

ACTA UNIVERSITATIS GOTHOBURGENSIS
GOTHENBURG STUDIES IN EDUCATIONAL SCIENCES 515

Powerful Vocational Mathematics Knowing

Hanna Knutson



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Abstract

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The aim of this thesis is to illuminate vocational mathematics knowing within the context of building and construction-work education. The overall purpose is to contribute to the understanding of how mathematics, as manifested in vocational practice, can be articulated in educational contexts in ways that acknowledge both conceptual and contextual depth

The thesis comprises three studies. The first, a phenomenographic interview study, explores mathematics teachers' and vocational teachers' various perceptions of educationally significant vocational mathematics knowing. The findings indicate that teachers, in different ways and to varying degrees, recognise the complexity of integrating aspects spanning a variety of domains. Teachers further describe mathematics as an essential component of students' holistic vocational 'Bildung', illuminating values such as critical thinking, confidence, creativity, and adaptability.

The second study aims to further explore how this kind of knowing is explicitly expressed in vocational problem solving. The empirical data consists of teachers' descriptions of the understanding required by students when approaching such problems, together with students' own discussions while working on construction-related mathematical tasks. Drawing on the conceptual framework developed within the phenomenography and variation theory research tradition (VITL), the results identify a set of critical aspects that are necessary to discern in order to understand and engage with mathematically rich construction-work situations in powerful ways. The study concludes that handling such problems requires coordinating both theoretical and practical forms of knowledge, originating from mathematics, other disciplinary areas, and vocational practice. The findings further highlight the importance of addressing both context-specific elements and conceptually coherent knowledge structures within vocational education.

Building on insights from the second study, a vocationally relevant mathematical concept (similarity) was selected for further analysis in the third study. This study aims to illuminate learning opportunities afforded in mathematics textbooks for Grade 10, using variation theory as an analytical tool. According to variation theory, students' opportunities to learn depend on patterns of variation. The analysis reveals that the textbooks make different sets of aspects possible to discern. While one textbook provides organised opportunities to practice a specific procedure for solving standardised tasks, another also makes it possible to discern essential aspects of the similarity concept.

Finally, when all three studies are considered together, the thesis argues that the learning opportunities afforded in mathematics textbooks for vocational students, particularly when the focus remains on delimited procedural skills and standard tasks, are insufficient for developing the kind of conceptual flexibility and contextual awareness that characterise powerful vocational mathematics knowing, as described and exemplified throughout the thesis.

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1. Introduction

In the field of construction-work, mathematics is widely recognised as playing a significant role (e.g. Boistrup et al., 2018; Moreira & Pardal, 2012; LaCroix, 2014; Saló i Nevado & Pehkonen, 2018; Zevenbergen & Zevenbergen, 2009). Despite this, the question of how mathematics should be understood and used within vocational activities, and consequently how it should be treated in vocational education, remains an important issue to be further explored (Bakker, 2014; FitzSimons & Boistrup, 2017; Straesser, 2015).

National reports indicate that failure in mathematics is one of the most common reasons why vocational students do not complete upper secondary education (Swedish National Agency for Education, 2024), and that mathematics is frequently perceived by students as abstract, theoretical, and disconnected from vocational experience (Swedish School Inspectorate, 2017). Assumptions about vocational students' lack of interest in theoretical tasks may in turn lead teachers to lower their ambitions and simplify the content (Rosvall et al., 2017; Swedish School Inspectorate, 2014). At the same time, employers argue that vocational students often lack sufficient mathematical knowledge to meet the demands of the workplace (Gallup, 2025; Muhrman, 2016; TYA, 2015). There is thus an evident risk that vocational students leave vocational education insufficiently prepared for the mathematical demands of vocational practice and that, at the same time, they come to perceive mathematics as an obstacle along the pathway towards vocational qualification rather than as an essential resource for vocational competence development.

For this reason, scholars emphasise the importance of clarifying the connections between mathematics taught in school and mathematics used in workplace settings, both in order to enhance students' motivation to learn mathematics and to promote their ability to discern and apply mathematics in vocational contexts (e.g. Bakker, 2014; Dalby et al., 2016; FitzSimons, 2014; Muhrman, 2016; Straesser, 2015). In Sweden, the mathematics syllabus for vocational students expresses such an intention (Swedish National Agency for Education, 2025). However, research within the field frequently highlights the difficulties, particularly for mathematics teachers without vocational experience, of

identifying mathematics in workplace activities and introducing workplace relevance into the classroom (Bessot, 2000; FitzSimons, 2014; Lindberg & Grevholm, 2013; Nicol & Crespo, 2002; Sundtjønn, 2021).

One reason for these difficulties is that mathematics used in workplace activities is intertwined with vocational practice, making it hard to discern from the vocational context (Christidis et al., 2024; Hoyles et al., 2010). Moreover, mathematical demands vary widely between workplaces, implying that such competencies must be investigated with close attention to the characteristics of specific vocations (Straesser, 2015). Such multifaceted, vocation-specific, and occasionally tacit forms of knowing are difficult to identify and articulate and therefore inevitably pose significant challenges for teaching and learning (FitzSimons, 2014; FitzSimons & Boistrup, 2017). It has also been pointed out that the area has received relatively limited attention within educational research partly due to this complexity (Bakker, 2014). It is therefore urgent to further investigate, not only how vocational mathematics should be taught, but also the content and nature of the knowledge to be taught (Straesser, 2015; FitzSimons & Boistrup, 2017). Against this background, the present thesis aims to explore what constitutes significant vocational mathematics knowing within a construction-work educational context.

Previous studies on vocational mathematics spans a broad research area ranging from studies of mathematics as used in workplace practices to studies of mathematics teaching in vocational education (e.g. Bakker & Akkerman, 2019; Dalby, 2016; Frejd & Muhrman, 2025; Hoyles, et al., 2001; LaCroix, 2014; Moreira & Pardal, 2012; Noss et al., 2000; Roth, 2014; Sundtjønn, 2021). At the same time, the field is complex and somewhat fragmented, as it draws on perspectives from several areas of research, and as studies often address different vocations, diverse educational systems, and different aspects of the relationship between mathematics, workplace practice, and vocational education. According to Bakker (2014) research on workplace mathematics, on vocational education, and on mathematics education have often developed in parallel rather than through studies of their interrelations. More recent research has contributed significantly to addressing this gap by focusing on the connection between mathematics in vocational practice and in education (e.g. Bakker & Akkerman, 2019; Frejd & Muhrman, 2020, 2025; FitzSimons & Björklund-Boistrup, 2017; Sundtjønn, 2021). At the same time, these studies consistently highlight that the interrelation between these domains still remains insufficiently understood.

The present thesis is positioned at the intersection of mathematics, vocational practice and education. It focuses specifically on vocational mathematics knowing as a form educational content in its own right and seeks to illuminate different ways in which such knowing may be understood and articulated within upper secondary construction-work education. In this sense, the study contributes to a broader picture of vocational mathematics knowing and the ways in which vocationally relevant mathematics may be conceptualised in education. It thereby responds to calls in the literature for further research clarifying how vocationally relevant mathematical knowing can be understood in relation to vocational practice and educational contexts, as well as to arguments emphasising the importance of teachers understanding the rationale underlying vocationally connected mathematics tasks (Bakker, 2014; FitzSimons & Björklund-Boistrup, 2017; Straesser, 2015; Sundtjønn, 2021).

Vocational mathematics is understood in this thesis as a distinct form of knowing with its own structures, demands, and educational significance. It is neither seen merely as formal mathematics applied in workplace situations nor as an implicit and tacit component of practical activity, but as a significant educational content in its own right. The notion of *powerful* vocational mathematics knowing, as expressed in the title of the thesis, is used to indicate that such knowing can be realised in qualitatively different ways. Accordingly, the study does not aim to define a minimal set of required skills, but rather to consider variation in ways of perceiving vocational mathematics knowing. In this sense, the thesis seeks to contribute to reflections on what powerful vocational mathematics knowing may entail and to discussions about the conditions under which such knowing might be developed in vocational education.

The thesis focuses on the didactical *what*-question. This is done through the lens of phenomenography and variation theory (VTI). Variation theory provides a distinctive approach to educational research by directing attention to objects of learning and the conditions under which *critical aspects* of that content may become possible to discern. Rather than focusing merely on teaching methods, learners' psychological processes, or the content as such, variation theory focuses on changes in learners' relation to what is being learned and on how such changes may be initiated through teaching (Marton, 2015). Approaching the object of research from this perspective provides tools for analysing educational phenomena in terms of what needs to be discerned and for discussing how the conditions for such discernment are constituted. By applying a theoretical perspective that has rarely been used to study vocational mathematics, the study contributes by opening

up new perspectives and insights, allowing new reflections to be made and new conclusions to emerge.

The thesis comprises three complementary studies that approach the object of research from different but related vantage points. Study 1 is a phenomenographic interview study that investigates mathematics teachers' and construction-work teachers' perceptions of what constitutes educationally significant vocational mathematics knowing, that is, the mathematical knowing that should be foregrounded and developed in education in order to support vocational knowing. Study 2 takes authentic workplace tasks as its point of departure and provides a concrete account of critical aspects associated with the integrated forms of knowing described in Study 1. Study 3, informed by the findings of Study 2, focuses on the learning opportunities afforded for a central mathematical concept, similarity, in vocational mathematics textbooks for Grade 10. The study provides an in-depth illustration of how specific aspects, of relevance for vocationally oriented task solving, may be made possible to discern in mathematics textbooks.

Taken together, the three studies do not merely examine isolated empirical instances but contribute to a broader understanding of vocational mathematics knowing. In the concluding discussion, insights from the three studies are synthesised and related to previous research in order to substantiate the concluding argument of the thesis concerning the conditions under which powerful vocational mathematics knowing may be developed. In this way, the thesis addresses the initial question of how mathematics should be understood and used within vocational activities by clarifying what constitutes vocationally relevant mathematical knowing and how it may be realised in vocational education.

Aim and research questions

The overall aim of this thesis is to illuminate and map educationally significant vocational mathematics knowing, by answering the following questions:

1. *In what qualitatively different ways do teachers perceive 'educationally significant vocational mathematics knowing'?*
2. *Which critical aspects do students need to discern in order to understand and engage with mathematically rich, construction-work tasks, in proficient ways?*
3. *What different learning opportunities are afforded through the ways in which similarity is presented in mathematics textbooks for vocational students?*
4. *How can powerful vocational mathematics knowing be understood within a construction-work educational context?*

How the research questions are addressed in the thesis

The research questions are addressed through the three studies and their synthesis. Together, the studies explore different but related aspects of educationally significant vocational mathematics knowing, which are brought together in addressing the fourth research question.

The first study is an interview study aiming to provide a multifaceted description of vocational teachers' and mathematics teachers' perceptions of 'educationally significant vocational mathematics knowing'. The study addresses the *first research question*. This question is explored using a phenomenographic approach, which focuses on variation in perceptions of the phenomenon. The findings are presented in terms of an outcome space consisting of categories that describe the qualitatively different ways in which teachers perceive of vocationally relevant mathematical knowing, and more specifically, how they experience what counts as meaningful mathematical knowing in education aimed at preparing students for competent professional action. The study both contributes to a deeper understanding of the phenomenon from a second-order perspective, and makes visible the variation in conceptions represented among teachers.

The second study complements the first by exploring how this kind of knowing is explicitly expressed in vocational task solving through an analysis of construction-related tasks. It thereby addresses the *second research question*. The empirical data consist of teachers' descriptions of the mathematical understanding required by students in vocational situations, together with students' own discussions while working on mathematically rich construction tasks. The results identify a set of critical aspects that need to be discerned in order to understand and engage with such tasks in proficient ways. While this study provides a more fine-grained and explicit account of vocational mathematics knowing than Study 1, it is limited by the fact that the identified critical aspects are tied to a specific vocational situation and to what is required to solve particular tasks. The first study, by contrast, allows for a broader perspective on what counts as relevant mathematical knowing across situations at a more general level. In this sense, the two studies complement each other.

The third study investigates learning opportunities in mathematics textbooks in relation to a mathematical concept that has been shown, in previous research as well as in the second study, to be relevant in vocational contexts, namely the concept of similarity. The study addresses the *third research question*. Rather than

focusing on the broader range of critical aspects identified in the second study, the analysis centres on a specific mathematical object of learning. This delimitation enables a systematic and in-depth comparison of learning opportunities in vocational mathematics textbooks in terms of the mathematical structure of a vocationally relevant concept. Since vocationally specific aspects are closely tied to particular practices and situations, they are less accessible to analysis at the level of textbooks. Moreover, previous research has repeatedly questioned whether teaching materials and classroom teaching have the capacity to adequately represent the full complexity of vocational relevance (e.g. Frejd & Muhrman, 2020; Coben & Weeks, 2014). By instead focusing on similarity, a concept that is both mathematically well-defined and has demonstrated vocational relevance, it becomes possible to conduct a meaningful analysis of mathematics textbooks while retaining relevance to vocational practice. This makes it possible to identify different ways in which textbooks may contribute to creating conditions for students to develop vocationally relevant mathematical knowing. Although learning a mathematical concept in itself is not sufficient for developing such knowing, as highlighted in Study 2, it constitutes a significant component.

By synthesising the outcomes of the three studies together with arguments derived from previous research and the theoretical framework, the thesis addresses the *fourth research question*. It thereby seeks to contribute to a more nuanced, fine-grained and in-depth understanding of vocationally relevant mathematical knowing. In doing so, it aims to provide a basis for discussing what powerful vocational mathematical knowing may entail and the conditions for its enactment within education.

2. Background

The Swedish educational system

A brief introduction to the Swedish educational system, with particular attention to the organisation and purpose of vocational programmes at the time when the thesis was conducted, is presented below. In addition, since older vocational mathematics textbooks are included as part of the empirical material, a brief historical overview of the role of mathematics in vocational education over the past century is provided. Finally, a short account of the ongoing discussion concerning the role of general subjects, particularly mathematics, within Swedish vocational education is presented in order to further contextualise the study's findings and arguments.

Education in Sweden is compulsory for ten years, from the age of six to sixteen. After completing compulsory school, students can choose to continue to upper secondary education, which is not mandatory but attended by the vast majority of young people. Upper secondary education in Sweden, known as 'gymnasieskolan', usually lasts for three years and offers both academic and vocational pathways. There are 18 national upper secondary school programmes, six of which prepare for higher education and 12 for vocational diploma. Students choose a programme based on their interests and future career plans (Swedish National Agency for Education, 2021). After a reform Gy 11 students at the vocational programmes are no longer automatically eligible for university studies, as they were between 1994 and 2011. However, provision to achieve basic eligibility for higher education could be obtained by taking additional courses. The twelve vocational national programmes focus on preparing students for specific vocational fields, for example: building and construction, vehicle and transport, health and care, hotel and tourism, etc. For each vocational programme there are national councils that link the programme content more closely to the world of work.

The Building and Construction Programme, which this thesis focuses on, aims to prepare students for professions in the construction sector, such as construction-worker, machine operator, painter, or sheet-metal worker. According to the programme objectives (Swedish National Agency for Education, n.d.), the

intends to develop students' knowledge and skills in construction, renovation, and infrastructure work, with an emphasis on safe, efficient, and environmentally sustainable building practices. The education also includes elements of entrepreneurship, planning, documentation, and quality assurance, as construction-work involves considerable economic responsibility. The construction-workplace is characterised by considerable variation in environments, materials, and processes. New construction methods, materials, tools, and technologies are continuously introduced, requiring workers to adapt to changing conditions. Such diverse and dynamic context demands adaptability, creativity, and a commitment to lifelong learning. Consequently, the education aims to foster students' initiative, problem-solving ability, and capacity for both independent action and flexible collaboration. Mathematics plays an important role in the programme, both as a general subject and as an integrated component of vocational competence (Swedish National Agency for Education, n.d.).

Vocational education and mathematics, historical overview

Vocational education in Sweden, and the role of mathematics within it, have changed over the past century. In the late 19th century, relatively general vocational education was followed by more specific training in the labour market.

In 1921, a vocational school charter was introduced to better organise vocational education and training (Hedman, 2001). It stated that education should include vocational theory which, depending on the profession, could involve arithmetic and geometric problem solving, drawing, and elements of mechanics, physics, and chemistry (Swedish Code of Statutes, 1921, 2 § 7). A further reform was implemented in 1955 (Swedish Code of Statutes, 1955). Mathematics textbooks, dedicated to these schools, could be found in library collections (see empirical material for Study 2). These textbooks include a rather comprehensive mathematical content, chosen based on vocational demands. The number of editions indicate that those textbooks were frequently used within vocational education at the time they were published.

In 1971, a uniform upper secondary school was introduced in Sweden, which was regulated by the Lgy70 curriculum (National Board of Education, 1970). The reform resulted in a unified upper secondary school with a variety of educational pathways, where the former vocational schools constituted the vocational technical tracks. The teaching of vocational subjects was regulated by detailed timetables. Mathematics was not included as a mandatory course at the vocational tracks but

could be taken as an optional course. The subject ‘vocational mathematics’ thus seems to have disappeared from vocational education as an independent subject at that time. However, the vocational subjects often contained substantial theoretical components that required mathematical understanding (National Board of Education, 1970).

In 1994 all programmes in upper secondary education, including vocational programmes, became three years long and provided access to university studies (Swedish National Agency for Education, 1994). A common course in mathematics (Ma A), then became mandatory for all students across all the upper secondary programmes, academic as well as vocational. This course was later divided into Mathematics 1a, 1b and 1c when a new curriculum was introduced in 2011 where the course given at the vocational programmes is Mathematics 1a (Swedish National Agency for Education, 2021).

The curriculum for the compulsory mathematics course at the vocational programmes, (Ma 1a)

The overall aim of the course is to enable students to work mathematically, i.e. to understand concepts and methods, develop problem-solving strategies, and use mathematics in vocational, social and life contexts. Instruction should foster creativity, highlight mathematics’ significance in other contexts, and strengthen students’ confidence. Students should thus become capable of understanding concepts, handling procedures and standard tasks, solving problems, formulating and evaluating models, conducting and following mathematical reasoning, and communicating mathematics. Mathematics 1a covers the following central themes (see Appendix 1):

- **Mathematics in vocational subjects and working life.** Including mathematical concepts relevant to working life, (e.g. proportionality, scale, similarity, angles, the Pythagorean theorem, percentages and ratios, index measures, profit margins, equilibrium, margins of error, symmetries, vectors, trigonometric functions), calculation and estimation methods, handling formulas, measurement/units (incl. errors and uncertainties), and practical tools (forms, templates, manuals, reference works).
- **Arithmetic, algebra, functions.** Including algebraic expressions (incl. factorisation), the idea of a function and its various representations

(algebraic expressions, tables and graphs), methods for solving linear equations, linear vs. exponential functions, change factors and multistep changes.

- **Probability & statistics.** Including independent, dependent and complement events, multi-step probabilities and the use of statistical ideas (significance, correlation, causation, sampling, error sources) in society and work.
- **Problem-solving, tools & applications.** Including spreadsheets for interest/amortisation; digital tools to support calculations and equation solving; modelling realistic situations; applications in working life, personal finance and society; evaluating models; problems linked to mathematics' cultural history.

The role of mathematics in current vocational programmes

The reform of Upper secondary school in Sweden Gy11 redefined the relationship between vocational and academic pathways. In this reform, vocational education became more distinctly separated from the higher education preparatory programmes. As a result, students attend mathematics courses at different levels depending on the programmes they enrol in. The mathematics course for vocational programmes Ma 1a corresponds to the lowest academic level. However, in addition to a general mathematical content, the Ma1a curriculum includes a potentially comprehensive vocation-specific part, encompassing concepts, and methods that are relevant to the particular vocation (see above). The proposal was based on evidence showing that vocational students often failed to achieve the learning goals in mathematics, and that they perceived mathematics teaching as abstract, overly theoretical, and disconnected from the development of vocational competence. However, although the syllabus emphasises vocational integration in mathematics teaching, this does not seem to be applied to any great extent, possibly due to limitations in teacher competence, curriculum materials, collaboration opportunities, and to the fact that the time allotted to the course Ma1, is not enough to accommodate both a breadth of general knowledge and vocational applications Lindberg and Grevholm (2013). Therefore, many students still report low motivation and a sense alienation toward the subject (Muhrman, 2016). At the same time, Muhrman (2016) points to the fact that employers frequently seem to find students' mathematical knowledge too poor for them to cope with work-specific calculations. Discussing the implications of Gy11

Lindberg and Grevholm (2013) argue that a narrower and more program-specific mathematics curriculum for vocational students, have weakened the academic and general content of mathematics in vocational education, which could be seen as a democratic threat. At the same time, the reform seems to have failed to afford genuine vocational relevance. Consequently, mathematics teaching at the vocational programmes became both less rigorous and insufficiently contextualised, through the reform, leaving students with limited access to higher education and little sense of how mathematics connects to their future vocations, according to Lindberg and Grevholm (2013). Another perspective advanced by Nylund et al (2018) draws attention to the prevailing perceptions, among teachers, students and parents, that vocational education is simple, with a low academic status. Something that, despite the fact that many vocational subjects require advanced mathematics and theoretical reasoning, leads to low expectations on vocational students, resulting in a circle of negative perceptions and expectations, which acts as a self-fulfilling prophecy. This is confirmed in Rosvall's (2017) study which shows that mathematics teachers' low expectations are reflected in less teacher-led instruction in vocational education compared to other programmes, and in a focus on procedural approaches to solving basic tasks, aimed at helping students pass the national tests in the simplest possible way. Challenges thus seems to remain regarding mathematics teaching at vocational programmes. Attempts to deal with certain identified problems, such as low motivation and perceptions of irrelevance, have not yielded the desired results and the reform of 2011 seems to have led to a lower academic level while the increased vocational relevance that has been sought, has not been fully achieved.

Concepts

Context

The concept of context is used and understood in different ways across educational research. Johansson (2015) identifies several uses of the concept in research literature. Context may refer to the situation in which knowledge is used, for example in school or in out-of-school contexts. Different school subjects may also be understood as distinct contexts, in relation to which students' expectations of how to approach the content may vary. Furthermore, context may refer to the background information that students are expected to draw on when engaging with a task, or to the domain or situation that the mathematical relationship is

intended to describe. The concept of context may also refer to both the situations in which knowledge is enacted and to those in which it is developed (Johansson, 2015).

Furthermore, different assumptions are made, in educational research, about the role of the context. According to some perspectives, context is understood primarily as a setting in which knowledge is applied, but not as part of the knowledge itself. This view is reflected in early research on transfer, where learning is seen as the acquisition of elements or principles that easily can be applied across contexts depending on their similarity (Thorndike, 1913; Judd, 1908). In contrast, sociocultural and situated perspectives conceptualise context as constitutive of knowledge and action. From this point of view, knowledge is understood as situated in practice and impossible to separate from the contexts in which it is used and made meaningful (Lave, 1988; Lave & Wenger, 1991). According to these perspectives, context is not external but forms an integral part of what is known. Several other perspectives likewise reject a fixed and static view of context and instead conceptualise it as dynamic and relational, where meaning is constructed through interactions between individuals and their contexts (Damşa et al., 2025).

VTL similarly adopts a dynamic view of context, in that the meaning of a phenomenon is understood as shaped by the context in which it is experienced (see Chapter 5). At the same time, context is defined in relation to the phenomenon as a background that is not in focal awareness and is therefore not considered part of the experienced phenomenon. The notion of context in VTL is thus also dynamic in terms of how the boundary between figure and ground is constituted. What constitutes the phenomenon and what constitutes the context is not fixed but depends on the focus of awareness and may therefore shift over time. As the focus changes, aspects that were previously part of the context may instead become part of the phenomenon (Booth, 1992).

In this thesis context is understood, in line with a VTL perspective, as constitutive of how phenomena are made meaningful. In contrast to how context is defined in VTL, however, context is seen here as a relatively stable framework that both shapes how a phenomenon is understood and that may also constitute a significant component of the experienced phenomenon. This means that it is not external to knowing but may form an essential part of the object of learning. Moreover, context refers not only to the physical setting, but more broadly to the framing conditions that are relevant for how something is made meaningful. When referring specifically to a limited physical environment, the term *setting* is used.

In this thesis, context is relevant at different levels. Vocational mathematical knowing is understood as mathematics within a *vocational context*, and the research focus is, in turn, directed to how such knowing is constituted in relation to an *educational context*.

Vocational

The term *vocational* is used to refer to a domain of knowledge and practice associated with skilled work. It encompasses both the practices through which such work is carried out and the ways in which knowledge is constituted, understood, and enacted within these practices, as well as in educational contexts preparing for them. This way of using the concept aligns with FitzSimons' (2014) definition: 'Vocational mathematics education by definition implies an automatic engagement of mathematics education with the world beyond the institution of formal education. The term *vocational* is taken to represent paid skilled work in our globalised economy' (FitzSimons, 2014 p 292). The present thesis primarily considers vocations that are typically accessed through upper secondary vocational education programmes. The term *workplace* is used to refer to the concrete settings and practices in which vocational work is carried out.

Mathematics

Mathematics is understood in this thesis as a system of premises, concepts and logical relations through which conclusions are drawn. Such logical structures can be conceptualised in different ways depending on underlying epistemological assumptions. From a realist perspective, logic is understood as a property of the world, reflected in the regularities and relations that structure reality (e.g. Devitt, 1991; Shapiro, 1997). In contrast, logic may be understood as arising from the ways in which experience is structured by the mind, giving rise to patterns and perceived regularities (e.g. Kant, 1781/1998). From a VTL perspective (see Chapter 5), logic is not understood as located solely in the world or in the mind, but in how the world is experienced and made meaningful. From this perspective, logic as structure and as meaning are not separate, but mutually constitutive within experience. Likewise, mathematics may be understood as located in the world, in the mind, or in the relation between the individual and the world, depending on the epistemological perspective. This thesis adopts the last one. However, while stable premises, logical structure, patterns, and predictability are general features of human experience, not all such structures could be described as mathematical.

In this thesis, mathematics is understood as those premises and logical relations that can be related to established mathematical domains (such as arithmetic, algebra, and geometry), and that are recognised as mathematics by the practitioner or from a researcher's perspective. This means that mathematical relationships may be expressed in different ways across contexts, including locally developed and practice-based forms (cf. ethnomathematics, as described in Chapter 3).

The term mathematics is thus used in a rather broad sense. To be more specific, if needed, the term *formal mathematics* is used to refer to mathematics as a discipline, characterised by established formalised language and explicit systems of axioms and logical deduction. *School mathematics* refers to the ways in which mathematics is typically structured, taught, and assessed within educational settings. *Vocational mathematics* refers to mathematics as it is understood and enacted in vocational contexts. However, as vocational mathematics is typically deeply intertwined with the context, vocational mathematical knowing cannot be enacted apart from it. Since mathematics is defined in terms of logical relations that can be related to established mathematical domains, both formal mathematics and school mathematics are encompassed by this broader conception of mathematics. Vocational mathematical knowing, however, necessarily involves more than such relations alone. In practice, such knowing includes, and cannot be separated from, aspects of the vocational context within which the mathematical relations are experienced and enacted.

Knowledge and knowing

In a similar way to mathematics, knowledge can be understood as located externally, internally, or in the relation between the individual and the world (Marton & Booth, 1997). In phenomenographic research one reality is considered, namely the reality of which we are aware and in which we participate. Our knowledge about this reality is understood as being shaped through our experiences and as emerging in the relation between the individual and the world (Marton & Booth, 1997; Olteanu, 2007). To mark a departure from a view of knowledge as something existing independently of the knower and to be acquired by the learner, the term knowing is used. It emphasises knowledge as emerging in the relation between the knower and the known (Carlgren et al, 2015).

From a sociocultural perspective, knowledge can be understood as a form of participation in practice, where individuals engage with and relate to the world through activity (e.g. Lave & Wenger, 1991). While focusing on experience rather

than action, VTL aligns with this perspective in the sense that knowing is understood as formed in the relation between the knower and the known through the intentionality of experience. Moreover, how something is experienced is regarded fundamental to how it is *acted* upon. (Carlgren et al. 2015; Marton & Booth, 1997; Marton et al., 2004).

Marton further describes *learning* as developing of more powerful ways of experiencing, where ‘more powerful’¹ refers to ways of experiencing that correspond to more effective or functional ways of *acting*. In this thesis *knowledge* is used in a broader way across epistemological perspectives, while *knowing* is used to indicate the perspective adopted in this thesis, as described above. The term *powerful* indicates the extent to which such knowing constitutes meaningful relational engagement.

¹ The term powerful is not used here in the sense developed within the ‘powerful knowledge’ tradition (e.g. Young, 2008). Rather, it refers to the extent to which knowing enables meaningful engagement. While these uses stem from different theoretical traditions, they may be seen as broadly compatible. The present study seeks to contribute to further nuancing discussions of what counts as valuable knowledge and why, particularly in relation to vocational contexts.

3. Previous research

Vocational mathematics knowing

Vocational mathematics education can be described as a specialised but important subfield of mathematics education in which context is crucial and which by definition implies real-world engagement (FitzSimons, 2014). According to Bakker (2014), the area has been underrepresented in mathematics education research. One suggested reason is that vocational education differs significantly between countries, which impedes a coherent understanding of the field. Another, is that the mathematics used in workplaces is situated and embedded in practice, making it a tacit and elusive part of vocational proficiency and hard to differentiate from the surrounding work-place context. Bakker (2014) further argue that a lack of interest from policymakers in supporting research on vocational mathematics, may have contributed to the limited development and visibility of the area within mathematics education. These conditions point to the need for further research that can contribute to a more clearly articulated understanding of vocational mathematics.

Seminal research within the field, has highlighted the contrast between school mathematics and out-of-school mathematics and emphasised the importance of investigating the relations between the two (e.g. Lave, 1988; Carraher, 1985). Doing mathematics at work and doing mathematics at school are two very different activities, implying different purposes, goals and priorities (FitzSimons 2014, FitzSimons & Boistrup, 2017; Sundtjønn, 2021). Considering this gap, researchers highlight that the research agenda in the field have shifted over time, from the assumption that mathematical knowledge readily transfers across contexts to an emphasis on the fundamental differences between formal and workplace mathematics, underscoring the challenges of bridging this gap (Bakker, 2014; Dalby & Noyes, 2016; Wake, 2014).

In recent years, however, researchers have increasingly begun to think beyond such dichotomies, directing their attention to what happens at the boundary where vocation and education meet (e.g. Dalby & Noyes, 2016; FitzSimons & Boistrup, 2017; Frejd & Muhrman, 2020, 2025; Sundtjønn, 2021). The present thesis builds

on this line of research by further exploring how vocationally relevant mathematical knowing can be understood within educational contexts.

The tension between different practices with regard to aims and characteristics of mathematics does not, however, exist only between school and workplace settings. Lampert (1990) discusses the relation between school mathematics and mathematics as a discipline, describing school mathematics as typically presenting knowledge as fixed and predetermined; content to be received and reproduced. The discipline of mathematics, on the contrary, presents knowledge as reasoned and open to critique. To know mathematics in this sense entails being able to question, evaluate, and participate in mathematical argumentation (Lampert, 1990). According to d'Ambrosio (1985) every cultural group develops its own mathematical knowing, referred to as 'ethnomathematics'. This perspective aligns with a social constructivist view of mathematics, which, in contrast to a traditionalist view, argues that mathematics is a social activity, with cultural limitations to its claims of certainty, and with meaning and authority arising from human agreement, rather than pure abstraction alone (Ernest, 2018). Mathematical practices in workplace settings are often described as an example of such culturally embedded mathematics, where mathematical reasoning emerges from the demands, norms, and problem-solving traditions of particular vocational communities (Muhрман, 2016; Wedege, 2010). However, a common assumption among people in general is that school mathematics, which is regarded a proxy for the discipline, is the only valid mathematics, and that school-based methods are the only correct ones (FitzSimons, 2014).

Also, the aims of mathematics education have been a question of debate, reflecting diverse perspectives. Mathematics may be viewed primarily as a practical tool for work and everyday life, or as a means to describe and model phenomena in the world. It could also be appreciated for its intrinsic value in terms of the aesthetics of pure mathematics. Furthermore, mathematics learning can be seen as a means of achieving social justice and empowerment (Ernest, 2018). For vocational mathematics education, these debates highlight that mathematics teaching is never neutral, it involves value choices about what kinds of knowledge, skills, and identities are developed, and which and whose interest mathematics ultimately serves. The aims and meanings of mathematics are thus rarely given as fixed or uniquely defined entities, in mathematics education in general, and even less so, in vocational education. This further underscores the need to explore vocational mathematics as a form of knowing that may vary in its nature and potential.

Against this background, the following section reviews key research on vocational mathematics and the implementation of vocational mathematics education, focusing on how mathematical knowledge is understood, taught, and used across contexts. The research reviewed in this chapter is used to situate the present study within the field of vocational mathematics education and to highlight central themes and tensions relevant to the object of research. Concepts drawn from previous research are used in the review mainly to give a fair account of the field, but do not function as theoretical constructs in the analysis. The review is organised around a number of overarching themes relevant to the thesis, including mathematics in the workplace, mathematics teaching in vocational education, and the relationships between school and workplace mathematics.

Mathematics in the workplace

Content, level and required precision

Even though the utility of mathematics in workplace settings is frequently presented as a key argument for its place within vocational education, the questions of what mathematics is actually needed in the workplace, and how it should be understood and learned, are not easily answered in an unambiguous way (Bakker, 2014; Muhrman, 2016; Straesser, 2015). Discussing ‘numeracy at work,’ Straesser (2015) problematises the common assumption that mathematical knowledge in vocational practice is confined to basic arithmetic. In contrast, research across various vocational fields reveals a much broader range of mathematical competences used in workplace settings (e.g., Bakker, 2014; Hoyles et al., 2010; Johansson et al., 2024; LaCroix, 2014; Muhrman, 2016; Straesser, 2015). Apart from basic arithmetic skills, however, the required mathematical qualifications vary widely between workplaces. Research must therefore be conducted for each vocation specifically, with close attention to the characteristics of the workplace in question (Straesser, 2015).

Such studies have been made in various vocational areas, such as trade, banking, healthcare, laboratories and industry, revealing a diversity of competences as well as common characteristics (e.g. Coben et al., 2010; Hoyles et al., 2010; Johansson et al., 2024; LaCroix, 2014; Masingila, 2009; Muhrman, 2016; Wake, 2014). The mathematical competencies identified in this body of research encompass a broad spectrum of skills, ranging from fundamental to more advanced forms of reasoning. These include, for example, the ability to interpret and use mathematical diagrams, statistical representations, and technical drawings

(e.g., Hoyles et al., 2010; LaCroix, 2014), to perform percentage calculations, mental estimations, and measurements of quantity (e.g., Muhrman, 2016), and to understand and apply proportional relationships, mathematical formulas, and geometric concepts (e.g., Johansson et al., 2024; Masingila, 2009; Wake, 2014). Taken together, these studies give an indication of the vast variations between vocations regarding the type of mathematics used, how it is applied, and to what extent.

In some workplace activities, rapid and precise calculations are essential, as in nursing practice, where errors may have fatal consequences (Coben et al., 2010, 2014; Hoyles et al., 2001; Noss et al., 2000; Weeks et al., 2000). For example, Coben and Weeks (2014) describe medication dosage calculation as a safety-critical practice in which even small miscalculations may result in overdosing or underdosing, with potentially severe or fatal consequences for patients. In other workplaces, practical rules, visual estimations, and specially designed tools may replace exact calculations and formal concepts (e.g. Moreira & Pardal, 2012; Roth, 2014; Zevenbergen & Zevenbergen, 2009). For example, in an empirical study of how electrician apprentices learn to bend electrical conduits in college and on the job, Roth (2014) shows that the knowledge required for successful performance differs between educational and workplace settings. In the classroom, formal trigonometry constitutes the primary reference, whereas in workplace practice, bending is guided by practical rules and considerations related to functionality. As a result, trainee electricians tend to rely on apprenticeship-based techniques rather than the trigonometric methods taught in school. Similarly, both masons (Moreira & Pardal, 2012) and boat builders (Zevenbergen & Zevenbergen, 2009) apply procedures and rules without connecting these practices to underlying geometric principles. Also in business, research findings indicate that the focus is typically not primarily on mathematical generalisability. Instead, procedures and terminology are used routinely, and sometimes inconsistently, in relation to formal mathematics (Hahn, 2000). Taken together, these studies indicate that in workplace situations, what counts as appropriate strategies and procedures are determined by the requirements that practical conditions impose on the desired outcome. Solution strategies are often pragmatic, allowing specific problems to be solved efficiently within the particular situation in which they arise (Hoyles et al., 2010).

However, studies also indicate that even in contexts where simplified or local strategies may lead to acceptable results, more advanced levels of mathematical understanding can promote more efficient and flexible vocational problem-solving

(Saló i Nevado & Pehkonen, 2018) as well as more reliable interpretations and evaluations (Gravemeijer et al., 2017; Hoyles et al., 2010). Scholars argue that in a digital society, automation reduces the need for manual computation but increases the need for workers to understand, interpret and communicate outputs from technological systems based on an understanding of underlying concepts (Gravemeijer et al., 2017; Hoyles et al., 2010). Therefore, it is argued that the focus of vocational mathematics education must shift toward competencies that complement, rather than replicate, what machines do, shifting the educational focus from procedural execution to conceptual understanding (Gravemeijer et al., 2017; Hoyles et al., 2010). Boistrup et al. (2018) further emphasise that mathematics is not only involved directly in performing vocational tasks but may also contribute to more advanced ways of understanding the vocation itself.

Taken together, these findings suggest that the role of mathematics in vocational practice cannot be reduced to either basic procedures or advanced formal knowledge alone. Rather, what counts as relevant mathematical knowing, in terms of content, precision, and theoretical coherence, varies across vocations and situations. Moreover, the level and nature of mathematical understanding may influence how vocational tasks are approached, how situations are interpreted, and more broadly, how the vocational situation is understood. In this way, it may affect the extent to which individuals are able to act flexibly, make informed decisions, and approach novel situations in meaningful ways.

While the studies above highlight substantial variation in the content and meaning of mathematics, they also point to another important dimension: the extent to which mathematics is made visible in practice; that is, not only what mathematics is used, but how it appears, or does not appear, to practitioners is highlighted as a critical issue for understanding vocational mathematics knowing.

Visibility

FitzSimons and Boistrup (2017) emphasise that some vocational problems require explicit calculations, meaning that they involve mathematical activities which are recognised as mathematics by the practitioners. In other activities, mathematical reasoning is integrated implicitly into vocational practice and intertwined with professional evaluations. Such problem solving might include creativity, questioning and innovation, sometimes using a vocation-specific mathematical language, which might not even be perceived as mathematics. In such activities, advanced mathematical reasoning might be used even though it is not thought about as mathematics (e.g. Noss et al., 2000). Coben and Weeks (2014) further

illustrate that, in nursing practice, mathematical activity is often embedded in professional action rather than explicitly presented as mathematics. They argue that competence in such vocations involves the ability to identify, interpret, and use mathematically relevant information within a situation, rather than merely performing calculations. In this sense, vocational mathematical knowing includes the capacity to ‘see’ what is mathematically relevant in a task, even when it is not explicitly formulated as mathematics.

Mathematics might also be implicit in the sense that it is hidden in tools and technological systems, by means of which tasks could be solved without understanding the underlying mathematical rationales (Hoyles et al., 2010). Gravemeijer et al. (2017) highlight the paradox that, in a digital society, mathematics becomes increasingly important as it lies at the core of what computers do, while at the same time becoming hidden inside these technological systems. However, as described above, vocational activities often require an ability to handle and interpret symbolic information in terms of values, tables and graphs, which are the result of mathematical models within these systems, and to understand how systematic information can be used to predict and improve processes, an ability that can be enhanced by increased mathematical awareness (Bakker & Akkerman, 2014; Hoyles et al., 2010).

This tendency for mathematics to be obscured within workplace practices is sometimes referred to as black boxing. Williams and Wake (2007) describe how mathematics might be hidden in ‘black boxes’, meaning that workers might be unaware of the mathematical rationales behind applied technology and procedures. Two kinds of black-boxing processes are identified. The first relates to instruments and routines. The second relates to division of labour, where workers are isolated from mathematics through workplace rules, norms, and expectations (Frejd & Vos, 2025; Williams & Wake, 2007). However, even though mathematical reasoning might be more or less implicitly integrated into vocational practice, teaching can serve to make it explicit for students (Boistrup et al., 2018).

This suggests that vocational mathematics education should not be limited to reproducing workplace practices as they appear. Rather, it may also involve creating opportunities to identify, articulate, and reflect on the mathematical structures and rationales that underpin these practices. In this sense, teaching can contribute not only to supporting students’ participation in vocational activities, but also to expanding their ways of understanding and engaging with those activities.

Significance of the vocational context

The synthesis above shows that mathematics in different workplace practices may be more or less explicit, precise, and advanced. However, what is characteristic across vocational settings is the central role of context. Mathematical problems that arise in workplace activities typically involve reasoning that is embedded in, constrained by, and interpreted through the specific situations in which they occur (FitzSimons & Boistrup, 2017; Johansson et al., 2024; LaCroix, 2014; Martin & LaCroix, 2008). Such tasks may involve both explicit and implicit mathematical reasoning of varying complexity, always combined with context-specific knowledge, such as interpreting representations (e.g. drawings, diagrams, and labels), performing accurate measurements, using appropriate tools, and selecting suitable strategies (e.g. Boistrup et al., 2018; Coben et al., 2010, 2014; Johansson et al., 2024; LaCroix, 2014; Zevenbergen & Zevenbergen, 2009).

Discussing numeracy in nursing, Coben et al. (2010, 2014) emphasise that competent performance extends beyond procedural accuracy. It involves the ability to judge which mathematical approaches are appropriate and how they should be applied in relation to the situation. In this sense, mathematical competence is inseparable from professional judgement, decision-making, and strategies that are safe precisely because they are attuned to the demands of the environment. Similarly, Hoyles et al. (2001) show that nurses employ different proportional reasoning strategies shaped by professional knowledge, patient needs, and situational constraints. Comparable patterns are found in other vocational domains. Johansson et al. (2024), for example, show that in electricians' work, mathematical knowledge, such as the use of formulas, is integrated with vocational knowledge, including understanding material properties and technical requirements. In such cases, mathematical reasoning cannot be reduced to the correct application of formulas alone; it must be coordinated with vocational knowledge in order to produce viable and appropriate solutions.

Understanding vocational mathematics thus entails paying attention not only to mathematical content as such, but to the ways in which mathematics and vocational knowing are intertwined in practice. From this perspective, vocational mathematics may be understood as a form of knowing that takes shape in the interplay between mathematical reasoning and the contextual conditions of vocational activity

Mathematics in construction-work vocations

Some studies have focused specifically on mathematics used in vocations within the area of building and construction-work. Across these studies, certain mathematical concepts consistently emerge as particularly important. For example, Masingila (2009) identifies a range of mathematical concepts used in carpet laying, including measurement (e.g. area and perimeter), ratio and proportion, and geometric concepts (e.g. constructing right angles). Similar components are identified by Boistrup et al. (2018), who investigate mathematical knowledge applied, more or less explicitly, at different stages of the construction-work process. They highlight activities such as calculating material requirements (including allowances for waste) and estimating the volumes of soil to be removed, taking volume changes during excavation into account. Bessot (2000) identifies two activities as fundamental in a construction-work context: measurement and graphical representations of spatial objects. She notes that the practice of reading plans on site involves linking two spatial systems, real space and its representation on a plan. Similarly, in the context of boat building and carpet laying, spatial visualisation, measurement concepts, and the ways in which these aspects are integrated are emphasised as playing a key role (Masingila, 2009; Zevenbergen & Zevenbergen, 2009). Across these studies, vocational mathematics in construction-related contexts is characterised by the integrated use of geometric concepts, ratio and proportion, measurement, and spatial reasoning.

In agreement with studies made within other vocational areas, Masingila (2009) further concludes that mathematical concepts are typically used in close connection with the constraints of the environment, and that considerations and arguments, obviously building on geometry, are sometimes made without being seen as mathematics. A similar conclusion is drawn by Moreira and Pardal (2012). In a study of masonry practice, they found that masons frequently apply mathematical knowledge, particularly geometric concepts, in an intuitive manner, using specific strategies to solve problems without always being aware that mathematical ideas are involved. For example, they may employ the Egyptian triangle (3:4:5) to produce 90-degree angles and determine the inclination of roofs without necessarily recognising it as mathematical knowledge, despite its clear relationship to formal mathematics. The difference between formal geometry and geometry in the workplace is also discussed by Bessot (2000), who argues that addressing real-world spatial problems involves geometrical properties that partly differ from those typically studied in formal mathematics. The concerns of

mathematicians, who seek truth and proof, differ from those of practitioners, who seek efficiency and workable solutions. Because of this divide, construction workers may struggle to apply school geometry to real construction-work tasks. At the same time, Bessot (2000) emphasises that geometry is not only used for solving practical tasks but also as part of a discourse that enables theoretical justification of the practices being taught. Similarly, Boistrup et al. (2018) describe mathematical knowledge within a construction-work context as an intersection between mathematics and vocational knowing, emphasising that mathematics is used both to solve practical problems and to explain aspects of the vocational context. Moreover, Saló i Nevado and Pehkonen (2018) show that, in cabinetmakers' practice, the ability to use advanced mathematics may enable more efficient problem-solving strategies. Although some cabinetmakers manage with only basic arithmetic, those who are able to use, for example, trigonometry regard it as an essential part of their vocational competence. Solving tasks without trigonometry, by contrast, tends to require trial-and-error approaches and increases the risk of wasting time and material. The article suggests that the combination of craftsmanship, creativity, and efficient problem-solving skills could enhance adaptation to future requirements of the labour market (Saló i Nevado & Pehkonen, 2018).

Problem-solving is also emphasised by Zevenbergen and Zevenbergen (2009) as an essential component of construction-work practice. They show that boat builders are frequently confronted with tasks that require novel approaches and strong problem-solving capacities. Problem-solving in boat-building practice is further described as a complex set of activities that demands holistic perspectives together with the simultaneous consideration of a wide range of variables. Even if the mathematical procedures involved in such problem-solving may be of a basic character, the problems require an understanding of how material properties and physical constraints mutually interact and affect the final outcome.

Similarly, Masingila (2009) describes vocational mathematics problem-solving as a multifaceted activity involving various aspects. It differs from school mathematics in the sense that the mathematical procedures are usually known. However, flexibility and adaptation to varying and unfamiliar constraints on the job sites are required. Practitioners need to be adaptive in their problem-solving and coordinate previous experiences, knowledge, and intuition with mathematical skills. It is this fusion of aspects that comprises the problem-solving skill required at the work site, according to Masingila (2009).

Another detailed exploration of the wide range of aspects involved in carpentry-related task solving was conducted by Diego-Mantecón et al. (2021). They analyse the mathematics involved when customers purchase carpentry materials for home projects in a real store context. The study shows that successful task solving depends both on the ability to use school-based mathematical techniques and on the ability to adapt these techniques to the specific logic of the carpentry context. Mathematical competence in a carpentry-related context depends not only on the correct use of mathematical procedures but also on contextual knowledge such as interpreting plans, accounting for material thickness, waste, friction, and assembly constraints. The findings therefore reinforce the view that vocationally relevant mathematics is shaped by the specific demands, norms, and rationales of the practice in which it is used.

The integrated character of mathematical activities within a pipe trade practice is further described by LaCroix (2014). Such activities involve mathematical calculations of varying difficulty, but also context-specific demands such as interpreting information (e.g. drawings, diagrams, and labels), performing accurate technical measurements, using appropriate tools, choosing suitable strategies, and understanding how the resulting answers should be interpreted within the context (LaCroix, 2014). The findings indicate that vocational mathematics activities are strongly shaped by semiotic tools and norms of practice. Moreover, the vast majority of the mathematics difficulties encountered by students were found to stem not from a lack of mathematical understanding, but from their novice-level understanding of workplace objects, practices, and conventions.

Martin and LaCroix (2008) focus on the problem-solving process as such. They suggest that students, while solving authentic problems, alternately focus on three levels or interfaces (named sets of images according to the framework used): images of the task, images of visible mathematics, and images of the workplace. Forming an image implies making sense and creating personal meaning (Marton & Booth, 1997). Martin and LaCroix (2008) found that apprentices moved between these three different interfaces as they worked on workplace tasks, starting from images of the task, which implies making sense of the task posed, prior to actually applying more formal mathematics to it. Subsequently, they focused alternately on images of visible mathematics (including conventional mathematical symbolism and representations) and images of the workplace (i.e. specific practices and ideas that govern the use and meaning of mathematics, including recognition of common practice and familiarity with equipment and specific calculations). Taking all of these images into account, Martin and LaCroix

(2008) propose that, in the context of vocational education, it is more appropriate to talk about the learning of mathematically rich tasks or mathematically rich workplace concepts than to simply talk about the learning of a particular mathematical concept.

Taken together, the reviewed studies illustrate that mathematics in construction-related vocations is typically deeply intertwined with the practical, material, and contextual demands of work. Across occupations such as carpet laying, cabinet making, boat building, masonry, and carpentry, mathematical activity emerges as situated, adaptive, and closely tied to the constraints and affordances of specific tasks and environments. While the mathematical procedures involved may range from basic arithmetic to advanced trigonometry or geometric reasoning, they are rarely applied in isolation. Instead, they are integrated with experiential knowledge, spatial visualisation, tool use, and an understanding of workplace norms and materials. The research further shows that vocational problem-solving typically requires flexible coordination of mathematical knowledge with intuition, prior experience, and the ability to respond creatively to novel or changing conditions. These results align with similar studies conducted within other vocational areas, as described above. Typical for the construction-work context, however, is the pronounced relevance of geometry and spatial reasoning. Several studies show that mathematical activity in construction-work often concerns interpreting and producing spatial representations, making accurate measurements, dealing with various geometric concepts, and understanding how material properties and physical constraints interact in three-dimensional space. Moreover, the spatial nature of construction tasks requires workers to shift flexibly between different representations of space: from the physical worksite, to technical drawings, to mathematical objects and relationships. This complexity explains why school geometry does not always transfer easily to vocational contexts, as noted by Bessot (2000).

The reviewed studies provide important and nuanced insights into the nature of mathematics in construction-related vocations. At the same time, it should be noted that they span a range of vocational contexts and have been conducted over an extended period of time. Many are based on small-scale or case study designs, focusing on specific occupations or local practices. While they offer rich and valuable accounts, their scope limits the extent to which the findings can be generalised across contexts. There is therefore a need for further research that both complements and extends this growing body of work.

Mathematics in vocational education

What is to be learned?

The review above shows that mathematical activity in vocational practice is deeply intertwined with practical action, spatial awareness, contextual knowing, and professional judgement. Rather than appearing as explicit and clearly delimited mathematics, it is often embedded in tools, routines, and problem-solving processes, and integrated with the specific demands of the vocational context. The review further indicates that even when simplified procedures and locally developed strategies are sufficient to handle specific situations, increased mathematical understanding and awareness may support more efficient, flexible, and reliable problem-solving, as well as more informed interpretations and decisions. This raises a fundamental question for vocational mathematics education: what, then, is to be learned?

There are several reasons to advocate a high level of general mathematics in vocational education. Access to theoretical knowledge could be regarded as a pedagogical and democratic right for all citizens. Formal mathematical competence could also be valued for providing access to higher education, and for affording flexibility in a rapidly changing occupational landscape (FitzSimons, 2014; Lindberg, 2013; Nylund, 2018). In addition, the kind of reasoning and structurally advanced thinking associated with high performance in mathematics is often highly valued by employers (Rosvall et al., 2017). A similar argument is raised by Wheelahan (2007), who argues that vocational students should have access to systematic and conceptually structured knowledge that enables them to think beyond immediate experience, recognise underlying structures, and engage with situations in informed and critical ways (sometimes referred to as *powerful knowledge*). From this perspective, critique is raised against competency-based curricula, which according to Wheelahan (2007) tend to restrict valued knowledge to what can be directly demonstrated in workplace performance, thereby marginalising broader conceptual understanding and the rationales that underpin vocational practice.

FitzSimons (2014) likewise emphasises the importance of conceptually coherent mathematical knowledge in vocational education, particularly in relation to problem-solving, innovation, and the ability to deal with unfamiliar situations. Vocational activities frequently involve mathematical reasoning that requires more than procedural competence within specific contexts. At the same time, FitzSimons (2014) cautions against approaches that privilege theoretical

knowledge alone. Vocational activity typically involves an interplay between theoretical and practical forms of knowledge, and these cannot be assumed to develop automatically from formal mathematics instruction alone. From this perspective, an exclusive emphasis on theoretical knowledge risks marginalising forms of knowing that are essential for effective participation in vocational practice.

In response to this tension, Carlgren et al. (2015, 2020) propose the notion of *powerful knowing*, shifting attention from cognitive understanding to knowing as a disposition to act, expressed through individuals' relation with the world. From this perspective, *powerful* refers to the extent to which such knowing enables learners to engage with situations in informed, flexible, and meaningful ways. This view aligns closely with the perspective on knowing and learning adopted in phenomenography and variation theory (see Chapter 5), and it provides a way of relating conceptual coherence to situated action without reproducing a sharp dichotomy between theoretical and practical knowledge. The perspective has implications for teaching. According to Carlgren (2020), knowing is not developed through the transmission of predefined knowledge, but through action in which the content becomes meaningful. At the same time, such development does not follow automatically from participation in practice. Carlgren (2020) argues that different kinds of experiences give rise to different kinds of knowing. Therefore, everyday, or vocational, experience alone may not necessarily provide sufficient conditions for developing more elaborated forms of knowing. This implies that teaching plays a crucial role, not by simply reproducing practice, but by organising experiences that enable learners to engage with content in ways that go beyond immediate practice.

Taken together, these perspectives suggest that the central question in vocational mathematics education is not only what mathematical content should be taught, but what forms of knowing should be developed. Vocational mathematics education cannot be reduced to either the transmission of theoretical knowledge or the reproduction of workplace practices. Rather, it must engage with the interplay between different forms of knowing and support learners in developing ways of understanding that are both conceptually grounded and practically meaningful.

Learning across contexts: transfer and boundary crossing

The relation between school mathematics and vocational practice has often been framed as a question of transfer. Yet within research on vocational education,

transfer is not understood in a uniform way. While some accounts describe it as the application of previously learned knowledge in new situations, more recent perspectives emphasise the reconstruction of meaning as learners move between different practices. Triantafillou and Potari (2014), who explore factors that may facilitate or constrain students' opportunities to use mathematics efficiently in work-place activities, define transfer as a sociocultural process of knowledge transformation, which occurs through participation in, and interaction between, different activity systems, involving reconstruction of meaning rather than the simple application of preexisting knowledge. Triantafillou and Potari (2014) found that opportunities for transfer depend on a combination of contextual, semiotic, cognitive, and motivational factors, and highlight that a context could function as a resource for learning. In the study students appeared to gain a more articulated, and thereby more elaborated understanding of mathematical objects through reflection on their actions within the workplace setting.

Also, Kilbrink and Bjurulf (2013) highlight communication, and reflection as key factors that can either enable or constrain transfer. Effective collaboration between teachers and workplace supervisors creates opportunities for students to connect school learning with authentic vocational practices. Kilbrink and Bjurulf (2013) emphasise that vocational education should not aim to cover all possible situations, but rather prepare students for future learning, enabling them to 'learn to learn' and adapt mathematical and practical knowledge to novel workplace challenges. This way of relating the concepts 'transfer' and 'learning' draws on the work of Marton (2006) who define transfer as 'relations between what people learn and can do in different situations' (p. 510). Marton (2006) argues that transfer is the ultimate aim of learning. At the same time, learning itself occurs through encountering different situations. Transfer is not primarily a matter of learning to do one thing in one situation in order to be able to do the same thing in another. Instead, it is the meaningful difference between various experienced situations, which may be drawn upon in subsequent situations; that is, transfer does not occur by repeating an action in similar situations, but by experiencing contrast between situations. For example, we are able to regard a person as tall only if we have seen people of different heights, and in order to discern the structure of a theory, one must encounter theories with different structures. What we could possibly make sense of in a situation is thus, according to Marton's (2006) line of reasoning, limited to the aspects along which we have previously perceived variation. (These ideas are further discussed in Chapter 5).

When discussing the relation between school mathematics and vocational mathematics, FitzSimons and Boistrup (2017) uses Bernstein's concept of recontextualization. They emphasise that knowledge is not carried unchanged from one setting to another; rather, it is transformed as it moves between contexts and takes on new meanings in relation to different institutional purposes, discourses, and practices. From this perspective, the task of vocational education is not only to help students apply school mathematics in workplace settings, but also to develop their capacity to recognise, interpret, and reshape mathematical meaning across contexts. Recontextualisation can therefore be seen as a central educational aim in its own right. Developing this kind of competence will according to FitzSimons and Boistrup (2017) give students a foundation for using mathematics in relevant and adaptable ways and become agents in a changing and evolving vocational practice.

Similar arguments have been put forward within physics education research. For example, Redish & Kuo (2015), suggest that what is to be transferred between settings is not a stable body of mathematical knowledge. Rather mathematical meaning is reconstructed in relation to the new context. In physics, mathematics is used to represent and make sense of physical systems, while at the same time its meaning is constituted in relation to those systems. This argument is further illustrated and confirmed by Johansson and Österholm's (2023) comparison of mathematics and physics textbooks for upper secondary education, which shows that algebra in physics is not merely applied school algebra, but appears in more complex and specified forms. The algebraic discourse in physics textbooks is generally more complex than in mathematics textbooks, involving longer symbolic sequences, more different algebraic objects, and more context-laden ways of referring to symbols. These results indicate that mathematics as it is used in other domains such as in physics, or in vocational contexts, does not simply involve discerning the same aspects in a different context, but entails awareness of a broader range of dimensions related to context, meaning, and use.

In relation to workplace contexts, Bakker and Akkerman (2019) further argue that traditional notions of transfer overlook the significant discontinuities between school and workplace practices. Instead of assuming the direct application of school knowledge to work-place tasks, they conceptualise learning across contexts as boundary crossing: the effort to establish continuity between activity systems that differ in goals, norms, tools, and representations. However, Bakker and Akkerman (2019) does not view differences between practices simply as obstacles to be overcome, but rather as potential resources for learning. In this sense,

'boundary crossing' aligns with Marton's (2006) view of transfer. However, while Marton highlights learner's discernment of variation, Bakker and Akkerman (2019) emphasise collaboration, communication, coordination, shared artifacts, and joint reflection as conditions that supports such learning. Taken together, the perspectives described above suggest that meaningful learning across school and workplace settings in vocational education is not achieved through the simple transfer of school mathematics to vocational practice, but rather through active participation and negotiation between practices, where differences become a foundation for enhanced mathematical and vocational learning.

One way, described by Bakker and Akkerman (2014), to bridge the gap between education and workplaces, and to establishing continuity in the interaction between the practices, is by means of boundary objects, i.e. artefacts that are kept stable across contexts and may therefore construct a link between them. This could for example be reports, tables, graphs, formulas, or spreadsheets, which have a meaning both in the school context and in the vocational context and therefore constitute a common ground for understanding. Bakker and Akkerman (2014) describe an intervention study in which boundary objects were introduced. They show that students' reasoning developed from a pragmatic work-related perspective without connection to theoretical reasoning, to an ability to integrate mathematical reasoning with vocational knowledge. Similarly, Coben (2014) describes how context-bound objects, such as symbols or equipment, can serve as a link between education and the workplace. Discussing boundary crossing as a mechanism for learning Jurdak (2016) further argue that tasks may function as boundary objects, providing a shared focus that allows students to mediate between contexts. For example, when dealing with the task of constructing a container both in the mathematics classroom and at the work site, students might get the opportunity to see differences and similarities in how the task is handled. The learning potential then comes from interacting with and reflecting on these differences, not from doing identical tasks. Jurdak (2016) views tasks as bridges across boundaries and argue that students only fully understand a task when they engage with both abstract mathematical tools and concrete constructions. The 'crossing' happens when learners connect symbolic and material aspects.

Authenticity and the role of context

The use of context-based activities, more generally, in mathematics education is an area which has been extensively explored (Vershaffel et al., 2020). While research

on mathematics teaching in vocational education often focuses on how the kind of mathematics used in workplaces could be described and developed in education (see Bakker, 2014), research on context-based tasks, to a greater extent, tends to appreciate context for its potential to enhance the learning of formal mathematics (see Verschaffel et al., 2020). Research in the field points to several roles that ‘contextualised’, ‘authentic’, or ‘real-life’ activities may play in school mathematics, including motivating students by making mathematics relevant, enhancing mathematical literacy, promoting a reinvention process in which informal knowledge develops into general understanding, suggesting possible solution strategies, and providing a conceptual anchor for mathematical reasoning (Meyer et al., 2001; Beswick, 2011; Verschaffel et al., 2020). However, the question of what authenticity actually means is not readily answered (Verschaffel et al., 2020).

Some researchers have sought to set up criteria for evaluating authenticity. Palm (2008) provides a detailed framework that breaks authenticity into five key aspects. The event, the question, the purpose, the language, and the information must all be realistic for a task to be considered authentic. Palm (2008) further suggests that increased authenticity might help students to use real-world knowledge more effectively, strengthening the connection between school mathematics and mathematics in ‘out-of-school’ practice. Other researchers, however, argue that it is generally impossible to assume that ‘real life’ can be represented in school mathematics (Gerofsky, 2010; Vos, 2018). Therefore, since differences will always remain between problems, presented and solved in a school setting and authentic real-life problems occurring outside school, Verschaffel et al. (2020) suggest that school mathematics tasks should rather be seen as relating to a ‘reality’ or ‘authenticity’ continuum.

It has also been discussed whether authenticity is actually desirable at all, and some scholars point to the potential disadvantage of introducing complex contextual references into school mathematics. Brantlinger (2022) argues that authentic problems in vocational education tend to entail a tension between vocational concerns and conceptual mathematical meaning. The increased complexity that a realistic scenario implies may, according to Brantlinger (2022), inhibit students’ opportunities to learn general mathematical relations. The study provides several examples which illustrate how either mathematics or vocational knowing are downgraded in context-based vocational word problems (Brantlinger, 2022). Hahn (2014) also highlights that problems situated in realistic contexts might encourage students to reason pragmatically and disregard mathematical aspects that could have been valuable in order to come to a valid result.

Furthermore, Lowrie (2011) found that, while students' familiarity with the context might allow them to draw on realistic considerations, it could also lead them to engage more with contextual details than with the task's essential mathematical aspects.

The other side of the coin is emphasised by Wedege (2010), who points to the fact that context-based tasks often require unrealistic assumptions and the neglect of important contextual aspects in order to become solvable. A similar point is also made by Lundberg and Kilhamn (2018), who exemplify context-based tasks in which important aspects of volume change when mixing materials need to be disregarded when solving the task. Likewise, Vos (2018) highlights that school mathematics typically offers inauthentic word problems which do not provide any opportunity to appreciate the authentic usefulness of mathematics. Like Palm (2008), Vos (2018) suggests that authenticity could apply to various aspects of a task. However, Vos (2018) argues that, while such criteria are intended to capture realism or authenticity, they could potentially be met by tasks which still are 'inauthentic' in the sense that they do not genuinely connect to real-world contexts or practices. Vos (2018) therefore suggests the term authenticity should be used for different aspects in a learning environment separately, such as contexts, questions, etc. This perspective emphasises that authenticity emerges through social interaction and validation, and that students can participate in recognising and confirming authentic aspects.

These arguments are confirmed in Sundtjøn's (2021, 2025) study, which illustrates that students themselves can make tasks more authentic by bringing their own experiences, knowledge, and vocational perspectives into the classroom and, in that way, bridging school mathematics with real vocational practice. Sundtjøn (2021) explores the role of vocationally connected tasks as potential boundary objects by investigating Norwegian vocational students' engagement with mathematics tasks connected to their future professions. Her findings suggest that students may recognise and engage with the real-world aspects of the tasks, drawing on their own experiences. In doing so, students may position themselves as 'knowers' of vocational practice and use this expertise in mathematical discussions. The students may, in this way, validate the authenticity of tasks and incorporate out-of-school knowledge. The study concludes that vocationally connected tasks can create space for discussing the role of mathematics in different practices and thereby promote authenticity and enhance transfer.

Taken together, these perspectives highlight that context is not a neutral feature but plays an active role in shaping how students interpret and engage with

problems. This perspective is further articulated by Boaler (1993), who demonstrates that context has the capacity to influence students' problem-solving and the strategies they use. Even when a context is only briefly referred to in word problems, it is shown to have a considerable influence on students' choice of solution strategy. Boaler (1993) demonstrates that small changes, such as the substitution of a single word, can alter students' perceptions of the task situation, their associations, and consequently the way they approach the problem. Moreover, students tend to perceive contexts differently based on their individual experiences and associations, and the relationship between the context and students' construction of meaning tends to influence their performance. In this way 'contexts have the power to form a barrier or bridge to understanding' (Boaler, 1993, p. 370). Similar results are reported by Johansson (2015), whose quantitative study show that students are more likely to solve tasks requiring creative mathematical reasoning when these are situated in a familiar contexts.

The potential power of context to serve as a bridge to understanding is specifically emphasised within 'realistic mathematics education' (RME) (van den Heuvel-Panhuizen, 2003). Several scholars have described how children, before they have been instructed in for example arithmetic, might be able to solve certain problem types involving these operations by means of a variety of informal strategies (e.g. Freudenthal, 1978; Carraher et al., 1985). Within RME, the educational potential of these strategies is emphasised, and context-based tasks are appreciated for their potential to serve as vehicles to build formal mathematical concepts and skills from these strategies (Gravemeijer, 1999; Verschaffel et al., 2020; van den Heuvel-Panhuizen, 2003). However, according to Verschaffel et al. (2020), 'the question of how this development can be optimally enhanced through instruction is still a matter of serious debate' (p. 5).

Taken together, the literature suggests that context plays a complex and sometimes contradictory role in mathematics education, simultaneously offering opportunities for meaning-making while also introducing challenges related to authenticity, complexity, and interpretation.

How context shapes engagement

One of the most common reasons to introduce connections to authentic contexts in mathematics education is to attract students' attention and curiosity and motivate them to learn and explore new mathematics (Beswick, 2011). Students who perceive mathematics as relevant to their vocational practice are, in turn, more likely to make generalisations and integrate their prior mathematical knowledge

with the workplace context (Triantafillou and Potari, 2014). Several studies have examined how mathematics teaching in vocational education can be made relevant to students' broader educational goals. A consistent finding across this body of research is that, when students are given opportunities to apply their mathematical knowledge in vocationally meaningful contexts, their understanding of mathematics improves, leading to increased competence and motivation (Boistrup, 2018; Dalby, 2016; Lindberg, 2010; Muhrman, 2016; Voss et al., 2025).

For example, Muhrman's (2016) comprehensive interview study, conducted within the agricultural programme, shows that integration between mathematics and vocational subjects can help students understand the applicability of mathematics in their vocational practice, thereby increasing their motivation and interest. Similarly, Voss et al. (2025) demonstrate that, when mathematics teaching is connected to authentic vocational problems and to teachers' own professional expertise, students' engagement, participation, and positive learning experiences increase. Also, Dalby et al. (2016), who conducted a study within English upper secondary vocational education comparing students' experiences of mathematics teaching with and without vocational relevance, found that students who had previously experienced negative attitudes toward school mathematics perceived the subject as more relevant when teachers regularly referred to applications within the vocation. The approach enhanced both engagement and attitudes toward mathematics.

The same conclusion is drawn by Lindberg (2010), who conducted a design research project aiming to develop the collaboration between mathematics teachers and vocational teachers. The approach proved to enhance students' motivation and learning. Likewise, Boistrup et al. (2018) conducted an action research project fostering collaboration between mathematics and vocational teachers. They illustrate how a clarified connection between school mathematics and vocational practice increased student engagement, as students planned and carried out real construction-work outdoors, enhancing both their understanding and involvement when they perceived the mathematical knowledge as relevant.

Although many studies highlight the benefits of vocational integration for enhancing students' motivation to learn mathematics, some findings raise concerns. For instance, Roth (2014) shows that semi-relevant tasks in the mathematics classroom can reinforce the perception that school mathematics is disconnected from professional practice, as they emphasise differences in how tasks are performed at school versus at work. Similarly, Hahn (2000) demonstrates that, even when calculations are similar, the connection between school and the

workplace can be disrupted if workplace calculations are performed in a localised, less rigorous, or non-generalisable way. In such cases, introducing a pseudo-professional context may compromise the mathematics teacher's credibility and decrease student interest.

An important factor identified by Frejd and Muhrman (2020) that may determine whether integration becomes successful is the location of vocational mathematics teaching. Their study shows that the positive effects of an embedded approach are most evident when mathematics is taught in vocational classrooms. The findings further indicate that well-established collaboration between mathematics and vocational teachers is essential for strengthening students' understanding in both subject areas and for helping them connect mathematical concepts to practical tasks in their vocational fields. The results also highlight that supportive organisational conditions and leadership are crucial for enabling successful subject integration (Frejd & Muhrman, 2020, 2025).

Summary

The review of research on mathematics in vocational education thus highlights a central tension: vocational students need access to mathematically powerful and conceptually structured knowledge, while mathematics teaching must also support the development of meaningful ways of engaging with mathematics in vocational practice. Authentic or vocationally connected tasks are often proposed as one way of supporting such learning. Such tasks may help students experience mathematics as meaningful by situating it in scenarios they recognise (e.g. Dalby, 2016; Voss et al., 2025). This sense of relevance may, in turn, enhance both mathematical learning and students' opportunities to use mathematics in new situations (Triantafyllou & Potari, 2014). In this way, vocationally connected tasks may function as boundary objects linking school mathematics to vocational practice (e.g. Jurdak, 2016). At the same time, research suggests that authentic vocational contexts alone do not guarantee opportunities for developing mathematically powerful vocational knowing (e.g. Carlgren, 2020; FitzSimons, 2014; Hahn, 2014; Wedege, 2010). Rather, such opportunities appear to depend on how teaching is coordinated across classroom boundaries through collaboration between mathematics and vocational teachers (Frejd & Muhrman, 2020; 2025).

Despite the growing body of research in this field, important questions remain regarding what students need to learn (i.e. which aspects of vocationally relevant mathematics are critical for students to discern), which of these aspects can

reasonably be included in mathematics teaching, and what opportunities different educational resources provide for making such aspects available for learning.

Mathematics textbooks

In recent years, there has been a growing international interest in textbook research and in scientific foundations for textbook development (Fan et al., 2013; Johansson, 2006; Qi et al., 2025; Rezat et al., 2021; Valverde et al., 2002). Textbooks are widely acknowledged to play a significant role in students' learning, serving as a bridge between the intended and the implemented curriculum (e.g., Fan et al., 2013; Johansson, 2006; Qi et al., 2025; Rezat et al., 2021; Valverde et al., 2002). Combined with other factors, the textbook creates conditions for the probable opportunities to learn mathematics and is sometimes referred to as the *potentially implemented curriculum* (Valverde et al., 2002). Although the role of textbooks is not fixed but rather shaped by individual teachers' practices, research findings indicate that textbooks influence not only students' individual training, but also teachers' choices of content and instructional strategies, as well as teacher–student interaction (Fan et al., 2000, 2021; Johansson, 2006).

Several studies confirm that textbook characteristics influence students' performance (e.g. Hadar, 2017; Sievert et al., 2021; Van den Ham & Heinze, 2018). For example, Van den Ham and Heinze (2018) investigated how different textbooks, although aligned with the same curriculum, influence students' mathematics achievement. Drawing on longitudinal data from 1,664 students, their multilevel analyses showed that the choice of textbook had a notable impact on students' performance in mathematics. The study also suggests that this effect may accumulate over the school years. Overall, the findings indicate that textbooks should be treated as an important covariate in educational research and that textbook selection is a relevant factor for educational practice. Similarly, Hadar (2017) investigated how the cognitive demands of mathematics textbooks affect student performance. Using data from 4,040 eighth-grade students, the study found that students using the textbook with higher cognitive demands achieved better results on standardised mathematics tests. The findings highlight that textbooks significantly influence student learning outcomes and thereby confirm that textbook analyses may provide valuable insight into potential learning opportunities.

Fan et al. (2025) argue that textbooks do not only support teachers' practices, but it also facilitates the maintenance of new curricula. The role mathematics

textbooks play depends, according to Fan et al., (2025), on various factors, such as design and pedagogical approaches. They conclude that more research is needed to understand how textbooks can be used effectively to support teaching and how the quality of textbooks can be evaluated.

Textbook analyses

Methods

Rezat and Straesser (2015) suggest that the field of research on mathematics textbooks can be divided into three areas: 1) research that focuses on the influences on textbooks; 2) research that focuses on the mathematics textbook itself; and 3) research on the use of mathematics textbooks and its impact. They further emphasise that research on mathematics textbooks usually relies on content analysis, either quantitative or qualitative, and either inductive or deductive. A quantitative content analysis involves systematically counting and categorising predefined elements in the material, often using established coding schemes, in order to identify patterns and frequencies. In contrast, qualitative content analysis focuses on interpreting meanings, themes, and underlying structures in the material. An inductive approach means that categories and patterns are derived from the data itself during the analysis, whereas a deductive approach starts from predefined categories or theoretical frameworks that guide the analysis.

Several frameworks have been developed for textbook analysis in order to investigate, for example, international differences (e.g., Brehmer et al., 2016; Charalambous et al., 2010; Jäder et al., 2020; Pepin & Haggarty, 2001; Sievert et al., 2021; Valverde et al., 2002). For example, Valverde et al. (2002) propose a framework for examining textbooks based on five categories: classroom activities proposed, the amount of content covered, the way the content is portrayed (i.e. either through a formally abstract system of text and symbols or through concrete examples and real-world representations), the physical characteristics of the textbook, and the complexity of the expectations of students' performance. Their findings highlight substantial cross-national differences and suggest that textbooks structure different opportunities to learn, particularly through the nature and demands of tasks.

Pepin and Haggarty's (2001) seminal study is a combined textbook analysis and interview study in which textbooks and their use are compared between countries. The material is analysed from various perspectives: mathematical intentions,

pedagogical intentions, sociocultural contexts, and cultural traditions. The results show that teaching materials in different countries vary in structure with regard to content, presentation, and the complexity and context of tasks. These differences may be linked to different pedagogical traditions and influence how mathematics is taught and experienced in classrooms.

Later, Charalambous et al. (2010) developed a framework to investigate learning opportunities in textbooks. A comparative analysis was conducted of how addition and subtraction of fractions are treated in textbooks from Cyprus, Ireland, and Taiwan. The textbooks were analysed from two perspectives: with respect to background information and overall structure (horizontal analysis), and through a more focused and in-depth examination of a single mathematical topic (vertical analysis). The vertical analysis examines how the mathematical content is communicated to students, what is required of them, and how connections are established within and beyond the topic. More specifically, it includes an analysis of the types of fraction constructs used (e.g., part-whole, operator), the nature and completeness of worked examples, and the representations employed. It also considers the cognitive demands of textbook tasks, ranging from routine procedures to more complex problem solving, and the types of responses expected from students, such as providing answers, writing mathematical expressions, or explaining reasoning. In addition, the analysis attends to how the content connects to other mathematical ideas and to real-world contexts. The findings reveal several similarities and differences among the textbooks and indicate that there may exist a recognisable 'textbook signature' within a country.

Other frameworks have been used to examine specific competences, such as problem solving and reasoning (Brehmer et al., 2016; Jäder et al., 2020) and modelling (Frejd, 2013; Jäder et al. (2020) developed a framework for analysing problem-solving opportunities, based on definitions of problem solving, that consider how strongly textbook tasks are connected to previously presented solution templates. They distinguish between tasks that directly follow a template, those requiring minor adaptations, and those in which students must construct a solution independently. In their analysis of more than 5,700 tasks from secondary mathematics textbooks in twelve countries, they found that most tasks could be solved by imitating available templates, while only a small proportion required genuine problem solving. Notably, textbooks that claimed to emphasise problem solving seldom provided tasks that actually demanded independent solution construction.

Frejd (2013) used content analysis based on definitions of the modelling cycle to examine opportunities to learn modelling in mathematics textbooks. The analysis follows an analytical scheme focusing on steps in the modelling process and concludes that modelling is rarely presented as a complete process. Instead, textbooks tend to isolate specific steps, typically intra-mathematical, while omitting key components such as task formulation and the systematisation of relevant objects and relations. As a result, students encounter modelling only in a fragmented form, far removed from how modelling operates in authentic vocational or professional contexts. Frejd (2013) argues that this partial representation restricts opportunities to learn modelling in school mathematics and contributes to a substantial gap between educational materials and real-world modelling practices.

Textbook studies with a specific emphasis on task context have also been conducted. For example, Wijaya et al. (2015) investigate the opportunity to learn to solve context-based mathematics tasks offered by Indonesian textbooks. An analytical framework was developed to examine textbook tasks from four perspectives: the type of context used, the purpose of context-based tasks, the type of information provided, and the level of cognitive demand. The type of context is divided into two subcategories: camouflage context and relevant and essential context. The purpose of context-based tasks is differentiated into application and modelling. The findings indicate that most tasks in the textbooks are decontextualised, requiring little mathematisation or modelling. When context is provided, it is typically of a ‘camouflage’ type, meaning that it does not influence the solution of the task.

Many textbook analyses thus rely on systematic content analysis, often guided by predefined categories grounded in prior research on the phenomenon under investigation. These studies usually adopt a deductive approach, where theoretical frameworks inform the selection of categories and guide the analysis, although many also incorporate inductive elements through the refinement or expansion of categories in response to empirical material. Across the field, both quantitative and qualitative approaches are used, sometimes in combination, enabling researchers to capture both frequencies and distributions of textbook features and also their meanings, structures, and pedagogical implications. Frameworks also vary in scope: some aim to provide broad, comparative descriptions of textbook systems across countries (e.g., Valverde et al., 2002; Charalambous et al., 2010), while others target specific competences or practices, such as problem solving, reasoning, or modelling (e.g., Jäder et al., 2020; Frejd, 2013). In addition, some

studies extend beyond the textbook as an artefact to consider its use in practice or its sociocultural and pedagogical context (Pepin & Haggarty, 2001), thereby linking textbook design to instructional traditions. Key findings indicate that textbooks often emphasise reproduction of methods, while opportunities for more complex engagement, such as independent problem solving, modelling, or reasoning, are less frequent or unevenly distributed (e.g. Frejd, 2013; Jäder, 2020).

Textbook analyses of geometric topics, including similarity

Differences in learning opportunities in geometry textbooks have been analysed to some extent, with a substantial part of this research focusing on reasoning and proof. For example, Jones and Fujita (2013) employ a structured, framework-based content analysis to examine how textbooks interpret and implement curriculum goals in geometry. Their analysis is guided by predefined analytical categories related to reasoning, proof, and problem solving, which are systematically applied to textbook material in order to identify patterns in how these elements are presented. By analysing the distribution and organisation of these components across the textbooks, the study reveals how different emphases may shape students' opportunities to engage with formal proof and problem solving. The findings highlight the importance of careful textbook design and the need for continued research on how different structures support or constrain students' development of geometric thinking.

Similarly, Otten et al. (2014) offer a detailed examination of the reasoning-and-proving opportunities embedded in secondary geometry textbooks. In this study, a qualitative textbook analysis is conducted using a category-based analytic framework. The analysis is primarily deductive, as it builds on an existing framework, which in turn draws on established theoretical perspectives on reasoning and proof. However, the framework is also adapted and extended based on pilot analysis and theoretical considerations, which introduces an inductive element to the analysis. The analysis indicates that, reasoning and proving opportunities are relatively abundant, and they differ significantly in nature across expository text and student tasks. The textbooks rarely position the reasoning-and-proving process itself as an explicit object of reflection. Otten et al., (2014) argue that this absence of meta-level attention has implications for students' development of deeper understanding.

Lo et al. (2024) analyse geometry textbooks using the 'mathematics curriculum as a story' framework, which conceptualises textbook content in terms of characters, actions, settings, and plot. The study employs a qualitative, theory-

driven analysis combined with inductively developed coding schemes for mathematical actions and characters. The results show that the textbooks construct different narrative structures, with differences in sequencing, task complexity, and use of representations. In particular, one textbook emphasises transformation-based reasoning and includes more complex tasks, while the other combines proof with real-life applications, suggesting differing learning opportunities for students.

A few studies of the similarity concept in mathematics textbooks have been made. Lo et al. (2006) compare the potential to develop conceptual understanding of similarity, in textbook series for Grades 6-8. The study adopt a qualitative, conceptually oriented curriculum analysis focusing on the concept of similarity. The analysis is theory-driven, drawing on the constructs of *concept definition* and *concept image*, and involves examining how definitions are presented, how activities are sequenced, and what kinds of concept images these may support. By mapping the progression of tasks and identifying different types of activities, the study analyses how textbooks afford opportunities for conceptual understanding. In contrast to structured, category-based content analyses, this approach does not rely on fixed coding schemes but instead uses theoretical constructs to guide an interpretive analysis of the material. The result point out extensive differences amongst textbooks, concerning, focused aspects and related concepts and context. Lo et al., (2006) argue that when students engage with activities in mathematics textbooks, they get certain opportunities to make sense of the concept of similarity. The nature and sequence of tasks and activities affect the development of concept images and the way in which the terms ‘similar figures’ or ‘scale drawings’ are perceived. Lo et al (2006) report on differences among textbooks regarding definitions (formal or informal). They also identify three types of activities involved in tasks, i.e. differentiating, measuring, and constructing. Differences between textbooks were found regarding the extent to which these activities were used. Concerning applications Lo et al. (2006) show that only one of the textbooks provides task in which students need to make the correct associations between corresponding parts. Lo et al. (2006) conclude that the differences between textbooks concerning the treatment of the similarity concept ‘highlight the mathematical complexity of this concept’ (p.227) and they suggest that further research should examine the effects of different curricula on the development of conceptual understanding of similarity.

The lack of learning opportunities regarding conceptual aspects of similarity is also highlighted by Wjayanti (2019) who analysed the treatment of the topic in

several college textbooks. The study employs a theory-driven and didactic textbook analysis based on ATD. More specifically, it combines epistemological analysis with an emergent categorisation of task types, which is then used for comparative and partly quantitative analysis. A number of common tasks-types were initially identified, and the textbooks were then analysed according to the categorisation. The study concludes that textbooks emphasise how to use similarity to solve tasks rather than how the concept is defined and understood. The analysis also reveals varying degrees of connection between similarity and proportional reasoning. Such connections are substantial but often implicit. Many techniques used in similarity tasks, such as finding missing side lengths using proportional reasoning, mirror those found in earlier arithmetic content, yet textbooks rarely make these relations explicit, and explanations of how proportional reasoning underpins similarity is generally lacking. As a result, students may not fully recognise how these concepts are connected across mathematical domains.

Zhang and Wong (2021) discuss the role of the topic of similarity in developing deductive reasoning. The study examines how similarity is treated in secondary mathematics textbooks in Hong Kong and describe how intuitive ideas can be gradually developed into formal definitions. The study employs a theory-driven qualitative content analysis with a historical and epistemological orientation. More specifically, it combines horizontal and vertical textbook analysis to examine how the concept of similarity is defined, arranged, and justified across different periods. They also conclude that few textbooks provide formal proofs of similarity and argue that students thereby lose the opportunity to understand the nature of the similarity concept, to understand what a proof is, and to understand how deductive reasoning works. Amaral and Hollebrands (2017) analysed context-based tasks on similarity and concluded that 9-29% of the similarity tasks in textbooks analysed from Brazil and the United States were context-based, many of them with low cognitive demand.

In summary, research on geometry textbooks employs a range of qualitative analytical approaches, often theory-driven but differing in structure and focus. Many studies use category-based frameworks, typically combining predefined analytical categories with systematic coding of textbook content (e.g., Jones & Fujita, 2013; Otten et al., 2014). These approaches are largely deductive, though often complemented by inductive refinements based on empirical material. Other studies adopt more interpretive and conceptually oriented analyses, guided by theoretical constructs rather than fixed coding schemes (e.g., Lo et al., 2024).

With respect to the concept of similarity, findings indicate that textbooks often provide limited opportunities for developing conceptual understanding. The treatment of similarity tends to emphasise procedural applications, such as calculating unknown side lengths, rather than supporting a deeper understanding of the concept itself (Wijayanti, 2019). Connections to related ideas, particularly proportional reasoning, are frequently present but remain implicit, which may hinder students' ability to see coherence across mathematical domains. Studies also show variation in how definitions are presented (formal or informal) and in the types of activities included (Lo et al., 2006). Tasks that require identifying corresponding parts or engaging in more advanced reasoning, including deductive arguments, appear to be relatively rare (Zhang & Wong, 2021; Amaral & Hollebrands, 2017). Overall, the research suggests that the organisation of textbook content offers constrained opportunities to develop a robust and connected understanding of similarity.

Textbook analyses of textbooks for vocational education

Textbook analyses of vocational mathematics textbooks are rare. Just a few such studies have been found. For example, different textbook series used in upper secondary education in the UK, was compared by Dowling (1996). At that educational level, students were divided into groups, depending on 'ability', using different textbook series. The analysis is a sociological analysis of mathematics textbooks, focused on the upper and lower series, 'Y' and 'G'. A framework, based on Bernstein's and Vygotski's works, were used to describe differences in the 'voice' used in the textbooks. Dowling (1996) states that in the G-series, mathematics is concerned with the everyday, 'public domain', which is known and in which students already participate. On the other hand, the Y-series is concerned with an 'esoteric domain' which addresses students' supposed curiosity about what is yet unknown.

Queiroz, et al. (2018), made a comparison between textbook tasks and authentic tasks in vocational practice, and identified some important differences. They found differences in the routines expected, differences in problems faced and variation in the use of technology. Authentic vocational tasks usually imply a focus on selecting relevant data and performing accurate measurements, using available technology in an appropriate manner, and understanding how the answer should be interpreted in the context, and making decisions of strategies based on practical conditions. Textbook tasks on the contrary are typically pseudorealistic, involving routines that include selecting models and mathematically consistent

calculation procedures. The use of technology is usually marginal, the input values given and the expected result unambiguous.

Rejeki et al. (2021), analysed types of task context in vocational mathematics textbooks based on the framework for task context developed by Wijaja et al., (2015), and concludes that there are few tasks with relevant, essential context, and the tasks typically contain precisely the information needed to find the solution.

In summary, the results from textbook analyses for vocational education, seems to indicate that the mathematical content tend to be restricted to a kind of every-day-use-level, and at the same time the connections and relevance to vocational practice tend to be vague.

4. Mathematical content areas

The review below concerns formal mathematical content areas with particular relevance for construction-work vocations. Geometry instruction is an area of special interest in vocational mathematics education because of its potential to serve as a tool to handle practical problems, as well as to enhance understanding of mathematics in general (Bessot, 2000). In this review special attention is paid to the similarity concept, which was selected for the textbook analysis in Study 3.

Geometry and spatial reasoning

Because of the potential of geometry to serve as a tool to handle practical problems it is an area of particular interest in vocational mathematics education (Bessot, 2000). Geometry is also one of the main pillars of mathematics. It draws on the capacity of human mind to visually handle a vast amount of information at one moment. Understanding the world we see relies heavily on such spatial intuition, which has thus become a well-developed part of human cognition (Atiyah, 2002). Geometry harnesses this capacity and is therefore both an important topic and a powerful tool for perceiving other areas of mathematics (Atiyah, 2002; Clements and Battista, 1992). Fujita and Jones (2002) discuss geometry, as both a theoretical domain and an area of practical experience, and points to the opportunities and difficulties that this dual nature can imply).

Formal geometry can be seen as a complex, interconnected network of concepts and theorems, a system of representations applicable for analysing and understanding spatial environments (Battista, 2007). Geometry and spatial reasoning are thus strongly related. Sinclair (2017) notes the increased attention given to the strength and effectiveness of spatial reasoning, which has led to a stronger emphasis on geometry as a central part of mathematics curriculum. However, while spatial reasoning comprises representations and manipulations of spatial objects and relationships, formal geometry, by contrast, formalises spatial intuitions into a system of mathematically defined objects and relationships governed by the axioms of geometry (Clements & Battista, 1992). Below these relationships are further discussed by first examining the nature and role of spatial

reasoning and then describing how the axiomatic structure of geometry formalises intuitive spatial concepts.

Spatial reasoning

Underlying most geometric thinking is spatial reasoning (Battista, 2007), which is an essential aspect of human cognition (Gardner, 2006) that involves mental manipulation, and organisation of spatial information (Uttal et al., 2013). The role and power of spatial reasoning in mathematics learning have received increasing attention in mathematics education research due to its' documented influence on mathematical performance (Sinclair, et al., 2017). Based on research in cognition, and neuroscience, Newcombe and Shipley (2015) suggest that spatial abilities could be categorised in relation to two major dimensions: intrinsic-extrinsic and static-dynamic. The intrinsic–extrinsic distinction concerns whether spatial information is object-based or environmental, and static characteristics focus on relationships, while dynamic characteristics includes transformations such as rotations, scaling and bending. (Uttal et al. 2013; Newcombe & Shipley 2014). Spatial thinking includes a variety of abilities such as understanding spatial arrangements and relations, navigating in space, and recognising configuration of objects, as well as performing mental transformations (Newcombe & Shipley, 2015). Such abilities are associated with a lot of different activities, such as discerning embedded figures, manipulating geometric objects mentally, visualising 3D in 2D projections, and combining these skills with the understanding of geometric concepts (Fujita et al., 2022; Seah, 2020; Sorby, 2009; Xi et al., 2019).

Spatial reasoning, particularly the skills that are associated with intrinsic-dynamic tasks (for example, performing mental visualisations and transformations in three dimensions), have been identified as key factors linked to success in engineering and science as well as in mathematics (Sorby, 2009; Xi et al., 2019; Newcombe, 2017; Sinclair, 2017). In addition, spatial skills have proved to be central to professional competence within various vocational fields, for example construction-related trades (e.g. Cuendet et al., 2014; Lee, 2019; Liao, 2017). For carpenters, the ability to relate 2D drawings and 3D objects, is of specific importance in vocational practice (Cuendet et al., 2014). In Cuendet's et al. (2014) study, carpenter apprentices demonstrated higher levels of spatial ability than other students at a similar educational level, a finding that might be explained by a combination of selection effects and the impact of vocational training (Cuendet et al., 2014). The study further show that carpenters' spatial skills improved over the

course of their apprenticeship, supporting the idea that spatial ability is trainable. Similarly, several other research findings confirm that spatial skills can be improved through education, and experience (e.g. Hawes et al. 2022; Uttal et al. 2013). For example, Sorby et al., (2013) shows that an intervention, aimed at enhancing spatial ability, produced measurable improvements in engineering students' performance. Notably, the intervention also led to transfer effects, including better outcomes in calculus. Moreover, Owens (2020) conclude that students' activities in their cultural context may enhance visuospatial reasoning in two- dimensional and three-dimensional geometry.

Spatial reasoning is an area which is essential both in relation to formal mathematical knowledge and to vocational practice. It is also reasonable to expect that spatial skills developed, for example in construction-work practice may enhance mathematical learning and vice versa.

The axiomatic system of geometry

Our commonly agreed intuition of space is formalised and represented through the axiomatic systems of geometry. The development of geometry requires a small number of simple, fundamental principles. These are called the axioms of geometry. From a certain set of axioms, theorems, which are logical consequences of the axioms can be derived using deductive reasoning. The theorems are about concepts, which are defined in terms of other concepts, of which the most fundamental are called primitives. Primitives are undefined objects, such as for example points and lines, which are related to each other through the axioms. The choice of the axioms of geometry has been discussed extensively, and several sets of axioms have been suggested (e.g. Hilbert, 1902/1910; Birkhoff, 1932). Around 300 B.C.E, the Greek geometer and logician, Euclid, established the foundations of geometry in *The Elements*. However, in the late 1800, an extensive discussion among mathematicians arose from the recognition of insufficiencies in Euclid's *Elements*. Several mathematicians attempted to develop the foundations of geometry from the very beginning, and therefore re-examined the axiomatic method, used by Euclid, in order to replace the established, and to some extent, imprecise approaches, used so far (Smith, 2010).

One attempt to choose a simple and complete set of independent axioms for geometry, and to deduce from these the most important geometrical theorems, was presented by Hilbert (1902), and has been used as a foundation for geometry teaching in several European countries. Hilbert's (1902) axiom system is created

based on three primitive concepts; point, line and plane, having certain mutual relations, such as connection, order, parallels, congruence and continuity, which are described in five groups of axioms. Another attempt to present the most fundamental geometric ideas was made by the American mathematician George Birkhoff (1932) who used four primitive concepts, points, lines, distance and angle. The system thus builds on properties of the positive numbers.

An Italian mathematician, Mario Pieri, also published a memoir, in which an axiomatisation of geometry was suggested, based on only two primitive concepts, points and motion (Smith, 2010). Drawing on Giuseppe Peano's pioneering work on the logic underlying geometry, Pieri replaced the axioms of congruence (see Hilbert, 1902) with the primitive concept, direct motion. According to the approach, a transformation is a motion that is defined by initial and final position. An isometry is a point transformation that preserves distance, and figures can be defined as congruent if some isometry maps one to the other. For Pieri the idea of defining geometric concepts through transformations and isometries had, to some extent, a pedagogical rationale. A transformational approach would according to Pieri facilitate the teaching of geometry by encouraging students to appreciate the duality between abstract mathematical objects and their concrete representations (Marchisotto & Millán Gasca, 2021).

The same principle was advocated by the German mathematician Felix Klein, who stated that the best way to understand geometry is through the study of invariants under certain groups of transformations. He developed the Erlangen program (1872) which led to a reconceptualisation of geometry, and also aimed to reorganise the school geometry curriculum according to transformational principles (Herbst et al., 2017; Klein, 1893/2008) (The Erlangen program (1872) is a unified way to describe various geometries (Euclidean geometry being one of them) in terms of collections of geometric transformations, declared to preserve similarity of figures.) Thereafter, mathematicians have discussed the benefit of focusing on transformations when learning geometry. For example, in the US, several states implemented the Common Core State Standards in 2010, which encourages a geometric transformations approach to learning geometric concepts such as congruence and similarity (Seago, 2013). All of these approaches are independent and different, yet equivalent in the sense that the same theorems can be proved using any of them. The different axiomatic systems, and their implications for the teaching and learning of geometry, are further discussed in relation to the concept of similarity.

The concept of similarity

Geometric similarity is a fundamental mathematical idea, which entails rich mathematical understanding, and is therefore considered a key topic in secondary mathematics (Lehrer et al., 2021). It is situated at the cross road of algebra and geometry, of intuitive perception and deductive reasoning, of geometry and number, and of proportional reasoning and geometric transformations (Cox, 2013; Kallia & Panagiotis, 2010; Simsek et al., 2021; Wang et al., 2015). Probably as a result, several research findings confirm that similarity is a mathematical concept that tend to be particularly difficult for students to comprehend (Lamon, 2007; Son, 2013). At the same time, the connection between geometry and algebra, affords opportunities to use geometric visualisations in algebraic problems and to utilise algebraic operations in geometric problems. In addition, properties of similar triangles, constitute a cornerstone of trigonometry and is an essential preparation for developing the concept of slope. Moreover, the measurements of length, area, etc., relies on the condition that congruent sets have equal measurements, and is therefore impossible to explain without referring to a mathematical definition of congruence (Wu, 2005).

There are also numerous practical and vocational applications of similarity, such as, scale drawings, surveying and inclination ratio. (Lo et al., 2006). In all of these examples, a conceptual understanding of similarity, based on a mathematically precise definition, is essential. Therefore, (Wu, 2005) argues that similarity and congruence are concepts that are good 'illustration of the need of definitions' (p 5). However, due to intuitive perceptions, based on informal, and everyday use of the similarity concept, many students fail to identify similar shapes, based on defining conditions (Vollrath, 1977; Wu, 2005). Because of the importance, and multifaceted role, of the similarity concept, in various domains, together with the reported tendency to rely on imprecise informal definitions, when teaching and learning similarity, an in-depth study of afforded opportunities to discern essential aspects of similarity in mathematics textbooks, is pertinent. In this chapter, important aspects of the similarity concept, with respect to both how the concept is defined in geometry as a mathematical field, and to previous research about teaching and learning similarity, are investigated and discussed.

Definitions

The concept of similarity is defined in different ways according to the different axiomatic systems, which, despite being equivalent, might affect the way similarity

is taught and understood, and consequently, the aspects that are potentially critical to discern in order to understand it. In the following sections, the treatment of the similarity concept in a few different, frequently used, axiomatic systems for geometry, are described and compared.

Congruence and similarity according to Hilbert (1910)

As described above Hilbert's axiomatic system is built upon three primitive concepts, (point, line, and plane), which are related and formalised in five distinct groups of axioms. The fourth group considers the idea of congruence and describes specific relations among segments and angles. Axioms IV, 1–3 contain statements concerning congruence of segments. Axioms IV, 4, 5 contain statements relating to the congruence of angles. Axiom IV, 6 establishes the connection between congruence of segments and congruence of angles:

AXIOM IV 6: If, in the two triangles ABC and $A'B'C'$ the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv \angle B'A'C'$ hold, then the congruences $\angle ABC \equiv \angle A'B'C'$ and $\angle ACB \equiv \angle A'CB'$ also hold. (p. 9)

As consequences of the axioms of congruence the following statements are made

DEFINITION: Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be congruent to one another when all of the following congruences are fulfilled:
 $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$,
 $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$. (p. 10)

THEOREM. (First theorem of congruence for triangles). If, for the two triangles $\triangle ABC$ and $\triangle A'B'C'$, the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle A \equiv \angle A'$ hold, then the two triangles are congruent to each other. (p. 11)

In contrast to Birkhoffs' (1932) approach to geometry, in which distance is used as a primitive concept, implying that properties of real numbers directly apply to geometric segments, Hilbert's approach, merely uses points, lines and planes as primitive concepts. Consequently, a method for calculating with segments, in which the rules for calculating with real numbers remain valid, must be introduced and proved. This is obtained by proving Pascal's theorem². Thereafter the word

² Theorem 21. (Pascal's theorem.) Given the two sets of points A, B, C and A', B', C' so situated respectively upon two intersecting straight lines that none of them fall at the intersection of these lines. If CB' is parallel to BC' and CA' is also parallel to AC' , then BA' is parallel to AB' . (Hilbert, 1910 p. 25)

‘congruent’ and the sign \cong , could be replaced by the word ‘equal’ and the sign $=$. Consequently, a theory of proportion could be established and the following statement could be made:

If a, b, a', b' are any four segments whatever, the proportion $a : b = a' : b'$ expresses nothing else than the validity of equation $ab' = ba'$. (Hilbert, 1910, p. 32)

After introducing the theory of proportion, Hilbert (1902) defines the similarity concept:

DEFINITION. Two triangles are called similar when the corresponding angles are congruent.

THEOREM. If a, b and a', b' are homologous sides of two similar triangles, we have the proportion $a : b = a' : b'$ (p.32)

Similarity and congruence according to Birkhoff

Birkhoff (1932), developed another system of four axioms (in this approach named postulates) using four primitive concepts; points, lines, distance and angle. The fourth postulate concerns similarity:

POSTULATE IV: If in two triangles ΔABC and $\Delta A'B'C'$ and for some constant $k > 0$, $d(A'B') = kd(A,B)$, $d(A'C') = k d(A,C)$, and $m(\angle B'A'C') = m(\angle BAC)$, then also $d(B'C') = k d(B,C)$, $m(\angle C'B'A') = m(\angle CBA)$, and $m(\angle A'C'B') = m(\angle ACB)$. (p. 332)

Where distance is denoted by d and angle-measure by m . The postulate is commonly named SAS (side-angle-side) and can also be expressed:

...if ΔABC and $\Delta A'BC'$ have two sides proportional and the corresponding included angle equal, they are similar. (p 332)

Based on the primitive concepts and postulates, several definitions are made. One of them concerning similarity and congruence:

DEFINITION: Any two geometric figures whatsoever may be termed similar if there exists a correspondence of points such that all corresponding distances are in proportion and corresponding angles are all equal or the negatives of one another. In an analogous manner two broken lines or triangles or polygons or figures may be said to be congruent if they are similar with ratio of proportionality $k=1$ of distances, so that corresponding distances are equal. (p. 333)

It could be noticed that, in this approach, in contrast to Hilberts' (1910), similarity is the concept that is defined initially, and then congruence is treated as a special case, for which $k=1$.

From the similarity postulate, other theorems can be derived:

THEOREM II: Two triangles ΔABC and $\Delta A'B'C'$ are similar if two pairs of corresponding angles are equals or negatives of one another. (p. 334)

THEOREM IV. Two triangles ΔABC and $\Delta A'B'C'$ are similar if their corresponding sides are proportional. (p. 335)

Which are named the AAA and SSS similarity theorems.

A transformation based approach (Pieri)

As previously described a transformations-based approach to geometry has been advocated by scholars such as Pieri and Klein (Smith, 2010). According to that approach congruence and similarity should be defined in terms of rotations, reflections, translations, and dilations (Seago, 2013). An attempt to formulate a transformational approach for use in upper secondary geometry courses, was made by Usiskin and Coxford (1972). They define congruence as follows:

DEFINITION: Let α and β be any two figures. Then α is congruent to β if and only if there is a composite of reflections mapping α onto β .

They continue by explaining:

In most advanced geometry courses, the definition of congruence includes the word isometry. An isometry can be defined as a composite of reflections, or it can be defined as a distance-preserving transformation. These two definitions are equivalent. (p. 25)

Similarity is then defined:

DEFINITION: Let α and β be any two figures. Then $\alpha \sim \beta$ if and only if there is a composite of reflections or size transformations mapping α onto β .

In slightly different words, a definition of similarity, according to a transformational approach, may read:

DEFINITION: A figure F_1 is similar to a figure F_2 if there exists a similarity transformation (composition of dilation and isometry) such that: $s(F_1)=F_2$.

(Vollrath, 1977, p. 211)

DEFINITION: A two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations

(Seago, 2013, p. 74)

From the properties of transformations, familiar statements about congruence and similarity, can then be deduced (instead of, as in Hilbert's (1910) approach, using establishing axioms of congruence):

In similar figures, corresponding angles have the same measurement (for both size transformations and reflections preserve angle measure) and corresponding sides are proportional. (Usiskin & Coxford, p. 28)

One can also prove:

Any two line segments are similar; Any two circles are similar; Two angles are similar only if their measurements are equal; Two circular arcs are similar only if their measurements are equal. (Usiskin & Coxford, p. 28)

The triangle-similarity theorems, (SSS), (AAA), and (SAS), for congruence and similarity can also be derived using properties of isometries and similarity transformations (Usiskin & Coxford, 1972). Seago (2014) argues that both a transformations-based approach, and other approaches, (termed static approaches) to similarity are mathematically precise. However, when teaching similarity, a transformations-based approach is, according to Seago (2014), more robust since it offers more clarity when visualising the corresponding parts of transformed figures.

Definitions of similarity in secondary school textbooks

In secondary level textbooks, various ways of defining similarity can be seen. For example, the following, or corresponding, definitions are commonly used:

Two polygons are similar if the two following conditions are satisfied:
 All pairs of corresponding angles are congruent.
 All pairs of corresponding sides are proportional.

For triangles, it is sufficient that one of the conditions is satisfied for the triangles to be similar.

(see e.g. Wijayanti, 2019; Lo et al., 2006)

Wijayanti (2019), who analysed the treatment of the similarity concept in several college textbooks, argues that the meaning of ‘corresponding’ is a challenge with regards to the definitions, given in many textbooks. With a semi-formal definition involving ‘corresponding’ sides or angles, examples are, according to Wijayanti (2019), likely to be the only explanation of what ‘corresponding’ means.

Informal and imprecise definitions of similarity are also commonly used in textbooks, according to several scholars (Lo et al., 2006; Wu, 2005). Such definitions usually involve visual ideas like figures that have ‘the same shape, but not (necessarily) the same size’, carried by a large range of examples, without any explicit definition of what ‘same shape’ really means. Students are in these cases assumed to be capable of determining whether two shapes are similar by simply looking at them (Lo, et al., 2006). Wu (2005) argues that informal definitions, based on intuitive perceptions of what ‘same shape’ means, can sound attractive, but does not explain the way in which certain figures are similar, and may cause students to misinterpret the concept according to every-day speech and intuitive perceptions.

Teaching and learning the concept of similarity

Informal definitions and intuitive perceptions of similarity

It has been documented that even young students can reason about similarity using visual perceptions of shape. However, the everyday use of the term “similarity” tends to evoke images that do not correspond to the geometrical concept. Therefore, students, even in secondary school, frequently associate the similarity concept with various properties based on intuitive personal perceptions, rather than a formal definition (Vollrath, 1977). In particular, it has been observed that imprecise ideas of similarity may cause students to regard all triangles or rectangle as similar, because they generally look alike (Chazan, 1988), or to consider figures, that are actually similar but oriented differently, as non-similar, since similar figures are typically drawn beside each other with the same orientation. (Seago, 2010; Kallia & Panagiotis, 2010)

Therefore, several scholars argue that a conceptual understanding, based on formal definitions and theorems, should be emphasised in secondary education (Haj-Yahya, 2021; Kallia & Panagiotis, 2010; Vollrath, 1977; Wu, 2005). However, even though visual and intuitive perceptions are insufficient in order for students to achieve a valid conceptual understanding of similarity, scholars argue that visual perceptions can support students’ reflections and improvement of numerical

strategies (Cox, 2013; Lehrer et al, 2002). Cox (2013) states that students can be encouraged to mathematise their visual perceptions in order to increase their ability to attend to the numeric relationships between shapes and argue that ‘the use of visual reasoning to blend numeric and geometric reasoning on similarity tasks is cognitively useful’.

Likewise, Lehrer et al. (2002) suggest that children’s intuitive ideas about similarity should be developed and expressed in terms of symbolic descriptions and algebraic rules, and subsequently further explored and generalised. The strategy of using intuitive spatial reasoning in parallel with formal definitions is also used by senior mathematicians. Wang (2015), who compared prospective teachers’ and mathematicians’ answers to similarity identification tasks, found that both groups used visual intuitive strategies at a first level of identification. However, when the prospective teachers were asked to explain and to justify claims, based on conceptions of similarity, a vague understanding of the concept was displayed. Mathematicians, on the contrary, used a wide variety of strategies. They were aware of properties of similarity at an abstract level, they were prepared to consider different approaches to verify similarity, and they looked for both internal ratios and between ratios.

It can thus be concluded that, intuitive, visual perceptions are useful to get a first idea of a geometric concept or a task but have to be developed and explained in terms of accurately defined relations.

Similarity of Triangles

Regarding similarity of triangles, Haj Yahara (2021) concludes that many students tend not to discern minimal conditions for similarity of triangles (i.e. the SAS, SSS and AAA similarity theorems). Moreover, it turns out that students more readily seem to accept minimal conditions based on proportional lengths (i.e SSS), than minimal conditions based on congruent angles (i.e AAA). The tendency to appreciate conditions based on proportional lengths rather than congruent angles is also described by DeJarnette et al. (2014), who studied the conceptual field, used by students, working on similarity tasks. They conclude that ratio and proportion, rather than angle measure, were the most commonly used concepts in action when students enlarge complex puzzle pieces.

Proportional reasoning in geometric similarity tasks

In research literature, the topic of similarity frequently appears, not primarily as an important geometric concept, but rather as a context for proportional reasoning

(Arican, 2018; Cox, 2013; Lamon, 2007). As such, several studies confirm that geometric similarity tasks, turns out to be the most challenging proportionality tasks (Lamon, 2007). Students who are capable of proportional reasoning in numeric contexts, may still be unsuccessful on similarity tasks (Arican, 2018; Cox 2013). Results of several studies also indicate that students' difficulties with solving similarity tasks, often stem from aspects of the geometric configuration, rather than the procedure to solve the proportion correctly (e.g. Chazan, 1987; Cunningham & Rappa, 2016; Seago, 2013; Ubah, 2021). Meanwhile, Son (2013) showed that even when students' errors derive from conceptual aspects of similarity, a majority of the preservice teachers in his study misinterpreted the errors as originating from procedural aspects of solving the proportion.

One of the most common sources of students' errors regarding proportionality in general, and geometric similarity in particular, appears to be the use of additive, rather than multiplicative, strategies (e.g. Cox, 2013; Lamon, 2007; Son, 2013), that is, students tend to focus on the difference between quantities rather than the multiplicative relationship. It has also been documented that students who use multiplicative strategies in one particular kind of similarity tasks, may still switch to additive strategies, when solving comparable tasks, with another geometric configurations (Chazan, 1988). Chazan (1988) therefore argues that aspects of the geometrical configuration in a similarity problem have to be taken into consideration when exploring the cause of students' incorrect use of scaling strategies, and that students' lack of proportional reasoning within a geometric similarity context, is closely related to the particular geometric aspects of the task (Chazan, 1988). Moreover, Ubah (2022), conclude that students' difficulties in similarity tasks frequently derives from difficulties to correlate visual and symbolic representations and reasoning. However, even if the geometric aspects, and the correlation between geometric and algebraic properties seems to infuse certain challenges for students, visual structure can, on the other hand, also afford powerful opportunities to enhance the understanding of abstract mathematical concepts.

Arican (2018) argues that an essential goal of mathematics education is to make connections among mathematical ideas, and that geometric similarity and proportional relationships are two such areas. Correspondingly, Lee and Yim (2014) argue that instruction ought to be designed to allow students to use and relate various representations of mathematical objects. For instance, the use of geometric similarity is a powerful means to visualise proportional relationships in general, and essential for being prepared to apprehend the idea of slope (for

example of linear functions, or of a roof). Seago et al. (2010) state that it is necessary for students to discern that the proportionality relation in similar figures can be expressed in different ways, i.e., either through internal or through external ratio (see Fig. 4.1).

Similarity transformations

As described above, some scholars argue that a transformations-based approach to geometry would facilitate students learning, and result in a more appropriate way of perceiving the concept. From research findings, which show that ‘setting up’ a proportion tend to be more challenging for students than solving the proportion, Seago et al. (2010) conclude that students’ difficulties in solving similarity tasks often stem from a lack of understanding of geometric transformations. It is, according to Seago et al. (2010) precisely in ‘setting up’ the proportion that knowledge of geometric transformations is critical. In particular, Seago et al. (2010) state that a static definition, makes it challenging for students to determine which parts of the given figures that correspond. By comparing figures through the underlying transformations, however, it becomes clear which sides and angles correspond and why (Seago, 2013). Also, Cunningham & Rappa. (2016), who compared prospective teachers answers to similarity problems, confirm that, while all teachers solved static problems correctly, only half of them succeeded to solve problems that required conceptions of transformations. Seago (2013) argues that a transformations-based approach to similarity can result in a more robust ability to apply the concept, and in the long term, open a door of opportunity mathematically.

Through a transformations-based approach, students are encouraged to visualise the specific transformations, needed to make two similar figures coincide, and to create classes of similar figures, through similarity transformations. Chazan (1988), who studied students’ perceptions of a few specific similarity tasks, found that it seems to be particularly difficult for students to solve similarity problems that involves rotated, reflected, and embedded similar triangles, which share sides. Such problems demand correct identification of correspondences and proportions among segments. These tasks, involve two steps. First similar figures must be identified, and then the correct correspondence have to be discerned in order to choose the correct proportions. Chazan (1988) performed an in-depth interview study, to identify students’ difficulties in these kinds of tasks. He concludes that it does not seem to be difficult for students to accept reflected triangles as similar, but rather to perform mental flipping of triangles in order to find a correct

correspondence. Triangles sharing sides and an angle, in embedded triangles also seem to cause particular difficulties to recognise the involved triangles and the similarity among them. Lo et al. (2006), who studied the treatment of similarity in three middle grade textbook series, conjecture that the lack of experience with identifying the corresponding parts, which was shown in the textbooks in their study, might contribute to the low success rate among high school students in seeing similar triangles in complex figures.

Ratio and proportion

The similarity concept relies on the more general concept of proportionality. Proportionality represents a body of knowledge that permeates the entire mathematics curriculum from elementary school to university level. Lamon (2007) asserts that the field of proportional relations is one of the most essential and extensive in mathematics education, and at the same time one of the most mathematically complex and cognitively challenging. Proportion in turn relies on the concept of ratio. The two concept are described below and then research on how proportional relationships are taught and understood, is reviewed.

Ratio

A ratio is defined as a multiplicative comparison between any two or more quantities. According to *'The Concise Oxford Dictionary of Mathematics'* a ratio is defined and explained as follows:

The quotient of two numbers or quantities giving their relative size. The ratio of x to y is written as $x : y$ and is unchanged if both quantities are multiplied or divided by the same quantity. So $2 : 3$ is the same as $6 : 9$ and $1 : 3/2$. When a ratio is expressed in the form $1 : a$ it is called a unitary ratio, and this form makes comparison of ratios much easier.

(Clapham, et al., 2009, online)

The Collins Dictionary of Mathematics gives the following definition:

Ratio: a QUOTIENT or PROPORTION of two numbers, magnitudes, quantities or expressions, such as a measure of the relative size of two classes; for example, the ratio of the side of a square to the diagonal is $1 : (\text{square root of } 2)$

(Borowski & Borwein, 1989, p 498)

A ratio can be denoted in a number of ways, for example:

‘a to b’

a:b

a/b, which may also be expressed as a percentage

A ratio may be represented either by the numbers a:b, or by the fraction a/b, that is, equal quotients correspond to equal ratios. It could be discussed whether the different expressions are exchangeable, and whether it is therefore unnecessary to make use of different notations. Scholars argue that the common notation between fractions in general, and ratios can make the relationship between them confusing, and that the important difference, for example, between ratios and part whole fractions, can become lost when the same notation is used (Lamon, 2007, Petit et.al., 2020). Therefore, colon or fraction notations are sometimes used alternately (for example in the US) depending on which characteristics of ratio that are intended to be highlighted (Lamon, 2007).

In Sweden today the : notation seem to have a limited use in secondary mathematics textbooks (according to my own reading). It is however used in many vocational applications of geometry, such as scale-drawings (a unit of measure on a map : the real distance) inclination ratio (horizontal distance : vertical distance) and the ratio of components in a mixture (i.e. in concrete, cement : sand : gravel).

There is a variety of different situations in which the concept ratio is used. For example, the ratio between number of boys and girls in a class, the ratio between ingredients in a recipe, the score of a football game, the ratio between the length and width of a rectangle, velocity, scale and population density. According to Freudenthal (1983) there are basically three types of ratios:

- Comparing magnitudes of different quantities. These comparisons are usually not called ratios, but rather rates or densities.
- Comparing parts of a single whole.
- Comparing magnitudes of two quantities that are related but not obviously considered as parts of a whole.

The ratio of lengths in similar figures, is an example of the last category.

Proportion

A proportion is a statement expressing the equivalence of two ratios A:B and C:D. According to the Collins Dictionary of mathematics, a proportion is:

1: A LINEAR relationship between two variable quantities or their inverse; corresponding elements of two sets that are in proportion, are in a constant ratio.

2: A relationship between four numbers or quantities in which the ratio of the first pair equals the ratio of the second pair; written $a:b=c:d$, or $a:b :: c:d$.

(Borowski & Borwein 1989, p. 474)

A proportion is thus a system of variables between which there exists a linear functional relationship. It describes situations that can be characterised by a constant ratio. A proportion can be expressed in different but equivalent forms.

$$a : b :: c : d$$

$$a : b = c : d$$

$$\frac{a}{b} = \frac{c}{d}$$

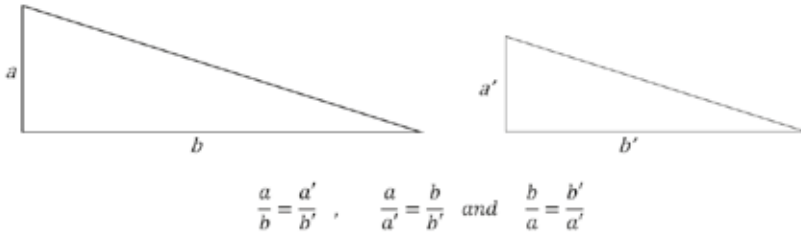
For a given proportion:

$$\frac{a}{b} = \frac{c}{d}$$

The following proportions are obviously also true:

$$\frac{b}{a} = \frac{d}{c} \text{ and } \frac{a}{c} = \frac{b}{d} \text{ and } a \cdot d = b \cdot c$$

In the case of similar figures this means that, since the ratio of corresponding side lengths are in proportion, the ratio of lengths within a figure are also equal to ratios of corresponding lengths in a similar figure.

Figure 4.1 Similar triangles internal and external ratio

Freudenthal (1978) made a distinction between *internal ratio* and *external ratio*. The comparison of ratios within the same measure space he called an internal ratio (e.g. $\frac{x_1}{x_2} = \frac{t_1}{t_2}$) and the comparison of ratios between different measure spaces is called external ratio³ (e.g. $\frac{x_1}{t_1} = \frac{x_2}{t_2}$). Regarding similarity, the expression ‘external ratio’ is used to denote the ratio of *different lengths* in the same figure ($\frac{a}{b} = \frac{a'}{b'}$), while ‘internal ratio’ is used to denote the ratio of *corresponding lengths* in similar figures ($\frac{a}{a'} = \frac{b}{b'}$). When it comes to vocational applications of similarity, *internal ratio* is thus used in the case of scale drawings, and *external ratio* is used in the case of slope, for example in contexts such as roofs, ramps, ditches, roads, or drainages. Moreover, a distinction is made between intensive and extensive variables. The term *intensive* is used to describe rates, i.e. quantities, such as speed, slope and density, whose specification implies a constant ratio relationship. *Extensive* variables comprise quantities such as length, time, or mass, which only depend on the size or extent of an object (Karplus, 1983).

Teaching and learning proportionality

In mathematics education, the ability related to detecting, expressing, analysing and explaining proportional relationships in different situations, is termed proportional reasoning. Lamon (2007) propose that proportional reasoning could be described as follows:

³ A confusion has arisen in research literature, concerning whether we should talk about ratios within and between systems (scientific definition) or within and between measure spaces (Freudenthal’s definition). These two perspectives motivate contradictory definitions of internal vs. external ratios (Lamon, 2007)

Supplying reasons in support of claims made about the structural relationship among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios and products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities. (p. 637, 638)

She continues by arguing that the ability to solve certain proportion tasks, does not guarantee that a proportional reasoning is taking place. Rather, proportional reasoning requires that the underlying multiplicative structure of a situation, including the invariant relationship between covarying quantities, is discerned.

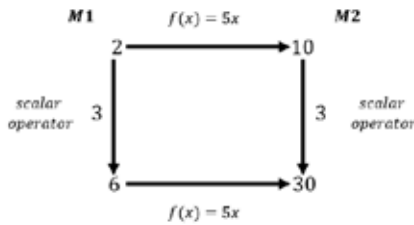
Research literature have identified three different types of proportionality problems (e.g. Vergnaud, 1983; Karplus et al., 1983; Lamon, 2007)

- Missing value problems, in which three out of four values in the proportion ($a/b=c/d$) are given, and the task is to find the missing value
- Comparison problems, in which all four values are given ($a, b, c,$ and d) and the task is to compare the ratios.
- Qualitative prediction and comparison problems, in which the task is to evaluate the effect on a ratio of a change in the quantities involved

Solving missing value tasks, or *isomorphism-of-measure* problems, involve discerning a structure, comprising a direct proportion between two measure-spaces $M1$ and $M2$ (Vergnaud, 1983). These tasks arise in many familiar contexts, such as equal sharing, constant price, uniform speed, constant density, and the preservation of ratios in similar figures. Vergnaud (e.g. 1983) developed a framework that illustrates how such problems are organised (Fig.4.2).

Figure 4.2 Direct proportion between two measure spaces.

(Inspired by Lamon, 2009)



The figure shows that, a linear functional relationship ($y=k*x$) links elements of two measure spaces (where k represents the external ratio), and a scalar operator is used to transform quantities corresponding to the same measure space (internal ratio). Proportional reasoning thus entails discerning both the constant ratio between elements of the same measure space and the functional relationship between measure spaces (Lamon, 2007) (for example discerning the invariance of both external and internal ratios in pairs of similar figures).

The framework also illustrates how the same proportional problem can be solved in various ways, either by applying a scalar operator, by applying a function operator, or by a missing-factor procedure. Research has demonstrated a variety of strategies that students employ when solving proportion problems (Arıcan et al., 2025; Karplus et al., 1983). Although these strategies can sometimes be used interchangeably, they represent different levels of sophistication. Some are more elementary and procedural, while others are more general and conceptually powerful (Vergnaud, 1983). The ability to relate, evaluate, and flexibly switch between strategies can therefore be regarded as an indicator of a more advanced understanding of proportional relationships.

Scholars have identified several aspects that are essential for developing an advanced understanding of ratio and proportion (e.g. Hilton et al., 2013; Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983; Weiland et al., 2021). A key aspect, emphasised in several frameworks, entails distinguishing multiplicative from additive relationships, and discerning which situations involve multiplicative relations (Hilton et al., 2013; Karplus et al., 1983; Lamon, 2007; Noelting, 1980). Moreover, the capacity to identify underlying logical and algebraic structures in proportional situations and thereby make informed decisions about efficient strategies reflects a more sophisticated level of understanding than merely solving proportion tasks by applying procedures or relying on context-bound reasoning (Ben-Chaim et al., 2012; Noelting, 1980; Vergnaud, 1983; Weiland et al., 2021).

A robust understanding of proportionality further includes viewing ratio as a comparison between two quantities, either within or across measure spaces (Karplus et al., 1983; Thompson, 1994; Weiland et al., 2021). This constant relationship typically gives rise to a third quantity that could be abstracted and conceptualised in its own right (called unit rate intensive quantity, or functional operator) (Hilton et al., 2013; Lamon, 2007; Thompson, 1994; Vergnaud, 1983). For example, the relationship between distance and time leads to speed, the relationship between corresponding sides in similar figures leads to scale and the relationship between rise and run leads to slope. Lamon (2007) highlights the

context-specific nature of the functional operator (the k constant in $y = kx$), emphasising that interpreting the meaning of this constant within a particular context is a challenge that extends beyond understanding proportional relationships as such. Moreover, differentiating and relating fractions and ratios, as well as recognising various representations of these (e.g. colon, fraction bar, decimal, or percentage), are identified as crucial aspects of proportional understanding (Lobato & Ellis, 2010).

Research findings further indicate that a teaching approach focusing on ratios may facilitate a deep understanding of proportions as well as of fractions. The contrast achieved by introducing the part-to-part perspective associated with ratios makes the part-to-whole perspective of fractions more explicit, a relationship that may remain implicit when fractions are taught in isolation (Howe et al., 2011). Finally, a robust understanding of proportional reasoning is considered to entail connecting proportionality to broader mathematical ideas such as geometric similarity and the slope of linear functions (Weiland et al., 2021).

Previous research has identified stages in the development of proportional reasoning, which represent qualitative shifts in understanding. Inhelder and Piaget (1958) first described a developmental progression from intuitive reasoning, through concrete and context-bound reasoning, to formal reasoning based on generalised mathematical principles. Later studies refined this model (e.g. Noelting, 1980) by identifying intermediate levels. Rather than a sharp transition from concrete to formal thinking, Noelting (1980) argues that proportional reasoning evolves through a sequence of developmental levels, reflecting a gradual development. Moreover, recent research (e.g. Vanluydt et al., 2020) demonstrates that the development of proportional reasoning begins earlier than previously assumed. Even young children, before being taught proportional relations, show emerging proportional awareness evolving from intuitive to more formal reasoning as they engage with diverse proportional situations.

According to Vanluydt et al. (2020), the development of proportional understanding depends on learners' prior experiences, the contextual features of tasks, and the inherent properties of the quantities being compared. Ben-Chaim et al. (2012) suggest that instruction in ratio and proportion should build on students' intuitive reasoning and concrete perception as foundations for developing a more formal understanding of proportionality. By first recognising and working with informal approaches, teachers can support learners in constructing the intuitive basis upon which formal proportional strategies can be developed (Ben-Chaim et al., 2012). Similarly, Jitendra et al. (2019) demonstrate that middle school students'

proportional reasoning might be enhanced through intervention, building on context-rich problems, and supporting students in recognising underlying problem structures, using visual-schematic representations, and developing metacognitive awareness.

The concept of slope

Closely connected to the concepts of proportionality and similarity is the concept of slope, a fundamental mathematical idea that appears in numerous contexts and representations throughout the curriculum (Nagle et al., 2019; Stump 1999). Slope is both a foundational prerequisite for advanced mathematics and a concept that is frequently applied across diverse contexts and disciplines (Nagle et al., 2013). It is an example of a geometric proportion and also a powerful means to visualise the functional operator in proportional relationships in general (Seago, 2014). Previous research has identified multiple ways of conceptualising slope. A framework compiled by Stump (1999) and further developed by Moore-Russo et al. (2011) outlines eleven distinct categories. These range from physical interpretations in real-world settings to trigonometric and calculus conceptions. In order to develop a comprehensive understanding of slope, Nagle et al. (2013) argues that students must be able to integrate and connect these various conceptualisations.

- Physical Property
- A general understanding of the ‘steepness’ or ‘slant’ of a line
- Algebraic Ratio
- Identified by the symbolic ratio: $\frac{y_2 - y_1}{x_2 - x_1}$
- Geometric Ratio
- Identified by determining the rise (vertical displacement) divided by the run (horizontal displacement) of a line
- Parametric Coefficient
- Identified by the coefficient m in the equation $y = mx + b$
- Functional Property
- Representing the rate of change between variables
- Trigonometric Conception
- Representing the tangent of a line’s angle of inclination
- Calculus Conception
- Relating to the tangent line to a curve at a point

- Real World Situation
- Relating to either static or dynamic situations involving lines
- Determining Property
- Being used to determine parallel, perpendicular
- Behavior Indicator
- An indication if a line is increasing or decreasing (or even horizontal)
- Linear Constant
- A constant property unique to 'straight' figures that is independent of representation

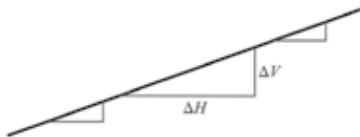
(Nagle et al., 2019, p 5)

Slope as a geometric ratio

Understanding slope as a geometric ratio means seeing it as vertical change ΔV (rise) over horizontal change ΔH (run). This ratio could be understood in a static, or in a dynamic way (Nagle et al., 2019). A dynamic approach entails a mental visualisation of a potentially infinite set of similar triangles, conceived as movable along a line, thereby forming equal ratios of ΔV to ΔH . Drawing on the concept of similarity such views enable students to realise that the slope does not depend on the particular triangle used to compute it and understand that parallel lines have equal slope (see Fig. 4.3). (Nagle et al, 2019; Seago 2013).

Figure 4.3 Geometric ratio and similar triangles

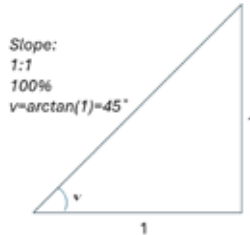
(Nagle et al., 2019, p 6)



Slope in real world situations

In real-world contexts, particularly in vocational settings, slope typically appears in two distinct types of situations: physical and functional. Functional contexts involve relationships between variables, where the slope represents a rate of change. In physical contexts, such as, roads, ditches, wheelchair ramps, roofs, and drainage system, slope is understood as a measure of steepness (Nagle et al., 2019). It can be described trigonometrically, as the tangent of a line's inclination ($k = \tan \theta$), or as a geometric ratio, either in the form of $\Delta V:\Delta H$ or as the calculated value $k = \Delta V/\Delta H$. This value is often expressed as a percentage and interpreted as centimetres of rise per metre of run.

Figure 4.3 Example of different representations for the slope of a surface



Teaching and learning the slope concept

Scholars have emphasised the importance of providing opportunities for students to see relationships in mathematics (e.g. Skemp 1976) including linking new ideas to prior knowledge, connecting conceptual and procedural understanding, and bridging between mathematical and real-world contexts. When slope is concerned, scholars argue that students should get opportunities to explore and connect its various representations: physical, functional, geometric, and trigonometric. This also involves becoming familiar with the specific symbol systems and rules used to describe slope in different contexts (Stump, 1999). While such connections can foster a sense of coherence across mathematical domains and support a deep understanding of the slope concept, the diversity of conceptualisations can also pose challenges for both students and teachers (Hoban, 2021; Nagle et al., 2013; Seago, 2013). In a study of students' understanding of slope, Nagle et al (2013) found that students even in university-level mathematics courses often display a procedural focus, and their conceptions of slope tend not to transfer between different types of problems.

Moreover, geometric interpretations of slope are often disconnected from conceptions of slope as a rate of change between variables. Relations which according to Nagle et al. (2013) are critical to address for a deep and coherent understanding of the concept. Stump (1999) suggests that this difficulty in making connections may stem from the way instructional materials, especially textbooks, typically present slope. If knowledge about slope, linear equations, rate of change, and trigonometric functions is introduced in isolated segments, with few explicit connections made across contexts or curriculum stages, it is likely that teaching

practices present these concepts as separate, thus limiting students' opportunities to develop a relational understanding (Stump, 1999).

Geometric measurement concepts

Measurements play a central role in geometric and spatial reasoning. It has multiple applications in vocational as well as everyday contexts and is fundamental to the understanding of various advanced mathematical concepts. Nevertheless, research findings indicate that students, even at secondary level, often lack a conceptual understanding of measurement, particularly of area and volume (Battista, 2007; Vergnaud, 1983). Vergnaud (1983) describes area and volume calculations as instances of multiplicative structures called 'product of measure problems'. In such problems two (or more) measure spaces M_1 and M_2 are composed into a third M_3 . Vergnaud (1983) found that, when solving such problems, students often struggle to coordinate the component measures. They tend to reason additively or confuse the product space with one of its dimensions. Such confusions can persist even at secondary level, making concepts like volume far more difficult than curricula typically assume (Vergnauds, 1983). This lack of understanding might be reflected in a procedural use of measurement formulas, without attention to their underlying meaning (Clements et al., 2018). Lamon (2007) argue that elementary curriculum typically handles measurement in a way that trains students to perform the act of measuring rather than developing a conceptual understanding. Such a poor conception of measurement might lead to errors, such as confusing area with perimeter or volume with surface area, and to difficulties with non-routine problems, and problems which require explanation (Smith et al., 2016).

A central explanation for the lack of conceptual understanding is, according to Battista (2007, 2017), a disconnection between spatial and measure-based reasoning. Working on non-routine measurement problems requires both a conception of the spatial structure and an ability to coordinate this structure with numerical calculations. An important aspect to consider in this regard is the conservation of length, area and volume over decomposition and transformation.

Another critical point, fundamental to the understanding of measurement concepts, that is emphasised by Battista (2007) concerns the required transition from discrete to continuous quantities. Understanding measurement entails applying the process of discrete unit counting to quantities such as length, area, or volume, which are continuous in nature. To understand area (or volume) measurements, students must first conceive of area as composed of iterated unit

areas. This involves identifying an appropriate unit square (cube) and understanding how coordinated iterations of such units, in line with the principle of additivity, represent the extent of a region (Battista, 2007, Clements et al., 2018). At the same time, learners need to reconcile the tension between area (or volume) as a continuous magnitude and the discrete practice of counting unit squares (cubes), which entails perceiving multiplication as a multiplicative composition, i.e. a structuring principle rather than merely repeated addition of units (Smith, 2016). This implies that students' understanding must develop from seeing a rectangle (prism) simply as an arrangement of rows and columns of unit squares (cubes) to a principled conception of area (volume) as the product of its side lengths.

At an advanced level of measurement conceptualisation, the idea of explicit unit iteration should thus recede into the background, in such a way that learners may operate on measurement numerically and inferentially and seamlessly integrate the understanding of geometric measurement concepts with non-measurement geometric reasoning, such as the principle of area (or volume) conservation under decomposition and rigid motions, described above (Battista, 2007). This conception of area and volume measurements is essential in vocational contexts, where measurement problems may include decomposing complex areas and volumes into standard shapes and applying measurement formulas in conjunction with additive reasoning.

Summary

The review above concerns mathematical concepts that are central in construction-related vocational practices (see Chapter 3). Taken together, the literature shows that developing a fundamental understanding of these concepts is far from self-evident. Furthermore, the concepts are mathematically connected at different levels, which adds another dimension to the complexity. A deep, relational and coherent understanding requires attending to such relationships. Such connections could be conceptual, as the concepts build on and presuppose each other. Similarity relies on proportionality, since it entails that corresponding lengths are in proportion. Slope, in turn, can be understood as a specific ratio between vertical and horizontal change, and can be visualised through similar triangles. Measurement concepts such as area and volume also depend on proportional and geometric relationships. In addition, several of these concepts function as bridges across mathematical domains. Similarity in particular connects geometry, proportional reasoning, transformations, and measurement. For this

reason, such concepts are both central and demanding. They require not only the application of rules, but the development of a conceptual and coherent understanding that establishes relationships and integrates different areas of mathematics.

5. Theoretical framework

Phenomenography

Phenomenography is a research approach aimed at identifying and describing the qualitatively different ways in which people experience aspects of the world around them (Marton, 1986). The approach was developed in the early 1970s with the intention of complementing other forms of research by focusing not on the world as such, but on how it is experienced (Marton, 1981). Initially, phenomenography emerged as an empirical approach grounded in general observations rather than in specific theoretical stances. When the framework was later summed up, however, differences and similarities between the fundamental assumptions made within phenomenographic research, and the epistemological and ontological premises underlying for example, phenomenology and gestalt psychology, were more explicitly recognised and explicated (Uljens, 1996).

The word 'phenomenography' which first appeared in the work of Marton (1981), is a compound from two Greek words 'phainemenon' and 'graphein' (Pang, 2003). 'Phainemenon' is a noun that derives from the verb phainseqai, which means 'to appear', or 'to become manifest'. Accordingly, 'phainemenon' means 'that which manifests itself' or 'that which can be revealed and apparent' while 'graphein' means 'to describe' (Kroksmark, 1987). In other words, phenomenographic research is concerned about describing various ways in which that, which is apparent, might appear.

Phenomenography, like phenomenology (e.g. Husserl, 1970), is grounded in the principle of intentionality (Marton, 1986; Pang, 2003; Uljens, 1996), which entails that awareness is always directed towards something. There is no experience without something being experienced, and a way of experiencing is understood as a relation between the experiencer and that which is experienced. According to phenomenography, it is through such relations that meaningful phenomena are constituted (Kroksmark, 1987; Marton, 1986; Marton & Booth, 1997). In accordance with gestalt psychology (e.g. Wertheimer, 1938), it is further assumed that, at any given moment, what is in the focus of attention tends to be perceived as a meaningful whole, a gestalt, identified and discerned against its surrounding

context (Marton & Booth, 1997). In phenomenography, as in phenomenology, such meaningful configurations are referred to as phenomena. However, while phenomenology focuses on the essence of experience, phenomenography focuses on variation in ways of experiencing (Kroksmark, 1987; Uljens, 1996). From this perspective, a phenomenon is best understood in terms of the qualitatively different ways in which it is experienced at a collective level. In phenomenographic research, such structured sets of perceptions are described in terms of categories of descriptions comprising a phenomenographic outcome space (Marton, 1981).

Ontologically, phenomenography neither takes the position that the world is entirely objective and possible to describe unambiguously in a true way, nor does it assume that the world is entirely determined by human thought and exists solely through human conception. In this sense, phenomenography is neither materialistic nor idealistic (Booth, 1992), but rather adopts a non-dualistic perspective (Uljens, 1996). The only reality to which we can meaningfully refer is an experienced reality. Consequently, a way of experiencing something cannot be compared to an objective reality, but only to other ways of experiencing. Qualitatively different ways of understanding, corresponding to different ways in which a phenomenon is constituted in human awareness, can be categorised and compared to each other. This mapping of variation constitutes the object of phenomenographic research (Kroksmark, 1987; Marton, 1986; Uljens, 1996).

Methodologically, this research focus entails a second-order perspective. The distinction between first- and second-order perspective is fundamental to phenomenographic research and implies that a phenomenon is described, not *per se*, but in terms of the range of qualitatively different ways in which it is experienced.

Another fundamental assumption in phenomenography is that there is only a limited number of qualitatively different ways in which a phenomenon can be experienced (Marton, 1981; Marton & Booth, 1997). This assumption is grounded in empirical observations showing that different perceptions of the 'same' phenomenon tend to cluster into a finite set of qualitatively distinct ways of understanding. Marton (1981) describes these as a 'level of modes of experiencing' that exists between the individual and the collective. The limitation in ways of experiencing can be understood in relation to ideas about the structure of awareness. According to Marton & Booth (1997), the way in which a phenomenon is experienced depends on the aspects that are simultaneously in the focus of awareness. Since human mind is limited in terms of its capacity, only a restricted number of aspects can be discerned at any given moment. Different ways of

experiencing arise from differences in which aspects are discerned and how they are related to each other and to the whole. As both the number of relevant aspects and the ways in which they can be combined are finite, the variation in ways of experiencing a phenomenon is limited (Marton, 1981; Marton & Booth, 1997).

Alternatively, it can be argued that it is precisely the combination of limited variation and a shared core that makes it possible to interpret different perceptions as referring to the 'same' phenomenon.

Phenomenographic research

Data in phenomenographic research is typically generated through semi-structured, open-ended interviews designed with the aim of exploring variation in meaning. The purpose of the interviews is not to elicit factual knowledge, but to access participants' ways of understanding (Marton, 1986; Marton & Booth, 1997). The interviewer adopts a flexible and responsive stance, encouraging participants to consider the phenomenon from different perspectives. At the same time, the direction of the interview is allowed to develop in response to the participants' contributions. Throughout the process the interviewer maintains a clear focus on the phenomenon under investigation (Bowden & Walsh, 2000).

The analysis typically begins with repeated readings of the interview transcripts in order to become familiar with the data and to gain an initial sense of variation in meaning. In this early stage, the researcher maintains an open stance, avoiding premature categorisation and allowing potential meanings to emerge from the data (Åkerlind, 2005). Subsequently, relevant excerpts are selected from the transcripts. When analysing empirical data, for example from interviews, the selected quotations from the data are considered to make up a 'pool of meaning', which forms the basis for the analysis. The researcher's attention is, then, not directed to individual subjects, but rather to the meaning expressed in the quotations.

Importantly, phenomenographic interviews do not provide direct access to participants' experiences. Rather, they generate accounts that are interpreted by the researcher, who seeks to describe the variation in ways of experiencing as expressed through participants' language (Runesson, 1999; Åkerlind, 2005). A phenomenographic outcome space does not emerge independently of the researcher, but rather through the researcher's intentional engagement with the empirical material. In line with a non-dualistic perspective, other people's experiences are understood as a part of the experienced reality. As with all aspects

of the world, they are not directly accessible or describable as such but are only encountered through interpretation and meaning making.

As a result of the interpretive analysis, quotations are gathered into categories on basis of their similarities, and categories are differentiated due to their differences. Through a process of comparison and contrast, similarities and differences in meaning are identified. The analysis proceeds through repeated cycles of grouping and regrouping excerpts based on these similarities and differences. Categories of description are gradually constituted, tested against the data, and refined through ongoing comparison between excerpts and between emerging categories (Marton & Booth, 1997; Åkerlind, 2005). In addition to identifying categories the researcher seeks to establish the structural relationships between them. The result is presented in terms of categories of description, which are described in terms of their core meanings and illustrated with quotations from the data (Marton, 1986).

Phenomenographic results

In phenomenography, perceptions and ways of experiencing phenomena are not seen as qualities attributable to individuals. Nor is an individual assumed to possess a single, fixed way of experiencing a phenomenon (Marton, 1981). During years of phenomenographic research, it has been observed that a single individual's statements may alternate between qualitatively different ways of understanding, for example during an interview (e.g. Dahlgren, 1979; Marton & Pong, 2005). Marton (1981) argues that this indicates the impossibility of making reliable statements about an individual's conceptions, utilisable to predict that individual's behaviour in other situations. However, even though individuals may move from one way of experiencing to another on different occasions, the set of categories could be considered stable, representing a reliable description of conceptions, at a collective, rather than individual, level. The outcome space, thus, presents a description on a collective level, at which individual voices are not distinguishable (Marton & Booth, 1997).

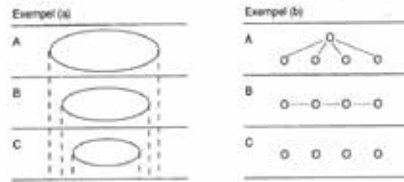
Moreover, what is communicated in an interview could not be considered fully consistent with the particular way the phenomenon is understood by a specific individual even at a certain moment. At the same time, it can at least be assumed to reflect the individuals' way of seeing. Since individuals could hardly express a conception that goes beyond their repertoire of ways of experiencing, their accounts provide instances, which, when analysed in relation to other instances,

make it possible to identify a set of qualitatively different categories of description (Kullberg & Ingerman, 2022). In this way, the set of categories can provide a reliable description of the phenomenon at a collective level, even if specific excerpts do not fully reflect a particular individual's way of experiencing. This is consistent with the aim of phenomenographic research, which is not to classify or make judgements of people, or to compare groups of people with each other, but rather to find and systematise forms of thought (Marton, 1981).

Phenomenographic outcome-spaces are typically hierarchical. This is based on the assumption that the capacity for action in a situation depends on how the situation is perceived. Ways of experiencing that enable more appropriate or flexible action could therefore be regarded as more powerful than others. Phenomenographic outcome space, are concequently frequently organised according to how advanced the categories are considered to be. (Marton et al., 2004; Pang & Ki, 2016). In this sense, qualitatively different ways of experiencing a phenomenon form a structured set, where more complex ways of seeing build on and integrate aspects that are present in less complex categories, while also adding new dimensions. Marton & Booth (1997) define a number of quality criteria for the set of categories, resulting from a phenomenographic analysis:

- The individual categories should each stand in clear relation to the phenomenon of the investigation so that each category tells us something distinct about a particular way of experiencing the phenomenon
- The categories have to stand in a logical relationship with one another, a relationship that is frequently hierarchical.
- The system should be parsimonious, which is to say that as few categories should be explicated as is feasible and reasonable, for capturing the critical variation in the data.

Uljens (1989) elaborates further on the structure of an outcome spaces and suggests that phenomenographic outcome spaces could be divided into a few different types: *horizontal*, *vertical*, and two kinds of *hierarchical* structures. A horizontal outcome-space is referred to as a set of categories which are equally advanced and not ranked mutually. In contrast, a vertical structure is based on increasing quality in relation to educational goals. The hierarchical structure could according to Uljens (1989) emerge in two ways: either when categories become increasingly inclusive (Example a, Fig 5.1) or when they represent more developed relationships between the parts and the whole (Example b, Fig 5.1).

Figure 5.1 Two types of hierarchical structures*(Uljens, 1989)*

As phenomenography, and later variation theory, evolved, the structure of the outcome space came to be understood in terms of the discernment of critical aspects (see definition below). The critical differences among categories were then attributed to different patterns of aspects that are simultaneously focused upon and internally and externally related (Marton et al., 2004; Pang & Ki, 2016).

Compared to Uljens' (1989) description, distinctions between categories in a horizontal outcome space can be associated with the discernment of different critical aspects. Hierarchical outcome spaces could be deduced from the discernment of an increasing number of critical aspects or from a more elaborated awareness of the relationships among aspects. However, the fundamental theoretical assumptions of VIL does not imply that an outcome-space must be either strictly hierarchical or entirely horizontal. Complex outcome spaces, structured through a combination of key principles would rather be expected.

Structural and referential aspects

Within the phenomenographic research tradition, the question of what experiencing something in a certain way entails have been thoroughly elaborated. According to Marton & Booth (1997), the way in which something is experienced depends on the structure and dynamic of awareness. How awareness is organised in a specific moment, constitutes a way of seeing. In alignment with the fundamental principles of gestalt psychology (see Werthimer 1938) Marton & Booth (1998) suggest that, to experience something as figural, that something must be perceived as a meaningful whole, which is discerned from and related to its context. Seeing a figure involves discerning its contours, its component parts, and the internal relations between the parts and the whole. However, for the parts to be discerned as parts of a figure, the figure must be experienced as a coherent and meaningful whole. To see something, as something, is to assign it meaning, and

what is seen as component parts of a figure is determined by how the whole is experienced and made meaningful (Marton & Booth, 1997; Pang 2003). According to Marton & Booth (1997), ‘Structure presupposes meaning and meaning presupposes structure. The two aspects, meaning and structure, are dialectically intertwined and occur simultaneously when we experience something’ (p 87). When we see something, we ascribe it a meaning, and the meaning we experience, in turn, determines what we see; that is, experience has a *structural aspect* and a *referential aspect* (which corresponds to meaning). In that sense structure and meaning are mutually constitutive in experience.

Moreover, the structural aspect could be divided into an internal and an external horizon. The internal horizon denotes the component parts, the relationship between them, and the relationship between the parts and the whole. The external horizon refers to ‘the way in which the phenomenon we experience in a certain way is discerned from its context and how it is related to its context’ (Marton & Booth, 1997, p. 89). Experiencing something in a certain way, involves discerning the whole from the context, as well as relating it to the context (Marton & Booth 1997, Pang 2003).

When analysing data, this implies that attention is paid to both the structure and the meaning of the expressed perception. The structural and referential aspects are assumed to be logically related and could therefore mutually elucidate each other during the process of analysis. Since the meaning an interviewee assign to a phenomenon, determines the role ascribed to the parts (Wertheimer, 1938), the analysis of how the parts, and their relations, are perceived, could contribute decisively to the researchers understanding of the meaningful phenomenon, as seen by the interviewee.

The field of consciousness and the dynamics of thought

When we focus on and delimit a phenomenon, other elements are necessarily perceived as background. The interplay between figure and ground is, according to phenomenography, central to how we experience the world (Marton & Booth, 1997). Drawing on Gurwitsch (1964), Marton and Booth (1997) describe the background as an external horizon that coexists with and influences the phenomenon. From this perspective, the experienced object is both delineated from and constituted in relation to the horizon against which it appears. The horizon constitutes a field of possible meanings, relations, and appearances within which an object is experienced, thereby shaping how the phenomenon can be

made meaningful in a given situation. Gurwitsch (1964) suggests a description of the structure of consciousness as organised into three domains; the *theme*, the *thematic field*, and the *margin*. The object of focal awareness is referred to as the theme. Separated from, and yet related to, the theme is the thematic field. The thematic field comprises elements which are connected to the theme through a sense of affinity and relevance, and which influences how the theme is perceived. In this sense every theme is surrounded and suffused by a halo of references, which are present in awareness in an inarticulate but still significant way. Things which are present to consciousness in a peripheral way, and which are not identified as relevant for the theme, constitutes the margin.

Gurwitsch (1964) further describes the *flow of consciousness* as a smoothly shifting focus, through which elements from the thematic field successively enters focal awareness and evolve into new themes. We move from something we know to something we become aware of as we pass through this stream of thought (Booth, 1992). Drawing on Gurwitsch (1964), Booth (1992) describes the flow of consciousness in terms of *pointing references*. During reasoning, the pointing references bind the theme and the thematic field together and guides the flow of the stream of consciousness. In this way a theme is replaced by another theme, and consequently, a thematic field by another thematic field, in a process which is directed and constrained by the pointing references. In this sense the content and structure of the thematic field, has a significant impact on the reasoning opportunities available. Booth (1992) discusses problem solving in programming, based on this model. Students' different approaches to writing a program are interpreted in terms of the different thematic fields to which the student initially relate the problem. This, in turn, depends on the body of knowledge and associated situations, previously experienced. The flow of consciousness then proceeds in a manner that is guided by the pointing references and constrained by students' conceptions of what is relevant to the considered problem.

Consciousness of identity

According to Gurwitsch (1964), the principle of perceived coherence applies not only synchronically but also diachronically. Individual, momentary perceptions are not experienced as isolated, but as organised into continuous, unified entities across time, that is, what appears across different moments is constituted as the same object, despite variations in how it appears (Gurwitsch, 1964). This tendency to synthesise changing perceptual content into a stable unity is what Gurwitsch (1964) refers to as the *consciousness of identity*, a fundamental structure of awareness.

A related idea is suggested in phenomenography, where the principle of perceived coherence is extended from the level of individual consciousness to a collective level of description. Although individuals may experience a phenomenon in qualitatively different ways, these differences are understood as ways of experiencing the same phenomenon (Marton, 1981; Marton & Booth, 1997). In this sense, not only one's own changing experiences, but also others' different ways of experiencing are interpreted as referring to a common object of experience. This can be understood as a necessary condition for shared meaning: in order to make sense of others' voiced experiences, we interpret them as variations of the same phenomenon rather than as entirely different phenomena. When a certain core structure is perceived as invariant, providing a stable reference against which variation can be recognised as meaningful and systematic, a phenomenon can be interpreted as the same, despite variation in how it is experienced, as long as the variation is perceived as logically structured in relation to what remains invariant.

Learning and critical aspects

Phenomenography, like phenomenology, sees behaviour as framed by experience. People do not act in relation to situations as they are, but in relation to the situations as they are perceived and understood. Consequently, ways of seeing that enable more appropriate, flexible or effective action may be regarded as more powerful than others. Learning could therefore be seen as the process of gaining more powerful ways of seeing, and teaching should make it possible for learners to develop the eye through which the world is seen (Marton et al., 2004). In order to understand teaching and learning, it is therefore of great importance to investigate how the difference between ways of seeing is constituted and what it takes for a learner to see something in a particular way. As the phenomenographic research tradition developed further, the main challenge therefore became to explain the observed differences between ways of understanding, and what those differences might stem from.

As stated above, differences in ways of experiencing correspond to differences in the structure and dynamic of awareness, and when something is learned the structure of awareness is changed (Marton & Booth, 1997). This could, according to Marton et al. (2004), be traced to the discernment of critical aspects. The way in which something is experienced depends on the aspects of the phenomenon that are simultaneously discerned and attended to, and how they are perceived to be related to each other and to the context (Runesson, 1999). The aspects that are

critical for experiencing something in a certain way, and that are non-trivial for the students to understand, are called *critical aspects* (Marton, 2015; Marton & Booth, 1997; Pang & Ki, 2016). These aspects are assumed to constitute the essential differences between ways of experiencing and are thus critical components in the transition from inability to proficient action (Marton, 2015; Pang & Ki, 2016). When someone attends to a phenomenon certain aspects of it will be discerned, and if two people discern different aspects of a phenomenon they will see the same thing in different ways (Marton et al., 2004). Discernment also includes the separation of objects from the context and parts of the object from each other and the whole. The pattern of aspects simultaneously focused upon and internally (and externally) related thus constitutes a way of seeing, and a difference within that structure implies a difference in the way a phenomenon is experienced (Marton et al., 2004; Pang & Ki, 2016). In accordance with the work of Gibson and Gibson (1955), phenomenography sees learning as differentiation rather than accumulation, that is, learning implies discerning more and more aspects and making finer and finer distinctions within the perceived whole (Kullberg & Ingerman, 2022).

This means that learning is all about discerning critical aspects, and when the intended learning does not occur, learners have not been able to discern the necessary aspect (Runesson, 1999). How teaching may create the necessary conditions for such discernment is addressed in the *variation theory of learning* (Marton, 2015).

Variation theory

The object of learning

Fundamental to variation theory is the statement that learning is always the acquired knowledge *of something*. There is no learning without something being learned. The verb ‘to learn’ must have an object, and that *object of learning* must always be kept in mind when planning and enacting teaching (Marton et al., 2004; Marton & Booth, 1997; Runesson, 2005).

As previously stated, phenomenography describes learning as the process of gaining more powerful ways of seeing, in order to be able to handle novel situations in more expedient ways. Consequently, the object of learning could be conceptualised as the way of experiencing a phenomenon that is considered to be

the most powerful (Pang & Ki 2016)⁴. This implies that the object of learning is not synonymous with learning objectives. It is not merely defined by the content being taught (such as formulas, mathematical tasks etc.), but also by the nature of the capability connected to the content (such as interpreting, remembering, understanding, discerning). These aspects are sometimes conceptualised as the *what* and the *how* aspects of the object of learning, or as the *direct* and the *indirect* object of learning (Marton & Pang, 2006). While the direct object refers to the content, the indirect object refers to *how* the content is understood (Marton & Booth, 1997).

Moreover, the object of learning could be considered from different points of view: the teachers', the researchers', and the learners'. The concepts *intended*, *enacted*, and *lived* object of learning are used to address the questions: What is intended to be learned? What is made possible to learn? and What is actually learnt? (Kullberg et al., 2017; Runesson, 2005). Teachers are supposed to focus both on what students are to learn and on the way in which students are supposed to master that which they are learning, that is, teachers are working toward an object of learning. What the teacher strives for is defined as an *intended* object of learning. The intended object of learning, which corresponds to the target way of seeing, is established by the teacher based on accepted disciplinary knowledge and on considerations of the capability appropriate for the learners to develop during a limited sequence of lessons (Marton et al., 2004; Marton & Pang, 2006; Pang & Ki, 2016). The *enacted* object of learning 'is described by the researcher from the point of view of what is afforded to the learners' (Runesson, 2005, p. 70), that is, it defines what is made possible to learn in the current setting, which depends on the extent to which certain necessary conditions for learning have been met. What is afforded to the learners is that which appears and comes into the foreground of their attention, the aspects that they discern and focus on. This means that, by analysing the enacted object of learning, in terms of constraints and possibilities for developing a certain capability, the researcher could say something about the learning opportunities that are afforded to the students. In the best-case scenario, learners focus on critical aspects and thereby learn what the teacher intends. Discerning new aspects of a phenomenon changes learner's awareness in a way that enables them to see and act upon novel situations in more powerful ways. These aspects constitute the *lived object of learning*. In other words, what students actually learned, the capability that they have developed as a result of learning, is

⁴ Marton & Pang (2006) describe the direct object of learning as the phenomenon per se, and the indirect object of learning as a target way of understanding that phenomenon.

the lived object of learning (Marton et al., 2004; Marton & Pang, 2006; Pang & Ki, 2016).

Patterns of variation and the space of learning

In variation theory, a fundamental point of departure is that variation is a necessary condition for learning (Marton, 2015). The idea that variation plays a crucial role in directing attention and enabling discernment is not unique to variation theory but is shared across several traditions in perception and cognition, where contrast and change are seen as fundamental to what is noticed (e.g. Gibson, 1979; Bruner et al., 1956).

More specifically, Marton (2015) argues that merely talking about or providing examples of an idea, concept, or principle is not sufficient to convey its meaning. Critical aspect could not be discerned unless meaningful variation in relation to the aspect has been experienced. For example, to be able to notice a certain feature (e.g. green) of an object, it is not enough to see a lot of green objects. As the colour is invariant it will remain in the background and be considered unimportant for sense making. Instead it is necessary to experience variation in the corresponding aspect (colour); one must observe something that is green and something that is not green if colour should become a discernible aspect. In a similar manner, it is impossible to understand the nature of a certain language unless one has heard another language spoken, or to understand a geometric concept without seeing examples which does not belong to the concept (non-examples). An aspect is, therefore, according to variation theory, defined as a dimension of variation (Marton, 2015). To enhance students learning it is not enough to tell them about critical aspects. Aspects must be discerned, by opening up corresponding dimensions of variation (e.g. Kullberg, 2010; Marton, 2015; Marton & Pang, 2006). It is, however, not just variations concerning values within aspects that are important for learning, but also variations that make it possible to separate aspects, and relate them to each other and to the whole (Marton & Pang, 2006; Runesson, 1999).

Patterns of variation

Marton (2015) suggests that the aspects, intended to be illuminated, should be treated according to a certain sequence of patterns of variation; starting with the undivided whole, (to get acquainted with the object of learning), followed by contrast, generalisation and finally fusion. *Contrast*, constitutes an opposite of the

classic, inductive way of teaching, and implies that learners experience contrasting values (features) within a dimension of variation against a background of invariance in others. This enables the learner to discern the corresponding aspect and separate it from other aspects. For example, to understand the concept of linear function $y = mx + b$, it is necessary to distinguish it from non-linear functions. Otherwise ‘linear function’ merely becomes a synonym for ‘function’ (Kullberg, et al., 2017). Similarly, the decimal system cannot be understood without another number system being introduced (Marton, 2015). The same is true of geometric concepts such as the altitude of a triangle, it is necessary to distinguish it from other line segments that do not represent altitudes, such as line segments which are not drawn from a vertex, or which are not perpendicular to the opposite side (Gu et al., 2017). Without contrasting examples the concept of altitude risks being reduced to any line drawn inside a triangle.

Contrast should be followed by *generalisation*, which implies that the focused aspect is kept invariant against a varying background of other aspects (Marton, 2015). A simple example can be found in the concept of colour. To generalise a particular colour, the colour itself must be kept invariant while other properties, such as shape, size, or material, vary. Then colour is discerned as a defining aspect independent of specific instances. Similarly when the same set of defining aspects of linear functions, are kept invariant across different functions, these aspects are separated from a particular instance, which enables a generalisation (Kullberg, et al., 2017). Likewise, to develop a generalised understanding of the geometric concept of altitude in a triangle, it is necessary to show that altitudes can be drawn from any vertex, and that an altitude does not always intersect the opposite side within the triangle but may instead intersect its extension. When the defining aspects of altitude are kept invariant while other aspects vary, the defining aspects are separated from a particular example.

Moreover, to understand a phenomenon, means to understand it as a whole. This implies a simultaneous discernment of critical aspects and the relation between them. Hence, the learner has to experience variation corresponding to several critical aspects simultaneously. This pattern of variation is named *fusion* and is the ultimate objective of learning. Aspects could be independent of each other or functionally/logically related, which gives rise to different constraints to the fusion pattern. As stated above, the described set of patterns of variation also indicates the preferred order between them. Generalisation must be preceded by contrast, and individual aspects need to be discerned and separated before they can be brought together and experienced simultaneously (Marton, 2015; Marton

& Pang, 2006; Olteanu, 2007). Olteanu (2007) further distinguishes between convergent and divergent sequences of patterns of variation, referring to whether variation is structured in a way that directs attention towards a coherent whole, or instead opens up multiple meanings without first establishing clear distinctions. In divergent variation, meanings are varied simultaneously, which may lead to fragmentation and confusion if aspects are not first separated and clarified. In contrast, in convergent variation, aspects are first made distinguishable and are then systematically related to each other and to the whole. This supports the discernment of structure and the formation of coherent understanding (Olteanu, 2007). In this way, teachers may create sequences of patterns of variation that constitute a space of variation within which learning becomes possible.

The space of learning

Runesson (2005) describes the patterns of variation and invariance, opened through interactions about a certain topic, as a *space of learning*. According to Marton & Tsui (2004) 'A space of learning comprises every number of dimensions of variation and denotes the aspects of a situation, or phenomena embedded in that situation, that can be discerned due to the variation present in the situation' (p. 21). The patterns of variation and invariance, thus constitute a space of learning that makes it possible to discern and relate critical aspects, which determines what is made possible to learn in the situation, (i.e. the enacted object of learning). This implies that the space of learning, can be analysed by means of the observed patterns of variation, in order to evaluate the afforded learning opportunities. (Marton et al., 2004; Kullberg, 2010; Runesson, 1999). Several studies have been reported, which illustrate that, what students learn, is dependent on the constituted patterns of variation and invariance. In many of them, all of the three objects of learning, mentioned above, are repeatedly studied and related, in order to explore the relation between teaching and learning (e.g. Kullberg, 2010; Marton, 2015; Marton & Pang, 2006; Olteanu, 2007). In these studies, the intended object of learning is defined by the teachers, and the critical aspects are found by means of investigation of students' previous knowledge, combined with knowledge of the object of learning within the discipline. Some studies, however, have focused solely on the enacted object of learning, and have explored conditions for students learning by analysing patterns of variation and the corresponding space of learning (e.g. Häggström, 2008; Kullberg, et al., 2017, Runesson, 2005). Importantly, according to variation theory (Marton, 2015), these conditions are necessary, but

not always sufficient, for learning. What is actually learned, the lived object of learning, depends on, but is not determined by, the space of learning.

Adjacent theoretical perspectives

When variation theory is applied in the context of learning mathematical concepts, through tasks and examples, the ideas to some extent coincide with other perspectives, and ways of using variation in order to highlight essential properties of concepts, and to visualise their limit. Especially in geometry, teaching with variation has been widely appreciated for providing opportunities to extract essential features of geometric concepts and simultaneously disregarded non-essential features (Sinclair, 2016). Watson and Mason (2005) developed and expanded the idea of VTL (Marton 2015), by introducing the notions dimensions-of-possible-variation and range-of-permissible-change, describing that mathematical concepts can be perceived by being aware of aspects that may possibly vary and the range of permissible change ‘within which an object remains an example of the concept’. Concerning tasks and examples. Watson and Mason (2006) exemplify how patterns of variation within well planned sets of tasks (plotting points and finding gradients) focus learners’ attention to certain mathematical structures, relationships and properties. (p132).

In a discussion of the role of examples Liz et al. (2006) argue that ‘generic examples’, ‘counterexamples’, and ‘non-examples’ are all important for how a student perceives a mathematical object. Where here ‘generic examples’ are taken to mean examples of the concept, ‘counter examples’ to mean examples which counters a hypothesis, and ‘non-examples’ to be examples that are used to clarify boundaries for a concept.

Tsamir et al (2008) studied childrens’ perception of the concept triangle, and discusses the use of non-examples in concept formation. They define examples to be instances of a concept, and argue that students should be exposed to a variety of non-intuitive examples and non-examples of a concept in order to generalise the concept and to see the boundaries within which certain aspects are permitted to change. Liz et al. (2006) argue that both the succession of examples and the aspects which are varied, affords access to key features of a concept. In comparison with VTL, the use of non-examples would correspond to a contrast, and non-intuitive examples would correspond to a generalisation (even if the VTL concepts ‘contrast’ and ‘generalisation has a broader use and meaning).

Geometric tasks in mathematics textbooks, have been analysed utilising a framework based on Gu's et al. (2004) theorisation of the practice of teaching through variation in mathematics (e.g. Zhang, et al., (2017)). Two different types of variation are described by Gu et al. (2004), namely conceptual and procedural variation. Through *procedural variation*, mathematical activity is developed step by step in a dynamic process through which a particular way of experiencing the topic gradually emerges. For example, preconditions of a simple task, may be systematically changed, in order to subsequently pose more challenging problems (hierarchical variation). A problem can also be varied by: 1. Using multiple solution methods. 2. Applying an invariant method in various contexts (Gu et al. 2004). *Conceptual variation*, on the other hand, aims to develop understanding of a concept from multiple perspectives. Two categories of conceptual variation are described. *Concept variation* is used to explore common features of different instances, and to draw attention to the essential characteristics of a concept, by differentiating them from non-essential features. For example by comparing prototypical figures (e.g. triangles with a horizontal base line) to non-prototypical figures (e.g. triangles with a non-horizontal base line). *Non-concept variation* is used to highlight essential features of a concept by displaying non-concept cases as counterexamples. For example, figures of vertical angles may be contrasted with figures of non-vertical angles. Similarly, Lo (2012) distinguishes between defining and non-defining aspects of a concept. Defining aspects are those that determine the nature of the concept, such as the size of the angle if the object of learning is a right angle, whereas non-defining aspects include features such as the length of the lines or the colour of the angle. Both defining and non-defining aspects can be critical for learning. According to Gu et al.'s (2004) framework, the former can be made discernible through non-concept variation, while the latter are made discernible as non-essential through concept variation.

Several examples of how conceptual variation might be used to enhance students learning of geometry are described by Gu et al. (2017). Drawing on results from a few experimental studies, they conclude that teaching through designed variation problems, using procedural and conceptual variation, is a more effective approach than teaching through repeated explanations of a definition. Gu's et al (2004) framework could be compared to that of VTL:

Table 5.1 Comparison between VTL and Gu's et al. (2004) framework

VTL-concepts	Concepts used by Gu et al. (2004)	
	Conceptual variation	Procedural variation
Contrast	Non-concept variation	The same task - different methods
Generalisation	Concept variation	Invariant method- different tasks
Fusion		Hierarchical variation

If Gu's framework is compared to VTL-concepts (see Table 5.1), the kind of procedural variation, that implies invariant problems-varying method, would correspond to a contrast, drawing attention to a procedure. A non-concept, conceptual variation would correspond to a different kind of contrast, focusing on aspects of the concept. A procedural variation that implies an invariant method, while tasks and contexts vary, would correspond to a generalisation of a certain procedure, and a concept-variation, would correspond to a generalisation of the concept. A hierarchical variation might be seen as a kind of fusion, that aims to create patterns of variation in a way which is similar to Olteanus (2007) description of convergent variation.

Marton and Häggström (2017) argue that the two frameworks agree on the principle that essential aspects of the object of learning can only be appropriated by the learners by means of separating *essential aspects* from *non-essential* aspects. Varying the essential aspects (through contrast) while letting the non-essential aspects remain invariant, makes the essential aspects possible to discern as essential. Letting the essential aspects remain invariant (through generalisation) while varying the non-essential aspects, makes the non-essential aspects possible to regard as non-essential. An important difference between Gu's, et al. (2004) framework and VTL, however, is the preferred order of contrast and generalisation. According to VTL, generalisation always has to be preceded by contrast. Seemingly, Gu. et al. (2004) make no such distinction. However, according to Marton and Häggström (2017), the significant issue is whether the aspects are *novel* to the learner. If learners have already acquired some basic perceptions of a concept, all aspects are not novel and does not necessarily have to be opened up by contrast.

Analyses of afforded learning opportunities

Some studies, based on variation theory, has analysed sequences of mathematical examples used by teachers in the classroom. These studies illustrate how the space

of learning, constituted by mathematical examples, could be described in terms of patterns of variation and invariance. In this way it is explored how systematic variation in and between sets of examples could affect what aspects student might discern and what mathematical structures and relationships they might see. Several researchers have claimed that the choice and sequencing of examples are crucial for students' learning and have argued that single or randomly chosen examples are insufficient to open up dimensions of variation and clarify relationships between aspects (Kullberg, et al., 2017; Kullberg & Skodras, 2018 Watson & Mason, 2006).

Kullberg and Skodras (2018) analyse mathematics teaching based on a variation theory framework, focusing on how sets of examples afford certain learning opportunities. The analysis examines sequences of examples (referred to as 'strings') by identifying patterns of variation. Particular attention is given to what changes and what remains the same across examples, in order to identify opportunities for learners to discern critical aspects of the mathematical content. In the presentation of the analysis, the examples are considered in the order in which they are introduced in the lesson, allowing the sequencing and internal structure of each set to be made visible. The analysis highlights how specific features are varied systematically while others are kept invariant. Commentary is provided in relation to each set of examples, drawing attention to patterns of variation and their potential to foreground mathematical relationships. This is followed by a discussion of what learners are afforded the opportunity to discern through these patterns. The results indicate that systematic variation within and between sets of examples enables learners to identify key mathematical structures and relationships, such as the distributive, commutative, and associative properties. The findings suggest that carefully sequenced examples, in combination with attention to variation and invariance, play a crucial role in supporting learners' movement from specific cases to general mathematical understanding.

Also, Pillay et al. (2022) used a variation theory framework to analyse a lesson at a detailed (micro) level. The researchers examined the example sets presented during the lesson by identifying patterns of variation and invariance both within episodes (horizontally) and across episodes (vertically), following Watson and Mason's (2006b) idea of analysing 'with and across the grain.' Particular attention was paid to what changes and what remains the same in the examples, as well as how examples were sequenced, paired, and juxtaposed, in order to identify how opportunities were created for learners to discern the critical aspect of the object

of learning. In the presentation of the analysis, all examples used during the lesson are set out in the order in which they were introduced. This enables an illustration of the sequencing and pairing of examples, which, according to variation theory, is significant for determining learning opportunities. Commentary is provided for each episode, drawing attention to varying and invariant features, followed by a discussion of the learning opportunities afforded through these patterns. The results indicate that the lesson was characterised by a systematic use of variation against a stable background, which supported learners in discerning key mathematical aspects, particularly the role of the exponent in distinguishing function types.

In a similar way, a few textbook studies use variation theory to analyse textbook tasks. Geometric tasks in mathematics textbooks, were analysed by Zhang, et al., (2017) using a framework based on Gu’s et al. (2004) theorisation of the practice of teaching through variation in mathematics (see above). They exemplify how variation can be used in textbook tasks to make learning possible. The study shows that in the selected textbooks, systemic variation tasks are used to develop mathematical concepts, principles and mathematical skills and mathematical thinking methods. They argue that the systematic use of variation tasks in Chinese textbooks (such as those in Fig 5.2, 5.3) may provide an explanation of Chinese students’ excellent performance of mathematics on international comparative assessment.

Figure 5.2 Zhang et al. (2017, p 217)

In the figures on the right, A, O, B are on a line and O, P are two points on line AB. Are $\angle 1$ and $\angle 2$ a pair of vertical angles? Why? Are $\angle 3$ and $\angle 4$ a pair of vertical angles? Why?


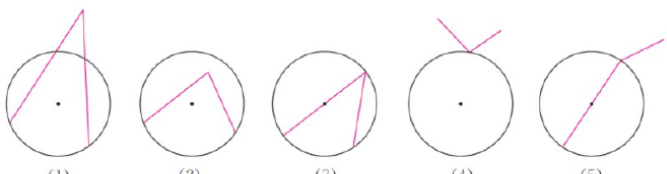


Figure 5.3 Zhang et al. (2017, p 224)

Judge whether the angles in the following figures are inscribed angle or not, and explain your statement.



The two tasks above are given as examples of how non-concept and concept variation can be used in textbook tasks to provide opportunities for students to discern essential features of the concepts. By providing examples of non-concept figures (in contrast to concept figures) and ask students to judge whether Figures belong to the concept or not, opportunities to discern essential aspects of vertical angles (Figure 5.2) as well as inscribed angles (Figure 5.3) are given (Zhang et al., 2017). Furthermore, by giving several examples of the concept while varying aspects that are not essential, students are provided the opportunity to generalise the concept, by disregarding non-essential aspects.

Similarly, Sun (2011) compares Chinese and American textbooks by analysing sets of subtraction/addition and fraction division problems. She identifies three patterns of variance and invariance, that are typically used in Chinese textbooks (c.f. Gu et al., 2004), ‘one problem, multiple solutions’, ‘one problem, multiple changes’ and, ‘multiple problems, one solution’, and demonstrates how these patterns provide opportunities for students to reflect, interrelate and generalise. In the sets of problems typical of US textbooks (and probably other Western textbooks), however, the exercise could be completed without engagement with the mathematical concepts (Sun, 2011).

In a later study, Sun (2019) develops the framework further by illustrating how variation problems with concept connection may contribute to the generalisation of addition, multiplication, and division principles from the familiar domain of whole numbers to that of fractions. Sets of problems in which, for example, the multiplicand varies between whole numbers and fractions, while other aspects remain invariant, direct attention to invariant relations across the two domains. In the analysis, Sun (2019) identifies patterns of variation in textbook problem sets, focusing on how variation is used to foreground critical aspects and invariant relationships between concepts, procedures, and representations. Rather than treating problems as isolated instances, the analysis considers them as parts of structured sets, where variation serves to connect different mathematical ideas into what Sun (2019) describes as a coherent ‘knowledge package.’ The results are presented through detailed examples of textbook tasks, in which patterns of variation are examined in relation to the concepts they are intended to make visible. Through a comprehensive interpretive and argumentative text, Sun (2019) demonstrates how variation can be used systematically to support the development of conceptual connections and to overcome well-documented learning difficulties, such as whole number bias in fraction learning.

Sun (2019) argue that although there is a robust literature on textbook comparison in the field of mathematics education, the main focuses is typically on visible, surface features, while there are few textbook analyses that focus on how textbook structures, such as the way features of the tasks vary across task sequences, could affect students opportunities to discern aspects and make connections.

Theoretical considerations -framing the object of research

This section outlines the theoretical considerations that guide the study and clarifies how the object of research is conceptualised and approached. As established in the introduction, the overall aim of the thesis is to illuminate and map educationally significant vocational mathematics knowing. This aim is addressed through three complementary studies, each approaching the object of research from different, but theoretically and methodologically connected perspectives. Taken together, the studies enable a progressively deepened understanding, and in the concluding synthesis, insights from the three studies are connected and compared in order to substantiate the thesis' concluding argument concerning the conditions under which powerful vocational mathematics knowing may be developed.

Throughout the thesis, the overarching focus is on students' learning. Learning is understood, in line with the phenomenography and variation theory research tradition (VTL), as a change in ways of experiencing. Experiencing is, in turn, conceptualised as an internal relation between the person and the world (Marton & Booth, 1997). What is required for such a change to occur is that appropriate conditions for learning are established. These conditions are conceptualised as affordances constituted in the relation between the learner and the environment (Marton, 2015; Gibson, 1979). In this sense, both knowing and learning are understood as relational. The present thesis seeks to illuminate such relations. This is done in different ways across the studies.

The first study focuses on how educationally significant vocational mathematics knowing is perceived and articulated by teachers. The phenomenon could, in turn, be understood in terms of how students are expected to make sense of what is to be learned. The second study investigates how vocational tasks are engaged with and interpreted, with a focus on the critical aspects that need to be discerned in order to understand and solve such tasks. The third study examines afforded learning opportunities in mathematics textbooks by analysing patterns of

variation and invariance. From a variation-theory perspective, such patterns form the space of learning and thereby constitute the conditions for learning (see Marton, 2015). Across the three studies, meaning, learning opportunities, and critical aspects are understood as relationally constituted rather than as intrinsic properties of individuals, learning objectives, tasks, or materials.

By the same rationale, the object of research itself, is not conceived as a purely external entity about which the researcher seeks to convey an objective truth. Rather, the object of research is constituted in the relation between the researcher and the empirical data through the very act of inquiry. It is through the interplay between the researcher's questions, perspectives, and prior understandings and the expressions of the empirical material, that the object of research emerges. In this process, research functions as an act of meaning-making and the object of research takes shape within the research process itself, through interpretation, reflection, and engagement with what is being studied. In this sense, the study follows an interpretivist research tradition rather than a positivist one (Cohen et al., 2018).

Study 1

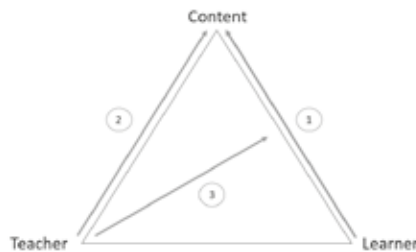
The phenomenon

In Study 1 the explored phenomenon could be described in terms of 'educationally significant vocational mathematics knowing', and the object of research is vocational -and mathematics teachers' conceptions of this phenomenon. To understand what the phenomenon is, one should, however, rather turn to the outcome-space, than paying attention to the researchers' initial view of the phenomenon. As described previously phenomenographic research is conducted from a second order perspective, which means that the phenomenon is described and defined by the outcome space, rather than predefined by the researcher. The phenomenon is what the interviewee sees. The settings and development of the interviews create conditions for a particular direction of the interviewees' attention, which in turn determines the phenomenon that appears to the interviewee. Therefore, the intention has been to make the interviews structured enough for the interviewees to focus on a common phenomenon, and at the same time flexible enough to encourage the interviewees to explore the phenomenon from different perspectives, and in relation to different external horizons.

Several phenomenographic interview studies have been carried out, exploring students' different ways of experiencing specific contents (e.g. Neuman 1987;

Booth, 1992). This means that, objects of research, corresponding to arrow 1 in Fig. 2, have been extensively investigated. In this study, however, it is not students' experiences of the content, nor is it teachers' experiences of the content (arrow 2, Fig.2), that is researched, but rather teachers' conceptions of the way in which students are supposed to understand the content (arrow 3, Fig 5.4).

Figure 5.4 The didactical triangle. The phenomenon under consideration corresponds to Arrow 3



Students' expected knowing is thus explored from their teachers' perspective, and the object of research is, accordingly, teachers' conceptions of appropriate knowing for students. Therefore, during the interviews, questions were asked about the content, how students are supposed to understand and use it, and what students' need to learn (and have not yet learned) in order to understand it in the intended way.

The categories of description

A relevant objection that could be raised, in relation to the current study is that it might be difficult to evaluate whether the categories of description really corresponds to different ways of experiencing the same phenomenon or whether the differences in ways of experiencing, rather stem from the fact that different phenomena are experienced. Uljens (1989) discusses this issue as a general methodological challenge in phenomenographic research and concludes that the question is rarely possible to resolve it in an indisputable manner.

From the principle of intentionality, the *what* and *how* aspects of experience are inseparable; experiencing something differently therefore, in a sense, necessarily implies experiencing something different. Consequently, what is referred to as the 'same phenomenon' may be understood as a construct established by the researcher, based on the interpretation of a shared and structured meaning across different ways of experiencing. This means that the question of whether we

describe different interpretations of the same object, or different interpretations of different objects, will always remain an issue in phenomenographic research. According to Uljens (1989) there are ways of handling this challenge in order for the researcher's interpretation of sameness to appear convincing.

Such credibility may be established by ensuring that the empirical material is focused on a shared domain, for example through the use of the same tasks or questions, and by demonstrating that the different ways of experiencing can be meaningfully related to one another within a structured outcome space. To further refine the analysis, Uljens (1989) suggests a step-wise analysis, in which, the question of *what* the interviewee refers to regarding the phenomenon is firstly investigated. Then the ways of seeing, that what is referred, is explored, and categories of descriptions are defined within a certain domain. In the present study, all of the interviews were based on a common set of tasks. Moreover, the interviewees were asked to describe the content area which they regarded the tasks to belong to, and to bring a task that they considered would complement the set of tasks. The answers given to these questions showed that all of the interviewees considered the tasks to represent vocational mathematics knowing, relevant in a construction-work context. For example:

M3: They [the tasks] are about construction-work... well...tasks that feel adapted to our students.

B4: They [the tasks] connect to different parts of the construction-work process, [...] But all of them are connected to things that you actually do in the workplace.

M4: It is applications that are relevant to their choice of vocation [...] actually applications that are relevant to their vocational path

It was also obvious that the tasks, which the interviewees brought to the interview, were mathematical tasks, clearly related to a construction-work context, but not constrained to a particular mathematical area.⁵ Moreover, when analysing the interviews, the parts of the interviews which kept a clear focus on appropriate learning outcomes were selected for the analysis. In line with this reasoning, it is therefore reasonable to claim that this study considers a delimited, common phenomenon, and that the phenomenon could be described as 'educationally significant vocational mathematics knowing'. This implies that, if one interviewee,

⁵ This means that the phenomenon, considered by the interviewees, differed slightly from the phenomenon which was initially intended to be investigated, namely the concept of geometric similarity in a construction-work vocational context.

during the discussion of a task, decides to relate the content mainly to a workplace situation, while another interviewee focuses on specific mathematical concepts, this is interpreted as expressions of different ways of understanding the same phenomenon (i.e. educationally significant vocational mathematics knowing) rather than that they are talking about different phenomena.

The outcome space

In line with a phenomenographic research approach, excerpts from the interviews are regarded to constitute a ‘pool of meaning’, in which a variation in ways of seeing the phenomenon emerges. The aim is thus not to describe and categorise individuals’ ways of experiencing a certain phenomenon, but rather to provide an outcome space that captures the meaning of the explored phenomenon through the variation in ways of seeing that phenomenon. This means that a way of seeing, which is manifested in some part of the interview, is not considered to fully represent individual interviewees’ conceptions of the phenomenon. A movement between different ways of seeing, during an interview is rather expected.

In the current study, it turned out that different ways of seeing, were conveyed by the same interviewee in different parts of the interview. Moreover, there seemed to be a tendency that some ways of seeing were more likely to occur when particular tasks or topics were discussed, and that the most complex ways of seeing were most likely manifested when the discussion concerned the tasks that the interviewees, themselves, brought to the interview. This might be regarded a reliability issue. A similar result was obtained by e.g. Dahlgren (1979) It turned out that the same interviewee could articulate different ways of understanding ‘price’ when considering different tasks, and that the design of the task seemed to affected the probability of a certain conception to be expressed, which might be regarded a reliability issue. However, Marton (1981) argues that the potential threat to reliability is only an issue concerning personal conceptions. At a collective level ‘conceptions of price’ could be described in a reliable way. According to Marton (1981) ‘The set of categories is thus stable and generalisable between situations, even if the individuals ‘move’ from one category to another on different occasions’ (p 195). In line with Marton’s (1981) reasoning, it is thus possible to argue that the set of categories is stable and that the entire outcome space as a whole, represents a reliable description of the phenomenon.

The structure of the outcome space

As described above, Uljens (1989) defines a few different kinds of outcome spaces: horizontal, vertical and hierarchical. In hierarchical outcome spaces, the ranking is based on the idea that some categories are more developed, advanced or extensive than others. With regard to critical aspects, hierarchical outcome spaces could be explained by the fact that some of the categories correspond to a larger number of discerned aspects, (and relations between them) than others. This does not imply that an outcome-space must be strictly hierarchical. The basics of the theory rather predict the opposite. Two categories could for example correspond to discerning different, but the same amount of aspects, or one category could be more inclusive, while another corresponds to more developed relations between parts. Therefore, it is reasonable to argue that the categories, in any outcome-space, could be arranged, based on increasing complexity. At the same time, a certain amount of uncertainty regarding the order of the categories must always be allowed, especially in the middle of an outcome-space.

In this study there is no predefined educational objective to compare the different ways of understanding to, in order to motivate a ranking of the of the categories. Therefore, no valuation of the categories is made in terms of better or worse. However, it is possible to regard some of the categories as more advanced than others in terms of inclusivity and complexity. Drawing on Uljens' (1989) categorisation and considering later developments of the theory (e.g. Marton, 2004; Pang & Ki, 2016), the outcome space in this study can be described as having a semi-hierarchical structure, based on a combination of horizontal and vertical shifts. More precisely, distinctions between categories are understood as representing either differences in the depth and quality of understanding and/or a difference in focus. The difference in focus may, in turn, involve a broader understanding and extended inclusivity (i.e. more aspects are discerned), and/or a distinction in which aspects of the phenomenon are foregrounded (e.g. either aspect A or aspect B is discerned). In this study, an increased depth or quality of understanding is referred to as a *vertical* distinction, while a shifting focus is described as a *horizontal* distinction. While both increased quality and extended inclusivity may justify a *hierarchical* ordering of categories, categories that primarily differ in terms of *which* aspects are foregrounded do not necessarily imply a hierarchical relationship.

Study 2

According to previous research, learning mathematics for vocational purposes is far more complex than simply acquiring particular mathematical concepts and procedures. As discussed in Chapter 3, Marin and LaCroix (2008) suggest that when students engage with mathematics in vocational activities, their attention shifts between different interfaces, such as the task as a whole, specific mathematical concepts, or the workplace setting. They further argue that, in vocational contexts, it may be more appropriate to speak of the *growth of understanding of mathematically rich tasks* than merely the learning of particular concepts. Similarly, Booth (1992) describes problem-solving as a process in which students' attention shifts between the task as a whole and its constituent parts. At any given moment, what is in focal awareness is perceived as a meaningful entity, a phenomenon. How this phenomenon is perceived (in terms of discerned critical aspects), and how references that guide the ongoing process are established, shape the possibilities for successful task solving.

In this study, a *mathematically rich vocational task* is understood as a delimited, vocationally situated activity in which mathematical reasoning constitutes an integral part that significantly influences the outcome. A task is further conceptualised as a coherent sequence of actions directed towards a specified practical objective. Defining the boundaries of such tasks is not self-evident, as vocational activities are often interconnected and embedded within broader processes. In the present study, what counts as task boundaries are established in relation to what teachers identify as relevant when describing the activity, under the premise that the focus is on vocational tasks that involve mathematical content.

As described previously in this chapter, learning is, according to VTL, essentially a matter of discerning critical aspects. The aim of Study 2 is therefore to explore which critical aspects students need to discern in order to understand and engage proficiently with mathematically rich vocational tasks. According to VTL, the human mind is limited to attending to only a few aspects at a time (Marton & Booth, 1997). However, during the process of solving a task, as described by Booth (1992), several phenomena—and consequently a large number of aspects—may be involved. Critical aspects of such tasks are thus understood as the totality of aspects that need to be discerned in order to understand and engage proficiently with the task and its constituent parts. These aspects may relate not only to formal mathematics, but also to other domains of knowledge and to the vocational context more broadly.

Focusing on critical aspects, however, primarily captures what is brought into focal awareness and does not fully account for how broader fields of relevance influence reasoning, and shape the problem-solving process (see Booth, 1992). From a phenomenological perspective (Gurwitsch, 1964), the thematic field may include relations, experiences, and knowledge that are not directly attended to, but nevertheless shape how the task is understood and approached. This may include, for example, knowledge and experience developed through education or vocational practice over time. While the present study does not analyse such background structures explicitly, it is reasonable to assume that a broader and more coherent understanding of, for example, the mathematical relationships involved, together with broader knowledge structures developed through vocational practice, may influence the problem-solving process.

By identifying critical aspects, the study contributes by making visible what needs to be discerned for meaningful engagement with vocationally relevant mathematical tasks. At the same time, the result may serve as a basis for further discussion of how these aspects relate to broader structures of understanding and practice, and how such structures might contribute to more powerful ways of understanding and acting.

Critical aspects

Critical aspects are understood as those elements that are essential and non-trivial for students to discern in order to understand a phenomenon in a coherent and meaningful way. In teacher interviews, such aspects are typically identified by teachers as elements that students need to understand, and that often constitute particular challenges in learning. Critical aspects further correspond to differences in the structure of awareness, and may become visible in situations where different ways of understanding are contrasted, challenged, or transformed. In discussions between students about particular tasks, such situations may appear as occasions where contrasting ways of understanding come into conflict or fail to align, thereby making differences in understanding visible. From a VTL perspective, critical aspects are understood as dimensions of variation. Analytically, this implies that an aspect must be possible to make visible through contrasting values along the corresponding dimension. By definition, a critical aspect represents a single dimension of variation and cannot be further decomposed into separate components that are independently necessary for understanding. All of these characteristics of critical aspects are analytically significant in the present study.

Study 3

In Study 3, textbooks are analysed to investigate afforded learning opportunities regarding the concept of similarity. As described above, there are several studies using variation theory, which investigate learning opportunities based on analysis of patterns of variation. However, since the enacted object of learning is constituted in the learning situation by interactions among learners and the teacher, the enacted object of learning could not be studied through textbook analyses alone. At the same time, the significant role of textbooks, and their influence on students learning is widely recognised. (e.g., Qi et al., 2025; Rezat et al., 2021; Johansson, 2006; Fan et al., 2013; Valverde et al., 2002). And the influence of textbook characteristics on students' performance have been repeatedly confirmed (e.g Van den Ham & Heinze, 2018, Sievert et al., 2021; Hadar, 2017). Combined with other factors, the textbook is therefore considered to create conditions for the probable opportunities to learn mathematics and is sometimes referred to as the 'potentially implemented curriculum' (Valverde et al., 2002). It is thus reasonable to assume that textbooks affect the enacted object of learning in a significant manner. In this thesis the concept 'afforded learning opportunities' is used, when analysing the potentially enacted object of learning, in mathematics textbooks. In accordance with the variation theory framework (Marton, 2015), patterns of variation are regarded to constitute affordances (see Gibson, 1979) in the interaction between learner and material. In this sense, the textbook can be understood as initiating activities in which particular aspects become available for discernment depending on how variation and invariance are structured in the material.

Afforded learning opportunities provided in task sequences, have been studied to some extent (e.g. Kullberg & Skodras, 2018; Pillay et al., 2022; Zhang et al., 2017). Similar to these studies the focus of the present study is on patterns of variation and the space of learning. More specifically, attention is paid to what is varied and what is held invariant, and how such patterns are organised within and across tasks. The analytical interest thus concerns how task sequences make it possible, or not, to discern, relate, and generalise aspects of the similarity concept. The concept of *critical aspects* is not used directly in this study, since the concept is closely connected to students' learning, and since the present analysis is limited to textbook material and therefore does not include empirical data on students' learning. Instead, the analysis focuses on *aspects that are made possible to discern* through patterns of variation in the tasks. A distinction is made between essential

and non-essential aspects, where *essential aspects* are understood as the aspects of a concept, or principle, that define its nature and must be discerned for the concept to be recognised and correctly understood. *Non-essential* aspects, in contrast, are features that may vary without altering the identity of the concept. This distinction aligns with the framework proposed by e.g. Gu et al. (2004) and Zhang et al. (2017), where essential aspects are made visible through contrast between examples and non-examples, and non-essential aspects are varied to support generalisation of the concept.

The analysis is not deductive in the sense that aspects are predefined. At the same time, it is conducted against a background of disciplinary preunderstanding of similarity, as well as previous research on the teaching and learning of the concept. Furthermore, following the analysis, the aspects that are analytically identified in the material, are related to findings from previous research in order to discuss their potential significance. Below, theoretical concepts and previous research are used to identify potential critical aspects of similarity. These are treated merely as possibilities. The empirical analysis, in turn, focuses on patterns of variation in the textbooks, in order to describe the learning opportunities afforded in the material. These afforded opportunities are then related to the potential critical aspects identified in previous research, making it possible to discuss the extent to which important aspects of similarity are made possible to discern.

Summary of research on similarity- potential critical aspects

Similarity could be defined in different ways according to different axiomatic systems. In all of them specific *conditions for similarity*, such as *congruent angles* and *proportional corresponding sides*, are emphasised, and *similarity theorems for triangles*, are derived (Hilbert 1902; Birkhoff, 1932; Smith, 2010). However, different axiomatic systems use different primitive concepts and makes different foundational statements, which affects the way in which similarity is introduced, and concequently also which aspects become essential. Especially important is the difference regarding the role of *transformations*, which has a significant impact on the way similarity is taught and understood (Seago, 2013). Previous research emphasises the importance of a conceptual understanding of similarity according to a valid and complete definition, and also point to aspects of similarity which are of considerable importance in relation to other areas of mathematics. It can be concluded that, even though the concept of similarity is easily apprehended, to some extent, by visual intuition, informal conceptions, of ‘the same shape’ may

imply perceptions of similarity, which deviate from a mathematically accurate definition (e.g. Seago, 2010; Vollrath, 1977; Wu, 2005).

Research findings also indicate that some aspects of similarity tend to be more difficult for students to discern than others. For example, congruent angles seem to be more difficult for students to appreciate as a condition for similarity, than proportional sides (Haj-Yahya, 2021). Moreover, to accept that transformations preserve similarity is not as complicated as to actually perform the *mental flipping* required to make figures coincide, and finding *the correct correspondence* in order to set up the right proportion seems to be more challenging than to solve the equation (Chazan, 1988). Different aspects of *proportional reasoning* are also important for students to discern. Of certain importance is the relationship between *internal and external ratios* (Lamon, 2007). Proportionality between corresponding sides implies that the relationships between sides within each figure are preserved, and conversely, consistent relationships within figures can establish proportionality between corresponding sides, provided that the correct correspondence is identified. Essential aspects concerning proportionality, may also include discerning that proportional *corresponding sides is both a condition for, and an implication of, similarity*. Research findings also indicate that minimal conditions for similarity of triangles, such as the (AAA) and (SSS) *similarity theorems* turn out to be difficult for students to accept as valid definitions (Haj-Yahya, 2021). Thus, an important aspect to discern is that, similarity of triangles, constitutes a special case, when it comes to conditions for similarity. Within more comprehensive geometry courses, three triangle similarity theorems are considered: (SSS), (AAA) and (SAS). The (SAS) triangle similarity theorem seems to be lacking in many secondary school textbooks (according to my own reading of about 20 textbooks).

The research review also shows that an important issue which has been discussed extensively regarding how similarity is taught and learned, is the treatment of *similarity transformations* (Seago, 2013). In a static approach to similarity, transformations are *non-essential aspects*, since they do not affect similarity. This means that, it is necessary to understand that they do not affect similarity. In a transformations-based approach, on the contrary, similarity transformations are *essential aspects* of the definition and are therefore necessary to discern. Another thing that has been suggested is that identification of *correct correspondence*, could be one of the most challenging parts of the solution to some similarity problems (e.g. Chazan, 1988). In many tasks, however, correspondence is visually obvious, and do not have to be in the focus of attention (Wijayanti, 2019). In Hilberts (1902) or Birkhoff (1932) approaches to define congruence and similarity, there is an

obvious relation between corresponding sides and corresponding congruent angles. In the definitions of congruence and similarity, described in previous section, the *order of the angles* A B and C in a triangle $\triangle ABC$ is an essential aspect. Once this order is established the correspondence is obvious. The definition implies that $\triangle ABC$ is not similar to $\triangle BCA$, (unless the triangle is isosceles). This is essential to discern in order to use the concept properly in geometric proofs. However, in secondary school textbooks (according to my own reading), the order of the angles in $\triangle ABC$ is an aspect that is usually not focused, (maybe for good reasons). However, this implies that an explicit explanation, relating corresponding angles to corresponding sides, is needed, which according to Wijayanti, (2019) is not always done. When a transformations-based approach is used, figures can be compared through the underlying transformations. In that way it becomes clear which sides and angles correspond and why (Seago, 2014). This kind of mental flipping seems to be challenging for students to perform (Chazan, 1988) and requires that *similarity-preserving transformations* have been discerned as *essential aspects*.

There are also aspects which might have been expected to be critical for student to discern, but which, according to previous research seems to be less challenging for student to apprehend. For example, aspects related to equation solving. Several researchers (Chazan, 1988; Cox, 2013; Seago, 2014; Son, 2013) argue that students' difficulties in solving similarity problems seem to be related to the geometric configuration, and therefore stem from a lack of geometric reasoning and conceptual understanding, rather than the equation solving process.

Taken together, these findings suggest that what may constitute critical aspects of similarity is not fixed but depends on how the concept is defined and approached. Different axiomatic systems and instructional approaches foreground different aspects as essential, which in turn influences what needs to be discerned in order to understand and use the concept. Moreover, the intended level of understanding affects which aspects become critical. Thus, what counts as critical aspects of similarity must be understood in relation to both the underlying axiomatic system, the target way of understanding and students pre knowledge.

Examples of potential patterns of variation

In the present study, it is assumed that these aspects may be made possible to discern through variation, in ways similar to those illustrated by Zhang et al. (2017). In their study, tasks in which students are asked to determine whether figures represent vertical angles or inscribed angles (Fig 5.2, 5.3) are used as examples of how variation can be used in tasks to make certain aspects possible to discern.

More specifically, tasks that juxtapose concept and non-concept cases are described as instances of non-concept variation (corresponding to contrast), while tasks that present multiple instances of a concept with varying non-defining features are described as instances of concept variation (corresponding to generalisation).

In a similar way, aspects of similarity could be made possible to discern through structured variation in textbook tasks. For example, the condition of equal angles may be opened up by contrasting figures that appear visually similar but lack angle congruence with figures that satisfy the condition, thereby making the role of angle equality discernible as an essential aspect of similarity. Likewise, the order of angles in $\triangle ABC$ could be made discernible as an essential aspect by the contrast: The fact that $\triangle A'B'C'$ is similar to $\triangle ABC$ *does not imply* that $\triangle A'B'C'$ is similar to $\triangle BCA$. Moreover, geometric transformations could be made discernible as non-essential through generalisation where the essential aspects of similarity are held invariant while the figures are rotated or reflected.

6. Method

This chapter builds on the previous chapter, outlining the theoretical points of departure, and presents the methodological approaches employed in the three studies comprising the thesis. All studies adopt qualitative research methods consistent with approaches commonly used within the phenomenography and variation theory (VTI) research tradition. This tradition permeates the thesis at several interconnected levels. As argued by Cohen et al. (2018), research methods cannot be reduced to technical procedures for generating knowledge; rather, they are informed by how the world is understood. Ontological assumptions inform epistemological assumptions, which in turn shape methodological choices, including what counts as relevant data and how such data is generated.

VTI belongs to an interpretive research tradition (Cohen et al., 2018), which in contrast to positivist ones, focuses on understanding experienced meaning, rather than trying to describe a measurable reality. In line with this perspective, the studies are exploratory in nature, focusing on how meaning is constituted in relations, such as how a phenomenon is experienced, what aspects need to be discerned, and what learning opportunities are afforded to the learner. Accordingly, data is generated through flexible and responsive methods, such as open-ended interviews and observations, with the aim of capturing participants' perspectives and understanding (Cohen et al., 2018).

The analytical processes used in the studies are guided by constructs and theoretical principles, developed within the VTI tradition (see Chapter 5). In the analysis, meaning is generated through systematic, exploratory and in-depth engagement with the empirical material, where interpretation is developed in close relation to the data and guided by the analytical principles of the theoretical framework. The theoretical framework serves to organise the analysis and make patterns visible, rather than to impose predefined categories or interpretations. In that sense the analysis is neither inductive nor deductive. Rather it might be classified as abductive in the sense that it takes its point of departure in what is not yet fully understood and, through an iterative movement between empirical material and theoretical perspectives, seeks to generate new knowledge (see Olteanu, 2007)

Across the studies the analyses are presented as extended analytical narratives, in line with typical result sections within the VTL research tradition (e.g. Runesson, 1999; Booth, 1992; Kullberg, 2010). Categories of description and critical aspects are articulated and substantiated through empirical excerpts and reasoned argumentation. In this way the analytical process is rendered explicit within the text, allowing the reader to follow how interpretations are developed and supported in relation to the empirical material. The credibility of the findings thus rests on the transparency and coherence of the analytical reasoning and the aim is not to provide unambiguous evidence, but to present well-grounded, transparent, and convincing arguments for the conclusions made, in such a way that the results appear credible to the reader (see Åkerlind, 2012).

Throughout the thesis, the theoretical approach is thus not treated merely as a general inspiration but as a framework that shapes the formulation of the research questions, the selection of empirical material, the logic of the analysis, and the presentation of the results.

Study 1

The first study aims to illuminate the qualitatively different ways in which teachers understand educationally significant vocational mathematics knowing. Phenomenography is an empirical, exploratory research approach concerned with identifying qualitative differences in how phenomena are experienced and understood (Marton & Booth, 1997). As such, it is particularly well suited to the present study, as it enables a systematic analysis of variation in teachers' understandings, thereby revealing multiple facets of the phenomenon and accounting for its complexity, nuances, and potential contradictions, rather than seeking to establish a singular or definitive description.

Kvale (2007) argues that the qualitative interview is a uniquely sensitive and powerful method for capturing experience and exploring meaning. In line with this view, interviews are understood not as neutral tools for collecting information, but as purposefully guided conversations through which knowledge is produced in interaction between interviewer and interviewee (Kvale & Brinkmann, 2009). Semi-structured, open-ended interviews are therefore a typical approach for generating data in phenomenographic research, as they allow participants to articulate their ways of experiencing a phenomenon in their own terms while enabling the researcher to probe and clarify meanings (Marton, 1986; Marton & Booth, 1997; Kvale & Brinkmann, 2009).

The interviews in Study 1 were designed to direct attention towards the phenomenon described in Chapter 5. In order to maintain a shared focus of attention the interviews were structured around a common set of tasks and initial questions. At the same time, the directions of the interviews were allowed to develop in response to the participants' contributions in order to explore the phenomenon from different perspectives as these emerged during the interview (see Bowden & Walsh, 2000; Marton, 1986; Uljens, 1989).

The tasks were designed with the intention to encourage teachers to talk about the phenomenon, and its' component parts, in a way that revealed the experienced meaning of the phenomenon. Therefore, task solutions as such were not the primary focus; rather, the emphasis was on students' appropriate knowing in relation to corresponding tasks in an authentic vocational context. Bowden and Walsh (2000) recommend starting from problem-based questions grounded in the field under study, rather than from abstract questions of the kind 'what is [the phenomenon]?'. This allows interviewees to direct their attention to aspects that appear most relevant to them, which is of importance for the subsequent analysis. By posing problems or referring to shared topics of discourse, there is a greater likelihood of establishing a shared understanding between the researcher and the interviewee of what is being discussed (Bowden & Walsh, 2000).

Empirical data

Participants and setting

Sixteen Swedish secondary school teachers were interviewed. The interviewees, who were employed at seven different upper secondary schools were teaching vocational subjects (labelled V) or mathematics (labelled M) in the vocational programme 'building and construction'. Within both groups of teachers (each comprising 8 teachers) there were also variations in gender and age. In line with phenomenographic principles, the aim was not representativeness in a statistical sense, but to capture as wide a variation as possible in qualitatively different ways of experiencing the phenomenon (Marton & Booth, 1997; Marton, 2015; Bowden & Walsh, 2000). To make contact with potential participants for the interview study, upper secondary schools in the region, where the building and construction program was provided, were mapped. Emails were sent out to principals at the schools, describing the planned study (see Appendix 2). Since this did not generate any response, teachers were instead phoned, or emailed directly, and the purpose of the study was explained. Once a teacher showed interest in participating in the

study, an email was sent out with further information about the study (see Appendix 3), and time and place for the interview was decided, usually at the interviewees school.

The sampling strategy can be described as purposive in that teachers were selected on the basis of teaching either mathematics or vocational subjects within the building and construction programme, and convenience-based in that participation depended on availability and willingness (Cohen et al., 2017).

Most of the interviews were conducted individually. In two cases, however, mathematics and vocational teachers were interviewed in pairs. This arrangement was made at the request of the participants, as they expressed a preference for being interviewed together. Allowing this format was considered important for creating a comfortable and open interview situation. From a phenomenographic perspective, the focus of analysis was not on individual consistency but on the variation of ways of experiencing the phenomenon (Marton, 1986). Therefore, dialogical interactions between participants were treated as analytically relevant expressions of meaning-making rather than as methodological disturbances.

The interview tasks

In order to ensure a shared point of departure across interviews while allowing for open-ended exploration, all interviews were structured around a common set of tasks (see Appendix 4). The tasks functioned as prompts for reflection and discussion (see Uljens, 1998; Bowden & Walsh, 2000). Four mathematical textbook-type tasks were designed by the researcher. The design was inspired by tasks drawn from the textbooks analysed in Study 2 and aimed to address different aspects of the concept of similarity, as described in Chapter 4, while situating these aspects within a construction-work context. The first task concerned scale drawings and involved reasoning about scale factors and area change. The second task addressed slope in the context of drainage, expressed both as a ratio (1:200) and as a percentage (0.5%). The third task involved the use of the SSS similarity theorem to generate right angles through enlargement of a 3:4:5 triangle. The fourth task concerned the construction of a porch roof matching the slope of an existing house roof. The task could be solved either by applying the converse SAS similarity theorem, or the slope concept.

Taken together, the tasks were intended to involve theoretical understanding of a constrained geometric topic (similarity), as well as potential relationships to a broader area of mathematics (geometry in general, proportional reasoning,

trigonometry, etc.) and to a vocational context. By giving rich opportunities to make connections both within formal mathematics and within the vocation, the tasks were considered to constitute an appropriate starting point for exploring how the interviewees construed educationally significant vocational mathematics knowing.

Data generation

The interviews lasted approximately one hour. At the beginning of each interview, participants were provided with a printed set of tasks and additional paper for drawings, calculations, or written explanations. The interviews were conducted in Swedish and were audio- and video-recorded, with the camera directed towards the workspace to capture written and drawn representations. The video recordings supported the interpretation of the transcripts in cases where written or visual representations formed an integral part of the participants' explanations. All interviews were transcribed verbatim. The excerpts presented in this thesis were subsequently translated into English by the author. Participation was voluntary, informed consent was obtained, and all data was anonymised in the reporting of the results.

The tasks were sent to the participants a few days prior to the interview, together with a brief description of the study and examples of guiding questions (see Appendix 4). During the interviews, open-ended questions were posed in relation to the tasks, inviting participants to characterise the content area, identify central tasks, reflect on students' appropriate understanding and what they typically find challenging. Follow-up questions were used to probe meanings, request clarification, and encourage elaboration, as well as to check the interviewer's understanding (see Marton, 1986; Kvale, 2007). The interviews did not aim to evaluate teachers' competence, but to explore ways of understanding 'educationally significant vocational mathematics knowing'. The interviewees' answers to the questions were assumed to reflect how they experienced and construed the phenomenon under investigation. Although the interview could be understood as an interaction in which meaning is constituted through dialogue, and which could consequently never be independent of the researcher (Kvale & Brinkman, 2009), the interviewer sought to maintain questions at a level that invited participants to articulate their own ways of seeing the phenomenon. Efforts were therefore made to avoid introducing interpretations not expressed by the participants themselves (see Bowden & Walsh, 2000).

Method of analysis

A phenomenographic approach was used for the analysis, drawing on for example Marton (1986), Booth (1992), Bowden & Walsh (2000) and Åkerlind (2005). Initially the transcripts were read several times, in order to become familiar with the data, and to get an overall impression of the material. Common themes addressed by the interviewees, when talking about the tasks in relation to students learning, were identified, and the possibility of defining these themes as phenomena in a phenomenographic sense, was investigated, by considering whether the themes were delimited in a similar way and seen from various perspectives. The sections of the interviews which were significant for and maintained a specific focus in relation to the studied phenomenon were selected as unit of analysis. The interviews as a whole and the selected quotations were read in detail in order to understand the meaning conveyed in each quotation, and to detect distinctive differences in meaning between quotations (see e.g. Marton, 1986; Booth, 1992; Bowden & Walsh, 2000; Åkerlind, 2005). In several phenomenographic studies it has been observed that the same person can express different or even contradictory perceptions of a phenomenon. The quotations were, therefore, not seen as expressions of individual perceptions, but rather as making up a 'pool of meaning' (Marton & Booth, 1997). This 'pool of meaning' formed the basis for the subsequent analysis. Hence, the researcher's attention was not merely directed to individual interviews, but rather to the meaning expressed in the quotations.

Categories were formed on the basis of similarities and differentiated on the basis of differences. The process of reading, comparing and forming categories was repeated so as to increase the coherence within categories and the differences between them. The aim was to form an outcome space of qualitatively distinct categories of description, which, in its entirety captures both the essence and the variational structure of the experienced phenomenon, and in which the categories stand in a clear and a logical relation to each other (see e.g. Marton & Booth 1997; Åkerlind 2005). To enhance credibility, other researchers were involved at different stages of the research process. During the process of analysis, preliminary results were discussed repeatedly at seminars and conferences.

Within the phenomenographic tradition a few different ways of analysing data have been described, all of them aiming to provide a set of categories of description that are empirically grounded, clearly distinguished, logically interrelated, and as few as possible (Åkerlind, 2005). The current analysis was

carried out focusing alternately on the ‘what’ and ‘how’ aspects (structural and referential aspects) of the voiced experience, in order to capture key features of the categories. Phenomenography, like gestalt theory, assumes that people tend to experience phenomena as meaningful wholes, and that the way they attend to specific parts, reflects how they make sense of the whole. This means that, by noticing ‘what’ teachers choose to focus on, and the ways in which these parts are discussed, conclusions could be made about ‘how’ the phenomenon is understood.

Six categories of description emerged as a result of the analysis, denoting different ways of experiencing ‘educationally significant vocational mathematics knowing’. Once categories were formed the result was described in a way that aimed to clarify the core meaning of each category, and the specific characteristics that constitute the critical differences between categories of description. The result was presented in terms of a descriptive text and illustrated with quotations from the data (see Marton, 1986).

Study 2

The aim of Study 2 is to identify what students need to learn in order to engage proficiently with mathematically rich construction-work tasks (see Marin & LaCroix, 2008). To address this aim VTL was adopted as an analytical framework. VTL was considered a suitable approach as it allows for a fine-grained analysis focusing on critical aspects, that is, the components that must be discerned in order to understand something in a powerful way (Marton, 2015). VTL has made significant contributions to mathematics education research by providing conceptual tools that support teachers’ and researchers’ in focusing on essential aspects of specific objects of learning (e.g. Marton, 2015). In this study, the articulation of critical aspects provides a detailed account of construction-work activities, focusing on what students need to learn. The analytical focus is thus the student–task relation, specifically the discernments required for proficient engagement with vocational mathematics tasks.

To articulate what this entails, the study draws on complementary methods and empirical sources, each contributing to the same analytical object. Following a research design suggested by Maxwell (2012), the methodology was developed during the research process in response to opportunities that emerged in the interview data. During the interviews, teachers introduced relevant vocational mathematics tasks, which they also described in detail. This generated rich and interesting material that was not fully captured through the phenomenographic

analysis alone yet did not constitute a sufficient empirical basis on its own. Maxwell (2012) argues that a research approach that is allowed to develop flexibly enables a consistent focus on the object of study, while methods and approaches may be modified in response to emerging insights. He further recommends integrating data from a variety of methods and sources in order to reduce the limitations of a single method. In line with Maxwell (2012) the empirical sources in this study include interviews with vocational teachers, together with tasks from vocational mathematics textbooks, as well as student discussions based on these tasks. These sources can be seen as constituting a ‘pool of meaning’ from which critical aspects, defined in accordance with the theoretical framework, are derived (see Marton, 2015).

The interview data form part of the same empirical material as Study 1. The interviews were thus designed as phenomenographic interviews (e.g. Marton, 1986). More generally, they were conducted as typical qualitative, open-ended interviews, allowing for flexible exploration of participants’ perspectives (e.g. Kvale & Brinkmann, 2009). The analytical focus of the study however differs from typical phenomenographic studies. The focus in this study is not on describing variation in understanding from a second order perspective, but on identifying critical aspects in relation to a third persons understanding. In other words, the student–task relation is viewed through vocational teachers’ perspectives. The interviews provide insights into teachers’ views and experiences of students’ understanding and potential challenges while solving mathematically rich vocational tasks, which in turn reflects which aspects are regarded as critical for successful task-solving.

To complement and broaden the perspective, structurally similar tasks to the ones described in the interviews, were analysed across a corpus of five vocational mathematics textbooks published over the last fifty years. This structural mapping served to scan for mathematical demands across a wider range of vocationally oriented tasks that are comparable in terms of their intended objectives and underlying structure. Older vocational mathematics textbook tasks make an important contribution in this respect, as they offer a large number of tasks, designed to represent vocationally relevant content, closely aligned with vocational contexts and practices. As such, they provide rich examples of vocationally oriented mathematical tasks and thereby contribute to broadening the scope of the empirical material. However, although the textbook tasks are vocationally related, they are typically stripped of contextual complexity and situational detail, and they

reflect vocational practices as represented at the time of publication. Consequently, textbook tasks alone could not constitute sufficient empirical material.

In the third phase, selected textbook tasks served as a basis for student discussions. This phase complements and extends the task analysis by returning the focus to the student–task relation as it emerges in students’ engagement with the tasks. The students’ discussions were not intended to provide a complete account of students’ learning, which the limited number of participants precludes. Rather, they function as instances through which the student–task relation is illuminated, while retaining a focus on critical aspects.

In this way, each data source contributes partial but complementary insights into the same analytical object, and the strength of the design lies in their combined contribution to capturing a wide range of potentially critical aspects, rather than in the completeness of any single source (see Maxwell, 2012)

In line with Studies 1 and 3, this study is oriented towards students’ learning. At the same time, this study, as well as the thesis as a whole, contains relatively few direct observations of students’ engagement with vocational mathematics, which may appear counterintuitive. This is, however, due to the nature of the object under investigation. Educationally significant vocational mathematics knowing does not yet constitute a clearly delimited and well-defined content area that can easily be isolated and observed in classroom interaction. Although such content is intended to be integrated into mathematics teaching, both research and teacher accounts indicate that this integration is often partial or uneven in practice, partly due to uncertainty regarding the intended learning outcomes (Lindberg et al., 2013). Observing students’ direct engagement with well-defined instances of vocational mathematics presupposes such content to be specified. This involves delineating relevant vocational tasks and articulating the mathematical demands they entail, both in terms of *what* needs to be known and *how* it should be understood. The studies in this thesis aims to contribute precisely to such a specification. For this reason, the studies approach students’ learning predominantly indirectly: through teachers’ articulations of intended learning, through structural analyses of vocationally oriented tasks, and through selected instances of student engagement.

Empirical data

Teacher interviews

The empirical data was generated through in-depth semi-structured interviews. Eight Swedish upper secondary school teachers were interviewed. The interviewees, who were employed at six different upper secondary schools, taught vocational subjects⁶ within the construction and building programme (labelled V). Within the group of teachers there was variation in both gender and age.

As described above (see Study 1), the sample was generated using purposive sampling combined with convenience sampling (Cohen et al., 2017, p. 218). The interviews lasted just over an hour in total and were audio- and video-recorded (with the camera directed downwards to capture only the paper) and subsequently transcribed verbatim. The empirical material is thus identical to the vocational teacher interviews analysed in Study 1, but it is examined here with a different analytical focus. In the two studies, partly different sections of the interviews are selected for analysis depending on what serves the specific purpose of each study. The excerpts used in this text were later translated into English by the author.

During the interviews, teachers were encouraged to provide examples of mathematically rich vocational tasks, to explain these tasks, and to describe which aspects would be essential and challenging for students to understand. The interview transcripts were read several times in order to identify recurring examples of workplace tasks. Based on this reading, four task types were selected for further analysis. Task types were identified on the basis of similarities in overall objective and underlying structure. These categories primarily serve to organise the analysis and presentation of the material. As such, the task type distinction does not in itself constitute a decisive analytical tool but rather functions as a pragmatic way of structuring the empirical material.

Textbooks

The textbook sample was selected using purposive sampling (Cohen et al., 2017). The selection was generated through systematic searches in national and university library catalogues, including the Swedish national library catalogue (LIBRIS) and the University of Gothenburg library catalogue, using predefined inclusion criteria related to course level (upper secondary or equivalent) and vocational orientation (building and construction). Since textbooks from the mid-20th century typically

⁶ The mathematics teacher interviews included in Study 1 are not included here

contain high proportion of vocation-specific tasks (see Chapter 2), they were used in this study. Even though the textbooks were identified through careful searches, the purpose of the selection was not to obtain a representative sample of vocational mathematics textbooks from a particular historical period. Rather, the textbooks were included as sources of vocationally oriented tasks, in order to broaden the opportunity to identify critical aspects in relation to selected task types, and to serve as a basis for the student discussions. Material of a similar kind, produced by the teacher (V2), was provided during one of the interviews. This material served to corroborate the relevance of the tasks identified in the selected textbooks; however, it is not explicitly included in the analysis, as it is difficult to reference properly, and since similar tasks could be found in the textbooks below:

- Yrkesräkning för byggnadsyrken (Schmale, 1959) Bonnier
- Yrkesräkning Yr2 Byggyrkesutbildning, (Svenska byggnadsindustiförbundet, 1978) Liber
- Förberedande yrkesmatematik, Yrkesdel Vekstads mekanik plåt-svets och bygg (Kristersson, 1987) Ekonomikonsult
- Byggmatematik (Plesner, 1995) Liber
- Matematik 1a, Byggmatematik (Bedrosian, 2024) LiVa läromedel

In addition to these textbooks, the smaller number of vocational mathematics tasks found in the textbooks constituting the empirical data in Study 3 were also used as a point of reference.

Overview of textbook content

All selected textbooks were read multiple times to gain a general understanding of the material as a whole. This reading revealed that the textbooks contained a considerable mathematical content applied within a construction-work context. Among the textbooks Schmale (1959) emerged as the most comprehensive. Task types identified in the teacher interviews were selected for further analysis. Below a brief description of the textbook contents are presented, to provide a background for the analysis:

Length

In all of the textbooks, there are numerous examples of tasks that address the concept of length in a variety of vocational contexts. The content spans a wide range of topics, from basic measurement calculations to more advanced geometric

reasoning. Examples include calculating the perimeter of shapes, dividing distances into equal segments, and using formulas for stair construction. Further applications involve unit conversions (e.g., feet and inches to metres) and working with length ratios such as scale, slope or gradients. The Pythagorean theorem is used in several tasks, often applied in reverse in order to establish right angles. For this aim the carpenters angle is frequently used, i.e. the 3:4:5 triangle, which is proved using the Pythagorean theorem and scaled, applying the SSS similarity theorem. Moreover, several tasks involve determining distances from scale drawings. These tasks often require understanding the concept of scale as well as understanding how spatial positions are represented and measured. This includes interpreting coordinates in two and three dimensions, recognising reference lines and zero levels, distinguishing between cumulative and baseline measurements, and relating local and national three-dimensional coordinate systems. The concept of slope is addressed in most textbooks, sometimes referred to as a ratio, sometimes as an angle. In some textbooks, the tangent function is introduced to connect these two interpretations.

Area

All of the textbooks include area calculations of shapes such as rectangles, triangles, circles, and composite surfaces. Typical tasks in these sections involve material estimations in contexts such as tiling, wall tiling, carpet laying, painting, wallpapering, roofing, paving, plastering, and panelling. The tasks involve different types of materials, such as materials supplied by the metre with a given width (e.g., panel boards, roofing felt), materials consisting of items with fixed dimensions, and materials supplied by volume measurements (e.g., plaster, paint). The tasks require understanding the concept of area as well as applying area formulas for simple and composite surfaces. Furthermore, these tasks requires understanding and using the multiplicative relationships between dimensions, units, and quantities, and using change factor to adjust for waste etc.

Volume

Volume calculations in different vocational contexts are also typical tasks in all of the analysed textbooks. The textbook tasks typically involve geometric solids such as cuboids, cylinders, cones, and spheres, and sometimes complex, composite solids. Such tasks demand an ability to decompose shapes, and to reason about structure before applying formulas. The tasks concern estimating the volume of containers, the amount of soil or material to excavate, or the space to be filled with materials like gravel, concrete or insulation. The tasks thus involve identifying

relevant geometric shapes in objects or drawings, selecting and applying correct volume formulas, and decomposing volumes of complex shapes into known sub shapes. In addition, some textbooks requires it to be considered that volumes of materials can change due to, for example, excavation, compaction or blending, which involves interpreting and using ratios between volumes.

Other concepts

The textbooks also address concepts such as mass, density, force, moisture content, and temperature, as well as the relationship and interaction between these quantities. For example, the tasks require understanding the relation between volume and mass, and density; and an understanding of that the density in turn might be affected by, temperature, compression etc. Moreover, the concept of force is considered, highlighting aspects, such as: force as vector, force as decomposable into components (using trigonometry), normal force, torque (defined as $\tau = r \times F$), static equilibrium conditions (i.e. the sum of forces and the sum of torques in each direction equals zero) and centre of gravity. These are all examples of aspects, which are typically considered belonging to the domain of physics rather than to formal mathematics.

Student discussion

In addition to the teacher interviews, a discussion between two students (denoted S1 and S2) solving corresponding tasks was recorded. The dialogue, which lasted approximately one and a half hour in total, took place on several occasions and was video recorded, with the camera directed towards the students' drawings and written work. The recordings were transcribed verbatim.

Based on the teacher interviews, textbook tasks were selected for the student discussions. The tasks were identical to those found in the textbooks and were accompanied by relevant supporting materials, such as drawings, paper and calculators. The students were asked to jointly discuss and solve the tasks.

The two students were familiar with each other prior to the study and participated voluntarily. They were selected to provide variation in prior knowledge: one student attended the first year of the construction programme, while the other had recently completed the technology programme. This composition was intended to create a productive contrast in perspectives, encouraging the students to explain their reasoning to one another while still allowing the tasks to be solved collaboratively. In practice, both students contributed constructively to the discussions and gradually reached a shared understanding of the tasks.

The discussions took place with the researcher present. The students were encouraged to solve the tasks freely and to discuss their reasoning while doing so. After the tasks had been completed, the students were also asked to explain their reasoning and solutions. During these explanations, subtle clarifying questions were posed by the researcher when necessary, allowing potential misunderstandings or errors to be addressed without interfering with the students' initial problem-solving process.

The student dialogue was analysed in its entirety. The primary purpose of this was not to provide a comprehensive account of students' learning, but rather to illuminate critical aspects of the tasks as they emerged in students' engagement with them. In this sense, the dialogue functions as an extended task analysis that directs attention towards the student–task relation and the aspects that become critical for understanding mathematically rich vocational tasks. All participants gave informed consent prior to the study, and pseudonyms are used throughout the presentation of the data.

Method of analysis

The transcribed teacher interviews and student dialogue were analysed through the conceptual lens of VTL, with a focus on identifying critical aspects. In this study, critical aspects are understood as aspects that learners need to discern in order to engage with a task in a more proficient way. The identification of critical aspects followed the theoretical definition outlined earlier (Chapter 5, Study 2). In brief, critical aspects are understood as elements that are essential and non-trivial for students to discern (Marton, 2015; Pang & Ki, 2016). Analytically, such aspects correspond to changes in the structure of awareness, manifested through contrasting or shifting ways of understanding (Marton & Booth, 1997; Pang & Ki, 2016). The characteristic of an aspect as constituting a dimension of variation also served as an important analytical criterion in the identification of critical aspects (Marton, 2015). These characteristics guided the identification and interpretation of potential critical aspects throughout the analysis.

The analysis proceeded in several steps. First, the interview transcripts and the student dialogue were read repeatedly to gain an overall understanding of the material. During these readings, episodes were identified where aspects of the tasks appeared to become focal in the participants' reasoning. These tentative aspects were subsequently examined in relation to the theoretical definition of a critical aspect. The analysis involved moving iteratively between the empirical material and

the theoretical framework, allowing the interpretation of potential aspects to be refined.

In the teacher interviews, critical aspects were identified as elements that teachers emphasised as central or challenging when describing the task and explaining how they would support students in approaching it. In the student dialogue, critical aspects were identified as aspects at the edge of students' understanding, manifested as focal points of discussion, where differences or shifts in understanding became apparent. When textbook tasks were analysed directly, the identification of critical aspects was based on the researcher's informed judgement of what could potentially be critical for students' learning. This involved analysing the tasks by solving them and examining the mathematical structure and demands involved. These interpretations were guided by the theoretical framework and informed by insights from previous research on students' learning of the relevant mathematical topics.

Using several types of empirical material contributed both to analytical robustness and to a more comprehensive account of the aspects involved. The identified aspects were therefore compared across teacher interviews, student dialogue, and textbook tasks. This comparison served partly to confirm overlapping aspects and partly to broaden the range of aspects identified.

Throughout the analytical process, interpretations were discussed with other researchers, providing opportunities for collegial feedback and refinement of the analysis.

Study 3

In Study 3, afforded learning opportunities related to geometric similarity in three different mathematics textbooks for vocational students are investigated and compared.

Different approaches to textbook analysis were exemplified in Chapter 3. Many of these studies rely on systematic content analysis, often guided by predefined categories based on previous research on the specific phenomenon under investigation (e.g., Charalambous et al., 2010 Jäder et al 2020; Frejd; Wijaya). In contrast, some studies use theoretical constructs to guide an interpretive analysis of the material rather than relying on fixed coding schemes. For example, Lo et al. (2006) adopt a theory-driven approach in which the analysis is guided by conceptual tools, focusing on concept development through definitions and concept images. The present study share similarities with Lo et al., (2006) in that

it is theory-driven and openly interpretive, meaning that it is grounded in theoretical principles without being operationalised into fixed coding categories. However, since the analysis in this study is guided by theoretical constructs developed within the VTL research tradition, particularly the notion of patterns of variation, it aligns more closely with research on learning opportunities based on VTL than with conventional textbook analyses. The use of VTL for the analysis was considered particularly powerful, as it provides a focused analytical framework grounded in well-established and clearly articulated educational principles (see Marton, 2015). Particularly in geometry education, teaching informed by variation theory has been widely recognised for its potential to support learners in discerning essential features of geometric concepts while simultaneously disregarding non-essential aspects (Sinclair et al., 2016).

As described in Chapter 5, several studies analyse sequences of mathematical examples or tasks in order to identify patterns of variation and invariance and the corresponding space of learning (e.g. Kullberg & Skodras, 2018; Pillay et al., 2022). The analytical focus in these studies is typically on variation and invariance within and across tasks, and on how such patterns create opportunities for learners to discern critical aspects. Methodologically, this involves examining sets of examples in their presented order, attending to how they are constructed, sequenced, and juxtaposed. Patterns of variation are identified both within tasks (horizontally) and across tasks or lesson episodes (vertically) (Pillay et al., 2022).

The results are typically presented in terms of ordered task sequences, followed by analytical commentary that highlights patterns of variation and their implications for what learners are afforded the opportunity to learn. Similar approaches are used in a few textbook studies. By contrasting concept and non-concept examples, and by varying non-essential aspects while keeping critical features invariant, tasks are designed to support learners in discerning defining characteristics and generalising mathematical concepts. (e.g. Zhang et al, 2017; Sun, 2019). The present study follows a similar approach to that of Sun (2019), focusing on variation that enables the discernment of essential aspects of geometric concepts, similar to those described by Zhang et al. (2017). In line with Sun (2019), the results are presented through detailed examples of textbook tasks, followed by a comprehensive interpretive and argumentative analysis that highlights identified patterns of variation and the aspects which thereby are made possible to discern. The study is limited to the section explicitly addressing geometric similarity, and the analysis therefore concerns the patterns of variation opened up within this section.

Empirical data

The textbook sample was defined using purposive sampling (Cohen et al., 2017). The selection was generated through systematic searches in library collections (e.g. LIBRIS), by visiting the teaching materials exhibition at the National Centre for Mathematics Education (NCM), and through internet searches for currently used teaching materials. The aim was to identify all textbooks for the course *Mathematics 1a* that were available and used in Swedish upper secondary schools during the period 2020–2022. Four textbooks from different publishers were identified: *Exponent* (Gleerups), *Matematik 5000* (Natur & Kultur), *Matematik 1a* (Liber), and *Origo* (Sanoma)⁷. The same textbooks were identified across all search strategies. The publishers were contacted to obtain permission to reproduce excerpts from the textbooks in the results section, which was granted.

Only printed textbook materials were included in the analysis. Although digital resources are available for these textbook series, such materials were excluded in order to maintain comparability between the analysed sources. All textbooks available on the market for the course during the specified period were examined. Of the four identified textbooks, three contained a section addressing the concept of geometric similarity. These three textbooks (here labelled A, B, and C) were further analysed. The concept of similarity was selected for the analysis. This concept was chosen due to its central role in geometry, its close relation to other mathematical domains such as proportional reasoning, and its relevance in vocational contexts such as construction-work. In these contexts, similarity underlies various practices, even when the concept itself is not always explicitly articulated (see results of Study 2, Chapter 8).

The analysed textbooks share a broadly similar structure, organising content into chapters and subchapters addressing specific mathematical topics. In each of the selected textbooks, a section of approximately two pages is devoted to similarity. The empirical material analysed in Study 3 consists of the full similarity section in each textbook, including explanatory text, worked examples, and tasks. The analysis focuses on identifying patterns of variation through which the concept of similarity is presented and made available for learning.

Textbook A, Origo (Sanoma Utbildning) publishes one textbook for vocational programmes and one for the natural science programme. Each chapter

⁷ In retrospect, I identified a book containing a substantial number of construction-related tasks. However, this source was not published at the time the study was conducted and was therefore not included in the analysis of Study 3. It should also be noted that the section on similarity in this book is limited, comprising only three tasks. (*Matematik 1a, Byggnatematik*, LiVa läromedel)

begins with an activity that students can do in groups or in pairs, in order to practice the concepts in a playful way (Olofson, 2018).

Textbook B, Matematik 5000 (Natur och Kultur) is a textbook series for upper secondary school. Two textbooks are published for Ma 1a. One textbook is aimed at students in service-oriented vocational programmes and one textbook is aimed at students in technically oriented vocational programmes. The books alternate explanatory theory sections, solved examples, tasks to be solved and theme sections (often related to vocational subjects) (Alfredsson et al, 2011).

Textbook C, Exponent (Gleerups) publishes one textbook for all vocational programmes. The general content is first presented in a number of chapters. At the end of the book there is a section for each vocational programme. In addition to theoretical reviews, solved examples and tasks there are also ‘challenges’ i.e. activities of problem-solving nature in which mathematical abilities are aimed to be developed (Johansson et al, 2011).

Revised versions of the textbooks were published during the course of the PhD project. The most substantial changes occurred in Textbook B. The revised editions of Textbooks A and C entailed only minor differences. In the revised version of Textbook B, the section on similarity was merged into another section. This section contains only a few tasks from the earlier similarity section. Since tasks addressing similarity are few in the revised version of Textbook B, this textbook is not included in the present analysis of the similarity concept.

Method of analysis

Variation theory was used as an analytical tool to identify learning opportunities afforded in mathematics textbooks. Such analysis builds on the idea that learning entails discerning critical aspects of a phenomenon. As described in Chapter 5, a fundamental point of departure in variation theory is that such discernment becomes possible only through experienced variation along the dimension that constitutes the critical aspect. Patterns of variation and invariance provided in a learning situation is regarded as constituting a *space of learning*, which determines what is made possible to learn (Runesson, 1999). This implies that conclusions about learning opportunities afforded in the material can be drawn through the identification of patterns of variation in tasks, examples, and sequences of tasks (Marton et al., 2004; Kullberg, 2010; Runesson, 1999).

A *task* is understood in this study as a textual configuration that initiates an action directed towards a determinate completion. The patterns of variation that

emerge in this activity are seen as constituting learning opportunities. However, in the analysis the task as such did not constitute the primary unit of analysis. Rather, the analysis focused on the dimensions of variation opened up in the overall configuration of activities initiated in the textbook section as a whole. In line with Watson and Mason (2006b) patterns of variation were considered both within and across tasks. Tasks were thus seen primarily as a structure of the material, while the analytical focus was directed towards patterns of variation.

Particular analytical focus was placed on whether contrasts and generalisations regarding essential aspects of the concept were foregrounded through the construction and formulation of the tasks. Contrasts may, for example, be created through the use of non-examples or conflicting cases, while generalisation can be supported through variation against a background of invariant defining aspects (see Figure 5.2 and 6.1). In this way, learners may be enabled to discern essential aspects and disregard non-essential aspects. In addition, attention was paid to whether variation is structured in a way that direct attention towards a coherent whole, or in a way, where multiple meanings are opened up without prior differentiation (Olteanu, 2007). Attention was therefore paid to the sequencing of patterns of variation where the initial variation of individual aspects against a background of invariance in others, constitutes an important component, as well as the subsequent fusion of aspects where several aspects are brought into simultaneous variation.

The analysis did not rely on a predefined coding scheme in which specific critical aspects were identified in advance and subsequently searched for in the material. However, the dimensions of variation identified through the analysis were later compared with relevant mathematical theory and educational research, as described in Chapter 4, and were articulated in the text using established mathematical terminology. In this sense, both the analysis and its presentation adheres to established methodological conventions within the VTL research tradition. The analysis proceeded as follows:

First, the selected chapter was read through to get an overview of its' structure and the type of tasks provided. The placement of the section in relation to other topics in the textbook was also noticed. Then the section was studied in detail, task by task. Patterns of variation were recognised, and the corresponding aspects, according to VTL, were identified. The analysis resulted in a comprehensive narrative description, focusing on the patterns of variation emerging in the text, tasks and sequences of tasks. The aspects were summarised in a table at the end of the section, and the afforded learning opportunities in the three textbooks, thereby

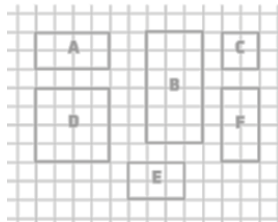
became possible to compare. Once dimensions of variation were identified, the learning opportunities afforded in the textbook were also compared to mathematical definitions and educational research (Chapter 4).

Example of analysis

In order to exemplify and explain the analysis one example from a textbook is presented below.

Figure 6.1 The task is to identify similar figures.

(*Olofsson & Gerholm, 2021, p 314*)



In Figure 6.1 some pairs of rectangles are similar (A–B–F and D–C) and the others are not. The task invites students to identify which rectangles are similar. This activity creates patterns of variation and invariance that allow certain aspects of the concept of similarity to be discerned, while others remain in the background. The table below illustrates what varies and what remains invariant, and outlines the conclusions drawn from these patterns based on variation theory:

Table 6.1 The table exemplifies and explains the analysis procedure

Example	Variation identified				Conclusion regarding learning opportunities		
	Identity shift (similar / not similar)	Pattern of variation	What varies	What is kept invariant	Aspect constituted as essential	Aspect constituted as non-essential	Aspect not focused
A-B (similar) A-E (not similar)	Yes (Non-concept variation)	Contrast	Proportional sides vs non-proportional sides	Angles	Proportional relationship between corresponding sides		Congruent angles
A-B-F (similar)	No (Concept variation)	Generalisation	Orientation Absolute size	Shape (The rectangles are similar)		Orientation Absolute size	Congruent angles

Explanation of the table

In the comparisons A–B vs A–E (first row) variation with respect to whether corresponding sides are proportional or not, is introduced. In A–B the sides are proportional, whereas in A–C they are not. This pattern is an example of *contrast* and brings the proportional relationship between corresponding sides into focus. Since variation in this aspect determines whether the rectangles are similar or not, the pattern of variation corresponds to *non-concept variation* in the framework proposed by Gu et al. (2004). According to the framework, variation in an aspect, which leads to an identity shift, meaning that a case no longer represents the concept, makes it possible to discern the varying aspect as essential for determining the concept. In this case, the pattern of variation, (i.e. contrast) serves to bring the proportional relationship between corresponding sides into focus as an essential aspect of similarity. Moreover, the framework emphasises that a particular aspect can only be discerned against a background of invariance in other aspects. In this case, the angles obviously remain invariant across the rectangles. Since this aspect does not vary ‘congruent angles’ is kept in the background in this task. This selective pattern of variation prevents too many aspects from being introduced simultaneously and reinforces the discernment of the proportional relationship between corresponding sides.

In the second set of comparisons (A–B–F), the rectangles remain similar while their orientation and absolute size vary. This pattern of variation constitutes a *generalisation*, as essential aspects remains invariant while other features vary. Since the instances are examples of the same concept (similarity) the variation corresponds to *concept variation* in the framework proposed by Gu et al. (2004). According to the framework, as the identity of the concept is preserved while certain aspects vary, these aspects are constituted as non-essential in relation to the concept. In this case, the variation serves to constitute orientation and absolute size as non-essential aspects of similarity.

Taken together, the task thus isolates the proportional relationship between corresponding sides as the critical aspect to be discerned. Other aspects are either varied in ways that support this discernment or are kept invariant.

In the analysis, patterns of variation initiated in interaction with the task are examined not only in terms of the internal structure of the task itself, but also in relation to its position within the surrounding sequence of tasks in the textbook. According to variation theory, the sequencing of patterns of variation plays a decisive role for the learning opportunities afforded. Critical aspects must first

become possible to discern through patterns of contrast and generalisation, where particular aspects vary against a background of invariance. Subsequently, several aspects should vary simultaneously, allowing learners to attend to their interrelations (fusion).

Viewed in the context of the similarity section as a whole the example presented above reveals that different aspects of similarity are introduced progressively across the task sequence, allowing learners first to discern individual aspects before later encountering tasks in which several aspects are focused simultaneously. This sequential organisation of variation is, according to the framework, essential in relation to the learning opportunities afforded to students, and is therefore, together with the task analysis exemplified above, an important part of the analysis.

Quality of the studies

In all forms of research, questions of validity, reliability, and generalisability are central. However, as emphasised by Cohen et al. (2018), these are interpreted differently across research paradigms, and must be understood in relation to underlying ontological and epistemological assumptions. Within qualitative and interpretive research traditions, validity is not primarily a matter of measurement accuracy but concerns the extent to which interpretations are well grounded, coherent, transparent, and warranted (Cohen et al., 2017). For this reason, several researchers have proposed conceptualisations of validity alternative to those used in quantitative and positivist research. For example, Lincoln and Guba (1986) suggest replacing criteria such as internal and external validity with credibility and transferability, and reliability with dependability and confirmability.

In this chapter, validity is discussed, in line with Cohen et al. (2018) and Lincoln & Guba (1986) through the concepts of *credibility*, *transferability*, and *dependability*. These categories provide an overarching structure for the discussion. At the same time, their interpretations are informed by the VTL theoretical framework. From this perspective, validity concerns primarily how convincingly the study illuminates the object of research and the relations it entails, and how well the interpretations can be defended in relation to the empirical material and the theoretical framework (Åkerlind, 2012; Cohen et al., 2018). Validity criteria such as *content related validity*, *methodological validity* and *communicative validity* have been proposed in phenomenographic research (Booth, 1992).

In line with a non-dualist ontology, and the view of knowledge as formed in the relation between the knower and what is known (e.g. Uljens, 1996), quality cannot be understood as an inherent property of the thesis, as such, but depends on what it affords, that is, the extent to which it may contribute to a richer and more nuanced understanding of the object of research. From this perspective, credibility rests on whether other researchers with similar interests are able to recognise the interpretations and conclusions as reasonable (see Collier-Reed et al., 2009; Åkerlind, 2012).

To this end the present thesis has sought to establish a shared frame of reference across its different parts, thereby supporting the development of an interpretive horizon within which the results can be understood. First, through the review of previous research, the object of research is situated within an existing field of knowledge (Chapter 3 and 4). Second, the theoretical points of departure are made explicit, and their implications for how the object of research is conceptualised are carefully articulated (Chapter 5). Third, the methodological section clarifies how the theoretical framework is interpreted in the study, and how this interpretation informs the generation of data and the analytical process (Chapter 6). In the results and discussion sections (Chapters 7-10) the analysis is presented through extended analytical narratives, in which interpretations are developed through systematic argumentation and substantiated by rich empirical examples, in order to allow the reader to follow how conclusions are reached and to confirm their plausibility. Finally, the thesis has been explicit about the nature of its claims, consistently emphasising that the aim has not been to establish a single, definitive account, but to contribute to a more powerful, developed and differentiated understanding of the object of research.

Kilpatrick (1995) argues that research in mathematics education gains its relevance by affording tools for thinking and reflection about educational practice, rather than by providing ‘recipes’ for teaching. The present thesis aims to contribute to such reflection by illuminating, from several complementary perspectives, what mathematical knowing students need to develop in vocational education in order to understand and use mathematics in a powerful way in vocational practice.

Credibility (Internal validity)

Internal validity concerns the extent to which interpretations and conclusions can be sustained by the empirical material (Cohen et al., 2018). In qualitative research,

this is closely related to credibility and evaluates whether there is a logically coherent and well-argued connection between research focus, empirical setup, methods, and conclusions; and whether interpretations are theoretically informed, empirically grounded, defensible, and transparent (Åkerlind, 2012; Booth, 1992; Collier-Reed et al., 2012).

In this thesis, credibility entails establishing coherence both within and across the studies. It further includes grounding the identified aspects, categories, and relations in appropriately selected and generated data and in well-defined theoretical constructs; and conducting the analytical process in a coherent, systematic, and transparent manner.

The three studies in the thesis approach the research object from different perspectives, all of which are grounded in a common theoretical framework and a consistent research focus. During the research process, however, the design and analytical focus of the individual studies were adjusted in response to emerging insights and to the areas in which the most significant findings appeared. For example, the mathematical focus in the interview study expanded beyond the initial intention, and the more limited focus adopted in the textbook analysis. Moreover, the textbook analysis revealed greater differences between textbooks for vocational programmes than initially expected, and this variation therefore became more interesting to explore than to confirm differences between textbooks for different upper secondary programmes, which was the original intention. While these developments led to a broader and more nuanced understanding of the object of research, they also repeatedly affected the coherence and progression of the thesis, at the expense of its initial structure and internal consistency.

Nevertheless, in line with Maxwell's (2012) notion of an interactive research design, such developments can be understood as necessary adaptations that enable a study to take account of empirically grounded insights as they emerge. Rather than strictly adhering to an initially defined design, the flexibility in methods and analytical focus may make it possible to capture a richer and more meaningful representation.

At the same time, a consistent focus on the underlying research problem has been maintained throughout the research process, which has meant that all of the studies, in different ways, have contributed to a deeper understanding of the research object.

Content-related validity

Discussing validity in relation to phenomenographic research Booth (1992) argue that such research must be grounded on a sound understanding of the researched area. The researcher should have a comprehensive understanding, of topics related to the considered phenomenon. To enhance the pre-knowledge of the mathematical topic and the current field of educational research, extensive reading of previous research regarding vocational mathematics as well as educational research related to relevant mathematical topics was carried out together with additional studies of geometry. This background understanding is presented in Chapter 4. Further experience and proficiency within the vocational domain would have strengthened the validity of the study. Bakker (2014) argues that research in vocational mathematics education is inherently complex due to its interdisciplinary nature, requiring expertise not only in mathematics and mathematics education, but also in the vocational practices in which the mathematics is embedded.

As this constitutes a potential limitation of the present study, interviews with vocational teachers form an important component of the empirical material, drawing on their professional expertise. In addition, a broad range of vocational mathematics textbooks was systematically reviewed by the researcher in order to gain an overview of the mathematical content and practices represented in the field, thereby providing a broader basis for interpreting the topic under consideration.

Even though such background understanding is essential scholars argue that, in phenomenographic research, the researcher's openness to ways of understanding that differ from both their own and those commonly accepted, may be even more important than possessing a firm prior knowledge of the field (Booth, 1992; Collier-Reed et al., 2009). As stated previously, the researcher's prior knowledge in the present thesis does not function as an analytical template. Rather, it forms an interpretive horizon within the analysis and plays a more direct role in the synthesis and discussion of the results.

Methodological validity

Booth (1992) further argues that the credibility of a study's methodology is essential. Methodological validity depends on the degree to which its design and execution are aligned with its aims. From this perspective, key considerations include the composition of the sample, the context in which data are generated, and the way in which the data are analysed.

In a phenomenographic study, it is crucial that the sample is relevant to the central research question under investigation and that sufficient variation in perspectives is represented (Collier-Reed et al., 2009). In **Study 1**, purposive sampling was used in order to capture variation in ways of understanding the phenomenon. To this end both mathematics and vocational teachers were interviewed. At the same time all of them were teaching at the building and construction programme, ensuring a common frame of reference (see Bowden & Walsh, 2000). Second, a shared focus on the phenomenon under investigation is essential (Uljens, 1989; Collier-Reed et al., 2009). Therefore, the interview context was structured around a shared set of tasks, providing a consistent focus. This supported the establishment of a shared understanding of what was being discussed. Third, in line with established methodology, the interviews were conducted as semi-structured and open-ended conversations, allowing participants to articulate their own ways of understanding while enabling the researcher to probe and clarify meanings, with care being taken to avoid leading questions and to support an open and reflective dialogue (Bowden et al., 2000; Marton, 1986). At the same time, in phenomenographic research, findings are never expected to be independent of either the interview context or the analytical process, rather meaning is seen as constituted within these interactions (Kvale & Brinkman, 2009; Åkerlind 2012). Moreover, the focus is not on describing individual interviewees' understandings, but on identifying a range of qualitatively different ways of understanding the phenomenon (Marton, 1986). However, to explore variation it is pertinent that the researcher's own perspective remains in the background. Particular attention was therefore paid to maintaining openness to different ways of understanding both in the interview situation and in the analysis, allowing meanings to emerge from the empirical material rather than being imposed by the researcher (see Collier-Reed et al., 2009). The analytical process further followed established methodological approaches (as described earlier in this chapter), involving repeated readings and comparisons in the search for meaning in the data, allowing interpretations to be refined in a systematic and theoretically informed manner (Collier-Reed et al., 2009; Åkerlind, 2012).

In **Study 2**, the use of multiple data sources was partly motivated by developments that emerged during the interview phase. Certain task types exemplified during the phenomenographic interviews were identified as particularly relevant and interesting in relation to the research focus. However, these tasks did not recur with sufficient frequency or consistency across the interviews to constitute a sufficient empirical basis for further analysis. This limited the possibility of exploring these task types in depth based on interview data alone.

It would have been advantageous to plan for several iterative cycles in which task types were first identified and then explored in greater depth. Such an approach was, however, not feasible within the scope of the present study. Instead, this limitation was addressed through the inclusion of additional data sources, including textbook tasks and student discussions. These complementary sources contributed to broadening and refining the empirical basis for the analysis, enabling a more robust identification of critical aspects. This design should not be understood as a methodological weakness, but rather as a consequence of an exploratory process. Since the analysis was guided by a well-defined theoretical construct, and the analytical focus was consistently directed towards the student–task relation, the study maintained a stable point of reference across the different data sources (see Maxwell, 2012).

The use of multiple data sources can thus be seen as an effective and, at the same time, methodologically robust way of identifying and corroborating a broad range of critical aspects, allowing interpretations to be examined across different forms of data, thereby strengthening the warrants connecting data and conclusions (see Cohen et al., 2018). The main weakness in this approach lies in the contextual framing of the tasks, where a richer interview material could have contributed additional context-specific aspects. At the same time, the existing material is sufficient to illustrate the complexity and contextual embeddedness of vocational mathematics knowing in terms of the critical aspects identified.

A further challenge relates to the nature of the object of research. Compared to more established forms of analysis, the present study addresses a more complex and less clearly delimited phenomenon. This means that the study could not rely solely on established methodological procedures. Instead, the analysis has been guided by theoretically informed interpretations, requiring continuous attention to the core definitions and assumptions of the theoretical constructs employed. In this sense, methodological rigour has been achieved not through the application of predefined techniques, but through maintaining a close and consistent relation between theoretical framework and empirical analysis.

In **Study 3** methodological validity primarily rests on the clear theoretical grounding of the analysis. The identification of patterns of variation and the interpretation of afforded learning opportunities are guided by established theoretical premises and constructs within the VTL framework (e.g. Marton, 2015). This provides a coherent basis for linking empirical observations to analytical claims and ensures that the analysis is conducted in a systematic and theoretically consistent manner.

The study analyses textbooks as representations of potential learning opportunities. While textbook analysis cannot capture the enacted object of learning as it emerges in classroom interaction, the role of textbooks in shaping teaching and learning is well documented (e.g., Fan et al., 2013; Johansson, 2006; Qi et al., 2025; Rezat et al., 2021; Valverde et al., 2002). Textbooks can therefore be understood as constituting an important part of the conditions for learning. From this perspective, the analysis provides a valid basis for comparing learning opportunities.

At the same time, certain limitations must be acknowledged. The sample is restricted to textbooks from a specific period, and both textbooks and educational practices are subject to change over time. Furthermore, the analysis is delimited to a single mathematical concept. However, since geometric similarity is a central concept in both mathematics and vocational contexts, it provides a meaningful and relevant basis for analysis (see Hoyles & Clark-Wilson et al., 2021; Lehrer et al., 2002). In addition, the textbooks analysed are all designed for the same course and target group, which allows for a focused comparison of variation in learning opportunities within an invariant curricular context.

Finally, it should be acknowledged that the validity of the study is dependent on the assumptions underlying the theoretical framework. The analysis rests on the premise that variation is a necessary condition for learning, an assumption supported by previous VTL research. However, if this assumption were to be fundamentally challenged, the validity of the conclusions drawn from the analysis would also be affected.

Taken together, internal validity in Study 3 rests on the coherence between theoretical framework, analytical procedures, and empirical material, as well as on the extent to which the analysis provides a transparent and defensible account of the learning opportunities afforded in the textbook material.

Construct validity

Methodological validity also depends on that analytical constructs are well defined (Cohen et al., 2017). Concepts such as *phenomenon*, *categories of description*, *critical aspects*, and *object of research* function as theoretical constructs that guide the analytical process. These concepts are well defined and carefully applied. At the same time, the specific phenomenon under investigation, the particular aspects identified and the categories described, are not predefined constructs but emerge through the analysis, as empirically grounded descriptions. The category titles should therefore not be understood as theoretical constructs, but rather as empirically derived ways

of describing qualitatively different understandings of the phenomenon under investigation. Mathematical concepts are, to some extent, drawn upon as theoretical constructs, and are therefore rather comprehensively described in Chapter 4; however, these do not function as analytical tools in the identification of aspects. Rather, they are used in the interpretation and articulation of the results. In this sense, the mathematical framing of the findings is grounded in established theoretical perspectives, including different axiomatic systems, and serves to clarify the aspects identified in the analysis.

Communicative validity

Communicative validity concerns the extent to which interpretations are presented in a way that allows them to be critically examined and judged by others, and the researcher's ability to argue persuasively for the interpretations made (Booth, 1992; Kvale, 1996; Collier-Reed et al., 2009; Åkerlind, 2005). In this thesis, communicative validity is primarily addressed through the transparent presentation of the analysis. Across the studies, interpretations are not presented as self-evident but are developed through extended analytical narratives in which the reader is enabled to follow how conclusions are reached. These narratives are not intended to provide proof, but to offer convincing arguments. Moreover, the findings and interpretations have been repeatedly discussed and scrutinised in different academic contexts, including seminars, research groups, and conferences, involving varying constellations of researchers. This ongoing dialogue has contributed to refining the interpretations and strengthening their communicative credibility (see Åkerlind, 2012).

While communicative credibility may also involve participants' recognition of interpretations, this form of validation is typically limited in phenomenographic research. The analysis is based on a collective 'pool of meaning', and the resulting categories of description do not represent individual understandings, but rather qualitatively different ways of experiencing the phenomenon across participants. Therefore, it is not necessarily desirable that participants agree with or recognise the categorisation (Booth, 1992; Collier-Reed et al., 2009). Moreover, as experiences are context-sensitive and may shift during the interview, participants could not be expected to recognise or confirm the final outcome space (Åkerlind, 2012). In this thesis, communicative credibility is therefore primarily established through the transparency, coherence, and defensibility of the analysis.

Dependability

Dependability refers to the consistency and transparency of the research process and the extent to which interpretations are grounded in a systematic and theoretically informed analysis (Collier-Reed et al., 2009; Lincoln & Guba, 1986; Åkerlind, 2012). Rather than expecting identical results to be reproduced, the emphasis lies on making the analytical process visible and defensible.

One threat to dependability concerns transcription. Efforts were made to ensure that the spoken content was represented as accurately as possible, while maintaining a level of detail appropriate for phenomenographic analysis. This implies a focus on meaning, whereby detailed accounts of how something is said, such as pauses, intonation, and other paralinguistic features, are not considered necessary (see Collier-Reed et al., 2009). As the interviews were conducted in Swedish and later presented in English, translation constitutes a potential threat to dependability. To address this, the original transcripts were used throughout the analysis, and care was taken to preserve the intended meaning in translation.

In the presentation of results, a deliberate choice was made not to include extended interview passages containing both interviewer and interviewee turns. While this may limit the reader's access to the full conversational context, it enhances readability and allows the most relevant aspects of the data to be foregrounded. It also made it possible to include a larger number of excerpts, thereby providing a broader empirical basis for the analysis. Efforts were also made to exclude passages in which the interviewees' responses explicitly mirrored the interviewer's formulations, in order to avoid presenting interpretations that reflected the interviewer's framing rather than the participants' own ways of understanding.

The analyses were conducted through repeated readings, iterative comparisons, and continuous refinement of interpretations, in line with established phenomenographic procedures (Bowden et al., 2000; Marton, 1986; Åkerlind, 2005). To strengthen dependability, the analysis was repeatedly discussed with supervisors and researchers experienced in phenomenography and variation theory, and, where relevant, compared to similar analyses in the field. Intersubjective scrutiny was primarily achieved through dialogic engagement rather than through strict coder agreement. Drawing on Collier-Reed et al. (2009) and Åkerlind (2005), dialogic dependability checks, where interpretations are examined and refined through discussion, were considered more appropriate than coder reliability checks.

Rather than aiming for replicability in a positivist sense, dependability in phenomenographic research rests on interpretive awareness, that is, the researcher's continuous effort to critically examine their own assumptions and to make the analytical process transparent (Collier- Reed et al., 2009). By providing detailed descriptions of the analytical procedures and extensive empirical exemplification, this study enables readers to assess the consistency and plausibility of the interpretations. To this end a relatively large number of excerpts compared to common practice in phenomenographic research, were included in the presentation of the results, thereby providing a broad empirical basis for the interpretations.

Transparency also involves acknowledging the role of the researcher. The thesis does not assume that meaning resides transparently in the empirical material, waiting to be extracted. Rather, meaning is understood as constituted in the relation between researcher and data. The researcher's prior understandings are therefore acknowledged as a significant part of the interpretative horizon. At the same time efforts have been made to avoid allowing these understandings to function as ready-made analytical templates.

Transferability

External validity traditionally concerns generalisability. In qualitative research, however, this is more appropriately understood as transferability, that is, the extent to which findings may be relevant in other contexts (Cohen et al., 2018; Lincoln & Guba, 1986). The value of the findings does not lie in their direct generalisability, but in their potential to contribute to a cumulative development of knowledge, where results from different studies are related, compared, and refined over time (Collier-Reed et al., 2009; Kilpatrick, 1995). Rather than aiming for direct generalisation, the present thesis seeks to contribute to a broader understanding of vocational mathematics knowing by offering analytically grounded insights that may be relevant in similar contexts. According to Larsson (2009), the value of qualitative research may lie not in generalising findings directly, but in contributing a piece to a larger pattern of research. The significance of such a contribution depends on the relevance of the case in relation to a broader context.

In Study 3, this argument is particularly applicable. Although the analysis is based on a limited number of textbooks, such materials are widely used and play a significant role in shaping teaching practices. It can therefore be argued that even

a limited empirical sample may have wide relevance since the particular textbooks influence the teaching of a large group of students.

In Study 2, the findings should be understood as contributing to a broader research field. Previous research suggests that appropriate vocational mathematical knowledge varies across vocational domains (Straesser, 2015), and that further research therefore is needed in different vocational areas. In construction work contexts, a number of studies have been conducted over recent decades (e.g. Bessot, 2000; Boistrup et al., 2018; LaCroix, 2014; Masingila, 2009; Moreira & Pardal, 2012; Saló i Nevado & Pehkonen, 2018; Zevenbergen & Zevenbergen, 2009). While these studies largely point in similar directions, most of them are based on relatively small samples. In this sense, the present study may constitute one important contribution to an emerging body of knowledge. At the same time, the analysis is closely connected to specific mathematical and vocational contexts, and the aim is not primarily to generate general results across domains. Instead, the focus is on identifying aspects that are critical within particular contexts.

Transferability is further supported through purposive sampling, which aims to maximise variation in the empirical material. In Study 1 participants were selected to represent a range of perspectives, thereby enabling a broader representation of the phenomenon under investigation. Although an outcome space cannot be regarded as exhaustive, but rather as complete in relation to the group studied, phenomenographic research assumes that there is only a limited number of qualitatively different ways in which a phenomenon can be experienced. Consequently, saturation may be approached with a relatively small sample, provided that the sample is sufficiently varied (Marton & Booth, 1997).

Another basis for transferability lies in contextual similarity (Larsson, 2009). In the present study, it may be argued that the teachers and students included share characteristics with others in similar educational and vocational contexts, which may support the relevance of the findings beyond the immediate sample.

Ethical considerations

Ethical considerations in this thesis concern both formal research ethics and broader issues related to the responsible production and presentation of knowledge (Cohen et al., 2018; Kvale & Brinkmann, 2009). These considerations have been addressed in relation to both the interview studies and the textbook analyses.

With regard to the interview studies, participation was voluntary and based on informed consent, obtained through written information provided to

participants (Swedish Research Council, 2024; Swedish Code of Statutes 2003:460). Participants were informed about the purpose and design of the study and were given the opportunity to ask questions prior to participation. They were also informed of their right to withdraw from the study at any point before the analysis phase, and that the data would be used for research purposes only (see Kvale & Brinkmann, 2009). The participants in the study were vocational and mathematics teachers, who may be regarded as experts in their respective domains. This reduces concerns related to power asymmetry in the interview situation (Kvale & Brinkmann, 2009). At the same time, care was taken to conduct the interviews in a manner that supported open and reflective dialogue. To protect participants' anonymity, identifying information such as names of individuals, schools, and locations has been removed or replaced with codes. The risk of indirect identification is considered minimal. All data have been handled in accordance with the University of Gothenburg's guidelines for information classification and storage, and no sensitive personal data, as defined by the Personal Data Act (Swedish Code of Statutes 1998:204), were collected.

The textbook studies involve publicly available material, which raises a different set of ethical considerations. While there is a potential risk that individual textbooks or publishers could be perceived negatively, this has not been the intention of the study. The analysis is limited to a specific mathematical domain and focuses on variation in learning opportunities within this domain. It cannot, and does not aim to, make general claims about the overall quality of particular textbooks. Care has been taken to represent the material fairly and to ground interpretations in systematic analysis rather than evaluative judgement (Cohen et al., 2018).

Ethical considerations in qualitative research also concern how empirical material is interpreted and presented (Cohen et al., 2018; Kvale & Brinkmann, 2009). In this thesis, participants' contributions are treated as analytically meaningful expressions in relation to the phenomenon under investigation, rather than as objects of evaluation. Efforts have been made to present findings transparently, enabling readers to assess the interpretations while ensuring that neither participants nor materials are misrepresented.

7. Result Study 1

This study aims to provide a multifaceted description of vocational teachers' and mathematics teachers' perceptions of 'educationally significant vocational mathematics knowing'. The research question thus reads:

In what ways do teachers experience 'educationally significant vocational mathematics knowing'?

This is explored using a phenomenographic approach, which focuses on variation in perceptions of a phenomenon, in this case 'educationally significant vocational mathematics knowing'. In this way several facets of a phenomenon could be exposed, taking its' complexity, nuances, and contradictions, into account. The findings of this study are presented as a set of phenomenographic categories describing qualitatively different ways in which teachers experience this phenomenon. The categories are then further analysed in terms of their structural and referential aspects, as well as the relationships among them. Finally, the overall structure of the outcome space is discussed in relation to typical structures identified in phenomenographic research.

Categories of description

The categories of description capture variation in how the phenomenon is understood and are analytically distinct yet relationally connected. Rather than representing a linear progression or a normative hierarchy, the categories should be understood as complementary ways of experiencing the phenomenon, each highlighting different aspects of vocational mathematics knowing. Together, they form an outcome space that illustrates the relations and contrasts between these ways of understanding. In this chapter the characteristics of each category are described and illustrated using representative interview excerpts. Significant passages in the excerpts are highlighted in **bold**.

Table 7.1 The phenomenographic outcome space: Different ways of understanding 'Educationally significant vocational mathematics knowing'

Procedural mastery -for completing vocational tasks
Context-bound understanding -for handling practical tasks
Conceptual mathematical understanding - for solving various tasks
Transferable mathematical understanding – for coordinating mathematical meaning across educational and workplace contexts
Integrated vocational mathematics knowing -for navigating complex vocational challenges
Powerful vocational mathematics knowing - for developing vocational 'Bildung'

A. Procedural mastery

-for completing vocational tasks

In this category, memorised procedures, facts, and rules, applicable to perform specific tasks, are in the foreground. The category is characterised by a focus on isolated techniques, to be learned by heart, without necessarily requiring an understanding of the underlying mathematical rationale. When considering what students need to understand the emphasis is on remembering what to do, and how to do. For example, how to read a table, which values to subtract, or which numbers to use. This indicates that appropriate learning outcome, from this perspective, is seen as procedural know-how, involving methods, which are internalised to the point where they become second nature. Remarks such as 'it's something you have to imprint', 'it should just be kept in mind', and 'it should become second nature', typify this category. Moreover, frequent references to basic arithmetic skills, as essential and sufficient mathematical knowledge, further confirms that underlying mathematical concepts and reasoning, which are implicitly integrated in the vocational task, are not appreciated as appropriate vocational mathematics knowing.

In the interviews, this perspective is manifested through a focus on specific techniques and procedures:

V1: Well, **they should know the measurement**. There are fixed numbers: 60, 80, and 1m is the diagonal, then. And so, you get 90 degrees. And then you have 120, 160, 2 m, for a larger angle. We talked about that as recently as yesterday (...) This is very useful (...) [They need to know] how to calculate! A carpenter... They should **know this by heart**: 60, 80, 1m, and 120, 160, 2 m. **It's just supposed to be kept in mind.**

V1's remark about knowing fixed proportions of sides in a triangle, 60:80:100, or 120:160:200, to get a right angle, makes clear that the focus is on recalling specific, useful measurements by heart. Neither geometric principles (such as the Pythagorean theorem and the concept of similarity) nor proportional relationships are highlighted by the interviewee. Later in the same interview, the same perception is voiced in a slightly different way:

V1: It's no difficult math in construction-work, but it's important that you know it, plus minus multiplication and division, it's sufficient for a construction-worker actually.

The remark that 'it's no difficult math' and that 'plus, minus, multiplication and division' are sufficient for construction-workers, further emphasises that geometric principles are not discerned as significant vocational knowing. A similar focus is portrayed in the excerpts below, where roof angles are considered.

V1: This is how you find the slope of a roof: You take the height measurements in cm [at a distance of 1 m], and then the difference. And then you get, in this table [shows a table, relating slope in percent to the angle of inclination], the roof slop (...). **They ought to be able to learn this:** If you know that you should have a **metre in level**, like that, and a **height there and a height there** [pointing at a picture of a roof]. That's how you get the slope of the roof.

VM5: Now I have to think... 1:2, 1:3, 1:4, 1:5...**I use to go through that with them...** (...) **how many degrees it is (...)** **We don't do trigonometry and such,** we don't.

The interviewees description of using tables or memorising specific ratios, instead of applying trigonometric functions and understanding the concept of slope, indicates that memory rules foregrounded rather than conceptual understanding. This indicates that the mathematical tools used are perceived as isolated facts and procedures to be memorised and applied, rather than as part of a broader mathematical framework that elucidates the mathematical rationale underlying these procedures. In the following excerpt the same perception is voiced when discussing scale drawings:

M3: We can see that they can learn, maybe, **certain procedures and things like that** (...) they can learn that **if they see 6.5 cm on a drawing** and they know the scale. **Then they can remember to multiply** by 50 or multiply by 100, for instance.

The argument that students might be able to learn ‘certain procedures’ and the suggestion that they could ‘remember to multiply’ indicates a focus on isolated skills in order to perform specific tasks without integrating broader mathematical reasoning. It reflects an approach to learning, in which the ability to execute specific procedures takes precedence over understanding how and why these procedures work.

Also, the quotation below indicate this perception in the excerpt below, where mathematics teaching in vocational education is described.

M7: We might perhaps have worked more integrated with different parts of what they are working on, but at the same time, **it is about addition and subtraction** to a very great extent (...) It’s about the **speed of solving tasks**. And really **doing it neatly!** (...) Improving penmanship and working on fine motor skills (...) I say: ‘Mathematics, and being good at mathematics, is also about **being quite fast at calculating**. ‘This is good to remember when you apply for a job later. If they see that you can calculate quickly (...) Wow! We should keep that guy!’ (...) I try to tell them, ‘Keep on calculating now, and you’ll speed up so that you can do more tasks, and the more tasks you can do, the better you’ll get at math. (...) **It’s really about how they’ve been drilled**. The multiplication table, for example. If you know the multiplication table really well, everything becomes so much easier.

This quotation focuses on speed, accuracy, and memorisation. M7’s emphasis on addition and subtraction and ‘the speed of solving tasks’ makes clear that being good at mathematics is associated with rapid routinised calculation and neatness, rather than understanding deeper mathematical concepts. The idea that ‘it’s really about how the have been drilled’ further reinforces the notion that vocational mathematics is perceived to involve repetitive practice and basic skills. The remark about impressing employers by calculating quickly confirms that appropriate vocational mathematical competence is experienced as an ability to perform practical tasks swiftly and accurately through procedural fluency. In this category, appropriate vocational mathematics knowing, is thus perceived more as a set of isolated, practical tools, aiming to solve certain vocational tasks, than as a coherent knowing.

*B. Context-bound understanding
-for solving practical tasks*

In this category ‘educationally significant vocational mathematics knowing’ is perceived as context bound understanding; a vocational competence, to be learned ‘by hand’, and used in specific vocational situations. The vocational task is in the

foreground, viewed as a unified process where the mathematical aspects are seamlessly integrated into the vocational context and not perceived as separate or abstracted from the context. The category is characterised by reference to examples in which the mathematical reasoning, and the understanding of concepts, are based on direct engagement with the task at hand. This means that, the focus is on solving practical problems, with little emphasis on mathematical principles outside of the immediate context. This indicates that the phenomenon is related to vocational practice rather than to vocational education. In contrast to Category A, however, the phenomenon is not referred to merely as remembering how to do but rather as understanding in practice, by ‘understanding it with the hands’. Reference to practice-based knowing, developed through the use of concrete tools and materials, typifies the category. Remarks like ‘when they work with it practically, then they understand’, and ‘Construction-work mathematics is about it needing to be logical (...) to see the construction in front of you’ further emphasises a perception of the phenomenon as hands-on understanding of vocational mathematics concepts.

This perception is for instance voiced in the excerpt below, in which a way of explaining the concept of slope, closely embedded in workplace practice, is suggested:

V3: They learn quite easily to calculate the metre. So, if the **pipe is 6 m long**, how much should it differ **from that end to this end?** So, then they **divide it into 1 m parts**. That’s what they learn most easily out there. (...) They usually work with 6-m long pipes, and then 1m parts. We have made it like that, so that it should become pretty easy for them (...). **We use our lasers as a point of departure**, and we work with the height difference. We **explain all the way from the laser**, and from there on we continue. So, when they have learned the laser, which they do fairly easy, then the other things follow as well.

The recommendation to use specific pipes of certain dimensions, together with the laser instrument, as a point of departure, when explaining how to handle slope calculations, exemplifies a focus on mathematical understanding, anchored in concrete, vocation-specific resources. This approach to teaching reflects an understanding of the phenomenon as embedded in practical, context-specific engagement, rather than in an abstract, transferable mathematical framework. As V3 further elaborates on the concept of slope, applied in a workplace setting, the lack of generalisation of the concepts across context is even more evident:

V3: We talk about it [slope] vocationally (...) Percent and per mille, that's what we always use when it comes to pipes. (...) And this [i.e. denoting slope 1:X] we only do when it comes to machine driving, when they're going to make a slope or something, using a digging bucket to make a ditch edge or something like that (...) Then we use 3 :1, and so on, in that situation (...) **but we never calculate them together**

The remark that slope is conceptualised differently when digging ditches compared to determining a drainage gradient, and that relating these contexts is irrelevant, highlights a focus on context-bound understanding, and underscores a lack of emphasis on generalised mathematical principles. In this way, the excerpt exemplifies the category's defining characteristic: vocational mathematics knowing, intrinsically tied to specific vocational contexts, rather than as part of a unified mathematical framework.

This perception is conveyed even more explicitly in M2's and M4's response, further reinforcing the category's orientation toward practical, context-specific reasoning:

M4: They usually have their methods of solving, which may not be very impressive mathematically, but they work. (...) They have no interest in intellectualising mathematics (...) **In the construction hall, they use simple quick methods, which I admittedly use myself when I calculate practically** (...) I can't see that the students would think: Well, this is indeed about similarity they would take a **comparable, more practical approach. And from that practical application, to take the step to the general, that is extremely difficult.** (...) They would do it practically (...) they see the yard, they measure (...) Then they get a correct measurement for volume. And then the next step is to check the weight of gravel. And so, they have that mass. **It's... very, very, practical.**

M2: Many of them say: 'When I'm in the workplace, then it's no problem... Then you just go for it', but **here [in the mathematics classroom] they don't really have anything to grasp on**, and even if I use a task that is about...well: 'You're in a construction site, you're doing this'. Well, it's still too theoretical, so that even if I try, to connect to the vocation, it is still considered too abstract and far away. When they're out **there [in the workplace] they solve it like...well, without thinking... or rather, because they're using their hands.**

The methods exemplified are referred to as non-impressive from a theoretical standpoint and yet regarded functional and powerful within the specific context. M2 and M4 thus articulate and acknowledge a context-bound way of reasoning, in contrast to 'intellectualising mathematics', even if they also recognise its limitations.

Also, when students' vocational mathematics learning is discussed in more general terms, practical learning, 'by hand', through logical reasoning in authentic contexts, is advocated. Once again, this supports the interpretation that educationally significant vocational mathematics knowing is perceived as a vocational skill, to be learned and used in a vocational context.

V7: Well, **they are hands-on people**. Many of those who are going to become carpenters are hands-on; **they see things visually in front of them**. Not everyone is a mathematician, who can just easily translate numbers in their head, like that. Instead, I think **construction-work mathematics is much more about being logical**. Yes, you see it in front of you. You need to **see the construction in front of you** (...) because then you also learn to see this in reality and connect it with numbers. It's probably obvious that I'm a hands-on person. **I'm not a mathematician** (...) I see it in a different way. (...) **I have to think logically** (...) I do have a few math experts in the classroom (...) They calculate quickly, like that (...) **I want to see it in front of me. I have to create an image in my head to be able to translate it into numbers.**

V3: They want to use a shovel, a hammer, a saw. They want to work with their hands. They do not want to do math. They do not want to do anything like that (...) This is their maths: **A basic simple math**, to cope with everyday life. (...) When they come to the workshop, to us, vocational teachers, and do maths there (...) **they can do it with their hands**. They get a **completely different understanding** of it, than when they are sitting with a book, number and pencil (...) Because there [in the classroom] they cannot visualise it. (...) They just: 'Ahh! No! Ough! What do you mean? What is this?' (...) But, **if they work with it practically**, it won't take long until they learn (...) **When they do it themselves, then they understand**. (...) So, you need to work with it there and then. Like: Now **we're working with this and explain it** (...) Then they do not see it as mathematics. Instead, they see it as part of their vocational competence, their vocational role, of what they will work with in the future (...) **Then it is not math. Then it is a vocation!**

These quotations exemplify, in an even more pronounced manner, the characteristics of this category. The emphasis is on a form of learning and knowing that qualitatively and essentially diverges from purely theoretical knowledge. Not necessarily by being less advanced, but rather by being different in nature. A logical, hands on, visual kind of understanding, integrated in practical experience, as illustrated by utterances like: when 'they can do it with their hands, they get a completely different understanding' and 'if they work with it practically (...) then they understand'. Here understanding is emphasised rather than just remembering, and understanding through hands-on, logical and visual, experience is contrasted to merely theoretical arguments, and explanations. In this way the quotation

highlights a perception that stands in contrast to both Category A and Category C, as Category A focuses on rote learning and Category C emphasises a more generalised, abstract mathematical framework.

*C. Conceptual mathematical understanding
- for solving various tasks*

In this category, theoretical understanding of vocationally relevant mathematical concepts is in the foreground. The category is characterised by a focus on mathematical understanding and conceptual connections, emphasising foundational principles and generalisability over rote learning and context-bound strategies. Reference is typically made to mathematical principles which are separated from and abstracted beyond a specific vocational context. Here the phenomenon is perceived as conceptual and universal mathematical understanding, applicable across diverse contexts, rather than as context-specific tools for specific vocational use. The focus on conceptual understanding and generalisation, contrasts with categories that focus on context-bound, hands-on understanding applied in specific situations, as well as with categories that emphasises rote learning and memorisation. However, in contrast to both Category B, and the coming categories, the relation to, and understanding of, the vocational context, is less focused in this category of description, and the ability to apply geometric concepts in a vocational context, is unproblematised, and taken for granted. In the interviews this perception is conveyed through the emphasis on the importance to ‘understand the essence’ and ‘explain the connections’ in contrast to just ‘learn things routinely’. The phenomenon is referred to as understanding ‘why’ rather than just ‘how to do’, and as ‘truly understanding concepts theoretically’ rather than ‘understanding in a context’. Basic arithmetic skills are, in contrast to previous categories, referred to as non-sufficient knowledge, which shows that the mathematical concepts used, are perceived as more than mere memory rules. When a concept is discussed, different situations in which the same concept is used, are exemplified, and essential aspects of concepts, are highlighted, which indicates an attention to mathematical concepts that are generalised and separated from the context. Moreover, remarks such as ‘if you really understand it, then you can apply it to different problems’ indicate that the ability to apply theoretical concepts in diverse contexts, is taken for granted. A strong example of the category is provided in the interview with M6:

M6: The students aren’t really into calculating things; using math to figure things out. **They have a lot of tricks**, which they’ve learned in the construction

hall (...) So they do solve these kinds of problems (...) but **not in a mathematical way, not by using the concept of similarity and proportion.** (...) **Not in the way that I, as a math teacher, would prefer.** (...) A floor in a bathroom should slope so that water drains off... it's sort of obvious for them... but (...) It doesn't automatically follow that they understand the concept of ratio and can apply it to other tasks... it's like **they learn it in a contextual way**, in a sense. (...) I mean... if you understand the concept of ratio... **if you really understand it, then you can apply it to different problems.** (...) If they understand the concept, they can solve many different problems that they'll encounter in their vocation later. (...) **To become a problem-solver, you need to understand the concept more theoretically.** To solve different kinds of problems (...), not just understand the concepts in one context, you need to **move towards something a bit more theoretical.**

This quotation explicitly voices a perception of the phenomenon as generalised theoretical understanding, fostering an ability to solve a variety of problems: 'to become a problem-solver, you need to understand the concept more theoretically'. The excerpt also indicates that transfer is taken for granted: 'if you really understand it, then you can apply it to different problems'. The quotation further reinforces this perception by contrasting his own perspective with both memorised skills and context-bound, through remarks, such as 'they have a lot of tricks' and 'they learn it in a contextual way', 'not in the way that I as a mathematics teacher would prefer'. In the excerpts below the same perception is voiced more implicitly. The emphasis on the importance to understand what the concept of scale means, shows that mathematical concepts are in the foreground.

V2: It is to understand what that [scale] means, because they don't know that (...) That they learn the concept of scale what it actually entails.

Below, when V2 considers Task 4, the similarity concept is in the foreground, rather than context-bound ways of reasoning or memorised procedures, as in Category A and B:

V2: 'The porch roof should have the same slope'... Yes, that's right... 'Calculate the height of the porch roof?' Well, that involves calculating the angles, and **it's related to the proportions.** I would probably have **started with the similarity**, to make it simple, and then worked out the proportions for the height,... or the legs of the triangle. I would have taken that approach. **If I were to explain it to them, I would use the proportions**

[Coming back to task4 later in the interview, when discussing vocational tasks:]

V2: (...) it would be more like this problem (Task 4). Well, you're going to build an extension somewhere, and then you need to have the same roof slope as the existing one. But then you don't actually know the slope of the existing roof either, so **you have to calculate the proportions. (...) it is actually about proportions, what this really means.**

Here the focus is on the geometric concept of similarity and on proportions. The remarks: 'I would probably have started with the similarity, to make it simple, and then worked out the proportions for the height' and 'it is actually about proportions, what this really means' shows a reliance on conceptual understanding to frame and solve the problem, rather than just applying procedures without understanding their meaning. V2 further highlights the importance of explaining why certain methods work: 'If I were to explain it to them, I would use the proportions', which aligns with the category's focus on deeper conceptual understanding rather than routine procedures.

Also, in the following excerpt this perception is expressed through a focus on a specific relevant concept, to be understood as a whole:

V4: [the task concerning scale] is perhaps the simplest, but it's, nonetheless, the one that's most fundamental. We do a lot of drawing reading with the students. If you don't understand scale, then you can't build the house either. (...) It's about being able **to really master maths at some kind of fundamental level (...)** [they need] **to understand it [the scale concept] as a whole.** Because usually that's probably what's most difficult, about maths.

This quotation voices a perception that prioritises fundamental understanding of mathematical concepts. The reference to the importance of being able to 'understand [the scale concept] as a whole' highlights the need for abstract understanding, where concepts like scale are not just applied in specific ways in isolated situations. In the following excerpt, the contrast between understanding concepts and just memorising procedures, is explicitly elaborated, and deep and surface approaches to learning are discussed. The importance of a theoretical understanding of concepts is emphasised and exemplified by highlighting certain aspects of the scale concept.

M2: I see that many of them may figure out how to do, but may still have difficulties to really understand what is happening. If you see the difference. So, I think that many of them would be able to solve the task, but there are probably many students who struggle to really, **deeply understand the essence of this relationship.** (...) Well, you can learn all kinds of knowledge very routinely by heart, and then you can learn things... well, **learn it deeply.** And that's really what we want. Well, also the surface learning because they also

need to be able to routinely solve tasks. But at the same time, we also prefer deep approaches to learning.

It's about **working with it from both directions**. To work with enlargements as well. (...) we're stretching or shrinking...we're not changing the shape. It's a bit like zooming in on the screen, and zooming out. Everything is the same (...) And also being able to go in the other direction, between image and reality.

This quotation illustrates a perception in which appropriate mathematical understanding is seen as something deeper than merely knowing 'how to do'. Here M2 explicitly emphasises the difference between 'figuring out how to do' and 'understanding what is happening' highlighting a preference for conceptual depth over rote learning. The contrast between 'surface learning' and 'deep approaches' explicitly reinforces the notion that vocational mathematics is not solely about completing tasks but also about understanding the underlying principles. Furthermore, the explanation of working with 'both directions' of scale, 'stretching or shrinking' without altering shape, is used as an example to encourage students to understand the concept of scale rather than only using it practically. In this way, the quotation exemplifies a preference for cultivating a general understanding of mathematical concepts that extends beyond vocational applications.

D Transferable mathematical understanding

- for coordinating mathematical meaning across educational and workplace contexts

In this category, appropriate vocational mathematics knowing is perceived as the ability to connect school mathematics with the mathematics used in workplace practice and recognise equivalent mathematical structures across contexts. Central to this perception is the focus on, and problematisation of transfer, i.e. the ability to apply of previously learned mathematical concepts in a vocational context. The awareness of the link between mathematics learned in the classroom and mathematics used in practice, is emphasised as an essential and non-trivial part of vocational mathematics understanding. Teachers, who voice this perception, frequently underscore students' difficulties to understand that school mathematics and mathematics used in workplace practice is 'the same mathematics', and highlight the 'Aha'-moment, when students suddenly understand how previously learned mathematics might be applied in a vocational context. The focus on these moments as instances in which new meaning is constituted, indicates that connecting school mathematics and workplace practice is perceived as an essential and non-trivial part of vocational mathematics knowing, fostering both an

enhanced understanding of mathematics and more powerful ways of handling vocational tasks. This way of understanding is further indicated through references to the need for guided transfer, through which teachers support students in developing new ways of experiencing the connection between school mathematics and workplace practice. For example, when asked what students need to understand regarding vocational mathematics, V4 highlights the importance of helping students to recognise the connection between the mathematics they learned through mathematics textbooks and the mathematics used in vocational contexts.

V4: You have to help them to **see that connection**. And then sometimes, suddenly: ‘Oh well! That’s right! It’s, kind of, **the same thing as in the maths textbook**, even though we’re now in the construction hall, on the floor!’ (...) It is all about making that connection ... Because regardless of whether they’re good at math or not, or if they think it’s fun or boring, they’ve been struggling with maths for many years (...) and when they get this: ‘Aha!’, as I said, the connection to the maths textbook, like: ‘Oh well! Okay! This is how we can use it! Now! Oh! Now I will calculate!...and that they, **in this way, really understand what they are doing**.

The excerpt highlight moments of realisation in which students suddenly come to experience that the mathematics they work with in a workplace setting is the same as the mathematics encountered in the textbook. By pointing to these ‘Aha!’ moments, an awareness of the learning potential of making connections between school mathematics and workplace practice explicit is manifested. This formulation typifies the category’s emphasis on recognising the relationship between school mathematics and workplace mathematics as central to developing vocational mathematical understanding. Likewise, VM5 refer to students’ sudden recognition that theoretical concepts have practical implications:

VM5: We use percentages, for example concerning humidity in concrete. And then suddenly a student was saying: *‘What the hell, it’s the same percentage as in maths!’* So, it is such a difference. It’s something that you do in math over there. But this is for real.

[through a cooperative teaching approach] we got rid of this: math over there and the math here. And they saw that I and this math teacher, we cooperate. We’re talking. I understood her, and she understood me (...) **that it is the same mathematics, it is actually not anything new**.

The excerpt points to students sudden understanding that ‘it’s the same percentage’, bridging the gap between ‘math over there’ and ‘math here’. The

quotation thus characterises the category as it highlights a breakthrough moment in which students realise that the mathematics they learn in mathematics lessons directly applies to vocational tasks. Also the quotations below exemplify the category's focus on the need to connect mathematics learned in the classroom with mathematics in workplace contexts:

V7: (...) you connect it [mathematics] with the practical stuff, (...) so that they can **connect their math lessons with the lessons here**. That would be good, because right now it's like construction-work [here] and math [there]. (...) Maybe **the math teacher can go a bit deeper** while we (...) [can] actually **do these examples outside**, so that (...) the students can see how math relates to what we're doing out in the workshop. Right **now, it's like two different worlds**.

M8: [Describing collaboration with a colleague] He'd been a construction teacher for a while, but also worked in the trade. (...). And when he did the calculations, he could bring in the methods he'd learned at work. And then they could see that **it lined up with the methods I teach** [in mathematics]. They mostly went hand in hand, even if we explained them in different ways (...) so they could see **it was actually the same thing**. Yeah, **it was exactly the same, just done in different ways**.

These excerpts underline the importance of bridging the gap between 'math lessons' and the practical tasks in the workplace setting, which typifies the category's focus on developing the ability to apply mathematical concepts learned in one context (classroom) to another (workplace). V7 points to the disconnect that currently exists between school mathematics and the hands-on, practical work being done in the workshop. The remark 'it's like construction-work [here] and math [there]' underscores the idea that, without a clarification of the connection, the two domains seem separate. M8 reinforces this idea by emphasising the importance of seeing that the methods taught in class 'line up' with those used in the workplace, even if the methods are explained differently. This overlap allows students to recognise that despite different approaches or explanations, the core concepts align. The phrase 'it was exactly the same, just done in different ways' underscores the need to make the connections between mathematics in the classroom, and mathematics in workplace settings, visible to students and exemplifies a learning outcome which entails a realisation that theoretical and practical methods are essentially similar just presented differently. V9 further pinpoints that, while the theoretical part is crucial to understand in its own right it is only through seeing and experiencing the application of mathematics on the construction site that the students fully understand the concept.

V9: (...) You **start with the theoretical part**. (...) During the first lesson, no one understands. But then, when you **go outside and do the practical part, talk about it the same way**, and then come back in and **repeat it on the board**, that's when they usually get it.

This approach reflects the essence of the category, where connecting school mathematics and mathematics in workplace practice is seen as a crucial step in improving students' vocational mathematics knowing.

*D. Integrated vocational mathematics understanding
-for navigating complex vocational challenges*

In this category, educationally significant vocational mathematics knowing is perceived as a dynamic and integrative process, where mathematical concepts, vocational understanding, and workplace experiences are interactively combined to address complex vocational challenges. While Category D emphasise the connection of practices this category focuses on connections of domains. This perception emerges in the interviews both through a clearly articulated understanding of how these various elements, i.e. vocational knowing, practical experience, and mathematical understanding, interact and support each other, and through concrete examples in which this type of knowing is manifested. The intricate nature of vocational mathematics knowing is underscored by highlighting the dynamic relationship between theory and practice, both when discussing students learning and when giving examples of complex vocational challenges. The focus on these various aspects, and their interconnections, indicate that vocational mathematics knowing is perceived as a synergetic fusion of mathematical and vocational theory and practice; that is, as a competence which involves a productive integration of various aspects, where the combined effect is more advanced and powerful than the sum of the individual parts.

This category thus portrays a form of knowing which, alongside a deep familiarity with mathematical concepts and vocational principles, also entails an keen awareness of how to apply these ideas adaptively in diverse vocational situations. In the interviews this perception is typically voiced by highlighting the vocational context as adding complexity to the task, making it far more challenging than simply performing straightforward calculations. For instance, when comparing authentic vocational mathematics tasks to those typically encountered in mathematics lessons, it is noted that 'this is not that simple', 'reality is more complex', and 'there are so many other factors to consider.' In other examples, the emphasis shifts to the importance of truly understanding mathematics, while the

exemplified tasks remain complex vocational challenges drawn from real-world contexts. Phrases like ‘then they have to understand mathematics first’ and ‘They had such an advantage! They really understood the math’ underscore the necessity of a deep understanding of mathematics. The excerpts below, drawn from the same interview, together illustrate this perception.

V9: They might understand something in math classroom, but then when they have to do the same thing in practice, everything feels new to them (...) I think it’s **the context that might be the most difficult part**. It becomes **another step (...)** it adds another dimension, like.

You need to have understood the math first (...) and then you have to connect that math with the drawings (...). **It’s actually kind of interesting to see how much there is that affects**. A result, an understanding, an understanding, a way of thinking. (...) **It’s really about problem-solving**, like in all practical professions (...) Here, you need to create a product to a person who’s ordered something and expects it to turn out just like they imagined

The quotation reflects a perception of vocational mathematics knowing as transcending the mastery of mathematical procedures, requiring an integrated understanding of mathematical concepts and the vocational context. The importance of mathematical understanding, and a problem-solving ability, is emphasised. At the same time aspects of the vocational context are recognised as critical components of vocational mathematics knowing. The remark that ‘it’s the context that might be the most difficult part’ since it ‘adds another dimension’ underscores this idea. V9 continues by underlining the importance of a theoretical depth together with an integrative understanding of theory and practice:

V9: [During mathematics lessons] they probably go more in-depth (...) and **we focus more on the matter at hand**. And, of course, **the depth is really important for them to understand right from the start**. Because, we, vocational teachers, we do not really have that education in math (...) [In the construction site] we have the **theoretical and practical parts at the same time**. (...) There, you can really see concretely what happens, you see things. **Even if you understand it in theory, you don’t get the full picture of what happens**. Here, you actually see what’s happening. (...)

[In the workplace] you might not get the chance to **try different methods or think in different ways**. When you get to the construction site, you’ll adopt that guy’s [a more experienced colleague], or woman’s, way of thinking (...) I think it’s **extremely important that math is included from the beginning [otherwise] you don’t really get the mathematical mindset, and [understand] that [the same idea] might apply in several places**. (...) Maybe you solve a certain task (...) and you may have used a similar approach [to solve another

task], but you don't make the connection (...) and then you have to solve it all over again.

V9's utterance that 'the depth is really important for them to understand' and that 'even if you understand it in theory, you don't get the full picture' together with the observation that vocational situations include 'theoretical and practical parts at the same time', indicate a simultaneous awareness of a complex of several aspects: mathematical and vocational; theoretical and practical. Moreover, the value of being able to solve the same problem in multiple ways, as well as the ability to recognise the same mathematical idea in different tasks, is emphasised. This further indicates an awareness of the importance of conceptual mathematical understanding that extends beyond simple, predetermined procedures and context-bound task solutions. The emphasis of all of these aspects, and their interconnections, indicates a perception of vocational mathematics knowing as integrated vocational mathematics thinking with the aim of solving complex vocational challenges. The same perception is voiced by M9, when comparing mathematical textbook tasks to real-world vocational mathematics problems:

M9: In the beginning (...) I had a lot of these ambitions, like, I wanted to make math necessary for them (...) I was quite ambitious about creating, like, reality-based problems. (...) **It becomes quite complex when you start from reality (...) because reality is a bit more complex.** I mean, often when you calculate things [in the classroom], you simplify everything.

We'd made a task (...) where you had to **consider all the different parts** (...) actually a real problem which they might have to face. However (...) this wouldn't show up on a [national] course test. (...) so they think 'Why are we learning this' while **I think 'This is so useful for your future!'** (...) These [vocational mathematics tasks] **requires problem-solving.** If you compare it to the tasks in the book, which are often just identical tasks but with different numbers, **these are more about thinking.** (...) **When you merge several parts together. Which is actually what they need to be able to do.**

M9's description of the challenge of creating reality-based problems, pointing at the complexity that arises when moving from theoretical mathematics to authentic vocational tasks, indicates a perception of vocational mathematics knowing as a domain beyond common school-mathematics. The remark that vocational tasks in contrast to structured textbook exercises, 'requires problem solving', and 'are more about thinking' reinforces this idea. Additionally, the mention of the students' difficulty in understanding the broader, interconnected nature of the task 'when you merge several parts together' reflects a focus on the necessity to

understand how various aspects of the task, both vocational and mathematical, must be integrated to form a coherent solution. Also, V4 convey this view by giving examples of typical vocational challenges:

V4: There is so many other things to take into account [beyond calculating the area]. **There are other, as I said, vocational skills, kind of, or vocational experiences**, in some way. Things you cannot learn by reading.

V4's remark that area calculations constitute just a small part of the solution to the task, and that several essential vocational skills and experiences, must be taken into account, typifies the categories' focus on the essential fusion of vocational and mathematical aspects. Since V4 particularly emphasises this integration when reflecting on the characteristics of vocational mathematics knowing, the excerpt underscores the fusion of vocational and mathematical knowing as an essential aspect of vocational mathematic knowing. This fusion is a key feature of the category. In the excerpt below, an example is given, in which, it is the level of required mathematical understanding, that is focused. The example as such, is similar to the one above, yet it is the importance of understanding mathematical principles, that is explicitly highlighted.

(...) Excel and such, I've started using that. (...) It's for making material calculations and work time estimates and such. **But they must understand the math first** in order to create their documents (...) How much material is actually needed? (...) What does each metre of timber cost, and how much should I add on? How long does it take, and what does an hour cost? And then some VAT at the end and... **because this is the kind of thing they'll actually be doing in practice.** (...) You write a formula into each cell where you want an answer. (...) And then you input... you have an area (...) often, when working with drawings in the construction industry, you use square metres. Square metres of walls, floors, ceilings, and then you convert that to running metres of timber or the number of sheets or whatever it is you're buying. So, in such an Excel program, you input calculations to figure out how much material is needed (...) and then add prices and sum it all up (...). [they need to understand math] to be able to create their documents. (...) I mean, **otherwise they can't create the formulas in the Excel program.** (...) **So if they don't understand maths!** (...) **To create such an Excel program, they actually have to understand how it is all connected, at a more intellectual level** (...) Because otherwise they just know that, *'if I do like this, it will be like this'*. **Here they really have to think a step further and actually understand why it is like that.**

In the excerpt above, vocational mathematics knowing is perceived as requiring the integration of advanced mathematical understanding with vocational knowing.

The teacher provides a comprehensive account of the vocational considerations underlying the task solution. A contrast is made between knowing how to do and actually understanding why, emphasising that students must ‘think a step further and actually understand why it is like that’. The excerpt as a whole indicates an emphasis on a robust understanding of mathematical concepts together with an extensive understanding of vocational conditions. This clearly indicates a perception of the phenomenon as an integrative form of vocational mathematics knowing, in which conceptual mathematical understanding and vocational considerations are inseparably intertwined in order to address complex and authentic workplace challenges.

A similar integration of mathematical and vocational aspects is also highlighted by V3 exemplifying how students’ understanding develops through the interplay between practical action and mathematical reasoning, while constructing a right angle:

V3: they often struggle when they have to make a square or a rectangle (...) They know the side lengths, but calculating the diagonal, that’s where they have difficulties. (...) Often it ends up as a parallelogram when they set it out. So then I say, ‘Now you need to correct it!’ But then they ask, ‘Which side should we move?’ (...). ‘Well, try moving one side and see what happens.’ (...) And then we also show calculation examples: ‘If you calculate this, you’ll get this diagonal measurement.’ There’s something about that **relationship, when all the parts fit together**. But that part is quite difficult for them to understand. However, once they’ve done it **practically a few times**, and **have moved things around themselves until the measurements are correct**, then they begin to understand. **They start to see the whole, and then they begin to understand how to calculate it.**

This excerpt illustrates how understanding emerges through the coordination of multiple aspects: physical manipulation, visual judgment, and mathematical relationships. The shift from trial-and-error adjustments to recognising ‘how all the parts fit together’ reflects a movement toward an integrated way of thinking, in which mathematical reasoning and vocational action are no longer treated as separate domains but as mutually informing components of problem solving. In this way, the example further reinforces the characterisation of vocational mathematics knowing as integrated vocational mathematics knowing.

*F. Powerful vocational mathematics knowing
- for developing vocational 'Bildung'*

In the final category, 'educationally significant vocational knowing' is perceived as a part of an individual's holistic vocational '*Bildung*' (see Klafki, 2000), involving, not only factual knowledge and vocational skills, but also critical thinking, confidence, creativity, adaptability, and development. This way of understanding is voiced in the interviews by referring to an increased level, or depth, of mathematical knowing, as 'another way of seeing' and as a 'a capacity for action'. In this category, the level of mathematical understanding and its impact on how vocational mathematics tasks might be understood and handled, is in the foreground. However, mathematical knowing is, according to this perception, not just a tool for solving specific vocational tasks, but rather a means to develop a more insightful and adaptive perspective on vocational challenges; a deeper level of mathematical knowing, which allows students to handle a variety of vocational challenges in innovative ways. In contrast to the focus of Category C, where the emphasis is on the mastery of mathematical concepts, this category emphasises the interconnection between vocational mathematics and broader educational views. Here, mathematics is seen as a gateway to enhancing vocational competence, as it provides the tools to understand and address diverse vocational challenges with greater depth and creativity.

VM5: You get another eye (...) It is a matter of confidence and security as well. *I have made the calculations, and it will be all right.* It's like a certificate, **it is about knowledge.** (...) There are different possible ways to go, but the more mathematics you understand... well, **you can see things in a different way.** (...). **Basically, that is what knowledge really is about.**

The remark that an increased mathematical understanding promotes another way of seeing, that you 'get another eye' and a 'stronger confidence' within the vocational role, suggests a perception of vocational mathematics knowing that emphasises mathematical knowing as a means to improve the ability to engage in vocational activities, by developing the eye through which the world is seen. In the next excerpt, a similar perception is conveyed. A deeper level of mathematical understanding is emphasised as an opportunity to understand and handle a variety of vocational tasks in adaptive ways through a more advanced way of seeing the vocational situation.

M2: As a mathematics teacher, I would find it valuable to understand the mathematics behind...As I see it, that would improve their problem solving

ability. If they encounter different types of problems in the workplace (...) **I believe it can help them to think more broadly**, when everything is not straight (...) We want this deep learning, just to enable them to use it for problem solving in reality. **If you really understand what you're doing, it's easier to adapt to various situations.**

This quotation refers to a 'problem-solving ability' and to 'really understand what you are doing' as prerequisites for being able to 'think more broadly when everything is not straight'. This reflects the perception that mathematical knowing is not just a tool for specific tasks but a means to develop broader competencies like flexibility, and creativity.

M8: (...) **The ability to see a problem and figure out how to solve it.** That skill stays with you'. (...) I really think it makes a big difference, when you get out into the working life. (...) It gives you a **capacity for action** (...) **You become more independent.** (...) As a result of this kind of problem-solving mindset, I absolutely think they become more willing to take on different kinds of roles. Because **then they have a different way of thinking. It's not just a one-track approach, like 'this is how it's done'**, but there's an understanding that there are **multiple ways into a problem**, not just one. Maybe not everyone will get that, and I don't expect them to. (...) But I still believe in the importance of being able to solve problems in various situations—it implies...well **they get more tools** to handle a task.

It is pinpointed the ability to see and solve a problem as an essential vocational skill, which implies a 'capacity for action' and a state of independence. The emphasis on the effect of problem-solving skills in terms of an ability to approach tasks with a flexible, creative, and open mindset indicates a perception of vocational mathematics knowing as a part of a broader educational 'Bildung', where mathematical competence not only enhances vocational proficiency but also fosters the development of critical thinking and adaptability. Also, M9 convey this perception as she describes a skilled carpenter's vocational mathematics knowing.

M9: It's probably not very common to ask: 'Are these of similar shapes?' (...) You just get the blueprint with all the measurements. Well, maybe some carpenters...like, we had one when we were renovating our house (...) We got a really good impression of that carpenter who built our house. Like, **he was attentive, he wasn't just blindly following what he was given, but actually thinking, 'Will this turn out right?'** And, obviously, in my opinion, that makes him a better carpenter, because he thinks things through. **You may not necessarily be a better carpenter just because you're good at math, but it probably gives you better conditions to... well, to be able to get better.**

This quotation highlights the value of critical thinking and questioning within a vocational context and reflects an idea that vocational mathematics knowing enables individuals to engage more deeply with their work and make more informed decisions. The carpenter's approach is seen as an example of independence and problem-solving, qualities that M9 emphasises as making him a better carpenter. While being good at math alone may not necessarily make someone a better carpenter, M9 suggests that mathematical thinking, provides better conditions for developing the skills needed to excel in the profession. This aligns with the idea that vocational mathematics knowing contributes to personal growth and professional competence, helping individuals to not only perform tasks but also to improve and adapt to challenges in their work, a perception that characterises this category. A relation between creativity and mathematical development is also exemplified in the quotation below.

V1: Those who are interested pick it up right away, the drawings, the scales, and how to calculate things. We definitely see a difference. (...) If you (...) *want* to build and create something, you approach the math in a completely different way as well.

Here, a curious and creative mindset is considered to affect the way in which mathematics knowing is valued and approached. The excerpt below, is an example of the same perception, but expresses it in a slightly different manner. The excerpt below highlights adaptable teaching for individual growth, and the role of mathematics in this context.

VM5: Maybe you won't solve this problem in this [particular] way, but you'll figure it out. You'll do it differently or, ask someone, that's fine too, you know. (...) This is really what it's about: maintaining their trust in our education system. **It's not about sorting people, but helping them grow, just in different ways.** (...) It [Math] is what... for the students I've had... when they get it, they **grow the fastest.** (...) **When they finally get it. It creates a huge hunger.** Suddenly, they've mastered something that others already knew (...) Some of them really wake up, you know, and want to try Ma2 as well, and that's where the real math begins. **Everything is new and just plain exciting.**

Here, a vision of a desirable educational system, is presented, one that aims to foster personal growth 'It's not about sorting people, but helping them grow, just in different ways.' In this development mathematics is seen as a significant component. A component that serves as an essential part of a holistic 'Bildung',

further developed through the motivation driven by the sense of personal growth. Here personal growth is seen as both an aim, and as a means for fulfilling the aim.

A final example of this category is presented below. In this excerpt, the interviewee describes the vocational mathematics knowing demonstrated by two of his former students. The students' ability to independently apply mathematical understanding in a demanding vocational situation is highly appreciated by the teacher, indicating a perception of vocational mathematics knowing as integrated, internalised, and enabling confident action in practice

V6: I had these two students, just sitting there calculating everything, and honestly, I had no clue how they were doing it. I figure they must've done such things in math class back then. It is not like that nowadays. (...) We were up in the roof trusses [and were supposed to find the angle formed at the valley between two intersecting roofs]. And, I was like, 'How the hell am I supposed to solve this?'. So, I started sketching on a plank and jumped down to see it. The students were just staring at me, like, 'Why are you doing that?'. **Then they started measuring. By the time I got back up, they already had all the measurements ready. It was insane!** Made me feel a bit... you know, I'm supposed to be the teacher here, right? But yeah, it was kinda funny. (...) **They had such an advantage!** (...) **They had absorbed their math knowledge, like... they really understood the math.**

In this excerpt, vocational mathematics knowing is perceived as a resource for independent, efficient, action, a way of seeing the vocational situation that allows for adaptable solutions. The students' ability to interpret the situation in mathematical terms, select appropriate strategies, and act confidently without explicit instruction highlights an understanding of mathematics as enabling autonomous and effective engagement with complex vocational tasks. The teacher's reaction underscores how this form of knowing is perceived as powerful vocational mathematics knowing.

Structural and referential aspects

Perceptions of a phenomenon can be described in terms of an interplay between its structural and referential aspects, as outlined in Chapter 5 following Marton Booth (1997). The perceptions in this study could be differentiated both with respect to the referential aspect and with respect to the internal and the external horizon, which together constitute the structural aspect.

Category A Procedural mastery

The structural aspect of Category A includes series of isolated, mathematical ‘tools’, without connections to underlying mathematical rationales. The emphasis is on memorised facts, fixed numerical values, basic arithmetic operations, and accurate measurements. Furthermore, speed, precision, and neatness are prioritised in vocational task solving. In the external horizon references are made mainly to vocational practice and to vocational education as a setting which values careful attention and diligence. The value or meaning ascribed to vocational mathematics knowing, i.e. the referential aspect, is thus associated with handling predetermined workplace tasks efficiently, fluently, and accurately, prioritising procedural mastery over broader understanding and adaptability to different situations.

Category B Context-bound understanding

The structural aspect of Category B focuses on practical, context-bound task solutions; mathematical methods, which are seamlessly integrated into vocational practice through engagement with concrete materials and tools. Particular vocational tasks are in the foreground together with associated context-specific resources, which are identified as key sources for learning and understanding. In the external horizon relations are established to vocational experience as well as practice, a place for simultaneous vocational performance and development. Regarding the referential aspect, Category B reflects the perception that relevant vocational mathematics knowing is based on visual, hands-on understanding, rather than abstract mathematical principles. This indicates that, the aim and meaning of vocational mathematics knowing is perceived as understanding vocational mathematics through practice, in order to engage effectively with specific vocational tasks.

Category C Conceptual mathematical understanding

From a structural perspective, generalised mathematical concepts and theoretical reasoning take centre stage in Category C. Conceptual understanding is prioritised over route learning and context-bound reasoning. The external horizon is now less constrained to a specific vocational domain and the boundaries between different vocations and workplaces are weaker. Regarding the referential aspect, appropriate vocational mathematics knowing is perceived as universal and transferable, with the aim of cultivating a deep understanding that enables problem-solving across

diverse contexts. The primary referential focus is thus on fostering theoretical comprehension in order to address problems beyond immediate vocational tasks.

Category D Transferable mathematical understanding

The structural aspect of Category D focuses on connections and transfer, ie. the process of applying mathematical concepts (learned in mathematics classroom) to vocational tasks, with the necessity of actively bridging the gap between mathematics learned in the classroom and mathematics used in vocational practice, particularly emphasised. In the external horizon is the idea of a coherent vocational education in contrast with the present separation between subjects, and between theory and practice. The referential aspect concerns the capability of solving and understanding vocational problems by means of an enhanced understanding of mathematical concepts, developed through a strengthened connection between school mathematics and mathematics in workplace settings.

Category E Integrated vocational mathematics understanding

The structural aspect of Category E, is multilayered, combining abstract concepts with hands-on application. It involves not just technical and vocational skills or isolated mathematical applications but a sophisticated fusion of various aspects of vocational and mathematical knowing. In the external horizon vocational education is viewed as a dynamic system where new knowledge is developed through a supportive alignment between different parts of the education, such as theoretical, learning, practical application, and workplace experience. The referential aspect of vocational mathematics knowing, transcends the immediate application of mathematical principles in workplace situations. It entails a synergetic understanding in which mathematical and vocational theory, and practical experience are mutually supportive, and encompasses the ability to use mathematics not just as a tool, but as an integral part of vocational challenge-resolution and decision-making.

Category F Powerful vocational mathematics knowing

The structural aspect of Category F focuses on adaptability, critical thinking, and innovation as outcomes of an in-depth mathematical understanding. Now the external horizon encompasses a holistic view of education that extends from specific contexts and immediate demands, to include personal growth and development. Regarding the referential aspect Category F highlights vocational mathematics knowing as integral to a persons' holistic vocational 'Bildung'

fostering readiness, not only for understanding vocational tasks, but also for critically questioning their prerequisites; not only for solving challenges, but also for challenging existing solutions, thereby contributing to further development of the profession and a collective vocational competence.

Relations among the categories

The phenomenographic categories describe different ways of understanding vocational mathematics and how it is applied in vocational practice. These categories are logically and empirically interrelated, as they represent different stages of advancement in both the structural and referential aspects of vocational mathematics knowing. The outcome space in this study reveals several dimensions along which contrasting perspectives emerge.

Routine procedures or deep understanding.

In Category A vocational mathematics knowing is experienced primarily as the retention of routine procedures, facts, and methods. Category B on the contrary focuses on practice-based understanding, embedded in specific vocational contexts. Category B thus differs from A by emphasising *logical reasoning* and *understanding* in and through vocational tasks rather than *routine procedures*. Category A foregrounds knowing what to do, that is, recalling and executing established procedures. In contrast, Category B foregrounds knowing through doing, where understanding is developed in and through engagement with vocational practice. Category C, (like the later categories), represents a further contrast within the same dimension by emphasising the importance of *understanding the essence* of mathematics, i.e. understanding underlying mathematical rationales and structures beyond contexts and methods.

Context-bound reasoning or mathematical generalisation

The outcome space further shows that vocational mathematics knowing might be experienced as inherently *context-bound reasoning*. Category B foregrounds embodied action, material engagement, and situational sensitivity and emphasises knowing that is tightly bound to specific tasks or contexts. In contrast, Category C (as well as later categories) value opportunities for *generalisation* beyond the immediate situation. Here, mathematical structures and rationales are foregrounded and appreciated for enabling application across a wider range of contexts. The development from B to C thus represents a shift toward theoretical thinking and

the ability to apply general mathematical concepts across various domains. At the same time Category C is characterised by a weaker orientation toward the vocational context and familiarity with concrete vocational tasks.

Isolated skills or integrative reasoning

Categories A, B and C emphasise foundational aspects of knowing, bound to specific tasks, procedures or concepts. Category A focuses on procedural retention (isolated facts and methods), while Category B focuses on practice-based understand, where knowing is embedded in specific vocational tasks Category C involves intra-mathematical connections, although contextual connections are relatively weak. Category D and E are the bridging categories that in different ways connect the theoretical understanding of Category C with the vocational activities of Category B. Here the appreciation of the vocational context is resumed and the challenge in vocational application of mathematics is recognised. Category D and E thus represent a movement beyond simple tool use, context-bound task-solving, and isolated mathematical concepts, and highlight the connection between theoretical knowledge and real-world tasks. While conceptual understanding is valued in Category C it is in Category D where the significance of applying that understanding in real-world vocational tasks is explicitly acknowledged. Categories D and E both emphasise the importance of connecting theoretical mathematical understanding with vocational activities. Category E differs from Category D, however, by representing an integrative way of understanding vocational mathematics knowing, in which mathematical, vocational, and experiential forms of knowing are brought together as an interconnected whole, emphasising the inseparable nature of these dimensions in solving complex vocational challenge, rather than focusing primarily on establishing connections between these domains.

Direct applicability or guided transfer

Another tension that emerges in the outcome space concerns the transfer of mathematical understanding across contexts. While Category B does not necessitate transfer, and Categories A and C takes it for granted, Category D explicitly problematises the assumption of direct transfer by portraying the connection between school mathematics and workplace practice as an essential and non-trivial aspect of vocational mathematics knowing, which consequently must be explicitly addressed in teaching. This aspect is also present in the later categories, where it forms an implicit part of a more comprehensive understanding of vocational mathematics, although it is less explicitly foregrounded.

Immediate task performance or holistic vocational 'Bildung'

Categories E and F build on the earlier categories but reflect more holistic, integrative and transformative perceptions. Category E represents a synergetic view of vocational mathematics knowing, in which theory, practice, and workplace experience are perceived as integrated into an adaptable system, highlighting the inseparable nature of these dimensions. Category F represents an even broader view where vocational mathematics knowing is seen as a key part of vocational 'Bildung.' It connects mathematical knowledge with personal and professional development, fostering creativity, confidence, and adaptability, qualities that are valuable across vocational tasks. Here the empirical data reflects a growing recognition that vocational mathematics knowing is not just a tool but an integral part of vocational identity.

Structure of the outcome space

As described in Chapter 5, the outcome space in this study could be described in terms of shifts and interplay between vertical and horizontal dimensions. For example, the difference between Category A: 'Procedural mastery' and Category B 'Context-bound understanding' can be described as a vertical distinction, characterised by a contrast between a focus on *rote learning* of isolated facts and procedures and an emphasis on practice-based *understanding* in vocational settings. At the same time it can be viewed as a horizontal reorientation; a shifting focus, where different aspects are emphasised. Category C 'Conceptual understanding' differs from Category B through a further vertical distinction, in which context-specific understanding contrasts with more abstract and *generalisable forms of knowing*. The difference between Category B and C also involves a horizontal reorientation, where the connection to immediate vocational contexts is less pronounced in Category C, while mathematical aspects are kept in the foreground and separated from the context. Category D 'Transferable understanding' bridges these ways of understanding through a horizontal broadening, reintroducing a stronger vocational focus while maintaining the explicit focus on mathematical aspects typical of Category C. Category D further represent a more advanced understanding as it also emphasises and *problematizes transfer* of school mathematics knowledge into workplace contexts. The difference between Category D and E 'Integrated knowing' represent a horizontal shift as *integration of knowledge domains* are focused rather than *connections between practices*. Category E is further characterised by an understanding of vocational mathematics as a synergetic and

adaptable integration of theory, practice, and workplace experience, which compared to previous categories represent a deepening of both mathematical and vocational knowing, as well as an expansion to encompass diverse perspectives and contexts, mathematical as well as vocational. Category F differs from the other categories by situating vocational mathematics knowing within a broader framework of personal and professional meaning. Here, vocational mathematics is understood in relation to *wider vocational aims and personal development*, extending the focus beyond immediate vocational or workplace concerns. Vertically, this category can be described as representing the highest level, in which vocational mathematics is experienced as having transformative potential.

8. Result Study 2

Study 2 examines characteristic features of vocational mathematics knowing, as it is expressed in authentic vocational task-solving. Vocational teachers' descriptions of students' difficulties and required understanding were analysed alongside students' discussions while solving vocational mathematics tasks. Using a phenomenography-variation theory framework, the study identifies critical aspects that students need to discern to engage with mathematically rich construction-work tasks in proficient ways. Study 1 highlight qualitatively different ways of experiencing vocational mathematics knowing in the context of construction-work practice and education. It is concluded that teachers, in various ways, acknowledge the complexity of integrating various aspects during vocational problem-solving. This chapter aims to further illustrate how this kind of knowing is explicitly manifested in vocational mathematics task-solving by addressing the research question:

Which critical aspects do students need to discern in order to understand and engage with mathematically rich, construction-work, vocational tasks, in proficient ways?

In this Chapter, five task types are presented. The presentation of each task type begins with a brief description of its characteristics. This is followed by an account of the identified critical aspects, illustrated with interview excerpts from which these aspects were derived. The organisation of the material into task types and the use of headings are pragmatic rather than analytical, intended to enhance the clarity and readability of the presentation. Some of the identified aspects are typical of the task type (e.g. even distribution of items), while others are specific to particular vocational situations (e.g. roof tiling or wall panelling). Critical aspects are indicated by text in *italics*, while significant passages in the excerpts are highlighted in **bold**.

Task type 1. Constructing even spacing

The first type of task analysed in this study focuses on creating evenly distributed spacing. It is discussed in several interviews and is also found in one of the

textbooks (Schmale, 1966). The task involves determining both the number of elements and the spacing between them, given a total length and a maximum allowable distance, in such a way that the elements are distributed evenly across the entire length. Typical applications include contexts such as panelling, roofing, and fencing⁸. Several similar tasks appear in the textbook. The example below was selected for the students' discussion:

250. The c/c distance between the roof trusses in a building must not exceed 120 cm. Calculate the number of roof trusses needed if the c/c distance between the two outermost trusses is 24.36 metres. What will the actual distance between the trusses be if they are evenly distributed?

(Schmale, 1966, authors' translation)

The general solution procedure for this type of task can briefly be described in terms of three main steps. First, the minimum number of intervals is determined by dividing the total length by the maximum allowed spacing and rounding the result up to the nearest whole number. Next, the exact spacing is calculated by dividing the total length by this number of intervals. Finally, the total number of items is established, which is typically one more than the number of intervals.

Making sense of the task as a whole

As outlined earlier, the initial phase typically involves approaching the task in its entirety, including considering its framing, that is, the intended outcome, and relevant conditions and constraints. This stage is particularly critical in authentic situations, where such aspects are not explicitly specified in advance. In the present case, this involves recognising that *the spacing should be evenly distributed along the full length*, that *the total length is fixed*, that *the spacing can be adjusted*, and that *the spacing is constrained within certain limits*. In addition, an initial, tentative solution strategy, consisting of a few key steps, needs to be anticipated at this stage (see example above). These aspects can be traced in the excerpts below:

V9: And then I'm thinking of something else that students really struggle with, and that is when we have to **divide up a panel surface**. Distributing boards, with a **certain spacing in between**. There is a certain way of calculating this (...) You have a surface, and then you're going to install **panels of a certain width**, and in the end it all has to **divide evenly**.

⁸ It should be noted that three different types of material are considered in this chapter: panels, roof trusses, and tile battens.

VM5: This whole idea of **dividing things up**, it's really difficult for them. For example, when installing panels on a wall. You have a panel that looks like this [shows], and base boards like that, and then you place a trim piece over it. (...) **it's that distance [between the base boards] that we can adjust** (...) We must **end up with a board here** [beside window] so that the trim covers any gap (...) We calculate things like this a lot, and I've made formulas; **I've developed ways of thinking.**

Both V8 and V5 highlight this type of task as vocationally relevant while also pointing to its difficulty from an educational perspective. They further indicate that the task can be approached through a *specific procedure*. In their descriptions, key aspects of the task framing are illuminated: First, the distribution must remain even across the entire surface 'it must divide evenly at the end' (V9). Second the spacing between boards is the adjustable variable 'it's that distance [between the base boards] that we can adjust' (VM5). Finally, the panel width functions as a constraint for the spacing.

A similar line of reasoning can be observed in the students' approach to the textbook task presented above. The students begin by interpreting the task, noting that the items must be evenly distributed, identifying a maximum allowable spacing, and recognising that the number of intervals must be determined. Based on this, S1 anticipates that the actual spacing will likely be slightly less than the maximum value:

S1: You need to have them **evenly spaced** (...) It can be a **maximum** of 120 cm (...) and then you're supposed to **see how many will fit** in 24.36 m. So, it'll **probably end up being a bit less than 120 cm** then.

Again, the initial focus is on making sense of the task, including its aim, conditions, and constraints. In this case, however, the task description includes several conditions that, in real-world situations, would also need to be identified as relevant.

Identifying initial measurements

In authentic vocational settings, such as those described in the teacher interviews, relevant measurements are typically not provided in advance. It is therefore essential to discern *which measurements that are essential, how they should be obtained, and how they may be influenced by physical constraints*, such as material dimensions.

Correct measurements

For example, aspects related to proper measuring practice are emphasised:

V6: [Concerning tile battens] You must take the **centre-to-centre measurement** [between the upper and lower tile batten] However, (...) students might take the **measurement between the edges of the tiles**. It's common. Then, they end up with the **wrong distance for the last item**.

Here, V6 highlights the importance of distinguishing the *correct centre-to-centre measurement* from the commonly misused practice of measuring between adjacent edges, which can lead to incorrect spacing in the final result.

Physical conditions and situation-specific constraints

The excerpt below underscores the need to *discern how physical conditions (such as wall configurations) impose context-specific constraints*:

V5: [Concerning panelling] We get a **certain measurement under the window**, and we get a completely **different measurement over there**, and then, possibly, above this window there is another window, but with an offset of 15 cm. There they get stuck!

In particular V5 points to the challenge of recognising how the wall's physical features generate varying fixed distances that must be taken into account.

In a similar manner, the excerpt below shows that, concerning the task of distributing roof tiles, it is necessary to discern the *maximum spacing as dependent on the slope of the roof*:

V6: Let's say this roof slopes at 27 degrees. I use the tile instructions (...) If the slope is 27 degrees, the spacing between the battens should be, say, 340 mm to 375 mm (...). **If the slope is less, the tiles need to overlap more**, to protect against wind and water.

In this particular context, it is therefore necessary to recognise *that the required degree of overlap, and consequently the maximum permissible spacing, is dependent on the roof's slope* ('if the slope is less, the tiles need to overlap more,' V6) These aspects relate to both general physical principles and vocation-specific requirements.

More implicitly, the excerpt also points to the *concept of slope* itself. The task may involve determining the roof angle, which requires interpreting slope as *rise over run* and understanding *its relationship to the angle via the tangent function*. In other contexts, different types of constraints may be relevant, such as safety regulations, material

properties, or workplace standards, making the identification of such constraints a critical aspect of these situations.

In this context, it becomes necessary to discern the *relation between the required overlap (and thus the maximum allowable spacing) and the slope of the roof* ('if the slope is less, the tiles need to overlap more,' V6). This also entails understanding *the rationale behind this relationship* ('to protect against wind and water,' V6) and *its numerical representation* (e.g., 340–375 mm). Such considerations connect general physical principles with trade-specific requirements.

At a more implicit level, the excerpt also draws attention to the *concept of slope* itself. The task may include determining the roof angle, which involves interpreting *slope as rise over run* and discerning the *relation between the ratio and the angle through the tangent function*. In other situations, different forms of constraints may come into play, such as safety regulations, material characteristics, or workplace standards, making the ability to identify and account for these constraints an essential aspect of the task.

Applying a general procedure

Multiplicative reasoning and rounding practices

Once the practical conditions have been taken into account and the relevant measurements established, the problem-solving process proceeds to determining the spacing. At this stage, recognising a *general solution procedure* becomes critical, with the procedure itself constituting an important dimension of variation. Understanding this procedure involves several interrelated aspects. A key aspect is the *multiplicative relationship between the spacing, the number of intervals, and the total distance*, a mathematical idea commonly encountered in school mathematics. However, successfully solving the task also requires an awareness of how numerical results must be adjusted to fit practical conditions. For instance, discerning *the number of intervals as necessarily a whole number*, and more specifically, *as obtained by rounding up the calculated value*. This, in turn, rests on understanding the *relation between the maximum allowable spacing and the minimum number of required items*. These aspects are illustrated in the following student discussion:

S2: All right... I know exactly how to do it! [calculates: $24.36 / 1.2 = 20.3$ i.e. total length divided by maximum spacing]

S1: 20.3. **Should there be 20.3 of those then?... (..) But should there be 20 roof trusses then?... No, then you have to use 21 roof trusses, because it can't be more than 120 cm between each, so you have to round up.**

S2: Yeah, I think so... or we make the house smaller [laugh]

S1: [calculates] Then we get 1.16. Yes, so then it becomes less than 120 cm.

The critical point of discussion here concerns the fact that the number of items must be an integer (i.e., not 20.3) and that rounding up is necessary to ensure compliance with the maximum spacing constraint (i.e., 21 rather than 20). As a contrast, it is concluded that rounding down would result in a spacing that exceeds the allowed maximum. Finally, using multiplicative reasoning again, the appropriate spacing is calculated as the total length divided by the number of intervals, and it is confirmed that the resulting spacing satisfies the constraint.

Another critical aspect emphasised by teachers and evident in the student discussion, is the *distinction between the number of items and the number of spacings*.

S1: Well... If you have 1.16 m between each and multiply by 21, you get 24.46 m, but then **you've only counted the spaces between, and you're missing the final truss.**

As emphasised by S1, dividing the total distance by the spacing gives the number of spacings, not the number of items.

Accumulation of errors

Having determined the appropriate spacing and number of items, the task then moves on to physically marking out their positions. At this stage, a critical aspect is the accumulation of error. More specifically, students need to *discern deviation at the final item as a consequence of repeated addition of rounded values* and the need of accounting for this. V6 highlights this issue and proposes a way of addressing it:

V6: I use the first measurement and mark it out. I use the [+] [+] [=] method [using the calculator in a specific manner] to add the same spacing again. By the end, I'll reach 4332 mm exactly. (...) **If you use the plus-plus method, you can go twelve miles with millimetre precision.**

It is thus critical to discern the *need to adjust for cumulative error* as well as a strategy to cope with it.

Summary Task type 1

As illustrated above, vocational tasks, such as distributing items evenly, requires a variety of different aspects to be discerned. Some of them concerns making sense of the specific task at hand, such as discerning aims and conditions, and distinguishing fixed and adjustable quantities. Other aspects involve understanding how relevant features of the vocational context and the physical environment affect the initial values to measure, and the interpretation of the result obtained. Such aspects might be general, or specific for the particular situation and could involve proper measuring practice as well as effects of specific physical constraints. Moreover, some aspects concern understanding the task structure. Once having made sense of the specific task, there might be a recognition that the problem is of a certain task type for which a typical task-solving strategy is applicable. Discerning and generalising an over-all strategy is thus a critical aspect as such. Understanding and performing the procedure, in turn, involves a number of interrelated aspects, such as, in this case, understanding multiplicative relations, and rationales for proper rounding practices. Finally, when applying the result in practice the effect of cumulative errors must be discerned, an aspect well established within formal mathematics, but often disregarded in context-based textbook tasks (according to the researchers review).

Task type 2. Tasks involving slope

The second task type selected for analysis concerns applications of the slope concept. A few tasks involving slope, were used as a starting point in the interviews and were repeatedly confirmed by the interviewees to have vocational relevance. The concept of slope is addressed in all of the textbooks. Here, slope refers to the steepness or incline of a surface, defined as the ratio of the vertical change to the horizontal change between two points ($\text{slope} = \text{rise} / \text{run} = \Delta S_{\text{vertical}} / \Delta S_{\text{horizontal}}$). It could be expressed as a ratio $\Delta S_v : \Delta S_h$ (e.g. 1:200), as a percentage (e.g. 0,5%), or as an angle, calculated using the tangent function ($\tan(\alpha) = \Delta S_v / \Delta S_h$). The tasks typically imply calculating rise, run, or slope when given the two other values, or converting between representations. Vocational contexts, such as pipe laying, ramp construction, ditch digging, and roofing are commonly used examples. During the interviews, the task presented below, was discussed, and other tasks were given as examples by the interviewees.

181. The drainage adjacent to a house should have a slope of 1:200.
- What is the slope expressed in per mille?
 - What is the total height difference in millimetres if the distance from the highest to the lowest point is 12.5 metres?
- (Plesner, 1995, p 108)

Making sense of the task as a whole

During the teacher interviews, vocational tasks, involving slope calculations, were discussed. Typically, the initial focus was directed to specific vocational situations in which the concept of slope is relevant. The examples below highlight desired outcomes (e.g. finding vertical distance or finding the slope), implications (e.g. the slope value influences material choices, safety, and design) and situation-specific constraints (e.g. maximum incline of a ramp for accessibility or required drainage slope). Furthermore, practical and physical rationales for these aims, implications and constraints are explicitly emphasised as essential aspects.

Permissible range of slope values

The excerpt below illustrates how practical experience and physical principles are essential components in making sense of the task. The permissible slope range draws on physical laws as well as practical constraints, where the acceptable gradient is determined by what is physically feasible and functionally required.

V1: I know this from personal experience... the inclination... I used to work as a carpenter, and we had a project where we built wheelchair ramps. There the incline was really important because **it shouldn't be too heavy to get up with a wheelchair**. I actually enjoyed Figuring out the measurements. (...) So in that case, it definitely mattered.

In this particular context it is thus important to discern *the dependence of maximum allowable slope on accessibility requirements*, and is connected directly to the physical effort involved, 'it shouldn't be too heavy to get up with a wheelchair' (V1), which, in turn, is governed by a fundamental physical principle, i.e. *the relation between steepness of the surface and the component of gravitational force acting along the surface*. Other situations might entail alternative conditions, all of which are grounded in the practical requirements of the particular situation. For example, the excerpt below highlights conditions and requirements, specific to drainage work.

V1: **What is the total drop in millimetres from the highest to the lowest point over the distance?** (...) This is extremely important — around a

house, it's crucial that it's done correctly. The slope has to be right. You can't just go over a stone or something; **it has to work all the way through.** (...) It's about **making sure that water doesn't collect anywhere** — not on the roof, not in the ground. It has to be **directed away from the house.**

Here, V1 refines the problem by emphasising the importance of ensuring *sufficient slope at each particular location* 'You can't just go over a stone or something; it has to work all the way through' (V1). This aspect implicitly draws on the contrast between *average slope* over a distance and *local slope* at a specific point.

Implications of the slope value

Situations involving slope calculations may also entail understanding how the slope affects other factors such as material choices, safety, and design. One such example was discussed in relation to Task 1 above (distance between tile battens). Further examples highlight the practical implications of slope values, as illustrated in the excerpts below:

V1: It's important to know **the angle of the roof**, it really is (...) below 14 degrees, you are not allowed to use tiles. We usually tell them that too.' But why?', 'Well, because then, if it's windy and raining, the wind pushes the water under the tiles, (...) and it has to do with **weight** as well, if you want a roof pitch of 14 degrees (...) then we may have to **reinforce the roof trusses**, since the weight will become completely different if you have a flatter roof. (...) It is important to understand that. **The flatter the roof, the lighter the material.**

V6: If the slope is less, the tiles need to overlap more, to protect against wind and water.

VM5: We have certain roofing materials, and they are built differently for different **angles or slope of the roofs** (...) You can't lay them as flat as you want because then they don't do the job. And the roofing felt we use on the roof is also different depending on... well, the density differs, so to speak. **Like, the flatter the roof, the denser it must be.**

Critical aspects, evident in all of the excerpts above concerns the *relation between slope of the roof and the choice and use of roofing materials* 'the flatter the roof the denser it [the roofing felt] must be' (V1), 'the flatter the roof, the lighter the material' (VM5) and 'If the slope is less, the tiles need to overlap more, to protect against wind and water' (V6). These insights highlight the interaction between slope and physical conditions, material properties, and construction regulations. It is also evident from the excerpts that explanations for why things must be constructed in

a certain way, are valued by teachers as essential aspects ‘we may have to reinforce the roof trusses, since the weight will become completely different if you have a flatter roof (...) if it becomes too flat it will start to bend’ (V1), that is, physical effects of a flatter roof are highlighted as explanations to why certain actions must be taken. (This remark implicitly draws on the physical principle: the flatter the surface, the greater the component of gravitational force acting perpendicular to the surface.)

In the analysed textbooks, these kinds of principles are further elaborated, focusing on the concept of force. In these tasks aspects of the force concept such as seeing *force as vector*, *force as decomposable into components*, *normal force as acting perpendicular to surface* and principles of *static equilibrium* etc., are emphasised. These are all examples of aspects which could be formalised through mathematical language, but which also involves theoretical knowledge beyond formal mathematics.

The examples illustrate that understanding the task goes beyond simply calculating the slope, it also involves reasoning about function, safety, and material suitability, grounded in practical experience and knowledge about general physical principles.

Understanding the slope concept

Solving vocational ‘slope problems’ inevitably involves making sense of the slope concept as such, within the specific context at hand. An analysis of the textbooks, alongside interview excerpts, offers a multifaceted view of potentially critical aspects in this regard. On the one hand, aspects related to a coherent understanding of the slope concept across different contexts are revealed, especially through the textbooks. On the other hand, the interviews highlight how the slope concept, and the mathematical representations used to describe it, depend on the specific vocational context in which the concept is applied. Together, these empirical sources provide complementary perspectives, considering formal consistency and conceptual connections, as well as the practical, context-sensitive nature of the slope concept in vocational task-solving.

Different representations of slope- Conceptual coherence

As described in Chapter 3, the concept of slope can be conceptualised and represented in various ways, with the connections between these conceptualisations developed to varying degrees in different textbooks, as well as by teachers and students. This variation is reflected in the empirical material of this

study. While all textbooks acknowledge slope as a key vocational mathematics concept, they differ in how they represent, apply, and connect it to broader conceptual mathematical structures. For instance, one textbook introduces slope in conjunction with the concept of scale, thus interpreting both as *instances of geometric ratios* (Schmale, 1960). Other textbooks present slope as a *practical application of trigonometric functions*, making explicit the *relationship between the rise-run ratio and the angle of inclination through inverse trigonometric functions* (Plesner, 1995). Though no single textbook fully promotes a coherent understanding of slope across its different representations (e.g., geometric ratio, percentage, and angle of inclination), nor systematically support conversion between all of these forms, these relations can be identified in the empirical material in as a whole.

A fundamental aspect emphasised across different representations, is *slope as representing the ratio of vertical to horizontal change*. In formal mathematical terms, this involves interpreting slope as $\Delta S_{\text{vertical}}/\Delta S_{\text{horizontal}}$, which also entails discerning the *correspondence between the steepness of the surface and the absolute value of this ratio*. Closely related to this is the discernment of *the invariance of the ratio between vertical and horizontal change for any right-angled triangles formed along the same line*. This property is drawn upon in several textbook tasks and follows from proportionality and the concept of similarity (i.e. the AAA similarity theorem implies that right angled triangles sharing the same angle are similar, and therefore the ratios between corresponding sides, such as $\Delta S_v/\Delta S_h$ is invariant).

The analysis of the totality of the empirical material further suggests that a coherent understanding of slope involves discerning *the equivalence of slope as a geometric ratio, as a percentage, and as the tangent of an angle of inclination*, that is, recognising these as *different representations of the same underlying relationship* (Schmale, 1960; Kristersson, 1987; Plesner, 1995).

Furthermore, developing a consistent conceptual framework requires understanding how these representations are interrelated, including understanding *the relation between angles, trigonometric concepts and the proportionality of sides in similar triangles*. In this regard, it may also be relevant to discern *the role of triangle similarity theorems in justifying and unifying trigonometric and proportional representations*, an aspect which would probably support a more integrated and transferable understanding of the slope concept.

Different representations of slope- Contextual dependence

In the interviews, the contextual dependence of the representations of slope are emphasised rather than the conceptual coherence across contexts. It is repeatedly

confirmed that different representations of slope are used in different contexts. In some contexts, slope is conceptualised as *incline in centimetres per metre* (i.e. a percentage), in others as *an angle of inclination*, or as a *ratio (1: X), where X represent horizontal distance that correspond to one unit of vertical depth*. For example, in the context of drainage system construction, both V1 and V3 conceptualise *slope as a vertical rise (measured in centimetres) per metre horizontally*, or more specifically, *a ratio derived from the relationship between height difference and horizontal distance, expressed per unit length*. Moreover, a critical aspect to discern in this context is the multiplicative structure of slope as a constant ratio, that is, *the proportional extension of a per-metre slope to a total height difference over a given distance*.

V1: It's about slope per metre and that kind of thing that we work with, in centimetres then. (...) People who install drainage systems today know the slope per metre. (...) So, if they know that a drainage system should **slope one centimetre per metre**, and **if the house is 12 metres long**, then it **should slope 12 centimetres**.

V3: Yes, and percentages and per mille are what we always use when it comes to pipes. And drainage is considered a type of pipe. So in that case, percentages and per mille are what apply. (...) Then you usually get: **here is the length, there is the height difference** (...) **What does that give per metre?** That's usually how we calculate it. (...) What is the height difference? How long is the distance? **That gives you the slope.**

Both V1 and V3 refer to slope in terms of centimetres per metre, a value which is explicitly interpreted as a percentage by V3. To calculate the slope V3 suggests a way of thinking: 'Then you usually get: here is the length, there is the height difference (...) What does that give per metre'. Moreover, V1 relates an incline of one centimetre per metre to a total incline of 12 centimetres over a distance of 12 metres, thereby demonstrating the proportional nature of slope as a constant ratio.

When discussing ditch construction, V3 conceptualises slope in a different way, now as a ratio on the form 1:ΔH:

V3: Then we use a 1:3 ratio in that situation (...) When they're going to build a slope or something (...) you usually get that it has to slope by that amount. So you just do it like that! **The ditch has a side slope of 1 to 3. So where is the bottom, basically, and how far out do you need to go?**

In this excerpt, given a certain slope 1:ΔH, it is suggested understanding ΔH as *how many times further horizontally you must go in relation to the vertical depth* 'where is the bottom, basically, and how far out do you need to go' (V3). More

specifically, the *multiplicative relation between vertical and horizontal distances*, is emphasised. Once again, the concept of slope is closely connected to, and interpreted in relation to, the specific vocational situation.

Furthermore, the interview material explicitly shows that the choice of representation, whether a ratio, percentage, or angle in degrees, depends on the specific vocational context.

V3: We always talk in per mille or percent when we're setting falls. (...) When we talk about slopes, like [e.g. 1:3], **then we're dealing with machine operation and side slopes of ditches. (...) In those situations, we use 1:3 and similar ratios. (...) Percent and per mille are what we always use when it comes to pipes.** Yes, we also count drainage as a type of pipe. (...) But you almost never convert something like 1:200 into percent.

V3: **We [in groundworks] never express it as an angle, that's just not how it's done there.** (...) Carpenters probably do that (...) **carpenters tend to work with angles.**

VM5: How slope is expressed and what these values mean is something we mainly work with when we're dealing with drainage. **But when it comes to roofs, then it's degrees instead. (...) Slopes on roads or hills are given in percent, not in degrees.**

It is explicitly explicated that different representations of slope are used in different contexts, indicating that the *contextual dependence of these representations* is critical to discern. When laying drainages, the slope is expressed as a percentage and when digging ditches it is expressed as a ratio. Slope of the roofs, on the other hand, are expressed as angles of inclination 'But when it comes to roofs, then it's degrees instead (...) Slopes on roads or hills are given in percent, not in degrees' (VM5). These statements illustrate how the representation of slope is shaped by the norms and practices of specific context.

Summary Task type 2

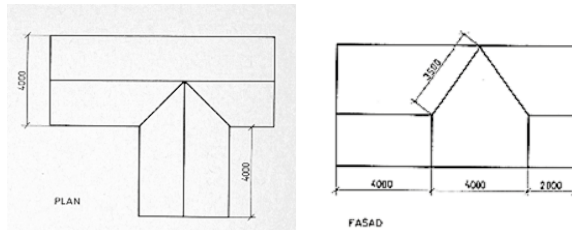
As illustrated above, vocational 'slope-tasks', requires a variety of different aspects to be discerned. Understanding such tasks typically entails being aware of how the slope value affects practical considerations such as material choice and reinforcement needs, and how practical constraints affect allowable slope values. Furthermore, physical and practical rationales are frequently emphasised (including physical principles, material constraints and vocational standards) which confirms that both knowing *how*, and understanding *why*, are deemed critical

components in vocational mathematics task solving. Taken together, the empirical material confirm that a comprehensive conceptualisation of slope involves conceptual coherence as well as contextual sensitivity. Thus, developing a robust vocational understanding of slope requires recognising both how the multiple, interchangeable representations of slope are related, and how they are appropriately used in specific vocational contexts. In particular, understanding slope in a vocationally relevant way involves discerning the invariance of the ratio between corresponding sides within similar figures, that is, understanding that the slope remains constant regardless of the particular triangle used to represent it, an insight grounded in proportionality and the concept of similarity.

Task type 3. Estimating material requirements for a surface.

The third task type selected for analysis entails calculating the amount of material needed to cover a specific surface. Several interviewees confirm that material estimation based on area calculations, is an essential issue in construction-work vocational practice. Vocational mathematics textbooks also contain a considerable number of tasks related to this theme. Based on technical drawings and information about the covering material (such as, dimensions, amount required per square metre etc.) students are expected to calculate the amount of material required to cover a specific surface. Situations such as painting, wallpapering, and roofing serve as examples. Four tasks were selected for student discussion:

Figure 8.1 Adapted and translated from Swedish mathematics textbook
(Schmale, 1960 Task 147)



Task 3.1 How many roof tiles are needed for the roof (Fig. 8.1), if 12 tiles are required per 1 m²? The valleys, which are covered with sheet metal, make

up 3% of the area. An allowance for waste is calculated as 8% of the tile-covered area

Task 3.2 A floor measures 5×7 m. It is to be covered with flooring that is 2 m wide. How should the cork be laid to best save material, how many metres of flooring need to be purchased, and how much will be left over?

Task 3.3. The floor in a square bathroom, with side length 3.92 m, is to be covered with clinker tiles measuring 0.15×0.25 m.

- a) What is the area of the room in m^2 ?
- b) How many clinker tiles are needed?

Task 3.4. In a rectangular room with a width of 4.75 m and a length of 7.20 m, ceiling tiles measuring 0.60×0.60 m are to be installed.

- a) What is the area of the room in m^2 ?
- b) How many ceiling tiles are needed if a 9% waste allowance is included for fitting against the walls?

(The tasks were constructed based on a review of common types of material estimation problems found in Swedish construction-work mathematics textbooks, e.g. Schmale 1960; Bedrosian, 2024; Plesner, 1995)

The procedure for solving the task type may differ between tasks depending on the material used, and how it is delivered. Either the areas of the surface and the covering items are compared, or the number of items needed along the length of a surface (rounded up to integer) is calculated and multiplied by the items needed along the width. Moreover, some tasks ask for the material amount in terms of number of items needed, others in terms of running metres, and others in terms of the required volume of material, which naturally also affects the calculation.

Making sense of the task as a whole

As in previous examples, the initial focus is typically on the task as a whole, focusing on its framing. Several interviewees emphasise the technical drawings as constituting a foundational structure, providing the basis for identification of relevant dimensions. Furthermore, a task solving procedure is suggested, typically following a common pattern. i.e. calculating a surface area and then estimate material requirements.

V2: Basically, mathematics in construction-work is closely **connected to drawings** (...) it's about areas, what I call **quantity calculations** (...) **square metres of walls, floors, and ceilings, and then converting that into running metres of paneling, or the number of sheets**, or whatever material you're buying. If you're on a small construction site and you run out

of material, you need to be able to say how many metres of roof boarding we need to cover the six square metres that the first delivery didn't cover.

Similarly, the students, solving Task 3.1 described above, initially focuses on the framing:

S1: So we're going to **calculate how many roof tiles are needed** to cover this entire roof... So first we probably need to calculate **how large the roof is**, so that we can work out how many tiles are required... and then we need to subtract a bit for the flashing and then add a bit again. (...) **So if we start by calculating this one here** [points/draws].

Considering the plan view, the student is pinpointing the desired outcome 'So we're going to calculate how many roof tiles are needed to cover this entire roof' (S1). They further propose an over-all solution strategy (i.e. calculate the roof area in order to find the material requirements) and an appropriate starting point 'so, let's start by calculating this one' (V1).

Deriving measurements from technical drawings

Distances and coordinate systems

In construction-work tasks, measurements used for area and material calculations are often obtained from technical drawings, making the ability to interpret drawings and derive such measurements an essential component.

V2: Basically, the math in construction-work is connected to the drawings.

V4: **You need to be able to read drawings in order to order materials without actually being on site** (...) you need to calculate materials, such as paint or wallpaper. (...) It can also be about calculating how large an area that you've painted or treated is. (...) **There you really need to be able to read the drawings.**

V6: In this drawing set, there are four drawings. An elevation — where they can see the cladding, the windows, and the roof. (...) Then they have a **floor plan** (...) a **foundation plan** (...) And then they get the fourth drawing, which is a **section drawing** with heights. (...) They can see the roof angle, the roof pitch, and then they have to calculate the height, so that they can calculate the amount of cladding on the gable end. **So they calculate the area of the entire house and then determine how many boards are needed.**

As shown above the teacher interviews make clear that measurements for area calculations in construction-work practice are typically derived from technical drawings. According to the analysed textbooks, length measurements in dimensioned drawings are determined either from cumulative values or reference line⁹ measurements, which consequently must be discerned and distinguished. Finding such measurements demand a clear understanding of how distances in technical drawings are referenced, represented and structured. For example, textbook tasks involving length measurements require discerning *distances as differences between reference line measurements in contrast to difference as sums of given distances between points*.

Two-dimensional projections of three-dimensional objects

A further critical component that is emphasised when discussing surface area calculations concern spatial reasoning, specifically the ability to discern *three-dimensional structures as represented in two-dimensional drawings*. This competence includes combining information from plan and sectional drawings, in order to identify relevant measurements for a given area or object.

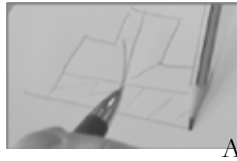
V4: For example, the roof height isn't shown on the floor plan, so then you have to look at a sectional drawing. (...) And that's where they start (...) first by locating the floor plan, and then they have to find the right building in the drawing, the right floor. **It can be difficult to orient yourself in a drawing.** And then they might find the measurements and be really satisfied, but **what they've actually found is just the floor's length and width.** (...). And then... **'Oh right, what we actually need is the wall!'**

The excerpt above confirms that the ability to orient oneself in a drawing and interpret how plan and sectional drawings are interconnected, are critical components of construction-work, area-related tasks.

Also, the students' discussion in the excerpt below illustrates that finding the relevant measurements by combining information from plan and façade drawing is one of the main challenges when solving Task 3.1 given at the beginning of the section:

⁹ Reference line measurements are distances measured from fixed reference lines that serve as zero levels within coordinate systems. Both local and national coordinate systems may be used, to which individual parts of a building are related. Distances between points, for example, the length of a certain wall, may also be given directly in the drawing.

Figure 8.2 Students drawing, roof area

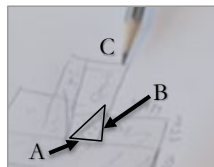


S1: Let's start by calculating this [dashed rectangle]
 S2: **So this one here [A] is 2000**, then [the short side of the rectangle is half the width of the house].
 S1: Yes, but **the roof slopes** as well (...)
 S2: **So that one is 3500** [dimension taken from the elevation drawing; see Fig. 6].
 S1: No, it isn't... well, yes... **it should be, since it's 4000 here as well** [the main building and the extension have the same width]... so then we already have that measurement there.

Considering the roof from the plan view (see Fig.8.2) S2 suggests that the side of the dashed rectangle is half the width of the main building (2000 mm), as dimensioned on the plan drawing (see Fig 8.1). In contrast S1 highlights that the roof slopes. Being aware of *three dimensions as represented in a two-dimensional drawing*, or more specifically understanding how a sloping surface in three dimensions is projected in two dimensions, thus turns out to be a critical aspect. Considering this aspect, S2 suggest that the measurement should instead be 3500, a measurement derived from the façade drawing. S1 first questions the assumption, but then confirms it, by intuitively referring to the SAS congruence theorem (“this one [main building] is also 4000 wide”)

Subsequently, the same aspect i.e. discerning the representation of three-dimensional objects in sets of two-dimensional drawings once more comes to the fore, now regarding the area where the two roofs intersect.

Figure 8.3 Students drawing, roof area



S1: So then we have (...) all these triangles [at the intersection of the roofs] (...) that one [B] is 3500, and this one [A] is half of that [the width of the house], so 2000 mm.

S2: So **this one here [C] is 3500 as well**, or what do you mean?

S1: I'm not completely sure about that, because (...) **the top side here [A] doesn't slope** (...) it's 2000 if it doesn't slope, basically. (...) **It just runs flat, it doesn't follow the slope of the roof**

Here, once again, it becomes evident that a critical aspect is the ability to discern how properties of a three-dimensional object (e.g. slope) are represented in two-dimensional drawings. Based on the plan view, the students attempt to visualise a three-dimensional sloping roof, particularly focusing on the *contrast between measurements parallel to the horizontal plane and those aligned with the sloping surface*: 'the other one is only 2000 because it is flat...it does not follow the slope of the roof.' (S1).

Taken together, these excerpts highlight spatial reasoning and the interpretation of technical drawings as prominent key components, revealing how mathematical procedures must be integrated with technical understanding, logical reasoning, and spatial visualisation in three-dimensional space.

Selecting procedure based on contextual factors

Material properties and rounding practices

Another key component of material estimation tasks involves selecting appropriate procedures, where the choice of method depends on contextual conditions and the physical properties of materials. It is emphasised by the interviewees and evident in the textbook tasks, that there are different approaches to solve material estimation problems, and that the appropriate method depends on practical conditions and properties of materials.

V6: The roof battens, for example. We can calculate the running-metres like this [multiplying the number of items (derived from Task type 1) by the length of the roof], or **we can use square metres**, where there is a number, for example 5.5 **[running] metres per square metre...**

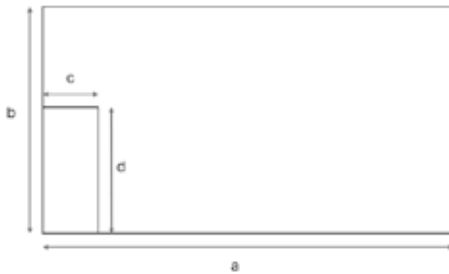
V4: Because otherwise, when you calculate wallpaper, for instance, you can't just go straight by square metres (...) you often have to calculate it in another way

Here, V6 emphasises the *different ways of solving the same problem*, and V4 highlights the *different approaches required*, and their *dependence on practical conditions such as*

material properties. An analysis of a large number of textbook tasks concerning material estimation in a construction-work context (see e.g. Task 3.2-3.4), reveals a number of alternative approaches. The choice of method depends on factors such as whether the material can be spliced, and reused when cut, whether the result should be given in units, running metres, or volume, and the specific dimensions of both the area to be covered and of the available material. Briefly, the approaches can be summarised as follows:

Figure 8.4 Surface area $a \times b$ covered by material of dimensions $c \times d$

Inspired by Bedrosian (2024)



The figure above shows a surface area $a \times b$ and some covering material of dimensions $c \times d$. One approach to calculate the number of items needed, entails calculating the areas $a \times b$, and $c \times d$, and forming the quotient $(a \times b)/(d \times c)$. Here it is also critical to discern the *number of boards as a positive integer*, and that it must be *larger than this quotient*, that is, the number is obtained by rounding up the quotient $n = \left\lceil \frac{a \times b}{d \times c} \right\rceil$. (This approach is typically combined with waste adjustments). An alternative approach is to calculate the number of boards needed along the length and width separately, rounding up each quotient and then multiply the two $n = \left\lceil \frac{b}{d} \right\rceil \times \left\lceil \frac{a}{c} \right\rceil$. The underlying calculation is obviously essentially the same. However, it involves different ways of perceiving the multiplicative structure of the task (see below).

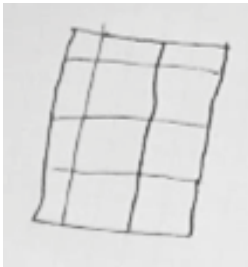
The suitability of each approach depends on the specific practical, material and physical constraints of the situation, including the properties of the surface and the characteristics of the material. Considering the full range of textbook tasks, several types of material coverage problems can be identified, for which different solution strategies are preferable. What is decisive for the choice of solution strategy is which of the intermediate quotients must be rounded up to the nearest

integer $\left\lceil \frac{a}{c} \right\rceil, \left\lceil \frac{b}{d} \right\rceil$ or (if $d > b$) $\left\lceil \frac{d}{b} \right\rceil$. This in turn depends on practical, material and physical conditions, such as how the material is delivered, how it might be spliced and cut and whether off-cuts might be reused¹⁰.

The analysis of textbook-tasks concerning material estimation thus indicate that, rather than recognising a common procedure applicable to all material estimation tasks, the students need to be able to adjust their approach to the particular conditions of the specific task at hand, by discerning *the solution strategies as influenced by rounding decisions* and how this *depends on physical material and practical conditions*.

These aspects are exemplified in students' task solving, for example, when solving Task 3.2. The discussion concerns the orientation of the carpet, that should be laid:

Figure 8.5 Student drawing Task 3.2



S2 I think this way might use a bit less material [shows the carpet laid crosswise]

S1: Should we just guess that, or calculate it?

S2: No, but it's obvious! **Either you get this much carpet left over** [shows along the long side], **or this much** [shows the part left over along the short side].

S1: But **if we split it in half, we could fit in an extra piece** with what we already have... but we could do that the other way too.

S2: In that case, we're done. [The requirements] are the same, and nothing is left over.

This exchange illustrates how students attend to material orientation, leftover pieces, and possibilities for reuse. In doing so, they consider *how cutting, rearranging, and reusing material affects the total requirement, and the approach to solving the task*.

Applying procedure for material estimation

Applying either of these different approaches, implies discerning a specific set of aspects. In one way or another those aspects relate to area calculations and multiplicative relations, combined with proper rounding considerations.

¹⁰ The covering material requirement might also be given in terms of a unit ratio: (running metre per square metre, number of boards per square metre, or volume per square metre).

Multiplicative relations

In the tasks solved by the students, the covering material is either given as a unit ratio, expressed in tiles per square metre (Task 3.1), or through the dimensions of the material (Tasks 3.3–3.4). The tasks may be solved using the different strategies described above. Although the tasks share a similar practical outcome, different solution strategies are suitable (see above).

The students' discussions of Tasks 3.1 to 3.4 indicate that different approaches involve attending to different aspects of the task. The approach of first determining the number of items along each side and then multiplying the results (used when solving Task 3.3) was readily accepted by both students and foregrounds the *structure of the surface as composed of rows and columns*. In this case, the geometric arrangement of the surface is directly visible and supports the calculation.

In Task 3.4, S1 proposed the alternative strategy based on calculating the total ceiling area, dividing it by the area of a single tile, and then adjusting for waste by applying a waste factor. The discussion regarding this approach revealed a discrepancy in how the students understood and related to the structure of the task. This suggests that discerning the *multiplicative relation between the total area and the area of a single item*, may be more challenging than working with length-based relations, possibly since the relevant structure is less directly visible and must be constructed by relating two areas.

More precisely, length-based multiplicative relations $((a/c) \times (b/d))$ and area-based multiplicative relations $((a \times b)/(c \times d))$, or equivalently $(a \times b) \times 1/(c \times d)$ differ in how readily they are discerned. Although mathematically equivalent, these approaches are experienced differently, as they bring different aspects to the fore. Length-based strategies make the *spatial structure of rows and columns* salient, whereas area-based strategies require discerning a *multiplicative relation between two-dimensional measurements*. In the students' discussion, the ability to use these different strategies appropriately and flexibly in different situations was identified as essential for solving different types of tasks, such as Tasks 3.1–3.4. Recognising the *relation between different types of multiplicative strategies for solving material estimation tasks, and when each is appropriate*, thus emerges as a set of critical aspects of solving material estimation tasks.

Waste factor

Accounting for waste is typically a critical component in material estimation tasks.

V3: You usually calculate about **20 or 25 percent, depending on the type of material**. So that's something you always factor into the calculations.

The excerpt above indicates that recognising *the need to adjust for waste*, is a critical aspect, along with discerning *the dependence of the waste factor on the type of material used*. Moreover, carrying out the calculations entails determining the total amount of material needed by multiplying the exact amount required to cover the surface by the waste factor. This represents an application of percentage calculation typically addressed within school mathematics. Students' discussion of Task 3.1 reveals critical aspects related to this kind of percentage application. In the task, 3 % of the roof area is covered by metal sheets instead of tiles.

S1: ...and 3% of the area should not have tiles, but some sheet metal instead. So then **we have to remove 3%**. Is it 0.97 that means removing 3%...?

S2: **Where did you get 0.97 from?**

S1: **That's what it means to remove 3%...**

S2: Yes, of course!

This excerpt indicate that discerning *the equivalence between multiplication of an area by $(1-a/100)$ and subtracting $a\%$ of the area* is a critical aspect 'Is it 0.97 that means removing 3%...?' An insight that draws on the understanding that removing 3% of the whole always leaves 97% of the whole, and that this is expressed multiplicatively as 0.97 times the whole. Aspects which in turn involve a combination of additive and multiplicative reasoning, together with an ability to distinguish between the two. Whether these aspects are educationally critical or not depends on students' pre knowledge.

Area measurement

The tasks involves calculating area measurement, which in itself according to V4, may constitute a challenge:

V4: **Calculating square metres of a wall, they find that difficult.** (...) I need to teach them how to find the amount of paint that they'll need. I've stood in our painting booth and drawn with pencil on the wall, **several times, and sometimes several times a day for the same student.** And many of

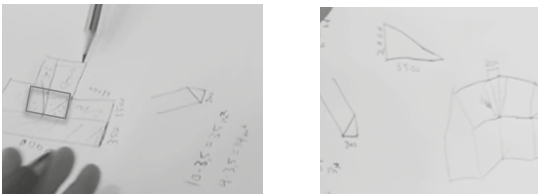
them say ‘oh, now I get it.’ When you’re on that site and doing it right there...
And then maybe they haven’t understood how to calculate square metres in a book.

The excerpt indicates that perceiving a rectangular *area as the product of its side lengths* may constitute a critical aspect for secondary school students. According to previous research (Battista, 2007), a conceptual understanding of this relationship is not self-evident. It requires an initial perception of area as composed of iterated unit areas, which gradually develop into an understanding of *area as a continuous two-dimensional quantity* that remains *conserved under decomposition*.

Surface decomposition

The *decomposition of an irregular area into rectangular or triangular regions*, is obviously a critical component in the task solution accomplished by the students (solving Task 3.1.), who starts by decomposing the roof into rectangular sub areas. ‘Let’s start by calculating this [rectangle]’ (S2). It is evident from the excerpt below that the task becomes considerably more complex as this aspects must be coordinated with spatial reasoning, for example when considering the area at the intersection of two sloping surfaces. While solving the exemplified problem, the students discuss whether or not it would be possible to calculate the area of the rectangle below (3500×4000), without dividing it into sub triangles.

Figure 8.6 Students’ drawing, relation between 2D drawing and 3D surface



S1: You mean if we were to calculate the whole thing... I think you have to calculate each triangle separately, actually... or?

S2: But wouldn't it be the same even if it were... because there's a corner here... but it's still the same thing, right? (...)

S1: I think it turns out like this [draws, see Fig. 8.6].

S2: Yes! I see it!

S2 thus suggests that it would be possible to calculate the area of the rectangle without decomposing it into sub triangles. S1 does not agree and convinces S2 that

the triangles must be treated separately, by making a sketch of the house from another angle (Fig. 8.6). However, having considered all triangles (which are congruent since the extension building has the same width as the main building) S2 later concludes that the suggested approach would actually have worked.

S2: So is it 3500 times 2000, then?

S1: Yes, and then you have to divide by two, since it's a triangle.

S2: But the two of them together are 3500 times 2000 (...) so why do you divide by two, I mean, we can just take these two triangles together plus the other two together?

S2 [later concludes]: I could have done 3500 times 4000 [which equals the area of the rectangle]. I didn't think it would work at first, but in the end it turned out that it did.

The excerpt illustrates that the task of decomposing a 3D surface is far more challenging than decomposing a plane area. Decomposing a surface is a challenge as such, but the task becomes far more demanding when the aspect must be combined with spatial reasoning, (in this case, discerned by recognising the contrast between *measurements parallel to the horizontal the plane vs. measurements along sloping surfaces*)

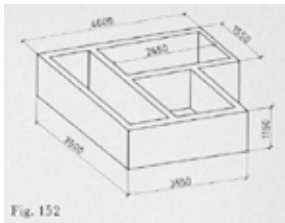
Summary Task type 3

Across the analysis of teacher interviews, textbook tasks and students' discussions, several critical aspects of material estimation tasks emerge. Mathematically, the tasks involve discerning different forms of multiplicative structures and the relationships between them, (for example length-based and area-based calculations). However, vocational mathematics tasks require more than basic calculations; they demand an integration of mathematical understanding with technical drawing literacy, spatial reasoning, and physical constraints. In addition to area calculations and multiplicative reasoning, solving material estimation tasks requires discerning how plan and façade drawings jointly represent a three-dimensional object (e.g. a roof), as well as how material properties influence rounding practices and, in turn, solution strategies. In this context, integer constraints are central, since material quantities must often be expressed as whole units, making rounding up a mathematically and practically significant step. Finally, accounting for waste involves percentage calculations, where students must discern the equivalence between subtracting a percentage and multiplying by a corresponding change factor.

Task type 4. Estimating volumes for filling and excavation

A task type closely related to the one above involves material estimations based on volume. For example, estimating the amount of filling material required or the volume of excavation masses to be removed. Such tasks often necessitate decomposing irregular shapes into standard forms, as well as making adjustments for volume changes due to processes such as mixing or compaction. Several interviewees confirm that ordering the correct volume of filling materials or calculating how many truckloads are needed to remove material, are critical issues in construction-work practices. A large number of volume-related tasks were found in the analysed textbooks

Figure 8.7 Task 246 (Schmale, 1960)



246. The foundation shown in Fig. 8.6 is to be made of concrete in the ratio 1:5. Calculate the required amounts of cement and aggregate (gravel), given that the wall thickness is 22 cm.

The textbook also specifies that for all of the tasks, involving concrete casting a 15% adjustment for waste should be included. Moreover, it is stated that the required amount of dry concrete components exceeds the volume of wet concrete by 40%.

Making sense of the task as a whole

Focusing on the task as a whole, including its' context, framing and desired outcome, the significance of accurate volume measurements is emphasised and an overall strategy is outlined:

VM5: And, like, when they order concrete... volume... it's for real. **If you order one cubic metre too little... it's really going to hurt!** You'll get so much trouble you wouldn't believe it!

V6: They have a construction drawing. Then they see the dimensions of the slab, so they know how much concrete is needed.. (...) You need to order concrete. Then you must be able to calculate, because it can't be two cubic

metres more than we need, or even worse, if it's too little. You can't correct that later.

V3: What they need is area and volume. That's basically the whole vocational part. (...) How many liters of concrete do I need to mix? (...) How much does a form hold? And so on. (...) We give them a task and say: **the area is this large. How many truckloads do you need** [to remove the soil]? (...) **Then we need to know the volume increase.** How much does the soil expand when we excavate it?

As exemplified above, several interviewees highlight the importance of accurate volume measurements, noting that the amount of concrete cannot be adjusted afterwards: 'If it's too little, you can't correct that later' (V6). Moreover, the statements illustrate an overarching strategy focused on by the participants. Starting from the construction drawings, relevant measurements are derived, enabling the calculation of the required volume: 'They have a construction drawing. Then they see the dimensions of the slab, so they know how much concrete is needed' (V6). It is also emphasised that the actual volume to be ordered or removed may differ from the calculated one, as materials such as soil and rock change in volume when excavated or blasted: 'We must know the volume increase. How much does the soil expand when we dig it up? How much more rock material will there be after blasting?' (V3).

Calculating the volume

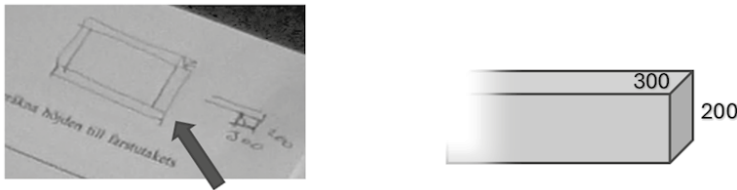
Volume decomposition into non-overlapping parts

Understanding *volume measurement as a three-dimensional extensive quantity*, that is, a measurable attribute representing the *amount of space occupied by an object*, are critical aspects of volume measurement, (which according to previous studies (e.g., Battista, 2007; Vergnaud, 1983) may be non-trivial tasks for primary and secondary students). *Decomposing an irregular three-dimensional shape into sub-shapes* clearly represents an even higher cognitive demand. Discussing the sources of students' difficulties, V6 identifies a common error in calculating the volume of, for example, a slab by multiplying the perimeter by the height and thickness, which results in double-counting the overlapping corner parts. The importance and challenge of being able to decompose a volume into non-overlapping parts, is thereby highlighted.

V6: We have a slab like this, it has length times width. Usually, the slab looks like this. [see below] There is a footing at the outer edge here (...) and then

we have the thickness of the slab, 100 mm. The footing here is 200×300 . And then the problem is... I can calculate the perimeter of this, but **often they count the corners twice**. You have to do this on the board several times before they understand. They have to **account for the full length and then subtract 300 mm from the short sides**.

Figure 8.8 Teachers' drawing, footing



This aspect (*the decomposition of a volume into non-overlapping parts*) is confirmed to be critical, in students' discussions of the exemplified task (see Fig. 8.6). The significance of discerning the *contrast between calculating volume as the total length multiplied by height and width, and the correct volume obtained by considering overlapping parts*, is highlighted in the excerpt below:

S2: [calculating the combined length of the walls] So then we can just multiply that by 1150 [the height].

S1: Yes, exactly, and then also multiply by 220 for the thickness. (...) **But the problem now, when we calculate the thickness, is that we're counting each corner twice!**

S2: **Right! We didn't think of that...**

S1: No... but could we not just ignore it? ...No, we can't do that!

S2: Then we have to think a bit (...) so for each corner we have to subtract 22... so the corners are 22 by 22... and then times 1150. [calculates $220 \times 220 \times 1150$ and subtracts all overlapping parts from the calculated volume]

S1: So that's the volume.

S2: **And now remember that you have to add 5 percent!** [calculates (total volume) $\times 1.05$]

The first suggestion made by the students is to calculate the volume by multiplying the total length by height and width. However, the error inherent in this approach is later recognised 'But the problem now, when we calculate the thickness, is that we're counting each corner twice!' (S2).

Another possible approach entails seeing the volume as the *outer volume of a solid minus the volume of the internal void*. Relating and comparing such alternative approaches may support a more advanced conceptual understanding of volume.

Volume conservation and volume change

The principle of *volume conservation under decomposition and rigid motion* is a critical aspect of the volume concept in general. In vocational practice it is critical to discern volume conservation under transformations. However, in vocational contexts it is also critical to discern the *change in volume of a given mass due to processes such as compaction or bulking*, or as a result of *temperature fluctuations, drying, or mixing*. These aspects are clearly emphasised in the excerpt below:

V3: You learn to think: OK, the area is this large, how much extra volume will I get? How many truckloads will I need? On this area, it's five truckloads, and when I excavate it, it becomes seven. That's something you always factor into the calculations. We have tables that show this, for example, **this type of soil increases by 20 percent, another by 5 percent. And when you blast rock, some volume increases can be as high as 80 percent.** So it depends.

It is evident that discerning that *the volume changes* under excavation or blasting work, and *dependence of the volume change on the excavated material* are critical aspects of construction-work volume tasks.

In a similar manner, when mixing concrete, it is essential to discern *the relation between the total volume of the individual components and the total volume of the mixed product*. Typically, the mixed volume is less due to the way smaller particles fill the voids between larger ones. In the task discussed by students, the volume change is approximated using a fixed ratio of 1:1.4 between wet (mixed) and dry (not mixed) concrete. In the excerpt below, this aspect turns out to be surprising for the students, and the way the ratio should be interpreted in the situation becomes a critical point of reasoning:

S1: **So it takes up less space when you pour in water?!** Ok. So this [calculated volume] is how much concrete we need to pour in, but **apparently we need 40% more** than that (...) so we have to multiply by 1.4 [calculates]. So this is the total amount we need (...) cement + gravel... so then we have to calculate how much is cement and how much is gravel (...) The ratio is 1:5, so that means 1/6 is cement and 5/6 is gravel.'

The student attempts to understand and explain the relation between the volume of the finished product and the volumes of the individual components required. To do so the *material behaviour* must be understood ('So it takes up less space when you pour in water?!') as well as how this is conceptualised mathematically within the specific situation ('this is how much concrete we need to pour in, but apparently we need 40% more, so we must multiply by 1.4'). In other words, the

student makes a *distinction between the volume to be poured into the form and the volume of the required materials*. Moreover, the *relation between these two quantities* is highlighted. As described in earlier examples, such calculations presuppose a solid understanding of *percentages*. In addition, in this task it is necessary to discern the *difference and relation between ratios and part-whole fractions* ('the ratio is 1:5, so then 1/6 is cement and 5/6 is gravel.')

Summary Task type 4

The analysis of teacher interviews, textbook tasks and students' discussions highlights several critical aspects of volume estimation in vocational mathematics tasks. As well as performing straightforward volume calculations, students must learn to decompose complex shapes into non-overlapping standard forms and apply principles of volume conservation. At the same time, vocational situations demand awareness of how volumes change through excavation, compaction, mixing or drying. Accounting for such changes require students to work with ratios, percentages, and multiplicative reasoning. Taken together, these aspects show that competence in volume estimation relies on both mathematical understanding and knowledge of materials and construction principles.

Task type 5. Constructing a right angle

The fifth task type selected for analysis concerns the construction of right angles. This type of task is described in several interviews as being carried out almost daily on construction sites, and it also appears frequently in the analysed textbooks. A number of complementary or interchangeable strategies are described in interviews and textbooks:

- Scaling a 3:4:5 triangle (applying the side–side–side similarity theorem).
- Calculating and constructing the hypotenuse of a right-angled triangle with given side lengths (using the converse of the Pythagorean theorem).
- Measuring and comparing the diagonals of a rectangle (drawing on the principle that equal diagonals constitute a sufficient condition for rectangularity).

Making sense of the task as a whole

As in the previous task types, an initial phase concerns making sense of the task as a whole. In this case, that involves discerning the intended outcome, namely perpendicularity, as well as the *required level of precision* and the *practical consequences of error*. It also involves recognising that *different methods may be more or less suitable* depending on scale, and the stage of the construction process.

Precision and amplification of errors

In the interviews, the practical importance of constructing right angles with high precision is strongly emphasised. The task is not merely to create an approximately right corner, but to ensure that the construction remains *correct across its full extent*. This highlights a critical *contrast between local accuracy at one point and accumulated deviation over distance*.

V1: 'One thing I've been thinking about is how to set out angles. There are **fixed measurements, sixty, eighty, and one metre for the diagonal, which gives ninety degrees** (...). They don't really understand why we do it, or why it's necessary... 'But we already have set squares!' ...'Yes, **but if you have a wall that's five metres long, you can't just use your small square that's thirty centimetres, the error will be huge at the far end.**

The excerpt points to the importance of using *proper methods and appropriately dimensioned tools* for constructing right angles, with as high an accuracy as possible. Students need to understand that the *suitability of tools and methods depends on properties of the construction*. For example, the importance of understanding that small tools such as set squares become insufficient for larger constructions is highlighted and explained: 'if you have a wall that's five metres long, you can't just use your small square that's thirty centimetres, the error will be huge at the far end' (V1). This aspect is emphasised even more clearly in the following excerpt:

V1: **It can't be off by even a millimetre down there at the bottom, because then it'll be four centimetres by the time you get up there at the top...** It has to be straight and true all the way up like this... 'but a millimetre isn't much, is it?' 'Yes, it is if you go up four metres, then the house ends up like this [shows a leaning house].'

In order to make sense of the task, students thus need to discern the *practical significance of precision*, as well as the aspect that *small angular or positional errors are*

amplified over distance. This way of reasoning implicitly draws on proportionality, slope, and similarity, even if these concepts are not explicitly named.

Combining and relating strategies

A second critical aspect concerns recognising *the availability of several mathematically related strategies for constructing right angles*, and their *complementary use in practice*. These include *cross-measuring*, applying the *converse of the Pythagorean theorem*, and *scaling the 3:4:5 triangle* (which, in turn, is proved using the Pythagorean theorem). This triangle is known in Sweden by many names, such as ‘the builder’s square’, the ‘carpenter’s square’, or the Egyptian triangle. The interviews indicate that vocational competence involves not only performing one method but also understanding *when and why different methods are useful*.

VM5: They understand how useful it is, that you can multiply the sides and get a larger triangle. We call it **the carpenter’s square**, that one is 60, 80, 100. (...) And of course, we also cross-measure, but that way they at least get as close to 90 degrees as possible.’

V6: The diagonal and so on, we use it all the time. I’ve got a project going on right now, their upper secondary project work, where they have to use it. They use the 3:4:5 (...) What we call the **builder’s square**. It has always been part of construction-work. (...) Sometimes you need to **scale it up** and use it in different sizes (...) Then I might use 60:80:100, or **divide or double them** (...) 1.20 and 1.60 (...) I just make it as big as I need it (...) If we can use **double the measurements or multiply them by three**, then we do that (...) So I think it’s really useful. I want it to be **as close to the actual measurement as possible**. And we also use **Pythagoras** (...) it’s essential when working with the diagonal. They’re starting their project work on Monday (...) And then **they’ll get measurements they’ve never worked with before. They need to be able to find the diagonal**, and I’ll come over once they’ve set up the profiles (...) And then everything has to match (...) within two or three millimetres. (...) It’s actually quite fun when it works, really enjoyable!

These excerpts point to several critical aspects. Students need to discern the *preserving the right angle when scaling of the 3:4:5 triangle*, a principle that draws on proportionality and triangle similarity. They also need to discern the *improvement of practical precision due to increased size of the triangle*. In addition, they need to recognise the *diagonal as a mathematically meaningful control measurement*, rather than merely an extra length to be checked. The interviews further show that the *relationships between the strategies* themselves may be an important aspect:

V3: We use it all the time, almost every other day when they're out working, so they can get things straight... at right angles. It comes up in many different situations, and we always **start with the Pythagorean theorem**. That's our starting point. And once they've got that, then we move on to this one (the 3:4:5 triangle). But we've actually noticed that if we **reverse the order and start with this one first [3:4:5], then they understand the Pythagorean theorem much more easily**. (...) We show the ratios first... and then we write down Pythagoras and show them that, well, **it's the same relationships, it's just written with letters instead**. And then they make the connection! After that they can calculate it, and everything just flows. But this one (...) they basically know it from day three. 3:4:5. (...) What I've noticed, though, is that they often struggle when they have to make a square or a rectangle and get it truly right-angled. They know the side lengths, but calculating the diagonal, that's where they have difficulties. (...) However, once they've done it practically a few times, and have **moved things around themselves** until the measurements are correct, then they begin to understand. They start to **see the whole**, and then they begin to understand how to calculate it.

This excerpt highlights that students need to discern both the *practical and conceptual relation between the 3:4:5 triangle and the Pythagorean theorem*. In other words, they need to recognise the scaled 3:4:5 triangle as a *particular case of a more general mathematical relation*. The excerpt also suggests that practical engagement may support conceptual understanding, as students come to 'see the whole' by physically adjusting a construction until the diagonals or sides correspond correctly.

Applying the method in practice

Once an appropriate strategy has been selected, the next part of the task involves physically setting out or checking the right angle. At this stage, further critical aspects arise.

One such aspect is the relation between abstract calculation and practical adjustment. As indicated in V3's description, students may know the required lengths but still struggle to move and align the construction physically until the measurements fit. This suggests that vocationally proficient task solving requires coordination between numerical result, measuring practice, and spatial judgement. Finally, as in other vocational tasks, students need to discern acceptable tolerances. In practice, constructions are not evaluated in purely theoretical terms, but in relation to permissible deviations 'Everything has to match (...) within two or three millimetres. (...) It's actually quite fun when it works, really enjoyable! (V6)'.

Understanding *what counts as 'close enough,' and when greater precision is necessary*, is therefore also a critical aspect of constructing right angles in vocational settings.

Summary Task type 5

The analysis of interview data indicates that constructing a right angle in vocational practice requires more than recalling a geometric rule. Students need to discern that several strategies may be used, how they are related and when they are appropriate. In addition, vocationally proficient task solving involves coordinating mathematical with measuring practice and an understanding of tolerances.

Summary Study 2

Taken together, the study shows that vocational mathematical competence involves recognising task structures; applying and adapting general procedures; understanding mathematical concepts and structures, and coordinating multiple forms of knowledge across mathematical, spatial, material, practical, and disciplinary domains. While each task type involves identifiable mathematical concepts and procedures, successful problem-solving depends on how these procedures are integrated with practical and physical constraints; spatial reasoning; and knowledge of materials, tools, and workplace norms.

A common feature across the task types is the central role of multiplicative relations, typically closely intertwined with geometric concepts. These relations are coordinated in different ways depending on the specific configuration of the task, the properties of the material, and the chosen solution strategy. Another key finding is the significance of spatial reasoning and the interpretation of representations. Many tasks require coordinating information across two- and three-dimensional representations, particularly when working with technical drawings. This spatial dimension also interacts with geometric reasoning, for example in understanding slope, similarity, and perpendicularity. The results also highlight the importance of making sense of the task as a whole including discerning relevant conditions and constraints. In authentic vocational contexts, such aspects are often not explicitly given but must be inferred from the situation. Moreover, recognising a task as an instance of a more general task type enables the use of general solution strategies, although these must be adapted to situational conditions. The task does thus not only involve aspects of a general procedure but also aspects specific to the task.

9. Result Study 3

Study 3 focuses on afforded learning opportunities, in Swedish mathematics textbooks for vocational students in the first year of upper secondary education in Sweden. The study aims to explore learning opportunities regarding the geometric concept of similarity, in these textbooks. Hence, the following research question is addressed:

What different learning opportunities are afforded through the ways in which similarity is presented in mathematics textbooks for vocational students?

In this study the section on the topic of similarity, was analysed using variation theory as an analytical framework. Mathematics textbooks for vocational education at upper secondary level (Ma1a), from three publishers were investigated. The selected textbooks were studied in detail, task by task, through an in-depth analysis. Patterns of variation were recognised, and the corresponding aspects, according to VTL, were identified. In this chapter learning opportunities for each Textbook are presented. The result is then summarised in a table at the end of the section.

All the selected textbooks are in Swedish and any quotations given here have been translated to English by the author. In the text critical aspects are highlighted in *italics* and the patterns of variation are highlighted in **bold**.

Textbook A

The subsection on Similarity is part of the chapter Geometry, which also contains the sections ‘Circumference, Area and Volume’ and ‘Right angled triangles and Trigonometry’. ‘Similarity and Congruence’ is a subsection of ‘Angles, Similarity and Symmetry’

Summary of learning opportunities in Textbook A

In this section several aspects are made possible to discern through patterns of variation. Conditions for similarity are opened up, both through **contrast** and

through **generalisation (conceptual variation)**. Similar figures are contrasted to non-similar figures, depending on either congruent angles or proportional sides (**non-concept variation**), making both *congruent angles and proportional sides as conditions for similarity* possible to discern. Sufficient conditions are **contrasted** to non-sufficient conditions, and essential aspects are separated from non-essential aspects (e.g. rotational position) through **generalisation (concept variation)**. It is thereby possible to discern that *a combination of uniform dilation, translation, rotation and reflection, preserves similarity*. It is also possible to discern *necessary conditions for similarity* of arbitrary figures, and of *triangles as a special case*. *Conditions for similarity and conditions for congruence* are **contrasted** and related. All of these aspects are elaborated throughout the set of tasks, and used in task solving in such a way that the complexity is increased through **fusion** of previously afforded aspects (**hierarchical variation**) rather than (as in Textbook B) by adding arbitrary aspects to a repeated procedure.

Defining similarity and congruence

In Textbook A Similarity is defined in the following way:

Explanation:

The large figure (in the example) is an enlargement of the smaller. This means that the figures have the same shape but different sizes. In mathematics we say that the figures are similar.

Definition:

For all similar polygons the following holds:

- Corresponding angles are of the same size
- The ratio of corresponding sides, are of the same measure

For triangles it is enough to ensure that one of the conditions is fulfilled to know that the triangles are similar

Congruence:

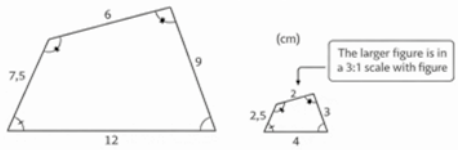
If two figures are identical in both shape and size, they are said to be congruent. This also applies if the figures are rotated and reflected.

This a general definition of similarity and congruence, according to a static approach (e.g. Birkhoff, 1932). The AAA and SSS similarity theorems are given. However, the SAS theorem is not mentioned. It is not explained how corresponding sides relate to corresponding angles.

Figure 9.1 Adapted and translated from a Swedish mathematics textbook
(*Olofsson & Gerholm, 2021, p 313*)

Similarity and Congruence

Similarity Here, we have drawn two quadrilaterals. The larger is an enlargement of the smaller figure. This means the figures have the **same shape** but different sizes. In mathematics, we say the figures are similar.



Similarity

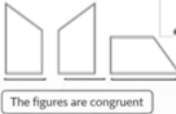
For all similar polygons, the following holds true:

- ▶ **Corresponding angles** are congruent
- ▶ The ratio of **corresponding sides** is the same


$$\frac{6}{2} = \frac{7.5}{2.5} = \frac{9}{3} = \frac{12}{4} = 3$$

For triangles, it is enough to check one of these conditions to determine that the triangles are similar.

Congruence If two figures are identical in both shape and size, then we say they are congruent. This also applies if we rotate or reflect the figures.



Example: Which of the figures are **congruent** and which are **similar**?



Solution: Triangles B and C are **congruent**. They are exactly like each other. Triangles A, B and C are **similar** because corresponding angles are congruent.

Congruent figures are also similar

Introduction to similarity, Textbook A

Initially a quadrilateral and an enlarged similar quadrilateral are presented. The scale factor is given. Conditions for similarity are exemplified using the figure. The ratio of corresponding sides is calculated. Since the particular pair of corresponding sides and their lengths **vary** while the ratio is **invariant**, it is possible

to discern the relationship between the covarying quantities, which means that *the proportional relation between corresponding sides in similar figures* is focused.¹¹

Then congruence is defined. In the picture beside this, three congruent quadrilaterals are shown. The second and the third are obtained by a rotation and a reflection of the first one, implying that shape and size are **invariant** while *orientation* vary. This constitute a **generalisation** of *shape* since the essential aspect is invariant while non-essential aspects such as transformations that preserves congruence, vary. Using Gu's et al (2004) terminology this is an example of a **concept variation**, through which *transformations are possible to discern as non-essential aspects*. However, since size and shape are **invariant**, no contrast is afforded here, and students' attention is, therefore not yet directed to aspects which are necessary conditions for congruence.

In the following example (fig 9.1), however, four triangles are shown. Two of them congruent (B, C) and three of them are similar (A, B, C). Between A and B, size and orientation vary while shape is invariant (similar). Between B and C, shape and size are invariant (congruent) while orientation varies, and in D the shape varies (not similar) compared to the other three triangles. This means that shape, orientation and size vary in a way that makes it possible for students to discern *conditions for similarity and congruence* (through **contrast (non-concept variation)** and **separate** them from non-essential aspects, i.e. *similarity preserving transformations* (through **generalisation**).

More precisely, congruent angles is opened up as an essential dimension of variation in relation to both similarity and congruence. The example constitutes a **contrast (non-concept variation)** which direct focus to the aspect *congruent angles as a condition for similarity* since the triangles are similar or non-similar, depending on whether the angles are congruent or non-congruent. It is also possible to relate and *distinguish the concepts congruence and similarity* through **contrast** in conditions, where *scale factor* is the dimension of variation that separates congruence and similarity

In this first example, no calculations have to be done, which makes it possible to focus only on defining aspects of similarity. Moreover, as congruent angles is a

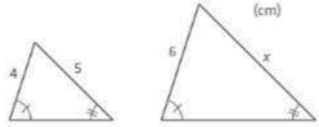
¹¹ From a variation theory perspective, it is somewhat difficult to determine precisely what is made possible to discern through this example. As only a single example is provided, the ratio does not emerge through contrast, nor is it made visible as a necessary condition for similarity through comparison with non-similar figures. However, the example may still make proportionality possible to discern as a structural feature of similarity, emerging through the invariance of the ratio against varying side lengths.

sufficient condition for similarity of triangles, proportional sides do not need to be taken into consideration, which limits the number of essential dimensions of variation in this example.

Figure 9.2 Adapted and translated from a Swedish mathematics textbook

(Olofsson & Gerholm, 2021, p 314)

Example: Calculate the length of the side marked with x .



Solution: The triangles have congruent angles marked. Since the angle sum in each triangle is 180° and the triangles match in two angles, they must also match in the third angle. Therefore, the triangles are similar. This means the ratio of corresponding sides is the same.

Method 1

$$\frac{6}{4} = 1,5 \quad \text{The larger triangle's sides are 1,5 times as long as the smaller triangle's sides}$$

$$1,5 \cdot 5 = 7,5 \quad \text{Side marked } x \text{ is 1,5 times as long as the side that is 5 cm}$$

Method 2

$$\frac{x}{5} = \frac{5}{4} \quad \text{The ratio of corresponding sides is the same}$$

$$x = 7,5$$

Answer: Side marked x is 7.5 cm.

In the second example (Fig. 9.2), proportionality of sides in similar triangles, is used to calculate a missing length. In this example the triangles have the same orientation, which makes it possible to focus only on the enlargement and how an unknown side might be calculated. Two different methods are exemplified, drawing students' attention to the fact that the same result may be obtained using different methods. This could be seen as an example of **procedural variation** (same problem, different methods). It is not stated explicitly in the problem that the triangles are similar, which motivates a discussion about conditions for similarity, aspects that have been possible to discern through patterns of variation and invariance in the previous example.

The introduction thus demonstrates two different procedures to find missing lengths in similar figures, making *procedure* a dimension of variation. The aspect

congruent angles as a condition for similarity is opened up by **contrasting** similar and non-similar triangles depending on congruent vs. not congruent angles (**non-concept variation**). A **contrast** is made between the concepts *similarity* and *congruence*. The similarity concept is **generalised** by varying non-essential aspects, i.e. transformations, preserving similarity (**concept variation**).

Figure 9.3 Adapted and translated from a Swedish mathematics textbook

(Olofsson & Gerholm, 2021, p 314)

Starter

Which of the rectangles are

a) similar b) congruent

LEVEL 1

8248 The figures are similar. Calculate the length of the side marked with x .

8249 Are the triangles

a) similar b) congruent

Answer: Side marked x is 7.6 cm.

Tasks in Textbook A

Figure 9.3 shows the first three tasks in the section. The first task resembles the first example in the introductory part. Similarity and congruence should be determined. Unlike the example, though, this task compares rectangles of different proportions and sizes. The task differs from the example in that all angles are obviously right angles; this aspect thus does not vary in this task. Instead, *proportional corresponding sides* becomes the essential aspect to determine similarity and congruence. Similar rectangles are **contrasted** to non-similar rectangles, depending on proportional vs. non-proportional sides which makes it possible to distinguish aspects that define similarity (in this case *proportional sides*) from aspects that does not affect similarity (in this case *orientation*) which constitutes a **generalisation (concept variation)**.

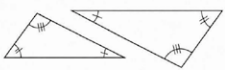
In Task 8248 two similar rectangles are shown and the fact, that similarity implies that the ratio of corresponding sides is constant, is used to calculate the missing length. In this task similarity is given, and the rectangle is just dilated and translated horizontally, which makes it possible to concentrate on the *scale factor* and the *procedure to calculate an unknown side*, without considering other aspects of the similarity concept.

In Task 8249 congruence and similarity should be determined in triangles that are reflected. This means that, conditions for similarity and congruence are once again focused. A **contrast** between congruence and similarity is introduced as the triangles are similar but not congruent. Either *congruent angles* or *proportional sides*, can be used to determine whether the triangles are similar, while equal size as a necessary condition for congruence, is made visible through **non-concept variation**. Furthermore, *reflection* must be discerned as a non-essential aspect. All of these aspects were possible to discern in previous tasks.

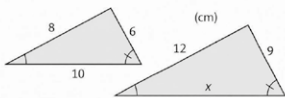
Figure 9.4 Adapted and translated from a Swedish mathematics textbook

(Olofsson & Gerholm, 2021, p 314)

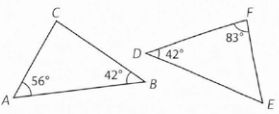
8250 Explain why the triangles are similar.



8251 The figures are similar. Calculate the length of the side marked with x .

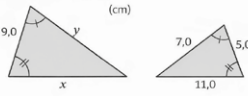


8252 Determine if the triangles are similar.

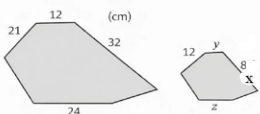


8253 The figures are similar. Calculate the length of the side:

a) marked with x
b) marked with y



8254 The polygons are similar. Calculate the length of the sides marked with x , y , and z .



In Task 8250 in Figure 9.4 similarity has to be established through congruent angles. Once again *congruent angles as a condition for similarity* is focused without drawing attention to other aspects such as ratio. The similarity transformation *rotation* is illuminated as a *non-essential aspect* through **concept variation**.

In Task 8251 the missing length, should be calculated. This task resembles 8248, except that all of the side lengths, except one, are given. An ability to identify the corresponding side among a few possible options, is thus required, making the identification of corresponding sides slightly more challenging. Also in this task, similarity is given and orientation is **invariant**, which implies that conditions for similarity are not focused in this task.

In Task 8252 similarity should be determined. In **contrast** to previous tasks the triangles are not similar since angles are not congruent, highlighting the aspect *congruent angles as a condition for similarity*. The triangles are both *reflected* and *rotated*. This task includes a **contrast** between visual perceptions and a *formal definition*, since the triangles looks similar, but are non-similar, according to the definition.

When considering Task 8249, 8250 and 8252 **as a sequence**, patterns of variation could also be detected. All of the tasks involve determining similarity in pairs of triangles, using the condition *congruent corresponding angles*. The three pairs of triangles are similar (8249, 8250) or non-similar (8252) depending on congruence of angles. This means that, a **contrast** is provided, implying that *congruent angles as a sufficient condition for similarity* (AAA) is opened up as a dimension of variation. Moreover, in the three tasks, the triangles are reflected (8249), rotated (8250) and finally both reflected and rotated (8252), which could be seen as a *hierarchical variation*, making it more and more difficult to see how the triangles might be similar and disregard the transformations. Other aspects are kept invariant in the set of tasks. This set of tasks, thus affords rich opportunity to discern that *congruent angles implies similarity* (AAA) and that certain *transformations preserve similarity*.

In Task 8253, students are asked to determine a missing length in a reflected, similar triangle by using the *proportionality of corresponding sides*, following the same *procedure* as in, for example, Task 8248. In the preceding tasks, students are given opportunities to experience similarity in reflected triangles through **generalisation**, as well as to identify and calculate corresponding sides in similar triangles that differ only by scaling. This means that the *procedure* is invariant compared to previous tasks while other aspects also need to be discerned simultaneously.

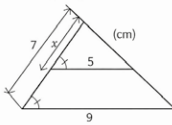
In Task 8254 a missing length should be calculated in a pair of similar, irregular polygons. Similarity is given. Compared to previous tasks this implies a **generalisation** as the *type of shape*, for which the similarity concept applies, varies (not only triangles and rectangles). Like in several other tasks *proportionality of sides* should be used as an *implication of similarity*, and the proportion has to be solved.

In Task 8255 (Figure 9.5) one of the similar triangles is embedded within the other, having one angle in common. This angle has to be used to justify similarity. It could be noted that while the task introduces the idea of a triangle that is contained inside another in this way, the triangles are neither rotated nor reflected.

Figure 9.5 Adapted and translated from a Swedish mathematics textbook
(Olofsson & Gerholm 2021, p 315)

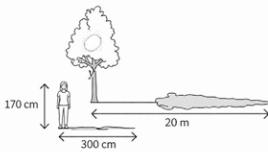
LEVEL 2

8255 In the figure, you can see two triangles.



- Are the triangles similar? Explain.
- Calculate the length of the side x .

8256 Gunilla is 170 cm tall. One day she casts a shadow that is 300 cm long at the same time as the shadow of the tree she is standing next to is 20 meters wide. How tall is the tree?



LEVEL 3

8257 Nina says that since all rectangles have equal angles, all rectangles are similar. Is Nina correct? Motivate.

8258 Do two triangles have to be congruent if their angles agree? Motivate.

8259 Calculate the length of segment x .



Task 8256 is the only context-based task in the section. It is difficult to determine the opportunity for students to use their understanding of the real-world situation in order to experience variation, when considering the situation. An important aspect, connected to the context, is the position of the light source¹², which must be ascertained to be situated sufficiently far away to assure that the rays are *parallel* enough, which in turn is essential when justifying similarity. If students discern this aspect, and also attends to the other conditions, such that if the flagpole is *vertical* and if the ground *horizontal*, as features in dimensions of possible variation, they will get an opportunity to experience a **contrast**. The triangles might be similar or non-similar depending on these conditions. This pattern of variation might afford an opportunity to develop a more advanced understanding of the concept. However, the task is probably most easily solved by students who ignore all of these aspects and assume the only relevant triangles to be similar. In this case

¹² This task illustrates how introducing a real-life context brings in aspects that are essential to the situation, but which can remain in the background and still allow the task to be solved.

students have previously been afforded the opportunity to discern the essential aspect, *congruent angles as a sufficient condition for similarity* (AAA) and to practice a *procedure where proportional sides is used to calculate missing length*.

In Task 8257 *conditions for similarity of rectangles*, are focused and the **contrast** to *conditions for similarity of triangles*, could be discerned. Students are asked to determine whether the fact that all rectangles have congruent angles implies that all rectangles are similar. (In previous tasks e.g. 8252, students had the opportunity to see that congruent angles is a sufficient condition for similarity of triangles). This means that, this task introduces a **variation** which makes it possible to discern *congruent angles as a sufficient condition* for similarity of triangles in **contrast** to a *non-sufficient condition* for similarity of other figures. (There are different ways of reasoning in this task. Either conditions for similarity, stated in the introductory section, might be referred to directly, or a *variety* of rectangles might be imagined, considering a *potential variation* of shapes of rectangles. In this way angles are kept invariant while shape varies, which makes it possible to discern that a sequence of equal angles in two rectangles still permits the shape to change.)

In Task 8258 *conditions for congruence in triangles* are further investigated in a similar way as in 8257. In this task a **contrast** between *conditions for congruence and for similarity* are foregrounded.

In Task 8259, a missing length should be calculated in a triangle, which is both *superimposed, reflected and rotated*, (sharing sides and an angle). Similar triangles have to be identified and similarity has to be established through *congruent angles* (AAA) Then the *procedure*, to use the *proportionality of sides as an implication of similarity* in order to find a missing length, has to be applied. This means that students have to regard several aspects **simultaneously**, in order to solve the task. In this section these aspects have been possible to discern, one at a time in previous tasks.

There is an exception, though. Corresponding sides have to be identified. However, in this task, finding the corresponding sides is not trivial, because of the complexity of the figure. It can for example not be done visually by inspection of side lengths, (since two of the sides are nearly of the same length). The opportunity to discern the necessary aspect, *corresponding sides* is however not afforded either by explanations or as a dimension of variation in this section.

There are seven tasks in this section, in which a missing length should be found in similar figures. When **considering these tasks as a sequence** (8248, 8251, 8253, 8254, 8255, 8256, 8259) patterns of variation could be seen. The same basic *procedure* could be used in all of the tasks. Meanwhile, the conditions are changed in a systematic way, by successively making the geometric configuration more

complex, in order to pose more and more challenging problems (**hierarchical variation**). For example, in one of the first tasks 8248, a missing length should be calculated. Similarity is given as a precondition, the figures have the same mutual orientation, and the only values given, are the ones that should be used. Task 8251 is almost identical, but in addition, superfluous values are given. In 8253, similar triangles are reflected. In 8255, the triangles are superimposed (sharing sides and angle) and similarity should be established. In 8259 the similar triangles are superimposed, reflected and rotated, compared to each other. It is therefore not obvious why and how the triangles are similar. Similarity, thus have to be established by means of congruent angles, an aspect that was afforded through patterns of variation in previous tasks (e.g. introduction, Figure 9.1, and 8250, 8252, Figure 9.4).

Afforded aspects in Textbook A

Opportunities to discern certain aspects are described below.

Conditions for similarity

Congruent angles is stated to be a *condition for similarity*. Triangles are pointed out as a special case, for which congruent angles is a sufficient condition and implies proportionality of lengths (introduction, Figure 9.1). Both in examples and in tasks, congruent angles, in **contrast** to non-congruent angles, is used to distinguish between similar and non-similar triangles (**concept variation**) (introduction, Figure; 68249 8250, 8252 Figure 9.3; 9.4). The condition is also made possible to discern as a *non-sufficient condition for similarity of rectangles*, in **contrast** to triangles (8257, Figure 9.3). In some of the most challenging tasks, at the end of the section, ‘congruent angles’ is a condition that is necessary to use in order to find similar triangles as sub-shapes of complex figures (8259, Figure 9.4).

In Textbook A, the aspect *proportional sides as a condition for similarity* is opened up as a dimension of variation. Proportional sides in **contrast** to non-proportional sides have to be used in order to determine similarity (starter, 8249, Figure 9.3). *Proportionality and scale-factor* are also focused as aspects that **separate** the concepts *similarity* and *congruence* (8249, 8258, Figure 9.3; 9.5). It is also possible to discern *proportionality as an invariant relation between pairs of corresponding sides*, and use this to solve missing value problems.

The ratio of corresponding sides is stated to be constant in similar polygons. Examples are given in which the ratio is **invariant** for different sets of

corresponding sides (introduction, Figure 9.1). It is thus possible to discern and use the fact that, *similarity implies invariant ratio, for different sets of corresponding sides in a pair of similar figures*. The condition is used in many of the tasks, in order to calculate missing length in similar figures.

Transformations that preserve similarity

Transformations such as *rotation* and *reflection* are introduced at the beginning of the section, and focused in such a way that it is possible, and necessary, to discern that they do not affect similarity. This means that, orientation **varies** while shape is **invariant**, which implies that essential characteristics of the similarity concept are explored by differentiating them from non-essential features (**generalisation or concept variation**) (e.g. introduction, Figure 9.1; 8250, 8252, 8253, Figure 9.3). These transformations are thus focused as non-essential aspects, which is important in order to **generalise** the similarity concept. However, neither reflection nor rotation are opened up as dimensions of variation, in such a way that it is possible to discern values within the dimensions (e.g. ‘rotated 90 degrees clockwise’). In order to determine similarity, this is not necessary either. In some of the tasks, however, corresponding sides have to be identified in complex figures where it is not intuitively obvious which sides that correspond (8259, Figure 9.5). Corresponding sides could be identified by seeing the particular transformations needed to map one triangle on the other. Then, reflections and rotations have to be discerned, not only as non-essential, but as essential aspects, which is not afforded in this section. That approach is, however, not the only possible way to find corresponding sides.

Corresponding sides

The concept *corresponding side* is present to a large extent in the section but not opened as a dimension of variation in a systematic way. The concept is not even discussed or defined. Corresponding sides have to be identified in many of the tasks. In some of them, only intuitive ideas are necessary. The corresponding sides could be identified as sides in corresponding positions, or as the longest sides in both figures, etc. Throughout the section, finding corresponding sides becomes increasingly tricky, since rotations reflections, overlap etc. are progressively introduced (e.g. 8259, Figure 9.5). This might have given students an opportunity to develop an intuitive idea into a more formal understanding, through **procedural variation**. However, students do not get any particular support in doing so. Moreover, disparate ways of experiencing the concept seems to be suitable in

different tasks¹³, which probably makes a successive development of the concept difficult..

Procedure

The way in which the sequence of tasks in this section, is structured, resembles Gu's et al. (2004) description of **hierarchical variation**. Features of a certain kind of tasks are systematically changed, into subsequent more challenging problems, by making the geometric configuration more and more complex in a systematic way. Moreover, multiple solution methods are demonstrated in such a way that *procedure* is opened up as a dimension of variation through **contrast** (introduction, Figure 9.1) This is an example of **procedural variation** (Gu et al. 2004)

Textbook B

In Textbook B, the section on Similarity is a part of the chapter Geometry, which also contains the sections 'Circumference and Area', 'Area and Volume', and 'Angles and Trigonometry'. Similarity is the second subsection of 'Scale and Similarity'. The section starts with an introductory explanation Figure 9.6.

Summary of learning opportunities in Textbook B

The section provides some opportunities to learn one single procedure to solve a certain kind of similarity task. This involves discerning that, *if figures are similar, then the ratio of corresponding sides is invariant* and could be calculated if one pair of corresponding sides is given. A procedure for finding a missing length in similar figures, when three side lengths are known, is demonstrated and practised: first determining the ratio of corresponding sides (the scale factor) and then using it to calculate the missing length. This implies discerning *proportionality as a relation between pairs of sides, scale factor as determined from one pair of corresponding sides*, and how this relation can be used to determine a missing length. The complexity of tasks increases somewhat throughout the section, but not in a systematic way (c.f. hierarchical variation). This means that, different aspects are added to a particular task variant, but since each aspect only appears in one task, the aspects are not opened up systematically through contrast, generalisation and finally fusion. The opportunity to separate aspects that define similarity (congruent angles and

¹³ Different ways of identifying corresponding sides: in task 8251-obvious since orientation is invariant, in 8253- determined by seeing the transformation, in 8259- defined as facing equal angles

proportional sides) from aspects that do not affect similarity, such as position, orientation and colour, is very limited, since those aspects are not focused on through systematic variation, and almost all of the similar figures, have the same mutual orientation. This means that, no generalisation of the similarity concept in relation to orientation, is afforded, and necessary conditions for similarity are not opened up as dimensions of variation, since there is no opportunity to contrast similar and non-similar figures.

Defining Similarity

In Textbook Ba Similarity is defined in the following way:

The Quadrilaterals (exemplified in Fig. 9.6) have the same shape. They are similar, which means:

- Corresponding angles are equal (same shape)
- The ratio (quotient) of corresponding sides are equal

Triangles constitutes a special case. The sum of all angles are equal. Two triangles are therefore similar if two angles are equal (and thus also the third)

The definition is related to one specific example, and therefore to some extent lacks **generalisation**. There is no connection between, or explanation of, the concepts corresponding sides and corresponding angles. The description of conditions for similarity of triangles only concerns the AA similarity theorem. It is pinpointed that triangles constitute a special case. However, the fact that two equal angles is a sufficient condition for similarity of triangles, is explained with ‘constant angle sum’, (which obviously applies to all polygons) rather than by the fact that each of the two conditions separately implies similarity of triangles, which is not the case for other polygons.

Introduction to similarity, Textbook B

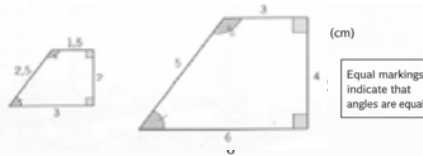
In the first example, in the introduction, a quadrilateral and an enlarged similar quadrilateral are presented. The scale factor is given and the ratio of corresponding sides is calculated. Since the particular pair of corresponding sides and their lengths vary while the ratio is invariant, it is possible to discern the relationship between covarying quantities, which means that *the proportional relation between corresponding sides in similar figures* is in focus. It is thus possible to discern *proportionality of sides as a consequence of the figures being similar*¹⁴.

¹⁴ This example is similar to the introduction in Textbook B. See footnote 10

Figure 9.6 Adapted and translated from a Swedish mathematics textbook
(Alfredsson, et al., 2011 p 248)

Similarity

A quadrilateral is enlarged so that all sides are doubled



The larger quadrilateral is an enlargement with scale factor 2:1.

Compare two corresponding sides to see that the ratio between them is

$$\text{the same: } \frac{5}{3} = \frac{4}{2} = \frac{3}{1.5} = \frac{5}{2.5} = 2$$

Similarity:

The quadrilaterals have the same shape; they are similar, which means

- Corresponding angles are equal (same shape)
- The ratio between corresponding sides is equal

Special case:

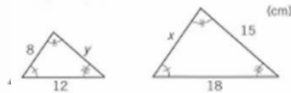
Triangles are a special case. The sum of angles is the same.

Two triangles are therefore similar if two angles are equal (and thus also the third)

5316

The triangles are similar.

Determine x and y .



Similarity gives that the ratios between corresponding sides are equal.

$$\frac{18}{12} = 1.5, \text{ i.e. the sides are 1.5 times longer in the right triangle}$$

which gives that:

$$x = 1.5 \cdot 8 \text{ cm} = 12 \text{ cm} \quad (\cdot 1.5 \text{ since } x \text{ is longer than } 8)$$

$$y = 15/1.5 \text{ cm} = 10 \text{ cm} \quad (/1.5 \text{ since } y \text{ is shorter than } 15)$$

Answer: $x = 12 \text{ cm}$ and $y = 10 \text{ cm}$.

In the solved example 5316, the ratio of corresponding sides is calculated and used to find missing length in similar triangles. One side in each triangle is calculated, which implies that the corresponding side should either be *multiplied or divided* by the scale factor, which constitutes a **variation**. According to the example, this depends on whether the corresponding side is *shorter or longer* than the unknown

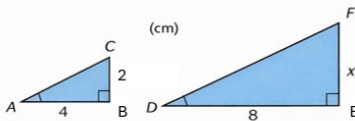
side. The **covariation** creates a connection between *shorter or longer* and *multiplication or division*, in a way that rather direct focus to a ‘memory-rule’ than to mathematical relations. The memory rule is, moreover, misleading since it only applies to ratios above 1.

The introduction thus makes it possible to learn that, once similarity is guaranteed, the ratio of corresponding sides is constant and can be calculated through one arbitrary pair of corresponding sides. A procedure to find a missing length in similar figures, is demonstrated, and focus is directed to an intuitive ‘memory rule’.

Figure 9.7 Adapted and translated from a Swedish mathematics textbook

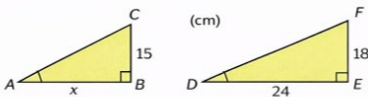
(Alfredsson, et al., 2011, p 249)

5317 The triangle on the right is an enlargement of the one on the left. The triangles are similar.



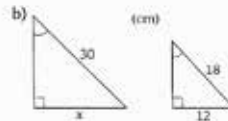
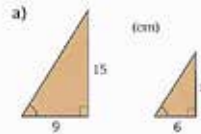
- a) Which side in the right triangle corresponds to side AB in the left triangle?
- b) How many times longer is the base of the right triangle compared to the base of the left triangle?
- c) Determine the length of side x.

5318 The triangles are similar.



- a) Which side in the right triangle corresponds to side BC in the left triangle?
- b) How many times longer is the altitude of the right triangle compared to the altitude of the left triangle?
- c) Determine the length of side x.

5319 The triangles are similar. Determine x.



Tasks in Textbook B

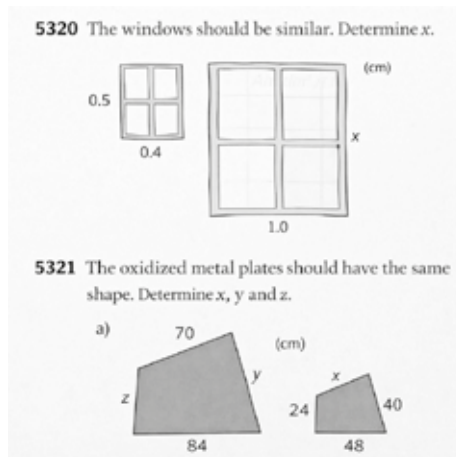
Figure 9.7. shows the first three tasks in the section (5317-5319). In all of the tasks, a missing length in similar triangles, should be calculated, in the same way as in the example in the introduction. The *procedure* is thus **invariant**. What varies between tasks is the values (scale and side-lengths), and the degree of guidance. According to Gu’s et al., (2004) framework, this type of variation could be regarded a type of **procedural variation**, in which the procedure is invariant while values vary. The procedure involves aspects such as *proportionality as an invariant relation between pairs*

of sides, the scale factor as determined from one pair of corresponding sides, and how this relation can be used to determine a missing length¹⁵. All of the triangles are right-angled and orientation is invariant. Similarity is given and emphasised by the colours of the triangles, which implies that congruent angles, does not have to be considered as a condition for similarity. Moreover, *colour* might falsely be discerned as a dimension of variation linked to similarity.

In Figure 9.8, the next two tasks are shown. In these tasks there is almost no changes made compared to previous tasks, except that the figures are no longer right-angled triangles. Since similarity is given and the **same procedure**, as in the previous tasks, may be applied, the fact that similarity of triangles is actually a special case, is not illuminated. Context is indeed added to these tasks, but no attention need to be paid to the context, in order to solve the tasks. (It does not matter for the solution whether the figures are stated to represent a metal plate or a window). The same **procedural variation** as in Task 5317-5319 is thus afforded.

Figure 9.8. Adapted and translated from a Swedish mathematics textbook

(Alfredsson, et al., 2011 p 249)



In Task 5321, more than two pairs of sides are given, which to a greater extent than in previous tasks, makes *corresponding sides* a potential dimension of variation. However, since the figures are not reflected nor rotated, it is not necessary to separate the concept from the idea of a side with a corresponding placement and similar orientation.

¹⁵ It could however be noted that the method suggested in the introduction rather introduces memory rules (see introduction)

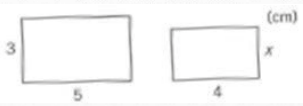
Figure 9.9 shows the last tasks in the section. The **same procedure** as in previous tasks can still be used. However, a few more aspects are introduced that have not been present earlier.

Figure 9.9. Adapted and translated from a Swedish mathematics textbook
 (Alfredsson, et al., 2011, p 249)


5322 A vertical flagpole casts a 21,1 m long horizontal shadow on the ground. At the same time a 0,70 m high vertical stick casts a horizontal shadow of 0,82 m.

How tall is the flagpole? (Draw a figure with two similar triangles.)

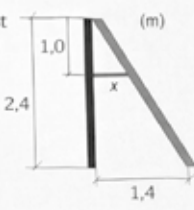
5323 True or false?
 "Similarity gives that x is 80% of 3 since 4 is 80% of 5."



5324 Determine the unknown sides x and y .



5325 Between a wall and a post a shelf is to be mounted according to the figure. What width, x , should the shelf have?



In Task 5322 a shadow is formed by sunlight, falling on an object, giving rise to similar triangles. (Regarding the opportunities to discern variation, see task 8258 above) Since no picture is provided here, interpreting the text and making a correct drawing, might be a challenge (but not related to the similarity concept). Once the correct picture is drawn, the same method as in the previous tasks, could be applied.

In Task 5323 the ratio of corresponding sides is given in percent. It is thus possible to discern that *ratio (or scale factor) could be denoted in different ways*, (a percentage as well as a quotient). This is the only task in which this aspect is considered.

Task 5324 is the only one in which the similar figures are reflected (rotation does not occur in any task). In relation to previous tasks this could be seen as a form of generalisation (or concept variation). However, the pattern is not frequent and not explicitly linked to the concept. In this task similarity is not given, which makes a justification of similarity, by means of congruent angles, necessary. (This may, however, probably be ignored by the students).

In Task 5325, students have to discern a similar embedded smaller triangle as sub-shape of a larger triangle. This task is the only one, involving sub-shapes. Moreover, in this task, one need to assume that the shelf and floor are parallel (horizontal), since the corresponding angles are not marked as congruent. It is thus not trivial to prove that the triangles are similar. However, as in Task 5322, this is probably ignored by many students who simply assume that the relevant triangles are similar.

All of the tasks in Figure 9.9 are in some way basically variants of the same task as in Figure 9.6. (find *corresponding sides*, calculate *ratio*, determine *missing length*). However, in each of the tasks on this level (Fig. 9.9), other aspects (connected to similarity or other areas of mathematics or context) are added as well. Those aspects have not been made possible to discern in terms of *dimensions of variation*, in previous tasks in the section.

When considering **all of the tasks as a sequence** it could be recognised that the basic *procedure* is **invariant** in all of the tasks, while certain conditions and values vary. Hence, procedure is not opened up through contrast, but rather **generalised** across tasks. This corresponds to one kind (no 2, see above) of **procedural variation**, namely keeping the procedure invariant in a variety of tasks (Gu et al., 2004). The complexity of the tasks increases somewhat throughout the section. This is done by adding different aspects to the basic task (ratio in percent, 5323, reflection 5324, embedded triangles 5325). Since each of these aspects, only appear in one task, the preconditions could not be said to change in a systematic way, and no hierarchical variation is afforded (Gu et al. 2004). Using a VTL terminology, the tasks involve **fusion** of aspects, that have not previously been opened up by contrast and generalisation.

Afforded aspects in Textbook B

Opportunities to discern certain aspects are described below.

Conditions for similarity

Congruent angles is stated to be a condition for similarity. The aspect is, however, not opened up as a dimension of variation. This means that, no examples or tasks are given in which a contrast is made between similar and non-similar polygons, due to congruent and non-congruent angles (non-concept variation). In most of the tasks similarity is given (e.g. 5317, 5318, 539, Fig. 9.7), and in tasks where it is not, it is not necessary to establish similarity (e.g. 5324, 5325 Fig. 9.9). Therefore, it is possible to arrive at correct answers to all of the tasks without paying attention to the angles at all. Even though the condition ‘congruent angles’ is mentioned to some extent in the section, I would therefore argue that it is an aspect that is neither afforded nor required. (An exception may be the context-based task (Task 5322, Fig. 9.9) If the context is taken seriously, it might provide an opportunity to discern angles as a dimension of variation and as a necessary condition for similarity.)

Proportionality. In Textbook B, the aspect proportional sides as a condition for similarity is not opened up as a dimension of variation. There is no **contrast** made between similar and non-similar figures, due to proportional or non-proportional sides. Nor is any **contrast** made to other concepts such as congruence, (for which equal, rather than proportional, sides is a condition) or between conditions for similarity of triangles and of other polygons.

The ratio of different pairs of corresponding sides in similar polygons is, however, demonstrated to be invariant (Introduction, Fig. 9.6). This condition is used in all of the tasks in the section. It is thus possible to discern and use *the fact that, once similarity is established, the ratio is invariant, for different sets of corresponding sides in a pair of similar figures.* This means that, it is possible to discern that *proportional sides* is an *implication of similarity.* This pattern of variation is interpreted as a **generalisation** in relation to the condition *proportional sides.*

It is claimed that congruent angles is a sufficient condition for similarity of triangles, unlike other polygons (Fig. 9.6), however, this condition is easy to ignore when solving the tasks. Similarity theorems of triangles in relation to conditions for similarity in other figures, is not possible to discern through patterns of variation (c.f. Textbook A).

Transformations preserving similarity

The opportunity to generalise the similarity concept by varying non-essential aspects, such as transformations that preserves similarity, is very limited. Only one task in the section, involves a reflection (5324, Fig. 9.9). In all of the other tasks, the similar figures have the same mutual orientation. It can also be seen that all of the similar figures are coloured similarly, while colours between pairs of similar figures vary. This may be interpreted as a pattern of variation, opening up a *connection between colour and similarity* which is obviously a ‘false’ aspect.

Corresponding sides

The concept ‘corresponding sides’ is central to the discussion but taken for granted. The concept is neither explained, nor opened up as a dimension of variation. Corresponding sides have to be identified in all of the tasks, but since almost all similar figures have the same mutual orientation, it is sufficient to have intuitive ideas of the concept.

Procedure

This section provides an opportunity to practice a *procedure to find missing length* in similar figures. In the introduction, the procedure to calculate the ratio of corresponding sides, and then use it to find a missing length, is advocated. No other procedures are exemplified and the procedure is applicable in all of the tasks, regardless of contexts, shapes of figures, etc. The *procedure* is thus *invariant*, while certain preconditions and values vary. This means that, *procedure* is not opened up through contrast, but **generalised** across tasks, which is an example of **procedural variation 2**, Gu et al. (2004). When the procedure is demonstrated in the introduction, a **contrast** between, *multiplying or dividing* by the scale-factor is presented (example, Fig. 9.6). This is however only related to the question ‘should x be shorter or longer than the corresponding side?’ which is some kind of a memory rule. One kind of **procedural variation** is thus afforded but not the other two. The procedure does not change, and the conditions does not change systematically.

Textbook C

Similarity is the third subsection in the chapter Geometry, which also contains the sections ‘Repetition, Geometry’, ‘Square root and Pythagorean theorem’, ‘Reduction and Enlargement’, ‘Trigonometry’, ‘Symmetry’ and ‘Vectors’.

Summary of learning opportunities in Textbook C

In Textbook C, a procedure to find missing length in similar figures, is briefly demonstrated in the introduction. The same procedure is applicable in the subsequent tasks. This means that, the section affords some opportunity to learn how to calculate a missing length in similar figures. The concept of similarity is introduced through several examples of similar figures. Conditions for similarity are not explicitly stated and not opened up as dimensions of variation through contrast. The concept is not generalised in relation to transformations in the introduction or in tasks at a lower level. In subsequent tasks, rotations occur and have to be perceived as a non-essential aspect. At the same time ‘congruent angles’ has to be discerned in order to justify similarity. The complexity of the tasks is thus increased throughout the section, by adding new aspects to a repeated method. These aspects are however, not possible to discern as dimensions of variation before they are crucial for the solution to a complex problem. There is, thus, a **fusion** of aspects, which are not opened up through contrast and generalisation.

Defining Similarity

In Textbook C. No explicit definition is provided. However, it is repeatedly stated that: ‘The figures, exemplified are of different size but have the same shape’. They are similar.’ This is an example of an informal definition. Angles, corresponding sides, proportionality or congruence are concepts that are not mentioned at all. Moreover, the concept ‘same shape’ is not explained, and it seems to be suggested that ‘different size’ is a condition for similarity.

Introduction to similarity, Textbook C

In the introduction, several examples of mutually similar figures are presented. It is claimed that similarity means that the shape is invariant while the size differs (whether similar figures might also have the same size is not clarified). There is, no variation in orientation, which makes it impossible to discern transformations, preserving similarity as a dimension of variation. It is to some extent possible to **generalise** the aspect *same shape*, however, any further explanation to what ‘the same shape’ means is not given, and no contrast is made to non-similar shapes. Congruent angles is not mentioned as a necessary condition for similarity, and angles are not marked. Neither is proportionality explicitly defined as a condition


for similarity. It is however, pointed out that, ‘if the height is doubled, the base is also doubled’.

Figure 9.10 Adapted and translated from a Swedish mathematics textbook


(Johansson & Olsson, 2021, p 208)

5.5 Similarity*


Consider the figures below.
The triangles are of different sizes, but they have the same shape.
They are similar.



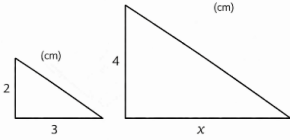
The rectangles are of different sizes, but they have the same shape.
They are similar.



The figures are of different sizes, but they have the same shape.
They are similar.



Here you see two triangles. They are of different sizes, but they have the same shape. They are similar.
How can we calculate how long x is?
The height in the large triangle (4 cm) is twice as large as the height in the small triangle (2 cm).
Since the triangles are similar, the base in the large triangle is also twice as long as the base in the small triangle.
That is, $2 \cdot 3 \text{ cm} = 6 \text{ cm}$.
You can see that you did not have to answer in one step, you calculated the length in the following way.



Enlargement, the scale factor of the illustrated large triangle is $\frac{4}{2}$ (= 2, also written 2:1).
The scale factor can also be written $\frac{x}{3}$. Thus $\frac{x}{3} = \frac{4}{2}$.
That means $2x = 12$ or $x = 6$. Answer: 6 cm.

Similarity

Calculate the segment x in the figure. The two rectangles are similar.

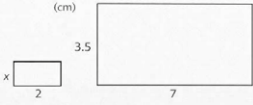
SOLUTION: $\frac{x}{3,5} = \frac{2}{7}$

$7x = 2 \cdot 3,5$

$7x = 7$

$x = 1$

ANSWER: The segment is 1 cm.



Tips: When writing the equation, start with the unknown segment. It tends to make solving the equation easier.

In the example, a procedure, for calculating a missing length, is exemplified. The condition, that the ratio of corresponding sides is constant in similar figures, is used, but not explained or focused in a VTL sense.

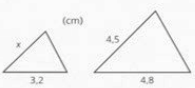
Tasks in Textbook C

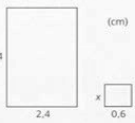
In the first three tasks (Fig. 9. 11), the *same procedure* as in the example, could be applied. Between tasks the shape, the scale factor and values **vary**, which, through **generalisation**, makes it possible to discern that the same procedure might be used regardless of the shape of the figure.

Figure 9.11 Adapted and translated from a Swedish mathematics textbook

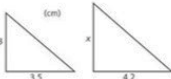
(Johansson & Olsson, 2021, p 209)

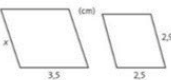
5116
Calculate the length of x . The figures are similar.

a) 

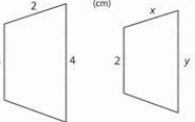
b) 

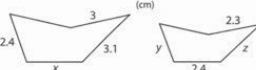
5117
Calculate the length of x . The figures are similar.

a) 

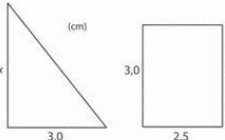
b) 

5118
Calculate the lengths of the sides marked with x , y and z . The figures are similar.

a) 

b) 

5119
Calculate the length of x . (Think carefully!)



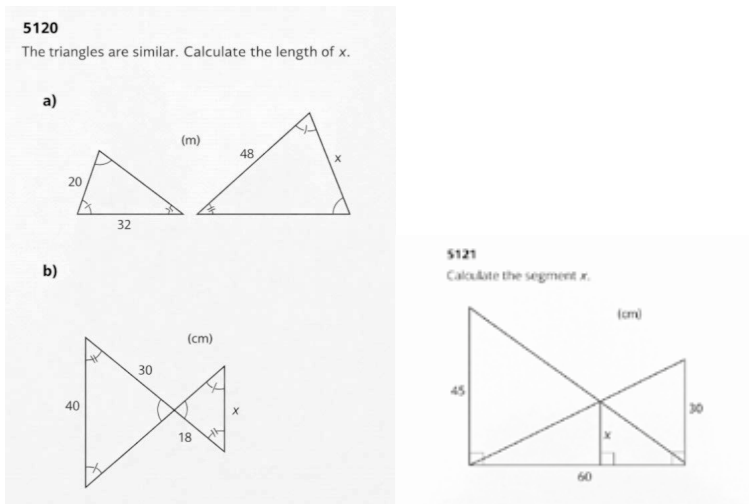
Within the tasks, the size of the figures varies, while shape and orientation both are invariant, which means that it is not possible to separate these aspects and discern conditions for similarity. No opportunity to contrast similar shapes with non-similar shapes, is given. Constant ratio of sides is a condition that has to be used, and corresponding sides have to be identified. This may quite easily be done in the first two tasks, since no other values, than those that should be used in the equation, are given. The word corresponding, is, however, never used, in the section. (It may also be noted that it is only stated that the figures are similar, not which sides and angles correspond, which actually implies that too little information is given to make the tasks possible to solve. See, for example 5117. The figures might be rotated or reflected. This would give a different result than if the figures are assumed to have the same orientation, which is assumed but not specified in the task.)

In Task 5118, it not immediately obvious which sides that should be used in the equation which to a greater extent makes *corresponding sides* a potential dimension variation.

In Task 5119 the shapes are non-similar, which have to be noticed by the student. In this case, *number of angles* becomes a dimension of variation through **contrast**, and *equal number of angles* is thereby focused *as a condition for similarity*. Equal size of angles is however, not mentioned as a necessary condition for similarity (and still less, opened up as a dimension of variation).

Figure 9.12 Adapted and translated from a Swedish mathematics textbook

(Johansson & Olsson, 2021, p 210)



In Task 5120 (Fig. 9.12) the same method should be applied as in the previous tasks. Here the triangles are rotated for the first time, making congruent angles an essential aspect to discern in order to establish similarity and find the corresponding sides. It could be noted that neither orientation nor size of angles has been in focus as dimensions of variation before they appear as essential aspects, required in this task. However, since the only values given are those that should be used, it might be possible to solve the task by guessing, which values to use in the equation.

In Task 5121, four (or actually six), pairwise similar, triangles have to be considered. Congruent angles have to be used to justify similarity of superimposed triangles, sharing sides and angles. Since embedded triangles appear for the first

time in this task and since there have been almost no opportunities to discern congruent angles as a necessary condition for similarity, the solution to this task requires discernment of several aspects that have not been afforded previously.

Afforded aspects in Textbook C

Opportunities to discern certain aspects are described below.

Conditions for similarity

Congruent angles is not illuminated as a condition for similarity. In the first examples and tasks, angles are not marked or mentioned to be of interest. At the same time the aspect is crucial to discern in order to identify similar triangles in complex figures in the last tasks in the section (5120, 5121, Fig. 9.12). The aspect *proportional sides* is not defined as a *condition for similarity*, it is not opened up through variation, and it is not necessary to use in order to determine similarity.

Proportional sides as an implication of similarity is in some way possible discern, or at least to use, even if it is not systematically opened up as a dimension of variation. For a particular pair of similar triangles, the height is stated to be doubled if the base is doubled. The same kind of condition has to be used in almost all of the tasks. No further explanations, statements or definitions, related to the aspect, are given.

Transformations that preserve similarity

In the introduction and in the first set of tasks, no reflections or rotations are present at all. In the second last task (5120, Fig. 9.12), the triangles are suddenly rotated, which, makes it necessary to, at least, discern the aspects as non-essential. At the same time angles are marked, for the first time in the section, without any further explanation. This means that several novel dimensions of variation are introduced simultaneously in complex tasks, and not in a way that makes it possible to discern essential aspects of the similarity concept, through for example a combination of concept and non-concept variations.

Corresponding sides

Corresponding sides have to be identified in all of the tasks. The concept is not even explicitly used in the section. In the first tasks and examples, no understanding (apart from intuitive ideas) of the concept is required. In the last two tasks (5120, 5121, Fig. 9.12), the identification of corresponding sides,

becomes more challenging. There might be different ways of figuring out which values to use in the proportionality equation, but no guidance based on an understanding of the similarity concept is provided.

Procedure

The procedure used in the example (introduction, Fig. 9.10) is to set up and solve the proportionality equation. No other procedure is exemplified and the advocated procedure is applicable in all of the subsequent tasks. Since the procedure is invariant, it is not opened up as a dimension of variation through contrast. It is however to some extent **generalised**, since it is invariant while certain conditions vary. The tasks become increasingly complex. However, the complexity is not increased in a systematic way, where aspects are first possible to discern separately and then progressively combined, but rather by adding novel aspects that are not afforded as dimensions of variation in previous tasks.

Summary

Comparison of learning opportunities in the textbooks

The results show significant differences between the textbooks regarding the aspects that are made possible to discern. In Textbook A, several aspects are opened up through patterns of variation. Properties of the similarity concept are made visible both through contrast and generalisation. Similar figures are contrasted to non-similar figures, depending on either congruent angles or proportional sides. Sufficient conditions for similarity of triangles (the AAA and SSS similarity theorems) are contrasted to the case of other figures, and the concepts of similarity and congruence are contrasted and related. It is thus possible to discern conditions for similarity of polygons in general, and of triangles as a special case. Moreover, learners get the opportunity to see similar shapes in different orientations even in tasks at the beginning of the section (e.g. Fig. 9.1), which affords an opportunity to separate defining aspects of similarity (e.g., congruent angles) from particular instances, and in that way generalise the concept. The tasks in the section are organised in such a way that the complexity increases through fusion of previously afforded aspects rather than (as in Textbook B and C) by adding new aspects to a repeated procedure.

In Textbook B mainly one particular procedure to solve a certain kind of similarity tasks, is practiced. Necessary conditions for similarity (congruent angles

and proportional sides) are specified in the introduction, but not opened up as dimensions of variation, since no contrasting examples of similar and non-similar figures, depending on the conditions, are given. Moreover, similarity is typically given, implying that it is hardly necessary to consider the criteria for similarity when solving the tasks. Moreover, almost all of the similar figures, in the section, have the same mutual orientation. The opportunity to separate aspects that define similarity from aspects that do not affect similarity, and thereby generalise the concept, is therefore limited.

In Textbook C, no definition of similarity is provided at all. Instead, students are supposed to understand the idea of ‘the same shape’ by considering several instances of similar figures, with the same mutual orientation. A procedure for finding a missing length, is briefly demonstrated. The same procedure is applicable in the subsequent tasks. Conditions for similarity are never mentioned. In particular, the aspect ‘congruent angles’ is almost absent. The aspect is not opened up as a dimension of variation by giving any examples of non-similar figures, and angles are not even pointed out as being of interest. In the last tasks however conditions for similarity according to definitions and theorems, must be discerned in order to determine similarity. This means that the aspects are not possible to discern before they are crucial for the solution to a complex problem. In summary, the VTL analysis shows that all textbooks make it possible to practice a procedure to use proportionality to calculate a missing length. However, when it comes to geometric aspects such as conditions for similarity, transformations, etc. there are major differences in learning opportunities between the textbooks.

The table below summarises learning opportunities in the three textbooks. The aspects emerged from the analysis, as afforded dimensions of variation. The table shows the aspects, the features that open up the dimension of variation, and the pattern of variation that is used. TbA, TbB and TbC denote the three textbooks. For each textbook and each aspect, the table shows whether the aspect is opened up as a dimension of variation or not.

Table 9.1. Afforded aspects in the three textbooks

Aspect	Features	Variation VTL	Variation Bianshi	Discernable in:
Congruent angles as a condition for similarity	Congruent vs. non-congruent angles	Contrast	Non-concept variation	TbA
Proportional sides as a condition for similarity	Proportional vs. non-proportional sides	Contrast	Non-concept variation	TbA
Proportionality as an invariant relation between corresponding sides in similar triangles	Ratio between corresponding sides is invariant while side lengths vary	Generalisation	Concept variation*	TbA, TbB, TbC
Similarity-preserving transformations	Invariant shape varying orientation	Generalisation	Concept variation	TbA
Orientation				
Size	Invariant shape varying size	Generalisation	Concept variation	TbA, TbB, TbC
Triangle similarity theorem (AAA)	(AAA) as sufficient for triangles vs. non-sufficient for other polygons	Contrast	Non-concept variation	TbA
Congruence (compared to similarity)	Congruent sides vs proportional sides	Contrast	Non-concept variation	TbA
Procedure				
Procedural steps**	Invariant procedure varying features and contexts	Generalisation	Procedural variation 1	TbA, TbB, TbC
Procedure as such	The same task various procedures	Contrast	Procedural variation 2	TbA
	Systematically increasing complexity***		Hierarchical variation	TbA, TbB

* This may be interpreted as a case of generalisation; however, it is not entirely clear whether it constitutes concept variation.

**e.g. Identify corresponding sides. Calculate scale-factor. Use scale-factor to calculate missing length.

***In TbB mainly by decreasing guidance, in TbA by adding conceptual aspects

Learning opportunities, in relation to previous research

Previous research, emphasise the importance of giving students in secondary education, the opportunity to develop conceptual understanding of similarity, built on defining properties (e.g., Son, 2013; Cox, 2013; Seago et al., 2014). As described above, result of the current study show that different patterns of variation are provided, and that the three analysed textbooks, therefore, afford different opportunities to learn. These results are in line with previous research (Lo et al., 2006) that found significant differences between textbooks regarding the emphases placed on different aspects of the similarity concept.

There are, however, also aspects of similarity, which according to previous research, may be important for students to discern, that are not focused on in any of the textbooks.

Previous research suggests that identification of *correct correspondence*, could be one of the most challenging parts of the solution to some similarity problems. The concept ‘corresponding’ is explicitly or implicitly used in all of the analysed textbooks, but neither not explained nor made visible through variation in any of the textbooks. Moreover, the research review indicate that similarity may be particularly difficult to recognise in triangles that share sides or angles. (see section 3.3). In many tasks, however, correspondence is visually obvious, and do not have to be in the focus of attention (Wijayanti, 2019). This might imply that students could solve several similarity problems correctly without reflecting on why and how certain sides or angles correspond. (In Textbook B there are only this type of problems.) When the correct correspondence is not immediately obvious, however, for example in tasks where triangles are rotated, reflected and embedded, students might fail to solve the problem due to a lack of developed strategies to identify corresponding sides. A more formal approach would be to realise that corresponding sides must face equal angles (only valid for triangles), a condition that is not explicitly stated anywhere in the section. In more advanced courses in geometry, correct correspondence follows directly from the fact that the order of angles is always considered (e.g. $\triangle ABC$) and similarity requires the same order of angles. In the current textbooks the order of the angles is not focused. Therefore, an explicit explanation, which relates corresponding angles to corresponding sides, ought to have been given. This is however, not done in any of the textbooks. According to variation theory, the aspect should also be opened up through variation, for example through contrast or hierarchical variation, that is, through a sequence of tasks in which rotations reflections and overlap are introduced separately, in order to progressively increase task complexity. This is to some extent done in Textbook A. But since the aspect is not opened up through dimensions of variation, nor even explained, this kind of variation is probably not enough to make the aspect discernible. In textbook C, the word is not used in the entire section, even though there are tasks in which finding the correct correspondence is necessary and non-trivial. When a transformations-based approach is used, Figures can be compared through the underlying transformations. In that way it becomes clear which sides and angles correspond and why (Seago et al., 2014). This kind of mental flipping seems to be challenging for students to perform (Chazen, 1987) and requires that similarity-performing

transformations have previously been discerned, not only as non-essential, but as essential aspects.

The research review shows that an important issue is the treatment of *similarity transformations* (Seago, 2010; 2014; Cunningham & Rappa, 2016). Transformations are made discernible as non-essential aspects through generalisation, in Textbook A, but not opened up through contrast in any of the textbooks, that is, the particular transformations that are necessary in order to visualise corresponding parts, are not possible to discern. In a static approach to similarity (e.g. Hilbert, 2050), transformations are non-essential aspects, since they do not affect similarity. This means that, students must get an opportunity to see that orientation may vary while shape is invariant (generalisation) in order to discern transformations as non-essential aspects, (which is done in Textbook A). It would also have been appropriate to contrast transformations that do preserve similarity from transformations that do not preserve similarity, in order to further highlight the aspect. In a static approach this is not absolutely necessary, however. In a transformations-based approach transformations are essential aspects, which would have made it necessary, both to discern transformations, preserving similarity in contrast to other transformations, and to discern specific features in these aspects (e.g. reflected and rotated 45°) which means that contrasting values within the aspects must be made possible to experience.

Regarding *triangle similarity theorems* (e.g. Hilbert, 1902), previous research indicate that students more readily seem to accept minimal conditions based on proportional lengths (SSS), than minimal conditions based on congruent angles (AAA) (Haj-Yahara, 2021; DeJarnette et al., 2014). This study show that similarity of triangles is pointed out as a special case, and contrasted to similarity in general, in Textbook A, which means that the AAA and SSS similarity theorems are possible to discern. However, none of the textbooks present the SAS similarity theorem, nor provide any proof. Within more comprehensive geometry courses (e.g. Hilbert, 1902), three triangle similarity theorems are exemplified, the (SSS), (AAA) and (SAS), and it is essential to discern conditions that guarantee similarity (or congruence) in contrast to conditions that do not (e.g. SSA, which is a false similarity case).

Proportional reasoning is an important component of similarity problems (Arıcan, 2018; Lee & Yim, 2014). In research literature, proportionality is often conceptualised in terms of both relations within a measure space and relations between measure spaces (Vergnaud, 1983; Lamon, 2007). Proportional reasoning thus entails discerning both the constant relation between corresponding

quantities (*within ratio*) and the functional relationship between them (*between ratio*). This dual nature of proportionality has been identified as a central aspect of developing a robust understanding of ratio and proportion (Karplus et al., 1983; Vergnaud, 1983; Lamon, 2007; Weiland et al., 2021). In all of the analysed textbooks, it is possible to discern that corresponding sides in similar figures are related by an invariant ratio. However, it is not made possible to discern that the ratios between sides within each individual figure (internal ratios) are also invariant, and can serve as an alternative way of expressing proportionality (see Lamon, 2007). In order to use the concept of similarity when subsequently introducing trigonometry or the concept of slope (either in mathematics or in vocational applications), this aspect is however, necessary to appreciate (Seago, 2014). A way of using dimensions of variation to open up this aspect, could have been to contrast ways of expressing the proportionality, either as ‘invariant, corresponding ratio of lengths, within similar figures’ or as ‘invariant ratio between corresponding sides in similar figures, keeping everything invariant except how the relation is expressed. Moreover, several triangles, of invariant corresponding internal ratio, but different sizes, could have been displayed. This would have been an advantage in preparing for the introduction of the concept of slope and also for the study of trigonometry.

There are also aspects which might have been expected to be critical for student to discern, but which, according to previous research seems to be less challenging for student to apprehend. For example, aspects related to *equation solving*. Several researchers (Chazen, 1987; Cox, 2013; Seago, 2014; Son, 2013) argue that students’ difficulties in solving similarity problems, seem to be related to the geometric configuration, and would therefore rather stem from a lack of geometric reasoning and conceptual understanding, than from difficulties regarding the equation solving procedure. In the textbooks in this study, no specific attention is paid to equation solving. It is however a part of the task solving procedure, which, in some of the textbooks (e.g. Textbook B) is the main aspect, that is repeatedly practiced.

Taken together, the results of this study both align with and extend previous research on the teaching and learning of similarity. Consistent with earlier findings, the analysis confirms that textbooks differ substantially in the extent to which they support the development of conceptual understanding, particularly with regard to defining properties and the role of proportional reasoning (Lo et al., 2006; Wjayanti, 2019; Zhang & Wong, 2021). In line with prior studies, the results also indicate a tendency for textbooks to emphasise procedural approaches, while leaving key conceptual relationships implicit. More specifically, the findings

reinforce previous claims that important aspects of similarity, such as identifying correct correspondence, recognising the role of transformations, and understanding proportionality as a relational structure, are often insufficiently addressed (Seago et al., 2014; Lamon, 2007).

At the same time, this study contributes by demonstrating how differences in textbook design can be understood as differences in the learning opportunities they afford. These differences are not only quantitative, in terms of task types or frequency, but qualitative, in terms of what becomes possible for students to experience and understand. As such, the findings highlight the importance of carefully designed sequences of tasks that systematically open up critical aspects of similarity.

10. Discussion

This chapter discusses the findings of the thesis in relation to its overall aim of illuminating vocational mathematics knowing in an educational context. It brings together insights from the three studies in order to articulate what constitutes powerful vocational mathematical knowing, how such knowing is expressed in authentic vocationally relevant tasks, and what learning opportunities are afforded through textbooks. First, the results of each study are discussed separately in relation to previous research. The results are then compared and synthesized to provide a more comprehensive understanding of the object of research and a basis for discussing what powerful vocational mathematical knowing may entail and the conditions for its enactment in education. Thereby the last research question is addressed:

How can powerful vocational mathematics knowing be understood within a construction-work educational context?

As outlined in the introduction, although mathematics is widely recognised as integral to construction-work, the understanding of vocationally relevant mathematical knowing is still limited and fragmented, particularly in terms of how it is constituted in specific vocational contexts and how it can be supported through education. Given that vocational mathematics remains underrepresented in educational research and that mathematical demands vary considerably across occupations, there is a substantial need for further vocation-specific investigations (e.g. Bakker, 2014; FitzSimons, 2014; Straesser, 2015).

The present thesis extends previous research by making this content more explicit and nuanced. Through detailed analyses of vocational tasks and teaching materials, it identifies and compares critical aspects required in authentic vocational tasks (Study 2) and those afforded in mathematics textbooks (Study 3). In addition, it reveals variation in teachers' perceptions of educationally significant vocational mathematical knowing, highlighting both tensions and complementarities across perspectives (Study 1). A central contribution lies in the systematic mapping of these perceptions and their interrelationships, and in demonstrating how they are actively present in teachers' sense-making.

Across the three studies, the findings suggest that vocational mathematics knowing can be understood as a hybrid form of knowing that extends beyond mathematics as a disciplinary domain. It involves the coordination of core mathematical ideas (e.g. measurement, proportionality, similarity, and slope) with visuospatial reasoning, technical representations, material behaviour, disciplinary principles (e.g. from technology and physics), contextual constraints, and vocational norms and regulations. This knowing is not limited to performing calculations correctly; it involves discerning which strategies are appropriate under given physical and situational conditions, and interpreting results in contextually meaningful ways. For mathematics education to function as a meaningful component of vocational education, teaching needs to account for this complexity.

Vocational mathematical knowing also emerges as educationally significant beyond immediate task completion. A deeper and more coherent understanding of mathematics is suggested to support students' vocational 'Bildung'; enabling more powerful ways of seeing vocational situations, and more flexible, creative, and responsible action in practice. While context-bound and procedural approaches may suffice for familiar tasks, an exclusive reliance on such approaches risks limiting students' ability to generalise, adapt, and reflect across varying situations. The synthesis of the findings suggests that powerful vocational mathematics knowing entails both the integration of aspects across domains and a firm understanding of mathematical concepts and relationships. Moreover, the findings indicate that the procedures emphasised in mathematics instruction, particularly through textbooks, often provide limited opportunities to discern the conceptual relationships that underpin vocational mathematical practice.

The present thesis contributes to the area of mathematics teaching in vocational education by identifying critical aspects of authentic construction-work tasks, thereby clarifying what needs to be learned to support meaningful vocational mathematics understanding. At the same time, further research is needed to map such critical aspects across a wider range of vocational fields and mathematical domains, and to explore how they can be enacted in teaching in ways that support conceptual mathematical coherence as well as contextual awareness.

Interpreted through a variation-theory framework, the findings indicate that teaching should not necessarily aim to reproduce the full complexity of workplace tasks directly in the school mathematics classroom. Instead, the concluding argument advocates carefully designed teaching sequences, supporting the gradual development of powerful vocational mathematical knowing. Such sequences might involve an initial phase in which critical aspects are made discernible within

each subject and in relation to coherent conceptual structures within each domain, followed by phases in which these aspects are integrated into more complex vocational situations. Coordinated teaching across mathematics and vocational subjects appears to be central in this process. The conclusions further point to the need for further research focusing on the design and analysis of interventions grounded in authentic vocational tasks, where critical aspects are systematically made discernible through coordinated teaching, with the aim of strengthening students' flexible and conceptually grounded vocational mathematical agency.

Ways of perceiving educationally significant vocational mathematics knowing

Study 1 aims to provide a multifaceted description of vocational and mathematics teachers' perceptions of 'educationally significant vocational mathematics knowing', answering the question:

In what qualitatively different ways do teachers perceive 'educationally significant vocational mathematics knowing'?

The outcome space of Study 1 reveals a range of contrasts and relations, represented in terms of categories of description. Rather than representing competing or mutually exclusive views, the categories should be understood as complementary facets that together illuminate the complexity and tensions regarding vocational mathematics knowing as experienced by teachers.

Routine procedures or conceptual understanding

The outcome space in Study 1 shows that vocational mathematics knowing is sometimes experienced primarily as the retention and execution of routine procedures, facts, and methods. Other ways of experiencing vocational mathematics knowing, on the contrary, emphasise the value of 'understanding the essence' and being able to understand underlying mathematical rationales. Previous research has repeatedly noted that vocational numeracy is often perceived as confined to 'basic arithmetic' and routine calculations (Straesser, 2015), despite evidence that vocational practice typically involves a much broader range of mathematical competences (e.g., Gravemeijer et al., 2017; Hoyles et al., 2010; LaCroix, 2014; Muhrman, 2016; Johansson et al., 2024; Williams & Wake, 2007). Importantly, this apparent contradiction is not merely a question about right and

wrong perspectives. Studies consistently show that mathematical activity in many workplaces is indeed enacted through rules of thumb, estimations, and routinised calculations, often without explicit conceptual articulation (Hoyles et al., 2010; Moreira & Pardal, 2012; Roth, 2014; Zevenbergen & Zevenbergen, 2009). However, from a vocational mathematics education perspective, this perception represents an implementation risk identified by, for example, FitzSimons (2014), Hoyles et al., (2010), Hahn (2014). Attempts to align vocational mathematics too closely with immediate usefulness may result in a narrow focus on pragmatic procedures while the underlying mathematical reasoning that would support adaptability, generalisation, and informed interpretations remain invisible. Moreover, the pragmatic and situation-specific procedures valued in vocational practice often differ from the formalised procedures emphasised in school mathematics (Hahn, 2000). A strong focus on isolated procedures may therefore reinforce, rather than bridge, the gap between mathematics classroom practices and workplace mathematics. From this perspective, the educational strength of perceptions that foreground deeper understanding becomes evident. These perspectives position conceptual understanding not merely as an abstract ideal, but as a resource that enables application across varying vocational situations.

Context-bound reasoning or mathematical generalisation

The outcome space in Study 1 further shows that vocational mathematics knowing might be experienced as an inherently *context-bound reasoning*. This perception foregrounds embodied action, material engagement, and situational sensitivity. The significance of such practice-based understanding is confirmed in workplace-mathematics research, emphasising that vocational competence is developed and expressed through participation in practice (Lave, 1988; Noss et al., 2000; Roth, 2014). In contrast, however, other ways of experiencing vocational mathematics knowing, value opportunities for generalisation beyond the immediate situation. This perception aligns with FitzSimons' (2014) argument that vocational mathematics requires more than context-bound competence and that access to conceptually coherent mathematical understanding is essential for addressing complex vocational problems. In a sense, these contrasting perceptions can be understood as complementary rather than competing views, emphasising situational responsiveness and conceptual continuity across situations. At the same time, an exclusive emphasis on context-bound reasoning in mathematics education entails the potential risk that students' understanding remains closely tied to local strategies and specific situations, limiting opportunities for flexible application,

reflection, and communication across contexts. An exclusive emphasis on mathematical concepts, on the other hand, implies a risk of neglecting situated, tacit, and practice-based dimensions of vocational mathematical knowing (FitzSimons, 2014).

Isolated skills or integrative understanding:

Taking both of these views into account, the later categories foreground contextual complexity while also acknowledging the importance of conceptual depth. While earlier categories frame vocational mathematics knowing as bound to isolated tasks and procedures. The later categories emphasise the importance of connecting different disciplinary domains and practices. Notably, in Category E, vocational mathematics knowing is perceived as a knowledge domain that extends beyond formal mathematics and comprises a fusion of multiple dimensions, in which mathematical, vocational, and experiential considerations are interwoven in order to address the complexity of vocational situations. This perception corresponds closely with previous research describing workplace mathematics as inherently multifaceted and shaped by the coordination of mathematical, vocational, contextual, and semiotic dimensions (Hoyles et al., 2010; LaCroix, 2014; FitzSimons, 2014). It also aligns with studies within construction-work vocations that describe geometry as intertwined with materials, measurement practices, tools, norms, and planning (Bessot, 2000; Boistrup et al., 2018; Martin & LaCroix, 2008). In doing so, it speaks directly to critiques of both extremes: it resists reducing vocational mathematics to context-bound reasoning, while also resisting an abstract conceptualism that neglects situated meaning-making.

Direct applicability or active transfer

Another tension that emerges in the outcome space concerns the transfer of mathematical understanding across contexts. Some perceptions entail an assumption that conceptual understanding is readily transferable between educational and vocational settings. Others explicitly problematise this ‘automatic transfer’ assumption by emphasising that the ability to recognise connections between school mathematics and workplace practice, constitutes an essential and non-trivial part of vocational mathematics knowing. Assumptions of direct transfer have been common in the educational discourse, but also repeatedly questioned by researchers (e.g. Bakker & Akkerman, 2019; FitzSimons & Boistrup, 2017; Wake, 2014) who emphasise that conceptual mathematical knowledge does not automatically translate into meaningful or actionable knowing in workplace

contexts. Instead, guided forms of transfer, supported through reflection, communication, and experience of contrasts between situations, can enable students to develop more articulated and flexible understandings of both the mathematical objects as such, and of how they are used in vocational contexts (Bakker & Akkerman, 2019; Kilbrink & Bjurulf, 2013; Marton, 2006; Triantafillou & Potari, 2014; Wake, 2014). This mechanism is particularly highlighted in one category of description, where the moments when students recognise the connection between school mathematics and workplace applications, are seen as moments of extended rather than merely applied mathematical understanding. Such perceptions imply important educational implications, as they open up learning across contexts as a potential resource for meaning reconstruction rather than simple application (cf. Bakker & Akkerman, 2019; FitzSimons & Boistrup, 2017; Kilbrink & Bjurulf, 2013; Marton, 2006; Triantafillou & Potari, 2014).

Immediate task performance or enduring vocational 'Bildung'

In contrast to earlier categories the last category extends the discussion of educationally significant vocational mathematics knowing beyond immediate task performance, and places vocational mathematics knowing within a broader educational perspective, involving confidence, judgement, adaptability, and critical thinking. Here, mathematical knowing is primarily framed not as a tool for solving specific vocational tasks, but as a means for developing a more insightful and adaptive perspective on vocational challenges. Vocational mathematics knowing is thus articulated as powerful not primarily because of its immediate utility, but because it enables students to act with autonomy and responsibility in vocational practice. This perception aligns closely with the arguments of Wheelahan (2007, 2010) and FitzSimons (2014) that access to theoretical and generalisable knowledge is crucial for enabling vocational students to act beyond immediate task performance and situated workplace demands, and to participate critically, productively, and developmentally in evolving workplace practices. The significance of this perception is also evident in studies that highlight the changing role of mathematics in workplaces where technology and digital systems increasingly take over routine computation. The value of mathematics then lies less in performing calculations and more in interpreting results, questioning assumptions, and making informed judgements (Gravemeijer et al., 2017; Hoyles et al., 2010). Theoretically, this way of understanding can be related to Booth's (1992) description of problem solving as a dynamic process shaped by the thematic fields surrounding each phenomenon. According to this view, learners' flexibility

and adaptability in problem solving depend not only on the aspects that are in immediate focus, but also on a broader field of related meanings, references, and potential connections. From this perspective, vocational mathematics knowing is strengthened when students possess a rich network of conceptual and experiential references that can be mobilised as situations change.

Critical aspects of vocational mathematics tasks

Study 2 deepens and concretises the thesis' overall argument by showing, in fine-grained empirical detail, what vocational mathematics knowing demands in practice when students engage with authentic construction-related tasks. Whereas Study 1 mapped qualitatively different ways in which teachers perceive educationally significant vocational mathematics knowing, Study 2 makes visible how such knowing is enacted as students engage with construction-related tasks. More specifically Study 2 addresses the question:

Which critical aspects do students need to discern in order to understand and engage with mathematically rich, construction-work, vocational tasks, in proficient ways?

A central claim of Study 2 is that successful engagement with construction-work tasks depends on the discernment and coordination of multiple interdependent aspects across domains, with the coordination of mathematical reasoning, practical constraints, and spatial structures constituting a central challenge. On the one hand, the findings suggest that the tasks presuppose a coherent, relational and conceptually grounded understanding of fundamental mathematical ideas. A kind of understanding that entails discerning how concepts, quantities and representations are constituted and related, rather than merely being able to compute numerical results. However, even though mathematical concepts such as measurement, proportionality, similarity, and slope are indispensable, they become meaningful only when interconnected with principles from physics and technology, visuospatial reasoning, vocational norms and contextual constraints. This position aligns closely with research describing workplace mathematics as embedded, interdisciplinary, and shaped by contextual constraints (e.g. FitzSimons, 2014; Hoyles et al., 2010; LaCroix, 2014; Straesser, 2015; Johansson et al., 2024). The present study adds to previous research through its fine-grained analysis of critical aspects, which clearly exemplifies and illustrates this multifaceted nature of vocational mathematics knowing. Boistrup et al. (2018)

describe vocational mathematical knowledge in construction-work education as content at the intersection between mathematics and vocational subjects, where mathematical reasoning is used both to solve practical problems and to explain aspects of vocational practice. This thesis confirms and extends this view by showing that vocational mathematics knowing represents a hybrid form of knowing, constituted at the intersection of fundamental mathematical ideas; vocation-specific knowing; logic, spatial and pragmatic reasoning in physical contexts; and theoretical knowledge within other disciplinary domains.

Measurement

Across several task types in Study 2, aspects related to area and volume measurement emerge as critical aspects. Aspects which previously have been pinpointed as essential in workplace practice by for example Masignila (1994) and Zevensbergen and Zevensbergen (2009). Research specifically addressing students' understanding of measurement concepts shows that students often lack a conceptual understanding of area and volume. Such understanding requires coordinating multiple measure spaces and integrating geometric measurement concepts with non-measurement geometric reasoning, such as recognising conservation under decomposition and rigid motion (Battista, 2007; Vergnaud, 1983). According to previous research (e.g. Battista, 2007) area calculations in non-routine situations, constitute challenges for many students. Even at secondary level, students frequently confuse measure-spaces (e.g. area and volume), rely on additive rather than multiplicative reasoning, and treat formulas procedurally rather than structurally (Clements et al., 2018; Vergnaud, 1983). The vocational tasks analysed in Study 2, (for example material estimation and calculations of volumes for filling and excavation) make such aspects explicit. These tasks require a firm understanding of measurement, including decomposing complex areas and volumes into standard shapes and applying measurement formulas in conjunction with spatial reasoning. Aspects which, according to the present study, in agreement with previous research, are non-trivial for many secondary school students (Battista, 2007; Vergnaud, 1983). The findings of Study 2 further illuminate how measurement-based, proportional, and spatial reasoning are deeply intertwined at different levels within and across vocational tasks. Rather than appearing as separate domains, these forms of reasoning co-occur and must be actively coordinated in making sense of vocational construction-work tasks.

Proportional relationships

Another set of critical aspects concerns proportional relationships. Across tasks involving slope calculations, similarity, material estimation, and volume change, students must discern proportional relationships. Proportional reasoning is widely identified as both central and challenging across the mathematics curriculum (Lamon, 2007), as well as in vocational contexts (e.g. Hoyles et al., 2001; Masingila, 1994; Muhrman, 2016). Research on proportional reasoning consistently shows that developing a robust understanding of proportionality is neither straightforward nor automatic. Even when students solve proportion tasks correctly, solutions may rely on context-bound procedures or rote formula use rather than on an understanding of invariant multiplicative relationships between covarying quantities, an understanding essential for identifying underlying structures and selecting efficient strategies (Ben-Chaim et al., 2012; Lamon, 2007; Noelting, 1980; Vergnaud, 1983; Weiland et al., 2021). This result is confirmed in Study 2, which shows that different multiplicative relations vary in their accessibility, and that students differ in their ability to recognise and relate different strategies, which in turn affects their flexibility in problem solving. Taken together, findings from Study 2 and previous research thus suggest that a structurally grounded understanding of proportionality is essential for engaging efficiently with vocational problems in construction-work practice. Without such understanding, tasks risk being reduced to rule-following or imitation; with it, students are better positioned to justify decisions and adapt strategies to changing conditions.

Similarity and slope

Research findings show that proportional reasoning is particularly demanding when tasks involve multiple representations or complex geometric configurations (Cox, 2013; Lamon, 2007; Son, 2013). Study 2 exemplifies various tasks, where proportional relationships are embedded in geometrical, spatial and physical considerations. This observation directs attention to the concept of geometric similarity which is closely related to the concept of slope. Several vocational tasks are exemplified in which mathematical aspects characteristic of similarity play a central, though often implicit, role. These include scaling practices such as using the Egyptian triangle (3:4:5), the notion of slope in contexts such as drainage systems, ramps, ditches, and roofs, cross-measuring methods, and the insight that small directional deviations can result in large displacements over distance due to proportional growth in similar figures. Although similarity is not explicitly

referenced, successful engagement with these tasks relies on aspects that are mathematically characteristic of similarity, such as; invariant external relationships (e.g. slope), invariant internal relationships (e.g. scaling), and links between different representations of slope, such as ratio, percentage, and angle of inclination. Research on teaching and learning similarity shows that these aspects are also challenging for students (Wu, 2005; Seago, 2013; Haj-Yahya, 2021). Studies of slope further indicate that students may apply procedures correctly in one context while failing to recognise equivalent meanings in another, indicating a lack of integrated understanding of proportionality and geometric similarity (Stump, 1999; Nagle et al, 2013; Seago, 2014).

Spatial reasoning

The findings of Study 2 also point to the central role of spatial reasoning. Across task types, students' understanding depends critically on their ability to flexibly navigate between physical space, technical representations, and mathematical relations. More specifically the study highlights critical aspects related to interpreting information from technical drawings, understanding how dimensions in plan, façade, and section drawings relate to one another, and discerning how sloping surfaces in three-dimensional objects are represented in two-dimensional drawings. Correspondingly, previous studies across construction-related vocations consistently show that mathematical activity at work is deeply intertwined with the interpretation of graphical representations, spatial visualisation, and the coordination of multiple representations of space. Rather than applying mathematical procedures in isolation, practitioners mathematical reasoning is dependent on and intertwined with properties of technical drawings and spatial configurations. (e.g. Bessot, 2000; Boistrup et al., 2018; LaCroix, 2014; Martin & LaCroix, 2008; Masingila, 2009). In turn, research on spatial reasoning frequently highlight its particular importance for success in engineering, science, and construction-related professions as well as in formal mathematics (ang, 2019; Sinclair et al., 2017; Sorby, 2009, 2013; Uttal, 2015). Research further indicate that construction-work students demonstrate high levels of spatial ability, which further improve during apprenticeship (Cuendet et al., 2014). Vocational practice itself thus seems to provide rich opportunities for developing spatial reasoning, an ability which, in turn, might enhance performance in mathematics (Clements, 1999; Fang, 2019; Newcombe, 2017; Sinclair et al., 2017; Sorby et al., 2009, 2013; Uttal, 2015). The result of Study 2 together with previous research, thus strongly

underscores the importance of recognising spatial reasoning as an essential component of vocational mathematics knowing.

Procedures

Several of the tasks described in Study 2 exemplify vocation-specific procedures that recur across different vocational situations. One such procedure concerns the even distribution of items, a type of procedure that appears in various construction-related tasks, such as panelling walls, placing battens, or distributing boards with equal spacing. According to Masingila's (2009) problem solving in vocational contexts often involves situations in which the mathematical procedures are already known, while the main challenge lies in adapting these procedures to unfamiliar and context-specific constraints. Williams and Wake (2007) describe how workers may apply established routines and procedures without necessarily being aware of the underlying mathematical principles. However, scholars also point to the advantage of revealing underlying rationales (e.g. Bakker & Akkerman, 2014; Hoyles et al, 2010) especially when learning the procedures (Johansson et al., 2024). Johansson et al., (2024) also highlight that mathematical procedures used in vocational practice might include aspects that are not typically emphasised in school mathematics. The present study confirms these findings by exemplifying procedures that are mathematically valid but still differ from typical school mathematics procedures.

Contextual constraints

Across the tasks considered, the analysis shows that successful engagement presupposes an initial understanding of the task's conditions and constraints, and that the mathematical reasoning involved in task solving is often inseparably intertwined with contextual considerations. This is illustrated, for example, in tasks involving even spacing, where students must discern that the total distance is fixed, that spacing is adjustable but bounded by a maximum value, and that it must remain uniform across the entire length. In slope-related tasks, students must similarly understand how the result relates to practical conditions such as drainage capacity, accessibility, or material choice. In material estimation tasks, solution strategies and rounding practices depend on material properties and dimensions.

Previous research on mathematics in construction-related vocations points to similar patterns. Studies emphasise the importance of initial sense-making, the adaptation of mathematical procedures to material and situational constraints, and an understanding of how physical properties interact to shape outcomes (Martin

& LaCroix, 2007; Masingila, 2009; Zevensbergen & Zevensbergen, 2009). Taken together, the present analysis and previous research suggest that discerning task-specific constraints and their implications for mathematical relations is a constitutive aspect of vocational mathematics learning. Such discernment shapes which strategies are valid, how results are interpreted, and how mathematical solutions are acted upon in practice.

Work-place norms and practice

Moreover, the findings in Study 2 highlight the role of workplace-specific aspects, such as norms, conventions, and regulations that shape how mathematics is used, interpreted, and valued in workplace practice. These include knowledge of appropriate measuring techniques, standard dimensions, tolerances, and practical conditions associated with specific tools or construction methods. For example, maximum spacing requirements, permissible slope values, standard material dimensions, delivery formats, waste allowances, and compaction factors. This pattern closely aligns with previous research showing that workplace mathematics is governed by vocational norms and conventions. Across vocational contexts, studies consistently report that successful mathematical engagement on understanding workplace objects, tools, regulations, and common practices (Clements, 1999; Fang, 2019; Newcombe, 2017; Sinclair et al., 2017; Sorby et al., 2009, 2013; Uttal, 2015). Taken together, the present study and previous research suggest that vocational mathematics knowing involves discerning vocation-specific aspects, which consequently must be addressed explicitly in vocational education if students are to develop the capacity to engage proficiently with authentic workplace tasks.

Disciplinary knowledge

Furthermore, the analysis demonstrates that vocational mathematics task solving involves disciplinary knowledge that extend beyond both mathematics and vocation specific domains, including areas such as physics, chemistry and technology. Examples from Study 2 include: density, compression, waste factors, force, pressure, and mixing ratios. While these phenomena are articulated and manipulated using mathematical relations, their conceptual meaning, interpretation and appropriate use depend on theoretical understanding from other disciplinary domains beyond mathematics. For instance, in tasks involving volume estimation, students must discern why, and how, the volume of a given mass of material changes as a result of physical processes such as excavation,

compaction, or mixing. Similarly, in slope-related tasks, disciplinary aspects from physics, such as force, and fluid flow are essential, as slope is repeatedly discussed in relation to phenomena such as water drainage, accessibility, material choice, and structural load. Previous studies, also exemplify that workplace mathematics often involves knowledge of material behaviour, technical considerations and physical constraints, (e.g. Boistrup et al., 2018; Hoyles et al., 2010; Johansson et al., 2024; LaCroix, 2014; Masingila, 2009). While these results acknowledge the role of physical, technical, or material considerations in vocational mathematics activity, previous research has predominantly described vocational mathematics in terms of a continuum between mathematics and vocational knowing. In a sense, everything that is involved in workplace task solving could be regarded as vocational knowing. However, in previous studies, mathematics is commonly treated as a discipline in its own right, intertwined with, yet analytically distinguishable from, vocational knowing. Other disciplinary domains, such as physics, chemistry, and technology, are not typically afforded the same analytical status, but are instead subsumed under vocation-specific knowledge. The findings of the present study indicate that analyses of vocational mathematics knowing might be enriched by giving greater analytical attention to forms of knowledge that exceed both mathematics as a discipline and vocation-specific knowing, including areas such as spatial reasoning, physics, chemistry, and technology.

Learning opportunities in mathematics textbooks

The comparative textbook analysis conducted in Study 3 aimed to investigate learning opportunities afforded in different mathematics textbooks for vocational education. The section on the topic of similarity, was analysed, using variation theory as an analytical framework. Thereby answering the question:

What different learning opportunities are afforded through the ways in which similarity is presented in mathematics textbooks for vocational students?

The results highlight differences in learning opportunities between textbooks in terms of the aspects of similarity that are made possible to discern, and illuminates how textbook structure, in terms of patterns of variation both within and across tasks, can give rise to markedly different objects of learning.

Scholars argue that it is essential for students in secondary education, to get opportunity to develop conceptual understanding of similarity, built on defining properties (e.g., Son, 2013; Cox, 2013; Seago et al., 2014). Moreover, research claim

that the succession of tasks and the patterns of variation provided in tasks and examples, affect and the learning opportunities afforded to students (Marton, 2015; Gu et al., 2017; Watson & Mason, 2006). The result of the current study show that different patterns of variation are provided, and that the three analysed textbooks, therefore, afford different opportunities to learn.

In two of the analysed textbooks, similarity primarily functions as an implicit precondition for task solving rather than as an explicit object of learning. Tasks focus mainly on developing a procedure, based on the proportional relation between corresponding sides, for calculating a missing length. At the same time limited attention is given to why such procedures are valid, what aspects define similarity, and how these aspects are related; dimensions that previous research has identified as essential for developing conceptual understanding (e.g. Lo et al., 2006; Seago et al., 2014; Son, 2013; Wu, 2005). This confirms earlier findings that textbooks tend to emphasise procedural applications at the expense of conceptual understanding of similarity (Lo et al., 2006; Wijayanti, 2019), and that the complexity of the similarity concept is reflected in vast differences in learning opportunities across textbooks.

By contrast, one textbook provides more systematic opportunities to discern essential aspects of similarity through structured patterns of variation. In this textbook, the conditions for similarity are opened up by contrasting similar and non-similar figures, with critical aspects, such as angle congruence and proportionality of sides, varied one at a time, making each of them discernible to students. Furthermore, similarity is related to congruence, and triangle similarity theorems are focused by contrasting sufficient conditions for similarity of triangles to necessary conditions for similarity in other figures. These aspects are progressively made discernible and subsequently required for problem solving, thereby supporting the development of a coherent and conceptually grounded understanding of similarity.

Also, the treatment of similarity transformations varies. One of the textbooks systematically opens up transformations as non-essential through patterns of variation. Initially, this is achieved through generalisation. Subsequently, transformations are gradually introduced across the sequence of tasks, increasing the overall complexity. In the other two textbooks however, transformations are not made possible to consider in a systematic manner.

The issue of identifying corresponding parts, has been highlighted as a key difficulty for students (Chazan, 1988; Wijayanti, 2019). Although the notion of correspondence is implicitly present in all textbooks, it is neither explicitly

explained nor systematically varied. In many instructionally simplified tasks correspondence is visually obvious meaning that students may solve tasks without understanding how or why sides and angles correspond. However, in more complex geometric configurations, including reflected, rotated and superimposed figures, corresponding parts may become a critical aspect.

Proportional reasoning, which underpins similarity, is present in all textbooks but is treated in a limited and fragmented manner. While the invariance of ratios between corresponding sides is emphasised, other important aspects, such as the relationship between internal and external ratios, are not made visible. Research has highlighted the importance of such relationships for developing a deeper understanding of proportionality and for connecting similarity to other mathematical domains, such as trigonometry and slope (Lamon, 2007; Seago et al., 2014). The absence of these connections in all of the textbooks, may therefore limit students' opportunities to develop a coherent and transferable understanding.

Taken together, these findings support earlier research indicating that differences in definitions, task types, and sequencing shape students' sense-making of similarity (Lo et al., 2006). It also confirms Wjayanti's (2019) findings, highlighting the lack of learning opportunities regarding conceptual aspects of similarity in mathematics textbooks. Like the present study they conclude that textbooks typically emphasise how to use similarity to solve tasks rather than how the concept is defined and understood; and that connection between similarity and proportional reasoning is usually lacking, resulting in limited opportunities to discern connections across mathematical domains.

Powerful vocational mathematics knowing

This section brings together findings from the three studies in order to address the overarching question of how powerful vocational mathematics knowing could be understood within a construction-work educational context. By synthesising insights from teachers' perceptions, analyses of students' engagement with vocational tasks, and examinations of learning opportunities afforded through teaching materials, the section develops a coherent account of the nature, content, and educational significance of vocational mathematics knowing.

The thesis contribute to ongoing discussions in vocational mathematics education concerning the relationship between mathematics in school and mathematics in workplace settings (e.g. Lave, 1988; Carraher et al., 1985; Hoyles et al., 2010; Wake, 2014; FitzSimons, 2014; Sundtjønn, 2021). Previous research has repeatedly shown

that doing mathematics at work and doing mathematics in school are disparate activities, shaped by different purposes, tools, norms, and forms of judgement. In response, the research agenda has gradually moved beyond a simple transfer assumption towards perspectives that emphasise boundary crossing, recontextualisation, and integration ((e.g. Bakker, 2014; Bakker & Akkerman, 2019; FitzSimons & Boistrup, 2017; Frejd & Muhrman, 2020; Triantafillou & Potari, 2014). The present thesis aligns with this development by conceptualising vocational mathematics as a form of knowing constituted at the boundary between school mathematics and workplace practice. It extends previous research by making this knowing analytically explicit through a systematic articulation of its constituent dimensions, as well as through a structured account of the qualitatively different ways in which vocational mathematics knowing is perceived. The synthesis of the findings suggests that powerful vocational mathematics knowing entails both the integration of aspects across domains and a firm understanding of mathematical concepts and relationships.

Powerful vocational mathematical knowing can be understood as a hybrid form of knowing. Rather than treating vocational mathematics as either applied school mathematics or implicit vocational routine, the thesis shows how powerful vocational mathematics knowing emerges through the coordination of conceptual mathematical ideas with spatial reasoning, disciplinary principles from physics and technology, material constraints, and vocational norms, thereby addressing calls to clarify not only how vocational mathematics is taught, but what constitutes its educational content and meaning (cf. Bakker, 2014; FitzSimons, 2014; FitzSimons & Boistrup, 2017; Straesser, 2015). This kind of knowing is articulated across the outcome space in Study 1; explicitly demonstrated through teachers' examples and students' discussions in Study 2; and further illuminated through the analysis of learning opportunities afforded in mathematics textbooks for vocational students in Study 3.

- *Powerful vocational mathematical knowing includes mathematical concepts and procedures.* The results of Study 1 and 2 point to the significance of developing a robust understanding of foundational mathematical ideas, such as measurement, proportional reasoning, similarity, and slope. Previous research indicates that such understanding is not self-evident among upper secondary students (e.g. Vergnaud, 1983; Lamon, 2007; Haj-Yahya, 2021; Chazan, 1988). More specifically, Study 2 demonstrates that vocationally relevant tasks in construction-work rely on aspects that are mathematically characteristic of similarity, including invariant internal and

external ratios, conditions for triangle similarity, and the role of similarity in structuring relationships between different representations of slope. Focusing on the concept of similarity, Study 3 highlights substantial differences in opportunities to understand this concept afforded in mathematics textbooks and identifies a range of potentially critical aspects. While some textbooks primarily support the application of procedures for solving standardised tasks, they offer limited opportunities to discern the underlying conceptual structures that are essential in vocational applications. This means that students may be able to perform procedures without having access to the concepts those procedures are based on. Also Study 1 reveals a tension between vocational mathematics knowing understood as the ability to apply simplified procedures and seen as a more relational and conceptually grounded form of understanding. While acknowledging the importance of procedural fluency, an exclusive focus on isolated techniques entails clear limitations, particularly in light of previous research indicating a mismatch between procedures used in workplace settings and those taught in school mathematics (e.g. Hahn, 2000). It is therefore reasonable to assume that practising procedures alone, whether in school mathematics or in workplace contexts, provides limited opportunities for students to apply their knowledge across settings. By contrast, forms of teaching that systematically open up critical aspects through contrast, generalisation, and increasing conceptual complexity provide more favourable conditions for developing the kind of relational understanding identified in Study 2 as central to powerful vocational mathematics knowing.

- *Powerful vocational mathematics knowing entails contextual awareness.*

Both Studies 1 and 2 highlight that engaging with vocationally relevant tasks requires sensitivity to the specific vocational conditions under which mathematical knowledge is applied. This includes engagement with material properties, tools, spatial configurations, workplace norms, and situational constraints. In Study 2, this is evident in how tasks are solved through the coordination of mathematical reasoning with contextual and practical considerations. Similarly, Study 1 shows that teachers emphasise the importance of understanding how mathematical reasoning is shaped by and embedded in vocational contexts. This aligns with previous research describing mathematics in workplace practice as situated, tool-mediated, and context-dependent (e.g. Masingila, 2009; Roth, 2014; Zevenbergen & Zevenbergen, 2009). However, while the efficiency of context-bound reasoning in specific workplace situations is clearly highlighted, its potential for generalisation

across situations is simultaneously problematised in Study 1. In line with previous research (e.g. FitzSimons, 2014), the findings suggest that approaches highly adapted to particular contexts may support successful task completion in familiar situations, but risk limiting students' ability to recognise underlying mathematical structures and to apply their knowledge in new or varied contexts.

- *Powerful vocational mathematical knowing involves the discernment of critical aspects and the ability to recognise them across contexts.* The findings further indicate, in agreement with previous research on transfer and recontextualisation (e.g. Bakker & Akkerman, 2019; FitzSimons & Boistrup, 2017; Marton, 2006; Triantafyllou & Potari, 2014), that the awareness of the relationship between mathematics learned in the classroom and mathematics used in vocational practice constitutes an essential and non-trivial component of vocational mathematics knowing. From this perspective, vocational mathematics knowing depends both on learners' prior discernment of critical aspects and on their ability to associate novel situations with those aspects. Marton (2006) argues that transfer is a central aim of learning, not as the repetition of the same action across similar situations, but as the ability to draw on experienced variation and contrast between situations. What can be made sense of in a situation depends on the discerned aspects (Marton & Booth, 1997). A range of such aspects is identified in Study 2. However, Booth (1992) emphasises that problem solving also depends on 'pointing references', that is, not only on direct discernment but also on the associations learners are able to make. This is further illuminated through the findings of Study 1 suggesting that the ability to recognise and relate aspects across contexts can be understood as a central component of powerful vocational mathematical knowing. Besides discerning critical aspects, vocational mathematics task solving thus entails the elicitation of these aspects into focal awareness, a process that can be regarded as non-trivial.

- *Powerful vocational mathematical knowing is characterised by the integration of critical aspects across domains.* Study 1 illuminates the complexity of vocational mathematics tasks, emphasising that they involve knowledge that extend far beyond formal mathematics. This view is further explored in Study 2, which confirm that vocational mathematics task solving requires coordinating mathematical aspects with contextual, vocational and physical considerations. Similar conclusions are drawn in several other studies (e.g. Johansson et al., 2024; Hoyles et al., 2001; LaCroix, 2014) and is the reason to why FitzSimons (2014) argue that the application of mathematics in vocational contexts is not merely a

question of transfer. It rather requires being able to connect mathematics to the contextual, perceptual, and functional aspects that dominate the vocational task. The result of Study 2 clearly illustrates this complexity and indicate that, mathematically rich work-place tasks does not only require the recall of mathematical aspects but the ability to integrate them into a broader system in which mathematics, together with aspects of various other domains become meaningful as part of a composite perception of the task as a whole.

- *Powerful vocational mathematical knowing enables flexible and meaningful action.*

Study 1 suggests that educationally significant vocational mathematics knowing might extend beyond the ability to discern and coordinate critical aspects in specific tasks. Rather, it involves more comprehensive forms of knowing that shape the range of possible interpretations and actions available to learners. This view aligns with arguments that, beyond immediate task performance, access to conceptually coherent knowledge enables vocational competence that goes beyond routine work and supports participation, interpretation, and development in evolving vocational practices (FitzSimons, 2014; Wheelahan, 2007; Hoyles et al., 2010; Bakker & Akkerman, 2014). The mathematical level and visibility may influence how vocational tasks are approached, how situations are interpreted, and more broadly, how the vocational situation is understood (Gravemeijer et al., 2017; Hoyles et al., 2010; Saló i Nevado & Pehkonen, 2018, 2020). Many of the aspects identified in Study 2 relate to broader conceptual structures within mathematics as well as within other disciplinary domains such as physics, economics, and technology. The perspectives expressed in Study 1, together with corresponding arguments in previous research (e.g. FitzSimons, 2014; Hoyles et al., 2010; Wheelahan, 2007, 2010, Gravemeier et al., 2017), and theoretical contributions concerning the significance of rich thematic fields for shaping dynamic meaning-making, and action (Booth, 1992), support the conclusion that access to such structures can substantially enhance students' opportunities to engage with vocational situations in creative, confident, and adaptive ways. A vocational mathematics education that aspires to support powerful vocational mathematics knowing should therefore prioritise opportunities for students to develop rich and relational conceptual understanding of key mathematical ideas, including how these ideas are defined, inherently related, and made meaningful within vocational practice.

Study 2 demonstrates that vocationally relevant tasks in construction-work rely on aspects that are also fundamental within formal mathematics. In line with

previous research (e.g. Otten, 2014; Lo et al., 2006; Stump, 1999), Study 3 confirms that the extent to which students are afforded opportunities to discern such aspects in school mathematics varies considerably depending on textbook design.

- *Powerful vocational mathematical knowing contributes to vocational 'Bildung'.*

The overall conclusion of this thesis is that powerful vocational mathematical knowing emerges, not simply from learning formal mathematics or from participation in vocationally relevant tasks in isolation, but from education that enable learners to discern, connect, and generalise key mathematical ideas across domains and situations. It involves the integration of conceptual mathematical understanding with contextual, spatial, and material aspects of vocational practice, as well as the ability to recognise and mobilise these relationships flexibly across contexts. Such knowing is not given by vocational experience alone, nor by procedural proficiency, or mathematical expertise, but depends on access to learning opportunities in which critical aspects from various domains are made visible and related through variation, contrast, and increasing conceptual and contextual complexity. It is through this access to structured, relational and synergetic knowledge that vocational mathematics education can support not only competent task performance, but flexible, critical, and adaptive participation in evolving vocational practices, thereby contributing to vocational 'Bildung'.

Didactical implications

This thesis highlights the complex, multifaceted and context dependent nature of vocational mathematics knowing. More specifically, it specifies critical aspects that students need to discern in order to engage with mathematically rich vocational tasks in proficient ways. As discussed above, the results reveal a dynamic interrelationship of aspects that span several domains. Mathematics education that aims to be relevant for vocational practice therefore needs to account for this complexity and provide opportunities for students to discern and relate these critical aspects. A central didactic question for vocational mathematics education is how teaching can be designed to effectively create such opportunities.

According to variation theory, teaching should systematically and progressively direct focus on critical aspects in ways that allow learners to discern individual aspects separately and subsequently relate them to one another (Marton, 2015). This process culminates in what variation theory terms fusion, where multiple aspects and their relations are simultaneously held in awareness. Importantly,

fusion is described as the final pattern of variation and presupposes that the aspects to be fused have been made discernible in advance. The findings of this study indicate that authentic, mathematically rich vocational tasks typically involve complex fusions of sets of aspects, among which aspects related to specific mathematical objects of learning constitute only a limited part.

This reveals a central didactic problem. Vocational students are expected to learn mathematics in ways that support the demands of their future vocational practice. However, the demands of authentic vocational mathematics tasks involve the fusion of aspects which extend beyond the scope of a school mathematics. In addition it includes spatial reasoning in relation to technical drawings; material behaviour; principles from physics, chemistry and technology; contextual conditions; and vocational norms and regulations. While these aspects are essential for authentic vocational task solving, giving them sufficient attention within the framework of a general mathematics course risks competing with the development of core mathematical concepts. Similar tensions have been identified in previous research on the use of authentic tasks in mathematics education, where concerns have been raised that increased contextual richness might obscure or dilute essential mathematical structures (; Brantlinger, 2022; Hahn, 2000; Wedege, 2010; Vos, 2018). Moreover, the results from Studies 1 and 2, together with previous research on workplace learning and vocational mathematics (e.g. FitzSimons, 2014; Masingila, 2009; Martin & LaCroix, 2008; Roth, 2014), indicate that vocational tasks can sometimes be solved through pragmatic, context-bound reasoning without explicit attention to general mathematical principles. Such reasoning may be sufficient for completing particular tasks in familiar situations and is therefore perceived by some teachers as sufficient understanding. However, from a broader educational perspective, this form of knowing is insufficient for developing vocational mathematics knowing that supports transfer, adaptation, and informed judgement across changing situations.

This concern aligns with research on boundary crossing and recontextualisation, which emphasises that learning across practices requires pedagogical mediation rather than the direct replication of either mathematical procedures or workplace activities (Bakker & Akkerman, 2019; FitzSimons, 2014; Wake, 2014;). Without such mediation, authentic tasks may anchor students' attention to situational details while leaving underlying mathematical relations implicit and unexamined.

This implies that an unreflective introduction of vocational authenticity into the mathematics classroom involves several risks:

- *Reinforcing local strategies* tied to specific contexts, tools, or routines, rather than fostering powerful vocational mathematics knowing that supports generalisation, judgement, and adaptability across situations.
- *Premature fusion of aspects*, where multiple aspects are simultaneously involved before students have had opportunities to discern them separately, rendering tasks too complex to function as effective learning situations
- *Displacement of mathematical focus*, as attention may be drawn to aspects, necessary for completing the task, but lying far outside mathematics as a disciplinary domain, thereby competing with the limited instructional space needed for developing core mathematical concepts.
- *Neglect or distortion of the vocational context*, where tasks are detached from or represent simplified or misrepresented versions of workplace practice, thereby losing relevance and authenticity and limiting students' opportunities to recognise the meaning and purpose of mathematics in vocational settings.

At the same time, authentic vocational task solving necessarily involves the coordinated mobilisation of multiple aspects (Hoyles et al., 2010; LaCroix, 2014). The fusion of such aspects cannot be assumed to develop automatically (Marton, 2015). Findings from Study 1 and 2 indicate that the coordination of mathematical ideas with practical constraints, physical properties, and spatial structures constitutes an essential and non-trivial part of vocational mathematics knowing. Similarly, previous research emphasises that transfer requires pedagogical mediation, where connections between formal mathematics and vocational purposes are made explicit and available for reflection (Bakker & Akkerman, 2019; FitzSimons & Boistrup, 2017; Triantafillou & Potari, 2014). From this perspective, vocational mathematics teaching must include deliberate opportunities for students to reflect on the mathematical structures underlying vocational tasks, rather than assuming that such connections will emerge implicitly through practice alone. The didactic challenge, therefore, is not to avoid authenticity, but to sequence and mediate it in ways that support sustained learning.

The theoretical and empirical insights developed throughout this thesis give rise to the argument that authentic task solving should be treated primarily as a goal of instruction, rather than as an immediate means for learning mathematics. By shifting the primary focus from authentic tasks to relevant aspects an appropriate instructional sequence might be developed. Rather than attempting to reproduce complex workplace tasks directly in the school mathematics classroom, vocational task solving might be decomposed into more delimited, conceptually

coherent objects of learning. Some of those aspects are clearly appropriate to address within mathematics education, such as core mathematical concepts, relations, representations, and structures (e.g. proportionality, similarity, measurement, and the interpretation of mathematical representations). Other aspects, while mathematically relevant, are more closely tied to vocational practice or related disciplinary domains. Once the critical aspects have been made discernible and related to broader objects of learning within each educational domain, students are better positioned to integrate them. At that point, the fusion of aspects that characterises authentic vocational task solving becomes educationally attainable.

This approach implies that meaningful vocational relevance in mathematics education is not achieved through task imitation, but through carefully designed instructional sequences that support the gradual development of powerful vocational mathematics knowing. Similar task sequences have been suggested by e.g. Gu et al, (2017) conceptualised as hierarchical variation. Designing such sequences within vocational mathematics education, necessitates coordination between mathematics and vocational subjects, including shared planning, explicit communication about educational goals, and agreement on how different aspects contribute to students' overall vocational competence.

Establishing such coordination benefits from insight into the qualitatively different perceptions and priorities, regarding vocational mathematics knowing, that may exist among teachers, as well as insight into the critical aspects that students need to discern in order to engage proficiently with vocationally relevant mathematical situations. By making these dimensions explicit, through articulating variation in teachers' ways of understanding and identifying critical aspects involved in vocational task solving, the present thesis aims to provide a common conceptual ground for dialogue and collaboration between mathematics and vocational teachers. Without such shared understanding, collaboration risks remaining superficial, limited to sporadic contextual references rather than coherent educational design.

The value of educational coherence and collaboration across subject boundaries has been demonstrated in previous research. For example, Frejd and Muhrman (2020; 2025), show that sustained collaboration between mathematics teachers and vocational teachers, supported by organisational structures and shared planning, enhances students' opportunities to connect mathematical concepts with vocational practice in meaningful ways. Their studies confirm that when teachers share responsibility for vocational mathematics education, students

are better supported in recognising the relevance of mathematics. Although the present study approaches vocational mathematical knowing from a different perspective, it arrives at similar conclusions.

Regarding the part of vocational mathematics knowing that is unambiguously situated within a formal mathematics curriculum, the synthesised results of the three studies point to the importance of providing opportunities to develop a conceptually coherent and relational understanding of central mathematical ideas, rather than primarily practicing isolated procedures. The textbook analysis in Study 3 shows that current learning materials offer significantly different conditions for developing the kind of understanding that vocational task solving relies on. A key step in strengthening powerful mathematical knowing among students is thus to place greater emphasis on conceptual understanding and the relational nature of central mathematical ideas. For mathematics textbooks, this implies moving beyond a predominant focus on practicing isolated procedures and instead offering sequences of tasks and examples that explicitly foreground defining conditions, conceptual relationships, and critical distinctions. More specifically, mathematics teaching needs to create systematic opportunities for students to discern essential aspects of concepts and structures. Through carefully designed variation and progression, textbooks can support the development of a relational understanding of mathematics that is applicable across vocational situations.

Methodological considerations and limitations

The methodological choices in this thesis were guided by the aim of exploring vocational mathematics knowing as it is perceived, enacted in task solving, and afforded in educational contexts. The combination of phenomenographic interviews, analyses of critical aspects of authentic vocational tasks, and textbook analysis, based on variation theory, enabled a multifaceted examination of the phenomenon. At the same time, each method entails specific limitations that must be acknowledged.

Strengths and limitations of the phenomenographic interview study

The methodological stance adopted in Study 1 is grounded in the phenomenographic research tradition as developed by, for example, Marton and Booth (1997). From a phenomenographic perspective, the aim is not to describe individuals' beliefs, attitudes, or practices, but to identify a set of qualitatively

different ways in which a phenomenon is experienced within a group. The focus is thus on variation in meaning rather than on frequency or distribution. Moreover, by adopting a second-order perspective, the study does not seek to determine what the phenomenon ‘really is’ in an objective sense, but to explore how it is understood and made meaningful by teachers in a vocational education context.

The second order perspective implies both advantages and limitations. A recognised limitation of the phenomenographic interview study is that it does not provide direct access to tacit or embodied dimensions of vocational knowing, but is restricted to what is available to awareness and, in some way, possible to articulate, either implicitly or explicitly. Teachers’ perceptions of what constitutes significant knowing do not necessarily correspond to their actual knowing as enacted in practice. This limitation is further accentuated by the fact that the analysis does not focus on how the vocational mathematics content is experienced, but rather on teachers’ perceptions of how such content ought to be experienced by students. This implies an additional analytical distance from vocational practice. Thereby some essential dimensions of vocational competence may remain outside the scope of the analysis.

At the same time, a central strength of the approach lies precisely in its focus on articulated experience. By analysing how teachers with either vocational or mathematical disciplinary competence, combined with teaching experience, perceive educationally significant vocational mathematics knowing, the approach efficiently provides access to forms of educationally relevant knowledge which would be considerably more demanding to obtain through direct observation of vocational performance alone. Moreover, how teachers experience and articulate content, constitutes a significant object of research in its own right, since ways of understanding shape ways of acting (cf. Marton, 2015). Teachers’ ways of understanding vocational mathematics knowing thus matter, not only as distinct perspectives, but as conditions that frame and constrain the learning opportunities that might be created in educational settings. Making the tensions, identified in the outcome space, explicit, thus enables a more reflective and deliberate approach to educational design.

Marton (1981) argues that studies adopting a second-order perspective do not replace studies taking a first-order perspective but rather complement them. Whereas much previous research on vocational mathematics has focused on describing mathematical practices as they occur in workplaces or on analysing mathematics teaching in vocational education, the present study contributes by examining how vocational mathematics knowing is understood, articulated, and

given educational meaning by teachers. In this sense, the study adds a complementary perspective to vocational mathematics research.

The limited sample size (16 interviews), makes it impossible to draw conclusions about prevalence or to make claims about causal relationships. However, the aim of the study is not to determine how widespread particular ways of experiencing a certain phenomenon are, nor to explain why individuals experience the phenomenon in a particular way, but to identify a variety of possible perspectives. In this regard the study could make valid claims. A central assumption within phenomenography is that there exists only a limited number of qualitatively different ways in which a phenomenon might be experienced (Marton & Booth, 1998). Consequently, saturation could be approached even with a relatively limited number of interviews, provided that the empirical material exhibits sufficient variation. The inclusion of both mathematics teachers and vocational teachers was a deliberate methodological choice aimed at maximising experiential variation and strengthening the analytical outcome space. Moreover, an advantage with the relatively limited sample size is that it enables close engagement with the data, allowing for careful analysis that preserves nuance and supports a detailed articulation.

Strengths and limitations of the task analysis

The contribution of the second study lies in its fine-grained analysis of vocational mathematics tasks, highlighting critical aspects (Marton, 2015) that must be discerned in order to engage with mathematically rich vocational tasks in proficient ways. Articulating these aspects provides a detailed account of such activities with a clear focus on what students need to learn. Over recent decades, phenomenography and variation theory (VTL) have made significant contributions to mathematics education research by offering conceptual tools that enable teachers and researchers to identify critical aspects of specific objects of learning (e.g. Marton, 2015). The focus on critical aspects implies a zoomed-in analytical perspective, in which the specific content plays a central role. This perspective allows for rich and detailed descriptions of how vocational mathematics knowing is constituted in particular tasks and learning situations. At the same time, the conceptual framework provides limited resources for describing overarching structural features of the empirical material. This has, to some extent, constrained the possibility of presenting the findings in a concise and abstract manner. However, mapping a number of critical aspects across tasks has made it

possible to inductively identify patterns among the identified aspects, which might have remained unnoticed had the analysis been guided by a predefined framework.

Moreover, in line with Study 1, it can be noted that while interviews provide important insights into vocational mathematics knowing, this approach comes at the cost of potentially losing some tacit or embodied dimensions of vocational mathematics knowing, that would possibly have emerged through direct observation of workplace practice. At the same time, interviews foreground aspects that are already recognised as educationally significant by teachers. These articulations are in turn grounded in participants' prior experience of vocational practice. In this way, the adopted approach makes it possible to capture important elements of vocational mathematics task solving, while maintaining a specific focus on aspects that teachers and students identify and confirm as educationally critical. Compared with direct workplace observations, this approach also entails methodological efficiency, allowing relevant aspects to be accessed without the logistical constraints associated with in situ data collection.

A further methodological consideration concerns the use of multiple data sources. Combining teacher interviews with task-based student dialogues might appear as a limitation in terms of consistency of data types. This potential inconsistency is partly addressed by the fact that the analysis across data sources was guided by the theoretically well-established concept of critical aspects, which contributed to analytical coherence. Moreover, the combination of data sources also enhances analytical robustness and allows for a more comprehensive account of the phenomenon. The identified aspects were compared across interviews, textbook tasks and student dialogues, both to confirm overlap and to capture a broader range of aspects. In line with established phenomenographic practice, this approach contributes to analytical saturation through variation rather than sample size

Although efforts have been made to capture a wide range of aspects, the analysis does not claim to be exhaustive. This constitutes a methodological limitation, but one that is consistent with common aims of qualitative analyses which typically seek analytical depth and richness rather than completeness. Moreover, the task types represent a selection and do not aim to provide a complete account of task types. A further limitation relates to the scope of the empirical material. As in Study 1, the claims made are analytical rather than statistical in nature. The analysis allow for identification of possible critical aspects but cannot determine the extent to which these aspects are critical across a broader population of students.

Another challenge concerned the organisation and presentation of the results. The complexity of the analysed vocational tasks meant that the material could plausibly be organised according to several different principles, none of which proved fully sustainable. Structuring the results by mathematical content areas was problematic, as many tasks simultaneously involved multiple mathematical domains. Attempts to organise the material according to broader vocational areas (such as roof construction or panelling work) similarly resulted in substantial overlap. Conversely, presenting each individual task separately produced an unwieldy, fragmented, and at the same time incomplete, account. The final decision was therefore to organise the results into task types, in which several tasks are treated as a unit based on a shared basic underlying structure. This categorisation was pragmatic rather than analytical, and the division into task types does not in itself constitute a central result of the analysis. This organisational choice nevertheless entailed certain inconvenience as some of the identified critical aspects are characteristic of particular task types, while others are more strongly tied to specific contexts. Since a central aim of the analysis was to capture the variety of aspects involved in authentic vocational task solving, both of these types of aspects were given analytical weight. This contributed to a more comprehensive account and unfortunately made the presentation of the results less concise and more demanding to navigate.

Taken together, these strengths and limitations reflect deliberate methodological choices aligned with the aim of the study. The approach prioritises analytical depth, variation, and educational relevance over breadth and representativeness. While other methods would be required to capture tacit dimensions of vocational knowing or general structures in the data, the chosen design is well suited to illuminate essential aspects and to supporting didactic reflection on what students need to learn in order to engage proficiently with mathematically rich vocational tasks.

Strengths and limitations of the textbook analysis

In the third study, variation theory is used as an analytical framework for the textbook analysis, implying a specific focus on patterns of variation. From a methodological perspective, this approach offers a strong and principled way of analysing learning opportunities. Variation theory posits that experiencing systematic patterns of variation is a necessary condition for learning. Accordingly, an analysis that examines the presence and organisation of such patterns can make

theoretically and analytically grounded claims about the learning opportunities afforded by instructional materials.

At the same time, this methodological choice entails clear limitations. While the analysis gains credibility and analytical weight through its reliance on a well-established theoretical perspective, it is also necessarily dependent on the validity of the assumptions, and the explanatory power of that theory. Consequently, the analysis is constrained by what the theoretical framework makes it possible to detect. Moreover, the analysis does not allow conclusions about how widespread certain learning opportunities are, nor about the extent to which they are realised in classroom practice. Instead, the study focuses on what kinds of learning are made possible by textbook design and on qualitative differences between textbooks. Nevertheless, such a mapping of learning opportunities can make a significant contribution to educational research. By showing how variation in task structure and sequencing affords different opportunities to discern critical aspects, the study clarifies how instructional materials may shape students' potential learning trajectories. The analysis thus illustrates how design choices matter: different patterns of variation foreground different aspects and relations, thereby supporting, or constraining, the opportunities for developing conceptual understanding. In this way, the study provides a principled basis for discussing the educational consequences of design decisions beyond questions of coverage or frequency, and thereby offers insights that are directly relevant for teachers and textbook authors. It supports more informed decisions about instructional design by clarifying how alternative ways of structuring content may lead to fundamentally different learning opportunities. In this sense, the mapping does not aim to rank or evaluate textbooks, but to contribute analytically grounded knowledge about how design features relate to learning opportunities.

Theoretical contributions

This study brings together a research area and a theoretical perspective in a way that has not previously been done. While this combination has posed analytical and methodological challenges, it has also created opportunities to approach the research object from new perspectives, to examine the limits of the theoretical framework, and to contribute to methodological development through the adaptation and refinement of analytical procedures. Therefore, particular emphasis has been placed on theoretical rigor, with the analysis consistently grounded in the core assumptions of the framework and in logically coherent implications of these

assumptions. In this way, the application of the theoretical perspective is argued to be both valid and methodologically innovative.

The textbook analysis

In previous research drawing on variation theory, the analytical focus has primarily been on the design of lessons and tasks (e.g. Watson & Mason, 2006), as well as on the analysis of enacted classroom teaching (e.g. Runesson, 1999) and student learning (e.g. Kullberg, 2010). In the present study, the theoretical framework is applied in a manner that is consistent with these established uses. At the same time, the study extends the line of work by applying variation theory to the analysis of mathematics textbooks. Such applications of the theory remain relatively limited, especially in vocational education contexts. Examining how critical aspects are made available in instructional materials is of particular significance given the central role of such resources in shaping teaching and learning. The textbook analysis in this thesis thus complements earlier research on lesson design and classroom interaction by illuminating how theoretical principles of variation appears in curricular materials, and how that may shape students' possibilities for discerning critical aspects.

The phenomenographic interviews

Study 1 contributes by applying the phenomenographic framework to the analysis of vocational mathematics knowing as experienced and valued from a vocational education perspective. In this sense, the framework is applied in a context and in a manner that is less typical for phenomenographic studies within educational research. As the phenomenon under investigation relates to an object of learning that is interdisciplinary in nature, the criteria for what constitutes appropriate knowing are less clearly defined than within a more established disciplinary tradition. This posed methodological challenges, particularly with regard to the structuring of the outcome space. Rather than exhibiting a clearly hierarchical structure, the analysis revealed qualitative differences that reflect variations in focus, integration, and inclusivity across ways of understanding. This led to the proposal of a semi-hierarchical outcome space, combining *vertical* distinctions in depth and quality with *horizontal* reorientations or broadening of focus. In this way, the present thesis makes a methodological contribution by offering a conceptualisation of how phenomenographic outcome spaces may be structured when addressing phenomena related to complex, interdisciplinary learning objects.

Moreover, the phenomenographic categories developed in the study, together with the relations among them, can be understood as contributing to theoretical development, beyond phenomenography, by offering a structured conceptualisation of qualitatively different ways of understanding vocational mathematics knowing. While previous research has addressed tensions related to context dependence and generality, transfer, and the relation between theory and practice, the overall structural pattern identified in teachers' ways of experiencing vocational mathematics knowing represents a novel contribution.

The task analysis

Study 2 examines complex vocational tasks in which the intra-mathematical object of learning constitutes only a limited part of the overall situation. Such contexts have received little attention in previous research informed by phenomenography and variation theory, and the analysis has therefore required the theoretical framework to be interpreted and applied in ways that are partly novel. In this work, particular emphasis has been placed on the concept of *critical aspects*, a well-established and central construct within the theoretical framework. In the analysis, this concept has been interpreted in line with the phenomenographic perspective, where critical aspects are understood as constituting qualitative differences in awareness, as well as from a variation-theoretical perspective, where they are conceptualised as dimensions of variation. At the same time, the concept has been applied to complex, practice-oriented vocational tasks in a manner that extends beyond its' more common uses. In this way, the study contributes to theoretical development by demonstrating how the concept of critical aspects can be meaningfully operationalised in interdisciplinary contexts.

However, a specific challenge in analysing what students need to discern to understand vocational tasks in proficient ways concerns the dynamic nature of vocational task solving (see Martin & LaCroix, 2008), and the role of more coherent mathematical conceptual structures in this process. Handling workplace problems involve not only the static perception of a phenomenon, but an ongoing elicitation and interrelation of aspects as students move back and forth between different views of the task. While variation theory explains how aspects become available for discernment and how they can be related and coordinated in the constitution of powerful ways of seeing, it does not fully address the dynamic and shifting nature of the task-solving process, in which new aspects are elicited and integrated over time. Moreover, the present analysis primarily addresses aspects

that are directly necessary for handling specific tasks. However, both this study and earlier research suggest that broader and more coherent mathematical knowledge structures could create conditions for more flexible, innovative, and powerful problem solving (e.g. FitzSimons, 2014, Saló i Nevado & Pehkonen, 2018). Such overarching structures are not readily captured in analyses that focus exclusively on task-specific critical aspects. In variation theory–informed analyses of intra-mathematical objects of learning, such structures are often treated as inherent to the object of learning itself. However, when it comes to relationships between multiple objects of learning, the framework offers more limited conceptual tools for analysing how such connections are constituted, developed, and coordinated in teaching. As a result, such analyses cannot fully account for the complexity of vocational task-solving. Addressing both this complexity and the required conceptual depth therefore calls for the incorporation of additional theoretical perspectives.

A model of problem solving, grounded in Gurwitsch’s version of Gestalt theory, was proposed by Booth (1992). Booth (1992) emphasises the role of the *thematic field*, (comprising elements which are not in the focus of awareness but which are related to the phenomenon through a sense of affinity and relevance) and *pointing references* (which connects the theme and the thematic field) in the problem solving process. In order to understand what gives rise to opportunities for association, creativity, and the formation of coherent knowledge structures, these ideas offer a valuable theoretical lens. From this perspective, a central component of the problem-solving competence lies in how thematic fields are constituted and how pointing references, between the theme and the thematic field, are established, selected, and organised. It is reasonable to assume that conceptual mathematical understanding could constitute an important component of such thematic fields and thereby enable alternative more creative and flexible actions, for example in vocational practice. Exploring how the theoretical perspective can be further developed and applied to address complex problem solving, and how the potential of deep mathematical understanding can be conceptualised in this context, therefore constitutes an important direction for further research.

Suggestions for further research

The findings of the present thesis point to several directions for further research. One important direction concerns the role of conceptual mathematical knowledge

structures in vocational problem solving. While the present analysis focused primarily on task-specific critical aspects, the findings also suggest that more a developed conceptual mathematical understanding may support more flexible, innovative, and transferable forms of vocational problem solving. Future studies could therefore examine how such overarching mathematical structures are formed and how they influence the ability to navigate complex vocational challenges in adaptive ways.

A second important direction for further research concerns the student perspective. While the present thesis primarily explores vocational mathematics knowing as experienced and articulated by teachers, further studies could focus more directly on students' ways of experiencing vocational mathematics knowing.

Methodologically, the approach used in Study 2 could be further developed and applied to a more extensive empirical material in order to provide a more comprehensive understanding of vocational mathematics task solving. In particular, more extensive task-based studies involving collaborative student problem solving could offer valuable insights into moments of shifting awareness, where critical aspects become salient. By combining such analyses with additional inductive or theory-driven approaches, future research may also be able to draw more general conclusions about what constitutes vocational mathematics knowing and how it is expressed in vocational practice.

Further textbook analyses could also extend the present work by examining other mathematical content areas in order to explore whether similar patterns emerge across different mathematical domains. In addition, comparative studies across different vocational programmes could investigate similarities and differences in how vocational mathematics knowing is constituted and valued in various vocational fields. Such studies would contribute to a broader understanding of the extent to which the findings of the present thesis are specific to construction-related vocations or reflect more general characteristics of vocational mathematics.

Finally, further research could investigate the didactical implications of the present findings, particularly how vocational mathematics education might better support the integration of conceptual understanding, procedural fluency, and contextual reasoning. Such work could inform the design of instructional materials and learning environments that intentionally foster both the discernment and integration of critical aspects across domains, as well as the development of coherent mathematical understanding applicable across vocational situations.

For example, intervention-based approaches in the form of learning studies, building on insights from the present thesis, could be enacted to explore how systematically designed patterns of variation may support students in gradually discerning and integrating mathematical, material, and contextual aspects in authentic vocational situations. Such work may contribute to the development of empirically grounded design principles for teaching that strengthen students' flexible and professional engagement with vocational mathematical challenges. An important component in the development of such interventions is to examine how collaboration between mathematics and vocational teachers, supported by a shared didactical understanding of vocational mathematical knowing, might lead to the formation of coordinated teaching approaches across subject domains, in which the object of learning is enacted through structured alignment between different parts of vocational education.

11. Svensk sammanfattning

Introduktion

Matematiken utgör en viktig del av den gymnasiala yrkesutbildningen i Sverige, både ur ett övergripande utbildningsperspektiv och mer specifikt som en central komponent i ett kvalificerat yrkeskunnande (t.ex. Boistrup m.fl., 2018; Muhrman, 2016; Skolverket, 2025). Samtidigt visar rapporter att elever ofta upplever matematiken som abstrakt och skild från sitt kommande yrke (SSI, 2017). Misslyckande i matematik är dessutom en starkt bidragande orsak till att många elever vid yrkesprogrammen inte uppnår yrkesexamen inom utsatt tid (Skolverket, 2024). Därmed riskerar matematiken att bli ett hinder på vägen mot yrkeskompetens och examen, snarare än en bärande grund för yrkeskunnande.

En tydligare integrering mellan matematik och yrkesämnen har i tidigare forskning visat sig vara en lovande väg för att stärka elevers motivation och förståelse av matematikens betydelse i yrkeslivet (Dalby m.fl., 2016; Frejd & Muhrman, 2025; Muhrman, 2016; Voss m.fl., 2025). Därför betonas vikten av att tydliggöra sambanden mellan skolans matematik och den matematik som används i yrket, inom ramen för matematikundervisningen, både för att stärka elevers motivation att lära sig matematik och deras förmåga att urskilja och tillämpa matematik i olika yrkessituationer (t.ex. Bakker, 2014; FitzSimons, 2014; Skolverket, 2025; Streefland, 2015; Triantafillou & Potari, 2014). Samtidigt är implementeringen av ett sådant arbetssätt komplicerad eftersom matematiken i yrkespraktiken ofta är svår att identifiera, genom att den är situationsbunden, mångfacetterad och delvis dold. Den skiljer sig också från skolmatematik vad gäller syften, representationer och arbetssätt (Coben & Weeks, 2014; FitzSimons, 2014; LaCroix, 2014; Masingila, 2009; Sundtjønn, 2021). Lärare på gymnasiets yrkesprogram ställs alltså inför den didaktiska utmaningen att inte bara avgöra hur de ska undervisa, utan också vad undervisningen ska innehålla och hur detta innehåll ska förstås, något som förutsätter en dubbel kompetens inom både matematik och det aktuella yrkesområdet. Mot denna bakgrund framträder ett behov av fördjupad kunskap om vad matematiskt yrkeskunnande innebär och hur detta kan göras tillgängligt i undervisningen. Trots att matematiken har en central

betydelse i många yrken har området fått relativt begränsad uppmärksamhet inom utbildningsforskning, delvis på grund av dess komplexitet (Bakker, 2014). Samtidigt som senare forskning i hög grad har bidragit till att adressera denna kunskapslucka genom att fokusera på relationen mellan matematiken i yrkespraktiken och matematikundervisningen inom yrkesutbildningen (t.ex. Bakker & Akkerman, 2019; FitzSimons & Boistrup, 2017; Frejd & Muhrman, 2020, 2025; Sundtjønn, 2021) framhålls också att kopplingen mellan dessa domäner fortfarande är otillräckligt klarlagd.

Därför fokuserar föreliggande avhandling på att, utifrån olika perspektiv, belysa vad ett kvalificerat matematiskt yrkeskunnande inom bygg- och anläggningsutbildning kan innebära. Fokus riktas mot den didaktiska vad-frågan: vilket innehåll behöver eleverna urskilja och vilket kunnande behöver de utveckla för att förstå och tillämpa matematik på ett professionellt sätt i olika yrkessammanhang? Yrkesmatematik betraktas här inte enbart som tillämpad skolmatematik eller som en implicit dimension av praktiskt arbete, utan som ett meningsfullt utbildningsinnehåll i sin egen rätt, där såväl konceptuella kunskapsstrukturer som kontextuellt och situerat kunnande utgör centrala komponenter.

Genom att synliggöra olika sätt att förstå vad matematiskt yrkeskunnande innebär, de kritiska aspekter som behöver urskiljas samt de sätt på vilka lärande möjliggörs och begränsas i läroböcker, avser avhandlingen att bidra till en fördjupad diskussion angående matematikens roll i yrkesutbildningen, yrkeskontextens roll i matematikundervisningen och angående vad som räknas som värdefull kunskap i relation till forandet av ett sammanhängande yrkeskunnande, det vill säga vad kraftfullt matematiskt yrkeskunnande kan innebära och villkoren för att ett sådant kunnande ska utvecklas inom yrkesutbildningen.

Syfte och forskningsfrågor

Avhandlingens övergripande syfte är att belysa och kartlägga matematiskt yrkeskunnande som utbildningsinnehåll, genom att besvara följande frågor:

1. På vilka kvalitativt skilda sätt uppfattar lärare matematiskt yrkeskunnande som utbildningsinnehåll?
2. Vilka kritiska aspekter behöver elever urskilja för att förstå och hantera bygg- och anläggningsrelaterade uppgifter med matematiskt innehåll på ett kvalificerat sätt?

3. Vilka olika lärandemöjligheter erbjuds genom de sätt på vilka likformighet behandlas i matematikläromedel för elever på gymnasiets yrkesprogram?
4. Hur kan kraftfullt matematiskt yrkeskunnande förstås inom en bygg- och anläggningsrelaterad utbildningskontext?

Avhandlingen består av tre delstudier, som besvarar var och en av de tre första forskningsfrågorna. Den första studien är en fenomenografisk intervjustudie som undersöker hur matematiklärare och yrkeslärare uppfattar matematiskt yrkeskunnande som utbildningsinnehåll. Studien synliggör variationer i hur relationen mellan skolmatematik och yrkespraktik förstås och vilket kunnande som betonas. Studie 2 preciserar detta innehåll genom att identifiera de kritiska aspekter som elever behöver urskilja för att förstå och hantera byggrelaterade uppgifter där matematik ingår. Studie 3 undersöker i vilken utsträckning sådana aspekter görs möjliga att urskilja i matematikläroböcker, genom analys av hur ett centralt matematiskt begrepp, med relevans för bygg och anläggningsyrken (likformighet) behandlas. Med variationsteori som analytiskt ramverk analyseras vilka lärandemöjligheter olika läromedel erbjuder.

I den avslutande diskussionen sammanförs resultaten från de tre studierna, med målet att bidra till en fördjupad och nyanserad förståelse av matematiskt yrkeskunnande som utbildningsinnehåll, och därmed besvara den fjärde forskningsfrågan.

Avhandlingen är avgränsad till bygg- och anläggningsprogrammet och till ett begränsat antal empiriska sammanhang, vilket innebär att resultaten inte utan vidare kan generaliseras till andra yrkesområden. Samtidigt pekar de identifierade mönstren på frågor som är relevanta även i ett bredare yrkesutbildningsperspektiv, särskilt vad gäller relationen mellan matematiska begrepp, yrkespraktik och undervisningens utformning.

Tidigare forskning

Forskning om matematik i yrkeslivet och om matematikundervisning i yrkesutbildning har beskrivits som ett mångfacetterat och relativt begränsat område inom matematikdidaktisk forskning (Bakker, 2014; FitzSimons, 2014). Matematiken i yrkespraktiken är ofta situerad och inbäddad i processer, verktyg och tekniska system, vilket gör den svår att avgränsa och synliggöra som ett explicit kunskapsinnehåll (Bakker, 2014; Gravemeijer m.fl., 2017; Hoyles m.fl., 2010; LaCroix, 2014). Den kan också vara dold i pragmatiska metoder, yrkesspecifika

minnesregler och ett språkbruk som skiljer sig från etablerad matematisk diskurs (Coben & Weeks, 2014; FitzSimons, 2014). Detta har bidragit till en allmän föreställning om att matematik i många yrken främst handlar om grundläggande räknefärdighet (Straesser, 2015). Samtidigt visar studier från olika yrkesområden att matematiska kompetenser i arbetslivet spänner över ett brett spektrum, från aritmetik och mätning till geometriska resonemang, proportionalitet, tolkning av representationer och hantering av formler (t.ex. Hoyles m.fl., 2010; Johansson m.fl., 2024; LaCroix, 2014; Muhrman, 2016; Straesser, 2015). Vilka kompetenser som krävs utöver grundläggande aritmetik varierar dock avsevärt mellan olika yrkesområden. Därför behöver forskning om matematiskt yrkeskunnande bedrivas på ett yrkesspecifikt sätt (Straesser, 2015). Samtidigt finns gemensamma drag mellan yrkesområden. Inom yrkespraktiken är matematikanvändning typiskt sett nära integrerad med förståelse av den fysiska miljön, kunskap om tekniska artefakter samt bedömningar av vad som är rimligt i ett givet sammanhang (Coben & LaCroix, 2014). Matematik framträder därmed sällan som en avgränsad aktivitet, utan är inbäddad i ett bredare sammanhang av yrkesmässiga överväganden (t.ex. Boistrup et al., 2018; Coben et al., 2010, 2014; Johansson et al., 2024; LaCroix, 2014; Zevenbergen & Zevenbergen, 2009). Även om matematiken i yrkeslivet ofta är dold bakom tekniska system eller lokalt etablerade metoder, betonar forskning att ett synliggörande av matematiska aspekter kan bidra till en fördjupad förståelse av yrkessammanhanget och en möjlighet till ett mer flexibelt agerande (Gravemeijer m.fl., 2017; Hoyles m.fl., 2010).

Inom bygg- och anläggningsrelaterade yrken framträder särskilt geometriska och spatiala aspekter som centrala (Bessot, 2000; Boistrup et al., 2018; Masingila, 2009). Matematiska aktiviteter inom dessa yrken innefattar ofta tolkning av ritningar och andra representationer, samt en förmåga att koppla samman matematiska och rumsliga aspekter, att förstå materialegenskaper och att bedöma rimligheten i resultat (Boistrup m.fl., 2018; Johansson m.fl., 2024; LaCroix, 2014). Flera studier indikerar dessutom att svårigheter i yrkesrelaterade matematikuppgifter inte enbart kan förstås som brist på formell matematik, utan ofta hänger samman med begränsad förståelse av yrkeskontexten (t.ex. LaCroix, 2014; Masingila, 2009; Zevenbergen & Zevenbergen, 2009).

En stor del av forskningen angående yrkesmatematik berör de möjligheter och svårigheter som uppstår då den matematik som lärs i skolan tillämpas i en yrkeskontext. Flera forskare beskriver att det under en period togs för givet att generella matematikkunskaper kan överföras problemfritt mellan olika sammanhang (Bakker, 2014; Dalby m.fl., 2016; Wake, 2014). Därefter kom

forskningen inom området att präglas av ett fokus på skillnaderna mellan formell skolmatematik och matematiken i yrkeslivet, och svårigheterna att överbrygga detta gap (Bakker, 2014; FitzSimons, 2014). På senare tid har forskare dock alltmer börjat tänka bortom sådana dikotomier och fokuserat på vad som händer i gränstrakterna där yrke och utbildning möts (t.ex. Bakker & Akkerman, 2019; Sundtjonn, 2021).

Enligt FitzSimons och Boistrup (2017) är förmågan att använda matematik i nya sammanhang central för att hantera dessa spänningar. Denna förmåga behöver utvecklas explicit inom utbildningen och kan i sig förstås som ett lärandeobjekt, snarare än som en automatisk följd av matematikundervisningen (FitzSimons & Boistrup, 2017). Skillnader mellan praktiker behöver emellertid inte enbart förstås som hinder att överbrygga. Flera forskare betonar att just dessa skillnader kan fungera som en resurs för lärande. Genom att kontrastera olika sätt att använda och förstå matematik synliggörs centrala aspekter (Bakker, 2014; FitzSimons, 2014; Kilbrink & Bjurulf, 2013; Marton, 2006). I detta perspektiv blir kopplingen mellan skola och yrke inte enbart en fråga om att överföra kunskap, utan om att skapa förutsättningar för meningsskapande i mötet mellan praktiker (Bakker & Akkerman, 2019).

Det finns flera motiv att förespråka en hög nivå av allmän matematisk kompetens inom yrkesutbildning. Det kan vara en demokratisk fråga om tillgång till abstrakt kunskap. Det kan också handla om tillträde till högre utbildning och flexibilitet i förhållande till personliga karriärvägar (FitzSimons, 2014; Lindberg, 2013). FitzSimons (2014) argumenterar även för att många yrkesuppgifter innefattar matematiska resonemang och problemlösningstrategier, som kräver djupare matematisk förståelse än att enbart kunna utföra kontextbundna procedurer. Samtidigt uppfattar många elever matematiken som meningslös och verklighetsfrånvärd (Skolinspektionen, 2017). Flera studier diskuterar därför möjligheten att undervisa matematik inom yrkesutbildning på ett sätt som gör att matematiken uppfattas som relevant i förhållande till utbildningen som helhet. En bild som framträder genom dessa studier är att eleverna, då de känner att de kan använda sina matematikkunskaper i ”verkliga” sammanhang, får ökad förståelse för matematik och därmed en ökad känsla av kompetens och motivation (Boistrup, 2018; Dalby m.fl., 2016; Lindberg, 2010; Muhrman, 2016).

Forskning om kontextbaserad och yrkesrelevant undervisning nyanserar samtidigt bilden av autentisk kontext som entydigt positiv (t.ex. Brantlinger, 2022). Autentiska uppgifter tenderar att uppmuntra pragmatiska strategier där matematiken reduceras till procedurer utan begreppslig förankring (Hahn, 2014).

Samtidigt finns en risk att yrkespraktiken missrepresenteras, eftersom sådana matematikuppgifter ofta förenklas för att bli lösbara genom orealistiska antaganden och genom att viktiga kontextuella faktorer försummas (Vos, 2018; Wedege, 2010). Detta kan i sin tur förstärka bilden av skolmatematik som fjärrad från yrkespraktiken (Hahn, 2000; Roth, 2014). En viktig aspekt, som kan vara avgörande för om ett integrerat arbetssätt ger de positiva effekter på elevers engagemang och lärande som eftersträvas, har visat sig vara platsen för undervisningen (Frejd & Muhrman, 2020). Undervisning i en autentisk yrkesmiljö där yrkeslärare och matematiklärare gemensamt utformar undervisning och innehåll skapar förutsättningar för ett starkare engagemang hos eleverna.

Sammantaget pekar detta på att frågan om yrkesintegrering i matematikundervisningen inte enbart handlar om ifall yrkeskopplingar bör göras, utan om hur den görs och vilka kunskapsdimensioner som synliggörs (Bakker, 2014; FitzSimons & Boistrup, 2017). Tidigare forskning angående matematiskt yrkeskunnande har alltså lyft fram matematik som en viktig del av yrkeskompetens och yrkesutbildning (e.g. FitzSimons, 2014; Hoyles et al., 2010). Samtidigt beskrivs matematiskt yrkeskunnande som ett komplext lärandemål där matematiska och yrkesmässiga aspekter samspekar, och där båda dessa områden innefattar såväl teoretiska som praktiska dimensioner (e.g. Bakker, 2014; Boistrup et al., 2018). Kopplingen mellan praktiker har också beskrivits som ett signifikant innehåll som explicit behöver uppmärksammas i planering och genomförande av undervisning, och som i högre utsträckning behöver synliggöras i forskning (FitzSimons & Boistrup, 2017). Yrkesmatematik framstår därmed som ett utbildningsinnehåll vars innebörd inte är entydigt preciserad, och ett forskningsområde som behöver fördjupas och utvecklas. Mot denna bakgrund finns ett behov av att inte bara studera hur yrkesmatematik bör undervisas, utan också vad matematiskt yrkeskunnande som utbildningsinnehåll innebär (FitzSimons & Boistrup, 2017; Straesser, 2015).

Teori

Fenomenografi

Fenomenografi är en empirisk och explorativ forskningsansats med fokus på att beskriva strukturella variationer i erfارande (Marton & Booth, 1997). Liksom fenomenologi (Husserl, 1995) bygger fenomenografi på antagandet att ett meningsbärande fenomen konstitueras i relationen mellan den som erfar och det

som erfars. Detta kan enligt fenomenografin ske på ett begränsat antal kvalitativt skilda sätt (Marton & Booth, 1997). Att beskriva, strukturera och inbördes relatera dessa olika sätt att erfara, är målet för en fenomenografisk studie. Detta beskrivs ofta som att forskaren intar ett andra ordningens perspektiv. I en fenomenografisk studie är fenomenet alltså inte fördefinierat utan beskrivs i termer av den variation av uppfattningar (eller sätt att erfara) som kan observeras inom en grupp. Resultat presenteras vanligen i form av ett utfallsrum med beskrivningskategorier, som representerar kvalitativt skilda sätt att erfara ett fenomen (Marton & Booth, 1997).

Lärande, enligt ett fenomenografisk synsätt, innebär att utveckla mer kraftfulla sätt att erfara eller förstå världen. En persons handlingsmöjligheter i en given situation antas bero av hur situationen uppfattas. Vissa sätt att erfara kan därmed betraktas som mer kraftfulla än andra, genom att de möjliggör mer ändamålsenliga eller flexibla sätt att agera (Pang & Ki, 2016). Enligt Marton och Booth (1997) beror skillnader i sätt att erfara på vilka aspekter av ett fenomen som urskiljs. Hur något förstås beror på vilka aspekter som samtidigt framträder i det fokala medvetandet. De aspekter som urskiljs och hur relationer skapas både internt inom fenomenet och externt i relation till dess omgivning, konstituerar ett visst sätt att erfara. Aspekter som är avgörande för att kunna erfara något på ett kraftfullt och ändamålsenligt sätt benämns *kritiska aspekter* (Marton & Booth, 1997; Pang & Ki, 2016). Lärande kan därmed förstås som en process där kritiska aspekter urskiljs och relateras.

Fundamentalt inom VTL är intentionalitetsprincipen, vilken innebär att medvetandet alltid är riktat mot något. Det som framträder i medvetandet tenderar att uppfattas som en meningsfull helhet (Marton & Booth, 1997). Sådana intentionala objekt benämns här *fenomen*. På liknande sätt förutsätter verbet 'att lära' ett objekt, det vill säga ett lärandeobjekt. Det är utifrån detta perspektiv meningslöst att tala om lärande utan att också definiera detta objekt (Marton & Booth, 1997). Ett lärandeobjekt kan förstås som det sätt att erfara ett innehåll eller en situation som bedöms vara eftersträvansvärt i en viss utbildningskontext (Pang & Ki, 2016). Lärandeobjektet motsvarar de kritiska aspekter som behöver urskiljas för att kunna förstå något på ett kvalificerat sätt. Dessa aspekter utgör avgörande skillnader i erfandet av ett fenomen och är därmed centrala i övergången från begränsad förståelse till kvalificerat kunnande (Marton, 2015; Pang & Ki, 2016).

När ett fenomen erfars framstår andra element som bakgrund, eller en extern horisont. Dessa element är inte i direkt fokus, men finns närvarande i medvetandet och bidrar till den relevansstruktur som ger fenomenet dess mening (Marton & Booth, 1997). Booth (1992) beskriver problemlösning i termer av ett skiftande

fokus där element från den externa horisonten successivt träder fram i fokus och därmed skapar nya fenomen. Eftersom människans förmåga att samtidigt hålla flera aspekter i fokus är begränsad (Marton & Booth, 1997), innebär problemlösning en dynamisk process där olika fenomen och aspekter successivt aktualiseras. Kritiska aspekter av ett problem består därmed av den samlade uppsättning aspekter som behöver urskiljas för att problemet ska kunna förstås och hanteras på ett kvalificerat sätt.

Variationsteori

Variationsteori har sitt ursprung i den fenomenografiska forskningstraditionen. En central utgångspunkt inom teorin är att variation är ett nödvändigt villkor för lärande. En aspekt definieras här som en dimension av variation och de värden en aspekt kan anta benämns 'egenskaper'. För att urskilja en viss egenskap måste elever först erfara kontrasterande värden inom den aktuella dimensionen. Exempelvis behöver en elev erfara olika värden inom aspekten färg (t.ex. grön, gul, röd) för att aspekten ska kunna urskiljas, vilket i sin tur är nödvändigt för att exempelvis en viss egenskap t.ex. 'grön' ska uppfattas som meningsfull (Marton, 2015). På motsvarande sätt kräver begreppsförståelse att inte enbart upprepade exempel av begreppet ges (t.ex. många exempel av likformiga figurer), utan att även icke-exempel presenteras (t.ex. figurer som i viss mån liknar varandra men inte är likformiga enligt definitionen). Lärande sker, enligt variationsteori, inte genom att kritiska aspekter enbart benämns eller förklaras; elever behöver ges möjlighet att urskilja dem genom variation (Gu m.fl., 2017).

Marton (2015) beskriver en optimal sekvens av variationsmönster som inleds med en odifferentierad helhet. Genom kontrast och generalisering öppnas successivt kritiska aspekter så att ett urskiljande blir möjligt. *Kontrast* innebär att värden inom en aspekt varierar medan övriga aspekter hålls invarianta, vilket gör den aktuella aspekten urskiljbar. *Generalisering* innebär att den kritiska aspekten hålls invariant medan andra aspekter varierar. På det sättet frikopplas aspekten från det specifika exemplet. Det övergripande målet är *fusion*, där flera kritiska aspekter samtidigt fokuseras och relateras, så att fenomenet kan erfaras som en differentierad och samtidigt sammanhållen helhet (Marton, 2015; Marton & Pang, 2006). Gu m.fl. (2017) beskriver en liknande princip genom begreppen *concept*- och *non-concept variation* och framhåller att betydelsefulla aspekter av exempelvis ett geometriskt begrepp kan synliggöras genom användning av icke-exempel. Det vill säga, genom exempel där figurer varierar mellan att tillhöra begreppet (t.ex. vara

likformiga) och att inte göra det (t.ex. vara icke-likformiga), beroende på om villkor såsom vinkelkongruens och proportionalitet mellan motsvarande sidor är uppfyllda (*non-concept variation*). När innebörden av ett begrepp väl har etablerats behöver elever få möjlighet att skilja betydelsefulla egenskaper från icke-betydelsefulla. Detta sker genom att betydelsefulla aspekter (t.ex. villkor för likformighet) hålls invariants medan icke betydelsefulla aspekter varierar (t.ex. rotation och spegling). Gu m.fl. (2017) benämner detta *concept-variation*.

Flera studier med variationsteori som utgångspunkt visar att valet av uppgifter så väl som deras inbördes struktur är avgörande för elevers lärande, och att enstaka eller slumpmässigt valda exempel inte räcker för att öppna upp centrala dimensioner av variation (Gu m.fl., 2017; Sun, 2011; 2019; Watson & Mason, 2006).

Analys och resultat

Studie 1

I Studie 1 beskrivs kvalitativt skilda sätt att erfa 'matematiskt yrkeskunnande, som utbildningsinnehåll'. Studien är en fenomenografisk intervjustudie, där yrkeslärare och matematiklärare intervjuats. Utfallsrummet representerar lärares kvalitativt skilda sätt att uppfatta detta innehåll. Syftet med studien är alltså inte att artikulera människors faktiska kunnande, utan att undersöka hur detta kunnande erfars. Begreppet 'kunnande' används här i linje med exempelvis Carlgren m.fl. (2015) för att beteckna en form av kunskap som inte enbart avser fakta och begrepp, utan även innefattar förmågan att använda, tolka och ge mening åt denna kunskap i olika sammanhang.

En fenomenografisk forskningsansats bedömdes vara särskilt ändamålsenlig i denna studie, eftersom den möjliggör en kategorisering som tar sin utgångspunkt i det empiriska materialet och som synliggör såväl gemensamma drag som betydelsefulla skillnader mellan uppfattningar.

Det empiriska material som används i en fenomenografisk analys behöver vara fokuserat och väl avgränsat kring ett gemensamt fenomen, men samtidigt rymma en variation av perspektiv (Marton & Booth, 1997). Därför genererades data genom semistrukturerade intervjuer. Åtta yrkeslärare (betecknade V) och åtta matematiklärare (betecknade M) deltog i studien. Samtliga intervjuer utgick från en gemensam uppsättning uppgifter, som konstruerats i syfte att belysa olika aspekter

av likformighetsbegreppet och samtidigt relatera till en bygg -och anläggningskontext.

Analysen genomfördes som en iterativ process där det empiriska materialet lästes, jämfördes och kategoriserades. Genom denna process utvecklades empiriskt grundade och tydligt avgränsade beskrivningskategorier. Målet var att etablera ett utfallsrum där kategorierna bildar en logiskt sammanhängande struktur och samtidigt representerar kvalitativt skilda synsätt. För att stärka studiens tillförlitlighet involverades även andra forskare i analysprocessen.

Resultatet presenteras i form av ett utfallsrum, bestående av sex kategorier. Dessa beskriver uppfattningar som sträcker sig från ett fokus på *procedurreproduktion och isolerade metoder*, via *praktikbaserad och kontextbunden förståelse* av specifika yrkesuppgifter, till *teoretisk och begreppslig förståelse* av yrkesrelevant matematik. Vidare identifieras uppfattningar där betoningen ligger på att synliggöra *kopplingar* mellan matematikklassrummet och i yrkespraktiken, samt på en *integrering* av matematiskt och yrkesmässigt kunnande. Den mest fördjupade förståelsen framställer matematiskt yrkeskunnande som en viktig komponent i elevens yrkesmässiga 'bildning', där matematisk förståelse bidrar till flexibilitet, kreativitet och nytänkande. Resultaten visar att lärare i varierande grad uppmärksammar såväl matematiska som yrkesmässiga dimensioner och relationen mellan dem. Matematiskt yrkeskunnande framträder här inte enbart som grundläggande aritmetik, utan som ett komplext och mångfacetterat fenomen. Tillsammans belyser kategorierna spänningar och komplexitet snarare än ömsesidigt uteslutande eller hierarkiskt ordnade perspektiv.

En central dimension rör spänningen mellan *rutinmässiga procedurer* och *begreppslig förståelse*. Inom vissa perspektiv betonas utförande av standardiserade metoder, medan andra perspektiv lyfter fram förståelse av underliggande matematiska principer och samband. Tidigare forskning visar att tumregler och rutinmässiga beräkningar kan fylla en viktig funktion på arbetsplatsen (Roth, 2014; Zevenbergen & Zevenbergen, 2009). Samtidigt riskerar ett ensidigt fokus på sådana procedurer att underliggande samband osynliggörs och möjligheten till flexibel problemlösning och välinformerade tolkningar försvåras (t.ex. Hoyles m.fl., 2010, Gravemeier).

En andra dimension gäller *kontextbundna resonemang* i relation till *matematisk generalisering*. Inom vissa kategorier ligger fokuset på kontextbunden, praktisk och materiellt förankrad kunskap, inom andra framträder vikten av att kunna generalisera bortom den specifika situationen som centralt kunnande. Tidigare forskning visar att ett ensidigt fokus på antingen matematik som generaliserbar

och abstrakt kunskap eller som situerad och praktikinära verksamhet riskerar att antingen begränsa möjligheterna till generalisering och överförbarhet, eller att man bortser från matematikens situerade och kontextberoende karaktär (jfr FitzSimons, 2014).

En tredje dimension handlar om *isolerade färdigheter* kontra *integrerat kunnande*. I de mer avancerade kategorierna betraktas matematiskt yrkeskunnande som en integrerad helhet där matematiska, yrkesspecifika och erfarenhetsbaserade aspekter samverkar. Detta ligger i linje med forskning som beskriver arbetsplatsmatematik som mångfacetterad och kontextuellt situerad (Hoyles m.fl., 2010; LaCroix, 2014).

En fjärde dimension rör *direkt tillämpning* i kontrast till aktivt *medierad transfer*. Vissa uppfattningar innebär ett förgivettagande att matematikkunskap enkelt kan överföras mellan skola och arbetsliv, medan andra fokuserar specifikt på denna överföring som något som inte är självklart utan som elever behöver bli medvetna om och aktivt utveckla. Studien ligger därmed i linje med tidigare forskning, som visar att meningsfull transfer utvecklas genom aktivt medierad reflektion och jämförelser mellan kontexter (Triantafyllou & Potari, 2014; FitzSimons & Boistrup, 2017; Marton, 2006).

Slutligen framträder en dimension mellan *omedelbar uppgiftslösning* och *långsiktig yrkesmässig bildning*. Inom de flesta kategorier är förmågan att hantera givna problem inom avgränsade och välkända situationer i fokus, medan en kategori ger uttryck för en bredare förståelse av matematiskt yrkeskunnande. Här förstås matematikkunskap som en resurs för att utveckla kvalificerat omdöme, flexibilitet och kreativt tänkande, vilket möjliggör ett självständigt och ansvarsfullt handlande i nya, komplexa och föränderliga arbetssituationer, samtidigt som det bidrar till en fördjupad och mer sammanhållen förståelse av yrket som helhet.

Sammantaget visar studien att matematiskt yrkeskunnande i en utbildningskontext kan förstås som ett spänningsfält mellan kvalitativ skilda, men samtidigt ömsesidigt kompletterande, dimensioner, där balansen mellan dem är avgörande för vilka former av kunnande som synliggörs och värderas i yrkesutbildningen.

Studie 2

Studie 2 undersöker hur matematiskt yrkeskunnande kan komma till uttryck i arbetet med autentiska yrkesuppgifter. Yrkeslärares beskrivningar av elevers svårigheter och den förståelse som krävs för att hantera sådana uppgifter

analyserades tillsammans med elevdialoger, där elever löser yrkesrelaterade problem i realistiska kontexter. Studien identifierar kritiska aspekter som elever behöver urskilja för att kunna förstå och hantera dessa situationer. Studien bidrar därmed till forskning inom yrkesmatematik genom en fördjupad beskrivning, som på detaljnivå synliggör vad elever behöver lära sig för att utveckla relevant matematiskt yrkeskunnande. Studien kompletterar även tidigare forskning inom VTL genom att tillämpa det teoretiska ramverket på autentiska yrkessituationer, en kontext som inte behandlats inom denna forskningstradition tidigare.

Det empiriska materialet består av lärarintervjuer och uppgiftsbaserade elevdialoger. Åtta yrkeslärare inom bygg (betecknade V1–V8) intervjuades om sina erfarenheter av elevers utmaningar vid lösning av autentiska yrkesmatematiska uppgifter. Därutöver spelades diskussioner mellan två elever (betecknade S1 och S2) in när de löste motsvarande uppgifter. Intervjuerna (cirka 1 timme vardera) och dialogen (cirka 1,5 timme, uppdelat på 4 tillfällen) transkriberades.

Under lärarintervjuerna ombads deltagarna att ge exempel på yrkesrelaterade matematikuppgifter samt att lyfta fram aspekter som de ansåg särskilt viktiga och utmanande för elever att förstå. Intervjutranskriptionerna lästes upprepade gånger i syfte att identifiera återkommande och likartade exempel på sådana uppgifter. Utifrån denna analys valdes fem uppgiftstyper (det vill säga uppgifter med liknande praktiskt syfte och underliggande struktur) ut för vidare analys. För att välja lämpliga uppgifter till elevdialogerna eftersöktes motsvarande uppgiftstyper i matematikläroböcker riktade mot byggrelaterade yrken. Eftersom äldre läroböcker erbjuder ett brett utbud av sådana uppgifter användes ett urval av sådana som underlag.

De transkriberade lärarintervjuerna och elevdialogen analyserades, med fokus på att identifiera kritiska aspekter. I lärarintervjuerna definierades kritiska aspekter som 'de aspekter som lärarna betonade som centrala och utmanande'. I elevdialogen identifierades kritiska aspekter genom analys av de sekvenser där elevernas förståelse var i förändring, det vill säga där de ännu inte fullt ut behärskade en aspekt men samtidigt visade tecken på begynnande förståelse. Dessa aspekter blev synliga i diskussionerna som centrala punkter där elevernas resonemang förändrades eller utmanades, och där skillnader i förståelse trädde fram.

I linje med resultat från tidigare studier (t.ex. Hoyles, 2010; Johansson m.fl., 2024; LaCroix, 2014; Martin & LaCroix, 2008) visar resultatet av den här studien att autentiska uppgifter innefattar aspekter från flera olika områden och belyser på så sätt den mångfacetterade karaktären hos yrkesmatematisk problemlösning. De

identifierade aspekterna tillhör olika domäner. Vissa ligger nära formell matematik (t.ex. proportionella relationer, geometriska begrepp t.ex. likformighet, och trigonometriska samband), medan andra representerar ett slags matematiskt tänkande som är sammanflätat med logiska resonemang i den praktiska yrkeskontexten. Detta innefattar exempelvis att göra rimlighetsbedömningar, tolka och anpassa beräkningar utifrån situationens förutsättningar samt att väga samman matematiska resultat med materialegenskaper och arbetsprocesser i den aktuella kontexten (t.ex. principer för lämplig avrundning och effekter av kumulativa fel).

Andra aspekter är mer direkt yrkesspecifika och kopplade till verktyg, normer, mätteknik och byggstandarder, såsom korrekt mätteknik, gränsvärden för avstånd och lämpliga lutningar. Därutöver framträder behovet av att urskilja aspekter av den fysiska miljön. Dessa kan relateras till matematik, men kräver även kunskap som går utöver det rent matematiska, exempelvis förståelse för hur och varför korrekt pannavstånd och val av takmaterial påverkas av taklutning, samt hur materialvolym kan förändras vid yttre fysisk påverkan. På motsvarande sätt kan uppgifterna innefatta situationer där geometriska begrepp, tolkning av ritningar och rumsligt resonemang behöver samordnas, exempelvis i urskiljandet av relationen mellan tredimensionella objekt och deras tvådimensionella representationer. Sammantaget tyder resultaten på att yrkesmatematisk kompetens inte enbart handlar om att 'överföra' skolmatematik till yrkespraktik, utan om att kunna urskilja och integrera kritiska aspekter över flera domäner och förstå hur matematiska principer samspelar med praktiska och fysiska rammar.

Studie 3

I studie 3 analyseras lärandemöjligheter i matematikläroböcker för årskurs 1 på gymnasiet yrkesprogram, med variationsteori som analytiskt verktyg. Likformighetsbegreppet valdes som analysobjekt på grund av dess matematiska tyngd och dess relevans inom bygg och anläggningsyrken (se t.ex. studie 2)

Läromedel har visat sig spela en betydande roll för elevers lärande genom att fungera som en brygga mellan den avsedda och den genomförda läroplanen (t.ex.; Valverde m.fl., 2002; Qi m.fl., 2025; Rezat, m.fl., 2021). Även om lärobokens roll inte är given utan formas av enskilda lärares praktik, visar forskningsresultat att läromedel både har en direkt påverkan både på elevers lärande och influerar lärares val av innehåll och undervisningsstrategier (Fan m.fl., 2021). Läromedelsanalyser kan därför ge värdefull kunskap om hur skilda lärandemöjligheter formas.

Flera olika ramverk har utvecklats för läromedelsanalys i syfte att exempelvis undersöka internationella skillnader (t.ex. Charalambous m.fl., 2010; Valverde m.fl., 2002; Pepin & Haggarty, 2001; Jäder m.fl., 2020). Studier av läromedel inom utbildningsvetenskaplig forskning omfattar analyser av läroböckers innehåll, deras påverkan, samt hur de används i undervisning (Rezat & Straesser, 2015). Många läromedelsanalyser bygger på systematisk innehållsanalys. Dessa studier använder ofta en deduktiv ansats, där fördefinierade kategorier grundade i tidigare forskning om det fenomen som undersöks, vägleder analysen. Samtidigt inkluderas ofta även induktiva inslag genom att kategorier förfinas eller utvidgas i relation till det empiriska materialet. Inom fältet används både kvantitativa och kvalitativa metoder, ibland i kombination, vilket möjliggör analys av frekvens och fördelning av exempelvis uppgiftstyper, innehåll, representationsformer tillsammans med kognitiva krav, strukturer och pedagogiska implikationer. Ramverken varierar i omfattning: vissa syftar till att ge breda, jämförande beskrivningar av läromedel i olika länder (t.ex. Valverde m.fl., 2002; Charalambous m.fl., 2010), medan andra fokuserar på specifika matematiska begrepp eller kompetenser, såsom problemlösning, resonemang eller modellering (t.ex. Frejd, 2013; Lo m.fl., 2006; Jäder m.fl., 2020).

I den här studien används variationsteori för analys av uppgifter och uppgiftssekvenser. Analysen utgår från teoretiska antaganden, men inte från på förhand fastställda kategorier eller beskrivningar av specifika kritiska aspekter. Dessa utgör i stället analysens resultat. Variationsteori har tidigare främst använts för undervisningsdesign och för att studera lärandemöjligheter i klassrumsmiljöer. Att använda variationsteori för läromedelsanalys är relativt ovanligt. Några studier har dock visat hur variationsmönster inom uppgifter och uppgiftssekvenser kan skapa lärandemöjligheter (Sun, 2019; Zhang m.fl., 2017; Watson & Mason, 2006). Föreliggande studie följer de analysmetoder av lärandemöjligheter, som är etablerade inom variationsteoretisk forskning, men skiljer sig i sitt utförande från mer typiska induktiva och deduktiva läromedelsanalyser.

Mer specifikt syftar den här studien till att undersöka lärandemöjligheter av det geometriska begreppet likformighet i matematikläroböcker som används i den första kursen inom gymnasiets yrkesprogram.

Tidigare studier visar att likformighet är ett begrepp som ofta är utmanande för elever att förstå (Lamon, 1993; Son, 2013). Läroböcker på gymnasienivå hanterar ofta begreppet på ett förenklat sätt där definitioner är semiformala eller helt informella, vilket gör att väsentliga aspekter utelämnas (Lo m.fl., 2006; Seago m.fl., 2014). Mot bakgrund av likformighetsbegreppets betydelse inom flera domäner,

såväl inom yrkespraktik som inom mer avancerad matematik, samt den dokumenterade tendensen att förståelsen ofta vilar på informella och otydliga föreställningar (e.g. Wu, 2005; Vollrath, 1977, Haj-Yahara, 2021), framstår en fördjupad studie av vilka möjligheter läromedel erbjuder för att urskilja kritiska aspekter av begreppet som särskilt relevant.

I Studie 3 analyseras matematikläroböcker för första kursen i gymnasiets yrkesprogram (Ma1a) från tre förlag: Lärobok A Origo 1a, 2021, Lärobok B (Matematik 5000 1a, 2011) och Lärobok C (Exponent 1a, 2021). Endast tryckt material analyseras. Samtliga läroböcker för kursen som fanns tillgängliga på marknaden under perioden 2020–2022 beaktades, och de böcker som innehöll ett avsnitt om likformighet valdes ut för analys. Såvitt författaren känner till är ingen av läroböckerna explicit utformad utifrån variationsteori. Analysen syftar inte till att utvärdera läromedel från olika förlag, utan att påvisa hur skillnader i lärandemöjligheter kan formas genom de variationsmönster som framträder inom uppgifter och över uppgiftssekvenser.

De tre läroböckerna analyserades uppgift för uppgift i syfte att identifiera kritiska aspekter. Variationsmönster, av det slag som beskrivs i teoriavsnittet, uppmärksammades i uppgifter och uppgiftssekvenser. De variationsdimensioner som läromedlet därigenom erbjuder kartlades, och de aspekter som på det sättet görs möjliga för elever att urskilja, identifierades. På så sätt möjliggjordes en jämförelse av de lärandemöjligheter som de olika läroböckerna erbjuder.

Resultatet visar stora skillnader i lärandemöjligheter mellan läroböckerna. Medan en lärobok huvudsakligen fokuserar på en specifik procedur för att lösa en särskild typ av likformighetsuppgifter, gör en annan lärobok flera väsentliga aspekter av likformighetsbegreppet möjliga att urskilja. I den tredje läroboken förekommer en mängd likformighetsuppgifter, men någon systematik i variationen är svår att identifiera.

Lärobok A: Här riktas fokus mot likformighetsbegreppet och hur det kan tillämpas. Analysen visar att ett flertal begreppsliga aspekter synliggörs genom variationsmönster. Läroboken fokuserar inte enbart på en procedur för att lösa typuppgifter, utan framställer likformighet som ett geometriskt begrepp, där villkor för likformighet, likformighetsbevarande transformationer och likformighetssatser för trianglar synliggörs. Liksom i Lärobok B förekommer typuppgifter (där ett saknat värde ska bestämmas) med ökande svårighetsgrad. Till skillnad från lärobok B sker denna progression genom att ytterligare begreppsliga aspekter successivt integreras. Dessa aspekter introduceras först separat genom kontrast och generalisering i tidigare uppgifter, vilket möjliggör ett stegvis urskiljande av olika

aspekter, som därefter kan samordnas i mer komplexa problem. Tillsammans skapar dessa uppgifter möjligheter att urskilja vilka aspekter som är avgörande för likformighet (genom kontrast), liksom vilka som inte påverkar likformighet (genom generalisering). Exempel på aspekter som synliggörs genom kontrast och generalisering är *kongruenta vinklar* och *proportionalitet mellan motsvarande sidor* som *villkor för likformighet*, samt *likformighetsbevarande transformationer*. Vidare synliggörs *tillräckliga villkor för likformighet hos trianglar* som ett särskilt fall.

Lärobok B: Här riktas fokus mot en specifik procedur för att lösa en viss typ av uppgift. Analysen visar att de variationsdimensioner som öppnas i Lärobok B främst synliggör *en specifik problemlösningsprocedur* (att använda proportionalitet för att beräkna en okänd sida i likformiga figurer). I den inledande uppgiftssekvensen guidas eleverna till en generell strategi som sedan kan tillämpas i samtliga senare uppgifter. Proceduren (d.v.s. identifiera motsvarande sidor, beräkna skalfaktor och använda den för att beräkna den okända längden) hålls invariant, medan numeriska värden och kontext varierar (en typ av generalisering). Detta sker på ett systematiskt genom att svårighetsgraden ökar och stödet minskar. Trots att nödvändiga villkor för likformighet (lika vinklar och proportionella sidor) nämns i introduktionstexten till avsnittet, öppnas inte motsvarande variationsdimensioner upp genom kontrast eller generalisering, det vill säga, det ges inga kontrasterande exempel på likformiga respektive icke-likformiga figurer kopplade till villkoren, och nästan alla likformiga figurer har samma inbördes orientering. Detta innebär att betydelsefulla aspekter av likformighet inte skiljs från icke-betydelsefulla aspekter som orientering och färg. I uppgifterna är likformighet ofta ett givet villkor, vilket gör att kriterierna för likformighet inte behöver beaktas för att lösa uppgifterna. Trianglar nämns kort som ett specialfall, men denna aspekt öppnas inte heller som en variationsdimension och likformighetssatser för trianglar är inte avgörande för att lösa uppgifterna.

Lärobok C innehåller ett flertal uppgifter om likformighet, men analysen visar få tecken på systematisk variation, varken gällande procedurträning eller begreppsförståelse. I introduktionen presenteras en följd av likformiga figurer med samma inbördes orientering, där eleverna förväntas förstå idén om likformighet genom upprepade exempel. Påståendet 'figurerna har samma form men olika storlek; de är likformiga' upprepas för varje par av likformiga figurer. Detta kan göra det möjligt att urskilja att likformighet gäller olika figurtyper (generalisering) och att likformiga figurer kan skilja sig i storlek. Däremot blir andra betydelsefulla aspekter, som proportionella sidor, lika vinklar eller likformighetsbevarande transformationer, inte urskiljbara.

En procedur för att finna en okänd sida demonstreras kort och kan användas i flera uppgifter, men uppgifternas komplexitet ökar inte systematiskt. I de flesta uppgifter är likformighet en given förutsättning och figurerna har samma orientering. I vissa uppgifter behöver dock villkor och likformighetssatser för trianglar användas för att identifiera likformiga trianglar. Till skillnad från i lärobok A är dessa aspekter varken urskiljbara eller ens omnämnda innan de blir avgörande för lösningen av mer komplexa problem. Varken vinklarnas betydelse för likformighet eller det faktum att vissa transformationer bevarar likformighet omnämns (än mindre synliggörs) innan kongruenta vinklar måste användas för att identifiera likformiga trianglar i komplexa figurer.

Sammanfattningsvis visar analysen att alla läroböcker i viss mån ger möjlighet att öva en procedur där proportionalitet används för att beräkna en saknad sidlängd i trianglar där likformighet är en given förutsättning. När det gäller begreppsliga aspekter finns däremot stora skillnader i vilka lärandemöjligheter som erbjuds.

Det finns även aspekter av likformighet som, enligt tidigare forskning, kan vara viktiga för elever att urskilja men som inte synliggörs i någon av läroböckerna. Exempelvis används begreppet 'motsvarande sidor' samtidigt som det varken förklaras eller synliggörs genom variation. Vidare görs transformationer urskiljbara i Lärobok B som icke-betydelsefulla genom generalisering, men de öppnas inte genom kontrast i någon av läroböckerna. Det innebär att de särskilda transformationer som kan behöva urskiljas för att visualisera motsvarande delar, inte görs synliga (jfr Seago m.fl., 2014). I alla läroböcker är proportionalitet mellan motsvarande sidor möjlig att urskilja, i betydelsen att sidlängder i den ena triangeln är en bestämd multipel av motsvarande sidlängder i den andra. Däremot synliggörs inte att förhållandet mellan två sidor inom en triangel är detsamma som förhållandet mellan motsvarande sidor i den andra triangeln, det vill säga att kvoterna mellan motsvarande sidor är lika (t.ex. $AB/AC = A'B'/A'C'$).

Resultatet visar alltså att de tre läroböckerna erbjuder olika variationsmönster och därmed olika lärandemöjligheter. Om detta resultat jämförs med resultaten från Studie 1 och 2, som visar att matematiskt yrkeskunnande förutsätter både begreppslig förståelse och förmåga att samordna matematiska, praktiska och yrkesmässiga aspekter, kan slutsatsen dras att dessa skillnader inte enbart har betydelse för elevers möjligheter att utveckla förståelse för likformighet som matematiskt begrepp. De påverkar också elevernas förutsättningar att använda likformighet i yrkesrelaterade problem på ett flexibelt, situationsanpassat och meningsfullt sätt.

Diskussion

Som framhålls i inledningen utgör matematik en betydelsefull komponent i bygg- och anläggningsarbete. Samtidigt är forskning angående yrkesmatematik och matematikundervisning inom yrkesutbildning underrepresenterad i utbildningsvetenskaplig forskning. Förståelsen av vad kompetent matematiskt yrkeskunnande innebär, hur detta kunnande kan ta sig uttryck i specifika yrkeskontexter och hur detta kan stödjas inom ramen för gymnasial yrkesutbildning är därför fortfarande begränsad (t.ex. Bakker, 2014; FitzSimons, 2014; Straesser, 2015). Mot denna bakgrund syftar föreliggande avhandling till att belysa matematiskt yrkeskunnande i en utbildningskontext på ett sätt som gör det mer explicit och analytiskt preciserat, och samtidigt öppnar för nyansering och reflektion. Genom detaljerade analyser av yrkesuppgifter och undervisningsmaterial identifieras och jämförs de kritiska aspekter som aktualiseras i autentiska yrkespraktiker (Studie 2) med dem som framträder i matematikläroböcker (Studie 3). Vidare synliggörs variation i lärares uppfattningar om vad som utgör ett utbildningsmässigt meningsfullt matematiskt yrkeskunnande, samt relationer mellan dessa uppfattningar i form av spänningar och ömsesidigt kompletterande perspektiv (Studie 1). Ett centralt bidrag är den systematiska kartläggningen av dessa uppfattningar och deras inbördes relationer.

Sammantaget visar resultaten från de tre studierna att ett kraftfullt matematiskt yrkeskunnande kan förstås som en hybrid form av kunnande som sträcker sig bortom matematik som disciplin. Det innefattar en samordning av grundläggande matematiska idéer (t.ex. mätning, proportionalitet, likformighet och lutning) med visuospatial aspekter; tekniska representationer; fysiska och materiella förutsättningar; yrkesmässiga normer samt principer från andra discipliner (t.ex. teknik och fysik).

Detta kunnande är inte begränsat till att utföra korrekta beräkningar, utan omfattar även att avgöra vilka strategier som är lämpliga under givna förutsättningar samt att tolka resultat på ett meningsfullt sätt i den specifika yrkeskontexten. För att matematikundervisning ska fungera som en meningsfull del av en sammanhängande yrkesutbildning behöver undervisningen ta hänsyn till denna komplexitet.

I avhandlingen framhålls att matematiskt yrkeskunnande kan förstås som ett utbildningsinnehåll vars betydelse sträcker sig bortom att hantera specifika uppgifter. En djupare och mer sammanhängande matematisk förståelse kan ses som en viktig del av elevers yrkesmässiga 'bildning', genom att möjliggöra mer

kraftfulla sätt att erfar yrkessituationer, som leder till ett mer flexibelt, kreativt och ansvarsfullt handlande i praktiken. Även om lokala och procedurbaserade metoder kan vara tillräckliga för att utföra välbekanta uppgifter, riskerar ett ensidigt fokus på sådana strategier att begränsa elevers möjligheter att generalisera, anpassa och reflektera i varierande situationer. Syntesen av resultaten pekar mot att ett kraftfullt matematiskt yrkeskunnande förutsätter både integration av aspekter från skilda domäner och en konceptuell förståelse av matematiska begrepp och samband.

Vidare indikerar resultaten att de procedurer som betonas i matematikundervisning, särskilt genom läroböcker, tenderar att ge begränsade möjligheter att urskilja de begreppsliga relationer som ligger till grund för matematiskt yrkeskunnande i praktiken. Som exempel kan nämnas att de aspekter av likformighet (t.ex. relationer mellan kongruenta vinklar och proportionella sidor; sambandet mellan inre och yttre kvot) som förenar olika tillämpningar inom yrket (t.ex. skala på ritningar; skalning av egyptiska triangeln; lutning som kvot och lutning som vinkel) endast i begränsad omfattning synliggörs i de läromedel som analyserades i Studie 3.

Resultaten av de ingående studierna sammantaget, tolkade genom ett variationsteoretiskt ramverk, pekar mot att matematikundervisningen inte nödvändigtvis bör sträva efter att direkt återskapa arbetslivets fulla komplexitet i matematikklassrummet. I stället argumenterar avhandlingen för noggrant designade undervisningssekvenser som stödjer en gradvis utveckling av kraftfullt matematiskt yrkeskunnande. Sådana sekvenser kan inledas med att kritiska aspekter görs urskiljbara inom respektive ämne och i relation till sammanhängande begreppsstrukturer, för att därefter integreras i mer komplexa yrkesrelaterade situationer. Samordnad undervisning mellan matematik och yrkesämnen framstår som central i denna process. Resultaten pekar också på behovet av fortsatt forskning där undervisning designas baserat på analys av autentiska yrkesuppgifter, på ett sådant sätt att kritiska aspekter systematiskt görs urskiljbara genom samordnad undervisning, i syfte att stärka elevers flexibla och begreppsligt förankrade matematiska kunnande i en yrkeskontext.

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Appendix 1

Centralt innehåll Matematik 1a (2011)

Undervisningen i kursen ska behandla följande centrala innehåll:

Taluppfattning, aritmetik och algebra

- Metoder för beräkningar med reella tal skrivna på olika former inom vardagslivet och karaktärsämnena, inklusive överslagsräkning, huvudräkning och uppskattning samt strategier för att använda digitala verktyg.
- Strategier för att använda hjälpmedel från karaktärsämnena, till exempel formulär, mallar, tumregler, föreskrifter, manualer och handböcker.
- Hantering av algebraiska uttryck och för karaktärsämnena relevanta formler samt metoder för att lösa linjära ekvationer, såväl med som utan digitala verktyg.

Geometri

- Egenskaper hos och representationer av geometriska objekt, till exempel ritningar, praktiska konstruktioner och koordinatsystem.
- Geometriska begrepp valda utifrån karaktärsämnenas behov, till exempel skala, vektorer, likformighet, kongruens, sinus, cosinus, tangens och symmetrier.
- Metoder för mätning och beräkning av storheter som är centrala för karaktärsämnena.
- Enheter, enhetsbyten och behandling av måttetal som är centrala för karaktärsämnena samt hur man avrundar på ett för karaktärsämnena relevant sätt.

Samband och förändring

- Fördjupning av procentbegreppet: promille, ppm och procentenheter.
- Begreppen förändringsfaktor och index. Metoder för beräkning av räntor och amorteringar för olika typer av lån med kalkylprogram.
- Begreppen förhållande och proportionalitet i resonemang, beräkningar, mätningar och konstruktioner.
- Skillnader mellan linjära och exponentiella förlopp.

Sannolikhet och statistik

- Beskrivande statistik med hjälp av kalkylprogram samt granskning av hur statistiska metoder och resultat används i samhället och i yrkeslivet.
- Begreppen beroende och oberoende händelser samt metoder för beräkning av sannolikheter vid slumpförsök i flera steg med exempel från spel och risk- och säkerhetsbedömningar.

Problemlösning

- Strategier för matematisk problemlösning inklusive modellering av olika situationer, såväl med som utan digitala verktyg.
- Hur matematiken kan användas som verktyg i behandlingen av omfattande problemsituationer i karaktärsämnena. Matematikens möjligheter och begränsningar i dessa situationer.
- Matematiska problem av betydelse för privatekonomi, samhällsliv och tillämpningar i andra ämnen.
- Matematiska problem med anknytning till matematikens kulturhistoria.

Centralt innehåll Matematik 1a (2021)

Undervisningen i kursen ska behandla följande centrala innehåll:

Matematik inom karaktärsämnen och yrkesliv

- Matematiska begrepp som är relevanta för karaktärsämnen och yrkesliv, till exempel proportionalitet, skala, Pythagoras sats, procent och andelar, indexmått, vinstmarginal, jämvikt, felmarginaler, symmetrier, vektorer, trigonometriska funktioner och matematiska begrepp som utvecklas under förskoleåldern.
- Beräkningsmetoder som är relevanta för karaktärsämnen och yrkesliv, till exempel uppskattningar, beräkningar på störningar eller mätfel, spill- och svinnberäkningar, överslagsräkning, avrundning, användning av kalkylprogram och metoder för kontrollberäkning.
- Hantering av formler som är relevanta för karaktärsämnen och yrkesliv.
- Mätning och hantering av storheter och enheter som är relevanta för karaktärsämnen och yrkesliv, till exempel enhetsbyten, avrundningsprinciper, tidsuppskattningar, beräkning av förbrukningsmaterial, kostnadsberäkningar, säkerhetsmarginaler, hantering av mätverktyg och hantering av mätosäkerheter.
- Hjälpmedel och verktyg som är relevanta för att hantera matematik inom karaktärsämnen och yrkesliv, till exempel formulär, mallar, tumregler, föreskrifter, manualer, referensverk och handböcker.

Aritmetik, algebra och funktioner

- Hantering av algebraiska uttryck, inklusive att faktorisera och multiplicera uttryck.
- Begreppet funktion. Representationer av funktioner i form av ord, funktionsuttryck, tabeller och grafer. Digitala metoder för att skapa funktionsgrafer.
- Metoder för att bestämma funktionsvärden. Grafiska metoder för att lösa ekvationer av typen $f(x) = a$.
- Begreppet linjär funktion och egenskaper hos linjära funktioner.
- Metoder för att lösa linjära ekvationer.
- Begreppet exponentialfunktion och egenskaper hos exponentialfunktioner, inklusive skillnader och likheter med linjära funktioner.
- Begreppet förändringsfaktor och beräkning av förändringar i flera steg.

Sannolikhet och statistik

- Begreppen oberoende och beroende händelse samt komplementhändelse. Metoder för att beräkna sannolikheter i flera steg, inklusive exempel från spel, risk- och säkerhetsbedömningar.
- Exempel på hur några statistiska begrepp används i samhälle och yrkesliv, inklusive signifikans, korrelation, kausalitet, urvalsmetoder och felkällor.

Problemlösning, verktyg och tillämpningar

- Användning av kalkylprogram för beräkning av ränta och amortering.
- Användning av digitala verktyg för att effektivisera beräkningar och komplettera metoder, till exempel vid ekvationslösning.
- Problemlösning som omfattar att upptäcka och uttrycka generella samband.
- Problemlösning som omfattar begrepp och metoder i kursen, med särskild utgångspunkt i yrkesliv, privatekonomi och samhällsliv.
- Tillämpning och formulering av matematiska modeller i realistiska situationer. Utvärdering av matematiska modellens egenskaper och begränsningar
- Matematiska problem med anknytning till matematikens kulturhistoria.

(The National Agency for Education, 2021)

Centralt innehåll (Matematik 1a, 2025)

Program- eller yrkesspecifikt innehåll

- Matematiska begrepp som är relevanta för arbetslivet, till exempel proportionalitet, skala, likformighet, vinklar, Pythagoras sats, procent och andelar, indexmått, vinstmarginal, jämvikt, felmarginaler, symmetrier, vektorer, trigonometriska funktioner och barns lärande inom matematik.
- Beräkningsmetoder som är relevanta för arbetslivet, till exempel uppskattningar, beräkningar på störningar eller mätfel, spill- och svinnberäkningar, överslagsräkning, avrundning, användning av kalkylprogram och metoder för kontrollberäkning.
- Hantering av formler som är relevanta för arbetslivet.
- Mätning och hantering av storheter och enheter som är relevanta för arbetslivet, till exempel enhetsbyten, mätning av vinklar, avrundningsprinciper, tidsuppskattningar, beräkning av förbrukningsmaterial, kostnadsberäkningar, säkerhetsmarginaler, hantering av mätverktyg och hantering av mätosäkerheter.
- Hjälpmedel och verktyg som är relevanta för att hantera matematik inom arbetslivet, till exempel formulär, mallar, tumregler, föreskrifter, manualer, referensverk och handböcker.

Aritmetik, algebra och funktioner

- Hantering av formler och algebraiska uttryck, däribland faktorisering och multiplicering av uttryck.
- Begreppet funktion. Representationer av funktioner i form av ord, funktionsuttryck, tabeller och grafer. Digitala metoder för att skapa funktionsgrafer.
- Metoder för att bestämma funktionsvärden. Grafiska metoder för att lösa ekvationer av typen $f(x) = a$.
- Begreppet linjär funktion och egenskaper hos linjära funktioner.
- Metoder för att lösa linjära ekvationer.
- Begreppet exponentialfunktion och egenskaper hos exponentialfunktioner. Skillnader och likheter med linjära funktioner.
- Begreppet förändringsfaktor och beräkning av förändringar i flera steg.

Sannolikhet och statistik

- Begreppen oberoende och beroende händelse samt komplementhändelse. Metoder för att beräkna sannolikheter i flera steg. Tillämpningar inom spel samt risk- och säkerhetsbedömningar.
- Exempel på hur några statistiska begrepp används i samhälle och arbetsliv, däribland signifikans, korrelation, kausalitet, urvalsmetoder och felkällor.

Digitala verktyg

- Användning av kalkylprogram för beräkning av ränta och amortering.
- Användning av digitala verktyg för att effektivisera beräkningar och komplettera metoder, till exempel vid ekvationslösning och problemlösning.

Problemlösning och tillämpningsområden

- Problemlösning som omfattar att upptäcka och uttrycka generella samband.
- Problemlösning med särskild utgångspunkt i arbetslivet samt privatekonomi och samhällsliv, däribland frågeställningar som berör hållbar utveckling och hur matematik kan användas för kritisk granskning av fakta och påståenden.
- Tillämpning och formulering av matematiska modeller i realistiska situationer. Utvärdering av matematiska modellers egenskaper och begränsningar.
- Orientering om något ur matematikens historia, till exempel hur ett matematiskt begrepp utvecklats, matematikens roll i något historiskt skeende, en betydande person inom matematiken eller ett historiskt matematiskt problem.

(The National Agency for Education, 2025, Nivå 1a - Matematik - GY25)

Hej!

Jag heter Hanna Knutson och är doktorand i ämnesdidaktik vid Göteborgs universitet. Jag skulle vilja fråga dig, som undervisar i matematik eller yrkesämnen på bygg -och anläggnings programmet, om du vill delta i en intervjustudie.

Mitt forskningsintresse handlar om matematikundervisningen på gymnasiets yrkesprogram.

Studien är framför allt en läromedelsanalys, och det område inom matematiken som jag valt att titta närmare på är begreppet likformighet och de yrkesmatematiska tillämpningarna, ritningar (skala) och lutningsförhållande (t.ex. av ett tak, en väg, etc.). För att få en bild av olika lärares erfarenheter av vad som kan vara särskilt viktigt eller svårt för elever att lära sig inom detta område, planerar jag att även genomföra en intervjustudie. Därför skulle jag vara väldigt tacksam om du vill träffa mig för en intervju! Intervjun kommer att ta omkring 60 min.

Hör av dig till mig om du har möjlighet att delta, så återkommer jag med mer detaljerad information om projektet och intervjun. Du är också välkommen att höra av dig med frågor. Jag nås enklast på

hanna.knutson@gu.se.

Huvudansvarig för projektet är min handledare
Prof. Angelika Kullberg

Vänliga hälsningar
Hanna Knutson

Appendix 3



Matematik i yrkesutbildning och arbetsliv

Information om projektet och vad det innebär att delta i studien

Jag heter Hanna Knutson och är doktorand vid Göteborgs universitet. Jag vill fråga dig om du vill delta i en forskningsstudie. I det här dokumentet får du information om projektet och vad det innebär att delta.

Forskningsprojektet

Den här studien handlar om matematikundervisningen på gymnasiets yrkesprogram och om kopplingar mellan matematik och yrkesämnen. Syftet med intervjun är att bättre förstå hur matematiska begrepp används, dels i matematikundervisningen och dels i yrkessammanhang, och vad som kan vara viktigt eller utmanande för eleverna att förstå. Intervjun kommer att utgå från några matematikuppgifter inom området geometri. Det material som samlas in under intervjun är dels ljudinspelningar, dels era eventuella anteckningar, dels videofilm från kamera riktad mot anteckningarna.

Din medverkan

Om du tackar ja till att delta i studien, bestämmer vi tid och plats för intervjun, förslagsvis träffas vi på din skola. Senast vid intervjutillfället behöver jag ditt underskrivna samtycke till att delta i studien (se nedan). Intervjun kommer att ta omkring 60 minuter.

Att delta i studien är naturligtvis helt frivilligt. Om du av någon orsak, innan eller i samband med intervjun vill avbryta din medverkan, meddelar du bara det, och du behöver då inte ange något skäl. Efter det att intervjun är genomförd och analysen av materialet har påbörjats, kommer denna möjlighet däremot att vara begränsad.

Datasäkerhet

Det material som samlas in under intervjun kommer inte att vara åtkomligt för obehöriga. Däremot behöver det i viss mån vara tillgängligt för andra forskare, till exempel för det syfte att man i framtiden vill kontrollera studiens resultat. I det skriftliga material som produceras

inom projektet kommer pseudonymer för lärare och skolor att användas. Utomstående kommer alltså inte att kunna knyta dessa texter till din identitet. De data och uppgifter som samlas in i studien förvaras kodade på en server hos huvudmannen Göteborgs universitet. Arbetsmaterial som inte är digitaliserat eller transkriberat kommer att förvaras inlåsta på institutionen i brandsäkra skåp. Data sparas i minst 10 år i enlighet med Arkivlagstiftningen vid Göteborgs universitet, och får bara användas på det sätt som du har gett samtycke till.

Ansvarig för personuppgifterna är Göteborgs universitet. Göteborgs Universitetet följer Vetenskapsrådets forskningsetiska principer och dataskyddsförordningen, GDPR, för att säkerställa att data skyddas mot obehörig insyn och olaglig behandling, och mot oavsiktlig förlust eller förstörelse. Du har rätt att kostnadsfritt få ta del av de uppgifter om dig som hanteras i projektet, och vid behov få eventuella fel rättade. Du kan också begära att uppgifter om dig raderas, samt att behandlingen av personuppgifterna begränsas. Rätten till radering och till begränsning av behandling av personuppgifter gäller dock inte när uppgifterna är nödvändiga för den aktuella forskningen.

Studiens resultat

Resultat av studien kommer att presenteras vid forskarkonferenser, samt i min doktorsavhandling. Avhandlingen, och eventuella artiklar i anslutning till den, kommer att finnas på nätet och på Göteborgs universitetsbibliotek. Du är också välkommen att kontakta mig om du har frågor om studien eller om du vill veta var du kan hitta resultatet.

Ansvariga för studien

Göteborgs Universitet är huvudman för forskningen och ansvarig forskare för doktorandprojektet är professor Angelika Kullberg, Göteborgs Universitet, Institutionen för didaktik och pedagogisk profession. angelika.kullberg@ped.gu.se

Dataskyddsombud nås på dataskydd@gu.se eller telefon: 031-786 00 00

Tack för hjälpen!

Hanna Knutson

Doktorand på Institutionen för didaktik och pedagogisk profession, Göteborgs Universitet
hanna.knutson@gu.se

Samtycke till att delta i studien

Jag har fått information om studien och har haft möjlighet att ställa frågor. Jag får behålla den skriftliga informationen.

Jag samtycker till att delta i studien *Matematik i yrkesutbildning och arbetsliv*. Jag har fått information om studiens syfte, hur undersökningen kommer att gå till och hur data kommer att hanteras.

Jag samtycker till att uppgifter om mig behandlas på det sätt som på det sätt som beskrivs i forskningspersonsinformationens.

Plats och datum
Underskrift
Namnförtydligande

Appendix 4

Intervju:

Intervjun kommer framför allt att utgå ifrån uppgifterna nedan. Läs gärna igenom dem innan intervjun. Några exempel på frågor som vi kommer att diskutera är:

- Hur skulle du beskriva det kunskapsområde som uppgifterna representerar?
- Välj ut den uppgift som du tycker är mest central. På vilket sätt stämmer innehållet i uppgiften överens med vad du anser att dina elever behöver lära sig?
- När du läser uppgifterna: Tycker du att det är något som saknas i sammanhanget? (Ta, i så fall, gärna med dig ett exempel på en uppgift, som kompletterar uppgifterna, nedan, i det avseendet.)

Uppgifter:

- ❖ De vanligaste skalorna på byggritningar är 1:50 och 1:100. Ett hus är 6,5 cm brett på en ritning i skala 1:200.
 - Hur brett är huset i verkligheten?
 - Vad blir husets bredd på en ritning i skala 1:50?
 - Vad blir husets bredd på en ritning i skala 1:100?
 - Vad innebär det för ritningens area om den görs i skala 1:100 i stället för skala 1:200?
- ❖ Dräneringen invid ett hus ska ha en lutning på 1:200.
 - Hur stor blir lutningen i procent?
 - Hur stor blir den totala höjdskillnaden i millimetre om sträckan från den högsta till den lägsta punkten är 12,5 metre?
- ❖ Man kan konstruera en rät vinkel genom att bilda en triangel med sidorna 3 cm, 4 cm och 5 cm (enligt fig.1).

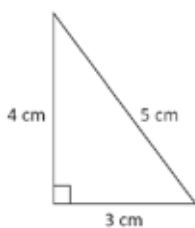


Fig.1

- Ge exempel på andra trianglar, likformiga med den i figuren, som kan vara lämpliga att använda för att konstruera en rät vinkel från en husvägg om man t.ex. vill bygga en altan eller ett plank.
- Varför är den triangel, som du föreslår, bättre att använda än den i fig.1?

The interview was structured around the tasks presented below. Participants were invited to read the tasks in advance. During the interview, discussion revolved around questions such as:

- How would you describe the knowledge domain represented by these tasks?
- Which task do you consider most central? In what way does its content correspond to what you believe students need to learn?
- When reading the tasks, do you find that something important is missing? If so, please provide an example of a complementary task.

The tasks addressed the following situations:

1. Scale drawings

The most common scales in construction drawings are 1:50 and 1:100. A house measures 6.5 cm in width on a drawing with scale 1:200.

- How wide is the house in reality?
- What would its width be on a drawing with scale 1:50?
- What would its width be on a drawing with scale 1:100?
- What happens to the area of the drawing if the scale is changed from 1:200 to 1:100?

2. Drainage slope

The drainage around a house must have a slope of 1:200.

- What is this slope expressed as a percentage?
- What is the total height difference in millimetres if the distance between the highest and lowest point is 12.5 metres?

3. Constructing a right angle

A right angle can be constructed using a triangle with side lengths 3 cm, 4 cm and 5 cm (see Fig. 1).

- Give examples of other triangles, similar to this one, that could be used to construct a right angle when building a porch or fence.
- Why would your proposed triangle be more suitable than the one in Fig. 1?

4. Constructing a porch roof

A porch is to be built according to the drawing (Fig. 2).

- The porch roof should have the same slope angle as the roof of the house. Calculate the height to the ridge of the porch roof (distance x).
- The window in the door is 40 cm high and 30 cm wide. Is this window similar to any of the other windows in the building?

– How much larger is the area of the window to the right of the door compared to the window in the door?

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What does it mean to know mathematics in a vocational context, and how can such knowing be constituted in an educational context?

This thesis explores vocational mathematics knowing, in the context of construction work, aiming to illuminate how mathematics, as it is embedded in vocational practice, can be understood and expressed as educational content. Drawing on phenomenography and variation theory, three studies investigate how teachers understand vocational mathematics knowing, how such knowing is expressed in authentic problem solving, and what learning opportunities mathematics textbooks for vocational programmes afford.

The findings illuminate vocational mathematics knowing as an educational content that extends beyond the mere application of school mathematics. It involves coordinating conceptual understanding with contextual awareness, practical reasoning, and vocational experience. At the same time, the thesis underscores mathematical coherence and conceptual depth as central components of this knowing. It further argues that the limited opportunities afforded in textbooks to discern essential aspects of key mathematical structures may constrain students' possibilities to develop the kind of knowing that entails a powerful way of understanding and engaging with the vocational context.



Hanna Knutson is a PhD student at the Department of Pedagogical and Professional Studies. Her research interest concerns mathematics teaching in vocational education.

