Similarity Problems

Which Groups Are Unitarizable?

Master's thesis in Mathematics

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Abstract

This thesis covers some theory on similarity of group representations to unitary representations. We discuss the notion of amenability and give some classes of groups that are amenable. We then prove the Dixmier-Day theorem, that states that a locally compact group G is unitarizable if it is amenable. We also investigate the converse of this statement, which is still an open problem. We will give some statements where we make some assumptions on the similarity that are equivalent to amenability. We will also investigate when bounded algebra homomorphism $A \to B(H)$, where A is a C^*-algebra, are similar to a *-homomorphism. We will present connections between the unitarizability of groups and unitarizability of group C^*-algebras, and this will be useful for some results about the converse of the Dixmier-Day theorem. We will also investigate the notions of completely positive and completely bounded maps and prove Stinespring's theorem for completely positive maps followed by Wittstock's theorem for completely bounded maps. We then prove Haagerup's theorem that states that unitarizability of homomorphisms is equivalent to the property of being completely bounded.

Keywords: amenability, completely bounded maps, Dixmier-Day theorem, Haagerups theorem, Kadison's problem, unitarizable groups, unitarizable representations

Contents

1 Introduction

In this thesis, we will explore when representations of locally compact groups, meaning group homomorphisms into the invertible, bounded linear operators on a Hilbert space, are similar to unitary representations. We will also investigate when an algebra representation $\pi : A \to B(H)$, where A is a C^*-algebra, are similar to a *-homomorphism. A key result regarding the first question is the Dixmier-Day theorem, that gives that if G is an amenable group, similarity to a unitary representation is equivalent to the property of being uniformly bounded. We will define the property of amenability of locally compact groups and prove some results about this property. To define the property of amenability, we need a measure on the group. We will define (up to multiplication by a constant) a non-zero left translation invariant Radon measure on the group, called *Haar measure*. This is necessary to define $L^\infty(G)$, and we will define amenability by the existance of a bounded linear functional $m \in L^\infty(G)^*$ such that $\langle 1, m\rangle = 1 = \|m\|$ and $\langle f, m\rangle = \langle L_s f\rangle$, where $L_s f(t) = f(s^{-1}t)$. We will prove that certain classes of groups are amenable. This includes abelian groups, finite groups and compact groups.

Groups that have the property that every uniformly bounded representation is similar to a unitary representation are called *unitarizable*. By the Dixmier-Day theorem, amenability implies unitarizablility. The converse is still an open problem, but we will prove that we can get a statement that is equivalent to amenability by adding conditions on the similarity S such that $S^{-1}\pi(\cdot)S$ is unitary. We will also prove that any non-abelian free group is not unitarizable, and that any discrete group that has a non-abelian free group is non-unitarizable.

Some results on the unitarizability of groups will use the theory of group C^*-algebras. The Kadison problem asks whether any bounded *-homomorphism $A \to B(H)$ is similar to a *-homomorphism. We will prove that an algebra homomorphism is equivalent to a *-homomorphism if and only if it is completely bounded. This reduces the Kadison problem to determine whether any bounded homomorphism is completely bounded. We will use this machinery to prove that G is unitarizable if and only if every countable subgroup is unitarizable.

2 Preliminaries

This section will cover some notions that are needed in this thesis.

2.1 Locally compact groups

We will begin by stating some definitions.

Definition 2.1. A *group* is a set G with a binary operation $(a, b) \mapsto ab$ such that:

- $a(bc) = (ab)c$ for all $a, b, c \in G$
- there exists an element in G, denoted by 1 such that $a1 = 1a = a$ for all $a \in G$.

- for all $a \in G$, there exists an element in G, denoted by a^{-1}, such that $aa^{-1} = a^{-1}a = 1$.

We call G abelian if $ab = ba$ for all $a, b \in G$.

Definition 2.2. A *subgroup* is a subset $H \subseteq G$ such that it is closed under the operation of G, and it forms a group with this operation. A subgroup is said to be normal if it is closed under $h \mapsto ghg^{-1}$ for all $g \in G$.

Definition 2.3. A *(left) coset* of a subgroup H is a set on the form $gH = \{gh : h \in H\}$. If N is a normal subgroup, the quotient group G/N is defined as the set of left cosets with operation $(gN)(hN) = (gh)N$. It can be verified that this operation is well defined.

The following definition is from [F, sec 2.1]

Definition 2.4. A *topological group* is a group G with a topology such that the mappings $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

A topological space X is called *locally compact* if every point $x \in X$ has a *compact neighborhood*, which is an open set U and a compact set K such that $x \in U \subseteq K$. A topological group is called locally compact if the underlying topology is locally compact and Hausdorff. We will define

$$AB := \{ab : a \in A \text{ and } b \in B\}, xA = \{xa : a \in A\}$$

and

$$A^{-1} = \{a^{-1} : a \in A\}.$$

The set Ax is defined in the obvious way. We say that a set $A \subseteq G$ is symmetric if $A = A^{-1}$.

Example 2.5. The following are examples of locally compact groups:

- any discrete group
- $(\mathbb{R}, +)$ with the standard topology
- $(\mathbb{R} \setminus \{0\}, \cdot)$ with the standard topology
- $GL_n(\mathbb{R})$ with the operator norm topology.

Example 2.6. The group $(\mathbb{R}, +)$ with the finite complement topology is not a topological group as the preimage of the open set $\mathbb{R} \setminus \{0\}$ under $(x, y) \mapsto x + y$ is $\mathbb{R}^2 \setminus \{(x, y) : x = -y\}$. The only open sets in the product topology are the sets where we remove a finite number of vertical or horizontal lines. Hence, addition is not continuous.

Example 2.7. The group $(\mathbb{Q}, +)$ is a topological group with the subspace topology inherited from $\mathbb{R}$ with the standard topology, but it is not locally compact. Pick $q \in \mathbb{Q}$ and pick a neighborhood U of q in $\mathbb{R}$. We get that $U \cap \mathbb{Q}$ is a neighborhood of q in $\mathbb{Q}$. We get that there is no compact set $K \supseteq U \cap \mathbb{Q}$ with respect to the topology on $\mathbb{Q}$ as we can pick open intervals approaching an irrational number in U to obtain an open cover without a finite subcover.

Another concept from group theory is the concept of group actions.

Definition 2.8. A *group action* of a group G on a set X is a group homomorphism $\phi : G \to Symm(X)$.

We write gx to denote $(\phi(g))(x)$. The relation $x \sim y$ if and only if there exists $g \in G$ such that $gx = y$ is an equivalence relation. The equivalence classes are called *orbits*.

2.2 Construction of Haar measure

On the real line, there exists a translation invariant measure normalizing the unit interval, Lesbesgue measure. It is not defined on all subsets of the real line, but it is defined on all Borel sets. The real line with the standard topology and addition is an example of a locally compact group and Lesbesgue measure is invariant under (left) translation. Hence it is natural to ask whether any locally compact group has such a measure. It will not be uniquely defined in the same way as Lesbesgue measure, as there is no obvious choice of which subset to normalize, but it will be a Radon measure and it will be unique up to multiplication by a constant. The aim of this section is to construct such a measure, that is called Haar measure, on locally compact groups. We will follow [F, Sec. 2.2]. We will now state the definition of a Haar measure.

Definition 2.9. A *Radon measure* is a measure such that every compact set has finite measure. A *Haar measure* μ on a locally compact group G is a non-zero Radon measure such that $\mu(gE) = \mu(E)$ for all $g \in G$ and for every Borel set $E \subseteq G$.

We now introduce some notation. We use $C_c(G)$ to denote the space of compactly supported functions $G \to \mathbb{R}$ and we define the set

$$C_c^+(G) := \{f \in C_c(G) : f \geq 0 \text{ and } f \not\equiv 0\}.$$

We will sometimes write C_c^+ when the group G is obvious from the context. We define the operations $L_y f(x) := f(y^{-1}x)$ and $R_y f(x) := f(xy)$. We will now state a definition.

Definition 2.10. A function $f : G \to \mathbb{C}$ is said to be *left uniformly continuous* if

$$\|L_y f - f\|_\infty \to 0, y \to 1.$$

We call f *right uniformly continuous* if the statement holds with R_y instead of L_y.

We will now state and prove a result that will be used later in the section.

Proposition 2.11. *Let $f \in C_c(G)$, then f is right and left uniformly continuous.*

Proof. We will give the proof from [F, Proposition 2.6] and prove right uniform continuity, left uniform continuity can be proven similarly. Let $K = supp(f)$ and pick $\varepsilon > 0$ arbitrarily. For every x there exist a neighborhood U_x of 1 such that

$$|f(xy) - f(x)| < \frac{1}{2}\varepsilon$$

for $y \in U_x$. It follows from the continuity of the product that there is a symmetric neighborhood V_x of 1 such that $V_x V_x \subseteq U$. We observe that $\{xV_x\}_{x \in K}$ is an open cover of K and by compactness it has a finite subcover $\{x_i V_{x_i}\}_{i=1}^n$. Let

$$V = \bigcap_{i=1}^n V_{x_i}.$$

It remains to prove that $\|f(xy)-f(x)\| < \varepsilon$ for $y \in V$. If none of x and xy is an element of K, the result follows trivially, so we only need to prove the result in two cases when $x \in K$ and when $xy \in K$. Fix $x \in K$. We easily observe that there is an i such that $x_i^{-1}x \in V_{x_i}$. We see that $xy = x_i(x_i^{-1}x)y \in x_i U_{x_i}$ by the definitions of the sets V_x and V. We use the triangle inequality to get

$$|f(xy) - f(x)| \leq |f(xy) - f(x_i)| + |f(x1) - f(x_i)| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The case where $xy \in K$ can be handled in the same way. □

We will now prove an existence theorem and a uniqueness theorem for Haar measure.

Theorem 2.12. *Every locally compact group has a Haar measure.*

Proof. We will give the proof from [F, Thm 2.10]. Let $f, \phi \in C_c^+(G)$ and define

$$(f : \phi) := \inf\{\sum_{i=1}^{n} c_i : f \leq \sum_{i=1}^{n} c_i L_{x_i}\phi \text{ for some } x_1, ..., x_n \in G\}.$$

We now pick $f_0 \in C_c^+(G)$ and define

$$I_\phi := \frac{(f : \phi)}{(f_0 : \phi)}.$$

We note that I_θ is subadditive and that additivity is the only criteria that fails for it to be a linear functional restricted to C_c^+. We note that

$$\begin{aligned} (L_y f : \phi) &= \inf\{\sum c_i : L_y f \leq \sum c_i L_{x_j}\phi\} \\ &= \inf\{\sum c_i : f(y^{-1}x) \leq \sum c_i \phi(x_j^{-1}yy^{-1}x)\}. \end{aligned} \tag{1}$$

Pick $y^{-1}x_j' = x_j$ and make the variable substitution $x' = y^{-1}x$. We easily see that I_ϕ is invariant under left translations. Before we proceed, we will need the following result.

Lemma 2.13. *For any f_1, f_2 in C_c^+ and for any $\varepsilon > 0$, there exists a neighborhood V of 1 such that for all ϕ with supp(ϕ) $\subseteq V$ we have that*

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \varepsilon.$$

Proof. Let $g \in C_c^+(G)$ be such that $g = 1$ on *supp*$(f_1 + f_2)$ and let δ be a positive real number. Let $h = f_1 + f_2 + \delta g$ and define $h_i(x) := f_i(x)/h(x)$ for all x such that $f_i(x) \neq 0$ and let $h_i(x) = 0$ otherwise and note that $h_i \in C_c^+(G)$, $i \in \{1, 2\}$. We apply Proposition 2.11 and get that there exists a neighborhood V of 1 such that

$$|h_i(x) - h_i(y)| = |h_i(y(y^{-1}x)) - h_i(y)| = |R_{y^{-1}x}h_i(y) - h_i(y)| < \delta$$

for $y^{-1}x \in V$. Let $\phi \in C_c^+(G)$ with *supp*$(\phi) \subseteq V$. If $h \leq \sum c_j L_{x_j}\phi$ we get that

$$\begin{aligned} f_i(x) &= h(x)h_i(x) \\ &\leq \sum c_j L_{x_j}\phi(x)h_i(x) \\ &= \sum c_j \phi(x_j^{-1}x)h_i(x) \\ &= \sum c_j \phi(x_j^{-1}x)(h_i(x) - h_i(x_j) + h_i(x_j)) \\ &\leq \sum c_j \phi(x_j^{-1}x)(h_i(x_j) + \delta). \end{aligned} \tag{2}$$

It is obvious that $h_1 + h_2 \leq 1$ and we observe that

$$(f_1 : \phi) + (f_2 : \phi) \leq \sum c_j(h_1(x_j) + \delta) + \sum c_j(h_2(x_j) + \delta) \leq \sum c_j(1 + 2\delta).$$

Moreover,

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) = \frac{(f_1 : \phi) + (f_2 : \phi)}{(f_0 : \phi)} \leq \frac{\inf \sum c_j(1 + 2\delta)}{(f_0 : \phi)} \\ = (1 + 2\delta)I_\phi(h) \leq (1 + 2\delta)(I_\phi(f_1 + f_2) + \delta I_\phi(g)). \end{aligned} \tag{3}$$

We observe that $(f : \phi) \leq (f : \psi)(\psi : \phi)$, as $f \leq \sum b_i L_{x_i}\psi$, and $\psi \leq \sum c_j L_{y_j}\phi$ implies that $f \leq \sum b_i c_j L_{y_j x_i}\phi$. We now get that $(f : \phi) \leq (f : f_0)(f_0 : \phi)$ and $(f_0 : \phi) \leq (f_0 : f)(f : \phi)$. This gives that $I_\phi(f) \in [(f_0 : f)^{-1}, (f : f_0)]$ and we get that it suffices to pick δ small enough for

$$2\delta(f_1 + f_2 : f_0) + (1 + 2\delta)\delta(g : f_0) < \varepsilon.$$

We get that

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) - I_\phi(f_1 + f_2) \\ \leq 2\delta(I_\phi(f_1 + f_2) + (1 + 2\delta)\delta I_\phi(g) \\ \leq 2\delta(f_1 + f_2 : f_0) + (1 + 2\delta)\delta(g : f_0) < \varepsilon. \end{aligned} \tag{4}$$

This completes the proof of the lemma. □

For every $f \in C_c^+(G)$ we define the interval

$$X_f := [(f_0 : f)^{-1}, (f : f_0)].$$

We denote the product of these intervals by X. It is trivial that X is Hausdorf, and it is compact by Tychonoff's theorem. We note that X is the set of functions from C_c^+ to $(0, \infty)$ such that f maps to an element of X_f. We note that I_ϕ is an element of this set for all $\phi \in C_c^+(G)$. We denote the closure of the set

$$\{I_\phi : supp(\phi) \subseteq V\}$$

in X by $K(V)$ and note that

$$\bigcap_{i=1}^{n} K(V_i) \supseteq K(\bigcap_{i=1}^{n} V_i).$$

Hence, $K(V)$ has the finite intersection property. Hence, as X is compact, let $I \in K(V)$ for every neighborhood V of 1. We note that for any $x_1, ..., x_n \in C_c^+(G)$, any $\varepsilon > 0$ and any neighborhood V of 1, there exists ϕ with $supp(\phi) \subseteq V$, such that $|I(x_i) - I_\phi(x_i)| < \varepsilon$ for all i. We pick V as in Lemma 2.13 and get that

$$\begin{aligned} I(f_1) + I(f_2) - I(f_1 + f_2) \\ \leq I_\phi(f_1) + I_\phi(f_2) - I_\phi(f_1 + f_2) + 3\varepsilon \leq 4\varepsilon. \end{aligned} \tag{5}$$

The additivity of I follows by letting $\varepsilon \to 0$. We observe that any $f \in C_c(G)$ can be written as $f = f_1 - f_2$ for $f_1, f_2 \in C_c^+(G)$. We define $I(f)$ as $I(f_1) - I(f_2)$. It is obvious that this extension is well defined. Haar measure is the measure μ implied by Riesz representation theorem, this is the step where local compactness is used.

□

Theorem 2.14. *Haar measure is unique up to multiplication by a positive constant.*

Proof. We follow the proof from [F, Thm 2.20]. Let μ and λ be two Haar measures and let $f, g \in C_c^+(G)$. We want to prove that

$$\frac{\int f d\mu}{\int f d\lambda} = \frac{\int g d\mu}{\int g d\lambda}.$$

Let V_0 be a symmetric compact neiborhood of 1 and define the sets $A = supp(f)V_0 \cup V_0 supp(f)$ and $B = supp(g)V_0 \cup V_0 supp(g)$. We get that A and B are compact by the continuity of the product and that $f(xy) - f(yx)$ is supported in A and $g(xy) - g(yx)$ is suppported in B. For any $\varepsilon > 0$ it follows from Proposition 2.11 that there is a symmetrict neighborhood $V \subset V_0$ of 1 such that

$$|f(xy) - f(yx)| = |f(xy) - f(x) - f(yx) + f(x)| \leq |f(xy) - f(x)| + |f(yx) - f(x)| < \varepsilon$$

and $|g(xy) - g(yx)| < \varepsilon$ for all x and for all $y \in V$. Let $h \in C_c^+(G)$ be such that $h(x) = h(x^{-1})$ and $supp(h) \subseteq V$. We now get that

$$\begin{aligned} \int h d\mu \int f d\lambda &= \int \int h(y) f(x) d\lambda(x) d\mu(y) \\ &= \int \int h(y) f(yx) d\lambda(x) d\mu(y). \end{aligned} \tag{6}$$

By using the assumption that $h(x) = h(x^{-1})$, we get that

$$\begin{aligned} \int h d\lambda \int f d\mu &= \int \int h(x) f(y) d\lambda(x) d\mu(y) \\ &= \int \int h(y^{-1}x) f(y) d\lambda(x) d\mu(y) \\ &= \int \int h(x^{-1}y) f(y) d\mu(y) d\lambda(x) \\ &= \int \int h(y) f(xy) d\mu(y) d\lambda(x) \\ &= \int \int h(y) f(xy) d\lambda(x) d\mu(y). \end{aligned} \tag{7}$$

We get that

$$\begin{aligned} &\left| \int h d\lambda \int f d\mu - \int h d\mu \int f d\lambda \right| \\ = &\left| \int \int h(y)(f(xy) - yx) d\lambda(x) d\mu(y) \right| \\ &\leq \varepsilon \lambda(A) \int h d\mu. \end{aligned} \tag{8}$$

By the same reasoning we get that

$$\left| \int h d\mu \int g d\lambda - \int h d\lambda \int g d\mu \right| \leq \varepsilon \lambda(B) \int h d\mu.$$

We multiply the first inequality by

$$\int h d\mu \int f d\mu$$

and the second inequality by

$$\int hd\mu \int gd\mu$$

and add the resulting inequalities. We get

$$\begin{aligned} &\varepsilon\left(\frac{\lambda(A)}{\int fd\mu}+\frac{\lambda(B)}{\int gd\mu}\right) \\ &\geq \left|\frac{\int hd\lambda}{\int hd\mu}-\frac{\int fd\lambda}{fd\mu}\right|+\left|\frac{\int gd\lambda}{\int gd\mu}-\frac{\int hd\lambda}{\int hd\mu}\right| \\ &\geq \left|\frac{\int gd\lambda}{\int gd\mu}-\frac{\int fd\lambda}{fd\mu}\right|. \end{aligned} \tag{9}$$

We complete the proof by letting $\varepsilon \to 0$. □

We will now give examples of Haar measure.

Example 2.15. The following are examples of Haar measure:

- Lesbesgue measure on $(\mathbb{R}^n, +)$;
- $\mu(A) = \frac{c(A)}{|G|}$ on finite groups, where c denotes the counting measure;
- $\frac{dx}{|x|}$ on $(\mathbb{R} \setminus \{0\}, \cdot)$, where dx denotes the Lesbesgue measure.

We note that any compact group has a Haar measure μ such that $\mu(G) = 1$. We call this Haar measure normalized.

2.3 Banach spaces and operators

Here, we will cover some concepts from functional analysis that are used in this thesis. We will now give some definitions.

Definition 2.16. A *normed vector space* is a vector space X with a map $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$, called a *norm*, such that for $c \in \mathbb{C}, x, y \in X$

- $\|cx\| = |c|\|x\|$,
- $\|x\| = 0$ implies $x = 0$,
- $\|x + y\| \leq \|x\| + \|y\|$.

A Banach space is a normed vector space such that every Cauchy sequence converges. An *inner product space* is a vector space Y with a map $\langle \cdot, \cdot \rangle$, called an *inner product*, that maps a pair of elements in Y to the underlying field, such that for $c \in \mathbb{C}, x, y \in X$

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An inner product induces a norm by $\|x\|^2 = \langle x, x\rangle$. A Hilbert space is an inner product space such that the induced norm gives a Banach space.

Recall the definition of a linear operator. We will now give some more definitions.

Definition 2.17. A linear operator is said to be *bounded* if $\sup_{\|x\|=1} \|Tx\| < \infty$. This supremum is denoted by $\|T\|$ and is called the *operator norm* of T. A *linear functional* is a linear mapping with the underlying field as it's codomain. The space of bounded linear functionals on a normed vector space X is denoted by X^* and is called the *dual space* of X. The space of bounded linear operators $X \to Y$ is denoted $B(X, Y)$. We write $B(X)$ instead of $B(X, X)$.

We have the following theorem.

Theorem 2.18 (Riesz representation)**.** *Let H be a Hilbert space and let T be a bounded linear functional. Then there exists a unique $y \in H$ such that $Tx = \langle x, y\rangle$ for all $x \in X$.*

Considering that we have this result for Hilbert spaces, it makes sense to use the notation $\langle x, y\rangle$ for $x \in X$ and $y \in X^*$, even if X only is a Banach space. The map $\langle \cdot, \cdot\rangle$ is called the *standard pairing.*

Example 2.19. Let $X = l^p$ and let $1/p + 1/q = 1$, $1 < p < \infty$. Then $X^* = l^q$ and the standard pairing is $\langle x, y\rangle = \sum x_i \overline{y_i}$.

We may of course use the norm topology on X^*, but another topology that we will use is the w^*-topology. Consider the topological space $Y = \prod_{x\in X} \mathbb{C}$ with the product topology. The dual space is clearly a subset of this, and we may consider this space with the subspace topology inherited from Y. This topology is the w^*-topology. We have the following result.

Theorem 2.20 (Banach-Alaoglu)**.** *Let X be a Banach space. Then the unit ball in X^* is w^*-compact.*

2.4 Basic theory of C^*-algebras

Here, we will present some theory about normed algebras. We will start by stating some definitions. We will start by defining an algebra ([A, Def. 1.3.1])

Definition 2.21. An algebra is a vector space A with a multiplication such that

- $a(\beta b + \gamma c) = \beta ab + \gamma ac$ and $(\beta b + \gamma c)a = \beta ba + \gamma ca$ for all $a, b, c \in A, \beta, \gamma \in \mathbb{C}$,
- $(ab)c = a(bc)$ for all $a, b, c \in A$.

An algebra A is called *commutative* if $ab = ba$ for all $a, b \in A$. If A has a norm such that $\|ab\| \leq \|a\|\|b\|$ we say that A is a normed algebra. If A is a Banach space with respect to that norm, we call A a Banach algebra.

We will give an important definition, more specifically [A, Def. 2.2.1].

Definition 2.22. An *involution* $*$ on an algebra A is a conjugate-linear map such that $x^{**} = x$

and $(xy)^* = y^*x^*$ for $x, y \in A$. A C^*-*algebra* A is a Banach algebra with an involution $*$ such that $\|x^*x\| = \|x\|^2$.

It is not required in the definition that algebras have a unit, but in this thesis we will assume that every algebra has a unit. If $x^* = x$, we say that x is *self adjoint.* If $x^*x = xx^*$, we say that x is *normal.* If $x^*x = x^*x = 1$, we say that x is *unitary.* An important definition is the definition of the spectrum.

Definition 2.23. Let A be an algebra. The *resolvent set* $\rho(a)$ of an element $a \in A$ is defined as

$$\rho(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is invertible}\}.$$

The complement of this set is denoted by $\sigma(a)$ and is called the *spectrum* of a. The *spectral radius* of a is defined as

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Remark 2.24. By the Stone-Weierstrass theorem, every continuous function defined on a compact set of complex numbers can be approximated by polynomials in z and $\overline{z}$. It can be proven that $\sigma(a)$ is compact. By identifying z with x and $\overline{z}$ with x^* we may define continuous functional calculus for normal elements of a C^*-algebra, so that $f(a) = \lim p_n(a)$, where p_n is a sequence of polynomials approximating $f \in C(\sigma(a))$.

We note the following

Proposition 2.25. *Let A be a C^*-algebra and let $a \in A$. Then we have the following:*

- *if A is a subalgebra of a C^*-algebra B, we have that $\sigma_A(a) = \sigma_B(a)$;*
- *$\sigma(f(a)) = f(\sigma(a))$ for all normal a and $f \in C(\sigma(a))$;*
- *$\sigma(a) \neq \emptyset$.*

The second point follows immediately from [A, Thm. 2.3.1]. The third point is [A, Thm. 1.6.3].

Example 2.26. The algebra $B(H)$ equipped with the operator norm, where H is a Hilbert space and where A^* is the adjoint operator of an operator A, is a C^*-algebra. More generally, any norm-closed subalgebra of $B(H)$ that is closed under the involution is a C^*-algebra. We call such C^*-algebras *concrete* C^*-algebras [A, p. 42].

In fact, every C^*-algebra turns out to be a concrete C^*-algebra by the following theorem.

Theorem 2.27 (Gelfand-Naimark)**.** *Any unital C^*-algebra is isometrically *-isomorphic to a concrete C^*-algebra.*

Before we proceed with the proof, we will need some theory.

Example 2.28. Let X be a compact Hausdorff space. By $C(X)$ we denote the space of functions $X \to \mathbb{C}$ with the usual pointwise operations. This space is a commutative C^*-algebra when equipped with the supremum norm and complex conjugation. In fact, by *Gelfand's theorem*, every commutative C^*-algebra is on this form up to an isometric *-isomorphism.

We will now give two definitions from [A, pp. 25-26].

Definition 2.29. The *Gelfand spectrum* of an algebra A, denoted by $sp(A)$, is the set of non-trivial algebra homomorphisms $A \to \mathbb{C}$. The *Gelfand map* is the map $A \to C(sp(A))$ given by $a \mapsto \hat{a}$, where $\hat{a}(\omega) = \omega(a)$.

We note that $sp(A)$ with the $w*$-topology is a compact Hausdorff space [A, Prop. 1.9.3].

Theorem 2.30 (Gelfand)**.** *Let A be a commutative C^*-algebra. Let $X = sp(A)$. Then the Gelfand map is an isometric *-isomorphism $A \to C(X)$.*

The proof of this result can be found in [A, Thm. 2.2.4]. We will now give some definitions, the first from [A, p.126], the second from [P2, p.9], the third from [A, p.122] and the fourth from [P2, p.26].

Definition 2.31. A *positive element* of a C^*-algebra is an element such that $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$.

Remark 2.32. Let x be self adjoint, then x^2 is positive.

Definition 2.33. Let A, B be C^*-algebras. A *positive map* is a linear map $\phi : A \to B$ that maps positive elements to positive elements.

Definition 2.34. A positive linear functional ρ such that $\rho(1) = 1$ is called a *state*.

We will now collect some results on positive elements and states.

Theorem 2.35. *Let A be a C^* algebra. Then a^*a is positive for all $a \in A$.*

Proof. We will give the proof from [A, Thm. 4.8.3]. Consider the continuous functions

$$f(t) = \begin{cases} \sqrt{t} \text{ if } t \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

and

$$g(t) = \begin{cases} 0 \text{ if } t \geq 0 \\ \sqrt{-t} \text{ otherwise} \end{cases} .$$

We observe that $fg = gf = 0$ and $f(t)^2 - g(t)^2 = t$. Let $x = f(a^*a)$ and $y = g(a^*a)$. We observe that x and y are self adjoint, $xy = yx = 0$ and $a^*a = x^2 - y^2$. We get that

$$(ay)^*ay = ya^*ay = y(x^2 - y^2)y = -y^4$$

is such that its spectrum is contained in $(-\infty, 0]$. We can now apply the following result

Lemma 2.36. *Let a be such that $\sigma(a^*a) \subseteq (-\infty, 0]$. Then $a = 0$.*

The proof of this lemma can be found in [A, Lemma 4.8.2]. This shows that $ay = 0$, implying that $y^4 = 0$ giving that $y = 0$ as y is self adjoint. We get that $a^*a = x^2$ where x is self adjoint, giving that a^*a is positive. ☐

Proposition 2.37. *Let ρ be a linear functional such that $\|\rho\| = \rho(1) = 1$. Then ρ is a state.*

Proof. We will give the proof from [A, Corr. 1,p. 128]. Let ρ be a linear functional with the assumed properties. We want to prove that $\rho(x) \geq 0$ for every positive x. We claim that $\rho(z)$ is in the convex closure of $\sigma(z)$ for z normal. Any positive element is self adjoint and hence normal, so this implies the desired result. It now remains to prove the claim. The sub-C^*-algebra B generated by 1 and x is clearly commutative and hence isometrically *-isomorphic to $C(X)$. The desired result follows from [A, Lemma 1.10.3], which is the claim for $A = C(X)$. □

Corrolary 2.38. *For all $x \in A$, there exists a state ρ such that $\rho(x^*x) = \|x\|^2$*

Proof. We will follow the proof from [A, Corollary 2, p.128]. Let B be the sub-C^*-algebra generated by $y = x^*x$ and 1. We claim that we can find $\omega_0 \in sp(B)$ such that $\omega_0(y) = \|y\|$. Indeed, Gelfand's theorem gives that $\sup_{\omega \in sp(B)} |\hat{y}(\omega)| = \|y\|$ and $\hat{y}$ is positive. As X is compact and $\hat{y}$ is continuous, $\hat{y}$ must have a maximum at some point ω_0. We use Hahn-Banach to extend ω to a linear functional $\rho : A \to \mathbb{C}$. We get $\|\rho\| = \rho(1) = 1$, completing the proof. □

One important result is the Gelfand-Naimark-Segal constgruction. As the construction is important for several results in the thesis, we will present it's proof here.

Theorem 2.39 (Gelfand-Naimark-Segal)**.** *Let ρ be a positive linear functional on a C^*-algebra. Then there exists a Hilbert space H, a *-homomorphism $\pi : A \to B(H)$ and an element $\xi \in H$ such that*

- $\overline{\pi(A)}\xi = H$
- $\rho(x) = \langle \pi(x)\xi, \xi \rangle$ *for all $x \in A$.*

We call the pair (π, ξ) a GNS-pair.

Proof. We will follow the proof from [A, Thm. 4.7.3]. Consider the positive semidefinite sesquilinear form $\langle a, b \rangle' := \rho(a^*b)$. As it is positive semidefinite, it follows from the Cauchy-Schwartz inequality that

$$N = \{b \in A : \rho(a^*a) = 0\} = \{b \in A : \rho(x^*b) = 0 \text{ for all } x \in A\}$$

is a left ideal. Consider the quotient space A/N with the inner product $\langle x + N, y + N \rangle = \rho(a^*b)$. The completion of this space is a Hilbert space. We now want to define a *-homomorphism. For each $a \in A$, define a map on $B(A/N)$ by $\pi'(a)(x + N) = ax + N$. This map is well-defined as for $n \in N$, we get $a(x + n) + N = ax + an + N = ax + N$ as N is a left ideal. We need to show that $\pi'(a)$ is a bounded operator for all $a \in A$. We claim that $\|\pi'(a)\| \leq \|a\|$. Without loss of generality, we assume that $\|a\| \leq 1$. We get

$$\langle \pi'(a)x + N, \pi'(a)x + N \rangle = \langle ax + N, ax + N \rangle = \rho(x^*a^*ax).$$

We observe that $\|a^*a\| = \|a\|^2 \leq 1$. This gives that $1 - a^*a$ is positive as $\lambda \in \sigma(1 - a^*a)$ if and only if $\lambda = 1 - \gamma$, where $\gamma \in \sigma(a^*a) \subseteq [0, 1]$, giving that $1 - a^*a$ has a non-negative spectrum. Let y be the positive square root of $1 - a^*a$. We get that

$$x^*x - x^*a^*ax = x^*(1 - a^*a)x = x^*y^2x = (yx)^*yx.$$

We get that
$$\rho(x^*x) - \rho(x^*a^*ax) = \rho((yx)^*yx) \geq 0.$$

We have now proven that

$$\langle \pi'(a)(x+N), \pi'(a)(x+N)\rangle \leq \rho(x^*x) = \langle x+N, x+N\rangle,$$

showing that $\pi'(a)$ is bounded. We observe that $\pi'(a^*) = \pi'(a)^*$ as

$$\begin{aligned}
\langle \pi'(a)(x+N), y+N\rangle &= \langle ax+N, y+N\rangle \\
&= \rho((ax)^*y) \\
&= \rho(x^*(a^*y)) \\
&= \langle x+N, a^*y+N\rangle = \langle x+N, \pi'(a^*)(y+N)\rangle.
\end{aligned} \tag{10}$$

For every $a \in A$, the operator $\pi'(a)$ can be uniquely extended to an operator $\pi(a) \in B(H)$, where H is the completion of A/N. It can be verified that $a \mapsto \pi(a)$ is a *-homomorphism $A \to B(H)$. Let $\xi = 1 + N$. It is obvious that $\overline{\pi(A)}(1+N) = H$. We also get that

$$\langle \pi(a)(1+N), 1+N\rangle = \langle a+N, 1+N\rangle = \rho(a),$$

showing that (π, ξ) is a GNS-pair and completing the proof. □

We can now prove the Gelfand-Naimark theorem.

Proof of Theorem 2.27. We will follow the proof from [A, p.128]. Pick $x \in A$ and let ρ be a state such that $\rho(x^*x) = \|x\|^2$, which exists by Corollary 2.38. Let (π_x, ξ) be a GNS-pair for ρ and denote the corresponding Hilbert space by H_x. By inserting $a = 1$, we get $\|\xi\| = 1$ and by inserting $a = x^*x$, we get
$$\|\pi_x(x)\xi\|^2 = \rho(x^*x) = \|x\|^2,$$
giving
$$\|\pi_x(x)\| = \|x\|.$$
We now let
$$H = \bigoplus_{x\in A} H_x$$
and
$$\pi = \bigoplus_{x\in A} \pi_x.$$
We get that π is injective. Hence $\pi : A \to \pi(A) \subseteq B(H)$ is a *-isomorphism. Any *-isomorphism must be isometric as $\|\pi(a)\| \leq \|a\|$ and $\|\pi^{-1}(T)\| \leq \|T\|$ but

$$\|a\| = \|(\pi^{-1} \circ \pi)(a)\| \leq \|\pi(a)\|,$$

completing the proof. □

Remark 2.40. This result induces a natural uniquely defined C^*-norm on the algebra $M_n(A)$ of $n \times n$ matrices with entries from A. We may treat A as a subalgebra of $B(H)$, giving that $M_n(A)$ is a subalgebra of the bounded operators on the direct sum of n copies of H, providing us with the obvious norm.

This characterisation of C^*-algebras as a norm closed *-subalgebra of $B(H)$ can be modified by requiring that the algebra is closed in the weak operator topology. We will now give a definition from [A, p.42].

Definition 2.41. A *von Neumann Algebra* is a weakly closed *-subalgebra of $B(H)$.

It is clear that any von Neumann-algebra is a C^*-algebra as the weak operator topology is coarser than the norm topology.

2.5 Group C^*-algebras

In this subsection, we will associate a C^*-algebra to a given discrete group G. We will give a construction from [R, p.240]. We note that $l^1(G)$ with pointwise addition, pointwise multiplication by scalars and convolution is an algebra. Indeed, we have the following.

Proposition 2.42. *Let $f, g \in l^1(G)$. Then $f * g \in l^1(G)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.*

Proof. We use the triangle inequality and Fubini's theorem to get that

$$\begin{aligned}\sum_{x\in G} |(f*g)(x)| &= \sum_{x\in G} |\sum_{y\in G} f(y)g(y^{-1}x)| \\ &\leq \sum_{x\in G}\sum_{y\in G} |f(y)g(y^{-1}x)| \\ &= \sum_{y\in G} |f(y)| \sum_{x\in G} |g(y^{-1}x)| \\ &= \sum_{y\in G} |f(y)| \sum_{x\in G} |g(x)| \leq \infty.\end{aligned} \tag{11}$$

This completes the proof. ☐

We have the following result.

Proposition 2.43. *The algebra $l^1(G)$ is commutative if and only if G is abelian.*

Proof. The only if direction is trivial. The if direction follows as

$$f * h = \sum_{i=1}^{\infty} f(g_i)\delta_{g_i} * \sum_{j=1}^{\infty} h(g_i)\delta_{g_i} = \sum_{i,j} f(g_i)h(g_j)\delta_{g_i g_j},$$

giving that f and h commute. ☐

The space $l^1(G)$ is a Banach *-algebra with involution defined by $f^* = \overline{\check{f}}$, $\check{f}(t) = f(t^{-1})$. However it is not a C^*-algebra in general. However, we can use it to create a C^*-algebra. Let $f \in l^1(G)$ and define $|f| := \sup\{\|f\|\}$, where the supremum is taken over all C^*-norms on l^1. We will prove that there exists a C^*- norm on $l^1(G)$ later. We note that

$$\|f\| = \|\sum_{i=1}^{\infty} f(g_i)\delta_{g_i}\| \leq \sum_{i=1}^{\infty} |f(g_i)|\|\delta_{g_i}\| = \|f\|_1$$

for all C^*-norms on $l^1(G)$, where we used that

$$\|\delta_g\|^2 = \|\delta_g^* * \delta_g\| = \|\delta_e\| = 1.$$

This shows that the supremum is finite. This is a norm as

- $|f| = \sup\{\|f\|\} = 0$ if and only if $\|f\| = 0$, which is equivalent to $f = 0$,
- $|cf| = \sup\{\|cf\|\} = c\sup\{\|f\|\} = c|f|$,
- $|f + g| = \sup\{\|f + g\|\} \leq \sup\{\|f\| + \|g\|\} \leq \sup\{\|f\|\} + \sup\{\|g\|\} = |f| + |g|$,
- $|f * g| = \sup\{\|f * g\|\} \leq \sup\{\|f\|\|g\|\} \leq \sup\{\|f\|\}\sup\{\|g\|\} = |f||g|$,
- $|f * f^*| = \sup\{\|f * f^*\|\} = \sup\{\|f\|\}^2 = |f|^2$.

We get that the completion of $l^1(G)$ is a C^*-algebra.

Remark 2.44. We may make a similar construction for general, locally compact groups.

3 Amenable Groups

We will introduce the notion of amenability, which will be very important later. This notion is connected to the Banach-Tarski paradox, and it turns out that in the discrete case, being amenable is equivalent to not being paradoxical. By $L^\infty(G)$ we mean $L^\infty(G, \mu)$, where μ denotes Haar measure. This section will follow [R, Sec. 1] and the results will be from this source unless another source is cited.

3.1 Definition and examples

We will now define amenability and give some examples of amenable groups.

Definition 3.1. We say that $m \in L^\infty(G)^*$ is a *mean* if $\langle 1, m\rangle = \|m\| = 1$.

We note that any mean is positive, meaning that $\langle 1, m\rangle \geq 0$ for any $f \geq 0$

Definition 3.2. We say that a mean m on G is left invariant if $\langle L_g\phi, m\rangle = \langle \phi, m\rangle$

Definition 3.3. Let G be a locally compact group. We say G is *amenable* if there exists a left invariant mean on G.

We make the following observation.

Remark 3.4. If m is a left invariant mean, then $\mu(A) = \langle \chi_A, m\rangle$ defines a left invariant finitely additive measure normalizing G on the Borel sets. To prove finite additivity we observe that

$$\langle \chi_{\bigcup_{i=1}^n A_i}, m\rangle = \sum_{i=1}^n \langle \chi_{A_i}, m\rangle.$$

Note that this finitely additive measure is not necessarily countably additive. We easily observe that $\sum_{i=1}^\infty \chi_{A_i}$, $\{A_i\}$ pairwise disjoint, does not converge in norm unless all A_i except for finitely many have

Haar measure 0. It will be proven later that the group $(\mathbb{R}, +)$ with the discrete topology is amenable, and hence has a left invariant finitely additive measure defined on all subsets normalizing the real line, but the only left invariant countably additive measure that assigns a finite measure to the real line is the zero measure.

We now formulate and prove a simple result.

Proposition 3.5. *Any compact group G is amenable.*

Proof. Let μ be the normalized Haar measure and define

$$\langle f, m \rangle = \int_G f d\mu, f \in L^\infty(G).$$

It is trivial that this is a left invariant mean and hence G is amenable. □

As any finite group is trivially compact, we get that finite groups are amenable with the mean $\langle \phi, m \rangle = \sum_{g \in G} \phi(g)/|G|$. We will now prove a result that can be used to prove that free groups on more than 1 generator are not amenable, but before we state the result, we will need to give a definition.

Definition 3.6. Let G be a group acting on a set X. We say that X is *G-paradoxical* if there are pairwise disjoint sets $A_1, ..., A_n, B_1, ..., B_m \subseteq X$ and group elements $g_1, .., g_n, h_1, .., h_m \in G$ such that

$$\bigcup_{i=1}^{n} g_i A_i = X = \bigcup_{i=1}^{m} h_i B_i.$$

A group G is said to be paradoxical if it is G-paradoxical with respect to the left multiplication.

We observe that a paradoxical discrete group can not be amenable as we get

$$1 = \langle 1, m \rangle = \sum_{i=1}^{n} \langle \chi_{A_i}, m \rangle + \sum_{j=1}^{k} \langle \chi_{B_i}, m \rangle = \sum_{i=1}^{n} \langle \chi_{g_i A_i}, m \rangle + \sum_{j=1}^{k} \langle \chi_{h_j B_j}, m \rangle = 2.$$

All the involved functions are measurable as all subsets of G are trivially Borel. In fact we have the following result, which is a part of [R, Corollary 0.2.11].

Proposition 3.7. *A discrete group G is amenable if and only if it is not paradoxical.*

To prove this, we use Tarski's theorem, the proof can be found in [R, Sec. 0.2].

Proof. By Tarski's theorem, there exists a left invariant finitely additive measure defined on the power set $\mathcal{P}(G)$ normalizing G if and only if G is not paradoxical. If there exists such a finitely additive measure μ, the mean is given by $\langle \phi, m \rangle = \int \phi d\mu$.

□

We will now give a definition.

Definition 3.8. Let S be a set. The *free group* generated by S is the set of reduced words consisting of letters from $\{s, s^{-1} : s \in S\}$. A word is reduced if it does not contain any pairs on the form ss^{-1} or $s^{-1}s$. The operation is concatenation followed by reduction.

Remark 3.9. Free groups are uniquely defined by the cardinality of the generating set. If $card(S) = card(S')$, we let $f : S \to S'$ be a bijection and let $\phi(s) = f(s)$ and $\phi(s^{-1}) = f(s)^{-1}$. Then ϕ is an isomorphism from the free group generated by S to the free group generated by S'. We will use F_n to denote the free group on n generators, and F_∞ to denote the free group generated by a countable set.

We now observe that the only possibly amenable free group is F_1. This follows from the following two results. The proofs are from $[\mathrm{R}, Sec.0.1]$.

Proposition 3.10. *Let G be a paradoxical group acting on a set X without non-trivial fixed points, then X is $G-$paradoxical.*

Proof. We invoke the axiom of choice and let M be a set of one element from each orbit. The sets gM form a partition of X as any element $x \in X$ has to belong to some orbit, and if $y \in gM \cap hM$ we get that $gx_1 = hx_2$ for some $x_1, x_2 \in M$ which implies $x_1 = x_2$ and $g = h$. Let $A_1, ..., A_n, B_1, ..., B_m \subseteq G$ and $g_1, .., g_n, h_1, ..., h_m \in G$ form a paradoxical decomposition of G. Then A_iM and B_iM together with the g_i and h_i's form a paradoxical decomposition of X. □

Proposition 3.11. *The free group on 2 generators is paradoxical.*

Proof. Let σ and τ generate F_2. Let W_l be the set of words with l as its first letter. Then

$$F_2 = W_\sigma \cup \sigma W_{\sigma^{-1}} = W_\tau \cup \tau W_{\tau^{-1}}.$$

□

As any subgroup acts on the whole group without non-trivial fixed points, any group that has a paradoxical subgroup is paradoxical. In particular, any group that has F_2 as a subgroup is paradoxical. As any free groups except for F_1 has F_2 as a subgroup, only $F_1 \sim (\mathbb{Z}, +)$ can be ameanable, and in fact, it is. We use $\hat{f}$ to denote the image of f under the obvious embedding of $L^1(G)$ in $L^\infty(G)^*$. We will introduce the set $P(G)$ which is the set of $L^1(G)$ functions such that $\hat{f}$ is a mean.

Lemma 3.12. *We have that $P(G) = \{f \in L^1(G) : f \geq 0, \|f\|_1 = 1\}$ and that the image $\hat{P}(G)$ of the set $P(G)$ under the obvious embedding of $L^1(G)$ into $L^\infty(G)^*$ is w^*-dense in the set K of all means.*

Proof. We follow the relevant parts of the proof of [P1, Prop. 0.1]. By definition of $\hat{f}$ we get

$$\langle \phi, \hat{f} \rangle = \langle f, \phi \rangle = \int f\phi d\mu.$$

By observation, we get that

$$P(G) = \{f \in L^1(G) : \|f\|_1 = 1 \text{ and } \int f\phi d\mu \geq 0 \text{ when } \phi \geq 0\}.$$

That $\int f\phi \geq 0$ for all $\phi \geq 0$ gives that $f \geq 0$ almost everywhere, and this completes the proof of the first part.

For the second part, we assume that the closure of $\hat{P}(G)$ is not K. We pick m_0 in $K \setminus P(G)$ and observe that there exists a w^*-neighborhood of m_0 that is disjoint from $\hat{P}$. This gives that there is a ϕ_0 such that $Re\langle \phi, \hat{f} \rangle \leq Re\langle \phi_0, m_0 \rangle - \varepsilon$ for all $f \in P(G)$. We may assume that ϕ_0 is real valued. Let $k_0 = ess \sup \phi_0$ and observe that there exists a set C of positive measure such that $\phi_0(x) \geq k_0 - \varepsilon/2$ for $x \in C$. Let $f = \chi_C/\mu(C)$. We observe that

$$\langle \phi_0, \hat{f} \rangle = \langle f, \phi_0 \rangle = \int f \phi_0 d\mu \geq k_0 - \frac{\varepsilon}{2}.$$

This gives that $\langle \phi_0, m_0 \rangle \geq k_0 + \varepsilon/2$ contradicting that $\langle \phi, m \rangle \leq ess \sup \phi$. □

Proposition 3.13. *The group* $(\mathbb{Z}, +)$ *is amenable.*

Proof. We will follow the proof from [P1, Ex. 0.3]. As the group is abelian, we do not need to distinguish between left and right invariance and we will simply write invariant. The idea of the proof is to find a suitable sequence $\{f_n\}$ in $P(G)$. Let

$$f_n = \frac{1}{2n+1} \sum_{-n}^{n} \delta_r.$$

For $\phi \in L^\infty(G)$ and $s \geq 0$ we get that

$$\begin{aligned}
&|\langle L_{-s}\phi, \hat{f}_n \rangle - \langle \phi, \hat{f}_n \rangle| \\
&= |\langle R_s\phi, \hat{f}_n \rangle - \langle \phi, \hat{f}_n \rangle| \\
&= \left| \frac{1}{2n+1} \sum_{r=-n}^{n} \phi(r+s) - \phi(r) \right| \\
&= \left| \frac{1}{2n+1} \left(\sum_{r=s-n}^{n+s} \phi(r) - \sum_{-n}^{n} \phi(r) \right) \right| \\
&= \frac{1}{2n+1} \left| \sum_{r=n+1}^{n+s} \phi(r) - \sum_{-n}^{s-n-1} \phi(r) \right| \\
&\leq \frac{2s\|\phi\|_\infty}{2n+1} \to 0.
\end{aligned} \tag{12}$$

The case where $s < 0$ can be handled similarly. We get that every w^*-cluster point of the sequence is an invariant mean, and hence $(\mathbb{Z}, +)$ is amenable. □

We can use the same idea for $(\mathbb{R}, +)$ with $f_n = \chi_{[-n,n]}/2n$. We note that this proof is a proof that $E_n = [-n, n] \cap \mathbb{Z}$ is a *Følner sequence* (see Definition 3.16), as $\mu(E_n) = |E_n| = 2n + 1$ and we may pick $\phi = \chi_{E_n}$.

More generally, we have the following result.

Proposition 3.14. *Any abelian locally compact group is amenable.*

Before we proceed, we will state a result that will be used in the proof.

Theorem 3.15 (Markov-Kakutani). *Let X be a locally convex topological vector space and let K be a compact, convex subset. Let S be a family of commuting continuous affine maps on K. Then the mappings in S have a common fixed point.*

In particular, this theorem holds for any family of commuting w^*-continuous linear maps on $L^\infty(G)^*$ that leave the set K of all means invariant. We will not prove this theorem, but a proof can be found in [P1, Prop. 0.14].

Proof. We observe that K is convex as for any means m, n we have that

$$\langle 1, (1-t)m + tn \rangle = 1 - t + t = 1$$

and

$$\|(1-t)m + tn\| \leq (1-t)\|m\| + t\|n\| = 1.$$

We note that K is w^*-compact as it is a closed subspace of the closed unit ball, that is w^*-compact by the Banach-Alaoglu theorem. We now define the linear operators

$$T_g : L^\infty(G)^* \to L^\infty(G)^*$$

by $\langle \phi, T_g y \rangle = \langle L_g \phi, y \rangle$, $\phi \in L^\infty(G), y \in L^\infty(G)^*$. As G is abelian, we observe that

$$\begin{aligned} T_{gh}y &= \langle \phi((gh)^{-1}x), y \rangle \\ &= \langle \phi(h^{-1}g^{-1}x), y \rangle = \langle \phi(g^{-1}h^{-1}x), y \rangle \\ &= \langle L_g L_h \phi, y \rangle = T_g T_h y. \end{aligned} \tag{13}$$

We observe that L_g is w^*-continuous for all g and that K is invariant, as $\langle 1, T_g m \rangle = \langle L_g 1, m \rangle = \langle 1, m \rangle = 1$ and

$$\|T_g m\| = \sup_{\|\phi\| \leq 1} |\langle L_g \phi, m \rangle| \leq \sup_{\|\phi\| \leq 1} |\langle \phi, m \rangle| = 1.$$

By Theorem 3.15, there is a $m \in K$ such that $T_g m = m$ for all g and hence, m is left a invariant mean. □

We have now proven the amenability of any abelian locally compact group.

3.2 Følner sequences

We will now introduce some properties that are equivalent to amenability. This subsection will follow [EW, Sec. 10]. In order to prove the amenability of more groups, we will need to give a definition.

Definition 3.16. We say that G admits *Følner sets* if we have that for all compact K and all $\varepsilon > 0$, there exists a measurable set E with non zero, finite measure such that

$$\frac{\mu(kE\Delta E)}{\mu(E)} < \varepsilon$$

for all $k \in K$. We say that G meets the *Reiter condition* in L^1 if we have that for all compact $K \subseteq G$ and all $\varepsilon > 0$, there exists $f \in P(G)$ such that

$$\|L_k f - f\|_1 < \varepsilon$$

for all $k \in K$. We say that $L^2(G)$ has an almost invariant vector if there exists $f \in L^2(G)$ such that

$$\|L_k f - f\|_2 < \varepsilon$$

for all $k \in K$.

Remark 3.17. If G is a $\sigma-$compact, metrizable group that admits Følner sets, there exists a sequence $\{E_n\}$ such that

$$\frac{\mu(kE_n \Delta E_n)}{\mu(E_n)} \to 0$$

for any fixed $k \in G$ and the convergence is uniform on compact sets. Such a sequence is called a *Følner sequence.* Let K_i be an increasing sequence of compact sets such that $G = \bigcup_{i \in \mathbb{N}} K_i$. Let E_n be the Følner set corresponding to K_n and $\varepsilon = 1/n$. The pointwise convergence is obvious. For every compact set K, it is a subset of K_n for some n, as if it is not, it can be written as $\bigcup_{i=1}^{\infty} K_i \cap K$ and $\{\overline{K_i^c}^c \cap (K_i \cap K)\}$ would be an open cover without a finite subcover. Hence

$$\sup_{k \in K} \frac{\mu(kE_n \Delta E_n)}{\mu(E_n)} < \frac{1}{n}$$

for n sufficiently large. It follows immediately that this is a Følner sequence. It is obvious that the existence of a Følner sequence implies that G admits Følner sets, as for any compact set K and $\varepsilon > 0$ we may pick K_n such that $K \subseteq K_n$ and $1/n < \varepsilon$. We then pick $E = E_n$ as the corresponding Følner set.

We can now state a result.

Theorem 3.18. *Let G be a discrete group or a σ-compact and metrizable group. Then the following are equivalent:*

- *G is amenable*
- *G admits Føllner sets*
- *G fulfills the Reiter condition in L^1*
- *$L^2(G)$ has an almost invariant vector.*

It is clear that $G = (\mathbb{R}, +)$ in the standard topology is $\sigma-$compact as it can be written as

$$\bigcup_{n \in \mathbb{N}} [-n, n].$$

By Proposition 3.14, G is amenable and by Theorem 3.18 and Remark 3.17, it has a Følner sequence. Indeed, $E_n = [-n^2, n^2]$ is a Følner sequence. We will now prove Theorem 3.18.

Proof. We will follow the proof given in [EW, Thm. 10.15]. If E is a Følner set for a compact K and $\varepsilon > 0$ we get that the Reiter condition is met by letting $f = \chi_E/\mu(E)$. We also get that $f_1 = \chi_E/\sqrt{\mu(E)}$ is an almost invariant vector in $L^2(G)$. If f_2 is such that $\|f_2\| = 1$ and $\|L_k f_2 - f_2\|_2 \leq \varepsilon$, we define $f = f_2^2$ and observe that $f \in P(G)$. We use the definition of the L^2 norm and the Cauchy-Schwarz inequality to get

$$\begin{aligned} \|L_k f - f\|_1 &= \|L_k f_2^2 - f_2^2\|_1 = \|(L_k f_2 - f_2)(L_k f_2 + f_2)\|_1 \\ &= \int_G |L_k f_2 - f_2||L_k f_2 + f_2| d\mu = \langle L_k f_2 - f_2, L_k f_2 + f_2 \rangle \qquad (14) \\ &\leq \|L_k f_2 - f_2\|_2 \|L_k f_2 + f_2\|_2 \leq \|L_k f_2 - f_2\|_2 (\|L_k f_2\|_2 + \|f_2\|) = 2\varepsilon. \end{aligned}$$

We now let G have the Reiter condition and we want to show that G admits Følner sets. Let K be a compact set. Without loss of generality, we may assume $\mu(K) > 0$. Fix $\varepsilon > 0$ and pick f that meets the Reiter condition. For every $\alpha > 0$ we define $E_\alpha = \{g \in G : f(g) \geq \alpha\}$. This set is obviously measurable as it is the preimage of a closed set under a measurable function. We get that

$$\begin{aligned} &\int_0^\infty \mu(E_\alpha) d\alpha \\ &= \int_0^\infty \int_G \chi_{E_\alpha} d\mu d\alpha \\ &= \int_G \int_0^\infty \chi_{E_\alpha} d\alpha d\mu \\ &= \int_G f d\mu = \|f\|_1 = 1. \end{aligned} \tag{15}$$

By very similar reasoning we get that

$$\int_G \mu(kE_\alpha \Delta E_\alpha) d\alpha = \|L_k f - f\|_1 < \varepsilon.$$

We now get that

$$\begin{aligned} &\int_0^\infty \int_K \mu(kE_\alpha \Delta E_\alpha) d\mu(k) d\alpha \\ &= \int_K \int_0^\infty \mu(kE_\alpha \Delta E_\alpha) d\alpha d\mu(k) \\ &< \int_K \varepsilon d\mu = \varepsilon\mu(K) \\ &= \int_0^\infty \varepsilon\mu(K)\mu(E_\alpha) d\alpha. \end{aligned} \tag{16}$$

This implies that there exists $\alpha \in (0, \infty)$ such that

$$\int_K \mu(kE_\alpha \Delta E_\alpha) d\mu(k) < \varepsilon\mu(K)\mu(E_\alpha).$$

If G is discrete, this implies that μ is a constant multiple of counting measure and that K is finite. We get that $|kE_\alpha \Delta E_\alpha| < \varepsilon|K||E_\alpha|$. In the other case, we note that what we have already proven implies that for all $\varepsilon > 0$ and $\delta > 0$ there is a measurable set $E = E_\alpha$ such that

$$\int_K \mu(kE\Delta E) d\mu(k) < \varepsilon\delta\mu(E) < \infty.$$

This shows that $\mu(N) < \delta$ where

$$N = \{k \in K : \mu(kE\Delta E) \geq \varepsilon\mu(E)\}.$$

We have now proven that for any compact K, any $\varepsilon > 0$ and $\delta > 0$, there exists a measurable $E = E_\alpha$ with finite measure and $N \subseteq K$ with $\mu(N) < \delta$ such that

$$\mu(kE\Delta E) < \varepsilon\mu(E)$$

for all $k \in K \setminus N$. Define $K_1 = K \cup K^2$ and let $\delta = \mu(K)/2$. We get that there is a measurable set E with finite measure and N with $\mu(N) < \mu(K)/2$ such that

$$\mu(kE\Delta E) < \varepsilon\mu(E) \tag{17}$$

for all $k_1 \in K_1 \setminus N$. Fix $k \in K$. We get that (17) holds for all $k_1 \in K$, where $\mu(K \setminus N > \mu(K)/2)$ and for all $kk_1 \in kK$, where $\mu(kK \setminus N) > \mu(K)/2$. We get that there exists a k_1 such that (17) holds for k_1 and kk_1. This gives that

$$\begin{aligned} \mu(kE\Delta E) &\leq \mu((kE\Delta kk_1E) \cup (kk_1E\Delta E)) \\ &< \mu(kE\Delta E) + \varepsilon\mu(E) < 2\varepsilon\mu(E). \end{aligned} \tag{18}$$

We have now proven that the last three statements are equivalent. We now want to show that the Reiter condition and amenability are equivalent. We assume that G fulfills the Reiter condition. Let f be a function given by the Reiter condition and let $\phi \in L^\infty(G)$. We get that

$$\langle f, L_{k^{-1}}\phi - \phi\rangle = \langle f, L_{k^{-1}}\phi\rangle - \langle f, \phi\rangle = \langle L_k f - f, \phi\rangle < \varepsilon\|\phi\|_\infty. \tag{19}$$

By embedding these functions into the set of means by the obvious embedding, we get that the set

$$A(\varepsilon, \overline{\phi}, \overline{k}) = \{\text{means } M : |\langle L_{k_j}\phi_i - \phi_i, M\rangle| \leq \varepsilon\|\phi_i\|_\infty \text{ for all } i, j\}$$

is non-empty for all $\varepsilon > 0$, $\overline{\phi} = (\phi_1, ..., \phi_l)$, $\overline{k} = (k_1, ..., k_n)$ and any l, n. These sets clearly have the finite intersection property, and as the set of means is w^*-compact, all sets on this form has at least one common element. This element is a left invariant mean. We now assume G is amenable and intend to prove the Reiter condition. Let m be a left invariant mean. We first handle the discrete case. We define $D : L^1(G) \to L^1(G)^K$ by $Df = (L_k f - f)_{k\in K}$. It can be verified that $D(P(G))$ is convex, and hence, by [EW, Cor. 8.74], the norm closure and weak closure coincide. We want to show that 0 is an element in the weak closure. For every $\phi_1, ..., \phi_n$, $\varepsilon > 0$ we want to find $f \in P(G)$ such that $|\langle L_k f - f, \phi_i\rangle| < \varepsilon$ for all i and $k \in K$. We observe that

$$\langle L_k f - f, \phi_i\rangle = \langle L_k\phi_i - \phi_i, \hat{f}\rangle.$$

We get that this expression is 0 for $\hat{f} = m$, and the result immediately follows from Lemma 3.12. In the non-discrete case we need the following result.

Lemma 3.19. *Let G be a σ-compact, metrizable, amenable group. Then there exists a mean M such that $\langle f * \phi, M\rangle = \langle \phi, m\rangle$ for all $f \in P(G)$. This mean is said to be topologically left invariant.*

The proof of this can be found in [EW, Lemma 10.17]. We fix $\phi_1, ..., \phi_n$ and define the operator D such that $Df = (\phi_i * f - f)_{i=1}^n$. We get that

$$\|\phi_i * f - f\|_1 < \varepsilon$$

for $i = 1, ..., n$. Let $\{g_1, ...\}$ be a dense countable subset of G such that $g_1 = 1$. Let $\phi_i = L_{g_i}\phi$. We can find a sequence $\{f_n\}$ such that $\|\phi_i * f_n - f_n\|_1 \to 0$. Let $K \subseteq G$ and fix $\varepsilon > 0$. By [EW, Lemma 3.74], there exist a neighborhood U of 1 such that $\|L_u\phi - \phi\|_1 < \varepsilon$ for all $u \in U$. As $\{g_1, g_2, ...\}$ is dense we get that $\bigcup_{i=1}^\infty g_iU = G$. By compactness of K, we get that any open cover has a finite subcover and hence we have

$$K \subseteq \bigcup_{i=1}^{l} g_iU.$$

Chose n large enough such that

$$\|L_{g_j}\phi * f_n - f_n\| < \varepsilon$$

for $j = 1, ..., l$. Let $k \in K$. We get that $k = g_ju$ for some $j \leq l$ and $u \in U$. We get

$$\|L_k\phi - L_{g_j}\phi\|_1 = \|L_{g_j}(L_u\phi - \phi)\| < \varepsilon.$$

We now get that

$$\begin{aligned}\|L_k\phi * f_n - \phi * f_n\| &= \|L_k\phi * f_n - L_{g_j}\phi * f_n + L_{g_j}\phi * f_n - f_n + f_n - \phi * f_n\| \\ &\leq \|L_k\phi * f_n - L_{g_j}\phi * f_n\| + \|L_{g_j}\phi * f_n - f_n\| + \|f_n - \phi * f_n\| < 3\varepsilon.\end{aligned} \tag{20}$$

It can be verified that $(L_k\phi) * f_n = L_k(\phi * f_n)$. Hence, the function $\phi * f_n$ satisfies the Reiter condition for $K \subseteq G$ and 3ε. This completes the proof. □

4 Unitarizable representations

In this section, we will investigate when group representations on Hilbert spaces are *unitarizable*. The aim of this section is to prove the Dixmier-Day theorem. Before we state and prove this theorem, we will need some preparations.

4.1 Some representation theory

We will now state some definitions.

Definition 4.1. A *representation* of a group G on a Hilbert space H is a map $G \to B(H)$, where $B(H)$ is the space of bounded linear operators on H, such that $\pi(1) = I$ and $\pi(xy) = \pi(x)\pi(y)$. A representation $\pi(\cdot)$ is said to be *unitary* if $\pi(g)$ is unitary for all g.

Example 4.2. Let $G = (\mathbb{T}, \cdot)$ be the multiplication group on the unit circle in the complex plane. Let

$$\pi(e^{i\theta}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This representation is unitary.

The left regular representation $\lambda : G \to B(l^2(G))$, defined by $\lambda(s) = L_s$ is another example of a unitary representation.

Let S be an invertible operator and let $\pi(\cdot)$ be a representation. It is clear that $g \mapsto S\pi(g)S^{-1}$ is a representation.

Definition 4.3. Two representations $\pi_1 : G \to B(H)$ and $\pi_2 : G \to B(K)$ are said to be *equivalent* if there exists an invertible operator $S : K \to H$ such that $\pi_1(\cdot) = S\pi_2(\cdot)S^{-1}$. We say that a representation $\pi(\cdot)$ is *unitarizable* if it is equivalent to a unitary representation.

Definition 4.4. A representation $\pi(\cdot)$ is said to be *uniformly bounded* if

$$\sup_{g\in G} \|\pi(g)\| < \infty.$$

We denote this supremum by $|\pi|$.

Unitary representations are important in physics and are in particular used in quantum mechanics. Mathematically, unitary representations have the very convenient property that they can be expressed in terms of *irreducible representations*, which can be seen by stating the following theorems.

Definition 4.5. An *invariant subspace* of a representation $\pi : G \to H$ is a subspace X of H such that $\pi(g)X \subseteq X$ for all $g \in G$. A representation is said to be *irreducible* if it has no non-trivial invariant subspace.

Lemma 4.6. *Let $\pi : G \to B(H)$ be a unitary representation on a Hilbert space H If a subspace X of H is invariant under the action of $\pi(\cdot)$, then $X^\perp$ is invariant.*

Proof. Let $x \in X$ and $y \in X^\perp$. We get that

$$0 = \langle x, y\rangle = \langle \pi(g)x, y\rangle = \langle x, \pi(g^{-1})y\rangle$$

for all $g \in G$, completing the proof. □

Theorem 4.7. *Let G be a group. Then every finite dimensional unitary representation is a direct sum of irreducible representations.*

Proof. We note that if $\pi(\cdot)$ is not irreducible, it can be written as $\pi(\cdot)|_X \oplus \pi(\cdot)|_{X^\perp}$, where X is a non-trivial invariant subspace. This procedure can be repeated until we have written $\pi(t)$ as a direct sum of irreducible representations. Note that this procedure will always terminate in a finite number of steps, as any one-dimensional representation is irreducible. □

Example 4.8. Let $G = (\mathbb{R}, +)$, $H = \mathbb{C}^2$ and

$$\pi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

This representation is not unitary. We see that the subspace

$$X = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} : t \in \mathbb{C} \right\}$$

is invariant, but $X^\perp$ is not.

Example 4.9. Let $\pi(\cdot)$ be defined as in Example 4.2. Any irreducible representation of an abelian group is one-dimensional, giving that $\pi(\cdot)$ is not irreducible.

We will restrict ourselves to representations of locally compact groups that are continuous with respect to the strong operator topology, and we need to check the continuity of $g \mapsto \langle \pi(g)\xi, \eta\rangle$. As the strong topology and the weak topology coincide on the ball, continuity of a uniformly bounded representation is equivalent to the continuity of $g \mapsto \pi(g)\xi$. We will use the word representation to mean a continuous representation.

4.2 Unitarizability

In this section we will follow [P4, Sec. 0]. For the remaining part of the section, we will require that representations are continuous. Let $\pi : G \to B(H)$ be a uniformly bounded representation, then

$$\sup_{g\in G} \|S\pi(g)S^{-1}\| \le (\|S\|\|S^{-1}\||\pi|) < \infty.$$

We get that uniform boundedness is invariant under equivalence. As any unitary operator trivially has norm 1, we get that any unitary representation, and hence any unitarizable representation is uniformly bounded. We now look at a concrete example.

Example 4.10. Let $G = (\mathbb{R} \setminus \{0\}, \cdot)$ an let $\pi(x) = x$. As this representation is not uniformly bounded, it is clearly not unitarizable.

It is noted in [P4, p. 5] that a representation is unitary if and only if $\sup_{g\in G}\|\pi(g)\| \leq 1$. This follows from the observation that $\pi(g^{-1}) = \pi(g)^{-1}$, which implies that $\pi(g)$ must be an invertible isometry for all g, as

$$\|x\| = \|\pi(g)\pi(g^{-1})x\| \leq \|\pi(g)\|\|\pi(g^{-1})\|\|x\| \leq \|x\|, x \in H,$$

and hence, π is unitary. We will now state and prove the main theorem of this section which gives a class of groups for which any uniformly bounded representation is unitarizable.

Theorem 4.11 (Dixmier-Day)**.** *Let G be amenable, then all uniformly bounded representations are unitarizable.*

Proof. We will follow the proof from [P4, Thm. 0.6]. Let π be a uniformly bounded representation. Define

$$f_{xy}(g) = \langle \pi(g^{-1})x, \pi(g^{-1})y\rangle.$$

Let m be a left invariant mean and define $\|x\|_\pi = \langle f_{xx}, m\rangle^{1/2}$. We note that

$$\|\pi(g)x\|_\pi^2 = \langle L_{g^{-1}} f_{xx}, m\rangle = \langle f_{xx}, m\rangle = \|x\|_\pi^2.$$

Hence π is unitary with respect to the inner product

$$\langle x, y\rangle_\pi = \langle f_{xy}, m\rangle.$$

We get that $\|x\|_\pi \leq |\pi|\|x\|$ by the definition of $\|\cdot\|_\pi$. We also get that

$$\begin{aligned}
\|x\|^2 &= \langle x, x\rangle \\
&= \langle \pi(t)\pi(t^{-1})x, \pi(t)\pi(t^{-1}x)\rangle \\
&\leq \|\pi(t)\|^2 f_{xx} \\
&\leq |\pi|^2 f_{xx}.
\end{aligned} \tag{21}$$

By the linearity and positivity of m, we get that

$$\|x^2\| = \langle \|x\|^2 1, m\rangle \leq |\pi|^2 \langle f_{xx}, m\rangle = |\pi|^2 \|x\|_\pi^2,$$

showing that $\|x\| \leq |\pi|\|x\|_\pi$. This equivalence of the norms will become important later. Let $\{e_i\}$ be an orthonormal basis in $(H, \langle\cdot,\cdot\rangle)$ and let $\{e_i'\}$ be an orthonormal basis in $(H, \langle\cdot,\cdot\rangle_\pi)$. Let S be the uniquely defined linear operator such that $Se_i' = e_i$. This S is clearly an invertible isometry and we get that $\|Sx\| = \|x\|_\pi$. Moreover, $S\pi(g)S^{-1}$ is unitary as $\|S\pi(g)S^{-1}x\| = \|x\|$ and we get that $S\pi(g)S^{-1}$ is an invertible isometry. □

Remark 4.12. Considering S as an operator $(H, \langle\cdot,\cdot\rangle) \to (H, \langle\cdot,\cdot\rangle)$, we get that $\|Sx\| = \|x\|_\pi \leq |\pi|\|x\|$ and $\|S^{-1}x\| = |\pi|\|S^{-1}x\|_\pi \leq |\pi|\|x\|$, giving that $\|S\|\|S^{-1}\| \leq |\pi|^2$.

We use this result and Proposition 3.5, Proposition 3.14 and Theorem 3.18 to immediately get the following.

Corrolary 4.13. *The following classes of groups are unitarizable:*

- *finite groups;*
- *compact groups;*

- *abelian groups;*

- *discrete or σ-compact, metrizable groups that admit Følner sets.*

4.3 Non-unitarizable groups

We have started to explore which groups have the property that every uniformly bounded representation is similar to a unitary representation. We will now explore some groups that do not have this property. It turns out that non-abelian free groups, and any discrete group that has a non-abelian free subgroup, are examples of groups that do not have this property. In this subsection, we will follow [P4, Sec. 2].

Definition 4.14. A group is called *unitarizable* if every uniformly bounded representation is unitarizable.

By Theorem 4.11, any amenable group is unitarizable. In this section, we want to prove that free groups are not unitarizable.

4.4 Coefficients of unitary representations

We will first introduce two spaces of functions. Let $B(G)$ be the set of functions $f : G \to \mathbb{C}$ such that there exists a Hilbert space H, a unitary representation π on H and $x, y \in H$ such that $f(t) = \langle \pi(t)x, y\rangle$.

Remark 4.15. If $\pi(t)$ is unitarizable, then $\langle \pi(t)x, y\rangle \in B(G)$.

We define $T_p(G)$ to be the set of functions $f : G \to \mathbb{C}$ such that there exist functions $f_1, f_2 : G\times G \to \mathbb{C}$ satisfying $f(st) = f_1(s, t) + f_2(s, t)$, where

$$\sup_{s\in G} \sum_{t\in G} |f_1(s, t)|^p < \infty$$

and

$$\sup_{t\in G} \sum_{s\in G} |f_2(s, t)|^p < \infty.$$

It is trivial that $T_p(G)$ is a vector space.

We define the norm on $B(G)$ by $\|f\|_{B(G)} = \inf\{\|x\| \cdot \|y\|\}$, and the norm on $T_p(G)$ by

$$\|f\|_{T_p(G)} = \inf \left\{ \sup_{s\in G} \left(\sum_{t\in G} |f_1(s, t)|^p \right)^{1/p} + \sup_{t\in G} \left(\sum_{s\in G} |f_2(s, t)|^p \right)^{1/p} \right\}.$$

Both these infimums are taken over all decompositions on the relevant form. We get that $\| \cdot \|_{B(G)}$ is a norm as we have the following

- $0 < |f(t)| = |\langle \pi(t)x, y\rangle| \leq \|x\|\|y\|$;

- $cf(t) = \langle \pi(t)cx, y\rangle$;

- $f+g = \langle \pi(t)x, y\rangle_H = \langle \pi_1(t)x_1, y_1\rangle_{H_1} + \langle \pi_2(t)x_2, y_2\rangle_{H_2}$ where $\pi = \pi_1 \oplus \pi_2$, $x = x_1 \oplus x_2$, $y = y_1 \oplus y_2$ and $H = H_1 \oplus H_2$.

The second and third point give that $B(G)$ is a vector space. The first point gives that $\|x\|_{B(G)} > 0$ for $x \neq 0$. The second point gives $\|cx\|_{B(G)} = |c|\|x\|_{B(G)}$ and the third point gives the triangle inequality. Before we prove this we note that $\langle \pi(t)x, y\rangle = \langle \pi(t)cx, y/c\rangle$ for any $c \in \mathbb{R} \setminus \{0\}$. Hence, we may chose x_1, y_1 and x_2, y_2 such that $\|x_1\| = \|y_1\|$ and $\|x_2\| = \|y_2\|$. Then

$$\begin{aligned} \|f+g\|_{B(G)}^2 &\leq \|x_1 \oplus x_2\|^2 \|y_1 \oplus y_2\|^2 \\ &= (\|x_1\|^2 + \|x_2\|^2)(\|y_1\|^2 + \|y_2\|^2) \\ &= (\|x_1\|\|y_1\| + \|x_2\|\|y_2\|)^2. \end{aligned} \tag{22}$$

and we get that $B(G)$ is indeed a normed space. We will focus on the discrete case and assume that G is discrete. In fact $B(G)$ is the dual space of $C^*(G)$, see [R, Thm. A.3.11]. We define the function

$$\delta_g(t) := \begin{cases} 1 \text{ if } t = g, \\ 0 \text{ otherwise.} \end{cases}$$

Remark 4.16. We have that $l^1(G) \subseteq B(G)$ as $\delta_s(t) = \langle L_t \delta_e, \delta_s \rangle$ implies that $\delta_s \in B(G)$ and $\|\delta_s\|_{B(G)} \leq 1 = \|\delta_s\|_1$

We get that $T_p(S)$ is a normed space as if $f(st) = a_1(s,t) + a_2(s,t)$ and $g(st) = b_1(s,t) + b_2(s,t)$, we get that

$$\begin{aligned} \|f+g\|_{T_p(G)} &\leq \sup_{s\in G} \left(\sum_{t\in G} |a_1(s,t) + b_1(s,t)|^p \right)^{1/p} \\ &\quad + \sup_{t\in G} \sum_{s\in G} \left(|a_2(s,t) + b_2(s,t)|^p \right)^{1/p} \\ &\leq \sup_{s\in G} \left(\left(\sum_{t\in G} |a_1(s,t)|^p \right)^{1/p} + \left(\sum_{t\in G} |b_1(s,t)|^p \right)^{1/p} \right) \\ &\quad + \sup_{t\in G} \left(\left(\sum_{s\in G} |a_2(s,t)|^p \right)^{1/p} + \left(\sum_{t\in G} |b_2(s,t)|^p \right)^{1/p} \right) \\ &\leq \sup_{s\in G} \left(\sum_{t\in G} |a_1(s,t)|^p \right)^{1/p} + \sup_{t\in G} \left(\sum_{t\in G} |b_2(s,t)|^p \right)^{1/p} \\ &\quad + \sup_{t\in G} \left(\sum_{s\in G} |a_2(s,t)|^p \right)^{1/p} + \sup_{s\in G} \left(\sum_{t\in G} |b_1(s,t)|^p \right)^{1/p}. \end{aligned} \tag{23}$$

Hence $\|f+g\|_{T_p(G)} \leq \|f\|_{T_p(G)} + \|g\|_{T_p(G)}$. The remaining axioms are trivial. We can now state a theorem.

Theorem 4.17. *Let G be a unitarizable discrete group. Then $T_1(G) \subseteq B(G)$.*

Before the proof, we will give a definition and state another result.

Definition 4.18. Let f and g be functions $G \to \mathbb{C}$ such that at least one of them has finite support. The *convolution* of f and g is defined as

$$(f * g)(t) := \sum_{s\in G} f(s) g(s^{-1}t).$$

Proposition 4.19. *Let $(a_{i,j})$ be an infinite matrix such that*

$$\sup_i \sum_j |a_{i,j}| \leq 1$$

and

$$\sup_j \sum_i |a_{i,j}| \leq 1.$$

Then the matrix defines a linear operator $A : l^2 \to l^2$ such that $\|A\| \leq 1$.

In particular, this result states that an operator A given by a matrix $(a_{i,j})$ with norm bounded by 1 in l^1 and l^∞, has norm bounded by 1 in l^2. The proof of this result is found in [P4, Prop. 2.15].

Proof. We will give the proof of the theorem from [P4, Thm. 2.1]. Let $f \in T_p(G)$. We get that there are a_1, a_2 such that $f(s^{-1}t) = a_1(s,t) + a_2(s,t)$ and there exists C with

$$\sup_{s\in G} \sum_{t\in G} |a_1(s,t)| < C$$

and

$$\sup_{t\in G} \sum_{s\in G} |a_2(s,t)| < C.$$

Let $A_1, A_2 : \mathbb{C}^{(G)} \to \mathbb{C}^G$, where $\mathbb{C}^{(G)}$ denotes the space of finitely supported functions $G \to \mathbb{C}$, be the linear operators given by the "matrices" a_1, a_2, which means that $(A_i g)(t) = \sum_{s\in G} a(t,s)g(s)$. For $f \in \mathbb{C}^G$ we set $\check{f}(x) = f(x^{-1})$ and define $\rho(f)g = g * \check{f}$. We get that

$$(\rho(f)g)(t) = \sum_{s\in G} g(s)\check{f}(s^{-1}t) = \sum_{s\in S} f(t^{-1}s)g(s), g \in \mathbb{C}^{(G)}$$

showing that $f(s^{-1}t)$ is the matrix for the linear operator $\rho(f)$ and that $\rho(f) = A_1 + A_2$. Moreover

$$\begin{aligned}(\rho(f)L_t g)(x) &= (g(t^{-1}y) * \check{f}(y))(x) \\ &= \sum_{y\in g} g(t^{-1}y)\check{f}(x^{-1}y) \\ &= \sum_{y\in g} g(y)\check{f}(x^{-1}ty) = (L_t\rho(f)g)(x),\end{aligned} \tag{24}$$

giving that $\rho(f)$ and L_t commute. Using $[a,b] := ab - ba$ to denote the *commutator* of the operators a, b, note that

$$0 = [\rho(f), L_t] = [A_1, L_t] + [A_2, L_t].$$

Let $D(a) := [A_1, a]$. Whenever $D(a)$, $D(b)$ and $D(ab)$ are defined, we have

$$D(a)b + aD(b) = (A_2a - aA_2)b + a(A_2b - bA_2) = A_2ab - abA_2 = D(ab),$$

meaning that D is a derivation. Let $H = l^2(G) \oplus l^2(G)$ and define

$$\pi(t) = \begin{bmatrix} L_t & D(L_t) \\ 0 & L_t \end{bmatrix}.$$

We observe that $D(L_t)$ is bounded as an operator on l^1 and l^∞ with norm not exceeding $2C$. Hence, it is bounded by $2C$ as an operator on l^2 by Proposition 4.19. We get that $\|\pi(t)\| \leq 1 + 2C$ as

$$\|\pi(t)\| \leq \left\| \begin{bmatrix} L_t & 0 \\ 0 & L_t \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & D(L_t) \\ 0 & 0 \end{bmatrix} \right\| = 1 + \|D(L_t)\| \leq 1 + 2C. \tag{25}$$

This is a representation of G by the fact that D is a derivation. Indeed, we have that

$$\pi(st) = \begin{bmatrix} L_sL_t & D(L_sL_t) \\ 0 & L_sL_t \end{bmatrix} = \begin{bmatrix} L_sL_t & D(L_s)L_t + L_sD(L_t) \\ 0 & L_sL_t \end{bmatrix} = \pi(s)\pi(t).$$

Let $x = (\delta_1, 0)$ and $y = (0, \delta_1)$. We get that

$$\begin{aligned} \langle \pi(t)x, y\rangle &= \langle D(L_t)\delta_1 \oplus \delta_1, \delta_1 \oplus 0\rangle \\ &= \langle D(L_t)\delta_1, \delta_1\rangle \\ &= \langle A_2L_t\delta_1 - L_tA_2\delta_1, \delta_1\rangle \\ &= \langle A_2\delta_t - L_tA_2\delta_1, \delta_1\rangle \\ &= \langle A_2\delta_t, \delta_1\rangle - \langle L_tA_2\delta_1, \delta_1\rangle \\ &= \langle A_2\delta_t, \delta_1\rangle - \langle A_2\delta_1, \delta_{t^{-1}}\rangle \\ &= a_2(1, t) - a_2(t^{-1}, 1). \end{aligned} \tag{26}$$

By Remark 4.15 and our assumption that every uniformly bounded representation is unitarizable, we get that $a_2(1, t) - a_2(t^{-1}, 1) \in B(G)$. As

$$f(t) = a_1(1, t) + a_2(1, t) = a_1(1, t) + (a_2(1, t) - a_2(t^{-1}, 1)) + a_2(t^{-1}, 1)$$

and $a_1(1, t) + a_2(t^{-1}, 1) \in l^1(G) \subseteq B(G)$ (see Remark 4.16), we obtain $f \in B(G)$, completing the proof. □

We will now prove the following result, from which we will immediately obtain that F_∞ is not unitarizable. Let $\{g_1, g_2, ...\}$ be the generators of F_∞. We use $|t|$ to denote the length of the reduced word t. We for example have $|1| = 0$, $|g_i| = 1$ $|g_ig_i| = 2$ and $|g_ig_j^{-1}| = 2$ for $i \neq j$.

Lemma 4.20. *Let $f : F_\infty \to \mathbb{C}$ be the indicator function of the set of reduced words of length 1. Then we have $f \in T_1(F_\infty)$ but $f \notin B(F_\infty)$.*

Proof. We will follow the proof given in [P4, Lemma 2.2]. We observe that $f \in T_1(F_\infty)$ as if we let

$$f_1(s, t) = \chi_{|st|=1, |t|>|s|}(s, t)$$

and

$$f_2(s, t) = \chi_{|st|=1, |t|<|s|}(s, t)$$

we get that

$$f(st) = f_1(s, t) + f_2(s, t).$$

We note that given a word t, there is exactly one word s such that st has length 1 and $|s| < |t|$, and given a word s, there is exactly one word t such that $|st| = 1$ and $|s| > |t|$. Hence, $f \in T_1(F_\infty)$. We now assume that $f \in B(G)$ and strive for a contradiction. Let H be a Hilbert space and let $\pi(\cdot)$, x and y be such that $f(g) = \langle \pi(g)x, y\rangle$. Hence

$$\langle \pi(g_j)x, y\rangle = \langle \pi(g_j)^*x, y\rangle = 1$$

and

$$\langle \pi(t)x, y\rangle = 0$$

for $|t| \neq 1$. Let $a_i = (\pi(g_i) + \pi(g_i)^*)/2$. For each $n \geq 1$ we let

$$R = \prod_{j=1}^{n} \left(I + \frac{i}{\sqrt{n}} a_j \right).$$

We get that

$$\begin{aligned}\left\|I + \frac{i}{\sqrt{n}}a_j\right\|^2 &= \|(I + in^{-1/2}a_j)(I + in^{-1/2}a_j)^*\| \\ &= \|I + a_j^2/n\| \le 1 + \|a_j^2\|/n \\ \le 1 + \|a_j\|^2/n &\le 1 + \|\pi(g_j) + \pi(g_j)^*\|^2/4n \\ \le 1 + (\|\pi(g_j)\| &+ \|\pi(g_j)^*\|)^2/4n = 1 + \frac{1}{n},\end{aligned} \tag{27}$$

giving

$$\|R\|^2 \le \left(1 + \frac{1}{n}\right)^n \le e.$$

We develop the product and get

$$R = \frac{i}{2\sqrt{n}} \sum_{j=1}^{n} \pi(g_j) + \pi(g_j)^* + \sum_{|t| \neq 1} \psi(t)\pi(t),$$

where ψ is some function with finite support. We use our assumption to get

$$\begin{aligned}\langle Rx, y\rangle = \frac{i}{2\sqrt{n}} \sum_{j=1}^{n} (\langle \pi(g_j)x, y\rangle + \langle \pi(g_j)^*x, y\rangle) + \sum_{|t|\neq 1} \psi(t)\langle \pi(t)x, y\rangle \\ = \frac{2in}{2\sqrt{n}} = i\sqrt{n}.\end{aligned} \tag{28}$$

By the Cauchy-Schwartz inequality we get

$$\sqrt{n} = |\langle Rx, y\rangle| \le \|Rx\|\|y\| \le \|R\|\|x\|\|y\| \le \sqrt{e}\|x\|\|y\|,$$

contradicting that $\sqrt{n}$ is unbounded. Hence, $f \notin B(F_\infty)$. □

The following result immediately follows from Theorem 4.17 and Lemma 4.20.

Corrolary 4.21. *The group F_∞ is not unitarizable.*

This will imply the non-unitarizability of F_n by the following result which shows that unitarizability passes to subgroups.

Theorem 4.22. *Let F be a subgroup of a discrete group G. Then F is unitarizable if G is unitarizable.*

Proof. We will follow the proof from [P4, Thm 2.8]. Given a uniformly bounded representation $\pi : F \to B(H)$, we intend to construct a Hilbert space $\hat{H} \supseteq H$ and a representation $\hat{\pi} : G \to B(\hat{H})$ such that $|\hat{\pi}| = |\pi|$, H is invariant under $\hat{\pi}(t)$ and $\hat{\pi}(t)$ extends $\pi(t)$ for all $t \in F$. Define

$$\pi_0(t) = \begin{cases} \pi(t) \text{ if } t \in F \\ 0 \text{ otherwise} \end{cases}.$$

We invoke the axiom of choice and let $\{s_j\}$ be a set of one representative of each element in G/F. For more convenient notation, we assume that G/F is countable so that we can index $\{s_j\}$ by the positive integers. Without loss of generality, we may assume that $s_1 = 1$. We define

$$\hat{H} = \bigoplus_{i \ge 1} H_i, H_i = H.$$

Let $\hat{\pi}(x)$ be the operator defined by the matrix $(\pi_0(s_i^{-1}xs_j))_{i,j}$. We make the following observations.

- $\pi_0(s_i^{-1}xs_j) = 0$ except if $s_iF = xs_jF$;
- $\hat{\pi}(e) = I, \hat{\pi}(xy) = \hat{\pi}(x)\hat{\pi}(y)$;
- If $t \in F$ and $i = 1$ or $j = 1$, then $\pi_0(s_i^{-1}ts_j) = 0$ except if $i = j = 1$;
- $|\hat{\pi}| = |\pi|$.

The first point is trivial. The second point holds as if $s_i^{-1}xys_j \in F$ we have $xys_j \in s_iF$, giving that $xys_jF = s_iF$ and $ys_jF = x^{-1}s_iF$ implying that $s_lys_jF = s_lx^{-1}s_iF$. We get that there exists a unique l such that $s_l^{-1}x^{-1}s_iF = F = s_l^{-1}ys_jF$ and

$$\pi_0(s_i^{-1}xys_j) = \pi(s_i^{-1}xys_j) = \pi(s_i^{-1}xs_l)\pi(s_l^{-1}ys_j) = \pi_0(s_i^{-1}xs_l)\pi_0(s_l^{-1}ys_j).$$

In the case that $s_i^{-1}xys_j \notin F$, we get that $s_i^{-1}xys_jF \neq F$ giving that

$$\pi_0(s_i^{-1}xys_j) = 0.$$

Pick l arbitrarily. We may assume that $\pi_0(s_i^{-1}xs_l) \neq 0$. This implies that $xs_lF = s_iF$. This gives that $xys_jF \neq xs_lF$, implying that $ys_jF \neq s_lF$ and that $\pi_0(s_l^{-1}ys_j) = 0$. We get that $\pi_0(s_i^{-1}xys_j) = \pi_0(s_i^{-1}xs_l)\pi_0(s_l^{-1}ys_j)$ for any l in this case. We get that

$$\hat{\pi}(x)\hat{\pi}(y) = \left(\sum_{k\geq 1} \pi_0(s_i^{-1}xs_k)\pi_0(s_k^{-1}ys_j)\right) = (\pi_0(s_i^{-1}xs_l)\pi_0(s_l^{-1}ys_j).$$

This gives that $\hat{\pi}$ is a representation. The third point follows immediately from elementary group theory and implies that H_1 is invariant under $\hat{\pi}(t)$ and the restriction of $\hat{\pi}(t)$ to H is equal to $\pi(t)$ for $t \in F$. The last point follows as

$$\begin{aligned} \|\hat{\pi}(t)\bigoplus_{l\geq 1} x_l\|^2 &= \|\bigoplus_{i\geq 1}(\sum_{j\geq 1} \pi_0(s_i^{-1}ts_j)x_j)\|^2 \\ &= \sum_{i\geq 1} \|\sum_{j\geq 1} \pi_0(s_i^{-1}ts_j)x_j\|^2 \\ &= \sum_{i\geq 1} \|\pi_0(s_i^{-1}ts_{l_i})x_{l_i}\|^2 \\ &\leq \sum_{i\geq 1} \|\pi_0(s_i^{-1}ts_{l_i})\|^2\|x_{l_i}\|^2 \\ &\leq \sum_{i\geq 1} |\pi_0|^2\|x_{l_i}\|^2 \\ &= \sum_{i\geq 1} |\pi|^2\|x_i\|^2 = |\pi|^2. \end{aligned} \tag{29}$$

where l_i is the unique index such that $\pi_0(s_i^{-1}ts_{l_i}) \neq 0$. This shows that $\|\hat{\pi}(t)\| \leq |\pi|$ for all $t \in G$, implying that $|\hat{\pi}| \leq |\pi|$. The other inequality is trivial as π is a subrepresentation of $\hat{\pi}|_F$ due to the third point. We have now proven that $\hat{\pi}(\cdot)$ has all the desired properties. As $\hat{\pi}(\cdot)$ is uniformly bounded, we get that $\hat{\pi}(\cdot)$ is unitarizable. We get that there exists S such that $S\hat{\pi}(\cdot)S^{-1}$ is unitary. If we let $S' : H \to W := SH$, $S'\xi = S\xi$, $\xi \in H$. Then S' is invertible with $S'\pi(t)(S')^{-1}$. We have $S\pi(t)S^{-1}W \subseteq W$ and $S\pi(t)S|_W = S'\pi(t)(S')^{-1}$. As $S\hat{\pi}(t)S^{-1}$ is unitary, so is $S'\pi(t)S'^{-1}$. □

We immediately get the following result.

Corrolary 4.23. *The group F_n is non-unitarizable for all $n \geq 2$.*

Proof. We use ideas from [P4, Lemma 2.7 iii)] to prove this result. Let σ and τ be two distinct generators of F_n. Let F be the subgroup of F_n generated by $\{\sigma^n \tau^n : n \in \mathbb{N}\}$. We easily observe that F is isomorphic to F_∞. We apply the contraposition of Theorem 4.22 to get the desired result. □

Remark 4.24. This result can be proven more directly by showing that the indicator function of $E \cup E^{-1}$ is in $T_1(G)$ but not in $B(G)$ if $E \subseteq G$ is an infinite free set, meaning that it generates a free group with the operation of G. For more details, see [P4, Lemma 2.7] and [P4, Thm 2.7].

We will now prove another similar result.

Proposition 4.25. *Let G be a discrete unitarizable group and let N be a normal subgroup. Then G/N is unitarizable.*

Proof. Let N be a normal subgroup, and $\pi(\cdot)$ be a uniformly bounded representation on G/N, then we can create a uniformly bounded representation on G in the folowing way. Let ϕ be the quotient map and define $\hat{\pi}(t) = \pi(\phi(t))$. This representation is clearly uniformly bounded, and we get that if G is unitarizable, then $\hat{\pi}(\cdot)$ is unitarizable, and this clearly gives that $\pi(\cdot)$ is unitarizable. We get that G/N is unitarizable, completing the proof. □

5 Similarity problems for C^*-algebras

In the previous section, we have seen some similarity results for group representations. We will now explore similar questions of when homomorphisms of C^*-algebras are similar to $*$-homomorphisms. One such problem is the still open *Kadison problem*[P4, p.168].

Let A be a C^*-algebra, H be a Hilbert space. This problem asks the question: "is every bounded homomorphism similar to a *-homomorphism?".

Before we proceed, we will need to repeat some theory of C^*-algebras. We will now present some theory about C^*-algebras from [A, Sec. 2]. Let A be a C^*-algebra and let H be a Hilbert space. We will now state and prove the following result.

Theorem 5.1. *A unital homomorphism $\pi : A \to B(H)$ is a *-homomorphism if and only if $\|\pi\| = 1$.*

Proof. We will follow the proof from [A, Thm. 2.5.5]. We may assume π is not degenerate. We get that $\pi(1) = I$. We observe that $\sigma(\pi(a)) \subseteq \sigma(a)$, as if $\lambda \in \rho(a)$, we get that $\pi(a - \lambda 1)^{-1} = (\pi(a) - \lambda I)^{-1}$, giving that $\lambda \in \rho(\pi(a))$. This implies that for the spectral radius, we have $r(\pi(a)) \leq r(a)$. For any a we get that $\pi(a^*a) = \pi(a)^*\pi(a)$ is self -adjoint and hence, it's norm coincides with it's spectral radius. We get

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = r(\pi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2.$$

This gives that $\|\pi\| \leq 1$ and $\pi(1) = 1$ gives $\|\pi\| = 1$.

Conversely, we note that if u is unitary, we have that $\|\pi(u)x\| \leq \|u\|\|x\| = \|x\|$ and $\|\pi(u^*)x\| \leq \|x\|$, but

$$\|x\| = \|\pi(u)^*\pi(u)x\| \leq \|\pi(u)x\|\|\pi(u)^*\| \leq \|\pi(u)x\|,$$

showing that $\|\pi(u)x\| = \|x\|$ and that $\pi(u)$ is unitary as it is an invertible isometry. We now want to prove that π maps self adjoint elements to self adjoint elements. We note that e^{itx} is unitary for all $t \in \mathbb{R}$ if and only if x is self adjoint. Indeed, from the Taylor series expansion of the exponential function, we get that $e^{itx^*} = e^{-itx^*}$. If x is self adjoint, we get that e^{itx} is unitary. If e^{itx} is unitary, we get that $I = e^{it(x-x^*)}$ for all $t \in \mathbb{R}$. If the standard limit

$$\frac{e^{ita} - 1}{t} \to a, t \to 0$$

holds in this case, we are done. In fact it does hold by inserting the Taylor series of the exponential function and observing that the constant term of the resulting series (not depending on t) is a. We get that π is a *-homomorphism as any $a \in A$ can be expressed as $a = x + iy$ for x, y self adjoint. $\square$

By the same reasoning as in the previous section, any algebra homomorphism that is similar to a *-homomorphism is bounded. The Kadison problem is to determine whether the converse of this statement is true. We will call any C^*-algebra for which this is true *unitarizable*.

5.1 Group C^*-algebras and similarity

In this section we will cover how to associate a C^*-algebra to a given group. We will denote this algebra by $C^*(G)$. We will investigate the connection between the unitarizability of a discrete group G and the unitarizability of $C^*(G)$. Recall that $C^*(G)$ is defined as the completion of $l^1(G)$ with respect to the norm we defined in Section 2.5. We will need the following results

Theorem 5.2. *Let u be a unitary representation of a discrete group G. Then u can be uniquely extended to a *-homomorphism $l^1(G) \to B(H)$ by*

$$\pi_u(\sum_{i=1}^{\infty} f(g_i)\delta_{g_i}) = \sum_{i=1}^{\infty} f(g_i)u(g_i).$$

Proof. We can clearly extend to $\mathbb{C}^{(G)}$ by linearity as

$$\pi_u(\sum_{i=1}^{n} f(g_i)\delta_{g_i}) = \sum_{i=1}^{n} f(g_i)u(g_i).$$

It is clearly a homomorphism. It also preserves the *–operation, as

$$\begin{aligned}
\pi(f^*) = \pi\bigg(\Big(\sum_{i=1}^{\infty} f(g_i)\delta_{g_i}\Big)^*\bigg) &= \pi\Big(\sum_{i=1}^{\infty} \overline{f(g_i)}\delta_{g_i^{-1}}\Big) \\
&= \sum_{i=1}^{\infty} \overline{f(g_i)}\pi(\delta_{g_i^{-1}}) \\
&= \sum_{i=1}^{\infty} (f(g_i)\pi(\delta_{g_i}))^* \\
&= \Big(\sum_{i=1}^{\infty} f(g_i)\pi(\delta_{g_i})\Big)^* = \pi(f)^*.
\end{aligned} \tag{30}$$

It is bounded as

$$\begin{aligned}\|\pi_u(\sum_{i=1}^{n} f(g_i)\delta_{g_i})\| &= \|\sum_{i=1}^{n} f(g_i)u(g_i)\| \\ &\leq \sum_{i=1}^{n} \|f(g_i)u(g_i)\| \\ &= \sum_{i=1}^{n} |f(g_i)| = \|f\|_1.\end{aligned} \tag{31}$$

As $\mathbb{C}^{(G)}$ is dense in $l^1(G)$ this *-homomorphism can be extended to a *-homomorphism $l^1(G) \to B(H)$ by continuity. We get

$$\pi_u(\sum_{i=1}^{\infty} f(g_i)\delta_{g_i}) = \sum_{i=1}^{\infty} f(g_i)u(g_i),$$

where the series on the right hand side converges in norm. This completes the proof. □

Proposition 5.3. *The left regular representation λ_G of G, can be extended to an injective *-homomorphism $l^1(G) \to B(l^2(G))$. We call this the left regular *-homomorphism.*

Proof. We will follow the proof of [P6, Thm. 2.2.5]. As λ_G is unitary, by Theorem 5.2. we extend it to $l^1(G)$ to get

$$\pi_{\lambda_G}(\sum_{i=1}^{\infty} f(g_i)\delta_{g_i})\xi = \sum_{i=1}^{\infty} f(g_i)\lambda_G(g_i)\xi, \xi \in l^2(G).$$

Pick $f = \sum_{i=1}^{\infty} f(g_i)\delta_{g_i} \in ker(\pi_{\lambda_G})$. Fix $g_0 \in G$. We get

$$\begin{aligned}0 &= \langle \pi_{\lambda_G}(f)\delta_{g_0}, \delta_1 \rangle \\ &= \langle \pi_{\lambda_G}\left(\sum_{i=1}^{\infty} f(g_i)\delta_{g_i}\right)\delta_{g_0}, \delta_1 \rangle \\ &= \sum_{h\in G}\sum_{i=1}^{\infty} f(g_i)\delta_{g_0}(g_i^{-1}h)\delta_1(h) = f(g_0).\end{aligned} \tag{32}$$

This shows that $f = 0$. □

Recall the definition of $|a|$, $a \in l^1(G)$ as the supremmum of $\|a\|$ over all C^*-norms on $l^1(G)$. We have the following equivalent definition.

Proposition 5.4. *We have that*

$$|a| = \sup\{\|\pi(a)\| : \pi \textit{ is a } * - \textit{homomorphism } l^1(G) \to B(H) \textit{ for some } H\}.$$

Proof. It is obvious that $|a| \geq \sup\{\|\pi(a)\|\}$, as $\sup\{\|\pi(a)\|\}$ defines a C^*-norm. It is a norm as we may pick π as the left-regular *-homomorphism. The Gelfand-Naimark theorem, gives that for every C^*-norm on $l^1(G)$, there exists an isometric *-isomorphism from the completion of $l^1(G)$ with that norm to B, where B is a norm closed subalgebra of $B(H)$ for some H. Hence we have that $|a| \leq \sup\{\|\pi(a)\|\}$. □

Remark 5.5. Instead of the norm we already defined, we may equip $l^1(G)$ with the norm $|a| := \|\lambda_a\|$. The completion with respect to this norm is called the *reduced C^*-algebra* of G and is denoted by $C^*_\lambda(G)$.

The next result gives a connection between *-homomorphisms of $l^1(G)$ and unitary representations of G.

Proposition 5.6. *We have that $\pi : l^1(G) \to B(H)$ is a *-homomorphism if and only if $\rho(g) := \pi(\delta_g)$ is a unitary representation of G.*

Proof. If ρ is unitary, then $\pi_\rho = \pi$ and π is a *-homomorphism by Theorem 5.2. Conversely, if π is a *-homomorphism, we have that

$$\rho(g)^* = \pi(\delta_g)^* = \pi(\delta_g^*) = \pi(\delta_{g^{-1}}) = \rho(g^{-1}).$$

This completes the proof. □

Considering the previous result, it may seem natural to believe that $C^*(G)$ is unitarizable if and only if G is unitarizable. We obviously get that $C^*(G)$ is unitarizable if G is unitarizable, but the converse is not true. The natural connection fails as if we try to extend a uniformly bounded representation to $\mathbb{C}^{(G)}$, this is not necessarily bounded. It is for example not possible to deduce that $C^*(F_2)$ is not unitarizable just because F_2 is not unitarizable.

We will now give some examples of $C^*(G)$ for concrete groups.

Theorem 5.7. *We have that $C^*(G) \simeq C(\hat{G})$, where G is a discrete abelian group and $\hat{G}$ is the group of homomorphisms $G \to \mathbb{T}$ with pointwise multiplication and it's topology is generated by sets on the form*

$$\{\chi \in \hat{G} : |\chi(g_i) - \chi_0(g_i)| < \varepsilon \text{ for all } i \in I\},$$

where I is finite.

Proof. We will follow the proof from [P6, Thm. 2.5.5]. We observe that $C^*(G)$ is a commutative C^*-algebra with unit δ_1. By Gelfand's theorem, $C^*(G)$ is isomorphic to $C(sp(C^*(G)))$ where the isomorphism is given by the Gelfand map. It suffices to show that $sp(C^*(G))$ and $\hat{G}$ are homeomorphic. Let $\phi \in sp(C^*(G))$. We define $\rho_\phi(g) = \phi(\delta_g)$. The mapping $f : sp(C^*(G)) \to \hat{G}$ given by $\phi \mapsto \rho_\phi$ is a continuous. It is injective as $C^*(G)$ is generated by $\{\delta_g\}$. It is surjective as if $\chi \in \hat{G}$, we get that χ is a unitary representation. Hence, we may extend it to a character of $l^1(G)$ and hence $C^*(G)$. As f is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. This completes the proof. □

Proposition 5.8. *We have that $C^*(\mathbb{Z}) \simeq C(\mathbb{T})$.*

Proof. This proof is based on the proof of [P6, Prop. 2.5.4]. We define a mapping $\phi : \hat{\mathbb{Z}} \to \mathbb{T}$ by $\chi \mapsto \chi(1)$. The mapping ϕ is clearly injective as $\mathbb{Z}$ is cyclic. It is obvious that for every $z \in \mathbb{T}$, there exists a homomorphism χ such that $\chi(1) = z$. Just pick $\chi(n) = z^n$. This is clearly a continuous and bijective map between compact Hausdorff spaces and hence it's a homeomorphism. We get $C(\hat{\mathbb{Z}}) \simeq C(\mathbb{T})$. The desired result follows from Theorem 5.7. □

The next theorem gives that $C^*(F_2)$ has a certain universality property.

Theorem 5.9. *For every C^*-alegbra A and for all unitary elements $u, v \in A$, there exists a unique *-homomorphism $\rho : C^*(F_2) \to A$ such that $\rho(\delta_\sigma) = u$ and $\rho(\delta_\tau) = v$. For every $n \in \mathbb{Z}^+$ there exists a surjective *-homomorphism $C^*(F_2) \to B(\mathbb{C}^n)$.*

Proof. We will follow the proof from [P6, Thm. 2.7.1]. Let $\pi : A \to B(H)$ be an injective *-homomorphism (see Theorem 2.27). We get that $\pi(u)$ and $\pi(v)$ are unitary. A basic result about free groups gives that there exists a group homomorphism $\phi : F_2 \to U(H)$ such that $\phi(\sigma) = \pi(u)$ and $\phi(\tau) = \pi(v)$. This is a unitary representation, and hence it extends to a *-homomorphism $\varphi : C^*(F_2) \to B(H)$. Let $\rho = \pi^{-1} \circ \varphi$. We complete the proof of the first statement by noting that the composition is well-defined. For the second statement, we pick

$$u = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & e^{2\pi i/n} & \dots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \dots & e^{2\pi i(n-1)/n} \end{bmatrix}$$

and

$$v = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

We note that the C^*-algebra generated by these matrices is $M_n(\mathbb{C})$ as u generates all diagonal matrices, and hence we can find a matrix in the C^*-algebra generated by u that maps a basis vector e_i of $\mathbb{C}^n$ to te_i for any complex t. It is obvious that there is a matrix in the C^*-algebra generated by v that maps e_i to e_j for any j, proving that any linear mapping $\mathbb{C}^n \to \mathbb{C}^n$, and hence any $n \times n$ matrix, is in the C^*-algebra generated by u, v. This gives that ρ is surjective in this case. ☐

6 Completely bounded homomorphisms

We will now investigate the theory of completely bounded maps. The aim of this section is to prove that a homomorphism $A \to B(H)$ is similar to a *-homomorphism if and only if it is completely bounded. The tools from this section, combined with results about group C^*-algebras will be used to prove a result that gives a necessary and sufficient condition for a group to be unitarizable.

6.1 Completely bounded maps

Definition 6.1. Let A be a C^*-algebra and let $\phi : A \to B$ be a linear map. We define $\phi_n : M_n(A) \to M_n(B)$ by $\phi_n((a_{i,j})) = (\pi_n(a_{i,j}))$. We say that ϕ is *completely positive* if ϕ_n is positive for all $n \in \mathbb{Z}^+$. We say that ϕ is *completely bounded* if $\sup_{n \geq 1} \|\phi_n\| < \infty$. We denote this supremum with $\|\phi\|_{cb}$. We say that ϕ is *completely contractive* if $\|\phi\|_{cb} \leq 1$.

Example 6.2. Consider the linear operator $\phi : M_2(\mathbb{C}) \to B(\mathbb{C})$ given by $M \mapsto M^t$. This operation is positive. However, ϕ_2 is not positive as

$$\phi_2 \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the input matrix is positive but the output matrix is not as it's characteristic polynomial is $(\lambda-1)^3(\lambda+1)$, giving that -1 is an eigenvalue.

We will now give a result that gives a characerization of completely positive maps.

Theorem 6.3 (Stinespring)**.** *Let A be a C^*-algebra and let $\phi : A \to B(H)$ be a completely positive map. Then there exists a Hilbert space K, a unital *-homomorphism $\pi : A \to B(K)$ and a bounded map $V : H \to K$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^*\pi(a)V.$$

Proof. We will give the proof from [P2, Thm. 4.1]. On the algebraic tensor product $A \otimes H$ we define a bilinear form by extending $\langle a \otimes x, b \otimes y\rangle := \langle \phi(b^*a)x, y\rangle_H$ to the entire space. We use $H^{(n)}$ to denote the direct sum of n copies of H. The bilinear form is positive semidefinite. Indeed, we have

$$\begin{aligned} \left\langle \sum_i a_i \otimes x_i, \sum_j a_j \otimes x_j \right\rangle &= \sum_{i,j} \langle a_i \otimes x_i, a_j \otimes x_j\rangle \\ & \sum_{i,j} \langle \phi(a_j^* a_i) x_i, x_j\rangle \\ &= \langle \phi_n((a_j^* a_i)_{i,j}) \oplus_i x_i, \oplus_j x_j\rangle_{H^{(n)}}, \end{aligned} \tag{33}$$

and hence it suffices to show that $(a_j^* a_i)_{i,j}$ is positive. We get

$$(a_j^* a_i)_{i,j} = \begin{bmatrix} a_1^* & 0 & ... & 0 \\ a_2^* & 0 & ... & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_n^* & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & ... & a_n \\ 0 & 0 & ... & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \end{bmatrix} \geq 0,$$

which completes the proof that the bilinear form is positive semidefinite. Therefore it satisfies the Cauchy-Schwartz inequality, i.e.

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle, u, v \in A \otimes H.$$

This gives that

$$N = \{u \in A \otimes H : \langle u, u\rangle = 0\} = \{u \in A \otimes H : \langle u, v\rangle = 0 \text{ for all } v\}$$

is a subspace of $A \otimes H$. We now consider the inner product space $(A \otimes H)/N$ with the inner product $\langle x + N, y + N\rangle = \langle x, y\rangle$. Let K be the completion of this space. We define $\pi(a) \sum(a_i \otimes x_i + N) = \sum(aa_i) \otimes x_i + N$. We observe that $(a_j^* a^* a a_i)_{i,j} \leq \|a^*a\|(a_j^* a_i)_{i,j}$. Indeed, we have

$$\|a^*a\|(a_j^* a_i)_{i,j} - (a_j^* a^* a a_i)_{i,j} = (a_j^*(\|a^*a\|1 - a^*a)a_i)_{i,j} \geq 0,$$

giving that $\phi_n(a_j^* a^* a a_i) \leq \|a^*a\| \phi_n(a_j^* a_i)$. This gives that

$$\begin{aligned} \Big\langle \pi(a)\Big(\sum a_i \otimes x_i + N\Big), \pi(a)\Big(\sum a_j \otimes x_j + N\Big)\Big\rangle \\ = \sum_{i,j} \langle a a_i \otimes x_i + N, a a_j \otimes x_j + N\rangle \\ = \sum_{i,j} \langle \phi(a_j^* a^* a a_i) x_i, x_j\rangle_H \\ = \langle \phi(a_j^* a^* a a_i) \oplus_i x_i, \oplus_j x_j\rangle_{H^{(n)}} \\ \leq \|a^*a\| \langle \phi(a_j^* a_i) \oplus_i x_i, \oplus_j x_j\rangle_{H^{(n)}} \\ = \|a^*a\| \sum_{i,j} \langle \phi(a_j^* a_i) x_i, x_j\rangle_H \\ = \|a\|^2 \Big\langle \sum a_i \otimes x_i + N, \sum a_j \otimes x_j + N\Big\rangle. \end{aligned} \tag{34}$$

This shows that $\pi(a)$ leaves N invariant, giving that it is a well defined operator on K, and it is bounded by $\|a\|$. We get that $\pi(a)$ can be extended to a unital *-homomorphism $A \to B(K)$(see the proof of the GNS construction). We now define $V : H \to K$ by $x \mapsto 1 \otimes x + N$. We note that

$$\|Vx\|^2 = \langle 1 \otimes x + N, 1 \otimes x + N\rangle = \langle \phi(1)x, x\rangle_H \leq \|\phi(1)\| \|x\|^2,$$

giving that V is bounded. We now get that

$$\langle V^* \pi(a) V x, y\rangle = \langle \pi(a) V x, V y\rangle = \langle \pi(a)(1 \otimes x + N), 1 \otimes y + N\rangle = \langle \phi(a)x, y\rangle_H$$

for all x, y, giving that $V^* \pi(a) V = \phi(a)$, completing the proof. □

We call the triple (π, V, K) a *Stinespring representation*. If the closed linear span of $\pi(A)VH$ is K, we call the Stinespring representation *minimal*.

Remark 6.4. This result implies the GNS-construction as if ρ is a positive linear functional, then ρ is completely positive by [P2, Prop. 3.8]. Let (π, V, K) be a minimal Stinespring decomposition. Let $\xi = V1$. We get $\langle \pi(a)V1, V1\rangle = V^* \pi(a) V = \rho(a)$ and $\overline{\pi(A)V1} = \overline{\pi(A)V\mathbb{C}}$, giving that (π, ξ) is a GNS-pair.

Two results that will be used later are the following.

Theorem 6.5. *Let X be a Banach space and let A be a von Neumann algebra. Then any bounded net $\{T_\lambda\}$ of linear maps $X \to A$ has a cluster point in the point-ultraweak topology.*

The proof of this result can be found in [BO, Thm. 1.3.7]

Proposition 6.6. *Let A, B be C^*-algebras and let $\phi : A \to B$ be a contractive completely positive map. Then we have the following:*

- *$\phi(a)^* \phi(a) \leq \phi(a^* a)$ for all $a \in A$.*

- *Fix $a \in A$. Then $\phi(ab) = \phi(a)\phi(b)$ and $\phi(ba) = \phi(b)\phi(a)$ for all b if $\phi(aa^*) = \phi(a)\phi(a)^*$ and $\phi(a^* a) = \phi(a)^* \phi(a)$.*

- *The set A_ϕ of $a \in A$ such that a satisfies the conditions in the previous point is a C^*-subalgebra of A.*

The proof of this result can be found in [BO, Prop. 1.5.7]. It is a natural question to ask whether any completely positive map defined on an operator system extends to the whole C^*-algebra. We have the following result.

Theorem 6.7 (Arveson's extension theorem)**.** *Let $S \subseteq A$ be an operator system and let $\phi : S \to B(H)$ be a completely positive map. Then there exists a completely positive map $\varphi : A \to B(H)$ such that $\varphi|_S = \phi$.*

The proof of this result can be found in [P2, Thm. 7.5].

Theorem 6.8. *Let A be a C^*-algebra and let ϕ be completely bounded. Then there exist completely positive maps $\phi_1, \phi_2 : A \to B(H)$ such that the map $\Phi : M_2(A) \to B(H \oplus H)$ given by*

$$\Phi\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = \begin{bmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{bmatrix}$$

is completely positive and $\|\phi_i\|_{cb} = \|\phi\|_{cb}$. If $\|\phi\|_{cb} \leq 1$, we may take ϕ_i such that $\phi_1(1) = \phi_2(1) = I_H$.

The proof of this result can be found in [P2, Thm. 8.3] We will now state and prove an analogous theorem to Theorem 6.3 for completely bounded maps.

Theorem 6.9 (Wittstock)**.** *Let A be a C^*-algebra and let $\phi : A \to B(H)$ be a completely bounded map. Then there exist a Hilbert space K, a *-homomorphism $\pi : A \to B(K)$ and bounded operators $V_1, V_2 : H \to K$ such that $\phi(a) = V_1^* \pi(a) V_2$ and $\|\phi\|_{cb} = \|V_1\|\|V_2\|$. If $\|\phi\|_{cb} = 1$, we may pick V_i as isometries.*

Proof. We will follow the proof from [P2, thm. 8.4]. Without loss of generality, assume that $\|\phi\|_{cb} = 1$. Let ϕ_1, ϕ_2 and Φ be as in Theorem 6.8. Let (π_1, V, K_1) be a minimal Stinespring representation of Φ. We get that Φ is unital, and we may pick V as an isometry. We now note the following.

Lemma 6.10. *There exists a Hilbert space K and a unital *-homomorphism $\pi : A \to B(K)$ such that $K_1 = K \oplus K$ and*

$$\pi_1\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = \begin{bmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{bmatrix}.$$

Proof of Lemma. We note that

$$\pi_1\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right] = U^* \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} U$$

for $U : K_1 \to K_1$ unitary and K a subspace of K_1, as

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is a projection. We use the same reasoning to get that

$$\pi\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

is unitarily equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & I_K \end{bmatrix}.$$

For

$$\pi\left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right] = \pi\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

giving that

$$U^*\pi\left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]U = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}.$$

We get that w is unitary $K \to K^\perp$ as

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ w^* & 0 \end{bmatrix} = \begin{bmatrix} ww^* & 0 \\ 0 & 0 \end{bmatrix}.$$

We also get

$$\begin{bmatrix} 0 & 0 \\ w* & 0 \end{bmatrix}\begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & w^*w \end{bmatrix}.$$

We have $K \cong K^\perp$ and $K_1 = K \oplus K$. Let

$$S = \begin{bmatrix} w & 0 \\ 0 & I \end{bmatrix}.$$

This operator is unitary. We get that

$$(US)^*\pi_1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}US = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}.$$

This gives that

$$(US)^*\pi_1\left[\begin{bmatrix} a1 & b1 \\ c1 & d1 \end{bmatrix}\right]US = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} a, b, c, d \in \mathbb{C}.$$

Let $a \in A$. We get that

$$\pi_1\left[\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} T_{1,1} & 0 \\ 0 & 0 \end{bmatrix}$$

as

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{1,1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We can use similar reasoning to show similar results for

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

etc. We conclude that

$$\pi_1\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = \begin{bmatrix} \pi_{1,1}(a) & \pi_{1,2}(b) \\ \pi_{2,1}(c) & \pi_{2,2}(d) \end{bmatrix}.$$

We get that

$$\begin{bmatrix}0 & \pi_{2,1}(a)\\ 0 & 0\end{bmatrix}\pi_1\left[\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix}\right] = \pi_1\left[\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right] = \begin{bmatrix}0 & \pi_{1,1}(a)\\ 0 & 0\end{bmatrix},$$

showing that $\pi_{2,1} = \pi_{1,1}$. It follows by a similar reasoning that all $\pi_{i,j}$ are equal. Moreover, we get

$$\begin{aligned}\begin{bmatrix}\pi_{1,1}(a_1a_2) & 0\\ 0 & 0\end{bmatrix} &= \pi_1\left[\begin{bmatrix}a_1a2 & 0\\ 0 & 0\end{bmatrix}\right]\\ &= \pi_1\left[\begin{bmatrix}a_1 & 0\\ 0 & 0\end{bmatrix}\right]\pi_1\left[\begin{bmatrix}a_2 & 0\\ 0 & 0\end{bmatrix}\right] = \left[\pi_{1,1}(a_1)\pi_{1,1}(a_2)\right],\end{aligned} \tag{35}$$

and

$$\begin{aligned}\begin{bmatrix}\pi_1(a^*) & 0\\ 0 & 0\end{bmatrix} &= \pi_1\left[\begin{bmatrix}a^* & 0\\ 0 & 0\end{bmatrix}\right]\\ &= \pi_1\left[\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\right]^*\\ &= \begin{bmatrix}\pi_{1,1}(a) & 0\\ 0 & 0\end{bmatrix}^* = \begin{bmatrix}\pi_{1,1}(a)^* & 0\\ 0 & 0\end{bmatrix}.\end{aligned} \tag{36}$$

This shows that $\pi_{1,1}$ is a *-homomorphism, completing the proof. □

We get that

$$h\oplus 0 = \begin{bmatrix}\phi_1(1) & 0\\ 0 & 0\end{bmatrix}(h\oplus 0) = V^*\begin{bmatrix}\pi(1) & 0\\ 0 & 0\end{bmatrix}V(h\oplus 0).$$

Let $V(h\oplus 0) = a\oplus b$, we observe that $b = 0$ as the fact that V is an isometry implies that

$$\|a\|^2 = \|h\|^2 = \|a\oplus b\|^2 = \|a\|^2 + \|b\|^2.$$

We get that there are isometries V_1 and V_2 such that $V(h\oplus 0) = V_1h\oplus 0$ and $V(0\oplus b) = 0\oplus V_2h$. We get

$$\begin{bmatrix}\phi_1(a) & \phi(b)\\ \phi^*(c) & \phi_2(d)\end{bmatrix} = V^*\begin{bmatrix}\pi(a) & \pi(b)\\ \pi(c) & \pi(d)\end{bmatrix}V = \begin{bmatrix}V_1^*\pi(a)V_1 & V_1^*\pi(b)V_2\\ V_2^*\pi(c)V_1 & V_2^*\pi(d)V_2\end{bmatrix}.$$

This completes the proof. □

6.2 Haagerup's theorem on completely bounded homomorphisms

We will now connect this to the previous section. It turns out that a homomorphism is equivalent to a *-homomorphism if and only if it is completely bounded. Before we proceed we need the following result.

Theorem 6.11. *Let A be a C^*-algebra and let $\rho : A \to B(H)$ be a completely bounded unital homomorphism. Then there exists an invertible operator S such that $\|S\|\|S^{-1}\| = \|\rho\|_{cb}$ and $S^{-1}\rho(\cdot)S$ is completely contractive.*

Proof. We will follow the proof from [P2, Thm. 9.1]. Let the Hilbert space K, the operators V_1, V_2 and the *-homomorphism π be as in Theorem 6.9. We define the seminorm $|\cdot|$ by

$$|h| := \inf\{\|\sum_{i=1}^n \pi(a_i)V_2h_i\| : \sum_{i=1}^n \rho(a_i)h_i = h, a_i\in A, h\in H, n\in\mathbb{N}\}.$$

This is a norm as it is equivalent to the usual norm on H. Indeed, we observe that

$$\|h\| = \| \sum_{i=1}^{n} \rho(a_i)h_i\| = \| \sum_{i=1}^{n} V_1^* \pi(a_i)V_2 h_i\| \leq \|V_1\|\| \sum_{i=1}^{n} \pi(a_i)V_2 h_i\|,$$

and hence $\|h\| \leq \|V_1\||h|$ We also get that $|h| \leq \|\pi(1)V_2 h\| \leq \|V_2\|\|h\|$. This gives that $|\cdot|$ is a norm. We now want to show that it is given by an inner product. It suffices to show that

$$|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2), x, y \in H.$$

The inequality

$$|x+y|^2 + |x-y|^2 \leq 2(|x|^2 + |y|^2)$$

follows from the parallelogram identity for the usual norm as

$$\begin{aligned} |x+y|^2 + |x-y|^2 &\leq \left\| \sum_{i=1}^{n} \pi(a_i)V_2 x_i + \sum_{j=1}^{n} \pi(a_j)V_2 y_j \right\|^2 \\ &+ \left\| \sum_{i=1}^{n} \pi(a_i)V_2 x_i - \sum_{j=1}^{n} \pi(a_j)V_2 y_j \right\| \\ &= 2\left(\left\| \sum_{i=1}^{n} \pi(a_i)V_2 x_i \right\|^2 + \left\| \sum_{j=1}^{n} \pi(a_j)V_2 y_j \right\|^2 \right). \end{aligned} \tag{37}$$

The other inequality follows by letting $x = u+v$ and $y = u-v$. We can now let $S : (H, |\cdot|) \to (H, \|\cdot\|)$ be the identity map. This is bounded with bounded inverse because of the equivalence between the $|\cdot|$-norm and the usual norm. This reduces the problem to showing that ρ is completely contractive with respect to the norm $|\cdot|$. We get that ρ is contractive as

$$|\rho(a)h| \leq \| \sum_{i=1}^{n} \pi(aa_i)V_2 h_i \leq \|a\|\| \sum_{i=1}^{n} \pi(a_i V_2 h_i\|$$

showing that $|\rho(a)h| \leq \|a\|\|h\|$. To show that ρ is completely contractive, we fix $n \in \mathbb{Z}^+$ and define a norm $|\cdot|'$ such that ρ_n is contractive on $H^{(n)}$ by

$$|\hat{h}|' = \inf\{\| \sum_{i=1}^{m} \pi_n(A_i)\hat{V}_2 \hat{h}_i\|_n : \sum_{i=1}^{m} \rho_n(A_i)\hat{h}_i = \hat{h}\},$$

where $\hat{V}_2$ is the direct sum of n copies of V_2. It suffices to show that $|\cdot|' = |\cdot|_n$. This follows as

$$\begin{aligned} |x|'^2 &= \inf\{\| \sum_{l=1}^{m} \pi_n(A_l)\hat{V}_2 \hat{h}_l\|_n^2 : \sum_{l=1}^{m} \rho_n(A_l)\hat{h}_l = x\} \\ &= \inf\{\sum_{i=1}^{n} \| \sum_{l=1}^{m} \sum_{j=1}^{n} \pi(a_{l,i,j})V_2 h_{l,j}\|^2 : \sum_{l=1}^{m} \sum_{j=1}^{n} \rho(a_{l,i,j})h_{l,j} = x_i\} \\ &= \sum_{i=1}^{n} |x_i|^2 = |x|_n^2. \end{aligned} \tag{38}$$

This completes the proof. □

Corrolary 6.12 (Haagerup). *A unital homomorphism $\pi : A \to B(H)$ is equivalent to a -homomorphism if and only if it is completely bounded.*

Proof. We will follow the proof of [P4, Corr. 4.4]. The only if direction is obvious as if a homomorphism $\rho(\cdot) = S^{-1}\pi(\cdot)S$ we have that $\rho_n(\cdot) = {S^{(n)}}^{-1}\pi_n(\cdot)S^{(n)}$, where $S^{(n)}$ is the direct sum of n copies of S. We now want to prove the if direction. It suffices to show that any contractive homomorphism is a $*$-homomorphism. This follows immediately from Theorem 5.1.

□

This result reduces the Kadison problem to determining whether or not every bounded homomorphism is completely bounded [P2, p.123]. We will now go back to the unitarizability of groups and use our new tools to prove that a discrete group is unitarizable if and only if every countable subgroup is unitarizable. We will now state and prove two results that will directly imply the desired result.

Theorem 6.13. *Let G be a discrete group and let π be uniformly bounded. The following are equivalent:*

- *π is unitarizable*
- *$\hat{\pi}(f) := \sum_{t\in G} f(t)\pi(t)$ defined on $l_1(G)$ extends to a completely bounded homomorphism $C^*(G) \to B(H)$.*

Proof. We will follow the proof from [P3, Thm. 0.9]. If π is unitarizable, we get that there exists $S \in B(H)$ such that $\pi = S^{-1}\rho S$ for a unitary representation ρ. We get that ρ extends to a *-homomorphism $\hat{\rho}$. We get that $\hat{\pi} = S^{-1}\hat{\rho}S$ and the desired property follows from Corollary 6.12. The converse is also a direct consequence of the same corollary. □

Corrolary 6.14. *A uniformly bounded representation of a discrete group is unitarizable if and only if the restriction to any countable subgroup is unitarizable.*

Proof. We will follow the proof from [P3, Corr. 0.10]. Any restriction of a unitarizable representation is clearly unitarizable. We will now prove the converse. We will prove this by contraposition. If π is not unitarizable, then $\hat{\pi}$ can not be extended to a completely bounded map $C^*(G) \to B(H)$. Then we can find $(a_{n,i,j})_{i,j} = a_n \in M_n(C^*(G))$, such that $\|a_n\| \leq 1$ and $a_{n,i,j} \in l^1(G)$(as $l^1(G)$ is dense in $C^*(G)$), such that $\|\hat{\pi}_n(a_n)\| \to \infty$. Let Γ be the countable subgroup generated by $\bigcup_{n,i,j} supp(a_{n,i,j})$, where $supp(f)$ denotes the support of the function f. We see that $\pi|_\Gamma$ is not unitarizable, completing the proof. □

From Corollary 6.14 and Theorem 4.22, we immediately get that G is unitarizable if and only if every countable subgroup is unitarizable. However, having to check every countable subgroup in order to use this result makes the criterion quite hard to use.

7 The converse of the Dixmier-Day theorem

We previously proved the Dixmier-Day theorem (Theorem 4.11), which states that any amenable group is unitarizable. A natural question is whether or not the converse of this is true. The converse is indeed true for almost connected locally compact groups, meaning that G/G_1, where G_1 is the connected component containing 1, is compact (see [P3, Remark. 0.8]). Due to this result, it is clear that the discrete case is among the more interesting cases as $G/G_1 = G$. Hence, we will focus on discrete groups. As unitarizability passes to subgroups (see Theorem 4.22), we get that the converse would be implied by the following conjecture by Von Neumann.

Conjecture 7.1. *A group G is amenable if and only if it does not have F_2 as a subgroup.*

Unfortunately, this conjecture has been disproved. There are non-amenable groups not containing F_2, some examples can be found in [M] and [P3, Remark 0.4]. Those groups are candidates that can witness the failure of the converse of the Dixmier-Day theorem. We will now state a result that gives us a weaker form of the converse of the Dixmier-Day theorem. As noted in Remark 4.12, the similarity S that we construct in the Dixmier-Day theorem has the additional property that $\|S^{-1}\|\|S\| \leq |\pi|^2$. The existence of such a similarity is one of the statements that turns out to be equivalent to amenability. Recall that the space $B(G)$ is defined as the space of functions $f : G \to \mathbb{C}$ such that there exists a Hilbert space H, a unitary representation $\pi : G \to B(H)$ and vectors $x, y \in H$ such that $f(t) = \langle \pi(t)x, y\rangle$. For $c > 0$, we define the space

$$B_c(G) := \{f : \text{ there exists } \pi \text{ with } |\pi| \leq c, x, y \in H : f(t) = \langle \pi(t)x, y\rangle\}.$$

We define the norm in the same way as on $B(G)$.

Theorem 7.2. *Let G be a discrete group. Then the following are equivalent:*

- *G is amenable.*
- *For every uniformly bounded representation π, there exists an invertible operator $S \in B(H)$ such that $S^{-1}\pi S$ is unitary and $\|S^{-1}\|\|S\| \leq |\pi|^2$.*
- *There exist constants K and $\alpha < 3$ such that for every uniformly bounded representation, there exists an invertible operator $S \in B(H)$ such that $S^{-1}\pi S$ is unitary and $\|S^{-1}\|\|S\| \leq K|\pi|^\alpha$.*
- *We have that $B_c(G)$ is a subspace of $B(G)$. For every $f \in B_c(G)$ and $c \geq 1$, we have that $\|f\|_{B(G)} = c^2\|f\|_{B_c(G)}$*
- *There is a K and $\alpha < 3$ such that for every $f \in B_c$ and for every $c \geq 1$, we have that $\|f\|_{B(G)} \leq Kc^\alpha\|f\|_{B_c(G)}$.*

It is stated in [P4, p.55] that this result can be generalized to arbitrary groups, but it is not a proof of the converse of the Dixmier-Day theorem as it does make assumptions about the similarity S that might not hold for a general unitarizable group.

Part one of the proof. We will follow the proof from [P3, Thm. 1.1]. The first point implies the second point by Theorem 4.11 and Remark 4.12. The second point implies the 3rd point by picking $K = 1$ and $\alpha = 2$, the fourth point implies the fifth point by the same reasoning. The third point implies the fifth point as for any $f \in B_c(G)$, we may express it as $f(t) = \langle \pi(t)x, y\rangle$, but then we may write it as $f(t) = \langle \rho(t)S^{-1}x, S^*y\rangle$, where ρ is unitary. We get that

$$\begin{aligned}\|f\|_{B(G)} &\leq \|S^{-1}x\|\|S^*y\| \\ &\leq \|S^{-1}\|\|S\|\|x\|\|y\| \\ &\leq K|\pi|^\alpha\|x\|\|y\| \leq Kc^\alpha\|x\|\|y\|.\end{aligned} \tag{39}$$

We obtain that $\|f\|_{B(G)} \leq K|c|^\alpha$ by taking the infimum over all representations of f on the form

$f(t) = \langle \pi(t)x, y \rangle$. By the same reasoning, the second point implies the fourth point. It remains to show that the fifth point implies the first point, but we need to make some preparations for this. $\square$

7.1 The spaces of multipliers

We will now make the necessary preparations before we proceed with the rest of the proof of Theorem 7.2. We will follow [P3, Sec. 2]. We define $M_d(G)$ as the set of functions $f : G \to \mathbb{C}$ such that there are bounded functions $\xi_i : G \to B(H_i, H_{i-1})$, where $\{H_i\}_{i=0}^d$ is a sequence of Hilbert spaces such that $H_0 = H_d = \mathbb{C}$, such that

$$f(t_1t_2...t_d) = \xi_1(t_1)...\xi_d(t_d)$$

for all $t_1, ..., t_d \in G$. This of course requires that we identify $B(\mathbb{C})$ with $\mathbb{C}$, which we can do as these spaces are clearly isomorphic. We note that $M_d(G)$ is a vector space as if $f(t_1...t_d) = \xi_{1,1}(t_1)...\xi_{1,d}(t_d)$ and $g(t_1...t_d) = \xi_{2,1}(t_1)...\xi_{2,d}(t_d)$, we have that

$$(f + g)(t_1...t_d) = \xi_1(t_1)...\xi_d(t_d),$$

where $\xi_d(t_d)z = \xi_{1,d}(t_d)z \oplus \xi_{2,d}(t_d)z$, $\xi_i = \xi_{1,i} \oplus \xi_{2,i}$ for $1 < i < d$. For $i = 1$, we let x be the representative of the bounded linear functional $\xi_{1,1} : H_i \to \mathbb{C}$ and y be the representative of $\xi_{2,1} : K_i \to \mathbb{C}$. Then ξ_1 is the bounded linear functional corresponding to $x \oplus y \in H_i \oplus K_i$. In the case where $d = 2$, we get that the condition can be rewritten as $f(t_1t_2) = \langle x(t_1), y(t_2) \rangle$, where x, y are bounded functions $G \to H$. The space $M_2(G)$ is called the space of *Hertz-Schur multipliers*. The norm on $M_d(G)$ is defined by

$$\|f\|_{M_d(G)} = \inf\{\prod_{i=1}^{d} \sup_{t_i \in G} \|\xi_i(t_i)\| : f(t_1...t_d) = \xi_1(t_1)...\xi_d(t_d)\}.$$

This is indeed a norm as we have the following.

- We have that $\|cf\|_{M_d(G)} = c\|f\|_{M_d(G)}$ as

  $$\|cf\|_{M_d(G)} \leq \sup \|c\xi_1(t_1)\| \prod_{i=2}^{d} \sup \|\xi_i(t_i)\| = |c|\|f\|_{M_d(G)},$$

  giving that $\|cf\|_{M_d(G)} \leq |c|\|f\|_{M_d(G)}$ for all $f \in M_d(G)$ and all c. We get the other inequality by $\|f\|_{M_d(G)} = \|cf/c\|_{M_d(G)} \leq \|cf\|_{M_d(G)}/c$.

- The triangle inequality holds. We will prove this in the case of $d = 2$ as $M_2(G)$ is the relevant space for the remaining part of this section. Let $f(st) = \langle x_1(s), y_1(t) \rangle$ and $g(st) = \langle x_2(s), y_2(t) \rangle$. Let $x(t) = x_1(t) \oplus x_2(t)$ and $y(t) = y_1(t) \oplus y_2(t)$. We use a similar argument as for $B(G)$ to pick x_1, y_1, x_2, y_2 such that $\sup \|x_1(t)\| = \sup \|y_1(t)\|$ and $\sup \|x_2(t)\| = \sup \|y_2(t)\|$. We get

  $$\begin{aligned} \|f + g\|_{M_2(G)}^2 &\leq (\sup \|x_1(t)\|^2 + \sup \|x_2(t)\|^2)(\sup \|y_1(t)\|^2 + \|y_2(t)\|^2) \\ &= (\sup \|x_1\| \sup \|y_1\| + \sup \|x_2\| \|y_2\|)^2. \end{aligned} \tag{40}$$

To complete the proof of Theorem 7.2, we will use the following result.

Theorem 7.3. *Let G be a discrete group, then the following are equivalent.*

- *G is amenable*

- $B(G) = M_2(G)$

- *The identity map* $I : (B(G), \|\cdot\|_{M_2(G)}) \to (B(G)\|\cdot\|_{B(G)})$ *is bounded.*

Proof. A sketch of the proof that the third point implies the first point can be found in [P3, Thm. 2.3]. We now want to prove that the first point implies the second point. We observe that $B(G) \subseteq M_2(G)$ for any group as if $f(t) = \langle \pi(t)x, y\rangle$ for a unitary representation π, we get $f(st) = \langle \pi(st)x, y\rangle = \langle \pi(t)x, \pi(s^{-1}y)\rangle$. In order to prove that $M_2(G) \subseteq B(G)$, pick $f \in M_2(G)$. By definition we have that $f(st) = \langle x(s), y(t)\rangle$ where $x, y \in l^\infty(G, H)$. We want to find a Hilbert space K, a unitary representation $\pi : G \to B(K)$ and vectors x, y such that $f(t) = \langle \pi(t)x, y\rangle$. Let m be a left invariant mean, which exists by our assumption that G is amenable. Recall that $\check{y}(t) = y(t^{-1})$. We note that for any $t \in G$

$$f(s) = f(tt^{-1}s) = \langle x(t), y(t^{-1}s)\rangle = \langle x(t), \check{y}(s^{-1}t)\rangle,$$

and $t \mapsto \langle x(t), \check{y}(s^{-1}t)\rangle$ is bounded. In particular we get that

$$f(s) = m(f(s)1(t)) = m(\langle x(t), \check{y}(s^{-1}t)\rangle).$$

We observe that $\langle x, y\rangle' := m(\langle x(t), y(t)\rangle)$ is a sesquilinear form on $l^\infty(G, H)$. Indeed, it is clearly linear in the first component and conjugate linear in the second component. Let $N = \{x : \langle x, x\rangle' = 0\}$. This is a subspace as Cauchy-Schwarz gives that

$$N = \{x : \langle x, x\rangle = 0\} = \{x : \langle x, y\rangle = 0 \text{ for all } y\}.$$

Let $K' = l^\infty(G, H)/N$. We define the inner product $\langle x+N, y+N\rangle_K = \langle x, y\rangle'$. Let K be the completion of K'. Define $\Lambda(s)(x(t) + N) = x(s^{-1}t) + N$. This is well defined as the left invariance of m gives that

$$m(\langle x(s^{-1}t), x(s^{-1}t)\rangle) = m(\lambda(s)\langle x(t), x(t)\rangle) = m(\langle x(t), x(t)\rangle),$$

giving that $x(s^{-1}\cdot) \in N$ for all $s \in G$ and all $x \in N$. We get that $\Lambda(s)$ is unitary with respect to the new inner product as $\Lambda(s)$ is an invertible isometry by the previous computation. We get that $\langle x, \Lambda(s)y\rangle_K = f(s)$, completing the proof that the first point implies the second point.

We now want to prove that the second point implies the third point. We get that the identity map $I' : (B(G), \|\cdot\|_{B(G)}) \to (B(G), \|\cdot\|_{M_2(G)})$ is bounded as

$$\begin{aligned} \|f\|_{B(G)} &= \inf\{\|x\|\|y\| : f(t) = \langle \pi(t)x, y\rangle\} \\ &= \inf\{\|\pi(t)x\|\|\pi(s^{-1})y\| : f(st) = \langle \pi(st)x, y\rangle\} \\ &= \inf\{\|\pi(t)x\|\|\pi(s^{-1})y\| : f(st) = \langle \pi(t)x, \pi(s^{-1})y\rangle\} \geq \|f\|_{M_2(G)}. \end{aligned} \tag{41}$$

This completes the proof, as the inverse of I' is I, giving that I is bounded by the bounded inverse theorem. □

We will need the following lemma ([P3, Lemma 2.5]), which we will leave without a proof.

Lemma 7.4. *For any* $d \geq 1$ *and any* $c \geq 2$, *we have that*

$$\|f\|_{B_c(G)} \leq 2\|f\|_{M_d(G)} + 2c^{-(d+1)}\|f\|_{B(G)}.$$

We can now complete the proof of Theorem 7.2.

Remaining part of proof. We assume that there exist K and $\alpha < 3$ such that $\|f\|_{B(G)} \leq Kc^{\alpha}\|f\|_{B_c(G)}$ for all $c \geq 1$. We use the previous lemma for $d = 2$. For every $c \geq 2$, we get that

$$\|f\|_{B(G)} \leq Kc^{\alpha}\|f\|_{B_c(G)} \leq 2Kc^{\alpha}\|f\|_{M_2(G)} + 2Kc^{\alpha-3}\|f\|_{B(G)}.$$

As $\alpha < 3$, $2Kc^{\alpha-3} \to 0$, $c \to \infty$, and we can pick c such that $2Kc^{\alpha-3} = \varepsilon$, $0 < \varepsilon < 2^{\alpha-2}K$. Theorem 7.3 gives us that G is amenable if there exists K such that for all $f \in B(G)$, we have the inequality $\|f\|_{B(G)} \leq K\|f\|_{M_2(G)}$. We immediately get such an estimate and this completes the proof. □

7.2 Strong unitarizability and nuclearity

Another notion that is equivalent to amenability is *strong unitarizablility*, which means that the similarity S may be picked such that it commutes with every unitary operator that commutes with the range of π. This equivalence is proven in [P5]. Recall that a *von Neumann algebra* is a (unital) C^*-subalgebra of $B(H)$ that is closed with respect to the weak operator topology. We will now follow [BO, Ch. 2] to make some preparations. We will give some definitions

Definition 7.5. A map $\theta : A \to B$ is said to be nuclear if there exists nets of contractive completely positive maps $\phi_n : A \to M_{k(n)}$ and $\psi_n : M_{k(n)} \to B$ such that

$$\|\psi_n \circ \phi_n(a) - \theta(a)\| \to 0$$

for all $a \in A$. A C^*-algebra is said to be nuclear if the identity map is nuclear.

A natural problem is to determine whether or not particular classes of C^*-algebras are nuclear. Any finite dimensional C^*-algebra A is nuclear as it is a norm closed subalgebra of $B(\mathbb{C}^k) \cong M_k$. We may chose ϕ_n as the inclusion map, and we may construct ψ_n by

$$\psi_n(e_{p,q}) = \begin{cases} e_{p,q} \text{ if } e_{p,q} \in A \\ 0 \text{ otherwise} \end{cases} .$$

for all n. Another important example is the class of commutative C^*-algebras. By Gelfand's theorem, it suffices to show the following.

Proposition 7.6. *Let X be a compact Hausdorf space. Then $C(X)$ is nuclear.*

Proof. We will follow the proof from [BO, Prop. 2.4.2]. Fix finite subset $M \subset C(X)$ and $\varepsilon > 0$. By uniform continuity, there is a finite open cover $\{U_i\}_{i=1}^n$ such that we for any $1 \leq i \leq n$ have that $\|f(x) - f(y)\| < \varepsilon$ for all $f \in M$ and all $x, y \in U_i$. We note that the set F of finite subsets of $C(X)$ is a directed set with respect to inclusion. Let $\{\sigma_i\}_{i=1}^n \subset C(X, [0, 1])$ be a partition of unity subordinate to the open cover, meaning that for every i, we have $supp(\sigma_i) \subseteq U_i$ and $\sum \sigma_i = 1$. Pick $y_i \in U_i$ arbitrarily. We define $\phi_M : C(X) \to \mathbb{C}^n$ by $f \mapsto (f(y_1), ..., f(y_n))$. This map is unital and even a *-homomorphism. It is completely positive as it is a positive map with it's domain being a commutative C^*-algebra (see [P2, Thm. 3.11]). We define $\psi_M : \mathbb{C}^n \to C(X)$ by

$$(d_1, ..., d_n) \mapsto \sum_{i=1}^{n} d_i\sigma_i.$$

This is a completely positive map by the same reason. It is unital as $\sum_{i=1}^{n} \sigma_i = 1$. We get that

$$\begin{aligned} \|f - \psi_M \circ \phi_M(f)\| &= \|\left(\sum_{i=1}^{n} \sigma_i\right) f - \sum_{i=1}^{n} (f(y_i)1)\sigma_i\| \\ &= \|\sum_{i=1}^{n} (f - f(y_i))\sigma_i\| \\ &\leq \varepsilon \sum_{i=1}^{n} \sigma_i = \varepsilon. \end{aligned} \tag{42}$$

We get that the nets $\{\phi\}_{M\in F}$ and $\{\psi\}_{M\in F}$ are such that

$$\|f - \phi_M \circ \psi_M(f)\| \to 0$$

for all $f \in C(X)$. □

We will now state a theorem that connects amenability of groups with nuclearity of group C^*-algebras.

Theorem 7.7. *Let G be a discrete group. Then G is amenable if and only if $C^*_\lambda(G)$ is nuclear.*

Before we prove this theorem, we will define $L(G)$ as the closure of $C^*_\lambda(G) \subseteq B(l^2(G))$ with respect to the weak operator topology.

Proof. We will follow the relevant parts of the proof from [BO, Thm. 2.6.8]. If G is amenable, we get that it admits Følner sets. We will look at the separable case. Let $\{E_i\}$ be a sequence of Følner sets, in the inseparable case, we would consider a net instead. Let P_k be the orthogonal projection of $l^2(G)$ on $span\{\delta_g : g \in E_k\}$. It is clear that $P_k B(l^2(G)) P_k$ can be identified with M_{E_k} of complex matrices indexed by E_k. We use $e_{p,q}$ to denote the matrix with 1 at index (p,q) and 0 everywhere else. We observe that

$$e_{p,p}\lambda_s e_{q,q} = \begin{cases} e_{p,q}, & \text{if } sq = p \\ 0 & \text{otherwise} \end{cases}.$$

We get that $P_k = \sum_{E_k} e_{p,p}$ giving that

$$P_k \lambda_s P_k = \sum_{p,q\in E_k} e_{p,p}\lambda_s e_{q,q} = \sum_{p\in E_k \cap sE_k} e_{p,s^{-1}q}.$$

We now define the map $\phi_k : C^*_\lambda(G) \to M_{E_k}$ by $x \mapsto P_k x P_k$. We observe that it is completely positive as if $P_k^{(i)}$ is the direct sum of i copies of P_k we get that $\phi_k^{(i)}(X) = P_k^{(i)} X P_k^{(i)} = P_k^{(i)} X P_k^{(i)^*}$, $X \in M_i(C^*_\lambda(G))$. We define $\psi_k : M_{E_k} \to C^*_\lambda(G)$ by

$$e_{p,q} \mapsto \frac{1}{|E_k|}\lambda_p \lambda_{q^{-1}}.$$

We now want to prove that this is a completely positive map. By Choi's theorem ([P2, Thm. 3.14]) it suffices to show that $(\psi_k(e_{p,q}))_{p,q}$ is a positive matrix. We get that

$$\frac{1}{|E_k|}(\lambda_p \lambda_q^*)_{p.q} = \frac{1}{|E_k|} \begin{bmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_p \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_p \end{bmatrix}^* \geq 0.$$

We get that $\{\lambda_s\}_{s\in G}$ spans a dense subspace of $C^*_\lambda(G)$, giving that it suffices to show that $\|\lambda_s - \psi_k \circ \phi_k(\lambda_s)\| \to 0$ for all $s \in G$. We get that

$$\begin{aligned}\psi_k \circ \phi_k(\lambda_s) &= \psi_k\bigg(\sum_{p\in E_k\cap sE_k} e_{p,s^{-1}p}\bigg)\\ &= \sum_{p\in E_k\cap sE_k} \frac{1}{|E_k|}\lambda_s \\ &= \frac{|E_k\cap sE_k|}{|E_k|}\lambda_s.\end{aligned} \tag{43}$$

We now get

$$\begin{aligned}\|\lambda_s - \psi_k\circ\phi_k(\lambda_s)\| &= \|\bigg(1-\frac{|E_k\cap sE_k|}{|E_k|}\bigg)\lambda_s\| \\ &= \|\bigg(\frac{|E_k\setminus(E_k\cap sE_k)|}{|E_k|}\bigg)\lambda_s\| \\ &\le \|\frac{|E_k\Delta sE_k)|}{|E_k|}\lambda_s\| \to 0,\end{aligned} \tag{44}$$

completing the proof of the only if direction.

We now want to prove that G is amenable if $C^*_\lambda(G)$ is nuclear. Let $\phi_n : C^*_\lambda(G) \to M_{k(n)}$ and $\psi_n : M_{k(n)} \to C^*_\lambda(G)$ be unital completely positive maps such that $\psi_n \circ \phi_n$ converges pointwise to the identity map. By Arveson's extension theorem, ϕ_n can be extended to a unital completely positive map $\varphi_n : B(l^2(G)) \to M_{k(n)}$. Let $\Phi_n := \psi_n \circ \varphi_n$. We get $\Phi_n(x) \to x$ for all $x \in C^*_\lambda(G)$. By picking a cluster point in the point-ultraweak topology, we get a map $\Phi : B(l^2(G)) \to L(G)$ such that $\Phi|_{C^*_\lambda(G)} = id_{C^*_\lambda(G)}$. We define $\tau(x) := \langle x\delta_e, \delta_e\rangle$. We note that this is a state and that $\tau(xy) = \tau(yx)$ as

$$\begin{aligned}\tau(\lambda(s)\lambda(r)) &= \langle \lambda(s)\lambda(r)\delta_e, \delta_e\rangle \\ &= \begin{cases}1 \text{ if } sr = e\\ 0 \text{ otherwise}\end{cases} \\ &= \langle \lambda(r)\lambda(s)\delta_e, \delta_e\rangle = \tau(\lambda(r)\lambda(s)).\end{aligned} \tag{45}$$

We define $m(f) := \tau(\Phi(f_i))$. The space $l^\infty(G)$ can clearly be embedded in $B(l^2(G))$ by $f \mapsto M_f$, where M_f is multiplication by f. By restricting this to $l^\infty(G)$, we get a mean. We get that

$$M_{\lambda(s)f}g = f(s^{-1}t)g(t) = f(s^{-1}t)g(ss^{-1}t) = (\lambda(s)M_f\lambda(s)^*)g.$$

One has

$$\Phi(\lambda(s)\lambda(s)^*) = \Phi(\lambda(s))\Phi(\lambda(s))^*$$

and

$$\Phi(\lambda(s)^*\lambda(s)) = \Phi(\lambda(s))^*\Phi(\lambda(s))$$

as Φ is the identity on $C^*_\lambda(G)$. By the same reasoning, we get that

$$\Phi(\lambda(s)^*(\lambda(s)^*)^*) = \Phi(\lambda(s)^*)\phi(\lambda(s^)*)^*$$

and

$$\Phi((\lambda(s)^*)^*\lambda(s)^*) = \Phi(\lambda(s)^*)^*\Phi(\lambda(s)^*).$$

We now use the second point of Proposition 6.6 to obtain that

$$\begin{aligned}\Phi(\lambda(s)M_f\lambda(s)^*) &= \Phi(\lambda(s))\Phi(M_f\lambda(s)^*)\\ &= \Phi(\lambda(s))\phi(M_f)\Phi(\lambda(s)^*) = \lambda(s)\Phi(M_f)\lambda(s)^*.\end{aligned} \tag{46}$$

We get that

$$m(\lambda(s)f) = \tau(\Phi(\lambda(s)M_f\lambda(s)^*)) = \tau(\lambda(s)\Phi(M_f)\lambda(s)^*) = \tau(\Phi(M_f)) = m(f).$$

This proves that m is left invariant, and completes the proof that G is amenable.

□

In particular, this result gives that $C^*_\lambda(F_2)$ is not nuclear. We can now state the following result that is [P5, Thm. 1].

Theorem 7.8. *A C^*-algebra A is nuclear if and only if any completely bounded homomorphism π is similar to a *-homomorphism and the similarity may be picked from the von Neumann algebra generated by $\pi(A)$.*

Corrolary 7.9. *Let G be a discrete group, then G is amenable if it is strongly unitarizable.*

Proof. We will follow the proof from [P5, Corr. 3]. Let $u : C^*_\lambda(G) \to B(H)$ be a bounded homomorphism. Then $\pi = u|_G$ is uniformly bounded. We get that $u(C^*_\lambda(G))$ and $\pi(G)$ generate the same von Neumann algebra as they generate the same C^*-algebra. As G is strongly unitarizable, we may pick the similarity S from the von Neumann algebra generated by $\pi(G)$. As S gives the similarity of u to a *-homomorphism, the previous theorem gives that $C^*_\lambda(G)$ is nuclear and hence G is amenable by Theorem 7.7. □

8 Conclusions and a related problem

In this thesis, we have presented some theory about two different similarity problems. We have proven the Dixmier-Day theorem that amenability implies unitarizablility, and we have given some statements that are equivalent to amenability.

For C^*-algebras, we have proven some results about the unitarizability of algebra homomorphisms and some C^*-algebras. There is, however, a related problem that was not covered. Recall that a derivation is a linear map δ that follows the product rule $\delta(ab) = \delta(a)b + a\delta(b)$. A related problem to Kadison's problem asks whether any derivation $A \to B(H)$, where A is a C^*-subalgebra of $B(H)$, is *inner*, meaning that there exists $T \in B(H)$ such that $\delta(a) = [a, T]$. This is called the *derivation problem* [P4, p.3]. In fact, the Kadison problem and the derivation problem are equivalent (see [P4, Thm. 10.1]).

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