb{T}_v$ they are

$$\mathcal{S}^+(G_v,K_v) = \left\{\omega_\lambda \in \mathcal{S}(G_v,K_v) \,\middle|\, \lambda \in i(0,\tfrac{1}{2}] \cup [0,\tfrac{\pi}{\log(v-1)}] \cup (\tfrac{\pi}{\log(v-1)} + i(0,\tfrac{1}{2}])\right\},$$

with $\lambda = i\frac{1}{2}$ corresponding to the constant spherical function.

Example 3.5.1. In Paper I we compute the spherical transform of indicator functions $\chi_{B_r(x_o)}$ on $\mathbb{H}^d$ to be

$$\widehat{\chi}_{B_r(x_o)}(\lambda) = \frac{2^{\frac{d+1}{2}}\Gamma(\frac{d}{2})}{(d-1)\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}\int_0^r (\cosh(r)-\cosh(s))^{\frac{d-1}{2}}\cos(\lambda s)ds\,,$$

and in Paper II we show that the spherical transform of $\chi_{B_r(x_o)}$ on regular trees $\mathbb{T}_v$ is

$$\widehat{\chi}_{B_r(x_o)}(\lambda) = \frac{\sin((r+1)\lambda\log(q)) + q^{-1/2}\sin(r\lambda\log(q))}{\sin(\lambda\log(q))}\,q^{-r/2}\,,$$

where again $q = v-1$ and $r \in \mathbb{N}_0$.

3.6 Spectral fluctuations

3.6.1 Spectral hyperuniformity

In Papers I and II we prove that no invariant locally square-integrable random measures real hyperbolic space or regular trees are never geometrically hyperuniform, in the sense that

$$\limsup_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{m_X(B_r(x_o))} > 0\,.$$

The intuition behind this phenomenon is that the volume of balls and there boundaries have the same exponential growth rate in the infinite radial limit, as opposed to the Euclidean setting. Thus this result can be seen as a generalization of Beck's Theorem to hyperbolic spaces and regular trees. In Paper III, we establish the existence and uniqueness of a Bartlett spectral measures $\sigma_\mu \in \mathscr{M}_+(\mathcal{S}^+)$ for invariant locally square-integrable random measures μ on proper separable metric commutative spaces X as previously described. The Bartlett spectral measure is uniquely determined by the identity

$$\mathrm{Var}_\mu(\mathbb{S}f) = \int_{\mathcal{S}^+} |\widehat{\varphi}_f(\omega)|^2 d\sigma_\mu(\omega)$$

for every $f \in \mathscr{L}_c^\infty(X)$, where $\varphi_f(g) = f(g.x_o)$. Similarly to the Euclidean case, we show in Paper III that the measure σ_μ assigns zero measure to the constant spherical function $\{1\} \subset \mathcal{S}^+$. Using the measure σ_μ and the explicit parameterizations of the positive-definite functions $\mathcal{S}^+$, we formulate a notion of spectral hyperuniformity for real hyperbolic spaces and regular trees in Paper I and II respectively. For the definition, we remark that the Bartlett spectral measure of the invariant Poisson point process is

$$d\sigma_{\mathrm{Poi}(m_X)}(\omega_\lambda) = \begin{cases} \chi_{(0,+\infty)}(\lambda)|c_d(\lambda)|^{-2}d\lambda & \text{if } X = \mathbb{H}^d \\ \chi_{(0,\pi/\log(v-1))}(\lambda)|c_v(\lambda)|^{-2}d\lambda & \text{if } X = \mathbb{T}_v\,, \end{cases}$$

where c_d, c_v are the *Harish-Chandra c-functions* for $\mathbb{H}^d$ and $\mathbb{T}_v$ respectively.

Definition 3.6.1. An invariant locally square-integrable random measure μ on

1. real hyperbolic space $\mathbb{H}^d$ is *spectrally hyperuniform* if

$$\sigma_\mu(i[0, \tfrac{d-1}{2})) = 0$$

and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\mu((0,\varepsilon])}{\sigma_{\mathrm{Poi}(m_X)}((0,\varepsilon])} = 0\,.$$

2. the v-regular tree $\mathbb{T}_v$ is *spectrally hyperuniform* if

$$\sigma_\mu(i[0, \tfrac{1}{2})) = \sigma_\mu(\tfrac{\pi}{\log(q)} + i[0, \tfrac{1}{2})) = 0$$

and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\mu((0,\varepsilon])}{\sigma_{\mathrm{Poi}(m_X)}((0,\varepsilon])} = 0\,.$$

An interpretation of this definition that is consistent with the Euclidean picture is that spectral hyperuniformity amounts to sub-Poissonian decay of the Bartlett spectral measure near the endpoint(s) of the support of $\sigma_{\mathrm{Poi}(m_X)}$. For real and p-adic Lie groups, this Poisson spectrum coincides with the so called *principal series* spherical functions, and remaining spherical functions are said to lie in the *complementary series*. We also formulate a notion of stealth for both geometries in Papers I and II respectively in terms of vanishing of the Bartlett spectral measure near the endpoints of the Poisson spectrum. Before moving on we also note that when the Bartlett spectral measure σ_μ is absolutely continuous with respect to $\sigma_{\mathrm{Poi}(m_X)}$, we can define the *structure factor* of μ to be the Radon-Nikodym derivative

$$\mathcal{S}_\mu(\lambda) = \frac{d\sigma_\mu(\lambda)}{d\sigma_{\mathrm{Poi}(m_X)}} = \begin{cases} \chi_{(0,+\infty)}(\lambda)|c_d(\lambda)|^2 \dfrac{d\sigma_\mu(\lambda)}{d\lambda} & \text{if } X = \mathbb{H}^d \\ \chi_{(0,\pi/\log(v-1))}(\lambda)|c_v(\lambda)|^2 \dfrac{d\sigma_\mu(\lambda)}{d\lambda} & \text{if } X = \mathbb{T}_v\,, \end{cases}.$$

For such random measures μ, spectral hyperuniformity is equivalent to the vanishing of the structure factor $\mathcal{S}_\mu$ at the endpoints of the support of the Poisson spectrum/principal series spherical functions.

3.6.2 Non-hyperuniformity of determinantal point processes

Let us consider an invariant determinantal point process μ_L in $X = \mathbb{H}^d$ or $\mathbb{T}_v$ defined by an equivariant kernel L as in Section 3.3. Assume in addition that the associated integral operator defines an orthogonal projection in $L^2(X)$ onto a closed subspace. As in the Euclidean case, L must satisfy

$$\int_X |L(x, x_o)|^2 dm_X(x) = L(x_o, x_o) = 1\,.$$

Moreover, in Paper III we compute the Bartlett spectral measure of μ_L, similarly to the Euclidean case, to be

$$d\sigma_{\mu_L}(\omega) = (1 - \widehat{\kappa}_L(\omega))d\sigma_{\mathrm{Poi}(m_X)}(\omega)\,,$$

where $\kappa_L(g) = |L(g.x_o, x_o)|^2$ for $g \in G$. We claim that μ_L is not spectrally hyperuniform. In order to prove this however, we will need to verify that non-trivial spherical functions for $\mathbb{H}^d$ and $\mathbb{T}_v$ have strict decay at the endpoints of the Bartlett spectrum for the invariant Poisson point process. The statement for regular trees is the following, and the analogous statement and proof for real hyperbolic spaces $\mathbb{H}^d$ can be found in Lemma 6.3(2) of Paper I.

Lemma 3.6.2. *If $\lambda \in [0, \frac{\pi}{\log(q)}]$, then the spherical function ω_λ on the v-regular tree $\mathbb{T}_v$ satisfies $\omega_\lambda(g) < 1$ for every $g \notin K_v$.*

The proof of this Lemma involves simple estimates using the formula for the spherical functions ω_λ in the v-regular tree case:

$$\omega_\lambda(g) = \frac{\sin((d(g.x_o, x_o)) + 1)\lambda \log(q)) - q^{-1}\sin((d(g.x_o, x_o) - 1)\lambda \log(q))}{(1 + q^{-1})\sin(\lambda \log(q))}\, q^{-d(g.x_o, x_o)/2}\,,$$

where $q = v - 1$. With this Lemma we can prove that the determinantal point processes μ_L in question are not spectrally hyperuniform.

Proposition 3.6.3. *Let $X = \mathbb{H}^d$ or $\mathbb{T}_v$ and let $L : X \times X \to \mathbb{C}$ be a continuous locally trace class positive-definite equivariant kernel that defines an orthogonal projection onto some closed subspace of $L^2(X)$. Then the determinantal point process μ_L is not spectrally hyperuniform.*

Proof. Let $X = \mathbb{H}^d$ and take $\lambda = 0$. Since $\mathrm{Im}(\lambda) < \frac{d-1}{2}$, then Lemma 6.3(2) in Paper I tells us that $|\omega_0(g)| < 1$ for all $g \notin K_d$. Since L is continuous, then $|L|^2$ is continuous and since L defines an orthogonal projection we get

$$\widehat{\kappa}_L(0) = \int_G |L(g.x_o, x_o)|^2 \omega_0(g) dm_G(g) < \int_G |L(g.x_o, x_o)|^2 dm_G(g) = L(x_o, x_o) = 1\,.$$

Similarly, if $X = \mathbb{T}_v$ and $\lambda = 0, \frac{\pi}{\log(v-1)}$, the Lemma above implies that $\omega_\lambda(g) < 1$ for all

$g \notin K_v$. By the same argument as in the hyperbolic case we get

$$\widehat{\kappa}_L(\lambda) < L(x_o, x_o) = 1\,.$$

$\square$

4. Summary of papers

4.1 Paper I

Paper I is a joint work with Michael Björklund. In it we derive general lower asymptotics of number variances in Euclidean and real hyperbolic spaces for invariant locally square-integrable random measures, as well as define spectral hyperuniformity for such random measures on real hyperbolic spaces.

In the Euclidean setting, we utilize that the number variance of an invariant locally square-integrable random measure μ on $\mathbb{R}^d$ can be written in terms of the Bartlett spectral measure σ_μ as

$$\mathrm{NV}_\mu(r) = (2\pi r)^d \int_{\mathbb{R}^d} J_{\frac{d}{2}}(r\|\xi\|)^2 \frac{d\sigma_\mu(\xi)}{\|\xi\|^d}$$

to prove the random measure analogue of Beck's Theorem in Theorem 2.2.2 (Theorem 1.1 in Paper I). The idea of the proof is first to observe that if

$$\liminf_{R\to+\infty} \frac{1}{R^d} \int_0^R \mathrm{NV}_\mu(r)dr > 0$$

then

$$\limsup_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{d-1}(\partial B_r(0))} > 0$$

as desired. Then one argues using the asymptotic expansion of Bessel functions from Equation 2.3.2 to show that there is a constant $C_d > 0$ such that

$$\frac{1}{R^d} \int_0^R r^d J_{\frac{d}{2}}(r\|\xi\|)^2 dr \geq \frac{C_d}{\|\xi\|}$$

for all $\xi \in \mathbb{R}^d \backslash \{0\}$. Making a cutoff at $\|\xi\| \geq \varepsilon_o > 0$ and applying Fubini yields

$$\liminf_{R\to+\infty} \frac{1}{R^d} \int_0^R \mathrm{NV}_\mu(r)dr \geq (2\pi)^d C_d \int_{\|\xi\|\geq\varepsilon_o} \frac{d\sigma_\mu(\xi)}{\|\xi\|^{d+1}} > 0\,.$$

In light of Beck's Theorem, we also prove a strengthening of Proposition 2.2.3 for the invariant random standard lattices $\mu_{\mathbb{Z}^d}$ in $\mathbb{R}^d$ when the dimension satisfies $d \equiv 1 \bmod 4$, for example $d = 5$. More precisely, we show that there is an *integer* sequence $r_j \to +\infty$

such that

$$\lim_{j\to+\infty}\frac{\mathrm{NV}_{\mu_{\mathbb{Z}^d}}(r_j)}{\mathrm{Vol}_{d-1}(\partial B_{r_j}(0))}=0\,.$$

Similarly to the proof of Proposition 2.2.3, the argument reduces showing for every $N\in\mathbb{N}$ that there is a sequence $r_j\to+\infty$ of integers such that

$$\sin^2(2\pi r_j\sqrt{n})\longrightarrow 0$$

as $j\to+\infty$ for every $n=1,\ldots,N$. This is done using that square-roots of distinct square-free integers are linearly independent over $\mathbb{Q}$, as proved by Besicovitch in [9, Thm 2, p.4], which allows for an equidistribution argument on tori to prove the desired convergence.

For real hyperbolic spaces $\mathbb{H}^d$ we prove a hyperbolic analogue of Beck's Theorem. Before stating it, we recall that the positive-definite spherical functions for $\mathbb{H}^d$ are given in Cartan coordinates by

$$\omega_\lambda(a_t)=\frac{2^{\frac{d-1}{2}}\Gamma(\frac{d}{2})}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}\sinh(t)^{2-d}\int_0^t(\cosh(t)-\cosh(s))^{\frac{d-3}{2}}\cos(\lambda s)ds$$

for $\lambda\in i(0,\frac{d-1}{2}]\cup[0,+\infty)$. The following is Theorem 1.2 in Paper I.

Theorem 4.1.1. *Let μ be a locally square-integrable invariant random measure on $\mathbb{H}^d$. Then*

$$\limsup_{r\to+\infty}\frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{\mathbb{H}^d}(B_r(x_o))}>0\,.$$

Moreover,

1. *if $\sigma_\mu(\{0\})>0$, then*

$$\liminf_{r\to+\infty}\frac{\mathrm{NV}_\mu(r)}{r^2\mathrm{Vol}_{\mathbb{H}^d}(B_r(x_o))}>0\,.$$

2. *if $\sigma_\mu(i[\delta\frac{d-1}{2},\frac{d-1}{2}))>0$ for some $\delta\in(0,1)$, then*

$$\liminf_{r\to+\infty}\frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{\mathbb{H}^d}(B_r(x_o))^{1+\delta}}>0\,.$$

Before getting into details about the proof, we note that the first inequality in the Theorem rules out the canonical definition of geometric hyperuniformity on $\mathbb{H}^d$ for invariant locally square-integrable random measures. Moreover, if the Bartlett spectral measure σ_μ assigns positive mass to the complementary series spherical functions $\omega_{i\lambda}$, $\lambda\in i[0,\frac{d-1}{2}]$, then the random measure μ is geometrically hyperfluctuating in analogy with the Euclidean definition.

The proof of Theorem 4.1.1 follows the same ideas as the proof for the Euclidean case, starting with a formula for the number variance in terms of the Bartlett spectral measure of μ,

$$\mathrm{NV}_\mu(r) = \int_0^\infty |\widehat{\chi}_{B_r(x_o)}(\lambda)|^2 d\sigma_\mu(\lambda) + \int_0^{\frac{d-1}{2}} |\widehat{\chi}_{B_r(x_o)}(i\lambda)|^2 d\sigma_\mu(i\lambda)\,.$$

The spherical transform $\widehat{\chi}_{B_r(x_o)}$ can be expressed in terms of a spherical function $\omega_\lambda^{(d+2)}$ for $\mathbb{H}^{d+2}$, and instead of providing a lower bound for averaging integrals of Bessel functions as in the Euclidean case, the core of the proof revolves around estimating means of the form

$$\frac{1}{r}\int_0^r \omega_\lambda^{(d+2)}(a_t)^2 \sinh(t)^{d-1} dt$$

from below. Such means turn out to exist in the limit $r \to +\infty$, see Proposition 6.11(2) in Paper I, and they are proportional to the square of the Harish-Chandra c-function for $\mathrm{SO}^\circ(1, d+2)$, $|c_{d+2}(\lambda)|^2$, on the principal series spherical functions ω_λ, $\lambda \in (0, +\infty)$. The quantity $|c_{d+2}(\lambda)|^2$ is positive for $\lambda \in (0, +\infty)$, and as in the Euclidean case we prove a slightly stronger statement of the form

$$\liminf_{r\to+\infty} \frac{1}{R}\int_0^R \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_{\mathbb{H}^d}(B_r(x_o))} dr \geq C_d \int_{\varepsilon_o}^\infty |c_{d+2}(\lambda)|^2 d\sigma_\mu(\lambda) > 0\,.$$

The proof of items (1) and (2) of Theorem 4.1.1 revolve around providing lower bounds on complementary series spherical functions $\omega_{i\lambda}$, which are strictly positive for $\lambda \in [0, \frac{d-1}{2}]$.

Lacking a canonical definition of geometric hyperuniformity, we define *spectral hyperuniformity* of an invariant locally square-integrable random measure μ on $\mathbb{H}^d$ in terms of vanishing of the Bartlett spectral measure σ_μ on $i[0, \frac{d-1}{2}]$ and sufficient decay near the *Harish-Chandra* Ξ*-function* $\Xi = \omega_0$, as mentioned in Definition 3.6.1. Moreover, stealth of μ is defined with the additional property of $\sigma_\mu((0, \varepsilon_o]) = 0$ for some $\varepsilon_o > 0$. In contrast to the Euclidean setting, there are examples of geometrically hyperfluctuating invariant random lattice orbits in $\mathbb{H}^d$, hence not spectrally hyperuniform either. However, we point out that Jenni in [16, Section 1.2, p.194] provides a cocompact lattice $\Gamma < \mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}^\circ(1, 2)$ for which the eigenvalues of the Laplace operator on the surface $\Gamma\backslash\mathbb{H}^2$ are strictly larger than 1/4, which corresponds to the Bartlett spectral measure σ_Γ having discrete support confined to $(0, +\infty)$. In particular, the associated invariant random lattice orbit μ_Γ is stealthy. This is however the only spectrally hyperuniform explicit random measure on real hyperbolic spaces that we are aware of at the moment.

In addition to the above we introduce i.i.d. perturbed invariant random lattice orbits $\mu_{\Gamma,\nu}$ for general proper separable metric homogeneous spaces as in Chapter 3 and prove that they are not spectrally hyperuniform in real hyperbolic spaces $\mathbb{H}^d$. The proof is similar to the determinantal case in Proposition 3.6.3.

4.2 Paper II

In Paper II we prove an analogue of Beck's Theorem for regular trees and provide invariant random lattice orbits whose number variances grows asymptotically slower than the volume of large metric balls along some subsequence of radii. We also define spectral hyperuniformity and stealth of invariant locally square-integrable point processes, similarly to the real hyperbolic setting. Moreover, we show that whenever the Bartlett spectral measure of an invariant locally square-integrable point process on the tree admits an atom at a "rational" spherical parameter, then the number variance grows asymptotically at least as fast as the volume of balls along every subsequence of radii.

For the $(q+1)$-regular tree $\mathbb{T}_{q+1}$ with $q \geq 2$, we show that the number variance of an invariant locally square-integrable (weighted) point process μ on $\mathbb{T}_{q+1}$ can be written in terms of the Bartlett spectral measure σ_μ as

$$\mathrm{NV}_\mu(r) = q^r \int_{\Lambda_q \setminus \{\frac{i}{2}\}} \frac{(\sin((r+1)\lambda\log(q)) + q^{-1/2}\sin(r\lambda\log(q)))^2}{\sin^2(\lambda\log(q))} d\sigma_\mu(\lambda)\,, \quad r \in \mathbb{N}_0\,,$$

where $\Lambda_q = i(0, \frac{1}{2}] \cup [0, \frac{\pi}{\log(q)}] \cup (\frac{\pi}{\log(q)} + i(0, \frac{1}{2}])$ is the parameterizing set for the positive-definite spherical functions for $\mathbb{T}_{q+1}$. A distinct difference from the Euclidean and hyperbolic settings is that for $\lambda = \frac{\pi}{\log(q)} + \frac{i}{2}$, the positive-definite spherical function

$$\omega_\lambda(g) = (-1)^{d(g.x_o, x_o)}$$

has modulus 1, and if σ_μ assigns positive mass to this spherical function then one shows that μ is maximally geometrically hyperfluctuating in the sense that

$$\limsup_{r\to+\infty} \frac{\mathrm{NV}_\mu(r)}{|B_r(x_o)|^2} = \frac{q-1}{q+1}\sigma_\mu(\{\tfrac{\pi}{\log(q)} + \tfrac{i}{2}\}) > 0\,.$$

There are interesting point processes μ that admit such an atom, so we prefer to regard this phenomenon as degenerate and instead consider the *sub-oscillatory number variance*

$$\mathrm{NV}^*_\mu(r) = q^r \int_{\Lambda_q^*} \frac{(\sin((r+1)\lambda\log(q)) + q^{-1/2}\sin(r\lambda\log(q)))^2}{\sin^2(\lambda\log(q))} d\sigma_\mu(\lambda)\,,$$

where $\Lambda_q^* = \Lambda_q \setminus \{\frac{i}{2}, \frac{\pi}{\log(q)} + \frac{i}{2}\}$.

The statement and proof of Beck's Theorem for invariant locally square-integrable (weighted) point processes in regular trees follows the same structure as the real hyperbolic analogue in Paper I, see Theorem 1.3 in Paper II. The main motivation for the paper is to provide an example of an invariant random lattice orbit μ_Γ on $(q+1)$-regular trees $\mathbb{T}_{q+1}$ for $q \geq 2$ satisfying

$$\liminf_{r\to+\infty} \frac{\mathrm{NV}^*_{\mu_\Gamma}(r)}{|B_r(x_o)|} = 0\,. \tag{4.2.1}$$

We emphasize that we do not have any example of an invariant point process in real hyperbolic space satisfying this. The lattices $\Gamma < \mathrm{Aut}(\mathbb{T}_{q+1})$ that we consider are realizations of fundamental groups of finite connected $(q+1)$-regular graphs $\mathfrak{X}$, where we think of the tree $\mathbb{T}_{q+1}$ as the universal simple covering graph of $\mathfrak{X}$ with the action of Γ by deck automorphisms. The Bartlett spectral measure of the associated invariant random lattice orbit μ_Γ is an atomic measure, and the sub-oscillatory number variance can be written as

$$\mathrm{NV}^*_{\mu_\Gamma}(r) = q^r \sum_{\lambda\in\Lambda^*_\Gamma} m_\Gamma(\lambda) \frac{(\sin(\log(q)\lambda(r+1)) + q^{-1/2}\sin(\log(q)\lambda r))^2}{\sin^2(\log(q)\lambda)}\,.$$

where $\Lambda_\Gamma^* \subset \Lambda_q^*$ is a finite set of spherical parameters corresponding to the eigenvalues of the adjacency matrix on $\mathfrak{X}$, and $m_\Gamma(\lambda) > 0$ are coefficients related to the multiplicity of the corresponding eigenvalue. We show that if $\mathfrak{X} = \mathfrak{K}_{q+2}$ is the complete graph on $(q+2)$ vertices with fundamental group $\Gamma = \pi_1(\mathfrak{K}_{q+2})$ acting by deck automorphisms on $\mathbb{T}_{q+1}$, then the invariant random lattice orbit μ_Γ satisfies 4.2.1. Moreover, we answer this question in the negative for complete bipartite graphs and in the positive for the 3-regular Petersen graph on 10 vertices. The proofs use (non-)equidistribution of spherical parameters in the unit circle similarly to the argument for the invariant random $\mathbb{Z}^5$-lattice in the Euclidean case.

Spectral hyperuniformity of invariant locally square-integrable point processes on $\mathbb{T}_{q+1}$ is defined as in Definition 3.6.1, and a spectrally hyperuniform point process μ is *stealthy* if the Bartlett spectral measure σ_μ completely vanishes near the ends of the principal series spectrum, in other words there is a $\varepsilon_o > 0$ such that

$$\sigma_\mu((0,\varepsilon_o]) = \sigma_\mu([\tfrac{\pi}{\log(q)} - \varepsilon_o, \tfrac{\pi}{\log(q)})) = 0\,.$$

With this definition we can show that the invariant lattice orbits coming from complete, complete bipartite and the 3-regular Petersen graph on 10 vertices are all stealthy.

Finally, we show that whenever the Bartlett spectral measure σ_μ of an invariant point process μ in $\mathbb{T}_{q+1}$ admits an atom in the subset of "rational spherical parameters",

$$\Lambda_q^{\mathrm{Rat}} = \begin{cases} (0, \frac{\pi}{\log(2)}) \cap \frac{\pi}{\log(2)}\mathbb{Q}\backslash\{\frac{3\pi}{4\log(2)}, \frac{5\pi}{12\log(2)}, \frac{11\pi}{12\log(2)}\} & \text{if } q = 2 \\ (0, \frac{\pi}{\log(3)}) \cap \frac{\pi}{\log(3)}\mathbb{Q}\backslash\{\frac{5\pi}{6\log(3)}\} & \text{if } q = 3 \\ (0, \frac{\pi}{\log(q)}) \cap \frac{\pi}{\log(q)}\mathbb{Q} & \text{if } q \geq 4\,, \end{cases}$$

then we must have that

$$\liminf_{r\to+\infty} \frac{\mathrm{NV}_\mu^*(r)}{|B_r(x_o)|} > 0\,.$$

This happens for example for invariant random lattice orbits of fundamental groups of complete bipartite graphs mentioned earlier. If $\lambda \in \Lambda_q$ is an atom of σ_μ, then

$$\begin{aligned} \frac{\mathrm{NV}_\mu^*(r)}{|B_r(x_o)|} &\asymp \int_{\Lambda_q^*} \frac{(\sin((r+1)\lambda\log(q)) + q^{-1/2}\sin(r\lambda\log(q)))^2}{\sin^2(\lambda\log(q))} d\sigma_\mu(\lambda) \\ &\geq \frac{(\sin((r+1)\lambda\log(q)) + q^{-1/2}\sin(r\lambda\log(q)))^2}{\sin^2(\lambda\log(q))} \sigma_\mu(\{\lambda\}) \end{aligned}$$

and the proof of the above statement reduces to determining rational solutions $0 < t < 1$ of equations of the form

$$\sin((r+1)t\pi) + q^{-1/2}\sin(rt\pi) = 0\,, \quad r \in \mathbb{N}_0\,.$$

The solutions to this equation are solutions to

$$\cos(2(r+1)t\pi) - q^{-1}\cos(2rt\pi) = 1 - q^{-1}\,, \quad r \in \mathbb{N}_0\,,$$

and the core ingredient for the proof is the following result of Berger in [8, Thm 1.2].

Theorem 4.2.1 (Berger). *Let $a_1, b_1, a_2, b_2 \in \mathbb{Z}_{\geq 1}$ such that $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$ and $\frac{a_1}{b_1} \pm \frac{a_2}{b_2} \notin \mathbb{Z}$. Then the following are equivalent:*

1. $\cos(\frac{a_1\pi}{b_1}), \cos(\frac{a_2\pi}{b_2})$ *are linearly dependent over* $\mathbb{Q}$,
2. $b_1, b_2 \leq 3$ *or* $(b_1, b_2) = (5, 5)$.

4.3 Paper III

Paper III is a joint work with Michael Björklund. In the paper we develop a general framework for studying invariant random measures on general proper separable metric commutative spaces. The main results of the paper are the following.

1. We prove the existence and uniqueness of Bartlett spectral measures σ_μ for invariant locally square-integrable random measures μ on proper separable metric commutative spaces $X = G/K$ as described in Section 3, a result which is used in Papers I and II.

2. For symmetric spaces $X = G/K$ realized as a quotient of a higher rank simple connected matrix Lie group G with finite center by a maximal compact subgroup $K < G$, we prove using techniques of Gorodnik-Nevo in [15] that there is a $\delta = \delta(G) > 0$ such that

$$\limsup_{r \to +\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_X(B_r(x_o))^{2-\delta}} > 0$$

   for any invariant locally square-integrable random measure μ on X.

3. We introduce the notion of *heat kernel hyperuniformity* for invariant locally square-integrable random (tempered) distributions μ on Euclidean and real hyperbolic spaces, and prove that this notion of hyperuniformity is equivalent to spectral hyperuniformity.

4. We compute Bartlett spectral measures for invariant determinantal point processes coming from equivariant kernels on commutative spaces. Using this we give a spectral proof of hyperuniformity for infinite polyanalytic ensembles of pure type, as well as prove heat kernel non-hyperuniformity of the random zero set of the standard Gaussian analytic function in the Poincaré disk.

For the existence and uniqueness of Bartlett spectral measures, we sketch the idea of proof here. We start out by letting μ be an invariant locally square-integrable random measure on a proper separable metric commutative space $X = G/K$ and consider the map $\alpha_0 : \mathscr{L}_c^\infty(G) \to L_0^2(\mu)$ sending a function $\varphi \in \mathscr{L}_c^\infty(G)$ to the statistic

$$\alpha_0(\varphi)(p) = \int_{X \times K} \varphi(s(x)k) d(p \otimes m_K)(x, k) - i_\mu \int_{X \times K} \varphi(s(x)k) d(m_X \otimes m_K)(x, k)\,,$$

where $s : X \to G$ is a *Borel section* (see Section 4 in Paper III) and m_K is the Haar probability measure on K. One verifies that if $\varphi_f(g) = f(g.x_o)$ for some $f \in \mathscr{L}_c^\infty(X)$ then

$$\mathrm{Var}_\mu(\mathbb{S}f) = \|\alpha_0(\varphi_f)\|^2_{L_0^2(\mu)}\,.$$

Such a map α_0 intertwines the left-regular representations on the respective function spaces in the sense that

$$\alpha_0(\varphi(g^{-1}\cdot))(p) = \alpha_0(\varphi)(g_*^{-1}p)\,,$$

for all $g \in G$. This provides us with positive-definite continuous functions

$$\eta_\varphi(g) = \int_{\mathscr{M}_+(X)} \alpha_0(\varphi)(g_*^{-1}p)\overline{\alpha_0(\varphi)(p)}d\mu(p)$$

on G for every $\varphi \in \mathscr{L}_c^\infty(G)$, and if $\varphi \in \mathscr{L}_c^\infty(G,K)$ is bi-K-invariant then η_φ is bi-K-invariant. Bochner's Theorem from Subsection 2.3.2 in the Euclidean setting was generalized to lcsc Gelfand pairs by Godement, see for example the proof in Barker's thesis [3, Theorem 2.5, p.58], which provides us with a spectral measure for each continuous positive-definite function η_φ.

Theorem 4.3.1 (Spherical Bochner's Theorem)**.** *Let $\eta \in C(G)$ be a positive-definite function. Then there is a unique positive finite Borel measure σ_η on the space $\mathcal{S}^+$ of positive-definite spherical functions such that*

$$\eta(g) = \int_{\mathcal{S}^+} \omega(g)d\sigma_\eta(g)$$

for every $g \in G$.

With this Theorem in mind we show using the Gelfand property that the measures $\sigma_\varphi := \sigma_{\eta_\varphi}$ for bi-K-invariant $\varphi \in \mathscr{L}_c^\infty(G,K)$ satisfy

$$|\widehat{\varphi}_1(\omega)|^2 d\sigma_{\varphi_2}(\omega) = d\sigma_{\varphi_1 * \varphi_2}(\omega) = d\sigma_{\varphi_2 * \varphi_1}(\omega) = |\widehat{\varphi}_2(\omega)|^2 d\sigma_{\varphi_1}(\omega)$$

as measures on $\mathcal{S}^+$ for every $\varphi_1, \varphi_2 \in \mathscr{L}_c^\infty(G,K)$. Lastly, an argument that can be found in Loomis' book [21, Section 26J-26K, p.98-100] yields a unique positive locally finite Borel measure σ_μ on $\mathcal{S}^+$ such that $d\sigma_\varphi(\omega) = |\widehat{\varphi}(\omega)|^2 d\sigma_\mu(\omega)$ as measures on $\mathcal{S}^+$. We then conclude for $\varphi_f(g) = f(g.x_o)$ as before that

$$\mathrm{Var}_\mu(\mathbb{S}f) = \|\alpha_0(\varphi_f)\|^2_{L^2_0(\mu)} = \eta_{\varphi_f}(e) = \int_{\mathcal{S}^+} d\sigma_{\varphi_f}(\omega) = \int_{\mathcal{S}^+} |\widehat{\varphi}_f(\omega)|^2 d\sigma_\mu(\omega)\,.$$

We also point out that this proof has nothing to do with random measures. One can just as well start out with a G-equivariant linear map $\alpha : \mathscr{A}(G) \to \mathcal{H}$ between some sufficiently large convolution algebra $\mathscr{A}(G)$ of integrable functions and a unitary G-representation $(\pi, \mathcal{H})$ with non-trivial K-invariant vectors. In particular, this allows for an extension to invariant locally square-integrable random *distributions*.

For item (2) above, we give a vague sketch of the proof. The first idea is to bound the number variance $\mathrm{NV}_\mu(r) = \mathrm{Var}_\mu(\mathbb{S}\chi_{B_r(x_o)})$ in terms of the variance of a regularized statistic $\mathrm{Var}_\mu(\mathbb{S}(\chi_{B_r(x_o)} * \rho_\varepsilon))$ for some positive approximate identity ρ_ε on G. This can be done using an admissibility property of balls $B_r(x_o)$ under small perturbations of the radius [15, Prop. 3.15]. The second idea is to use the Bartlett spectral measure σ_μ to rewrite the variance in question as

$$\mathrm{Var}_\mu(\mathbb{S}(\chi_{B_r(x_o)} * \rho_\varepsilon)) = \int_{\mathcal{S}^+} |\widehat{\chi}_{B_r(x_o)}(\omega)|^2 |\widehat{\rho}_\varepsilon(\omega)|^2 d\sigma_\mu(\omega)\,.$$

The main tool in the proof is the following spectral gap property for higher rank Lie groups, see [14, Theorem 4.1].

Lemma 4.3.2. *Let G be a higher rank simple connected matrix Lie group and $K < G$ a maximal compact subgroup. Then there is a $q \in [2, +\infty)$ and a constant $\Omega_q > 0$ such that every non-trivial positive-definite spherical function $\omega \in \mathcal{S}^+ \backslash \{1\}$ satisfies*

$$\int_G |\omega(g)|^q dm_G(g) \leq \Omega_q .$$

By Hölder's inequality, $|\widehat{\chi}_{B_r(x_o)}|^2 \leq \mathrm{Vol}_X(B_r(x_o))^{2(q-1)/q}\Omega_q^{2/q}$, so that

$$\begin{aligned}\mathrm{Var}_\mu(\mathbb{S}(\chi_{B_r(x_o)} * \rho_\varepsilon)) &\leq \mathrm{Vol}_X(B_r(x_o))^{2-\frac{2}{q}}\Omega_q^{2/q} \int_{\mathcal{S}^+} |\widehat{\rho}_\varepsilon(\omega)|^2 d\sigma_\mu(\omega) \\ &= \mathrm{Vol}_X(B_r(x_o))^{\frac{2(q-1)}{q}}\Omega_q^{2/q}\mathrm{Var}_\mu(\mathbb{S}\rho_\varepsilon) .\end{aligned}$$

Lastly, one makes an explicit choice of ρ_ε and $\varepsilon > 0$ to arrive at a universal constant $\delta > 0$ and the conclusion

$$\limsup_{r \to +\infty} \frac{\mathrm{NV}_\mu(r)}{\mathrm{Vol}_X(B_r(x_o))^{2-\delta}} > 0 .$$

We define heat kernel hyperuniformity of a random tempered locally square-integrable random distribution μ on $\mathbb{R}^d$ to mean that

$$\lim_{t \to +\infty} t^{d/2}\mathrm{Var}_\mu(\mathbb{S}h_t) = 0 ,$$

where $h_t(x) = (4\pi t)^{-d/2}\mathrm{e}^{-\frac{\|x\|^2}{4t}}$ is the classical heat kernel on $\mathbb{R}^d$. The following Proposition, which is proved in the Appendix of Paper III, illustrates a general correspondence between spectral hyperuniformity and the decay of variances for linear statistics of radial Schwartz functions on $\mathbb{R}^d$.

Proposition 4.3.3. *Let $F : \mathbb{R} \to \mathbb{R}$ is an even Schwartz function which does not vanish on $[-1, 1]$ and let $\sigma \in \mathscr{M}_+((0, +\infty))$ be a tempered measure. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\sigma((0, \varepsilon])}{\varepsilon^\alpha} = 0$$

if and only if

$$\lim_{s \to +\infty} s^\alpha \int_0^\infty F(s\lambda)^2 d\sigma(\lambda) = 0 .$$

In particular, taking $\alpha = d$, $\sigma = \sigma_\mu$ the (radial part of the) Bartlett spectral measure and $F(\|\xi\|) = \widehat{h}_1(\xi) = \mathrm{e}^{-\|\xi\|^2}$ for $x \in \mathbb{R}^d$ allows us to show that heat kernel and spectral hyperuniformity are equivalent. This Proposition also applies to Bartlett spectral measures for certain random distributions on real hyperbolic spaces, and we show that spectral hyperuniformity is equivalent to the following notion of heat kernel hyperuniformity,

$$\lim_{t\to+\infty} t^{3/2}\mathrm{e}^{\frac{(d-1)^2}{2}t}\mathrm{Var}_\mu(\mathbb{S}h_t)\,,$$

where the heat kernel is the bi-SO(d)-invariant function

$$h_t(g.x_o) = \int_0^\infty \mathrm{e}^{-t(\frac{(d-1)^2}{4}+\lambda^2)}\omega_\lambda(g)\frac{d\lambda}{|c_d(\lambda)|^2}\,.$$

for $g \in \mathrm{SO}^\circ(1,d)$.

For invariant determinantal point processes μ_L on commutative space $X = G/K$ as before coming from equivariant kernels as in Section 3.3, we prove that whenever the integrability condition

$$\int_X |L(x,x_o)|^4 dm_X(x) < +\infty$$

holds, then the Bartlett spectral measure of μ_L is given by

$$d\sigma_{\mu_L}(\omega) = (L(x_o,x_o) - \widehat{\kappa}_L(\omega))d\sigma_{\mathrm{Poi}(m_X)}(\omega)\,.$$

Here, $\sigma_{\mathrm{Poi}(m_X)}$ is the Bartlett spectral measure on $\mathcal{S}^+$ of the invariant unit intensity Poisson point process on X and $\kappa_L(g) = |L(g.x_o,x_o)|^2$. For the infinite polyanalytic ensembles $\mu_{2\pi,n}$ of pure type on the complex plane $\mathbb{C} = \mathbb{R}^2$ from Section 2.1.2 we observe that

$$\widehat{\kappa}_{2\pi,n}(z) = \mathcal{L}_n(\pi|z|^2)^2\mathrm{e}^{-\pi|z|^2}\,,$$

where $\mathcal{L}_n$ is the n'th Laguerre polynomial. In particular, $\widehat{\kappa}_{2\pi,n}(0) = L(0,0) = 1$, so $\mu_{2\pi,n}$ is spectrally hyperuniform. Lastly, for the standard Gaussian analytic function on the unit disk, Peres and Virág famously proved in [22] that its random zero set is an SU$(1,1)$-invariant determinantal point process with respect to the *Bergman kernel*

$$L(z_1,z_2) = \frac{1}{\pi(1-z_1\overline{z}_2)^2}\,,\quad z_1,z_2\in\mathbb{D}\,.$$

We prove that this point process is not spectrally hyperuniform using a calculation by Unterberger and Upmeier in [25, Prop. 3.39, p.591]. Alternatively, this follows from some modifications of L to produce an orthogonal projection, followed by Proposition 3.6.3 in Section 3.6.2.

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