UNIVERSITY OF GOTHENBURG

Low-lying zeroes of L-functions attached to modular forms

Master's thesis in Mathematics

Alf Söderberg

Department of Mathematical Sciences
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Supervisor: Anders Södergren, Department of Mathematical Sciences
Examiner: Christian Johansson, Department of Mathematical Sciences

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Department of Mathematical Sciences
University of Gothenburg
SE-412 96 Gothenburg
Telephone +46 31 772 1000

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Alf Söderberg
Department of Mathematical Sciences
University of Gothenburg

Abstract

We study the family of L-functions attached to Hecke newforms of weight k and level N and their low-lying zeroes. First, we recall the Density Conjecture of Katz and Sarnak and how it predicts the behaviour of the low-lying zeroes of any natural family of L-functions. Then, we review some basic theory of modular forms as an appropriate background to the subsequent investigations. Next, we follow the article [ILS00] by Iwaniec, Luo and Sarnak in their treatment of the 1-level density of our family at hand. From them we recover that the Density Conjecture holds for bounded support of $\widehat{\phi}$ when $kN \to \infty$ and N is squarefree, conditional on the Generalized Riemann Hypothesis. Also, following Miller [Mil09] we find a term of lower order when k is fixed and $N \to \infty$ through the primes. Lastly, we study the 1-level density through the Ratios Conjecture. The prediction of the Ratios Conjecture allows any compact support of $\widehat{\phi}$, as well as agreeing with the explicit calculations down to a power-saving error term.

Keywords: Number theory, L-functions, Modular forms, Newforms, Low-lying zeros, 1-level density, Density Conjecture, Ratios Conjecture.

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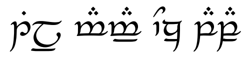
This report will most likely be read in its entirety by exactly three people. I offer my sincere apologies to all of you.

Alf Söderberg, Gothenburg, 2024-03-18



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Introduction

In this chapter, we outline the subject landscape which the report belongs to. Starting with standard topics from number theory, we recall the Riemann zeta function and examples of more general L-functions, their properties and applications. We also introduce the idea of *families* of L-functions, i.e. certain collections of L-functions with similar characteristics. Since our report will have a special focus on L-functions attached to *modular forms*, we devote a section of this chapter to them as well.

1.1 The Riemann zeta function and L-functions

A central topic in number theory, both today and historically, is the study of prime numbers. Being the multiplicative building blocks of the positive integers, much effort has been spent on investigating how prime numbers behave. A basic observation is that, as we count upwards from 1, roughly every other positive integer will be composite, since 2 is a prime number. Similarly, every third positive integer will be composite since 3 is prime, and so on. Hence one might suspect that the probability that a positive integer N is prime becomes smaller as N grows larger. Is it possible to quantify this behaviour more precisely? To a certain extent this question is answered by the prime number theorem (PNT), which was first shown independently by Hadamard and de la Valle Poussin in 1896. We define the *prime counting function* $\pi(x)$ as the number of primes less than or equal to x, i.e.

$$\pi(x) := \#\{p : p \leq x\}.$$

Then, the PNT states that [Dav00, Ch. 18]

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log x}}\right), \quad \text{where} \quad \mathrm{Li}(x) := \int_2^x \frac{1}{\log t} dt \tag{1.1}$$

and c is some positive constant.

Most proofs of the PNT involve a fair amount of complex analysis. In particular, the ones by Hadamard and de la Valle Poussin involves the study of the *Riemann zeta function*, defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{1.2}$$

This is our first example of an L-function, and it was studied already by Euler when $s > 1$ is real. He computed $\zeta(2) = \pi^2/6$ and more generally the numbers $\zeta(2k)$ for $k = 1, 2, ...$ in terms of Bernoulli numbers. He also showed that one may write $\zeta(s)$ as a product over primes.

Additional fundamental facts about $\zeta(s)$ were discovered by Riemann and published in his famous memoir [Rie59]. It was he who first regarded $\zeta(s)$ as a function of a complex variable as indicated in (1.2). The argument of Euler extends to this situation and shows that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1.$$

Riemann showed that $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$, which is analytic everywhere except at $s = 1$, where there is a simple pole with residue 1. He also found a functional equation relating $\zeta(s)$ to $\zeta(1 - s)$. To state it, we define the *Riemann ξ-function* by

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Riemann showed that this is entire and satisfies

$$\xi(s) = \xi(1 - s).$$

From this one may deduce that ξ has only zeroes in the *critical strip* $\{s \in \mathbb{C} : 0 \leq \Re(s) \leq 1\}$. These are also the zeroes of ζ, except for the so-called *trivial zeroes* at the negative even integers, which arise due to $\Gamma(s/2)$ having poles there. The Riemann Hypothesis (RH) states that all nontrivial zeroes of $\zeta(s)$ lies on the *critical line* $\{s \in \mathbb{C} : \Re(s) = 1/2\}$. To this day, the RH remains unproven. Numerical investigations, on the other hand, has yet to find a single counterexample to the RH among the (very many) known nontrivial zeroes.

Let us return to the PNT. A key step in the proofs by Hadamard and de la Valle Poussin is the fact that $\zeta(1 + it) \neq 0$ for $t \neq 0$. As a rule of thumb, the larger the region for which we can establish that $\zeta(s)$ do not vanish, the smaller the error term in the prime number theorem becomes. By the functional equation and since $\zeta(\overline{s}) = \overline{\zeta(s)}$, the nontrivial zeroes of $\zeta(s)$ are located at mirroring positions through the critical line. That is, if $1/2 + a + it$ is a zero of $\zeta(s)$ where a and t are real numbers, then $1/2 - a + it$ also is. Therefore the RH asserts the best possible zero-free region of $\zeta(s)$, and consequently the best possible error term in the PNT, in a sense. In 1901, von Koch proved that this error term would be of size $O(x^{1/2} \log x)$ [Koc01], which is a significant improvement of (1.1).

The Riemann zeta function is an example of an L-function. L-functions is a large and checkered multitude of functions arising in various situations, and which are central to many topics in modern number theory. We consider some other examples in order to illustrate this. As a second one, let $\chi(n)$ be a Dirichlet character modulo N. That is, $\chi : \mathbb{Z} \to \mathbb{C}$ is a periodic, completely multiplicative function which satisfies $\chi(n) = 0$ if and only if $(n, N) > 1$. Then the *Dirichlet L-function* $L(s, \chi)$ is defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1.$$

The only Dirichlet character of modulus $N = 1$ is the function which is constant and equal to 1 for all n. By letting $N = 1$, we thus recover $\zeta(s)$ as a particular case of a Dirichlet L-function. Since $\chi(n)$ is completely multiplicative, we may show that $L(s, \chi)$ has an Euler product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \Re(s) > 1.$$

in the same way as for $\zeta(s)$. Other properties of $\zeta(s)$ that translates to the setting of Dirichlet L-functions is the meromorphic continuation of $L(s, \chi)$ to $\mathbb{C}$ (which may or may not be entire), and functional equations. Also in this context the location of zeroes do matter. Indeed, one may show that for certain Dirichlet characters χ the L-function $L(s, \chi)$ do not vanish at $s = 1$. This can be used to show the existence of infinitely many primes in arithmetic progressions. In [Dav00] Davenport provides an excellent exposition of the subject.

As a third and final example, we may attach L-functions to number fields, that is, field extensions of $\mathbb{Q}$ of finite degree. If K is a number field, then the *Dedekind zeta function* of K is defined as

$$\zeta_K(s) := \sum_{\mathfrak{a} \subseteq O_K}^{\infty} \frac{1}{N(\mathfrak{a})^s}, \quad \Re(s) > 1.$$

Here, the summation ranges over all nonzero ideals $\mathfrak{a}$ of the ring of integers O_K of K, and $N(\mathfrak{a})$ is the ideal norm defined by $N(\mathfrak{a}) := |O_K/\mathfrak{a}|$. When $K = \mathbb{Q}$, we have $O_{\mathbb{Q}} = \mathbb{Z}$ and so any nonzero ideal is of the form $n\mathbb{Z}$ where n is a positive integer. Its norm is given by $|\mathbb{Z}/n\mathbb{Z}| = n$, so by letting $K = \mathbb{Q}$, we recover $\zeta_{\mathbb{Q}}(s) = \zeta(s)$. Following the unique factorization of ideals and complete multiplicativity of the norm N, we have the Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}, \quad \Re(s) > 1$$

where the product ranges over all non-zero prime ideals $\mathfrak{p}$ of O_K. The meromorphic continuation to $\mathbb{C}$ except the simple pole at $s = 1$ and the functional equation of $\zeta_K(s)$ was established by Hecke in 1917 (see also [Bom10, p. 30]). The general theory of Dedekind zeta functions and related subjects may be found in [Neu99].

We have now seen three different examples of how L-functions may arise. There are many more still. Their diversity begs the question whether there is any clear-cut definition of what an L-function is. An attempt to axiomatize the theory of L-functions was made by Selberg, when he introduced the class of L-functions bearing his name (see [Sel91]). The most important properties of such L-functions are summarized in [Per05], including having a Dirichlet series representation for $\Re(s) >$

1, an analytic continuation, an Euler product representation for $\Re(s) > 1$ and a functional equation relating the value at s to the value at $1-s$. As we have seen, all of the above examples have these properties. General conjectures of Langlands assert that all L-functions are finite products of so-called *principal* L-functions attached to cuspidial automorphic representations; these conjectures are very far reaching and far from proven in many cases. We will not delve into this, but we mention that the *Langlands program* is the the subject of much research.

1.2 Families of L-functions and random matrices

We have seen that L-functions may arise in many different situations. If a collection of L-functions arise from the same situation (such as being attached to Dirichlet characters or number fields) it may make sense to consider them as members of the same *family*, which we denote by $\mathcal{F}$. The underlying idea is that the average behaviour of the L-functions may reveal something intrinsic to the family at hand. There is no exhaustive account of families of L-functions, nor a universally accepted definition of what a family of L-functions is. However, numerous concrete examples have been found and investigated. One of the most well understood families is the family of Dirichlet L-functions attached to real primitive Dirichlet characters. It has been studied in [Rub01; Mil08; Gao05; FPS17] and [FPS18], for instance. Families of L-functions attached to field extensions and elliptic curves have been studied in [You06; FPS16] and [SST19], for instance.

Much interest lies in understanding the zeroes of various kinds of L-functions. Again, even though individual L-functions may be interesting in and of themselves, it is often hard to understand their zeroes. Thus a common approach is to consider a whole family $\mathcal{F}$ and study how the zeroes behave on average. Even though many of the fundamental problems such as the RH and its generalisation remain unanswered, we are not entirely left in the dark. A method that has turned out fruitful is to model the zeroes of L-functions after the behaviour of random matrices, conditional on the GRH for the relevant L-functions. The eigenvalues of random $N \times N$ matrices follow certain distributions as N tend to infinity, and these turn out to be the same as the distributions of the zeroes of L-functions as the size of the family $\mathcal{F}$ tend to infinity.

In this report, we study a particular family of L-functions and try to show that it fits into the larger framework outlined above. The family at hand is the family of L-functions $L(s, f)$, where f is a modular cusp forms of weight k and level N. All these concepts will be introduced in due time. The weight and level govern the size of the family in this situation. The study of this family was initialized in [ILS00], and we will review some of the classical results from there. Serving as a starting point, the subject has since been expanded and many results refined. In particular, we will show how to obtain lower order terms of the 1-level density as outlined in [Mil09], and how to refine the results even further by means of the Ratios Conjecture.

1.3 Modular forms

In this report we are interested in L-functions attached to a modular cusp form f. We are going to develop the necessary terminology and tools to define and study them below, but for now it suffices to think of a modular form as a holomorphic function defined on the upper half plane which satisfies some additional properties. A modular form f has a Fourier series expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n z},$$

where the Fourier coefficients with negative indices vanish. Two important quantities associated with a modular form f are its *weight* and its *level*. These are denoted by k and N, respectively, and will be introduced below. The set of modular forms with fixed weight and level turn out to be a finite dimensional vector space over $\mathbb{C}$.

A relatively concrete class of modular forms are called *Eisenstein series*. Eisenstein series may be defined as certain series over pairs of integers where not both integers are 0. Even though they play no significant role in this report, we briefly review Eisenstein series for the sake of illustration and completeness in the next chapter. After suitable renormalization, these Eisenstein series have Fourier coefficients equal to known arithmetic functions, such as the sum of divisors function $\sigma(n)$.

Let us for simplicity fix the level to be equal to 1. The modular forms for which $a_f(0) = 0$ in the Fourier expansion occupy a special position. Such forms are called cusp forms, and in the space of modular forms of fixed weight they constitute a subspace. The condition on $a_f(0)$ implies that they decay rapidly as $\Im(z)$ tends to infinity. This has many consequences, one of which is that we may endow the space of cusp forms with fixed weight with an inner product. This allows us to use results from linear algebra when we analyze the structure of this space. The notion of cusp forms can be generalized to higher levels with the same consequences, although the definition is slightly more technical.

Given a vector space of modular forms with fixed weight and level, we can decompose it into the two subspaces of Eisenstein series and cusp forms. A natural thing to do is to try to find a basis for the space, and this decomposes into the parallel tasks of finding bases for the spaces of Eisenstein series and cusp forms, respectively. Doing so for the space of Eisenstein series is relatively straightforward, while the situation is more involved in the case of cusp forms.

The task of finding a basis for the space of cusp forms is an important motivation behind the introduction of so-called *Hecke operators* and *newforms*. A Hecke operator $T_N(n)$ is a linear operator on the space of modular forms of fixed weight k and level N, indexed by the positive integers. Hecke operators preserve the subspace of cusp forms. A newform can be described roughly as a cusp form of level N which does not arise from a cusp form of lower level $M|N$, and thus can be seen as "new" in a sense.

It turns out that newforms are eigenvectors of all the Hecke operators $T_N(n)$. The eigenvalues turn out to be multiplicative, considered as an arithmetic function of the index n. The eigenvalues encode arithmetic information about f, and are used to define an attached L-function $L(s, f)$. This L-function satisfy the standard properties of having a Dirichlet series representation, an Euler product, an analytic continuation and a functional equation.

1.4 Outline of the report

The main objective of this report is to study the low-lying zeroes of L-functions attached to modular forms, more precisely newforms. In Chapter 2 we outline the background to this objective. In particular, we introduce the Katz-Sarnak heuristic of modelling the distributions of zeroes of L-functions after eigenvalues of random matrices. The main tool to study the low-lying zeroes is the 1-level density, which is defined in Section 2.2. The *Density Conjecture* predicts the asymptotical behaviour of the 1-level density, and all the main results of the report are related to verifying this conjecture under certain conditions. We also introduce the *Ratios Conjecture*, another tool by which the 1-level density may be studied.

In Chapter 3 we review some basic theory about modular forms. Our aim here is to introduce all the necessary concepts and results in preparation of the rest of the report. In Chapter 4 we perform many of the technical computations necessary to study the 1-level density, the focus being sums of Hecke eigenvalues of newforms. The computations of the 1-level density is performed in Chapter 5, where we first deduce the main term of the Katz-Sarnak heuristic for bounded support. We then extend the result down to a power-saving error term. In Chapter 6 we study the 1-level density through the Ratios Conjecture, and verify that it correctly predicts the shape of the 1-level density down to an error term of power-saving size.

1.5 Notation

Here we gather some notation (which may or may not be familiar to the reader).

For any $z \in \mathbb{C}$, we denote

$$\exp(z) := e^z \quad \text{and} \quad e(z) := e^{2\pi i z}.$$

This notation will mostly be used when the exponent is too cumbersome for the right hand sides to be readable.

The Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

The greatest common divisor and least common multiple of the integers $m_1, m_2, ..., m_k$ are denoted by

$$(m_1, m_2, ..., m_k) \quad \text{and} \quad [m_1, m_2, ..., m_k],$$

respectively. We shall only be interested in the case when $k = 2$ or 3.

The Möbius function $\mu(n)$ is the multiplicative function defined on prime powers as

$$\mu(p^m) := \begin{cases} 1, & m = 0, \\ -1, & m = 1, \\ 0, & m \geq 2. \end{cases}$$

The Euler φ-function $\varphi(n)$ is the arithmetic function defined by

$$\varphi(n) := \#\{m \in \mathbb{Z} : 1 \leq m \leq n, (m,n) = 1\}.$$

The Euler φ-function is multiplicative, and satisfies $\varphi(p^m) = p^m - p^{m-1}$ for prime powers.

The sum of divisors function is the arithmetic function

$$\sigma(n) := \sum_{\substack{d|n \\ d>0}} d.$$

This is the particular case $k = 1$ of the more general

$$\sigma_k(n) := \sum_{\substack{d|n \\ d>0}} d^k.$$

The case $k = 0$ is the number of divisors function, and is denoted by $\tau(n)$[1]. Another generalisation of $\tau(n)$ is $\tau_k(n)$, which is the number of ways of writing n as a product of k positive integer factors. In this language, $\tau(n) = \tau_2(n)$; apart from this, only $\tau_3(n)$ will be relevant for us.

The first Chebyshev function is

$$\theta(x) := \sum_{p \leq x} \log p.$$

We define the integral along a vertical line $\{s \in \mathbb{C} : \Re(s) = c\}$ by

$$\int_{(c)} f(s)ds := \int_{c-i\infty}^{c+i\infty} f(s)ds = \lim_{t\to\infty} \int_{c-it}^{c+it} f(s)ds.$$

Of course, we only use the notation provided that the integral exists.

[1]Not to be confused with the *Ramanujan τ-function*, which is a prominent function in the subject of modular forms, but not relevant in this report.

2

Preliminaries

In this chapter, we outline the framework in which the subject of the report should be understood. We start by recalling some random matrix theory and how it is employed to understand the theory of (families of) L-functions. Next, we introduce the main object of study, namely the 1-level density. The 1-level density views the low-lying zeroes of a family of L-functions through the lens of an even Schwartz test function. The expected asymptotical behaviour of the 1-level density is the content of the *Density Conjecture*, due to Katz and Sarnak [KS99b] and refined by Sarnak, Shin and Templier in [SST16]. Finally, we introduce the *Ratios Conjecture*, which is a tool for studying the 1-level density from another perspective.

2.1 Random matrix theory

We recall some of the theory of random matrices necessary to understand and predict the behaviour of (low-lying) zeroes of families of L-functions. The reviewed material follows the presentation in [KS99b] and [Con+05]. A thorough exposition of the subject, its origins and applications can be found in [Meh04].

An $N \times N$ matrix A is said to be *unitary* if $AA^* = I$, where A^* is the conjugate transpose of A. The set of unitary $N \times N$ matrices is denoted by $U(N)$. A unitary matrix A has n eigenvalues (counted with multiplicity) which all have absolute value 1. We write them as $e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_N}$, where we order the arguments $0 \leq \theta_1 \leq \theta_2 \leq ... \leq \theta_N < 2\pi$. Somewhat abusively, by the term *eigenvalue* we sometimes mean the argument θ of the eigenvalue $e^{i\theta}$. The exact meaning in each particular situation will hopefully be clear from the context.

The characteristic polynomial of a matrix $A \in U(N)$ is

$$P_A(s) := \det(I - sA^*) = \prod_{j=1}^{N}(1 - se^{-i\theta_j}),$$

and it has many properties mirroring those of L-functions. These have been elaborated upon in e.g. [Con+05, Section 1.2]. The following is a sample from there. Denoting

$$\det A = \prod_{j=1}^{N} e^{i\theta_j} = e^{i\theta}$$

for $\theta = \sum_j \theta_j$ a real number, we may write

$$P_A(s) = (-1)^N e^{-i\theta} s^N P_{A^*}(1/s).$$

This relates the value of P_A at s to the value of P_{A^*} at $1/s$. The point $s = 1$ is fixed under the change of variables $s \mapsto 1/s$, suggesting a special importance.

The motivation behind introducing random matrices in this report lies in the fact that their eigenvalues may be used to model the zeroes of L-functions belonging to various families. The eigenvalue distributions may depend on certain properties of the matrices, and it therefore makes sense to identify certain subsets of $U(N)$ defined by these properties. The subsets of interest to us are the *orthogonal* (with the subdivision according to parity) and *symplectic* matrices, defined in table Table 2.1. These subsets are associated with their respective so-called *symmetry types*. When studying families of L-functions, we assign to them a symmetry type depending on what distributions their zeroes obey. It is believed that the zeroes of any natural family of L-functions can be modeled after one of the symmetry types of table Table 2.1 in this way.

Symmetry type G	Random matrix realization $G(N)$
U	The group $U(N)$ of unitary $N \times N$ matrices A, satisfying $AA^* = I$.
O	The group $O(N)$ of orthogonal $N \times N$ matrices, that is, real matrices satisfying $A^T A = I$.
SO(even)	The group $SO(2N)$ of orthogonal $2N \times 2N$ matrices with determinant equal to 1.
SO(odd)	The group $SO(2N+1)$ of orthogonal $(2N+1) \times (2N+1)$ matrices with determinant equal to 1.
Sp	The group $USp(2N)$ of symplectic unitary $2N \times 2N$ matrices, that is unitary matrices satisfying $A^t J A = J$ where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$.

Table 2.1: Symmetry types and their realizations as sets of random matrices.

The distributions associated with each symmetry type G are revealed when investigating how the eigenvalues of a typical matrix $A \in G(N)$ becomes distributed as N tend to infinity. There are many different aspects one can investigate, some of which do not depend on the specific symmetry type G and some that do. As an example of the former, one may consider spacings between appropriately re-scaled eigenvalues. These were investigated by Katz and Sarnak in [KS99a] and were found to follow the so-called *GUE distributions* (see [KS99b, pp. 7-8]). As the analysis does not give any priority to eigenvalues depending on their distance from the central point $s = 1$, we may speak of *high-lying* eigenvalues. In short, the high-lying eigenvalues have a well-understood behaviour which is independent of the symmetry type.

Another aspect is how eigenvalues close to the point $s = 1$ behaves. Contrary to the above, this will indeed depend on the specific symmetry type. These eigenvalues are referred to as *low-lying*, since $s = 1$ is the central point of the functional equation. For this we can study the distribution of the eigenvalue (whose argument is) nearest, second nearest etc. to 1. Again these quantities were investigated in [KS99a], and the results summarized in [KS99b]. Precisely, one may count the number of scaled eigenvalues

$$\Delta(A)[a,b] := \#\left\{\theta : e^{i\theta} \text{ is an eigenvalue of } A \text{ and } \frac{\theta N}{2\pi} \in [a,b]\right\}$$

contained in intervals $[a, b]$. Since the average spacing between the eigenvalues θ is $2\pi/N$, the re-scaled eigenvalues have average spacing equal to 1. Integrating $\Delta(A)$ over $G(N)$, we obtain the average

$$W(G(N)) := \int_{G(N)} \Delta(A)dA.$$

Here, dA denotes a volume measure, also known as a *Haar measure*, which is invariant under conjugation by unitary, orthogonal or symplectic matrices, respectively (see [Meh04, Sections 2.3-2.5]). The interpretation of $W(G(N))$ is that it looks at the frequency of a typical eigenvalue occurring in a given interval. In other words, this is the random matrix analogue to the 1-level density to be introduced in the next section. The asymptotical behaviour of $W(G(N))$ as N tend to infinity will thus constitute our basic intuition about the corresponding behaviour of families of L-functions. In [KS99a], Katz and Sarnak showed the existence of distributions $w(G)$ such that

$$\lim_{N\to\infty} W(G(N))[a,b] = \int_a^b w(G)dx, \tag{2.1}$$

where w depend on the symmetry type G according to

$$w(G) = \begin{cases} 1, & G = U \\ 1 + \frac{\delta_0}{2}, & G = O, \\ 1 + \frac{\sin 2\pi x}{2\pi x}, & G = SO(\text{even}) \\ 1 + \delta_0 - \frac{\sin 2\pi x}{2\pi x}, & G = SO(\text{odd}), \\ 1 - \frac{\sin 2\pi x}{2\pi x}, & G = Sp. \end{cases} \tag{2.2}$$

Intuitively, the term δ_0 occurring in $w(SO(\text{odd}))$ is due to the fact that one eigenvalue of orthogonal matrices of odd dimension always is equal to 1 (the eigenvalues come in conjugate pairs and their product is equal to 1). The density $w(O)$ can be obtained by averaging the densities $w(SO(\text{even}))$ and $w(SO(\text{odd}))$.

2.2 Families of L-functions and the 1-level density

Having introduced random matrices and distributions of their eigenvalues, we now turn our attention to families of L-functions. We will restrict ourselves to L-functions of the form $L(s, f)$, where $f \in \mathcal{F}$. In the end we shall be interested in when f is a cusp form, but for the sake of generality, it now suffices to think of f abstractly as an interesting arithmetic object. For any f we may define a certain positive integer

c_f, called the *analytical conductor* of f. The analytical conductor serves as a way to order the objects $f \in \mathcal{F}$. Precisely, following the notation in [KS99b], we let X be a positive real number and define

$$\mathcal{F}_X := \{f \in \mathcal{F} : c_f \leq X\}.$$

The point is that we would like to analyze the average behaviour of the functions $L(s, f)$ where $f \in \mathcal{F}$. However, in many cases the family $\mathcal{F}$ is infinite, and so it is not clear what this should mean. Therefore, we are going to assume that $|\mathcal{F}_X|$ is finite and tends to infinity as X tends to infinity. We can then investigate the average behaviour of the functions $L(s, f)$ where $f \in \mathcal{F}_X$, and let X tend to infinity. As a rule, this will require us to know the asymptotical behaviour of $|\mathcal{F}_X|$.

Recall that the local re-scaled spacings of eigenvalues were distributed independently of symmetry type. It turns out that the same laws applies to zeroes of L-functions. Indeed, after appropriate rescaling, Rudnick and Sarnak showed in [RS96] that the n-level correlations, which determine the local spacing laws, follow the GUE predictions. Rudnick and Sarnak considered principal L-functions, which according to the Langlands conjectures are the multiplicative building blocks of all L-functions. Thus the high-lying zeroes of L-functions are distributed independently of symmetry type, in analogy with high-lying eigenvalues.

We turn our attention to the low-lying zeroes. To analyze them, we let ϕ be an even Schwartz function whose Fourier transform has compact support. As we shall see, the quality of the results one can obtain often depend explicitly on the exact size of the support of $\widehat{\phi}$. We will therefore assume that $\widehat{\phi}$ is supported in an interval of the form $(-\sigma, \sigma)$. The 1-level density for a single function $L(s, f)$ where $f \in \mathcal{F}$ is defined by

$$D(f, \phi) := \sum_{\rho_f} \phi\left(\left(\rho_f - \frac{1}{2}\right) \frac{\mathcal{L}}{2\pi i}\right), \tag{2.3}$$

where $\mathcal{L} := \log c_f$ and ρ_f ranges over the non-trivial zeroes of $L(s, f)$ counted with multiplicity (we always assume these lie in the critical strip). We often denote

$$\left(\rho_f - \frac{1}{2}\right) \frac{1}{i} = \gamma_f, \quad \text{so that} \quad \rho_f = \frac{1}{2} + i\gamma_f.$$

The Generalized Riemann Hypothesis (GRH) states that all γ_f are real, and we will assume it for all the relevant L-functions throughout the report. Then it makes sense to order the zeroes according to height, and by the rapid decay of ϕ, only the low-lying zeroes contribute to the 1-level density. However, even without assuming the GRH the sum (2.3) makes sense, as ϕ may be extended to an entire function which decays rapidly if $\widehat{\phi}$ is compactly supported, see e.g. [Rud87, Section 19.1] and [Cho+22, eq. 4.17].

The factor $\mathcal{L}/2\pi$ in (2.3) re-scales the height γ_f so that the spacings are approximately equal to 1. Morally, this means that if we consider test functions ϕ for which the contribution of zeroes larger than some fixed constant is negligible, then $D(f, \phi)$ measures the occurrence of zeroes whose height is of order $1/\mathcal{L}$. We shall see in Chapter 5, when we compute the 1-level density, that this is an appropriate re-scaling to make.

Although it is possible to obtain some information about $D(f, \phi)$ for a single function $L(s, f)$, the real interest lies when one considers an average or weighted average of several densities $D(f, \phi)$, where f ranges over some natural family $\mathcal{F}$. We thus define

$$\mathcal{D}(X, \mathcal{F}, \phi) := \frac{1}{|\mathcal{F}_X|} \sum_{f \in \mathcal{F}_X} D(f, \phi), \tag{2.4}$$

which is investigated as X tends to infinity. The definition (2.4) is quite general and can be applied to a variety of situations. The family which we will investigate in the Chapters 4-6 is the set of newforms. For any newform, and more generally any modular form of weight k and N, we define its analytical conductor by $c_f := k^2 N$. The set of newforms of weight k and level N is denoted by $H_k^*(N)$ (all these concepts will be carefully explained in due time). The 1-level density of a single newform $f \in H_k^*(N)$ is then defined as in (2.3). However, in this case the family $H_k^*(N)$ turns out to be finite, which allows us to define the 1-level density over the whole family $H_k^*(N)$ by

$$\mathcal{D}_{H_k^*(N)}(\phi) := \frac{1}{|H_k^*(N)|} \sum_{f \in H_k^*(N)} D(\phi; f). \tag{2.5}$$

This is not exactly the same setting as in (2.4). However, as kN tends to infinity, so does $|H_k^*(N)|$, and we can compute the asymptotic behaviour of how fast the growth is. Thus it is meaningful to investigate the asymptotic behaviour of $D_{H_k^*(N)}(\phi)$ as kN tend to infinity. This is the analogue to when X tends to infinity in the general setting. The Density Conjecture elaborated upon in the next section makes precise predictions about the behaviour of the 1-level density for different families $\mathcal{F}$.

2.3 The Density Conjecture

Recall that in the random matrix setting, the distribution of eigenvalues close to the central point $s = 1$ was investigated and found to depend on the symmetry type according to (2.2). In the L-function setting, the central point of the functional equation is $s = 1/2$. The natural analogy would therefore be to understand the distribution of zeroes near this point, which indeed is the purpose of the 1-level density. The *Density Conjecture* of Katz and Sarnak [KS99b, p. 20] states that

$$\mathcal{D}(X, \mathcal{F}, \phi) \to \int_{-\infty}^{\infty} \phi(x) w(G)(x) dx \tag{2.6}$$

for any natural family $\mathcal{F}$ and even Schwartz test function ϕ, where $\widehat{\phi}$ has compact support, as $X \to \infty$. Here, $w(G)$ may be any of the distributions in (2.2). This is the L-function analogy of (2.1) in the random matrix theory setting. The distribution $w(G)$ for a particular family $\mathcal{F}$ determines its symmetry type.

The full Density Conjecture is not proven, but one can obtain partial results conditional on the GRH. Typically, one then need to place some restrictions on ϕ in order to make rigorous proofs. The usual requirement is that $\widehat{\phi}$ is supported in $(-\sigma, \sigma)$ for some $\sigma > 0$, such as $\sigma = 1$ or 2. Katz and Sarnak [KS99b, Section 4] have compiled a list of families that have been studied, which symmetry they have, and some values of σ for which (2.6) holds (although many new results have been reached since they published it). Examples from there include the family of Dirichlet L-functions with real primitive characters, which has symplectic symmetry, and L-functions attached to newforms of weight k and level N, which has orthogonal symmetry. Furthermore, one may divide the latter family into two subfamilies $\mathcal{F}^{\pm}$ depending on the sign in the functional equation of f. It can then be shown that $\mathcal{F}^{+}$ follows the even orthogonal symmetry and that $\mathcal{F}^{-}$ follows the odd orthogonal symmetry.

When encountering a new family of L-functions, one may of course wonder what ways there are to determine its symmetry type. Sometimes, the family may have a function field analogue. In this case there are certain geometrical tools to determine the symmetry type of function field families. The symmetry type of the number field family is generally expected to agree with that of its function fields analogue.

Besides searching for function field analogues, analytic number theorists have had few methods of determining the symmetry type of a given family. In the paper [SST16] by Sarnak, Shin and Templier, they revised and updated the original Density Conjecture, and also introduced some indicators of how to decide the symmetry group of a particular family (see [SST16, p. 538]). These are given by integrating certain conjugate-invariant maps on an n-dimensional torus against the so-called *Sato-Tate measure*. We will not investigate this further.

2.4 The Ratios Conjecture

When analyzing the 1-level density the first step is to use the argument principle to write the sum (2.3) as a contour integral. The integrand will then naturally involve the logarithmic derivative L'/L, so we are interested in sums of the shape

$$\frac{1}{|\mathcal{F}_X|} \sum_{f \in \mathcal{F}_X} \frac{L(1/2+\alpha, f)}{L(1/2+\gamma, f)},$$

which upon differentiation with respect to α and evaluation at $\alpha = \gamma = r$ turns into

$$\frac{1}{|\mathcal{F}_X|} \sum_{f \in \mathcal{F}_X} \frac{L'(1/2+r, f)}{L(1/2+r, f)}.$$

The *Ratios Conjecture* is a conjecture, or rather a family of closely related conjectures, which predict the behaviour of these types of sums. We are primarily interested in applying the Ratios Conjecture to the analysis of the 1-level density. Several other applications are presented in [CS07].

The formulation of a Ratios Conjecture follows a relatively fixed recipe, first given by Conrey, Farmer and Zirnbauer in [CFZ08]. We are going to adapt it to our situation when studying L-functions attached to newforms, following [Mil09]. In some computations we diverge from his approach, but we reach essentially the same conclusions.

We emphasize that many of the steps involved are heuristical in nature. One such is to use an approximate functional equation for the attached L-function $L(s, f)$, where the error term is simply ignored. Another is to replace various quantities, such as sums of eigenvalues by their expected value when averaged over the family. Despite this, we will see that the Ratios Conjecture accurately predicts not only the main term of the 1-level density, but also lower order terms down to a power-saving error term. The accuracy of the Ratios Conjecture has been documented in other instances as well (see e.g. [CS07; FPS16; FPS18]), which given its heuristic nature is quite remarkable.

3

Basic theory of modular forms

In this chapter we gather definitions, notation and results from the theory of modular forms in order to create an appropriate foundation for the computations in later chapters. We start by introducing the *modular group* and its *congruence subgroups* in Section 3.1. These objects are essential to define some properties, such as weight-k invariance, which in turn are used when defining what a modular form is. This is carried out in Section 3.2. We also distinguish two different types of modular forms, namely *Eisenstein series* and *cusp forms*. The latter will receive more attention, both since cusp forms are more elusive than Eisenstein series, but also since the so-called *newforms*, which is the type of form to which we will attach L-functions, are cusp forms. Cusp forms can be studied by tools from linear algebra due to the *Petersson inner product*, which is introduced in Section 3.3. Another ingredient in our analysis of cusp forms are the *Hecke operators*, which are introduced in Section 3.4. We are particularly interested in finding cusp forms which are simultaneous eigenvectors to all Hecke operators at a given level. For this purpose we review the *subspaces of old- and newforms* in Section 3.5. With all this in place we can attach an L-function to any newform f by defining a Dirichlet series whose terms contain the Hecke eigenvalues of f. This is done in Section 3.6. In the last section we introduce two auxiliary L-functions whose properties also come in handy in Chapter 4.

The main source of the contents of this chapter is [DS05], and we will try to refer to it when possible. Another good reference is [CS17]. Much of the material in the last two sections comes from [ILS00, Section 3].

3.1 The modular group and congruence subgroups

The *general linear group* $\mathrm{GL}_2^+(\mathbb{Q})$ is the group of 2×2 matrices with rational entries and positive determinant. The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2×2-matrices with integer entries and determinant 1, that is

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

We may define an action of a matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ through the linear fractional transformation

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This means, and one may check, that the action of the identity matrix I is that of the identity transformation and that any two matrices $\alpha, \beta \in \mathrm{GL}_2^+(\mathbb{Q})$ satisfy

$$\alpha(\beta(z)) = (\alpha\beta)(z)$$

for any $z \in \widehat{\mathbb{C}}$. Two matrices $\alpha, \beta \in GL_2^+(\mathbb{Q})$ correspond to the same linear fractional transformation if and only if $\alpha = a\beta$ for some rational number a; if $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$, then they correspond to the same linear fractional transformation if and only if $\alpha = \pm\beta$.

The *upper half plane* $\mathcal{H}$ is the set of complex numbers with positive imaginary part,

$$\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

If $z \in \mathcal{H}$, then

$$\Im(\alpha(z)) = \det \alpha \frac{\Im(z)}{|cz+d|^2}, \quad \text{where} \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}). \tag{3.1}$$

This implies that linear fractional transformations map $\mathcal{H}$ into itself.

Let N be a positive integer. The *principal congruence subgroup of level* N is the set

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Here and from now on, we interpret congruence of matrices entrywise, so in this case we have $a \equiv d \equiv 1 \pmod{N}$ and $b \equiv c \equiv 0 \pmod{N}$. A *congruence subgroup* is a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma \supseteq \Gamma(N)$ for some N. Moreover, we say that Γ is a congruence subgroup of *level N*. The most important examples of congruence subgroups besides $\Gamma(N)$ are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\} \text{ and}$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where the stars are placeholders for any admissible integer.

The index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$ turns out be finite; indeed, it is outlined on [DS05, p. 21] how one may show that

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \simeq \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$$

and subsequently

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = |\,\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Hence, the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is finite for any congruence subgroup Γ, since

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma][\Gamma : \Gamma(N)].$$

We may thus write

$$\mathrm{SL}_2(\mathbb{Z}) = \bigcup_j \Gamma\alpha_j, \tag{3.2}$$

where $\{\alpha_j\} \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a finite set of coset representatives of Γ in $\mathrm{SL}_2(\mathbb{Z})$. In particular, we have (see [DS05, p. 21] again)

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

This is a multiplicative function of N, which we will denote by $\nu(N)$.

A congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ acts on $\widehat{\mathbb{C}}$ in the same way as the full modular group, and by (3.1) also on $\mathcal{H}$. It therefore partitions $\mathcal{H}$ into cosets

$$\Gamma\backslash\mathcal{H} = \{\Gamma z : z \in \mathcal{H}\}$$

consisting of Γ-equivalent points. A *fundamental domain* for the action of Γ on $\mathcal{H}$ is a connected set $\mathcal{D} \subseteq \mathcal{H}$, such that it contains exactly one point z from each of the cosets Γz. A common choice for a fundamental domain when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is the set

$$\mathcal{D} := \{z \in \mathcal{H} : |\Re(z)| \leq 1/2, |z| \geq 1\}. \tag{3.3}$$

Strictly speaking, this is not a fundamental domain since there are distinct points on its boundary belonging to the same coset. This will not cause any problems in our computations since the boundary has measure 0. For details, see [DS05, Lemmas 2.3.1 and 2.3.2].

We are interested in adjoining ∞ (or rather $\Gamma\infty$) to the quotient $\Gamma\backslash\mathcal{H}$. However, since ∞ may be mapped to any rational number by a matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ we cannot only adjoin ∞ to $\mathcal{H}$, but also have to adjoin all of $\mathbb{Q}$ as well. Doing so gives us the *extended upper half plane*

$$\mathcal{H}^* := \mathcal{H} \cup \mathbb{Q} \cup \{\infty\},$$

on which Γ acts. More precisely, Γ acts on the set $\mathbb{Q} \cup \{\infty\}$, partitioning it into equivalence classes. These are called the *cusps* of Γ and there are always finitely many of them, since the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is finite. Indeed, we saw how ∞ may be mapped to any rational number by a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This means that the full modular group $\mathrm{SL}_2(\mathbb{Z})$ has a single cusp. Then, from the decomposition (3.2) we may conclude that the cusps of any congruence subgroup Γ is a subset of the set (not necessarily equal to)

$$\{\Gamma\alpha_j(\infty)\},$$

where $\{\alpha_j\}$ is the chosen set of coset representatives of Γ in $\mathrm{SL}_2(\mathbb{Z})$. If $\mathcal{D}$ is a fundamental domain for the action of Γ on $\mathcal{H}$, then a fundamental domain for the action of Γ on $\mathcal{H}^*$ is obtained by adjoining the cusps of Γ to $\mathcal{D}$.

3.2 Modular forms

A modular form is a holomorphic function defined on $\mathcal{H}$ with some additional conditions imposed. Before we describe these conditions, we need to introduce some concepts and notation. Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathcal{H}$. Define the *factor of automorphy* of α at z by

$$j(\alpha, z) := cz + d,$$

where α has bottom row $(c \;\; d)$. Also, let $k \in \mathbb{Z}$ and define the *weight-k operator* $[\alpha]_k$ on the set of functions $f : \mathcal{H} \to \mathbb{C}$ by

$$f[\alpha]_k(z) := \frac{(\det \alpha)^{k-1}}{j(\alpha, z)^k} f(\alpha(z)). \tag{3.4}$$

Commonly, we have $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and so the determinant is 1. We shall be exclusively interested in the case when k is positive and even. Some of the basic facts of the factor of automorphy and the weight-k operator are collected in the following result.

Proposition 3.1 ([DS05, Lemma 1.2.2])**.** *For all $\alpha, \alpha' \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$, we have*

1. $j(\alpha\alpha', z) = j(\alpha, \alpha'(z)) j(\alpha', z)$,
2. $[\alpha\alpha']_k = [\alpha]_k [\alpha']_k$,
3. $\frac{d\alpha(z)}{dz} = j(\alpha, z)^{-2}$.

Remark 3.2. All three properties generalizes to when $\alpha, \alpha' \in \mathrm{GL}_2^+(\mathbb{Q})$, provided that we multiply the right hand side by $\det \alpha$ in the third one.

Remark 3.3. In the literature, one sometimes encounter the alternative definition

$$f[\alpha]_k(z) := \frac{(\det \alpha)^{k/2}}{j(\alpha, z)^k} f(\alpha(z)). \tag{3.5}$$

We shall not make use of this ourselves, but note that some sources use this convention. Property 2 extended to $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ in Proposition 3.1 remains valid with this convention.

A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is said to be *weakly modular* of weight k with respect to a congruence subgroup Γ if

$$f[\alpha]_k = f \quad \text{for all} \quad \alpha \in \Gamma. \tag{3.6}$$

Explicitly, this means that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \;\text{ for all }\; \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\text{ and }\; z \in \mathcal{H}.$$

The property (3.6) is referred to as *weight-k invariance* with respect to Γ.

The last notion we need to introduce before we can define modular forms is holomorphicity at the cusps of Γ. This is defined in terms of holomorphicity at ∞. To understand what holomorphicity at ∞ is, we have the following (see [DS05, pp. 3 and 16]). If f is weakly modular with respect to Γ, then f is periodic with some period h, since Γ contains a matrix α_h of the form

$$\alpha_h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

for some minimal $h \in \mathbb{Z}^+$. Precisely, $f(z) = (f[\alpha_h]_k)(z) = f(z+h)$ from weak modularity. With $D = \{z \in \mathbb{C} : |z| < 1\}$ being the open unit disc and $D' = D \setminus \{0\}$ the punctured open unit disc, the map $z \mapsto e^{2\pi i z/h}$ maps $\mathcal{H}$ holomorphically into D'. We define the map $g : D' \to \mathbb{C}$ by

$$g(q) := f(h(\log q)/2\pi i); \tag{3.7}$$

this makes sense, since the value of the right hand side is independent of which branch of $\log q$ we choose. As f is holomorphic on $\mathcal{H}$, the function g is holomorphic on D' and has a Laurent expansion

$$g(q) = \sum_{n \in \mathbb{Z}} a_f(n) q^n \tag{3.8}$$

around the origin. Now, if $q = e^{2\pi i z/h}$, then as $\Im(z)$ tends to infinity, q tends to 0. Hence, we define f to be *holomorphic at* ∞ if the function g extends holomorphically to D, that is if the coefficients $a_f(n) = 0$ for all $n < 0$. If this is the case, combining (3.7) and (3.8) gives us

$$f(z) = g(e^{2\pi i z/h}) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n z/h}$$

for any $z \in \mathcal{H}$, giving us a Fourier series of f.

Now, for any $s \in \mathbb{Q} \cup \{\infty\}$ there is some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha(\infty) = s$. Hence, we would like to say that f is holomorphic at the cusp s if $f(\alpha(z))$ is holomorphic at ∞. However, we have not formally defined what this means. Therefore, we instead require that the closely related function $f[\alpha]_k$ is holomorphic at infinity. This notion makes sense with the tools we have introduced; if f is holomorphic on $\mathcal{H}$ and weakly modular of weight k with respect to Γ, then $f[\alpha]_k$ is holomorphic on $\mathcal{H}$ and weakly modular of weight k with respect to the congruence subgroup $\alpha^{-1}\Gamma\alpha$. Indeed, to check that $\alpha^{-1}\Gamma\alpha$ is a congruence subgroup, note that if $\Gamma \supseteq \Gamma(N)$, $\gamma \in \Gamma(N)$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then $\alpha\gamma\alpha^{-1} \in \Gamma(N) \subseteq \Gamma$. Hence $\gamma = \alpha^{-1}(\alpha\gamma\alpha^{-1})\alpha \in \alpha^{-1}\Gamma\alpha$, and $\Gamma(N) \subseteq \alpha^{-1}\Gamma\alpha$. To check that $f[\alpha]_k$ is weight-k invariant, we have

$$(f[\alpha]_k)[\alpha^{-1}\gamma\alpha]_k = f[\gamma\alpha]_k = f[\alpha]_k$$

for any $\gamma \in \Gamma$ by weak modularity. Thus all notions are in place in order for the holomorphicity of $f[\alpha]_k$ at ∞ to make sense.

If we already know that a holomorphic function f is weakly modular, then its holomorphicity at ∞ follows if $f(z)$ has a limit, or even is bounded, as $\Im(z)$ tends to infinity. This will rule out the existence of any nonzero Fourier coefficients with negative indices, since such terms would blow up as $\Im(z)$ tend to infinity.

We summarize all the properties of modular forms and cusp forms into the following definition.

Definition 3.4. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and k an (even positive) integer. A *modular form of weight k with respect to* Γ is a function $f : \mathcal{H} \to \mathbb{C}$ satisfying:

1. f is holomorphic on $\mathcal{H}$,
2. f is weakly modular of weight k with respect to Γ,
3. $f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

If in addition to the above three conditions, we have that

4. $a_g(0) = 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, where $g = f[\alpha]_k$,

then we say that f is a *cusp form of weight k with respect to* Γ. The sets of modular forms and cusp forms of weight k with respect to Γ are denoted by $\mathcal{M}_k(\Gamma)$ and $S_k(\Gamma)$, respectively.

If there can be no misunderstanding of the intended weight or congruence subgroup, we will naturally speak of a *modular form* or *cusp form*, respectively. The set $\mathcal{M}_k(\Gamma)$ constitute a vector space over $\mathbb{C}$ with $S_k(\Gamma)$ being a subspace. Also, if $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$ then $fg \in \mathcal{M}_{k+\ell}(\Gamma)$. We shall mainly be occupied with the case $\Gamma = \Gamma_0(N)$, for which we use the shorthand notation $\mathcal{M}_k(\Gamma_0(N)) = \mathcal{M}_k(N)$ and $S_k(\Gamma_0(N)) = S_k(N)$. Our main source [DS05] focuses instead on the more general case when $\Gamma = \Gamma_1(N)$. For the most part, the results we cite are transferable by restricting to $\Gamma_0(N)$.

It is enough to check condition 3 and 4 for α_j running through a set of coset representatives of the space $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$. This is the case since if $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then by (3.2) we have $\alpha = \gamma\alpha_j$ where $\gamma \in \Gamma$ and α_j is an coset representative. The claim now follows since

$$f[\alpha]_k = f[\gamma\alpha_j]_k = f[\alpha_j]_k$$

by weak modularity of f.

With the definition of modular forms in place, it is natural to look for examples. We will provide a few in the next subsection.

3.2.1 Eisenstein series

The space of modular forms $\mathcal{M}_k(\Gamma)$ has a natural decomposition into the cusp forms $S_k(\Gamma)$ and the space of *Eisenstein series* of weight k, denoted by $\mathcal{E}_k(\Gamma)$. Eisenstein series can be defined quite explicitly in many cases. As a first example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we may define the series

$$G_k(z) := \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k},$$

where $k \geq 4$ to ensure absolute convergence. We may also assume that k is even, since otherwise the series vanishes due to the terms belonging to pairs (c, d) and

$(-c,-d)$ cancel each other out. By reordering terms according to greatest common divisor, we have

$$G_k(z) = \sum_{n=1}^{\infty} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=n}} \frac{1}{(cz+d)^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^k} = \zeta(k) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^k}.$$

We may define the normalized Eisenstein series by

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^k}. \tag{3.9}$$

For each pair (c,d) of two relatively prime integers, we may form a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $(c \;\; d)$ as its lower row. This matrix is unique up to multiplication from the left by a matrix $\alpha \in P_+$, where

$$P_+ := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

is the group of translational matrices. Hence, we may regard γ as an element of the quotient $P_+ \backslash \mathrm{SL}_2(\mathbb{Z})$. Finally, we note that the two matrices $\pm\gamma$ give the same term when k is even, so if we regard γ as a linear fractional transformation rather than a matrix, we see that the normalized Eisenstein series may be written as

$$E_k(z) = \sum_{\gamma \in P_+ \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{1}{j(\gamma, z)^k} \tag{3.10}$$

We will shortly present variations of this type of sum. In all instances we regard the sum as being over linear fractional transformation rather than matrices. Sometimes (3.10) is taken as the definition of an Eisenstein series rather than G_k. An advantage is that showing weight-k invariance involves less tedious calculations.

To see that E_k is a modular form, we note first that it is holomorphic on $\mathcal{H}$ as an absolutely and locally uniformly convergent series of holomorphic functions. Secondly, for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ we have that $j(\gamma, \alpha(z)) = j(\gamma\alpha, z)/j(\alpha, z)$, whence

$$E_k[\alpha]_k(z) = \frac{1}{j(\alpha, z)^k} \sum_{\gamma \in P_+ \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{j(\alpha, z)^k}{j(\gamma\alpha, z)^k} = \sum_{\gamma \in P_+ \backslash \mathrm{SL}_2(\mathbb{Z})\alpha} \frac{1}{j(\gamma, z)^k} = E_k(z),$$

since $P_+ \backslash \mathrm{SL}_2(\mathbb{Z})\alpha = P_+ \backslash \mathrm{SL}_2(\mathbb{Z})$. The steps to prove that $E_k(z)$ is holomorphic at ∞ are outlined on [DS05, p. 8] (where they consider G_k), and so $E_k(z)$ is indeed a modular form.

The representation (3.10) allows for variations which will yield other kinds of Eisenstein series. For instance, we can obtain modular forms with respect to the congruence subgroup $\Gamma_0(N)$, if we replace $P_+ \backslash \mathrm{SL}_2(\mathbb{Z})$ by $P_+ \backslash \Gamma_0(N)$ in (3.10). This makes sense, as P_+ acts on $\Gamma_0(N)$ in the same way as on $\mathrm{SL}_2(\mathbb{Z})$ for any N. The resulting series is

$$\sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \frac{1}{j(\gamma, z)^k}, \tag{3.11}$$

and it has an explicit representation similar to the right hand side of (3.9), where we in addition restrict the summation to $N|c$. The verification that (3.11) is holomorphic and weight-k invariant with respect to $\Gamma_0(N)$ is similar to $E_k(z)$. Holomorphicity at the cusps can be shown by computing the Fourier series.

Another variation of (3.10) is the *non-holomorphic Eisenstein series of weight 0*

$$E(z,s) := \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \Im(\gamma(z))^s = \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \left(\frac{\Im(z)}{|j(\gamma,z)|^2} \right)^s. \tag{3.12}$$

This is not holomorphic in z because of the appearance of $\Im(z)$ in the right hand side. However, for $z \in \mathcal{H}$ fixed it is holomorphic in s for $\Re(s) > 1$ [CS17, Proposition 5.2.12]. It can be shown that $E(z,s)$ has a meromorphic continuation to $\mathbb{C}$ in the variable s, with the only pole in the half plane $\{s : \Re(s) \geq 1/2\}$ being simple and located at $s = 1$. Its residue is equal to $3/(\pi\nu(N))$ independently of z [CS17, Corollary 8.5.9], which stems from a certain volume computation (see (3.19) for definitions). Being weight-0 invariant simply means that $E(\alpha(z),s) = E(z,s)$ for all $\alpha \in \Gamma_0(N)$, which can be swiftly deduced from the middle of (3.12).

As a final variation, we let $m \geq 0$ be an integer and consider the *Poincaré series*

$$P_m(z) := \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \frac{e(m\gamma(z))}{j(\gamma,z)^k}. \tag{3.13}$$

Holomorphicity and weak modularity is proved in a manner similar to the Eisenstein series (3.11), which also is what we recover if we let $m = 0$. If $m > 0$, the Poincaré series is a cusp form with respect to $\Gamma_0(N)$ (despite the title of this section). This can be shown by computing its Fourier series of $P_m[\alpha]_k(z)$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. We will compute the Fourier series of $P_m(z)$ in Section 4.2 as a part of the proof of the Petersson trace formula.

3.2.2 Dimension of the space of modular forms

Recall that $\mathcal{M}_k(\Gamma)$ is a vector space over $\mathbb{C}$. A natural question would be whether its dimension is finite, and if one can compute it in that case. The answers to both these questions are in the affirmative, and we will describe formulas for $\dim \mathcal{M}_k(\Gamma)$ and $\dim S_k(\Gamma)$ in this section. A connection to the next chapter is that the formulas provide expressions for the number of terms when we sum over bases of $\mathcal{M}_k(\Gamma)$ or $S_k(\Gamma)$. We will not delve too deep into the details of the proofs here, since it would interfere with the scope of the report.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The quotient $\Gamma\backslash\mathcal{H}^*$ may be viewed as a compact Riemann surface with a Hausdorff topology. As a topological surface, we can associate a certain nonnegative integer g to it, called the *genus* of the surface. When supplying the surface with local coordinates, one runs into some issues if a point $\Gamma z \in \Gamma\backslash\mathcal{H}$ has nontrivial *isotropy subgroup*, i.e. if there are matrices $\alpha \in \mathrm{SL}_2(\mathbb{Z}), \alpha \neq \pm I$, such that $\alpha(z) = z$. If this is the case, then z and the corresponding point Γz, are called *elliptic points*. It turns out that for each elliptic point z its isotropy subgroup is finite and cyclic [DS05, Corollary 2.3.5]. The *period* of an elliptic point z is defined as the size of its isotropy subgroup, possibly divided by 2 if the isotropy subgroup contains $-I$. As an example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then there are two elliptic points, namely i and $e^{2\pi i/3}$. Their periods are 2 and 3, respectively [DS05, Corollary 2.3.4]. For a general congruence subgroup Γ, these are the only periods an elliptic point can have [DS05, p. 67]. We denote the number of elliptic points of order 2 and 3 by ε_2 and ε_3, respectively. Together with the number of cusps of Γ, which we denote by ε_∞, these will make an explicit appearance in the dimension formulas to follow.

If $\Gamma_1 \subseteq \Gamma_2$ are two congruence subgroups, there is a natural map

$$f : \Gamma_1 \backslash \mathcal{H}^* \to \Gamma_2 \backslash \mathcal{H}^*$$

given by

$$\Gamma_1 z \mapsto \Gamma_2 z.$$

To the map f (and more generally to any nonconstant holomorphic map between two compact Riemann surfaces) we may associate a number d called the *degree* of f. The degree satisfies $|f^{-1}(\Gamma_2 z)| = d$ for all $\Gamma_2 z \in \Gamma_2 \backslash \mathcal{H}^*$ except finitely many (see [DS05, pp. 65-66] for details). With this at our disposal, the genus g of $\Gamma \backslash \mathcal{H}^*$ may be computed by the formula

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}$$

(see [DS05, Theorem 3.1.1]).

In the special case when $\Gamma_1 = \Gamma_0(N)$ and $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})$, we have

$$d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = \nu(N) \tag{3.14}$$

(cf. [DS05, p. 66]).

The dimensions of $\mathcal{M}_k(\Gamma)$ and $S_k(\Gamma)$ are computed in [DS05, Section 3.5]. They are summarized in the theorem [DS05, Theorem 3.5.1], which we now quote.

Theorem 3.5. *Let k be an even integer and Γ be a congruence subgroup. Let $g, \varepsilon_2, \varepsilon_3$ and ε_∞ denote the genus of $\Gamma \backslash \mathcal{H}^*$, the number of elliptic points of Γ with period 2 and 3, and the number of cusps of $\Gamma \backslash \mathcal{H}^*$, respectively. Then,*

$$\dim \mathcal{M}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty, & k \geq 2, \\ 1, & k = 0, \\ 0, & k < 0, \end{cases} \tag{3.15}$$

and

$$\dim S_k(\Gamma) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \left(\frac{k}{2} - 1\right) \varepsilon_\infty, & k \geq 4, \\ g, & k = 2. \\ 0, & k \leq 0. \end{cases} \tag{3.16}$$

As an example, when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is the full modular group, then the genus g is equal to 0. We have already seen that $\varepsilon_2 = \varepsilon_3 = 1$. Finally, $\varepsilon_\infty = 1$ since for any $s \in \mathbb{Q}$ there is a matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $s = \alpha(\infty)$. Inserting these values in (3.15) and (3.16) gives us the dimension of $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ and $S_k(\mathrm{SL}_2(\mathbb{Z}))$.

In Chapter 4-6 we shall be exclusively interested in the case when $\Gamma = \Gamma_0(N)$. In this instance, the numbers $d, \varepsilon_2, \varepsilon_3$ and ε_∞ can be computed explicitly. The case $N = 1$ is covered by the previous example, and the remaining cases are given by the following:

$$\begin{aligned}
d &= \nu(N), \\
\varepsilon_2 &= \begin{cases} \prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right), & N \geq 2,\ 4 \nmid N, \\ 0, & N \geq 3,\ 4|N, \end{cases} \\
\varepsilon_3 &= \begin{cases} \prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right), & N \geq 3,\ 9 \nmid N, \\ 0, & N \geq 3,\ 9|N, \end{cases} \\
\varepsilon_\infty &= \sum_{d|N} \varphi((d, N/d)).
\end{aligned}$$

Here, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol if p is odd, $\left(\frac{-3}{2}\right) = -1$ and $\left(\frac{-1}{2}\right) = 0$. The expression for d is from (3.14), and the remaining equations are quoted from [DS05, Corollary 3.7.2 and p. 103]. Together, these formulas give an explicit description of the dimensions of $\dim \mathcal{M}_k(N)$ and $\dim S_k(N)$.

3.3 The Petersson inner product

Let Γ be a congruence subgroup. Much of the analysis of the vector space $S_k(\Gamma)$ relies on endowing it with an inner product, called the *Petersson inner product*. This is defined as

$$\langle f, g\rangle_\Gamma := \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\Im(z)^k d\mu(z), \tag{3.17}$$

where we understand the integral as being over a fundamental domain for the action of Γ on $\mathcal{H}$, and $d\mu(z)$ is the *hyperbolic measure*

$$d\mu(z) := \frac{dxdy}{y^2}, \quad z = x + iy \in \mathcal{H}.$$

There are several ingredients in showing that the definition (3.17) makes sense. Indeed, we may define

$$\int_{\Gamma\backslash\mathcal{H}} \varphi(z)d\mu(z)$$

for any continuous, bounded and Γ-invariant function $\varphi : \mathcal{H} \to \mathbb{C}$ (Γ-invariance is simply weight-0 invariance with respect to Γ). To see that such a definition makes sense, note that the hyperbolic measure is invariant under the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathcal{H}$ (see [DS05, p. 182]), hence under the action of $\mathrm{SL}_2(\mathbb{Z})$. Hence, we may define

$$\int_{\Gamma\backslash\mathcal{H}} \varphi(z)d\mu(z) := \sum_j \int_{\alpha_j(\mathcal{D})} \varphi(z)d\mu(z) = \sum_j \int_{\mathcal{D}} \varphi(\alpha_j(z))d\mu(z), \tag{3.18}$$

where $\{\alpha_j\}$ is a set of coset representatives such that

$$\mathrm{SL}_2(\mathbb{Z}) = \bigcup_j \{\pm I\}\Gamma\alpha_j,$$

and $\mathcal{D}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. The definition is independent of the choice of coset representatives, by the Γ-invariance of φ and $d\mu$. The individual integrals on the right hand side converges since the integrand is bounded and the *total volume* of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$,

$$\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} d\mu(z) = \int_{x=-1/2}^{1/2}\int_{y=\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}dydx = [\arcsin x]_{-1/2}^{1/2} = \frac{\pi}{3}$$

is finite. In general, the volume of $\Gamma\backslash\mathcal{H}$ is given by

$$\mathrm{vol}(\Gamma\backslash\mathcal{H}) := \int_{\Gamma\backslash\mathcal{H}} d\mu(z) = [\mathrm{SL}_2(\mathbb{Z}):\Gamma]\,\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}) = \frac{\pi}{3}[\mathrm{SL}_2(\mathbb{Z}):\Gamma]. \tag{3.19}$$

Now we wish to ensure that all relevant criteria are satisfied for the choice $\varphi(z) = f(z)\overline{g(z)}\Im(z)^k$, where $f, g \in S_k(\Gamma)$. Continuity is immediate, and Γ-invariance follows from (3.1) and the weak modularity of f and g. This also means that to check that φ is bounded on $\mathcal{H}$, it is enough to check that $\varphi\circ\alpha$ is bounded on a fundamental domain $\mathcal{D}$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. The crucial point then is the rapid decay of the factor

$$e^{2\pi i z/h} = O(e^{-2\pi\Im(z)/h}),$$

which can be factored out in the Fourier expansions of f and g due to $a_f(0) = a_g(0) = 0$. The exponential decay dominates any power $\Im(z)^k$ and ensures that φ is bounded (see [DS05, p. 183] for details). This is also the reason why the Petersson inner product cannot be defined on the whole space $\mathcal{M}_k(\Gamma)$. However, $\langle f, g\rangle_\Gamma$ can be defined if one of f and g is a cusp form and the other an Eisenstein series.

Once we have verified that the integral (3.17) is well defined, we may check that the Petersson inner product is indeed an inner product with relative ease. We shall chiefly be interested in the case when $\Gamma = \Gamma_0(N)$, for which we write $\langle\cdot,\cdot\rangle_{\Gamma_0(N)} = \langle\cdot,\cdot\rangle_N$.

3.4 Hecke theory

Now that we have endowed $S_k(\Gamma)$ with the Petersson inner product, we are interested in linear operators on this space. We restrict our focus to $S_k(N)$. An important class of such operators, and the main object of study of this section, is formed by the *Hecke operators*. Each Hecke operator is indexed by a positive integer n. The Hecke operator $T_N(n)$ of level N and index n is the linear operator on $\mathcal{M}_k(N)$ defined by

$$(T_N(n))f(z) := \frac{1}{\sqrt{n}} \sum_{\substack{ad=n\\(a,N)=1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b \pmod d} f\left(\frac{az+b}{d}\right). \tag{3.20}$$

These operators are normal, and when $(n, N) = 1$ we shall see that they are self-adjoint. To analyze them, we begin by introducing an alternative interpretation of this seemingly intricate definition.

3.4.1 Double coset operators

When $n = p$ is prime, then the re-scaled Hecke operator $p^{(k-1)/2}T_N(p)$ may be expressed more concisely by means of a so-called *double coset operator*. This is the route taken in [DS05, Ch. 5], and is advantageous when showing that Hecke operators are normal. For this reason, we recall the basic facts about double coset operators, and cite the necessary results.

Let $\Gamma_1, \Gamma_2 \subseteq \mathrm{SL}_2(\mathbb{Z})$ be two congruence subgroups and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. A *double coset* is a set of the form

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}.$$

There is an action of Γ_1 on $\Gamma_1\alpha\Gamma_2$ given by $\gamma_1'(\gamma_1\alpha\gamma_2) = (\gamma_1'\gamma_1)\alpha\gamma_2$, where $\gamma_1' \in \Gamma_1$. This partitions the double coset into orbits $\Gamma_1\beta$, with $\beta \in \Gamma_1\alpha\Gamma_2$. If $\{\beta_j\} \subseteq \Gamma_1\alpha\Gamma_2$ is a set of orbit representatives, that is

$$\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta_j,$$

then we define the *weight-k $\Gamma_1\alpha\Gamma_2$ operator*

$$[\Gamma_1\alpha\Gamma_2]_k : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$$

by

$$f[\Gamma_1\alpha\Gamma_2]_k := \sum_j f[\beta_j]_k. \tag{3.21}$$

The definition is independent of the choice of coset representatives. Indeed, if $\Gamma_1\beta = \Gamma_1\beta'$ then $\beta' = \gamma\beta$ for some $\gamma \in \Gamma_1$, and $f[\beta']_k = f[\beta]_k$ by weak modularity of f. To see that $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{M}_k(\Gamma_2)$, we note first that $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic on $\mathcal{H}$. Secondly, if $\{\beta_j\}$ is a set of coset representatives and $\gamma \in \Gamma_2$, then $\{\beta_j\gamma\}$ is a set of coset representatives as well. Hence

$$f[\Gamma_1\alpha\Gamma_2]_k[\gamma]_k = \sum_j f[\beta_j\gamma]_k = f[\Gamma_1\alpha\Gamma_2]_k,$$

showing weak modularity with respect to Γ_2. Finally, we note that for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, when applying the weight-k operator $[\alpha]_k$ to $f[\Gamma_1\alpha\Gamma_2]_k$, each term on the right hand side of (3.21) is holomorphic at ∞ since f is (see [DS05, p. 24] for details). Hence $f[\Gamma_1\alpha\Gamma_2]_k[\alpha]_k$ is holomorphic at ∞ as well, being a sum of functions which are. This claim can be proven by considering the Fourier expansions of the individual terms.

If Γ is a congruence subgroup, $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{SL}_2(\mathbb{Z})$, then the weight-$k$ operator $[\alpha]_k$ is a linear operator from $S_k(\Gamma)$ to $S_k(\alpha^{-1}\Gamma\alpha)$. The following result determines its adjoint and the adjoint of $[\Gamma\alpha\Gamma]_k$ with respect to the Petersson inner product.

Lemma 3.6 ([DS05, Proposition 5.5.2]). *Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\alpha' = (\det\alpha)\alpha^{-1}$. Suppose $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{SL}_2(\mathbb{Z})$ and $f \in S_k(\Gamma), g \in S_k(\alpha^{-1}\Gamma\alpha)$. Then*

$$\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha']_k\rangle_\Gamma, \tag{3.22}$$

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle_\Gamma = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle_\Gamma. \tag{3.23}$$

That is, the adjoint of $[\alpha]_k$ *is* $[\alpha']_k$*, and the adjoint of* $[\Gamma\alpha\Gamma]_k$ *is* $[\Gamma\alpha'\Gamma]_k$*.*

The proof of the first assertion relies on the definition of the Petersson inner product and the properties of the weight-k operator. The proof of the second assertion relies on some technical results about finding coset representatives which we have not included, hence we omit it (for details, see [DS05, Lemma 5.5.1]).

We shall primarily be interested in the case when $\Gamma_1 = \Gamma_2 = \Gamma_0(N)$. For

$$\alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \tag{3.24}$$

it turns out that a set of coset representatives of $\Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N)$ is (see [DS05, Proposition 5.2.1] and the subsequent remark)

$$\begin{cases} \left\{ \beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} : j = 0, 1, ..., p-1 \right\}, & p \mid N, \\ \left\{ \beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} : j = 0, 1, ..., p-1 \right\} \cup \left\{ \beta_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\}, & p \nmid N. \end{cases} \tag{3.25}$$

From this and the definition of the weight-k operator, we may now "translate" the Hecke operator $T_N(p)$ as follows.

Lemma 3.7. *The re-scaled operator* $p^{(k-1)/2}T_N(p)$ *satisfies*

$$p^{(k-1)/2}T_N(p) = [\Gamma_0(N)\alpha_p\Gamma_0(N)]_k$$

where α_p *is given by* (3.24)*.*

With the connection to between double coset operators and Hecke operators established, we turn our attention to the properties of the latter.

3.4.2 Hecke operators

Our first goal is to understand how Hecke operators affect the shape of the Fourier coefficients of a given modular form. This is the content of the next result.

Lemma 3.8. *Let* $f \in \mathcal{M}_k(N)$ *and suppose the Fourier series of* f *is*

$$\sum_{m=0}^{\infty} a_f(m)e^{2\pi imz}.$$

Then the Fourier series of $T_N(n)f$ *is*

$$\sum_{m=0}^{\infty} p_f(m,n)e^{2\pi imz}$$

where

$$p_f(m,n) = \frac{1}{n^{(k-1)/2}} \sum_{\substack{d\mid(m,n) \\ (d,N)=1}} d^{k-1}a_f\left(\frac{mn}{d^2}\right).$$

Proof. This is a straightforward calculation. Inserting the Fourier series of f into the definition of $T_N(n)f$ give us

$$\begin{aligned}(T_N(n))f(z) =& \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,N)=1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} a_f(m) e\left(m\left(\frac{az+b}{d}\right)\right) \\ =& \frac{1}{n^{(k+1)/2}} \sum_{\substack{ad=n \\ (a,N)=1}} a^k \sum_{m=0}^{\infty} e^{2\pi i \frac{m}{d} az} a_f(m) \sum_{b=0}^{d-1} e^{2\pi i \frac{m}{d} b}.\end{aligned}$$

The innermost sum on the second row is equal to d if $d|m$, and 0 otherwise. We thus write $m = d\ell$ and get

$$\begin{aligned}& \frac{1}{n^{(k-1)/2}} \sum_{\substack{ad=n \\ (a,N)=1}} a^{k-1} \sum_{\ell=0}^{\infty} e^{2\pi i \ell a z} a_f(d\ell) \\ =& \frac{1}{n^{(k-1)/2}} \sum_{m=0}^{\infty} \Bigg(\sum_{\substack{a|(m,n) \\ (d,N)=1}} a^{k-1} a_f\left(\frac{mn}{a^2}\right) \Bigg) e^{2\pi i m z}.\end{aligned}$$

In the last step, we wrote $a\ell = m$, from which it follows that $a|(m,n)$. After changing the order of summation we relabel $a \mapsto d$ to achieve concordance with the notation of the statement. □

From Lemma 3.8 it follows that the Hecke operators map the space $S_k(\Gamma)$ into itself, since $p_f(0,n) = 0$ if $a_f(0) = 0$. It also allows us to find a relation between Hecke operators of different indices. Precisely, we have the following result.

Corollary 3.9. *The Hecke operators $T_N(m)$ and $T_N(n)$ satisfy*

$$T_N(m) T_N(n) = \sum_{\substack{d|(n,m) \\ (d,N)=1}} T_N\left(\frac{mn}{d^2}\right). \tag{3.26}$$

Proof. We apply both sides of (3.26) to an arbitrary $f \in \mathcal{M}_k(N)$ and check that the Fourier coefficients agree. We use Lemma 3.8 on several occasions. Starting with the Fourier expansion

$$f(z) = \sum_{\ell=0}^{\infty} a_f(\ell) e^{2\pi i \ell z},$$

we have that

$$T_N(n) f(z) = \sum_{\ell=0}^{\infty} p_f(\ell, n) e^{2\pi i \ell z},$$

where the Fourier coefficients are given by

$$p_f(\ell, n) = \frac{1}{n^{(k-1)/2}} \sum_{\substack{e|(\ell,n) \\ (e,N)=1}} e^{k-1} a_f\left(\frac{\ell n}{e^2}\right).$$

Applying $T_N(m)$ gives

$$T_N(m)T_N(n)f(z) = \sum_{\ell=0}^{\infty} q_f(\ell,m,n)e^{2\pi i\ell z},$$

with Fourier coefficients

$$\begin{aligned} q_f(\ell,m,n) &= \frac{1}{m^{(k-1)/2}} \sum_{\substack{d|(\ell,m)\\(d,N)=1}} d^{k-1} p_f\left(\frac{\ell m}{d^2}, n\right) \\ &= \frac{1}{m^{(k-1)/2} n^{(k-1)/2}} \sum_{\substack{d|(\ell,m)\\(d,N)=1}} \sum_{\substack{e|(\ell m/d^2,n)\\(e,N)=1}} d^{k-1}e^{k-1} a_f\left(\frac{\ell mn}{d^2e^2}\right). \end{aligned} \tag{3.27}$$

Applying an individual operator $T_N(mn/d^2)$ to f give us

$$T_N\left(\frac{mn}{d^2}\right) f(z) = \sum_{\ell=0}^{\infty} p_f\left(\ell, \frac{mn}{d^2}\right) e^{2\pi i\ell z},$$

and applying the whole right hand side of (3.26) to f give us

$$\left(\sum_{\substack{d|(n,m)\\(d,N)=1}} T_N\left(\frac{mn}{d^2}\right)\right) f(z) = \sum_{\ell=0}^{\infty} r_f(\ell,m,n)e^{2\pi i\ell z},$$

with coefficients

$$\begin{aligned} r_f(\ell,m,n) &= \sum_{\substack{d|(m,n)\\(d,N)=1}} p_f\left(\ell, \frac{mn}{d^2}\right) \\ &= \frac{1}{(mn)^{(k-1)/2}} \sum_{\substack{d|(m,n)\\(d,N)=1}} \sum_{\substack{e|(\ell,mn/d^2)\\(e,N)=1}} d^{k-1}e^{k-1} a_f\left(\frac{\ell mn}{d^2e^2}\right). \end{aligned} \tag{3.28}$$

To finalize the proof, we argue along the following lines. For a positive integer E such that $(E,N)=1$, let

$$B(\ell,m,n,E) = \{(d,e) \in (\mathbb{Z}^+)^2 : de = E,\ d|(\ell,m),\ e|(\ell m/d^2, n)\}.$$

Then, the sums on the right hand sides of (3.27) and (3.28) are

$$\sum_{\substack{(E,N)=1\\(d,e)\in B(\ell,m,n,E)}} E^{k-1} a_f\left(\frac{\ell mn}{E^2}\right) \quad \text{and} \quad \sum_{\substack{(E,N)=1\\(d,e)\in B(n,m,\ell,E)}} E^{k-1} a_f\left(\frac{\ell mn}{E^2}\right),$$

respectively. The set $B(\ell,m,n,E)$ is treated by Cohen and Strömberg in [CS17, Lemma 10.2.8]. They show that its cardinality is independent of permutations in the arguments ℓ, m and n (and in particular that the cardinality is 0 for E large enough). Thus there is a bijection between $B(\ell,m,n,E)$ and $B(n,m,\ell,E)$, and we may conclude that $q_f(\ell,m,n) = r_f(\ell,m,n)$. This finishes the proof. □

It immediately follows from Corollary 3.9 that all Hecke operators at the same level commute. Also, if m and n are relatively prime, then (3.26) reduces to $T_N(m)T_N(n) = T_N(mn)$, i.e. Hecke operators are multiplicative with respect to the index. The following recursive property

$$T_N(p^r) = \begin{cases} T_N(p)T_N(p^{r-1}) - T_N(p^{r-2}), & p \nmid N, \\ T_N(p)T_N(p^{r-1}) = ... = T_N(p)^r, & p|N, \end{cases} \tag{3.29}$$

where $r \geq 2$, also follows directly from (3.26). This allow us to reduce many results about Hecke operators to the case when $n = p^r$ is a prime power, or even a prime.

We are interested in determining the adjoint of $T_N(n)$ with respect to the Petersson inner product. In fact, when $(n, N) = 1$ then $T_N(n)$ is self-adjoint. By (3.29) and the commutativity of Hecke operators, it is enough to show that $T_N(p)$ is self-adjoint when p is a prime with $p \nmid N$. We recall that the re-scaled operator $p^{(k-1)/2}T_N(p)$ is the double coset operator $[\Gamma_0(N)\alpha_p\Gamma_0(N)]_k$, where

$$\alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

By Lemma 3.6, the adjoint is $[\Gamma_0(N)\alpha_p'\Gamma_0(N)]_k$. Observing that $\alpha_p' = \beta_\infty$ from the set of coset representatives (3.25), we conclude that

$$\alpha_p' \in \Gamma_0(N)\alpha_p\Gamma_0(N).$$

Hence, we have that

$$\Gamma_0(N)\alpha_p\Gamma_0(N) = \Gamma_0(N)\alpha_p'\Gamma_0(N),$$

finishing the claim that $T_N(p)$ is self-adjoint when $p \nmid N$.

A cusp form $f \in S_k(N)$ which is an eigenvector for a Hecke operator $T_N(n)$ is called an *eigenform for the operator* $T_N(n)$. If f is an eigenform for the Hecke operator $T_N(n)$, then we denote its eigenvalue by $\lambda_f(n)$. With this in place we can draw several conclusions when $T_N(n)$ is self-adjoint. First, one observes that the eigenvalue $\lambda_f(n)$ is real. Moreover, by a standard spectral theorem from linear algebra, there exists an orthogonal basis of the space $S_k(N)$, consisting of eigenforms for *all* the Hecke operators $T_N(n)$ where $(n, N) = 1$ (see [CS17, Lemma 10.2.10] for a proof). This observation is the first step towards finding a basis for $S_k(N)$.

If f is an eigenform for all the relevant Hecke operators, then (3.26) translates into a relation for the corresponding eigenvalues, namely

$$\lambda_f(n)\lambda_f(m) = \sum_{\substack{d|(n,m) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right). \tag{3.30}$$

In particular, $\lambda_f(n)$ is multiplicative when considered as an arithmetic function of n. The consequence of (3.30) corresponding to (3.29) is

$$\lambda_f(p^r) = \begin{cases} \lambda_f(p)\lambda_f(p^{r-1}) - \lambda_f(p^{r-2}), & p \nmid N, \\ \lambda_f(p)\lambda_f(p^{r-1}) = ... = \lambda_f(p)^r, & p|N, \end{cases} \tag{3.31}$$

when $r \geq 2$.

If N is squarefree and $p|N$, then

$$\lambda_f^2(p) = \frac{1}{p} \tag{3.32}$$

(see [ILS00, p. 72]). This will be used in the calculations on several occasions.

3.5 The subspaces of oldforms and newforms

If $N > 1$, it turns out that some cusp forms at level N arises from cusp forms of lower level. The notions of subspaces of oldforms and newforms is intended to capture this intuition. The subject is treated in [DS05, Section 5.4], although they do it slightly more generally, considering the space $S_k(\Gamma_1(N))$. However, most of the results are translatable into our setting by restriction.

There are two natural ways in which a cusp form can be regarded as "old", i.e. arising from lower levels. To specify these, let M, N be positive integers with $M|N$ and $M < N$. As $\Gamma_0(N) \subseteq \Gamma_0(M)$ we have the reverse inclusion $S_k(M) \subseteq S_k(N)$. Secondly, if $f(z) \in S_k(N)$ and r is a positive integer with $r|N/M$, then $f(rz) \in S_k(rM) \subseteq S_k(N)$. The *oldforms at level N* are defined as the linear subspace of $S_k(N)$ spanned by all the cusp forms arising from cusp forms of lower level in these two ways. We denote this space by $S_k^{\text{old}}(N)$. The space of *newforms at level N* is defined as the orthogonal complement of the $S_k^{\text{old}}(N)$ with respect to the Petersson inner product. We denote this space by $S_k^{\text{new}}(N)$. For the sake of completeness, we also define

$$S_k^{\text{old}}(1) := \{0\} \quad \text{and} \quad S_k^{\text{new}}(1) := S_k(1)$$

for the level $N = 1$.

To facilitate computations, we may describe the space of oldforms at level N as follows (see [DS05, Section 5.6]). If $d|N$ with $d > 1$, then any $f \in S_k(N/d)$ belongs to $S_k^{\text{old}}(N)$. Also, if we let

$$\alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.33}$$

then if $g \in S_k(N/d)$, we have that $g[\alpha_d]_k \in S_k^{\text{old}}(N)$. Thus upon letting

$$i_d : S_k(N/d)^2 \to S_k(N), \quad (f, g) \mapsto f + g[\alpha_d]_k,$$

we may write

$$S_k^{\text{old}}(N) = \sum_{p|N} i_p(S_k(N/p)^2). \tag{3.34}$$

One could suspect that the sum should range over all divisors $d|N$, but it turns out this does not add anything new to the space, compared to if we restrict prime divisors. Indeed, for a composite divisor $pd|N$ where p is prime and $d > 1$, we let

$$\begin{aligned} i_p &: S_k(N/p)^2 \to S_k(N), \\ i_d &: S_k(N/pd)^2 \to S_k(N/p), \\ i_{pd} &: S_k(N/pd)^2 \to S_k(N), \end{aligned}$$

and consider the pair $(f, g) \in S_k(N/pd)^2$. It is mapped to $f + g[\alpha_{pd}]_k$ by i_{pd}. On the other hand, the pair $(0, g)$ is mapped to $g[\alpha_d]_k$ by i_d, and then the pair $(f, g[\alpha_d]_k)$ is mapped to $f + g[\alpha_{pd}]_k$ by i_p, since $\alpha_d \alpha_p = \alpha_{pd}$. Thus

$$i_{pd}\left(S_k(N/pd)^2\right) \subseteq i_p\left(S_k(N/p) \times i_d(S_k(N/pd)^2)\right) \subseteq i_p(S_k(N/p)^2),$$

since $i_d(S_k(N/pd)^2) \subseteq S_k(N/p)$. Hence the space of oldforms at level N is obtained by restricting the sum (3.34) to prime divisors of N.

A natural question is how Hecke operators act on the spaces of old- and newforms. We claim that they are preserved, i.e. that any Hecke operator $T_N(n)$ map $S_k^{\text{old}}(N)$ and $S_k^{\text{new}}(N)$ into themselves. The claim holds for all n, but we will only prove it for when $(n, N) = 1$, since we have not discussed the adjoint of $T_N(n)$ for $(n, N) > 1$ (see [DS05, Proposition 5.6.2] for the full proof). Again, it is then enough to show the claim when $T_N(q)$ has prime index $q \nmid N$. We turn to the case of $S_k^{\text{old}}(N)$. If p is a prime with $p|N$, we have that $T_{N/p}(q) = T_N(q)$ from the definition. Also, we have that $T_N(q)$ commute with $[\alpha_p]_k$. This follows from

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & q \end{pmatrix} = \begin{pmatrix} 1 & pj \\ 0 & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

so that we have

$$\begin{aligned} q^{(k-1)/2} T_N(q)(f[\alpha_p]_k) &= \sum_{j \pmod q} f\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & q \end{pmatrix}\right]_k + f[\alpha_p \alpha_q]_k \\ &= \sum_{j \pmod q} f\left[\begin{pmatrix} 1 & pj \\ 0 & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k + f[\alpha_q \alpha_p]_k = q^{(k-1)/2}(T_N(q)f)[\alpha_p]_k. \end{aligned}$$

From this it follows that whenever $f, g \in S_k(N/p)$, we have

$$T_N(q)(i_p(f, g)) = i_p(T_{N/p}(q)f, T_{N/p}(q)g)$$

and as a consequence $T_N(q)$ maps any of the subspaces $i_p(S_k(N/p)^2)$ into $S_k^{\text{old}}(N)$. Hence it preserves $S_k^{\text{old}}(N)$. Since $T_N(q)$ is self-adjoint, it also preserves $S_k^{\text{new}}(N)$, finishing our claim.

Since both $S_k^{\text{old}}(N)$ and $S_k^{\text{new}}(N)$ are preserved, we may conclude that they each have orthogonal bases consisting of eigenforms for the Hecke operators $\{T_N(n) : (n, N) = 1\}$. In the next section we will pursue this idea further and see what it means for $S_k^{\text{new}}(N)$.

3.5.1 Newforms and a partially orthogonal decomposition of $S_k(N)$

We now focus on the space of newforms at level N. A central result due to Atkin and Lehner is the following result, often referred to as the *main lemma* in the literature.

Theorem 3.10 (Main lemma, [DS05, Theorem 5.7.1])**.** *Let $f \in S_k(N)$ have Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}.$$

If $a_f(n) = 0$ whenever $(n, N) = 1$, then

$$f = \sum_{p|N} \iota_p f_p,$$

where $f_p \in S_k(N/p)$ and $\iota_p f(z) : f(pz)$ for all p.

In particular, this means that a cusp form $f \in S_k(N)$ with $a_f(n) = 0$ whenever $(n, N) = 1$ lies in $S_k^{\text{old}}(N)$.

The proof of Theorem 3.10 is quite long, and we therefore omit it (see [DS05, Section 5.7] for a proof due to David Carlton). However, its consequences for the analysis of $S_k(N)$ are many and important. One of them is [DS05, Theorem 5.8.2], which we now present. Suppose $f \in S_k(N)$ is an eigenform for all operators $T_N(n)$ such that $(n, N) = 1$. That is, we write

$$T_N(n) f = \lambda_f(n) f \tag{3.35}$$

for some numbers $\lambda_f(n)$. Lemma 3.8 yields that the Fourier coefficient of index 1 on the left hand side of (3.35) is equal to

$$p_f(1, n) = \frac{1}{n^{(k-1)/2}} a_f(n).$$

Hence, by comparing the Fourier coefficient of index 1 on both sides of (3.35), we obtain

$$a_f(n) = n^{(k-1)/2} \lambda_f(n) a_f(1).$$

If $a_f(1) = 0$, then $a_f(n) = 0$ whenever $(n, N) = 1$, and therefore we have that $f \in S_k^{\text{old}}(N)$ by the main lemma. This means that if $f \in S_k^{\text{new}}(N)$ is as above with $a_f(1) = 0$, then we have $f = 0$, since it lies in the orthogonal complement of $S_k^{\text{new}}(N)$ as well. Hence, if $f \neq 0$, then $a_f(1) \neq 0$ and we may normalize f so that $a_f(1) = 1$. In this case, let m be any positive integer and consider the cusp form

$$g_m = T_N(m) f - \frac{1}{m^{(k-1)/2}} a_f(m) f.$$

This is an element of $S_k^{\text{new}}(N)$, since all Hecke operators preserve $S_k^{\text{new}}(N)$. It is also an eigenform for all $T_N(n)$ such that $(n, N) = 1$, since Hecke operators commute. Finally, we have that

$$a_{g_m}(1) = p_f(1, m) - \frac{1}{m^{(k-1)/2}} a_f(m) a_f(1) = 0,$$

from which it follows that $g_m = 0$. All in all, we have that f is an eigenform for *any* Hecke operator $T_N(n)$, with eigenvalue

$$\lambda_f(n) = \frac{1}{n^{(k-1)/2}} a_f(n).$$

A cusp form $f \in S_k(N)$ which is an eigenform for all Hecke operators $T_N(n)$ is simply called an *eigenform*. If in addition $a_f(1) = 1$, then f is called a *normalized eigenform*. A normalized eigenform in the space of newforms at level N is called a *newform*.

If $f, g \in S_k^{\text{new}}(N)$ are nonzero eigenforms with $\lambda_f(n) = \lambda_g(n)$ whenever $(n, N) = 1$, then $g = \lambda f$ for some complex number λ. To see this, it is enough to show that if both f and g are normalized, then they are equal. Indeed, then $h = f - g \in S_k^{\text{new}}(N)$ with $a_h(1) = 0$, whence $h = 0$ by the main lemma. This is referred to as the *multiplicity one principle*. In other words, if $\{\lambda(n) : n \in \mathbb{Z}^+\}$ is a sequence of complex numbers which are Hecke eigenvalues for a nonzero eigenform f, then the eigenspace

$$\{g \in S_k^{\text{new}}(N) : T_N(n)g = \lambda(n)g, \;\; n \in \mathbb{Z}^+\}$$

is one-dimensional.

The multiplicity one principle have several consequences. Recall that $S_k^{\text{new}}(N)$ has a basis of eigenforms, each with their own sequence of eigenvalues. Upon normalizing, this basis is unique, since the eigenspace to each sequence of eigenvalues is one-dimensional. Hence we may speak of *the* set of newforms at level N, which we denote by $H_k^*(N)$. Another consequence is that if H is a linear operator on $S_k^{\text{new}}(N)$ which commutes with all Hecke operators $T_N(n)$ when $(n, N) = 1$ and f is a newform, then $T_N(n)Hf = \lambda_f(n)Hf$, and so $Hf \in S_k^{\text{new}}(N)$ is an eigenform with the same eigenvalues as f. Therefore, $Hf = \lambda f$ for some constant λ. In short, the newform f is an eigenform with respect to H as well.

There is also a *Strong Multiplicity one* principle, which says the following: suppose $M, M'|N$ and $f \in S_k^{\text{new}}(M)$ and $g \in S_k^{\text{new}}(M')$, f and g both nonzero. Suppose also that $\lambda_f(n) = \lambda_g(n)$ whenever $(n, N) = 1$. Then $M = M'$ and $g = \lambda f$ for some complex number λ. Strong multiplicity one plays a role in the semi-orthogonal decomposirion (3.36) below. Its proof is well beyond the scope of this report; see e.g. [Miy06, Theorem 4.6.19] for details.

Recall that the eigenvalues $\lambda_f(n)$ for which $(n, N) = 1$ are real when f is a newform. By the above, this holds for all $\lambda_f(n)$. Indeed, the function $g(z) = \overline{f(-\bar{z})}$ (which acts by conjugating the Fourier coefficients of f) is also a newform (see [CS17, Proposition 10.3.14 a)] for the proof of weak modularity), whose eigenvalues agree with those of f whenever $(n, N) = 1$. By the multiplicity one principle they also agree in general, and the claim follows.

In the next chapter, we will be interested in finding an orthogonal basis for $S_k(N)$. Since the newforms of level N constitute an orthogonal basis of the space of newforms, we are left to consider the space of oldforms. If $N = 1$ there is nothing to prove, and if $N = p$ is prime, then the space of oldforms at level N are spanned by forms of the shape $f(z)$ and $f(pz)$, where $f \in S_k(1)$. Thus, for a general level N and $p|N$ a prime, assuming that the spaces $S_k(N/p)$ are spanned by forms of the shape $f(z)$ and $f(pz)$ where f is a newform of a level dividing N/p, the decomposition

$$S_k(N) = S_k^{\text{new}}(N) \oplus \sum_{p|N} i_p(S_k(N/p)^2)$$

shows by means of induction that $S_k(N)$ is spanned by the set

$$\bigcup_{ML=N} \{f(\ell z) : f \in H_k^*(M) \ \text{ where } \ \ell | L\}.$$

Linear independence follow from the Strong Multiplicity one principle, and so this set in fact constitute a basis for $S_k(N)$. This is [DS05, Theorem 5.8.3].

This basis displays a semi-orthogonality which we now describe. Suppose f, g are newforms at level M, M', respectively, where $M, M' | N$. By the Strong Multiplicity one principle, there is at least one index n with $(n, N) = 1$ such that $\lambda_f(n) \neq \lambda_g(n)$. We now have

$$\lambda_f(n)\langle f, g\rangle_N = \langle T_N(n)f, g\rangle_N = \langle f, T_N(n)g\rangle_N = \lambda_g(n)\langle f, g\rangle_N.$$

Since $\lambda_f(n) \neq \lambda_g(n)$, we conclude that $\langle f, g\rangle_N = 0$. This extends to when we consider forms $f(\ell z), g(\ell' z)$, where f and g are as before and $\ell | N/M$, $\ell' | N/M'$, since $T_N(n)$ commute with $[\alpha_\ell]_k$ and hence with ι_ℓ. If f and g are distinct newforms of the same level, we draw the same conclusions about $f(\ell z)$ and $g(\ell' z)$ since $\langle f, g\rangle_N = 0$ by construction. All in all, we obtain that

$$S_k(N) = \bigoplus_{LM=N} \bigoplus_{f \in H_k^*(M)} S(L; f) \tag{3.36}$$

where

$$S(L; f) = \operatorname{span}\{f_{|\ell} : \ell | L\} \text{ and } f_{|\ell}(z) = \ell^{k/2} f(\ell z).$$

The reason for re-scaling $f(\ell z)$ by the factor $\ell^{k/2}$ is purely technical. We note that $f_{|\ell} = f[\alpha_\ell]_k$ with the alternative convention (3.5) and

$$\alpha_\ell = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

3.6 The L-function of a newform

Let $f \in H_k^*(N)$. The *L-function attached to f* is defined as

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \quad \Re(s) > 1. \tag{3.37}$$

The Hecke eigenvalues grow slowly in n; indeed, we have the bound $|\lambda_f(n)| \ll \tau(n)$. This is not a trivial result, and was first proven by Pierre Deligne in [Del74] (our formulation is [ILS00, eq. 2.4]). Now, since $\tau(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$ (see [HW08, p. 343]), the function $L(s, f)$ converges absolutely for $\Re(s) > 1$. By multiplicativity of $\lambda_f(n)$ and the fundamental theorem of arithmetic, one has by standard arguments that

$$L(s, f) = \prod_p L_p(s, f), \quad \Re(s) > 1, \tag{3.38}$$

where

$$L_p(s, f) := \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^\alpha)}{p^{\alpha s}}$$

is the *local factor at p*. If $p|N$, we have $\lambda_f(p^\alpha) = \lambda_f(p)^\alpha$ by (3.31), from which it follows that

$$L_p(s,f) = \left(1 - \frac{\lambda_f(p)}{p^s}\right)^{-1}$$

by geometric summation. If $p \nmid N$, we have

$$\begin{aligned} L_p(s,f) &= 1 + \frac{\lambda_f(p)}{p^s} + \sum_{\alpha=2}^{\infty} \frac{\lambda_f(p^\alpha)}{p^{\alpha s}} \\ &= 1 + \frac{\lambda_f(p)}{p^s} + \sum_{\alpha=2}^{\infty} \frac{\lambda_f(p)\lambda_f(p^{\alpha-1}) - \lambda_f(p^{\alpha-2})}{p^{\alpha s}} \\ &= 1 + \frac{\lambda_f(p)}{p^s} + \frac{\lambda_f(p)}{p^s} \sum_{\alpha=1}^{\infty} \frac{\lambda_f(p^\alpha)}{p^{\alpha s}} - \frac{1}{p^{2s}} \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^\alpha)}{p^{\alpha s}} \\ &= 1 + \frac{\lambda_f(p) L_p(s,f)}{p^s} - \frac{L_p(s,f)}{p^{2s}}, \end{aligned}$$

by (3.31). Solving for $L_p(s,f)$, we obtain

$$L_p(s,f) = \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

when $p \nmid N$. All in all, we can write

$$L_p(f,s) = \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}, \tag{3.39}$$

where χ_0 is the trivial character modulo N.

The *local factor at infinity* is defined by

$$L_\infty(s,f) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right).$$

By the duplication formula for the Gamma function [Dav00, p. 73], we have

$$L_\infty(s,f) = \left(\frac{2^k}{8\pi}\right)^{1/2} \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right). \tag{3.40}$$

The *competed L-function*

$$\Lambda(s,f) := L_\infty(s,f) L(s,f)$$

has an analytic continuation to $\mathbb{C}$, which satisfies the functional equation

$$\Lambda(s,f) = \varepsilon_f \Lambda(1-s,f) \tag{3.41}$$

(see [ILS00, Section 3]). In particular, there is no pole at $s = 1$. Here, $\varepsilon_f = i^k \eta_f$, where η_f is the eigenvalue of the *Fricke involution* W_N, where

$$W_N f(z) := \frac{1}{N^{k/2} z^k} f\left(\frac{-1}{Nz}\right).$$

With the alternative convention (3.5), we have $W_N = [\alpha_N]_k$, where

$$\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

For details about how η_f appears in the functional equation, see for instance [Iwa97, Theorem 7.2 and eq. 7.17]. The fact that W_N is an involution follows from $\alpha_N^2 = -NI$. Also, f is an eigenfunction to W_N by the multiplicity one principle, since W_N commutes with every Hecke operator $T_N(n)$, where $(n, N) = 1$ (see [Iwa97, Theorem 6.27 and Section 6.8]). Since W_N is an involution, we must have $\eta_f = \pm 1$. Moreover, η_f can be computed in terms of the Hecke eigenvalue $\lambda_f(N)$ (see e.g. [Iwa97, Theorem 6.29]), leading to the expression

$$\varepsilon_f = i^k \eta_f = i^k \mu(N) \lambda_f(N) N^{1/2} = \pm 1 \tag{3.42}$$

when N is squarefree (see [ILS00, eq. 2.23]). The number ε_f is called the *root number* of f. An alternative formulation of the functional equation is

$$L(s, f) = \varepsilon_f X_L(s) L(1-s, f), \tag{3.43}$$

where

$$X_L(s) := \frac{L_\infty(1-s, f)}{L_\infty(s, f)}. \tag{3.44}$$

The Gamma function has its poles located at the nonpositive integers. This implies that $L(s, f)$ never has a pole at $s = 1$, and that X_L has its poles located at positive half-integers of the form $n + (k+1)/2$ where n is a nonnegative integer.

The local factor at a finite prime p factors as

$$L_p(s, f) = \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \tag{3.45}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are the *local coefficients at* p. They satisfy

$$\begin{cases} \alpha_f(p) + \beta_f(p) = \lambda_f(p), \\ \alpha_f(p)\beta_f(p) = \chi_0(p). \end{cases} \tag{3.46}$$

In the case when $p|N$, we adopt the convention $\alpha_f(p) = \lambda_f(p)$ and $\beta_f(p) = 0$. In the case when $p \nmid N$, we have (see [ILS00, p. 82]) that $\alpha_f(p) = \overline{\beta_f(p)}$ (This is a deep result known as the *Ramanujan conjecture*, and was proven by Pierre Deligne in [Del74]). Consequently, we have

$$|\alpha_f(p)| = |\beta_f(p)| = 1 \tag{3.47}$$

In any case, with the convention that $0^0 = 1$ we have

$$\lambda_f(p^m) = \sum_{\ell=0}^{m} \alpha_f(p)^\ell \beta_f(p)^{m-\ell} \tag{3.48}$$

for any nonnegative integer m. This can be seen by expanding the right hand side of (3.45) as two geometric series, and identify terms corresponding to the same prime power p^m on both sides.

Recall that the Generalised Riemann Hypothesis (GRH) states that all zeroes of an L-function lies on the critical line. A useful consequence in our setting is [IK04, Theorem 5.17], which says that the logarithmic derivative $L'(s,f)/L(s,f)$ grows slowly in s as $\Im(z)$ tends to infinity, whenever $1/2 < \Re(s) \leq 5/4$ is fixed. The growth rate is of the order of $\log s$.

3.7 Auxiliary L-functions

Let $f \in H_k^*(N)$. For technical reasons we introduce the functions

$$L(s, f \otimes f) := \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)}{n^s}, \quad \Re(s) > 1,$$
$$Z(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}, \quad \Re(s) > 1.$$

As with $L(s,f)$, these functions have meromorphic continuations to all of $\mathbb{C}$, with possibly a simple pole at $s = 1$. When $\Re(s) > 1$, then their Euler products are

$$L(s, f \otimes f) = \prod_p L_p(s, f \otimes f),$$
$$Z(s, f) = \prod_p Z_p(s, f)$$

where

$$L_p(s, f \otimes f) := \sum_{\alpha=0}^{\infty} \frac{\lambda_f^2(p^\alpha)}{p^{\alpha s}},$$
$$Z_p(s, f) := \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^{2\alpha})}{p^{\alpha s}}.$$

If $p|N$, then $\lambda_f^2(p^\alpha) = \lambda_f(p^{2\alpha}) = \lambda_f(p)^{2\alpha}$ and

$$L_p(s, f \otimes f) = Z_p(s, f) = \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p)^{2\alpha}}{p^{\alpha s}} = \left(1 - \frac{\lambda_f^2(p)}{p^s}\right)^{-1} = \left(1 - \frac{1}{p^{s+1}}\right)^{-1}, \quad (3.49)$$

where we used (3.32) in the last step. If $p \nmid N$, then

$$L_p(s, f \otimes f) = \sum_{\alpha=0}^{\infty} \frac{\lambda_f^2(p^\alpha)}{p^{\alpha s}} = \sum_{\alpha=0}^{\infty} \sum_{\gamma=0}^{\alpha} \frac{\lambda_f(p^{2\alpha-2\gamma})}{p^{\alpha s}}$$
$$= \sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\infty} \frac{\lambda_f(p^{2\beta})}{p^{\beta s+\gamma s}} = Z_p(s, f) \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (3.50)$$

where we used (3.30) in the second step, and substituted $\alpha - \gamma = \beta$ in the third step. Hence,

$$L(s, f \otimes f) = Z(s, f)\frac{\zeta(s)}{\zeta_N(s)} \tag{3.51}$$

where

$$\zeta_N(s) := \sum_{n|N^\infty} \frac{1}{n^s} = \prod_{p|N} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{3.52}$$

is the local zeta function (at N). It turns out that $L(s, f \otimes f)$ has a simple pole at $s = 1$, whose residue will be relevant in the proof of Lemma 4.7 below, and further computations. This means that $Z(1, f)$ is a finite number, although the series representation of $Z(s, f)$ only converges conditionally when $s = 1$. The number $Z(1, f)$ will be present in some of the sums over Hecke eigenvalues in the next chapter.

Closely related to $Z(s, f)$, we define the *symmetric square L-function*

$$L(s, \mathrm{sym}^2(f)) := \frac{\zeta(2s)}{\zeta_N(2s)} Z(s, f).$$

This L-function is not central to the report, but in Chapter 4 we will rely on estimations which follow from assuming the GRH for $L(s, \mathrm{sym}^2(f))$ on a couple of occasions.

We are interested in finding closed form expressions for $L_p(s, f \otimes f)$ and $Z_p(s, f)$. By (3.49) and (3.50), it is enough to do so for $Z_p(s, f)$ when $p \nmid N$. Then, we have

$$Z_p(s, f) = 1 + \sum_{\alpha=1}^{\infty} \frac{\lambda_f(p^{2\alpha})}{p^{\alpha s}} = 1 + \frac{1}{p^s} \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^{2(\alpha+1)})}{p^{\alpha s}}.$$

By (3.31), the last series is

$$\sum_{\alpha=0}^{\infty} \frac{\lambda_f(p)\lambda_f(p^{2\alpha+1}) - \lambda_f(p^{2\alpha})}{p^{\alpha s}} = \lambda_f(p) S - Z_p(s, f),$$

where

$$\begin{aligned} S &= \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^{2\alpha+1})}{p^{\alpha s}} \\ &= \lambda_f(p) + \sum_{\alpha=1}^{\infty} \frac{\lambda_f(p)\lambda_f(p^{2\alpha}) - \lambda_f(p^{2\alpha-1})}{p^{\alpha s}} \\ &= \lambda_f(p) Z_p(s, f) - \frac{1}{p^s} S. \end{aligned}$$

From this we deduce

$$S = \frac{\lambda_f(p)}{1 + p^{-s}} Z_p(s, f)$$

and

$$Z_p(s, f) = 1 + \frac{\lambda_f(p)^2}{p^s + 1} Z_p(s, f) - \frac{1}{p^s} Z_p(s, f).$$

Solving for $Z_p(s,f)$ yields

$$Z_p(s,f) = \left(1 - \frac{\lambda_f^2(p)}{p^s+1} + \frac{1}{p^s}\right)^{-1} = \left(\left(1+\frac{1}{p^s}\right)\left(1 - p^s\left(\frac{\lambda_f(p)}{p^s+1}\right)^2\right)\right)^{-1}. \quad (3.53)$$

4

The Petersson trace formula and sums of Hecke eigenvalues

The overarching goal of this chapter is to prepare the ground for the computations of the 1-level density in Chapter 5. The main focus is the study of various sums, the first of which is $\Delta_{k,N}(m,n)$ (defined in (4.10)). The main tool is the *Petersson trace formula*, which we prove in Section 4.2. Before we do so, we need two prerequisites, namely *Kloosterman sums* and *Bessel functions*, which are introduced in Section 4.1. After we have proven the Petersson trace formula, we seek to obtain an orthogonal basis for $S_k(N)$ in Section 4.3. This will allow us to evaluate $\Delta_{k,N}(m,n)$, which serves as the starting point of our analysis. Lastly, we will also study the pure sum of Hecke eigenvalues $\Delta^*_{k,N}(n)$ (defined in (4.42)). Among other things, this yields an asymptotical expression for the size $|H_k^*(N)|$ of the family in consideration. The proof of the Petersson trace formula follow the contents in [IK04, Section 14.2]. The results in the last three sections are from [ILS00, Ch. 2], and we follow the exposition there quite closely.

4.1 Kloosterman sums and Bessel functions

While Kloosterman sums and Bessel functions have a rich theory in and of themselves, here we will mostly be interested in the results relevant to the applications in later sections. Most of the material on Kloosterman sums can be found in [IK04, Section 1.4].

Let $m, n, c \in \mathbb{Z}^+$. The *Kloosterman sum* $S(m,n;c)$ is defined by

$$S(m,n;c) ;= \sideset{}{^*}\sum_{x \pmod c} e\left(\frac{mx + nx^{-1}}{c}\right), \tag{4.1}$$

where the starred sum means that we sum over invertible congruence classes modulo c, and x^{-1} is the inverse of x modulo c. If x runs through the invertible elements modulo c, then so does x^{-1}. Hence, replacing x by x^{-1} in each term of (4.1) gives

$$S(m,n;c) = S(n,m;c). \tag{4.2}$$

Also, if x runs through the invertible elements modulo c and ℓ is relatively prime to c, then ℓx also runs through the invertible elements, with $\ell^{-1}x^{-1}$ being the inverse.

From this observation we get

$$S(m\ell,n;c)=\sum_{x \pmod c}^{*} e\left(\frac{m\ell x+nx^{-1}}{c}\right)=\sum_{x \pmod c}^{*} e\left(\frac{m\ell x+n\ell\ell^{-1}x^{-1}}{c}\right)$$
$$=\sum_{x \pmod c}^{*} e\left(\frac{mx+n\ell x^{-1}}{c}\right)=S(m,n\ell;c), \quad (4.3)$$

where we made the change of variables $\ell x \mapsto x$ in the third step.

A third property of Kloosterman sums displays a multiplicative behaviour in the modulus c. To state it, suppose that $(c,d)=1$. Then, we have

$$S(md^{-1},nd^{-1};c)S(mc^{-1},nc^{-1};d)$$
$$=\sum_{x \pmod c}^{*} e\left(\frac{md^{-1}x+nd^{-1}x^{-1}}{c}\right)\sum_{y \pmod d}^{*} e\left(\frac{mc^{-1}y+nc^{-1}y^{-1}}{d}\right)$$
$$=\sum_{x,y}^{*} e\left(\frac{m(dd^{-1}x+cc^{-1}y)+n(dd^{-1}x^{-1}+cc^{-1}y^{-1})}{cd}\right).$$

We emphasize that x^{-1},d^{-1} denote inverses modulo c and y^{-1},c^{-1} denote inverses modulo d. From an elementary variant of the Chinese remainder theorem (see e.g. [Ros14, Theorem 4.13]), we have that as x and y runs through the invertible elements modulo c and d, respectively, then $a=dd^{-1}x+cc^{-1}y$ runs through the invertible elements modulo cd. We claim that $b=dd^{-1}x^{-1}+cc^{-1}y^{-1}$ is the inverse to a modulo cd. To see this, note that

$$a\equiv x \pmod c,$$
$$a\equiv y \pmod d,$$
$$b\equiv x^{-1} \pmod c,$$
$$b\equiv y^{-1} \pmod d.$$

Hence $ab\equiv 1 \pmod c, ab\equiv 1 \pmod d$ and so $ab\equiv 1 \pmod{cd}$. Thus

$$S(md^{-1},nd^{-1};c)S(mc^{-1},nc^{-1};d)=S(m,n;cd). \quad (4.4)$$

An important result, due to André Weil, is the bound (see [IK04, Corollary 11.12])

$$|S(m,n;c)|\leq (m,n,c)^{1/2}c^{1/2}\tau(c), \quad (4.5)$$

where τ is the number of divisors function. Using this bound and the properties (4.2), (4.3) and (4.4), Iwaniec, Luo and Sarnak obtain the slightly stronger bound

$$|S(m,n;c)|\leq (m,n,c)\min\left(\frac{c}{(m,c)},\frac{c}{(n,c)}\right)^{1/2}\tau(c) \quad (4.6)$$

(see [ILS00, eq. 2.13]).

Now we turn to the second of the prerequisites of this chapter, namely Bessel functions. The *Bessel function of order* ν is defined by

$$J_\nu(z) := \frac{z^\nu}{2^\nu}\sum_{k=0}^{\infty}\frac{z^{2k}}{2^{2k}\Gamma(\nu+k+1)}, \quad |\arg z| < \pi.$$

Here, ν may be a positive real number, or an integer. Bessel functions admit a representation (see [GR07, eq. 8.412.2])

$$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left(t+\frac{z^2}{4t}\right)dt. \tag{4.7}$$

The notation means that the integral is over the contour C, which goes from $-\infty$ to $-\varepsilon$, moves around the origin in a positively oriented circle with radius ε, and then goes back to $-\infty$.

There exist several upper bounds for $J_\nu(z)$ in various regions. The one we shall be interested in is (see [ILS00, eq. 2.11‴])

$$J_{k-1}(x) \ll 2^{-k}x, \tag{4.8}$$

which is valid for integers $k \geq 2$ and real numbers $0 < x \leq k/3$.

4.2 The Petersson trace formula

Let $f \in S_k(N)$ have Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi i n z}.$$

We define

$$\Delta_{k,N}(m,n) := \sum_{f\in\mathcal{B}_k(N)} \overline{\Psi_f(m)}\Psi_f(n), \tag{4.9}$$

where

$$\Psi_f(n) := \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\right)^{1/2}\frac{a_f(n)}{||f||}$$

is the *normalized Fourier coefficient*, $||f||^2 = \langle f, f\rangle_N$, and $\mathcal{B}_k(N)$ is any orthogonal basis of $S_k(N)$. In the proof of Petersson's trace formula, it will become clear that the definition of $\Delta_{k,N}(m,n)$ is independent of the choice of basis. In our case, all the basis elements will come from a newform at some level $M|N$. We may then reformulate (4.9) as follows. We define the *harmonic weight* (or *Petersson weight*) $\omega_f(N)$ as

$$\omega_f(N) := \frac{\Gamma(k-1)}{(4\pi)^{k-1}||f||^2}.$$

Then, since $a_f(n) = \lambda_f(n)n^{(k-1)/2}$ we have

$$\Delta_{k,N}(m,n) = \sum_{f\in\mathcal{B}_k(N)} \omega_f(N)\lambda_f(m)\lambda_f(n), \tag{4.10}$$

where we also used that Hecke eigenvalues of newforms are real.

We now state this section's central result.

Proposition 4.1 (Petersson's trace formula)**.** *For $m, n \geq 1$ and $k \geq 2$ and even, we have*

$$\Delta_{k,N}(m,n) = \delta(m,n) + 2\pi i^k \sum_{\substack{c \equiv 0 \pmod N \\ c>0}} c^{-1} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{4.11}$$

where $\delta(m,n)$ is the Kronecker delta symbol, $S(m,n;c)$ is the Kloosterman sum and $J_{k-1}(x)$ is the Bessel function of order $k-1$.

Showing the Petersson trace formula will require some steps, but the overarching strategy is simple. Consider the Poincaré series $P_m(z)$, introduced in Section 3.2.1. Since $P_m(z)$ is a cusp form, for any orthogonal basis $\mathcal{B}_k(N)$ of $S_k(N)$ we have

$$P_m(z) = \sum_{f \in \mathcal{B}_k(N)} \frac{\langle P_m, f\rangle_N}{||f||^2} f(z). \tag{4.12}$$

The equality (4.11) follows from comparing Fourier coefficients on the left and the right hand sides of (4.12).

Remark 4.2. With the tools we have introduced this strategy only works for $k \geq 4$, since otherwise $P_m(z)$ does not converge absolutely. This makes weak modularity a more intricate issue. The proof we present therefore excludes the case when $k = 2$, although the trace formula remains valid also then. In this case, one needs to modify the definition of $P_m(z)$ to ensure weak modularity. This involves something called *Hecke's trick*, and we will not delve into it here.

First we compute the Fourier coefficients on the left hand side of (4.12) directly. This is the content of the following result.

Lemma 4.3 ([IK04, Lemma 14.2])**.** *For $m \geq 1$, the Poincaré series $P_m(z)$ has the Fourier expansion*

$$P_m(z) = \sum_{n=1}^{\infty} p(m,n) e^{2\pi i n z} \tag{4.13}$$

where

$$p(m,n) = \left(\frac{n}{m}\right)^{(k-1)/2} \left(\delta(m,n) + 2\pi i^k \sum_{\substack{c \equiv 0 \pmod N \\ c>0}} c^{-1} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right).$$

Proof. We start with the definition (3.13). The first step is to find a set of coset representatives of $P_+\backslash\Gamma_0(N)$. Remember that, in this context, we regard the elements of $\Gamma_0(N)$ as fractional linear transformations rather than matrices. That is, we identify matrices with opposite sign. Hence, we may assume that the equivalence class $P_+\alpha$ is represented by a fractional linear transformation $z \mapsto \alpha(z)$, given by

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.14}$$

where $c \geq 0$. If $c = 0$, then $\alpha \in P_+$ and $P_+\alpha = P_+$. Thus the case $c = 0$ is covered by choosing the coset representative as the identity transformation I. Now suppose that $c > 0$ and $N|c$. Let D be a complete residue system modulo c. We claim that any transformation

$$\alpha' = \begin{pmatrix} a' & b' \\ c & d' \end{pmatrix} \in \Gamma_0(N)$$

lies in the same coset modulo P_+ as a transformation of the form

$$\alpha\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \tag{4.15}$$

where $d \in D$ and n is an integer. Moreover, we claim that the choice of $\alpha\beta$ is unique up to the choice of the integers a and b, which may be chosen arbitrarily as long as $ad - bc = 1$. Explicitly, this means that we can find integers m, n and a, b, d such that $ad - bc = 1$ and

$$\begin{pmatrix} a' & b' \\ c & d' \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & an+b \\ c & cn+d \end{pmatrix}. \tag{4.16}$$

In particular, we must find d and n such that $d' = cn + d$. But this can be done; just pick d as the representative of the congruence class which d' belongs to, and an appropriate n so that the equation is satisfied. Thus we have reduced (4.16) to the equation

$$\begin{pmatrix} a' & b' \\ c & d' \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & an+b \\ c & d' \end{pmatrix}, \tag{4.17}$$

which must be solved in a, b and m. This can also be done, by picking *any* a, b such that $ad - bc = 1$ (which is possible since $(c, d) = 1$). Then, the last transformation on the right hand side of (4.17) is an element in $\Gamma_0(N)$. But any two transformations in $\Gamma_0(N)$ with the same bottom row belongs to the same left coset modulo P_+. Hence, we can find an m so that (4.17) is satisfied. In total, we have shown that $P_+\alpha' = P_+\alpha\beta$ for some $\alpha\beta$ of the form (4.15).

To show uniqueness, suppose $P_+\alpha\beta = P_+\alpha'\beta'$, where

$$\alpha\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha'\beta' = \begin{pmatrix} a' & b' \\ c & d' \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

both are transformations of the form (4.15), where $d, d' \in D$. We need to show that $d = d'$ and $m = n$. To see this, write out $\alpha'\beta' = \gamma\alpha\beta$ for some $\gamma \in P_+$, and observe that the equality between the lower right entries reads $d' = c(m-n) + d$. Uniqueness now follows, since if $d \equiv d' \pmod{c}$, then $d = d'$ and $m = n$ (remember that $c > 0$).

All in all, we obtain that any fractional linear transformation $\alpha' \in \Gamma_0(N)$ with fixed nonzero lower left entry is equivalent to precisely one transformation of the form (4.15). Thus a set of coset representatives of $P_+\backslash\Gamma_0(N)$ is

$$\{I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : c > 0, c \equiv 0 \pmod{N}, (c, d) = 1, n \in \mathbb{Z} \right\}. \tag{4.18}$$

Now, suppose $\gamma = \alpha\beta$ is of the form in the right set in (4.18). That is, α is as in (4.14) and $\beta \in P_+$. Then,

$$\gamma(z) = \alpha(z+n) = \frac{a(z+n)+b}{c(z+n)+d} = \frac{a}{c} - \frac{1}{c(c(z+n)+d)}.$$

Moreover, we have

$$j(\gamma,z) = j(\alpha,\beta(z))j(\beta,z) = j(\alpha,z+n) = \frac{1}{c(z+n)+d},$$

by Proposition 3.1. The Poincaré series can therefore be explicitly written as

$$\begin{aligned} P_m(z) &= \sum_{\gamma\in P_+\backslash\Gamma_0(N)} \frac{e(m\gamma(z))}{j(\gamma,z)^k} \\ &= e(mz) + \sum_{\substack{c\equiv 0 \pmod N \\ c>0}} \sideset{}{^*}\sum_{d \pmod c} \sum_{n\in\mathbb{Z}} \frac{1}{(c(z+n)+d)^k} e\left(m\left(\frac{a}{c} - \frac{1}{c(c(z+n)+d)}\right)\right). \end{aligned} \tag{4.19}$$

The term $e(mz)$ will give rise to the Kronecker delta term in Lemma 4.3. Next, we apply Poisson summation to the sum over $n \in \mathbb{Z}$. The Fourier transform of the function

$$f(n) = \frac{1}{(c(z+n)+d)^k} e\left(m\left(\frac{a}{c} - \frac{1}{c(c(z+n)+d)}\right)\right)$$

evaluated at the integer n is equal to

$$\widehat{f}(n) = \int_{-\infty}^{\infty} f(v)e^{-2\pi inz}dv = \int_{-\infty}^{\infty} \frac{1}{(c(z+v)+d)^k} e\left(\frac{am}{c} - \frac{m}{c(c(z+v)+d)} - nv\right) dv.$$

Hence, the inner sum in (4.19) is equal to

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{(c(z+v)+d)^k} e\left(\frac{am}{c} - \frac{m}{c(c(z+v)+d)} - nv\right) dv \\ &\qquad = \sum_{n\in\mathbb{Z}} \left[e\left(\frac{am}{c} + \frac{dn}{c}\right) \int_{-\infty+iy}^{\infty+iy} \frac{1}{(cv)^k} e\left(-\frac{m}{c^2v} - nv\right) dv \right] e(nz). \end{aligned}$$

Here, we wrote $z = x + iy$ and used the substitution $c(z+v)+d \mapsto cv$. The next step is to analyze the integral

$$\int_{-\infty+iy}^{\infty+iy} \frac{1}{(cv)^k} e\left(-\frac{m}{c^2v} - nv\right) dv. \tag{4.20}$$

We claim that it vanishes for $n \le 0$. To see this, note that we are allowed to integrate along any line parallel to the real axis by Cauchy's theorem and the decay of v^{-k} as $|\Re(v)| \to \infty$. When $n \le 0$, the real part of $2\pi i(-m/c^2v - nv)$ works out to be negative, whence the integral is of the order of

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^{k/2}} dx \ll \int_{-\infty}^{\infty} \frac{1}{x^2+y^2} dx = \frac{\pi}{y},$$

which goes to 0 as $y \to \infty$. Hence our claim follows.

We now consider (4.20) when $n \ge 1$. We claim that it is equal to

$$2\pi i^{-k} c^{-1} \left(\frac{n}{m}\right)^{(k-1)/2} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{4.21}$$

A direct evaluation of the representation (4.7) yields that (4.21) is equal to

$$(2\pi in)^{k-1}\int_{-\infty}^{(0+)}\frac{1}{(ct)^k}\exp\left(t-\frac{4\pi^2mn}{c^2t}\right)dt. \tag{4.22}$$

We want to transform (4.20) into this expression. We begin by moving the integral downwards to the contour

$$(-\infty,-\delta]\cup\{\delta e^{-i\theta}:-\pi\le\theta\le 0\}\cup[\delta,\infty). \tag{4.23}$$

The contour is oriented in the right direction, and $\delta>0$ is thought of as small. Now we make the substitution $-2\pi inv=t$, which gives us the same integrand as in (4.22). The difference is that we integrate along the contour

$$(-i\infty,-i\delta]\cup\{\delta e^{i\theta}:-\pi/2\le\theta\le\pi/2\}\cup[i\delta,i\infty) \tag{4.24}$$

instead, which is oriented upwards, so that the circular arc is a part of a positively oriented circle (the radius δ need not be the same as in (4.23)). We need to argue why we may deform (4.24) into the one in (4.22). For this, consider the integral over the positively oriented curve

$$[-R,-\delta]\cup\{\delta e^{-i\theta}:-\pi\le\theta\le-\pi/2\}\cup[i\delta,iR]\cup\{Re^{i\theta}:\pi/2\le\theta\le\pi\}.$$

Here, $R>0$ is thought of as large. By Cauchy's integral theorem, the integral over this curve is 0. As the Section $\{Re^{i\theta}:\pi/2\le\theta\le\pi\}$ lies where $\Re(t)\le 0$, for R large enough the real part of the exponent

$$\Re\left(t-\frac{4\pi^2mn}{c^2t}\right)=x-\frac{4\pi^2mnx}{c^2(x^2+y^2)}=x-\frac{4\pi^2mnx}{c^2R^2}\le 0,$$

where $t=x+iy$ ($R\ge 2\pi\sqrt{mn}/c$ should suffice). This means that the integral over the large circular arc is of order $O(1/R)$, and hence goes to 0 as $R\to\infty$. A similar argument in the lower half plane shows that we indeed are allowed to deform the contour, and finishes the claim that (4.20) is equal to (4.21).

We now reorder the sums in (4.19), and write out what we have learned. Thus, we are at

$$\begin{aligned}P_m(z)=e(mz)+\sum_{n=1}^{\infty}\Bigg[\left(\frac{n}{m}\right)^{(k-1)/2}&2\pi i^k\\ \times\sum_{\substack{c\equiv 0\pmod N\\ c>0}}c^{-1}&\sum_{d\pmod c}^{*}e\left(\frac{am}{c}+\frac{dn}{c}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Bigg]e(nz).\end{aligned}$$

Since $ad-bc=1$, a is the inverse of d modulo c. Hence, the starred sum over d is equal to the Kloosterman sum $S(m,n;c)$. This finishes the proof. ☐

Remark 4.4. To show that $P_m(z)$ is a cusp form, one may compute the Fourier series of

$$P_m[\alpha]_k(z)=\sum_{\gamma\in P_+\backslash\Gamma_0(N)\alpha}\frac{e(m\gamma(z))}{j(\gamma,z)^k}$$

for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. The procedure is similar to the above; one finds a set of coset representatives analogous to (4.18), with the identity transformation I being present if and only if $\alpha \in \Gamma_0(N)$, and the requirement on the integers c, d be that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\alpha$$

(see [Iwa97, Proposition 2.7]). Then, one proceeds as above. The integral (4.20) is still present in this new setting, and its vanishing for $n \leq 0$ is what implies that $P_m(z)$ is a cusp form. See [Iwa97, Section 3.2] for details.

Next, we seek to compute the Fourier coefficients on the right hand side of (4.12). To do so we need to compute the Petersson inner product $\langle f, P_m \rangle_N$ for any cusp form f.

Lemma 4.5 ([IK04, Lemma 14.3])**.** *Let $f \in S_k(N)$ with*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}.$$

Then,

$$\langle f, P_m \rangle_N = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m).$$

Proof. Let F be a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$. The weight-k invariance of f and (3.1) implies that

$$\begin{aligned} \langle f, P_m \rangle_N &= \int_F f(z) \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \overline{j(\gamma, z)}^{-k} \overline{e(m\gamma(z))} \Im(z)^k d\mu(z) \\ &= \int_F \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} f(\gamma(z)) \overline{e(m\gamma(z))} \Im(\gamma(z))^k d\mu(z) \\ &= \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \int_{\gamma(F)} f(z) \overline{e(mz)} \Im(z)^k d\mu(z). \end{aligned}$$

Now we use an unfolding argument due to Rankin and Selberg. We choose a set of coset representatives $\{\alpha_j\}$ of $P_+ \backslash \Gamma_0(N)$. Then the integral turns into

$$\sum_j \int_{\alpha_j(F)} f(z) \overline{e(mz)} \Im(z)^k d\mu(z),$$

which can be viewed as an integration over the set

$$\bigcup_j \alpha_j(F)$$

(cf. the definition (3.18)). This set can be identified with a fundamental domain $\mathfrak{F}$ of the action of P_+ on $\mathcal{H}$, up to some boundary identifications. This follows since if $z \in \mathcal{H}$ and $\alpha \in \Gamma_0(N)$ map $z_0 \in F$ to z, then γ map $\alpha_j(z_0) \in \alpha_j(F)$ to z, where $\alpha = \gamma\alpha_j$ and $\gamma \in P_+$. We choose to integrate over the fundamental domain $\mathfrak{F} = [0,1) \times (0,\infty)$. This gives us

$$\begin{aligned}\langle f, P_m\rangle_N =& \int_0^1 \int_0^\infty f(z) e^{-2\pi i m\overline{z}} y^{k-2} dy dx \\ =& \sum_{n=1}^\infty a_f(n) \int_0^1 \int_0^\infty e^{2\pi i (nz - m\overline{z})} y^{k-2} dy dx \\ =& \sum_{n=1}^\infty a_f(n) \int_0^1 e^{2\pi i x(n-m)} dx \int_0^\infty y^{k-2} e^{-2\pi y(n+m)} dy \\ =& a_f(m) \int_0^\infty y^{k-2} e^{-4\pi m y} dy\end{aligned}$$

after inserting the Fourier expansion of f. The integral of $e^{2\pi i x(n-m)}$ is equal to $\delta(m,n)$ by simply evaluating the different cases. Finally, integrating by parts $k-2$ times followed by a direct evaluation yields

$$\int_0^\infty y^{k-2} e^{-4\pi m y} dy = \frac{\Gamma(k-1)}{(4\pi m)^{k-2}} \int_0^\infty e^{-4\pi m y} dy = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}},$$

finishing the proof. □

Now we can see that the n:th Fourier coefficient on the right hand side of (4.12) is

$$\frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{f \in \mathcal{B}_k(N)} \frac{\overline{a_f(m)} a_f(n)}{||f||^2},$$

and letting this be equal to the coefficient $p(m,n)$ from lemma 4.3, we obtain the Petersson trace formula after dividing by $(n/m)^{(k-1)/2}$ (recall that $a_f(n) = n^{(k-1)/2}\lambda_f(n)$).

Having established Proposition 4.1, the next step is to estimate the sum of Kloosterman sums and Bessel functions. The quality of results that one can obtain is often directly correlated to the quality of these estimates. We cite a result from [ILS00, Ch. 2].

Corollary 4.6 ([ILS00, Corollary 2.2])**.** *For any $m, n \geq 1$ it holds that*

$$\begin{aligned}\Delta_{k,N}(m,n) =& \delta(m,n) \\ &+ O\left(\frac{\tau(N)}{N k^{5/6}} \frac{(m,n,N)\tau_3((m,n))}{((m,N)+(n,N))^{1/2}} \left(\frac{mn}{\sqrt{mn} + kN} \right)^{1/2} \log 2mn \right).\end{aligned}$$

The interpretation is that when m, n are small, then $\Delta_{k,N}(m,n)$ is approximately equal to the Kronecker delta symbol. The proof involves the bound (4.6) as well as a crude bound (see [ILS00, p. 2.11])

$$J_{k-1}(x) \ll \min\left(1, \frac{x}{k}\right) k^{-1/3}.$$

4.3 Obtaining an orthogonal basis of $S_k(N)$

Recall the partially orthogonal decomposition (3.36). We wish to find an orthogonal basis for $S_k(N)$ in order to evaluate $\Delta_{k,N}(m,n)$. To do so, it is enough to find an orthogonal basis for the subspace $S(L;f)$, where $LM = N$ and $f \in H_k^*(M)$. The set of cusp forms $\{f_{|\ell} : \ell | L\}$ is a basis for $S(L;f)$, but not necessarily an orthogonal basis. We therefore wish to find an orthogonal basis by considering suitable linear combinations of these forms. In order to evaluate whether these new cusp forms are orthogonal or not, we need to investigate inner products of the shape $\langle f_{|\ell_1}, f_{|\ell_2}\rangle_N$.

Lemma 4.7 ([ILS00, Lemma 2.4])**.** *Let $N = LM$ be squarefree, $\ell_1, \ell_2 | L$ and $f \in H_k^*(M)$. Then*

$$\langle f_{|\ell_1}, f_{|\ell_2}\rangle_N = \lambda_f(\ell)\nu(\ell)^{-1}\sqrt{\ell}\langle f, f\rangle_N$$

where $\ell = \ell_1\ell_2/(\ell_1,\ell_2)^2$.

Proof. As on [ILS00, p. 72], we introduce the inner product

$$F(s) = \langle E(z,s)f(\ell_1 z), f(\ell_2 z)\rangle_N$$

where $E(z,s)$ is the non-holomorphic Eisenstein series defined in (3.12). Explicitly, we have

$$F(s) = \int_F \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \Im(\gamma(z))^s f(\ell_1 z)\overline{f(\ell_2 z)}\Im(z)^k d\mu(z),$$

where F is a fundamental domain of the action of $\Gamma_0(N)$ on $\mathcal{H}$. The expression

$$f(\ell_1 z)\overline{f(\ell_2 z)}\Im(z)^k$$

is invariant under the change of variables $z \mapsto \gamma(z)$, since $f \in H_k^*(M)$. Indeed, if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

then

$$\gamma_i = \begin{pmatrix} a & b\ell_i \\ c/\ell_i & d \end{pmatrix} \in \Gamma_0(M)$$

for $i = 1, 2$, and

$$f(\ell_i\gamma(z)) = f(\gamma_i(\ell_i z)) = j(\gamma_i, \ell_i z)^k f(\ell_i z) = j(\gamma, z)^k f(\ell_i z).$$

Combined with (3.1), we get

$$f(\ell_1\gamma(z))\overline{f(\ell_2\gamma(z))}\Im(\gamma(z))^k = f(\ell_1 z)\overline{f(\ell_2 z)}\Im(z)^k$$

as claimed. Hence, we have

$$F(s) = \int_F \sum_{\gamma \in P_+ \backslash \Gamma_0(N)} \Im(\gamma(z))^s f(\ell_1\gamma(z))\overline{f(\ell_2\gamma(z))}\Im(\gamma(z))^k d\mu(z).$$

Now, we use the same unfolding argument as in the proof of Lemma 4.5, which gives us

$$F(s) = \int_{y=0}^{\infty} \int_{x=0}^{1} y^{s+k-2} f(\ell_1 z)\overline{f(\ell_2 z)} dx dy.$$

The next step is to insert the Fourier expansions of $f(\ell_1 z)$ and $f(\ell_2 z)$. After reordering integration and summation, we have

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} a_f(n_1)\overline{a_f(n_2)} \int_{y=0}^{\infty} y^{s+k-2} e^{-2\pi y(n_1\ell_1+n_2\ell_2)} dy \int_{x=0}^{1} e^{2\pi i x(n_1\ell_1-n_2\ell_2)} dx$$
$$= \sum_{n_1\ell_1=n_2\ell_2} a_f(n_1)\overline{a_f(n_2)} \int_{y=0}^{\infty} y^{s+k-2} e^{-4\pi y n_1 \ell_1} dy.$$

After the variable substitution $u = 4\pi n_1 \ell_1 y$, we have

$$F(s) = (4\pi)^{1-s-k}\Gamma(s+k-1) \sum_{n_1\ell_1=n_2\ell_2} a_f(n_1)\overline{a_f(n_2)}(n_1\ell_1)^{1-s-k}. \tag{4.25}$$

Now, ℓ_1 and ℓ_2 are fixed. We can rewrite the integer $n_1\ell_1 = n_2\ell_2 = [\ell_1, \ell_2]n$, for some integer n. Since

$$[\ell_1, \ell_2] = \frac{\ell_1\ell_2}{(\ell_2, \ell_2)},$$

this means that we have $n_1 = \ell'' n, n_2 = \ell' n$, where $\ell' = \ell_1/(\ell_1, \ell_2)$ and $\ell'' = \ell_2/(\ell_1, \ell_2)$. Thus the sum in (4.25) translates into a sum over n, which is equal to

$$(\ell_1\ell_2)^{(1-k)/2}[\ell_1, \ell_2]^{-s} \sum_{n=1}^{\infty} \frac{\lambda_f(\ell' n)\lambda_f(\ell'' n)}{n^s}.$$

We denote the series by $R_f(\ell'\ell''; s)$; it depends only on $\ell = \ell'\ell''$ by (3.30) and converges absolutely when $\Re(s) > 1$. Now, for any

$$n = \prod_{p|n} p^{\alpha_p},$$

we may write $n = n'n''$ where $(n', \ell) = 1$ and $n''|\ell^\infty$ (that is, n' contains all prime factors of n not dividing ℓ, and all prime factors of n'' divide ℓ). Then, any prime $p|n''$ divide exactly one of ℓ' and ℓ'', since ℓ is squarefree by assumption. This also means that

$$\lambda_f(\ell' n)\lambda_f(\ell'' n) = \prod_{p|n'} \lambda_f^2(p^{\alpha_p}) \prod_{p|n''} \lambda_f(p^{\alpha_p+1})\lambda_f(p^{\alpha_p}). \tag{4.26}$$

Specifically, if $p|n'$, then the factor $\lambda_f(p^{\alpha_p})$ is present in both factors on the left hand side of (4.26). If $p|n''$ divides ℓ', then the left factor on the left hand side of (4.26) contributes with the factor $\lambda_f(p^{\alpha_p+1})$ on the right hand side, and the right factor on the left hand side contributes with the factor $\lambda_f(p^{\alpha_p})$ on the right hand side. If $p|\ell''$, then the roles are reversed. This means that we may factor $R_f(\ell; s)$ as

$$\prod_{p\nmid\ell} L_p(s, f\otimes f) \prod_{p|\ell} \left(\sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^{\alpha+1})\lambda_f(p^\alpha)}{p^{\alpha s}} \right),$$

where $L_p(s, f \otimes f)$ was defined in section 3.7. If $p|\ell$, then $p \nmid M$, and by (3.31) we have

$$S := \sum_{\alpha=0}^{\infty} \frac{\lambda_f(p^{\alpha+1})\lambda_f(p^\alpha)}{p^{\alpha s}} = \lambda_f(p) + \sum_{\alpha=1}^{\infty} \frac{\lambda_f(p)\lambda_f^2(p^\alpha) - \lambda_f(p^\alpha)\lambda_f(p^{\alpha-1})}{p^{\alpha s}}$$
$$= \lambda_f(p) L_p(s, f \otimes f) - \frac{1}{p^s} S,$$

implying that

$$S = \lambda_f(p) \left(1 + \frac{1}{p^s}\right)^{-1} L_p(s, f \otimes f)$$

and

$$R_f(\ell; s) = \lambda_f(\ell) \prod_{p|\ell} \left(1 + \frac{1}{p^s}\right)^{-1} L(s, f \otimes f).$$

Gathering all factors, we now have

$$F(s) = (4\pi)^{1-k-s}\Gamma(s+k-1)(\ell_1\ell_2)^{(1-k)/2}[\ell_1, \ell_2]^{-s}\lambda_f(\ell) \prod_{p|\ell} \left(1 + \frac{1}{p^s}\right)^{-1} L(s, f \otimes f). \tag{4.27}$$

This holds *a priori* for $\Re(s) > 1$ and by analytical continuation also in a punctured disc around $s = 1$. The final step of the proof consists of taking the residue at $s = 1$ on both sides of (4.27). We multiply both sides with $(\ell_1\ell_2)^{(k-1)/2}$, which on the left hand side gives

$$\mathrm{Res}_{s=1}\langle E(z, s) f_{|\ell_1}, f_{|\ell_2}\rangle_N = \langle \mathrm{Res}_{s=1} E(z, s) f_{|\ell_1}, f_{|\ell_2}\rangle_N = (\mathrm{Res}_{s=1} E(z, s))\langle f_{|\ell_1}, f_{|\ell_2}\rangle_N. \tag{4.28}$$

Now, if $\ell_1 = \ell_2 = 1$, then (4.27) reduces to

$$L(s, f \otimes f) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \langle E(z, s) f, f\rangle_N, \tag{4.29}$$

and from this it follows that

$$\mathrm{Res}_{s=1} L(s, f \otimes f) = \frac{(4\pi)^k}{\Gamma(k)} \langle f, f\rangle_N \,\mathrm{Res}_{s=1} E(z, s). \tag{4.30}$$

Hence, the residue on the right hand side of (4.27) is

$$(\ell_1\ell_2)^{(1-k)/2}[\ell_1, \ell_2]^{-1}\lambda_f(\ell) \prod_{p|\ell} \left(1 + \frac{1}{p}\right)^{-1} \langle f, f\rangle_N \,\mathrm{Res}_{s=1} E(z, s). \tag{4.31}$$

After we multiply (4.31) by $(\ell_1\ell_2)^{(k-1)/2}$, we equate the expression with the right hand side of (4.28). This finishes the proof. □

Recall that the residue of $E(s, z)$ at $s = 1$ is $3/(\pi\nu(N))$ independently of z, and the residue of $L(s, f \otimes f)$ at $s = 1$ is $Z(1, f)/\zeta_M(1)$, by (3.51). Equating the two in (4.30), we obtain the following result.

Lemma 4.8 ([ILS00, Lemma 2.5])**.** *If $f \in H_k^*(M)$ where $M|N$ and N is squarefree, then we have*

$$\langle f, f\rangle_N = (4\pi)^{1-k}\Gamma(k)\frac{\nu(N)\varphi(M)}{12M}Z(1,f).$$

Now, we make an ansatz

$$f_d := \sum_{\ell|L} x_d(\ell) f_{|\ell}, \quad x_d(\ell) \in \mathbb{C},$$

with the intention of making $H_k^*(L;f) := \{f_d : d|L\}$ an orthogonal basis of $S(L;f)$. *A priori*, $x_d(\ell)$ is only defined for $\ell|L$, but we can extend the definition to all integers by letting $x_d(\ell) = 0$ otherwise. Consider now the number

$$\delta_f(d_1,d_2) := \frac{\langle f_{d_1}, f_{d_2}\rangle_N}{\langle f, f\rangle_N}.$$

Our goal is accomplished if this is equal to the Kronecker delta $\delta(d_1,d_2)$. We thus want to compute $\delta_f(d_1,d_2)$ and see what conditions we need to impose on $x_d(\ell)$ in order to achieve orthogonality. We proceed as in [ILS00, Proposition 2.6]. Sesquilinearity of $\langle\cdot,\cdot\rangle_N$ and Lemma 4.7 gives

$$\delta_f(d_1,d_2) = \sum_{\ell_1|L}\sum_{\ell_2|L} x_{d_1}(\ell_1)\overline{x_{d_2}}(\ell_2)\frac{\langle f_{|\ell_1}, f_{|\ell_2}\rangle_N}{\langle f,f\rangle_N} = \sum_{\ell_1|L}\sum_{\ell_2|L} x_{d_1}(\ell_1)\overline{x_{d_2}}(\ell_2)\lambda_f(\ell)\nu(\ell)^{-1}\sqrt{\ell}.$$

We collect the common factors of ℓ_1 and ℓ_2 in the factor a. Thus $\ell_1 = a\ell', \ell_2 = a\ell''$ with $(\ell',\ell'') = 1$. Then, $\ell = \ell'\ell''$ and we get

$$\delta_f(d_1,d_2) = \sum_{a|L}\sum_{\substack{\ell',\ell''|L\\(\ell',\ell'')=1}} x_{d_1}(a\ell')\overline{x_{d_2}}(a\ell'')\lambda_f(\ell')\lambda_f(\ell'')\frac{\sqrt{\ell'\ell''}}{\nu(\ell')\nu(\ell'')}.$$

We use Möbius inversion in the form

$$\sum_{d|n}\mu(d) = \begin{cases}1, & n=1\\ 0, & \text{otherwise,}\end{cases}$$

to remove the condition $(\ell',\ell'') = 1$ and obtain

$$\delta_f(d_1,d_2) = \sum_{a|L}\sum_{\ell',\ell''|L}\sum_{b|(\ell',\ell'')} \mu(b)x_{d_1}(a\ell')\overline{x_{d_2}}(a\ell'')\lambda_f(\ell')\lambda_f(\ell'')\frac{\sqrt{\ell'\ell''}}{\nu(\ell')\nu(\ell'')}.$$

With the substitutions $\ell' \mapsto b\ell'$ and $\ell'' \mapsto b\ell''$, where $b|L$ (to save notation), by multiplicativity of λ_f and ν we get

$$\delta_f(d_1,d_2) = \sum_{a|L}\sum_{b|L}\mu(b)b\left(\frac{\lambda_f(b)}{\nu(b)}\right)^2$$
$$\times\sum_{\ell'|L} x_{d_1}(ab\ell')\lambda_f(\ell')\frac{\sqrt{\ell'}}{\nu(\ell')}\sum_{\ell'|L}\overline{x_{d_2}}(ab\ell'')\lambda_f(\ell'')\frac{\sqrt{\ell''}}{\nu(\ell'')}.$$

We now collect terms where $ab = c$. We only need to consider the case when $c|L$ since $x_d(c\ell) = 0$ otherwise. This motivates the introduction of

$$\rho_f(c) := \sum_{b|c} \mu(b) b \left(\frac{\lambda_f(b)}{\nu(b)} \right)^2 = \prod_{p|c} \left(1 - p \left(\frac{\lambda_f(p)}{p+1} \right)^2 \right)$$

and

$$y_d(c) := \sum_{\ell|L} x_d(c\ell) \lambda_f(\ell) \frac{\sqrt{\ell}}{\nu(\ell)},$$

so that we currently are at

$$\delta_f(d_1, d_2) = \sum_{c|L} \rho_f(c) y_{d_1}(c) \overline{y_{d_2}}(c). \tag{4.32}$$

We have expressed the coefficients $y_d(c)$ by means of the coefficients $x_d(c\ell)$. Möbius inversion allows us to reverse this relation and write

$$x_d(\ell) = \sum_{a|L} x_d(a\ell) \lambda_f(a) \frac{\sqrt{a}}{\nu(a)} \sum_{c|a} \mu(c).$$

Writing $a = ce$ and changing the order of summation yields

$$x_d(\ell) = \sum_{c|L} y_d(c\ell) \mu(c) \lambda_f(c) \frac{\sqrt{c}}{\nu(c)}. \tag{4.33}$$

Remember that we want $\delta_f(d_1, d_2)$ to be equal to the Kronecker delta $\delta(d_1, d_2)$. If we consider the matrix

$$Y = \left(y_d(c) \sqrt{\rho_f(c)} \right)_{\substack{d|L \\ c|L}},$$

then (4.32) tells us that we should require that $YY^* = I$. We are free to choose Y as long as this condition is satisfied, and we choose $Y = I$. This means that $y_d(c) = \rho_f(d)^{-1/2}$ if $c = d$ and 0 otherwise. Inserting in (4.33) gives

$$x_d(\ell) = \begin{cases} \mu(c) \lambda_f(c) \frac{\sqrt{c}}{\nu(c) \sqrt{\rho_f(d)}}, & d = c\ell \\ 0, & \text{otherwise.} \end{cases}$$

Finally, this means that

$$f_d(z) = \left(\frac{d}{\rho_f(d)} \right)^{1/2} \sum_{c\ell = d} \mu(c) \lambda_f(c) \nu(c)^{-1} \ell^{(k-1)/2} f(\ell z). \tag{4.34}$$

For this choice of f_d, the set $H_k^*(L; f)$ constitutes an orthogonal basis of $S(L; f)$. The union

$$\bigcup_{LM=N} \bigcup_{f \in H_k^*(M)} H_k^*(L; f) \tag{4.35}$$

is an orthogonal basis for $S_k(N)$.

4.4 Evaluation of $\Delta_{k,N}(m,n)$

Having obtained an orthogonal basis, the natural next step is to insert it into the definition for $\Delta_{k,N}(m,n)$. To do so, we seek the Fourier coefficients $a_{f_d}(n)$. An insertion of the Fourier series of $f(\ell z)$ into (4.34) reveals that these are equal to

$$a_{f_d}(n) = \left(\frac{d}{\rho_f(d)}\right)^{1/2} \sum_{c\ell=d} \frac{\mu(c)}{\nu(c)} \lambda_f(c) \ell^{(k-1)/2} a_f\left(\frac{n}{\ell}\right),$$

where $a_f(n)$ are the Fourier coefficients of f, and the convention is that $a_f(x) = 0$ if $x \notin \mathbb{Z}$. Equivalently, we require $\ell | n$. Division by $n^{(k-1)/2}$ yields that

$$\lambda_{f_d}(n) = \left(\frac{d}{\rho_f(d)}\right)^{1/2} \sum_{\substack{c\ell=d \\ \ell|n}} \frac{\mu(c)}{\nu(c)} \lambda_f(c) \lambda_f\left(\frac{n}{\ell}\right)$$

and a direct insertion into (4.10) (where $\overline{\lambda_{f_d}(m)} = \lambda_{f_d}(m)$ since f is a newform at some level) yields

$$\Delta_{k,N}(m,n) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{LM=N} \sum_{f\in H_k^*(M)} \frac{1}{||f||^2} \sum_{d|L} \frac{d}{\rho_f(d)}$$
$$\times \left(\sum_{\substack{c_1\ell_1=d \\ \ell_1|m}} \frac{\mu(c_1)}{\nu(c_1)} \lambda_f(c_1) \lambda_f\left(\frac{m}{\ell_1}\right)\right) \left(\sum_{\substack{c_2\ell_2=d \\ \ell_2|n}} \frac{\mu(c_2)}{\nu(c_2)} \lambda_f(c_2) \lambda_f\left(\frac{n}{\ell_2}\right)\right). \quad (4.36)$$

It is difficult to proceed from here without further assumptions on m, n or N. Since $c_1\ell_1 = c_2\ell_2$, it would be convenient if $(\ell_1, \ell_2) = 1$. This is the case if $(m, n, N) = 1$, since any prime dividing ℓ_1 and ℓ_2 need to divide both m, n and L. We write $d = b\ell_1\ell_2$ and obtain

$$\left(\sum_{\substack{c_1\ell_1=d \\ \ell_1|m}} \frac{\mu(c_1)}{\nu(c_1)} \lambda_f(c_1) \lambda_f\left(\frac{m}{\ell_1}\right)\right) \left(\sum_{\substack{c_2\ell_2=d \\ \ell_2|n}} \frac{\mu(c_2)}{\nu(c_2)} \lambda_f(c_2) \lambda_f\left(\frac{n}{\ell_2}\right)\right)$$
$$= \sum_{\substack{b\ell_1\ell_2=d \\ \ell_1|m,\ell_2|n}} \left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 \frac{\mu(\ell_1\ell_2)}{\nu(\ell_1\ell_2)} \lambda_f(\ell_1\ell_2) \lambda_f\left(\frac{m}{\ell_1}\right) \lambda_f\left(\frac{n}{\ell_2}\right). \quad (4.37)$$

The conditions

$$\begin{cases} d|L, \\ b\ell_1\ell_2 = d, \\ \ell_1|m, \\ \ell_2|n, \end{cases} \quad \text{are equivalent to} \quad \begin{cases} \ell_1|(m,L), \\ \ell_2|(n,L), \\ b|(L/\ell_1\ell_2), \\ b\ell_1\ell_2 = d \end{cases}$$

(note that $\ell_1\ell_2|L$ automatically since the ℓ_i:s are relatively prime and both divide L). We may reorder the sum on the right hand side of (4.37) and get

$$\sum_{d|L}\frac{d}{\rho_f(d)}\sum_{\substack{b\ell_1\ell_2=d\\ \ell_1|m,\ell_2|n}}\left(\frac{\lambda_f(b)}{\nu(b)}\right)^2\frac{\mu(\ell_1\ell_2)}{\nu(\ell_1\ell_2)}\lambda_f(\ell_1\ell_2)\lambda_f\left(\frac{m}{\ell_1}\right)\lambda_f\left(\frac{n}{\ell_2}\right)$$
$$=\sum_{\ell_1|(m,L)}\ell_1\frac{\mu(\ell_1)}{\nu(\ell_1)}\lambda_f(\ell_1)\lambda_f\left(\frac{m}{\ell_1}\right)\sum_{\ell_2|(n,L)}\ell_2\frac{\mu(\ell_2)}{\nu(\ell_2)}\lambda_f(\ell_2)\lambda_f\left(\frac{n}{\ell_2}\right)$$
$$\times\frac{1}{\rho_f(\ell_1\ell_2)}\sum_{b|\frac{L}{\ell_1\ell_2}}\frac{b}{\rho_f(b)}\left(\frac{\lambda_f(b)}{\nu(b)}\right)^2. \quad (4.38)$$

We temporarily denote the inner sum on the right hand side of (4.38) by $s(L/\ell_1\ell_2)$. It is a multiplicative function, and one can show that

$$s(L/\ell_1\ell_2)=\frac{1}{\rho_f(L/\ell_1\ell_2)}. \quad (4.39)$$

Indeed, it is enough to verify that $s(p)=1/\rho_f(p)$ when p is prime, since L is squarefree. Then, we have

$$s(p)=1+\frac{p}{\rho_f(p)}\left(\frac{\lambda_f(p)}{\nu(p)}\right)^2=\frac{\rho_f(p)+1-\rho_f(p)}{\rho_f(p)}=\frac{1}{\rho_f(p)},$$

as claimed. We define the sum

$$A_f(m,L):=\sum_{\ell|(m,L)}\ell\frac{\mu(\ell)}{\nu(\ell)}\lambda_f(\ell)\lambda_f\left(\frac{m}{\ell}\right),$$

and by (4.38) and (4.39), we arrive at the equality

$$\sum_{d|L}\frac{d}{\rho_f(d)}\left(\sum_{\substack{c_1\ell_1=d\\ \ell_1|m}}\frac{\mu(c_1)}{\nu(c_1)}\lambda_f(c_1)\lambda_f\left(\frac{m}{\ell_1}\right)\right)\left(\sum_{\substack{c_2\ell_2=d\\ \ell_2|n}}\frac{\mu(c_2)}{\nu(c_2)}\lambda_f(c_2)\lambda_f\left(\frac{n}{\ell_2}\right)\right)$$
$$=\frac{1}{\rho_f(L)}A_f(m,L)A_f(n,L).$$

We have

$$\lambda_f(\ell)\lambda_f\left(\frac{m}{\ell}\right)=\sum_{\delta|(\ell,m/\ell)}\lambda_f\left(\frac{m}{\delta^2}\right)$$

by (3.30) (the condition $(\delta,M)=1$ is always satisfied in this context, since $\ell|L, LM=N$ and N is squarefree). Inserting this into the definition of $A_f(m,L)$, we observe that the conditions $\ell|(m,L),\delta|(\ell,m/\ell)$ are equivalent to $\delta^2|(m,\delta L),\ell'|(m,L)/\delta,\delta\ell'=\ell$. Hence, we may switch order of summation and get

$$A_f(m,L)=\sum_{\delta^2|(m,\delta L)}\lambda_f\left(\frac{m}{\delta^2}\right)\frac{\delta\mu(\delta)}{\nu(\delta)}\sum_{\ell'|(m,L)/\delta}\frac{\ell'\mu(\ell')}{\nu(\ell')}.$$

Similarly to before, we identify the inner sum as a multiplicative function of the argument $(m,L)/\delta$, and denote it temporarily by $t((m,L)/\delta)$. Here, one may check that

$$t(p)=1+\frac{p\mu(p)}{\nu(p)}=1-\frac{p}{p+1}=\frac{1}{p+1}=\frac{1}{\nu(p)},$$

so that

$$\sum_{\ell'|(m,L)/\delta} \frac{\ell'\mu(\ell')}{\nu(\ell')} = \frac{1}{\nu((m,L)/\delta)}$$

when L is squarefree. Hence, we obtain

$$A_f(m,L) = \frac{1}{\nu((m,L))} \sum_{\delta^2|(m,\delta L)} \mu(\delta)\delta\lambda_f\left(\frac{m}{\delta^2}\right).$$

The evaluation of $||f||^2$ from Lemma 4.8 and (4.36) now gives

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f\in H_k^*(M)} \frac{A_f(m,L)A_f(n,L)}{\rho_f(L)Z(1,f)}.$$

By (3.49) and (3.53), we have upon evaluation at $s=1$ that

$$Z_p(1,f) = \begin{cases} \left(1+\frac{1}{p}\right)^{-1}\rho_f(p)^{-1}, & p\nmid M \\ \left(1+\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-1}, & p|M. \end{cases} \tag{4.40}$$

Hence, the local factor $Z_N(1,f)$ satisfies

$$Z_N(s,f) := \prod_{p|N} Z_p(1,f) = \frac{MN}{\varphi(M)\nu(N)\rho_f(L)},$$

from which we obtain

$$\Delta_{k,N}(m,n)\frac{12}{(k-1)N} \sum_{LM=N} \sum_{f\in H_k^*(M)} A_f(m,L)A_f(n,L)\frac{Z_N(1,f)}{Z(1,f)}.$$

Until now, we have assumed $(m,n,N)=1$. The factors A_f would simplify if only the term corresponding to $\delta=1$ occurred. This is the case if no prime factor of N occurs in m or n with greater exponent than 1. A natural condition to impose is therefore $(mn,N^2)|N$. Under this assumption, we have $A_f(m,L) = \lambda_f(m)/\nu((m,L))$ and consequently

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)N} \sum_{LM=N} \sum_{f\in H_k^*(M)} \frac{\lambda_f(m)\lambda_f(n)}{\nu((mn,L))}\frac{Z_N(1,f)}{Z(1,f)}, \tag{4.41}$$

which is [ILS00, Lemma 2.7].

4.5 Sums of Hecke eigenvalues

In this section, we analyze sums of Hecke eigenvalues. From (4.41), we are motivated to introduce the weighted mixed sum

$$\Delta^*_{k,N}(m,n) := \sum_{f\in H_k^*(N)} \lambda_f(m)\lambda_f(n)\frac{Z_N(1,f)}{Z(1,f)}.$$

Our goal is to make statements about the unweighted pure sum

$$\Delta^*_{k,N}(n) := \sum_{f\in H_k^*(N)} \lambda_f(n). \tag{4.42}$$

From the local factor computation (4.40), we have $Z_N(1,f) = \zeta_N(2)$. The next step is to express the two sums in terms of the original sum $\Delta_{k,N}(m,n)$. This is [ILS00, Propositions 2.8 and 2.11], valid when $(m,N) = 1$ and $(n,N^2)|N$. We start with $\Delta^*_{k,N}(m,n)$. We claim that

$$\Delta^*_{k,N}(m,n) = \frac{k-1}{12}\sum_{LM=N}\frac{\mu(L)M}{\nu((n,L))}\sum_{\ell|L^\infty}\frac{1}{\ell}\Delta_{k,M}(m\ell^2,n). \tag{4.43}$$

A direct insertion of (4.41) in the right hand side of (4.43) gives

$$\sum_{LM=N}\frac{\mu(L)}{\nu((n,L))}\sum_{\ell|L^\infty}\frac{1}{\ell}\sum_{KM'=M}\sum_{f\in H_k^*(M')}\frac{\lambda_f(m\ell^2)\lambda_f(n)}{\nu((n,K))}\frac{Z_M(1,f)}{Z(1,f)}.$$

The sum of $\lambda_f(\ell^2)/\ell$ over $\ell|L^\infty$ is $Z_L(1,f)$ by definition, and since $LM = N$, we get $Z_N(1,f)$ in the numerator. Also, (n,L) and (n,K) are relatively prime since K and L are, and their product equals (n,KL). Hence, we have

$$\sum_{LM=N}\sum_{KM'=M}\sum_{f\in H_k^*(M')}\frac{\mu(L)\lambda_f(m)\lambda_f(n)}{\nu((n,KL))}\frac{Z_N(1,f)}{Z(1,f)}.$$

Now $KLM' = N$ and $KL = N/M'$. The conditions $LM = N, KM' = M$ and $f \in H_k^*(M')$ are equivalent to $M'|N, f \in H_k^*(M'), M|N$ and $M'|M$. Reordering the sum gives

$$\sum_{M'|N}\sum_{f\in H_k^*(M')}\frac{\lambda_f(m)\lambda_f(n)}{\nu((n,N/M'))}\frac{Z_N(1,f)}{Z(1,f)}\sum_{\substack{M|N\\ M'|M}}\mu\left(\frac{N}{M}\right).$$

In the inner sum, we recall that $KM' = M$ and change the condition to $K|N/M'$. Möbius inversion tells us that the sum is 1 if $N = M'$ and 0 otherwise. Hence, the only term surviving is

$$\sum_{f\in H_k^*(N)}\lambda_f(m)\lambda_f(n)\frac{Z_N(1,f)}{Z(1,f)} = \Delta^*_{k,N}(m,n),$$

as we wished to show.

Turning to $\Delta^*_{k,N}(n)$, the approach is similar. Here, we claim that

$$\Delta^*_{k,N}(n) = \frac{k-1}{12}\sum_{LM=N}\frac{\mu(L)M}{\nu((n,L))}\sum_{(m,M)=1}\frac{1}{m}\Delta_{k,M}(m^2,n). \tag{4.44}$$

A direct insertion of (4.41) in the right hand side of (4.44) yields

$$\sum_{LM=N}\frac{\mu(L)}{\nu((n,L))}\sum_{(m,M)=1}\sum_{KM'=M}\sum_{f\in H_k^*(M')}\frac{\lambda_f(m^2)\lambda_f(n)}{m\nu((n,K))}\frac{Z_M(1,f)}{Z(1,f)}.$$

The sum of $\lambda_f(m^2)/m$ over $(m,M)=1$ is by definition $Z(1,f)/Z_M(1,f)$. Hence, we get

$$\sum_{LM=N}\mu(L)\sum_{KM'=M}\frac{1}{\nu((n,KL))}\sum_{f\in H_k^*(M')}\lambda_f(n).$$

We denote $KL=L'$ so that $L'M'=N$. Reordering the sum gives

$$\sum_{L'M'=N}\frac{1}{\nu((n,L'))}\sum_{f\in H_k^*(M')}\lambda_f(n)\sum_{L|L'}\mu(L).$$

The inner sum is 1 if $L'=1$ and 0 otherwise. Thus the only surviving term is when $M'=N$, which is

$$\sum_{f\in H_k^*(N)}\lambda_f(n)=\Delta_{k,N}^*(n),$$

as we wished to show.

We are now interested in estimating $\Delta_{k,N}^*(n)$ with a main term and an error term. This will, for instance, allow us to estimate the size of the family $|H_k^*(N)|$. However, in (4.44) terms with L and m large may cause some problems. We therefore follow the procedure and notation in [ILS00, p. 79], where Iwaniec, Luo and Sarnak split the sum as

$$\Delta_{k,N}^*(n)=\Delta_{k,N}^{'}(n)+\Delta_{k,N}^{\infty}(n), \tag{4.45}$$

where

$$\Delta_{k,N}^{'}(n)=\frac{k-1}{12}\sum_{\substack{LM=N\\L\le X}}\frac{\mu(L)M}{\nu((n,L))}\sum_{\substack{(m,M)=1\\m\le Y}}\frac{1}{m}\Delta_{k,M}(m^2,n), \tag{4.46}$$

and $\Delta_{k,N}^{\infty}(n)$ is the complementary sum. First, we want to estimate $\Delta_{k,N}^{\infty}(n)$. From the GRH for $L(s,f)$, Iwaniec, Luo and Sarnak deduce that

$$\sum_{(q,nN)=1}\lambda_f(q)a_q\ll(nkN)^{\varepsilon},$$

where

$$a_q=\begin{cases}\log p/\sqrt{p}, & q=p\le Q,\\ 1, & q=1,\\ 0, & \text{otherwise}\end{cases}$$

(see [ILS00, eqs. 2.65 and 2.66]). Here, Q is a fixed upper bound satisfying $\log Q\ll\log kN$. Then, it follows from the GRH for $L(s,f)$ as well as $L(s,\mathrm{sym}^2(f))$ that

$$\sum_{(q,nN)=1}\Delta_{k,N}^{\infty}(nq)a_q\ll(n,N)^{-1/2}kN(X^{-1}+Y^{-1/2})(nkNXY)^{\varepsilon}, \tag{4.47}$$

where $(n,N^2)|N$ (there are several steps in this calculation. See [ILS00, Lemma 2.12] for details.). Looking at the first term $q=1$, we have the estimation

$$\Delta_{k,N}^{\infty}(n)\ll(n,N)^{-1/2}kN(X^{-1}+Y^{-1/2})(nkNXY)^{\varepsilon}. \tag{4.48}$$

Next, we turn to the main sum $\Delta_{k,N}^{'}(n)$. Here, we apply the Petersson trace formula directly to every $\Delta_{k,N}(m^2,n)$. We will then get a main term from the Kronecker delta $\delta(m^2, n)$, provided that $n = m^2 \le Y^2$ and that $(n, N) = 1$. This term is equal to

$$\frac{k-1}{12\sqrt{n}} \sum_{\substack{LM=N \\ L\le X}} \mu(L)M = \frac{k-1}{12\sqrt{n}} \left(\varphi(N) - \sum_{\substack{LM=N \\ L>X}} \mu(L)M \right)$$
$$= \frac{(k-1)\varphi(N)}{12\sqrt{n}} \left(1 + O\left(\frac{\tau(N)N}{\varphi(N)X} \right) \right),$$

where we used the equality

$$\varphi(N) = \sum_{LM=N} \mu(L)M,$$

and the fact that $M = N/L \le N/X$ when estimating the sum over $L > X$. Inserting the sum of Klooserman sums and Bessel functions give

$$\Delta'_{k,N}(n) = \frac{k-1}{12} \frac{\varphi(N)}{\sqrt{n}} \left(1 + O\left(\frac{\tau(N)N}{\varphi(N)X} \right) \right)$$
$$+ \frac{k-1}{12} \sum_{\substack{LM=N \\ L\le X}} \frac{\mu(L)M}{\nu((n,L))} \sum_{\substack{(m,M)=1 \\ m\le Y}} \frac{2\pi i^k}{m} \sum_{c\equiv 0 \pmod{M}} \frac{1}{c} S(m^2, n; c) J_{k-1}\left(\frac{4\pi m\sqrt{n}}{c} \right),$$

where the main term exists only if $n = m^2 \le Y^2$ and $(n, N) = 1$. We now bound the second sum by the error in Corollary 4.6. We obtain the upper bound

$$k^{-1/3} \left(\frac{n}{(n,N)N} \right)^{1/2} \sum_{\substack{LM=N \\ L\le X}} \frac{(L(n,L))^{1/2}}{\nu((n,L))} \sum_{\substack{(m,M)=1 \\ m\le Y}} \tau(M)\tau_3((m,n)) \log 2m^2 n$$
$$\ll k^{-1/3} \left(\frac{n}{(n,N)N} \right)^{1/2} (nN)^{\varepsilon} \sum_{\substack{LM=N \\ L\le X}} \frac{(L(n,L))^{1/2}}{\nu((n,L))} \sum_{\substack{(m,M)=1 \\ m\le Y}} m^{\varepsilon},$$

where we used that $\tau(M) = M^{\varepsilon} \ll N^{\varepsilon}$ and $(n, N) = (n, M)(n, L)$. The inner sum over $m \le Y$ is bounded by $Y^{1+\varepsilon}$ by an integral comparison. We estimate the sum over L by the largest possible term, which is $X^{1/2}$ since $1/\nu((n.L)) \le 1/(n, L)$. Multiplied by the number of terms, which is at most $\tau(N) \ll N^{\varepsilon}$, the end result is that

$$\Delta'_{k,N}(n) = \frac{k-1}{12} \frac{\varphi(N)}{\sqrt{n}} \left(1 + O\left(\frac{\tau(N)N}{\varphi(N)X} \right) \right)$$
$$+ O\left(k^{-1/3} \left(\frac{nXY^2}{(n,N)N} \right)^{1/2} (nkNXY)^{\varepsilon} \right)$$

(c.f. [ILS00, eq. 2.70]).

Now, we choose values for the parameters X and Y. The presence of the factor $X^{-1} + Y^{-1/2}$ in $\Delta^{\infty}_{k,N}(n)$ suggest that we put $X = Y^{1/2}$ to minimize it. We now choose this value to be equal to $n^{-1/7}k^{8/21}N^{3/7}$. Then, for ε small enough, we obtain from (4.48) that

$$\Delta^{\infty}_{k,N}(n) \ll (n,N)^{-1/2}n^{1/7+\varepsilon}k^{13/21+\varepsilon}N^{4/7+\varepsilon} \ll (n,N)^{-1/2}n^{1/6}(kN)^{2/3}.$$

The main term comes from $\Delta'_{k,N}(n)$ and is unchanged. The first error term from $\Delta'_{k,N}(n)$ is bounded by

$$kn^{-1/2}N^{1+\varepsilon}X^{-1} \ll (n,N)^{-1/2}k^{13/21}n^{1/7}N^{4/7+\varepsilon} \ll (n,N)^{-1/2}n^{1/6}(kN)^{2/3}.$$

The second error term is

$$(n,N)^{-1/2}n^{1/7+\varepsilon}k^{13/21+\varepsilon}N^{4/7+\varepsilon} \ll (n,N)^{-1/2}n^{1/6}(kN)^{2/3}$$

as well. This finishes the proof of the following result.

Proposition 4.9 ([ILS00, Proposition 2.13])**.** *Let N be squarefree and $(n,N^2)|N$. Then*

$$\Delta^{*}_{k,N}(n) = \frac{(k-1)\varphi(N)}{12\sqrt{n}} + O\left((n,N)^{-1/2}n^{1/6}(kN)^{2/3}\right), \tag{4.49}$$

where the main term exists only if n is square and $(n,N)=1$.

When $n=1$, the sum $\Delta^{*}_{k,N}(n)$ simply counts the number of newforms in the family $H^{*}_{k}(N)$. Thus

$$|H^{*}_{k}(N)| = \frac{(k-1)\varphi(N)}{12} + O\left((kN)^{2/3}\right). \tag{4.50}$$

The asymptotic means that when we average by the cardinality of the family $H^{*}_{k}(N)$, we should divide by the main term. The error from doing so will as a rule be absorbed into already existing error terms.

5

Computing the 1-level density

We now turn to one of this report's main tasks, which is to compute the 1-level density (2.5), introduced Chapter 2. First, we will show the *explicit formula* in Section 5.1, where the 1-level density of a single newform is rewritten as an integral and a sum over primes. We then follow the results from [ILS00], in which Iwaniec, Luo and Sarnak show that the main term follow the orthogonal symmetry type for limited support of $\widehat{\phi}$. That is, when inserting the density $w(O)$ in the limit (2.6), we expect the main term in the limit to be equal to $\widehat{\phi}(0)+\phi(0)/2$. This is the subject of Section 5.2, and the key is to identify which parts of the prime sum contribute to the main term, and which do not, when we average over the family. In this investigation the Petersson trace formla is crucial. Having obtained the main term, we turn to the analysis of a lower order term of order $1/\mathcal{L}$ as done in [Mil09]. This is done in Section 5.3, and will be compared with the Ratios Conjecture prediction in the next chapter.

5.1 The explicit formula

Let $f \in H_k^*(N)$. Recall the definition (2.3) of $D(f,\phi)$, that is

$$D(f,\phi) = \sum_{\rho_f} \phi\left(\left(\rho_f - \frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right).$$

We wish to evaluate the sum using Cauchy's residue theorem. To do so, we observe that if ρ_f is a zero of $L(s,f)$ with multiplicity m, then it is a simple pole of $\Lambda'(s,f)/\Lambda(s,f)$ with residue m. Now let c be such that all the zeroes of Λ lie in the strip

$$\{s \in \mathbb{C} : 1-c < \Re(s) < c\};$$

unconditionally, by the Euler product and functional equation for $\Lambda(s,f)$, all its zeroes lie in the critical strip, and we may choose any $c > 1$. Assuming the GRH we may choose any $c > 1/2$. We now integrate

$$\frac{\Lambda'(s,f)}{\Lambda(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)$$

over the positively oriented rectangle with corners at $1-c \pm iT, c \pm iT$ (recall that $\mathcal{L} = \log(k^2N)$, and that $\Lambda(s,f) = L_\infty(s,f)L(s,f)$ was defined in Section 3.6).

Letting T tend to infinity with the requirement that T is not the imaginary part for a zero of Λ, we pick up all the terms of $D(f,\phi)$ as residues, giving

$$D(f,\phi) = \frac{1}{2\pi i}\left(\int_{(c)} - \int_{(1-c)}\right)\frac{\Lambda'(s,f)}{\Lambda(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds. \tag{5.1}$$

The functional equation (3.41) implies that

$$\frac{\Lambda'(1-s,f)}{\Lambda(1-s,f)} = -\frac{\Lambda'(s,f)}{\Lambda(s,f)},$$

from which the substitution $s \mapsto 1-s$ in the second integral of (5.1) gives us

$$D(f,\phi) = \frac{1}{2\pi i}\int_{(c)} 2\frac{\Lambda'(s,f)}{\Lambda(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds.$$

Since $\Lambda(s,f) = L_\infty(s,f)L(s,f)$, we get two terms

$$\begin{aligned} D(f,\phi) &= \frac{1}{2\pi i}\int_{(c)} 2\frac{L'(s,f)}{L(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds \\ &+ \frac{1}{2\pi i}\int_{(c)} 2\frac{L'_\infty(s,f)}{L_\infty(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds, \end{aligned} \tag{5.2}$$

where we integrate the logarithmic derivative of each factor.

For the first term in (5.2), we let $c > 1$, and may then use the factorization (3.45) to obtain

$$\log L_p(s,f) = -\log\left(1-\frac{\alpha_f(p)}{p^s}\right) - \log\left(1-\frac{\beta_f(p)}{p^s}\right)$$

and

$$\frac{L'_p(s,f)}{L_p(s,f)} = -\left(\frac{\alpha_f(p)/p^s}{1-\alpha_f(p)/p^s} + \frac{\beta_f(p)/p^s}{1-\beta_f(p)/p^s}\right)\log p = -\sum_{\nu=1}^{\infty}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu s}}\log p.$$

From this, we get

$$\frac{L'(s,f)}{L(s,f)} = \sum_p \frac{L'_p(s,f)}{L_p(s,f)} = -\sum_p\sum_{\nu=1}^{\infty}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu s}}\log p.$$

The first integral in (5.2) is therefore

$$-2\sum_p \log p\sum_{\nu=1}^{\infty}(\alpha_f^\nu(p)+\beta_f^\nu(p))\frac{1}{2\pi i}\int_{(c)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)p^{-\nu s}d.$$

Assuming the GRH for $L(s,f)$, we may shift the integral to $\Re(s) = 1/2$ due to the rapid decay of ϕ. We then parametrize $s = 1/2 + it$ and substitute $t\mathcal{L}/2\pi = u$ to get

$$- 2\sum_p\sum_{\nu=1}^{\infty}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu/2}}\frac{\log p}{\mathcal{L}}\int_{-\infty}^{\infty}\phi(u)e^{-2\pi iu\nu\log p/\mathcal{L}}du$$
$$= -2\sum_p\sum_{\nu=1}^{\infty}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu/2}}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}}. \quad (5.3)$$

We turn to the second term of (5.2). By (3.40), we have

$$\frac{L_\infty'(s,f)}{L_\infty(s,f)} = \log\left(\frac{\sqrt{N}}{\pi}\right)+\frac{1}{2}\psi\left(\frac{s}{2}+\frac{k+1}{4}\right)+\frac{1}{2}\psi\left(\frac{s}{2}+\frac{k-1}{4}\right), \quad (5.4)$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. We can also move the second integral of (5.2) to $\Re(s)=1/2$ and apply the same parametrization and substitution as before. Doing so gives

$$\frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\left[2\log\left(\frac{\sqrt{N}}{\pi}\right)+\psi\left(\frac{1}{4}+\frac{k\pm1}{4}+\frac{\pi iu}{\mathcal{L}}\right)\right]\phi(u)du. \quad (5.5)$$

For brevity, we let $\pm$ denote the presence of both ψ-terms on the right hand side of (5.4). In total, we have

$$D(f,\phi)=\widehat{\phi}(0)\frac{2\log\left(\sqrt{N}/\pi\right)}{\mathcal{L}}+\frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\psi\left(\frac{1}{4}+\frac{k\pm1}{4}+\frac{\pi iu}{\mathcal{L}}\right)\phi(u)du$$
$$-2\sum_p\sum_{\nu=1}^{\infty}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu/2}}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}}. \quad (5.6)$$

This is referred to as the *explicit formula*. It serves as a first step for most of our calculations.

5.2 The main term

In this section, we follow [ILS00, Sections 4 and 5] to obtain the main term of the 1-level density in accordance with the Density Conjecture when $\widehat{\phi}$ has limited support. In this context error terms of any size are allowed, as long as they tend to 0 when kN tend to infinity, in particular errors of size $1/\log(kN)$.

In the general context of the Katz-Sarnak Density Conjecture our family is predicted to have orthogonal symmetry. This means that what we want to show is that

$$\mathcal{D}_{H_k^*(N)}(\phi)\to\widehat{\phi}(0)+\frac{\phi(0)}{2},$$

as kN tends to infinity.

5.2.1 The 1-level density of a single newform $D(f,\phi)$

We wish to manipulate the integral on the right hand side of (5.6) in order to remove the u-dependence of ψ. For this purpose, we use the Stirling approximation

$$\psi(z)=\log z+O\left(\frac{1}{|z|}\right)$$

(see e.g. [Dav00, Ch. 10]). The integrand now becomes

$$\begin{aligned}
&\psi\left(\frac{1}{4}+\frac{k\pm 1}{4}+\frac{\pi i u}{\mathcal{L}}\right)\\
=&\log\left(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi i u}{\mathcal{L}}\right)+\log\left(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi i u}{\mathcal{L}}\right)+O\left(\frac{1}{k}\right)\\
=&\log\left(\frac{1}{4k}+\frac{k+1}{4k}+\frac{\pi i u}{k\mathcal{L}}\right)+\log\left(\frac{1}{4k}+\frac{k-1}{4k}+\frac{\pi i u}{k\mathcal{L}}\right)+O\left(\frac{1}{k}\right)+\log k^2.
\end{aligned}$$

The integral over the first three terms on the third row is bounded, since ϕ decays rapidly, and yield error terms of size $O(1/\mathcal{L})$. All in all, we have

$$\frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\psi\left(\frac{1}{4}+\frac{k\pm 1}{4}+\frac{\pi i u}{\mathcal{L}}\right)\phi(u)du=\widehat{\phi}(0)\frac{\log k^2}{\mathcal{L}}+O\left(\frac{1}{\mathcal{L}}\right).$$

This implies that we have

$$D(f,\phi)=\widehat{\phi}(0)\frac{\log k^2 N}{\mathcal{L}}-2\sum_p\sum_{\nu=1}^{\infty}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu/2}}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}}+O\left(\frac{1}{\mathcal{L}}\right). \quad (5.7)$$

At this stage, the reason we re-scale γ_f by $\mathcal{L}/2\pi$ becomes clear. The first term $\widehat{\phi}(0)$ will be a part of the main term as predicted by the Katz-Sarnak heuristic.

Next, we analyse the summation over primes, splitting it into terms where $\nu=1,2$ and $\nu\geq 3$, respectively. Recall that by (3.46), (3.48) and (3.47), we get

$$\begin{aligned}
\alpha_f(p)+\beta_f(p)&=\lambda_f(p),\\
\alpha_f^2(p)+\beta_f^2(p)&=\lambda_f(p^2)-\chi_0(p),\\
|\alpha_f^\nu(p)+\beta_f^\nu(p)|&\leq 2,
\end{aligned}$$

respectively. When $\nu\geq 3$ the sum over primes converge absolutely, yielding an error of size $1/\mathcal{L}$. When $\nu=1,2$ we split the sum depending on whether $p|N$ or $p\nmid N$. We have

$$\sum_{p|N}\frac{\lambda_f(p)}{p^{1/2}}\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}}\ll\frac{1}{\mathcal{L}}\sum_{p|N}\frac{\log p}{p},$$

and

$$\sum_{p|N}\frac{\lambda_f(p^2)}{p}\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}}\ll\frac{1}{\mathcal{L}}\sum_{p|N}\frac{\log p}{p^2}$$

(here, we used (3.32)). By a Stieltjes integration, we have

$$\sum_{p\leq x}\frac{\log p}{p}=\int_1^x\frac{1}{t}d\theta(t)=\frac{\theta(x)}{x}+\int_1^x\frac{\theta(t)}{t^2}dt\ll 1+\log x, \quad (5.8)$$

since $\theta(t)\ll t$ by the PNT. Here, $\theta(t)$ is the Chebyshev function. We now split the sum on the right hand side depending on whether $p\leq\log 3N$ or not[1]. By (5.8), we have

$$\sum_{\substack{p|N\\ p\leq\log 3N}}\frac{\log p}{p}\ll\sum_{p\leq\log 3N}\frac{\log p}{p}\ll 1+\log\log 3N,$$

[1] I owe this idea to V. Ahlquist.

and the remaining sum is bounded by

$$\sum_{\substack{p|N \\ p>\log 3N}} \frac{\log p}{p} \leq \frac{1}{\log 3N} \sum_{\substack{p|N \\ p>\log 3N}} \log p \leq \frac{\log N}{\log 3N} = O(1).$$

Therefore, we have

$$\sum_{p|N} \frac{\log p}{p} \ll 1 + \log\log 3N, \tag{5.9}$$

and the same of course holds if we replace p by p^2. Introducing these error terms in (5.7), we have

$$\begin{aligned} D(f,\phi) = \widehat{\phi}(0) - \sum_{p\nmid N} \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}} \\ - \sum_{p\nmid N}(\lambda_f(p^2)-1)\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{p\mathcal{L}} + O\left(\frac{1+\log\log 3N}{\mathcal{L}}\right) \end{aligned}$$

(cf. [ILS00, p. 87]). The second sum can be split into two terms, the second of which involves no Hecke eigenvalues. In this sum, we may add back the terms corresponding to $p|N$ at a cost of the admissible error term $\log\log 3N/\mathcal{L}$, by (5.9). Doing so gives us

$$\sum_p \widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{p\mathcal{L}} = \frac{2}{\mathcal{L}}\int_1^\infty \frac{1}{t}\widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right) d\theta(t)$$

by a Stieltjes integration. Now,

$$\frac{2}{\mathcal{L}}\int_1^\infty \frac{1}{t}\widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right) dt = \int_0^\infty \widehat{\phi}(u)du = \frac{\phi(0)}{2}$$

through the substitution $2\log t/\mathcal{L} = u$. So we have

$$\frac{2}{\mathcal{L}}\int_1^\infty \frac{1}{t}\widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right) d\theta(t) = \frac{\phi(0)}{2} + \frac{2}{\mathcal{L}}\int_1^\infty \frac{1}{t}\widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right) d(\theta(t)-t).$$

For the sake of showing that the main term of the 1-level density agrees with the Katz-Sarnak prediction, we only care about $\phi(0)/2$ and wish to bound the remaining integral by $O(1/\mathcal{L})$. This is possible by the PNT, by which

$$\theta(t) - t \ll te^{-c\sqrt{\log t}}$$

for some $c > 0$. By Stieltjes integration, we have

$$\begin{aligned} \int_1^\infty \frac{1}{t}\widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right) d(\theta(t)-t) &= -\int_1^\infty (\theta(t)-t)d\left(\frac{1}{t}\widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right)\right) \\ &= -\int_1^\infty \left(\frac{2}{\mathcal{L}}\widehat{\phi}'\left(\frac{2\log t}{\mathcal{L}}\right) - \widehat{\phi}\left(\frac{2\log t}{\mathcal{L}}\right)\right)\frac{\theta(t)-t}{t^2}dt. \end{aligned}$$

and by the PNT, the integral is bounded independently of $\mathcal{L}$.

All in all, we have

$$D(f,\phi) = \widehat{\phi}(0) + \frac{\phi(0)}{2} - P_1(f,\phi) - P_2(f,\phi) + O\left(\frac{1+\log\log 3N}{\mathcal{L}}\right), \tag{5.10}$$

where

$$P_1(f,\phi) = \sum_{p\nmid N} \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}},$$
$$P_2(f,\phi) = \sum_{p\nmid N} \lambda_f(p^2)\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{p\mathcal{L}}.$$

Already in (5.10), we can discern the predicted main term of the Katz-Sarnak heuristic. Before we can draw any conclusions, we need to handle the two sums $P_1(f,\phi)$ and $P_2(f,\phi)$. We will do this by averaging over the family $H_k^*(N)$, and use the Petersson trace formula.

5.2.2 Averaging $D(f,\phi)$ over the family $H_k^*(N)$

Now, we consider the averaged 1-level density

$$\mathcal{D}_{H_k^*(N)}(\phi) = \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)} D(f,\phi),$$

which we defined in (2.5). The term $\widehat{\phi}(0)+\phi(0)/2$ of $D(f,\phi)$, as well as the error term $O((1+\log\log 3N)/\mathcal{L})$, is independent of f and stays unchanged through averaging. What remains is to consider the average of the sums $P_1(f,\phi)$ and $P_2(f,\phi)$. The sum $P_1(f,\phi)$ requires the most careful analysis, due to two reasons. Firstly, we divide by $\sqrt{p}$ rather than p in each term, and secondly, we also have the quantity $\log p/\mathcal{L}$ rather than $2\log p/\mathcal{L}$ in the argument of $\widehat{\phi}$, leading to more terms than in $P_2(f,\phi)$. Therefore the first sum will determine the conditions for which the Density Conjecture is verified. The second sum is then handled similarly.

We thus start by considering the unaveraged sum

$$\mathcal{P}_k^*(\phi) = \sum_{f\in H_k^*(N)} P_1(f,\phi). \tag{5.11}$$

By definition of $P_1(f,\phi)$, we have

$$\mathcal{P}_k^*(\phi) = \sum_{f\in H_k^*(N)}\sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}} = \sum_{p\nmid N}\Delta_{k,N}^*(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}}. \tag{5.12}$$

We are again interested in splitting $\Delta_{k,N}^*(p)$ as in (4.45), possibly with other values for the parameters X and Y. This will split the sum on the right hand side of (5.12) in two. The latter is equal to

$$\sum_{p\nmid N}\Delta_{k,N}^{\infty}(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}}.$$

This can be estimated by (4.47), with $n = 1$, since the compact support of $\widehat{\phi}$ forces $\log p \ll \log kN$. We get an upper bound

$$\sum_{p \nmid N} \Delta^{\infty}_{k,N}(p) \widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right) \frac{2 \log p}{\sqrt{p}\mathcal{L}} \ll \frac{kN^{1-\varepsilon'}}{\mathcal{L}} \ll \frac{k\varphi(N)}{\mathcal{L}},$$

where we choose $X = Y = (kN)^{\varepsilon}$ with ε such that N has exponent $1 - \varepsilon'$. The estimation $N^{1-\varepsilon'} \ll \varphi(N)$ follows from [HW08, Theorem 327]. What is left is to consider (5.12) with $\Delta^*_{k,N}(p)$ replaced by $\Delta'_{k,N}(p)$. Following the steps in [ILS00, p. 91-92], we reorder the sum as

$$\mathcal{P}^*_k(\phi) = \frac{k-1}{12} \sum_{\substack{LM=N \\ L \le X}} \mu(L) M \sum_{\substack{(m,M)=1 \\ m \le Y}} \frac{1}{m} \sum_{c \equiv 0 \pmod{M}} \frac{1}{c} Q^*_k(m; c) + O\left(\frac{k\varphi(N)}{\log kN}\right), \tag{5.13}$$

where

$$Q^*_k(m; c) := 2\pi i^k \sum_{p \nmid N} S(m^2, p; c) J_{k-1}\left(\frac{4\pi m \sqrt{p}}{c}\right) \widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right) \frac{2 \log p}{\sqrt{p}\mathcal{L}}$$

contains the Kloosterman sum, the Bessel function and $\widehat{\phi}$. It is at this stage that we will see how the support of $\widehat{\phi}$ affects the size of $Q^*_k(m; c)$, and consequently $\mathcal{P}^*_k(\phi)$. We will assume that $\widehat{\phi}$ is supported in $(-\sigma, \sigma)$. We may then assume $p \le P = c_f^{\sigma'}$ where $\sigma' < \sigma$. We bound the Kloosterman sums by the Weil bound (4.5). We would like to use the bound (4.8) on the Bessel functions. This is allowed if

$$\frac{4\pi m \sqrt{P}}{c} \le \frac{k}{3},$$

since $p \le P$. Recalling the conditions $m \le Y$ and $c \ge M = N/L \ge N/X$, we impose that

$$12\pi XY\sqrt{P} \le kN. \tag{5.14}$$

With this, we have that

$$\frac{4\pi m \sqrt{P}}{c} \le \frac{4\pi XY\sqrt{P}}{N} \le \frac{k}{3}$$

as we wish to. We write out X, Y and P in terms of k and N and absorbing $12\pi XY$ into kN, we get

$$(k^2 N)^{\sigma'/2} \le (kN)^{1-\varepsilon},$$

and we see that this is the case if we choose

$$\sigma = 2\frac{\log kN}{\log k^2 N} = 1 + \frac{\log N}{\log k^2 N}. \tag{5.15}$$

Writing out the upper bounds (4.5) and (4.8), we get

$$Q^*_k(m; c) \ll 2^{-k} c^{-1/2} \sum_{\substack{p \nmid N \\ p \le P}} (m^2, p, c)^{1/2} \tau(c) m \ll 2^{-k} c^{\varepsilon - 1/2} \sum_{\substack{p \nmid N \\ p \le P}} m^2.$$

Since $m \le Y = (kN)^\varepsilon$ and $c \ge M \ge NX^{-1} = k^{-\varepsilon}N^{1-\varepsilon}$, we have $N \ll k^\varepsilon c$ and $m \ll (kc)^\varepsilon$. Estimating the sum over $p \le P$ by P gives us

$$Q_k^*(m;c) \ll 2^{-k}k^\varepsilon P c^{\varepsilon-1/2} \tag{5.16}$$

(cf. [ILS00, p. 92]). With $c = \ell M$ where $\ell \in \mathbb{Z}^+$, the inner sum over ℓ in the main term of $\mathcal{P}_k^*(\phi)$ in (5.13) is $O(1)$. Hence, the main term itself is bounded by

$$2^{-k}k^{1+\varepsilon}P \sum_{\substack{LM=N\\ L\le X}} M^{\varepsilon-1/2} \sum_{\substack{(m,M)=1\\ m\le Y}} \frac{1}{m} = 2^{-k}k^{1+\varepsilon}PN^{\varepsilon-1/2} \sum_{\substack{LM=N\\ L\le X}} L^{1/2-\varepsilon} \sum_{\substack{(m,M)=1\\ m\le Y}} \frac{1}{m}.$$

We bound each term by X and 1, respectively, and bound the sums by their number of terms. All in all, we obtain

$$\mathcal{P}_k^*(\phi) \ll 2^{-k}k^{1+\varepsilon}PN^{\varepsilon-1/2} + \frac{k\varphi(N)}{\log kN}.$$

If the former term can be absorbed into the latter, then the sum $\mathcal{P}_k^*(\phi)$ gives no contribution to the main term when we average by $|H_k^*(N)| \asymp k\varphi(N)$. Since 2^{-k} decays faster in k than any polynomial, we may ignore all k-dependencies. The relevant factors in N from both terms are $N^{\sigma'-1/2+\varepsilon}$ in the main term, and $N^{1-\varepsilon'}$ in the second term. We thus see that we would need

$$N^{\sigma'-1/2+\varepsilon} \ll N,$$

which is the case if $\sigma \le 3/2$. This finishes the analysis of $\mathcal{P}_k^*(\phi)$.

We now briefly describe the analysis when replacing $P_1(f,\phi)$ by $P_2(f,\phi)$ in (5.11). The factor $\lambda_f(p)$ is replaced by $\lambda_f(p^2)$, there is an extra factor 2 in the argument of $\widehat{\phi}$, and the factor $\sqrt{p}$ in the denominator is replaced by p. We reorder the sums and split $\Delta_{k,N}(p^2)$ as before. The complementary sum $\Delta_{k,N}^\infty(p^2)$ is estimated by (4.47) similarly to before, when replacing $n = p$ by $n = p^2$. The sum $\Delta_{k,N}'(p^2)$ may now contain a main term not present in the previous case, due to the square p^2. However, this will still leaves an error of size $k\varphi(N)/\mathcal{L}$. The modified version of $Q_k^*(m;c)$ is

$$Q_k^*(m;c) = 2\pi i^k \sum_{p\nmid N} S(m^2,p^2;c) J_{k-1}\left(\frac{4\pi mp}{c}\right)\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{p\mathcal{L}},$$

which we estimate by the same bounds as before. The condition (5.14) is now replaced by

$$12\pi XYP \le kN,$$

where $P = c_f^{\sigma'/2}$, the division by 2 being due to the extra factor 2 in the argument of $\widehat{\phi}$. From here, we deduce the same requirement on σ as in (5.15).

We obtain the same bound (5.16) on $Q_k^*(m;c)$, and therefore the same bound on $P_k^*(\phi)$ as before. However, P has a new meaning now, giving the requirement that $\sigma \le 3$, rather than $3/2$. The extra factor 2 essentially stems from occurring in the argument of $\widehat{\phi}$. Hence, the requirements on σ are laxer in this case, and are thus determined by $P_1(f,\phi)$.

In total, we have proved the following result, which is a part of [ILS00, Theorem 5.1] (slightly modified).

Theorem 5.1. *Let $k \geq 2$ be an even integer, $N \geq 1$ be a squarefree integer and ϕ be an even Schwartz function whose Fourier transform $\widehat{\phi}$ is supported in $(-\sigma, \sigma)$ for*

$$\sigma = \min\left(1 + \frac{\log N}{\log k^2 N}, \frac{3}{2}\right). \tag{5.17}$$

Assuming the Riemann hypothesis for $L(s, f)$ and $L(s, \mathrm{sym}^2(f))$, we have that

$$\lim_{kN \to \infty} \mathcal{D}_{H_k^*(N)}(\phi) = \int_{-\infty}^{\infty} \phi(x) w(O)(x) dx$$

where

$$w(O) = 1 + \frac{\delta_0}{2}.$$

Remark 5.2. There are several directions in which Iwaniec, Luo and Sarnak expands this result. One is that they split the family $H_k^*(N)$ into two families $H_k^{\pm}(N)$ depending on whether the root number $\varepsilon_f = \pm 1$. The basic observation then is that

$$1 \pm \varepsilon_f = 1 \pm i^k \mu(N) \lambda_f(N) N^{1/2} = 2\delta_{f \in H_k^{\pm}(N)}.$$

From this, we recognise that to sieve out the newforms with constant sign, we have

$$2\Delta_{k,N}^{\pm}(m, n) = \Delta_{k,N}^*(m, n) \pm i^k \mu(N) N^{1/2} \Delta_{k,N}^*(m, nN)$$

if $(n, N) = 1$. Here, $\Delta_{k,N}^{\pm}(m, n)$ is defined similarly to $\Delta_{k,N}^*(m, n)$, with the difference that we sum over $H_k^{\pm}(N)$ rather than $H_k^*(N)$. Hence some of the analysis of $H_k^{\pm}(N)$ boils down to the corresponding analysis of $H_k^*(N)$, and this sometimes can be carried out in parallel to the analysis conducted here. The end result is that the 1-level densities for the families $H_k^+(N)$ and $H_k^-(N)$ are equal to $w(SO(\text{even}))$ and $w(SO(\text{odd}))$, respectively.

Another expansion of Theorem 5.1 is to possibly enlarge the support for $\widehat{\phi}$. By a more careful analysis of sums of Kloosterman sums, Iwaniec, Luo and Sarnak manage to improve Theorem 5.1 to hold for $\sigma = 2$. Similar results hold when splitting the family according to sign. See [ILS00, Sections 6 and 7] for details.

Remark 5.3. Theorem 5.1 is flexible in the sense that we have not made any assumptions on k and N other than k being even and N being squarefree. It is not uncommon to distinguish between the cases when k tends to infinity and $N = 1$, and when k is fixed and N tends to infinity. These stronger conditions may lead to stronger results. We will take this approach in the next section.

5.3 Lower order terms for the family $H_k^*(N)$

Having arrived at Theorem 5.1, we now turn to a closer analysis of the terms occurring in the explicit formula. We wish to find an expression with a power-saving error term, which is significantly smaller than $1/\log kN$ allowed in the main term computation. The reason for this is that we want to make a prediction of the behaviour of the 1-level density using the Ratios Conjecture in the next chapter. As having an error term of size $N^{-1/2+\varepsilon}$ is a part of the Ratios Conjecture, we aim for an error term of comparable size in our number-theoretical analysis.

As observed in Remark 5.3, when we aim for stronger results we might have to make stricter assumptions to prove them. Indeed, in this section we will assume that k is fixed and that N will tend to infinity through the primes. An advantage of this is that the sums over $p|N$ will now contain only one term. Another virtue is that all dependencies on k in the error terms may be viewed as absolute.

In this section we follow the procedure of [Mil09, Section 5]. We start from the explicit formula (5.6) as before, and turn to the analysis of the prime sum. As N now is prime, we have $|\alpha_f^\nu(N)+\beta_f^\nu(N)| = |\lambda_f^\nu(N)| = 1/N^{\nu/2}$ by (3.32) and the convention directly after (3.46). Therefore, the term where $p=N$ is of order

$$\sum_{\nu=1}^{\infty}\frac{1}{N^\nu} \ll \frac{1}{N}.$$

For $p\nmid N$ and $\nu\geq 2$, we have $\alpha_f^\nu(p)+\beta_f^\nu(p) = \lambda(p^\nu)-\lambda_f(p^{\nu-2})$, by (3.48). Splitting the sum over primes into whether $\nu=1,2$ or $\nu\geq 3$ with our new conditions imposed, we get

$$\begin{aligned} D(f,\phi) = {} & \frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\left[2\log\left(\frac{\sqrt{N}}{\pi}\right)+\psi\left(\frac14+\frac{k\pm1}{4}+\frac{\pi i u}{\mathcal{L}}\right)\right]\phi(u)du \\ & -2\sum_{p\neq N}\frac{\lambda_f(p)}{\sqrt{p}}\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}} \\ & -2\sum_{p\neq N}\frac{\lambda_f(p^2)-1}{p}\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}} \\ & -2\sum_{p\neq N}\sum_{\nu=3}^{\infty}\frac{\lambda_f(p^\nu)-\lambda_f(p^{\nu-2})}{p^{\nu/2}}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{\log p}{\mathcal{L}}+O\left(\frac1N\right). \end{aligned}$$

Now, we average over $H_k^*(N)$. Adopting the notation of Miller (see [Mil09, eq. 4.5]), we have

$$\begin{aligned} \mathcal{D}_{H_k^*(N)}(\phi) = {} & \frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\left[2\log\left(\frac{\sqrt{N}}{\pi}\right)+\psi\left(\frac14+\frac{k\pm1}{4}+\frac{\pi i u}{\mathcal{L}}\right)\right]\phi(u)du \\ & +\sum_{p}\frac1p\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}-S_1(\phi)-S_2(\phi)-S_3(\phi)+O\left(\frac1N\right), \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} S_1(\phi) &= \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\sum_{p\neq N}\frac{\lambda_f(p)}{\sqrt{p}}\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}, \\ S_2(\phi) &= \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\sum_{p\neq N}\frac{\lambda_f(p^2)}{p}\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}, \\ S_3(\phi) &= \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\sum_{p\neq N}\sum_{\nu=3}^{\infty}\frac{\lambda_f(p^\nu)-\lambda_f(p^{\nu-2})}{p^{\nu/2}}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}. \end{aligned}$$

We will analyze each of these terms in turn, as done by Miller. The main tool is Proposition 4.9.

5.3.1 The sum $S_1(\phi)$

We turn our attention to the first sum $S_1(\phi)$. As previously noted, this sum requires the most careful analysis, due to its relatively many and large terms. After reordering the sum, we get the inner sum $\Delta^*_{k,N}(p)$. This is split into the two sums (4.45), with the parameters X and Y restricting the sum $\Delta'_{k,N}(p)$ as before. This splits $S_1(\phi)$ into two terms

$$S_1(\phi) = \frac{1}{|H_k^*(N)|}\sum_{p\neq N}\Delta'_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}} + \frac{1}{|H_k^*(N)|}\sum_{p\neq N}\Delta^{\infty}_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}} \tag{5.19}$$

which we analyze in turn. For the first term, we observe that if $X < N$, then

$$\Delta'_{k,N}(p) = \frac{k-1}{12}\sum_{\substack{LM=N\\ L\leq X}}\frac{\mu(L)M}{\nu((p,L))}\sum_{\substack{(m,M)=1\\ m\leq Y}}\frac{1}{m}\Delta_{k,M}(m^2,p) = \frac{(k-1)N}{12}\sum_{\substack{(m,N)=1\\ m\leq Y}}\frac{1}{m}\Delta_{k,N}(m^2,p),$$

since N is prime. Also, since p is not a square, from $\Delta_{k,N}(m^2,p)$ we only get the Bessel-Kloosterman sum in the Petersson trace formula. After reordering, we have the first term of (5.19) being equal to

$$\frac{(k-1)N}{12|H_k^*(N)|}\sum_{\substack{(m,N)=1\\ m\leq Y}}\frac{1}{m}\sum_{c\equiv 0 \pmod N}\frac{1}{c}Q_k^*(m;c),$$

where similarly to before,

$$Q_k^*(m;c) = 2\pi i^k\sum_{p\neq N}S(m^2,p;c)J_{k-1}\left(\frac{4\pi m\sqrt{p}}{c}\right)\widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right)\frac{2\log p}{\sqrt{p}\mathcal{L}}.$$

To proceed, we will reuse the bound (5.16), that is

$$Q_k^*(m;c) \ll 2^{-k}k^{\varepsilon}Pc^{\varepsilon-1/2}.$$

Recall that this holds when $p \leq P = c_f^{\sigma'}, \sigma' < \sigma$ and $4\pi m\sqrt{P}/c \leq k/3$ (where $\widehat{\phi}$ is supported in $(-\sigma,\sigma)$). We deduce that the first term of (5.19) is bounded by

$$2^{-k}k^{\varepsilon}\sum_{\substack{(m,N)=1\\ m\leq Y}}\sum_{c\equiv 0 \pmod N}\frac{(k^2N)^{\sigma'}}{c^{3/2-\varepsilon}}.$$

We write $c = N\ell$ and note that the resulting sum over $\ell \in \mathbb{Z}^+$ converges. Estimating the outer sum by its number of terms and removing all k-dependencies, we see that we get a contribution of order

$$N^{\sigma'-3/2+\varepsilon}Y \tag{5.20}$$

from the first term of (5.19).

We turn to the second term of (5.19). From the compact support of $\widehat{\phi}$, we may restrict the sum to $p \le c_f^{\sigma'}$, where $\sigma' < \sigma$. Recall (4.47), which gives

$$\sum_{\substack{p \neq N \\ p \le c_f^{\sigma'}}} \Delta_{k,N}^{\infty}(p) \frac{\log p}{\sqrt{p}} \ll kN(X^{-1} + Y^{-1/2})(kNXY)^{\varepsilon}$$

when $n = 1$. Writing the second term of (5.19) as a Stieltjes integral and using partial integration, we have

$$\begin{aligned}
&\frac{1}{|H_k^*(N)|} \sum_{p \neq N} \Delta_{k,N}^{\infty}(p) \widehat{\phi}\left(\frac{\log p}{\mathcal{L}}\right) \frac{2 \log p}{\sqrt{p}\mathcal{L}} \\
&\quad = \frac{-1}{|H_k^*(N)|\mathcal{L}} \int_1^{c_f^{\sigma'}} 2 \sum_{\substack{p \neq N \\ p \le t}} \Delta_{k,N}^{\infty}(p) \frac{\log p}{\sqrt{p}} d\left(\widehat{\phi}\left(\frac{\log t}{\mathcal{L}}\right)\right) \\
&\quad = \frac{-1}{|H_k^*(N)|\mathcal{L}} \int_1^{c_f^{\sigma'}} 2 \sum_{\substack{p \neq N \\ p \le t}} \Delta_{k,N}^{\infty}(p) \frac{\log p}{\sqrt{p}} \widehat{\phi}'\left(\frac{\log t}{\mathcal{L}}\right) \frac{1}{t\mathcal{L}} dt \\
&\quad \ll \frac{1}{kN\mathcal{L}} \int_1^{c_f^{\sigma'}} kN(X^{-1} + Y^{-1/2})(kNXY)^{\varepsilon} \frac{1}{t\mathcal{L}} dt \\
&\quad \ll N^{\varepsilon}(X^{-1} + Y^{-1/2}).
\end{aligned}$$

In total, we therefore have the bound

$$S_1(\phi) \ll N^{\sigma' - 3/2 + \varepsilon} Y + N^{\varepsilon}(X^{-1} + Y^{-1/2}). \tag{5.21}$$

Since X only occurs with a negative exponent, we choose it as large as possible under the requirement $X < N$, namely as $N - 1$. This means that the term involving X will be of order $N^{-1+\varepsilon}$. There are two errors involving Y, one of which grows and one of which shrinks as Y grows or shrinks. To balance them, we set them to be equal. From this, we get $Y = N^{(3-2\sigma')/3}$. Inserting this in (5.21) yield

$$S_1(\phi) \ll N^{-(3-2\sigma')/6+\varepsilon}.$$

We see that the largest possible choice is $\sigma = 3/2$ in order for the error to be of power-saving size. We also need the bound (5.15), in order to use the estimation (5.16) of $Q_m^*(m; c)$. Hence, the conditions on σ arising from the analysis of $S_1(\phi)$ are the same as in Theorem 5.1. We may assume that $\sigma = 3/2$, since in the end we want N to tend to infinity.

5.3.2 The sums $S_2(\phi)$ and $S_3(\phi)$

We turn to the analysis of the two remaining sums $S_2(\phi)$ and $S_3(\phi)$. They are handled similarly, and we start with $S_3(\phi)$. Here, the factor $\Delta_{k,N}^*(p^{\nu}) - \Delta_{k,N}^*(p^{\nu-2})$ appears after reordering. We recall the two contributions from (4.49), a main and an error term. A main term occurs only for those prime powers where the exponent ν is even. When we divide by $|H_k^*(N)|$, we should replace $\lambda_f(p^\nu)$ by $1/p^{\nu/2}$. Doing so gives the main contribution

$$\sum_{\substack{\nu\geq 4\\ \nu\equiv 0 \pmod 2}} \sum_{p\neq N} \frac{1-p}{p^\nu}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}. \tag{5.22}$$

The main term occur if $p^\nu \leq Y^2$, where Y is the second parameter restricting $\Delta'_{k,N}(n)$. We therefore need to estimate the tail of the sum (5.22), where $p^\nu > Y^2$, or $p > \sqrt{Y}$ since $\nu \geq 4$. The tail is of order

$$\sum_{\substack{\nu\geq 4\\ \nu\equiv 0 \pmod 2}} \sum_{\substack{p\neq N\\ p>\sqrt{Y}}} \frac{1-p}{p^\nu}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}$$
$$\ll \sum_{\substack{\nu\geq 4\\ \nu\equiv 0 \pmod 2}} \sum_{n>\sqrt{Y}} n^{\nu-1} \ll \sum_{\substack{\nu\geq 4\\ \nu\equiv 0 \pmod 2}} \frac{1}{Y^{\nu/2-1}} \ll Y^{-1},$$

which with the choice of Y above will be absorbed into the other error terms.

Recall that the error term of $\Delta^*_{k,N}(p^\nu)$ in (4.49) is of size $p^{\nu/6}(kN)^{2/3}$. To estimate the contribution to $S_3(\phi)$ arising from this error term, we need to specify the conditions imposed on ν and p from the compact support of $\widehat{\phi}$. The first observation is that we must have

$$\frac{\nu\log p}{\mathcal{L}} \leq \sigma'.$$

From this, we draw two conclusions. On the one hand, fixing $p = 2$ gives the restriction $\nu \leq \sigma'\mathcal{L}/\log 2$ on the exponent ν. On the other hand, we have

$$p \leq c_f^{\sigma'/\nu} \leq c_f^{\sigma'/3},$$

since $\nu \geq 3$. Hence, we obtain the upper bound

$$\frac{(kN)^{2/3}}{kN} \sum_{3\leq\nu\leq\sigma'\mathcal{L}/\log 2} \sum_{p\leq c_f^{\sigma'/3}} \frac{p^{\nu/6}}{p^{\nu/2}} \ll \frac{\mathcal{L}}{(kN)^{1/3}} \sum_{p\leq c_f^{\sigma'/3}} \frac{1}{p} \ll \frac{\mathcal{L}^2}{(kN)^{1/3}} \ll N^{-1/3+\varepsilon}.$$

In total, we have from $S_3(\phi)$ a contribution

$$S_3(\phi) = \sum_{\substack{\nu\geq 4\\ \nu\equiv 0 \pmod 2}} \sum_{p\neq N} \frac{1-p}{p^\nu}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}} + O\left(N^{-1/3+\varepsilon}\right).$$

We turn to the analysis of $S_2(\phi)$, which works similarly except that we have no sum over indices $\nu \geq 3$, but instead a single term corresponding to $\nu = 2$. The main term is

$$\sum_{p\neq N} \frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}.$$

Also here, to motivate the inclusion of all the terms, we need to bound the tail of this series. In this case, the primes are bounded below by Y. The procedure is similar to before, and the tail is again bounded by Y^{-1}. The error term from $S_2(\phi)$ is bounded by

$$\frac{(kN)^{2/3}}{kN}\sum_{p\le c_f^{\sigma'/2}}\frac{1}{p^{2/3}} \ll \frac{1}{(kN)^{1/3}}\int_1^{c_f^{\sigma'/2}}\frac{1}{t^{2/3}}dt \ll N^{-1/3+\sigma'/6}.$$

Hence, from $S_2(\phi)$ we have the contribution

$$S_2(\phi) = \sum_{p\neq N}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}} + O\left(N^{-(2-\sigma')/6}\right).$$

Grouping together the main term in $S_2(\phi)$ with the term

$$\sum_p \frac{1}{p}\widehat{\phi}\left(\frac{2\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}}$$

in (5.18), we get a term of the same shape as in $S_3(\phi)$, corresponding to the index $\nu = 2$. After we collect the error terms, we arrive at the following result.

Proposition 5.4. *For k fixed and N tending to infinity through the primes, we have*

$$\begin{aligned}\mathcal{D}_{H_k^*(N)}(\phi) &= \frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\left[2\log\left(\frac{\sqrt{N}}{\pi}\right) + \psi\left(\frac{1}{4}+\frac{k\pm 1}{4}+\frac{\pi i u}{\mathcal{L}}\right)\right]\phi(u)du \\ &+ \sum_{\substack{\nu\ge 2\\ \nu\equiv 0 \pmod 2}}\sum_{p\neq N}\frac{p-1}{p^\nu}\widehat{\phi}\left(\frac{\nu\log p}{\mathcal{L}}\right)\frac{2\log p}{\mathcal{L}} + O\left(N^{-(3-2\sigma')/6+\varepsilon} + N^{-1/3+\varepsilon}\right) \quad (5.23)\end{aligned}$$

for any $\sigma' < \sigma = 3/2$.

6

The 1-level density through the Ratios Conjecture

In this chapter, we analyze the 1-level density through the Ratios Conjecture. We will see that the Ratios Conjecture correctly predicts the shape of the 1-level density as computed in Section 5.3, down to the power saving error term obtained there. Indeed, the Ratios Conjecture goes further than that, and predicts an error term of size $N^{-1/2+\varepsilon}$, without any specific requirements on the support of $\widehat{\phi}$ other than compactness. From this, we may draw two conclusions. First, the agreement with previous calculations support the claim that the Ratios Conjecture is a reasonable heuristic. Second, we see that if the Ratios Conjectrue should turn out to be true, then it would imply the full Density Conjecture.

The actual computations are carried out in Section 6.3. Before we do them, we review the general Ratios Conjecture recipe in Section 6.1, and adapt it to our specific setting in Section 6.2. The first ones to formulate the Ratios Conjecture recipe were Conrey, Farmer and Zirnbauer in [CFZ08]. This chapter mostly follows the material presented by Miller, in particular [Mil09, Section 5].

6.1 The Ratios Conjecture recipe

The Ratios Conjecture is a heuristic method of making conjectures about sums of ratios of L-functions $L(s, f)$, where f belongs to a natural family $\mathcal{F}$. Recalling the notation of Section 2.2, it is common to consider the truncated finite family $\mathcal{F}_X$, and then let X tend to infinity. We will restrict to the situation where the ratios have precisely one factor in both the numerator and the denominator, although more general situations are possible (see e.g. [CFZ08, Section 5.1]). For the sake of concordance with the rest of the report, we will also only consider the case where we take the unweighted averages over the family, although it is possible to consider weighted averages as well. That is, we are studying sums of the form

$$R_{\mathcal{F}}(\alpha, \gamma) := \frac{1}{|\mathcal{F}_X|} \sum_{f \in \mathcal{F}_X} \frac{L(1/2+\alpha, f)}{L(1/2+\gamma, f)}. \tag{6.1}$$

The main reason for our interest in the sum in this report stems from its partial derivative

$$\frac{\partial}{\partial\alpha} R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=s} = \frac{1}{|\mathcal{F}_X|}\sum_{f\in\mathcal{F}_X}\frac{L'(1/2+s,f)}{L(1/2+s,f)},$$

which is useful when computing the 1-level density. The following conditions on α and γ are standard:

$$-1/4 < \Re(\alpha) < 1/4, \tag{6.2}$$

$$1/\log|\mathcal{F}| < \Re(\gamma) < 1/4, \tag{6.3}$$

$$\Im(\alpha), \Im(\gamma) \ll_\varepsilon |\mathcal{F}|^{1-\varepsilon} \text{ for any } \varepsilon > 0. \tag{6.4}$$

The requirement that the real parts of α and γ lie in the interval $(-1/4, 1/4)$ ensures that the products over the primes which we obtain in the computations below converges absolutely. We also need to keep $\Re(\gamma)$ positive in order to avoid the zeroes of $L(1/2+\gamma, f)$.

The process of making a conjecture about the asymptotic behaviour of a sum of the form (6.1) follow a more or less established recipe, which we now outline.

1. Write down the approximate functional equation for $L(s, f)$, and discard the error term. This will give two finite sums whose length are governed by two parameters x and y, where $xy \sim c_f$.
2. Write the factor
$$\frac{1}{L(s,f)} = \sum_{h=1}^{\infty}\frac{\mu_f(h)}{h^s}$$
as a Dirichlet series for a suitable multiplicative function μ_f.
3. Expand the sum (6.1) by using the representations of $L(1/2+\alpha, f)$ and $1/L(1/2+\gamma, f)$ from the steps 1 and 2, and replace the summands by their expected values when averaging over the family $\mathcal{F}_X$.
4. Complete the resulting expressions by letting the parameters x and y tend to infinity.
5. The prediction of the Ratios Conjecture is that the original sum is equal to the obtained expression, up to an error term of size $|\mathcal{F}_X|^{-1/2+\varepsilon}$.

We emphasize that the Ratios Conjecture is not an exact method, but a heuristic tool to get an idea of what results one might expect. There are a lot of error terms which we ignore in various steps, and the fact that the obtained conjectures seem to be able to accurately predict the behaviour of ratios of L-functions is quite remarkable.

In [CFZ08], the third step includes the act of replacing the root numbers ε_f with their average over the family. This is not included here, and this modification was consciously made by Miller. Recall that if $N = 1$, then $\varepsilon_f = i^k$ is constant over the family. In this case, the average is equal to ε_f and does not change anything. However, when $N > 1$, then around half of the newforms have $\varepsilon_f = \pm 1$ each (see [ILS00, Corollary 2.14]). Hence, the average would be equal to 0, and the term involving ε_f would vanish. We will not apply this average, but we will see that the term will be very small when N tends to infinity, namely of order $1/N$. As noted by Miller, this supports the claim that it is reasonable to take the average of the root numbers as a part of the recipe.

6.2 Following the recipe

We wish to adapt the Ratios Conjecture to our specific setting. Recall that the family of L-functions we investigate are those attached to newforms $f \in H_k^*(N)$. This family is finite, and the sum which we study is therefore

$$R_{H_k^*(N)}(\alpha,\gamma) := \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\frac{L(1/2+\alpha,f)}{L(1/2+\gamma,f)}, \tag{6.5}$$

and its partial derivative

$$R'_{H_k^*(N)}(s,s) := \frac{\partial}{\partial\alpha}R_{H_k^*(N)}(\alpha,\gamma)\Big|_{\alpha=\gamma=s} = \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\frac{L'(1/2+s,f)}{L(1/2+s,f)}.$$

We follow the exposition in [Mil09, Section 5.3]. Miller's main goal concerns the setting where the sums are weighted by the (modified) harmonic weights $\omega_f(N)$. However, the modifications needed for the unweighted setting do not change the situation too much, and they are briefly outlined by Miller. Here, we will flesh out the details of the unweighted computations. In concordance with Section 5.3, we focus on the case where k is fixed and N tend to infinity through the primes.

6.2.1 The approximate functional equation

Let $f \in H_k^*(N)$. Recall the functional equation (3.43), that is

$$L(s,f) = \varepsilon_f X_L(s) L(1-s,f),$$

where

$$\varepsilon_f = i^k\mu(N)\lambda_f(N)N^{1/2}$$

is the sign of the functional equation of f, and $X_L(s)$ was defined in (3.44). More precisely, we have

$$\begin{aligned} X_L(s) = \frac{L_\infty(1-s,f)}{L_\infty(s,f)} &= \left(\frac{\sqrt{N}}{2\pi}\right)^{1-2s}\frac{\Gamma((1-s)/2+(k-1)/2)}{\Gamma(s/2+(k-1)/2)} \\ &= \left(\frac{\sqrt{N}}{\pi}\right)^{1-2s}\frac{\Gamma((1-s)/2+(k-1)/4)\Gamma((1-s)/2+(k+1)/4)}{\Gamma((s/2+(k-1)/4)\Gamma((s/2+(k+1)/4)}. \end{aligned} \tag{6.6}$$

The last step is the duplication formula for the Gamma function. Differentiating $\log X_L(s)$ yields

$$-\frac{X'_L(s)}{X_L(s)} = 2\log\left(\frac{\sqrt{N}}{\pi}\right) + \frac{1}{2}\psi\left(\frac{1-s}{2}+\frac{k\pm 1}{4}\right) + \frac{1}{2}\psi\left(\frac{s}{2}+\frac{k\pm 1}{4}\right), \tag{6.7}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$.

The *approximate functional equation* takes the shape

$$L(s,f) = \sum_{m\leq x} \frac{\lambda_f(m)}{m^s} + \varepsilon_f X_L(s) \sum_{n\leq y} \frac{\lambda_f(n)}{n^{1-s}} + \text{ Error}, \tag{6.8}$$

where x and y are parameters satisfying $xy \sim c_f$ (this version is from [Mil09, Lemma 2.3]. See [IK04, Theorem 5.3] for a smooth version.). Recall that $c_f = k^2N$. It is common to choose $x = y \sim \sqrt{k^2N}$, and we will follow this convention. As a part of the recipe, we will discard the error term in our subsequent use of the approximate functional equation.

6.2.2 Expanding the L-function in the denominator

Recall the Euler product

$$L(s,f) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}$$

of the L-function attached to $f \in H_k^*(N)$, where χ_0 is the trivial Dirichlet character modulo N (see (3.38) and (3.39)). Inverting and expanding the resulting product yields

$$\frac{1}{L(s,f)} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right) = \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^s}, \tag{6.9}$$

where $\mu_f(n)$ is the multiplicative function defined on prime powers by

$$\mu_f(p^m) = \begin{cases} 1, & m = 0, \\ -\lambda_f(p), & m = 1, \\ \chi_0(p), & m = 2, \\ 0, & m \geq 3. \end{cases} \tag{6.10}$$

This holds *a priori* for $\Re(s) > 1$, where the convergence is absolute. Assuming the GRH for $L(s,f)$, the series (6.9) converges conditionally for $\Re(s) > 1/2$ (due to cancellations in $\mu_f(n)$; see [IK04, Proposition 5.14, part (3)]).

6.2.3 Averaging over the family

Inserting (6.8) (with the error term removed) and (6.9) into (6.5) gives us

$$R_{H_k^*(N)}(\alpha,\gamma) = \frac{1}{|H_k^*(N)|} \sum_{f\in H_k^*(N)} \left(\sum_{m\leq x} \sum_{h=1}^{\infty} \frac{\mu_f(h)\lambda_f(m)}{h^{1/2+\gamma} m^{1/2+\alpha}} \right.$$
$$\left. + i^k \mu(N) N^{1/2} X_L\left(\frac{1}{2}+\alpha\right) \sum_{n\leq y} \sum_{h=1}^{\infty} \frac{\mu_f(h)\lambda_f(N)\lambda_f(n)}{h^{1/2+\gamma} n^{1/2-\alpha}} \right). \tag{6.11}$$

In the computations to follow, it is useful to remember that $(m, N) = 1$ since N is prime and $m \leq x \asymp \sqrt{N}$, and similarly for n. We treat the two parts of (6.11) separately, starting with the first. To be able to distinguish which terms have significant contribution and which have not, we write the inner sum as an Euler product. As factors, we get sums over prime powers p^m, p^h (to save notation). In total, we write

$$\sum_{m \leq x} \sum_{h=1}^{\infty} \frac{\mu_f(h)\lambda_f(m)}{h^{1/2+\gamma} m^{1/2+\alpha}} \quad \text{as} \quad \prod_p \sum_{m,h} \frac{\mu_f(p^h)\lambda_f(p^m)}{p^{(1/2+\gamma)h+(1/2+\alpha)m}}.$$

Here, we have suppressed the exact conditions on m. We have to be careful with calling this an equality, since the sum over h only converges conditionally. However, for our purposes of distinguishing significant contributions, it will give the correct result.

We now consider a general factor in the Euler product for which $p \leq x$. By definition of μ_f, the only values of h for which $\mu_f(p^h)$ is nonzero are $0, 1$ and 2. Hence, we have

$$\prod_{p \leq x} \left(1 - \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1+2\gamma}}\right) \left(\sum_m \frac{\lambda_f(p^m)}{p^{(1/2+\alpha)m}}\right) \tag{6.12}$$

(remember that $\chi_0(p) = 1$, since $p \leq x \asymp \sqrt{N}$). The shape of the corresponding factors when $p > x$ could be investigated, but we are not interested in them since we will let x tend to infinity anyway, per step 4 in the recipe. However, we may only do so *after* we have replaced the summands by their expected values in step 3. This is what we turn our attention towards now.

For us, the relevant quantities to be averaged are the Hecke eigenvalues $\lambda_f(p^m)$. Thus, we use Proposition 4.9 to keep terms with significant contribution, and discard negligible ones (see also Remark 6.2). To apply the formula, we need to ensure that we are dealing with sums of pure eigenvalues. One such term is directly obtained from (6.12), namely

$$\prod_{p \leq x} \left(1 + \frac{1}{p^{1+2\gamma}}\right) \left(\sum_m \frac{\lambda_f(p^m)}{p^{(1/2+\alpha)m}}\right).$$

In the remaining term

$$-\prod_{p \leq x} \frac{\lambda_f(p)}{p^{1/2+\gamma}} \left(\sum_m \frac{\lambda_f(p^m)}{p^{(1/2+\alpha)m}}\right),$$

we have a product $\lambda_f(p)\lambda_f(p^m) = \lambda_f(p^{m-1}) + \lambda_f(p^{m+1})$ when $m \geq 1$, by (3.30). The contribution from $m = 0$ is discarded, since p is not a square. Shifting the resulting sum yields

$$-\prod_{p \leq x} \frac{1}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\lambda_f(p^m) + \lambda_f(p^{m+2})}{p^{(1/2+\alpha)m}}.$$

In Proposition (4.9), the main term occurs only if n is a square. Hence, we should only keep the powers p^m where $m = 2k$ is even, and discard the other terms. This means that we get a total contribution of

$$\prod_{p \leq x} \left(\left(1 + \frac{1}{p^{1+2\gamma}}\right) \sum_{k=0}^{\infty} \frac{\lambda_f(p^{2k})}{p^{(1+2\alpha)k}} - \frac{1}{p^{1+\alpha+\gamma}} \sum_{k=0}^{\infty} \frac{\lambda_f(p^{2k}) + \lambda_f(p^{2k+2})}{p^{(1+2\alpha)k}}\right)$$

from the first term of (6.11). When dividing by $|H_k^*(N)|$, $\lambda_f(n)$ is replaced by $1/\sqrt{n}$ when n is a square. This means that in our case, $\lambda_f(p^{2k})$ is replaced by $1/p^k$. This leaves us with geometric sums. When these are computed, we may let the parameter x tend to infinity and obtain the contribution

$$\prod_p \left(1 - \frac{1}{p^{2+2\alpha}}\right)^{-1} \left(1 + \frac{1}{p^{1+2\gamma}} - \frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right) \tag{6.13}$$

from the first term in (6.11).

We turn to the second term in (6.11). It is handled similarly and we do not need to do the calculations from scratch. The essential differences are that the factor $\lambda_f(N)$ is present, and that we replace α by $-\alpha$. When we wrote the first sum as an Euler product, the factors where $p > x$ where ignored. Now, there is one such factor which we need to take into account, namely the one where $p = N$. In this case, there can be no factors coming from $\lambda_f(n)$ since $n \leq y$ and $y \sim \sqrt{N}$, but the factors coming from $\mu_f(h)$ are present and equal to

$$1 + \frac{\mu_f(p)}{p^{1/2+\gamma}} + \frac{\chi_0(p)}{p^{1+2\gamma}}\bigg|_{p=N} = 1 - \frac{\lambda_f(N)}{N^{1/2+\gamma}}.$$

Multiplying this by $\lambda_f(N)$, we get two terms. The first involves $\lambda_f(N)$ and we therefore get $\Delta_{k,N}^*(N)$ when letting the sum range over $H_k^*(N)$. This term is discarded, since N is not a square. The other term involves a sum of the squared Hecke eigenvalue $\lambda_f(N)^2 = 1/N$ (see (3.32)), which is independent of f. Averaging over $H_k^*(N)$ therefore does not change the value of the summand. Replacing α by $-\alpha$, we see that the second part of (6.11) leaves a contribution

$$-\frac{i^k\mu(N)}{N^{1+\gamma}} X_L\left(\frac{1}{2}+\alpha\right) \prod_p \left(1 - \frac{1}{p^{2-2\alpha}}\right)^{-1} \left(1 + \frac{1}{p^{1+2\gamma}} - \frac{p+1}{p}\frac{1}{p^{1-\alpha+\gamma}}\right).$$

In order to facilitate the differentiation computation in the next section, we factor out $\zeta(1+2\gamma)$, as well as $1/\zeta(1-\alpha+\gamma)$. The first factor isolates a polar behaviour at the origin, while the second makes the computations easier (cf. [CS07, eq. 2.55]). We obtain

$$-\frac{i^k\mu(N)}{N^{1+\gamma}} X_L\left(\frac{1}{2}+\alpha\right) \frac{\zeta(2-2\alpha)\zeta(1+2\gamma)}{\zeta(1-\alpha+\gamma)} A(-\alpha,\gamma), \tag{6.14}$$

where

$$A(-\alpha,\gamma) = \prod_p \left(1 + \frac{1}{p^{1+2\gamma}} - \frac{p+1}{p}\frac{1}{p^{1-\alpha+\gamma}}\right)\left(1 - \frac{1}{p^{1+2\gamma}}\right)\left(1 - \frac{1}{p^{1-\alpha+\gamma}}\right)^{-1}$$
$$= \prod_p \left(1 + \frac{p^{1+\gamma} + p^{\gamma} - p^{1+3\gamma} - p^{1-\alpha}}{p^{2+3\gamma}(p^{1-\alpha+\gamma} - 1)}\right). \tag{6.15}$$

Combining (6.13) and (6.14) finishes the deduction of the following conjecture.

Conjecture 6.1. *Let $\varepsilon > 0$ and suppose that α, γ satisfies conditions* (6.2)-(6.4). *Assuming the GRH for $L(s, f)$, we have*

$$R_{H_k^*(N)}(\alpha,\gamma) = \zeta(2+2\alpha)\prod_p \left(1 + \frac{1}{p^{1+2\gamma}} - \frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right)$$
$$- \frac{i^k\mu(N)}{N^{1+\gamma}} X_L\left(\frac{1}{2}+\alpha\right)\frac{\zeta(2-2\alpha)\zeta(1+2\gamma)}{\zeta(1-\alpha+\gamma)} A(-\alpha,\gamma) + O\left(|H_k^*(N)|^{-1/2+\varepsilon}\right), \quad (6.16)$$

where $A(-\alpha, \gamma)$ is given by (6.15).

Remark 6.1. Sometimes, one is interested in the asymptotical behaviour as k tends to infinity, rather than N. In this setting it is common to choose $N = 1$, meaning that we consider the full modular group. Then, the Ratios Conjecture is similar except for the absence of the factor $\mu(N)/N^{1+\gamma}$.

Remark 6.2. The size of the error term is of order $N^{-1/2+\varepsilon}$, and is a part of the Ratios Conjecture recipe. However, when replacing the eigenvalues $\lambda_f(p^m)$ by their expected values, we simply discard the error term in Proposition 4.9. Since $(p, N) = 1$ and $p \le x \asymp \sqrt{N}$, these could be as large as $O(N^{-1/4})$, which is qualitatively larger than the predicted error term. The interpretation is that there are significant cancellations between terms that must occur in order for the Ratios Conjecture to hold. At present, this phenomenon is not understood very well.

6.2.4 Computing the derivative

Next, we wish to differentiate the two sides of (6.16) with respect to α and evaluate them at $\alpha = \gamma = s$. This will be used in predicting the shape of the 1-level density in the next section.

Lemma 6.3 ([Mil09, Lemma 2.9, modified])**.** *Denoting*

$$R'_{H_k^*(N)}(s,s) = \frac{\partial}{\partial\alpha} R_{H_k^*(N)}(\alpha,\gamma)\Big|_{\alpha=\gamma=s}$$

and assuming Conjecture 6.1, we have

$$R'_{H_k^*(N)}(s,s) = \sum_p \frac{(p-1)\log p}{p^{2+2s}-1}$$
$$+ i^k\frac{\mu(N)}{N^{1+s}} X_L\left(\frac{1}{2}+s\right)\zeta(2-2s)\zeta(1+2s)A(-s,s) + O\left(|H_k^*(N)|^{-1/2+\varepsilon}\right) \quad (6.17)$$

for $\Re(s) > 0$, where

$$A(-s,s) = \prod_p \left(1 + \frac{p^{1+2s} + p^{2s} - p^{1+4s} - p}{p^{2+4s}(p-1)}\right). \quad (6.18)$$

Proof. We differentiate the two terms of (6.16) separately. Turning to the first, the product rule gives

$$2\zeta'(2+2\alpha)\prod_p\left(1+\frac{1}{p^{1+2\gamma}}-\frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right)$$
$$+\zeta(2+2\alpha)\frac{\partial}{\partial\alpha}\prod_p\left(1+\frac{1}{p^{1+2\gamma}}-\frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right),$$

where the second part deserves the most attention. By

$$\frac{d}{d\alpha}\log f(\alpha)=\frac{f'(\alpha)}{f(\alpha)}, \tag{6.19}$$

the derivative of the product is

$$\prod_p\left(1+\frac{1}{p^{1+2\gamma}}-\frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right)\frac{\partial}{\partial\alpha}\sum_p\log\left(1+\frac{1}{p^{1+2\gamma}}-\frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right)$$
$$=\prod_p\left(1+\frac{1}{p^{1+2\gamma}}-\frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right)$$
$$\times\sum_p\left(1+\frac{1}{p^{1+2\gamma}}-\frac{p+1}{p}\frac{1}{p^{1+\alpha+\gamma}}\right)^{-1}\left(\frac{p+1}{p}\frac{\log p}{p^{1+\alpha+\gamma}}\right),$$

by (6.19). The first product in the right hand side equals $1/\zeta(2+2s)$ when $\alpha=\gamma=s$. Upon evaluation at $\alpha=\gamma=s$, we thus obtain that the derivative of the first term on the right hand side of (6.16) is

$$2\frac{\zeta'(2+2s)}{\zeta(2+2s)}+\sum_p\left(1-\frac{1}{p^{2+2s}}\right)^{-1}\left(\frac{p+1}{p}\frac{\log p}{p^{1+2s}}\right)=2\frac{\zeta'(2+2s)}{\zeta(2+2s)}+\sum_p\frac{(p+1)\log p}{p^{2+2s}-1}.$$

Here, the part $2\zeta'(2+2s)/\zeta(2+2s)$ is recognized as the derivative of $\log\zeta(2+2\alpha)$ evaluated at $\alpha=s$, which turns out to be

$$-\sum_p\frac{2\log p}{p^{2+2s}-1}.$$

In total, we see that the first term of $R'_{H_k^*(N)}(s,s)$ is

$$\sum_p\frac{(p-1)\log p}{p^{2+2s}-1}.$$

We now turn to the second term of $R'_{H_k^*(N)}(s,s)$. The presence of $\zeta(1-\alpha+\gamma)$ in the denominator of this expression allows us to use the following (cf. [CS07, eq. 2.13]). If $f(z,w)$ is analytic at $(z,w)=(s,s)$, we have

$$\frac{\partial}{\partial\alpha}\frac{f(\alpha,\gamma)}{\zeta(1-\alpha+\gamma)}=\frac{f'_\alpha(\alpha,\gamma)\zeta(1-\alpha+\gamma)+f(\alpha,\gamma)\zeta'(1-\alpha+\gamma)}{\zeta^2(1-\alpha+\gamma)}. \tag{6.20}$$

The first term in the right hand side vanishes ar $\alpha=\gamma=s$, due to the pole of $\zeta(s)$ at $s=1$. A Laurent expansion of $\zeta(s)$ around $s=1$ also reveals that

$$\left.\frac{\zeta'(1-\alpha+\gamma)}{\zeta^2(1-\alpha+\gamma)}\right|_{\alpha=\gamma=s}=-1.$$

Hence, the right hand side in (6.20) is equal to $-f(s,s)$ upon evaluation at $\alpha = \gamma = s$. We let f be equal to (6.14), with $1/\zeta(1-\alpha+\gamma)$ removed. The derivative of the second term on the right hand side of (6.16) is therefore equal to

$$i^k \frac{\mu(N)}{N^{1+s}} X_L\left(\frac{1}{2}+s\right)\zeta(2-2s)\zeta(1+2s)A(-s,s),$$

where

$$A(-s,s) = \prod_p \left(1 + \frac{p^{1+2s}+p^{2s}-p^{1+4s}-p}{p^{2+4s}(p-1)}\right).$$

This finishes the proof. □

Remark 6.4. In our calculations, the order of the error term stays unchanged through differentiation. This is of course not true in general. In our case, however, the error term in Conjecture 6.1 is an analytic function $g(\alpha,\gamma)$, since the left hand side and main term in Conjecture 6.1 both are. We can then use Cauchy's integral formula for $g'_\alpha(\alpha,\gamma)$ at $\alpha = \gamma = s$ to see that it is of the same order as $g(\alpha,\gamma)$, by integrating around a circle of small enough radius.

6.3 Computing the 1-level density from the Ratios Conjecture

We now analyze the 1-level density by means of the Ratios Conjecture, the chief tool of which is Lemma 6.3. Recall that by the argument principle and the functional equation (3.41), we obtained (5.2), that is

$$\begin{aligned} D(f,\phi) = &\frac{1}{2\pi i}\int_{(c)} 2\frac{L'(s,f)}{L(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds \\ &+ \frac{1}{2\pi i}\int_{(c)} 2\frac{L'_\infty(s,f)}{L_\infty(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds. \end{aligned}$$

Of these terms, the second was computed and found to be equal to (5.5). When averaging over $H_k^*(N)$, we thus obtain

$$\begin{aligned} \mathcal{D}_{H_k^*(N)}(\phi) = &\frac{1}{|H_k^*(N)|}\sum_{f\in\mathcal{H}_\|^*(\mathcal{N})}\frac{1}{2\pi i}\int_{(c)} 2\frac{L'(s,f)}{L(s,f)}\phi\left(\left(s-\frac{1}{2}\right)\frac{\mathcal{L}}{2\pi i}\right)ds \\ &+ \frac{1}{\mathcal{L}}\int_{-\infty}^{\infty}\left[2\log\left(\frac{\sqrt{N}}{\pi}\right)+\psi\left(\frac{1}{4}+\frac{k\pm 1}{4}+\frac{\pi i u}{\mathcal{L}}\right)\right]\phi(u)du. \quad (6.21) \end{aligned}$$

Here, we choose c such that $1/2+1/\log|H_k^*(N)| < c < 3/4$, to satisfy the conditions (6.2) and (6.3). This is allowed, as we assume the GRH.

6.3.1 The $L'(s,f)/L(s,f)$-term

We now treat the first sum in (6.21). It is here that our calculations will differ from those in Chapter 5. After changing the order of integration and summation and performing the change of variables $s \mapsto s - 1/2$, we obtain

$$\frac{1}{2\pi i}\int_{(c')} 2\frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\frac{L'(1/2+s,f)}{L(1/2+s,f)}\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds.$$

Here, $c' = c - 1/2$ satisfies $1/\log|H_k^*(N)| < c' < 1/4$. Since we assume the GRH, values of $\Re(s)$ in this range are allowed. The averaged sum of ratios suggests using Lemma 6.3, but this is only allowed when $\Im(s) \ll |H_k^*(N)|^{1-\varepsilon}$, by condition (6.4). Hence, we split the integral in two pieces according to this condition, replace the averaged sum by the right hand side of (6.17) in the allowed region, and extend the new integral to the whole line $\Re(s) = c'$. Doing so will produce two error terms, consisting of tail integrals over the part where $\Im(s) \gg |H_k^*(N)|^{1-\varepsilon}$. We now estimate these.

The first tail integral, where we integrate the averaged sum of logarithmic derivatives, leaves an admissible error term due to the Riemann hypothesis for $L(s,f)$. Indeed, from [IK04, eq. 5.7 and Theorem 5.17], we have that the logarithmic derivative is bounded by

$$\frac{L'(1/2+s,f)}{L(1/2+s,f)} \ll \frac{1}{2c'-1}\log\left(N(|s|+|k|+3)^2\right).$$

In the other tail integral, it is enough to show that each of the terms of the right hand side of (6.17) grow at most polynomially in $|s|$. First, we have

$$\sum_p \frac{(p-1)\log p}{p^{2+2s}-1} = O(1),$$

since $\Re(s) = c' > 0$. Second, the Stirling approximation

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}$$

(see [IK04, eq. 5.113]) is valid in vertical stripes when $\sigma > 0$ and $|t| \geq 1$. It implies that

$$X_L\left(\frac{1}{2}+s\right) = O(1) \tag{6.22}$$

as well. Third, since

$$\zeta(s) = O(|t|) \tag{6.23}$$

for $\Re(s) \geq \frac{1}{2}$ and s bounded away from the pole at 1 (see [Tit86, eq. 2.12.2]), the factors $\zeta(1+2s)$ and $\zeta(2-2s)$ are both polynomially bounded. Finally, from (6.18) we have

$$|A(-s,s)| \leq \prod_p \left(1+\frac{3p+1}{p^2(p-1)}\right) = O(1). \tag{6.24}$$

Hence, the tail of the integral is small due to the rapid decay of ϕ. We obtain an error term of size $O(|H_k^*(N)|^{-1+\varepsilon}) = O(N^{-1+\varepsilon})$ from the two tail integrals, which is absorbed into the error term $O(N^{-1/2+\varepsilon})$ from the recipe. In total, we have

$$\frac{1}{2\pi i}\int_{(c')}2\frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)}\frac{L'(1/2+s,f)}{L(1/2+s,f)}\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds=\frac{2}{2\pi i}\int_{(c')}\left[\sum_p\frac{(p-1)\log p}{p^{2+2s}-1}\right.$$
$$\left.+\frac{i^k\mu(N)}{N^{1+s}}X_L\left(\frac{1}{2}+s\right)\zeta(2-2s)\zeta(1+2s)A(-s,s)\right]\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds+O(N^{-1/2+\varepsilon}).\tag{6.25}$$

6.3.2 The prime sum

Let us consider the first term of the right hand side of (6.25), that is

$$\frac{1}{2\pi i}\int_{(c')}2\sum_p\frac{(p-1)\log p}{p^{2+2s}-1}\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds.$$

We note that

$$\frac{1}{p^{2+2s}-1}=\sum_{j=1}^{\infty}\frac{1}{p^{(2+2s)j}}$$

with convergence being absolute and uniform on compact subsets, since $c'>0$. Hence, it is justified to interchange the order of summation and integration, yielding

$$\sum_p\sum_{j=1}^{\infty}\frac{2(p-1)\log p}{p^{2j}}\frac{1}{2\pi i}\int_{(c')}\frac{1}{p^{2js}}\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds.\tag{6.26}$$

We focus on the integral in (6.26). After the change of variables

$$s\frac{\mathcal{L}}{2\pi i}=u,\tag{6.27}$$

we get

$$\frac{1}{2\pi i}\int_{(c')}\frac{1}{p^{2js}}\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds=\frac{1}{\mathcal{L}}\int_C\phi(u)e^{-2\pi iu(2j\log p/\mathcal{L})}du,$$

where C is the horizontal line $\Im(u)=-c'\mathcal{L}/2\pi$. We wish to move the contour of integration to the real axis. Consider the positively oriented rectangle with corners at $\pm T,\pm T-ic'\mathcal{L}/2\pi$, where $T>0$ is large. By the residue theorem, the integral over this rectangle is 0. We need to argue that the contribution on the vertical segments tends to 0 as T tends to infinity. To do this, we follow the corresponding computation in [FPS18, p. 1143]. Recall that since $\widehat{\phi}$ has compact support, the function ϕ may be extended to an entire function

$$\phi(z)=\int_{-\infty}^{\infty}\widehat{\phi}(x)e^{2\pi ixz}dx.$$

Integration by parts yields

$$\phi(z)=-\frac{1}{2\pi iz}\int_{-\infty}^{\infty}\widehat{\phi}'(x)e^{2\pi ixz}dx$$

for $z\neq 0$. We have

$$|e^{2\pi ixz}|\leq\begin{cases}e^{c'x\mathcal{L}}, & x\geq 0,\\ 1, & x<0,\end{cases}$$

uniformly for $-c'\mathcal{L}/2\pi \le t \le 0$. Hence,

$$|\phi(T+it)| \le \frac{1}{2\pi|T|}\int_{-\infty}^{\infty} |\widehat{\phi}'(x)| \max\left(1, e^{c'x\mathcal{L}}\right)dx = O(|T|^{-1})$$

in this range. We conclude that the vertical contributions indeed vanish as we let T tend to infinity. The shifted integral is equal to

$$\frac{1}{\mathcal{L}}\int_{-\infty}^{\infty} \phi(u)e^{-2\pi iu(2j\log p/\mathcal{L})}du = \frac{1}{\mathcal{L}}\widehat{\phi}\left(\frac{2j\log p}{\mathcal{L}}\right).$$

In the end, we have that (6.26) is equal to the series

$$\sum_p \sum_{j=1}^{\infty} \frac{2(p-1)\log p}{p^{2j}\mathcal{L}}\widehat{\phi}\left(\frac{2j\log p}{\mathcal{L}}\right). \tag{6.28}$$

Identifying $2j$ with ν, we see that this equals the last non-error term in (5.23). This means that the Ratios Conjecture prediction agrees with the computations in Section 5.3. provided that the second term from (6.25) can be absorbed into already existing error terms. Moreover, the Ratios Conjecture does not assume any bound on σ, except that $\sigma < \infty$. From our previous value $\sigma = 3/2$, this is a significant improvement indeed. The remainder of Section 6.3 is devoted to show that the remaining integral from (6.25) is of order $O(1/N)$.

6.3.3 A singular integral

We now consider the second integral from the right hand side of (6.25), that is

$$\frac{1}{2\pi i}\int_{(c')} 2\frac{i^k\mu(N)}{N^{1+s}}X_L\left(\frac{1}{2}+s\right)\zeta(2-2s)\zeta(1+2s)A(-s,s)\phi\left(s\frac{\mathcal{L}}{2\pi i}\right)ds. \tag{6.29}$$

One would like to to move this integral to the imaginary axis. However, we cannot do so directly, due to $\zeta(1+2s)$ having a pole at $s=0$. In order to evaluate the integral, we replace the factors of the integrand by their respective Taylor/Laurent expansions around $s=0$.

6.3.3.1 Taylor expansions

First, we approximate $X_L(1/2+s)$ by its first order Taylor polynomial. Since $X_L(1/2) = 1$, we have

$$X_L\left(\frac{1}{2}+s\right) = 1 + X_L'(1/2)s + O\left(|s|^2\right).$$

By (6.7), one may compute

$$X_L'(1/2) = -\left(2\log\left(\frac{\sqrt{N}}{\pi}\right) + \psi\left(\frac{1}{4}+\frac{k\pm 1}{4}\right)\right).$$

Next, we expand

$$\zeta(2-2s) = \zeta(2) - 2\zeta'(2)s + O\left(|s|^2\right).$$

The factor $\zeta(1+2s)$ has the standard Laurent expansion

$$\zeta(1+2s) = \frac{1}{2s} + \gamma + O\left(|s|\right)$$

(see [Tit86, eq. 2.1.16]), where γ is the Euler-Mascheroni constant. Finally, we turn to the expansion of the product $g(s) = A(-s,s)$. We aim for a Taylor polynomial of the first order, and therefore need to compute the values $g(0)$ and $g'(0)$. Evaluation at $s = 0$ yields

$$g(0) = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}.$$

To compute the derivative of $A(-s,s)$, we differentiate $\log A(-s,s)$. This yields

$$\begin{aligned}\frac{d}{ds}\log A(-s,s) &= \frac{d}{ds}\sum_p \log\left(1 + \frac{p^{1+2s}+p^{2s}-p^{1+4s}-p}{p^{2+4s}(p-1)}\right)\\ &= \sum_p \left(1 + \frac{p^{1+2s}+p^{2s}-p^{1+4s}-p}{p^{2+4s}(p-1)}\right)^{-1} \frac{2(2p-p^{1+2s}-p^{2s})\log p}{p^{2+4s}(p-1)}.\end{aligned}$$

Inserting $s = 0$ yields the series

$$T = \sum_p \frac{2\log p}{p^2-1},$$

which we recognize as

$$2\sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{2k}} = -\frac{2\zeta'(2)}{\zeta(2)}.$$

Hence the derivative of $A(-s,s)$ at $s=0$ equals $T/\zeta(2)$, by (6.19). In total, $A(-s,s)$ is expanded as

$$A(-s,s) = \frac{1}{\zeta(2)}(1+Ts) + O\left(|s|^2\right).$$

Now, it is at hand to multiply the expansions to obtain the expansion of the integrand in (6.29). Doing so yields

$$\begin{aligned}&X_L\left(\frac{1}{2}+s\right)\zeta(2-2s)\zeta(1+2s)A(-s,s)\\ &\qquad= \left(1 + X_L'(1/2)s + O\left(|s|^2\right)\right)\left(\zeta(2) - 2\zeta'(2)s + O\left(|s|^2\right)\right)\\ &\qquad\quad\times\left(\frac{1}{2s} + \gamma + O\left(|s|\right)\right)\left(\frac{1}{\zeta(2)} + \frac{Ts}{\zeta(2)} + O\left(|s|^2\right)\right)\\ &\qquad= \frac{1}{2s}\left(1 + \left(2\gamma + X_L'(1/2) - 4\frac{\zeta'(2)}{\zeta(2)}\right)s + O\left(|s|^2\right)\right). \qquad (6.30)\end{aligned}$$

6.3.3.2 Computing the integral

With the Taylor/Laurent expansions in place, we can proceed with our computation of (6.29). We follow steps closely resembling those in [FPS18, Lemma 4.6]. First of all, we use the substitution (6.27), which turns the integral (6.29) into

$$\frac{i^k\mu(N)}{N}\frac{2}{\mathcal{L}}\int_C \frac{1}{N^{2\pi iu/\mathcal{L}}}X_L\left(\frac{1}{2}+\frac{2\pi iu}{\mathcal{L}}\right)$$
$$\times\zeta\left(2-\frac{4\pi iu}{\mathcal{L}}\right)\zeta\left(1+\frac{4\pi iu}{\mathcal{L}}\right)A\left(-\frac{2\pi iu}{\mathcal{L}},\frac{2\pi iu}{\mathcal{L}}\right)\phi(u)du. \quad (6.31)$$

Here, C is the same horizontal line $\Im(u) = -c'\mathcal{L}/2\pi$ as before. The goal would be to move the integral to the real axis, but we have to avoid the singularity at 0. Therefore, we move the integral (6.31) to the contour $\mathcal{C} = C_0 \cup C_1 \cup C_2$, where

$$\begin{aligned}C_0 &= \{z \in \mathbb{C} : \Im(z) = 0, |z| > \mathcal{L}^\varepsilon\},\\ C_1 &= \{z \in \mathbb{C} : \Im(z) = 0, \eta < |z| \le \mathcal{L}^\varepsilon\},\\ C_2 &= \{z = \eta e^{i\theta} \in \mathbb{C} : \theta \in [-\pi, 0]\},\end{aligned}$$

and $\eta > 0$ is small. The change of contours is justified by arguments similar to those in Section 6.3.2. Indeed, on the vertical segments where $\Re(u) = \pm T$ and $-c'\mathcal{L}/2\pi \le \Im(u) \le 0$, we use the bounds

$$\begin{aligned}X_L\left(\frac{1}{2}+\frac{2\pi iu}{\mathcal{L}}\right) &= O(1),\\ A\left(-\frac{2\pi iu}{\mathcal{L}},\frac{2\pi iu}{\mathcal{L}}\right) &= O(1)\end{aligned} \quad (6.32)$$

in addition to the bound (6.23). These follow from the corresponding bounds (6.22) and (6.24), which are valid on vertical strips.

With the change to the contour $\mathcal{C}$ complete, we begin by bounding the integral over C_0. We can reuse the bounds (6.23) and (6.32) when $\Im(u) = 0$. From this, we obtain an upper bound of the integral over C_0 of order

$$\frac{1}{N\mathcal{L}}\int_{C_0}(|u|+1)^{-2/\varepsilon}du \le \frac{2}{N\mathcal{L}^2}\int_{\mathcal{L}^\varepsilon}^\infty(|u|+1)^{-1/\varepsilon}du \ll \frac{1}{N\mathcal{L}^2}.$$

On the contour segment $C_1 \cup C_2$, we replace the factors in the integrand of (6.31) with their Taylor/Laurent expansions, since then $s = 2\pi iu/\mathcal{L}$ is small. The integral over $C_1 \cup C_2$ thus is equal to

$$\frac{i^k\mu(N)}{N}\frac{2}{\mathcal{L}}\int_{C_1\cup C_2}\frac{1}{N^{2\pi iu/\mathcal{L}}}\frac{\mathcal{L}}{4\pi iu}$$
$$\times\left(1+\left(2\gamma+X_L'(1/2)-4\frac{\zeta'(2)}{\zeta(2)}\right)\frac{2\pi iu}{\mathcal{L}}+O\left(|u/\mathcal{L}|^2\right)\right)\phi(u)du$$
$$=\frac{i^k\mu(N)}{N}(I_1+I_2)+O\left(N^{-1}\mathcal{L}^{-2+\varepsilon}\right),$$

where

$$I_1 = \frac{1}{2\pi i}\int_{C_1\cup C_2}\frac{\phi(u)}{u}e^{-2\pi iu\log N/\mathcal{L}}du$$

and

$$I_2 = \frac{2\gamma+X_L'(1/2)-4\zeta'(2)/\zeta(2)}{\mathcal{L}}\int_{C_1\cup C_2}\phi(u)e^{-2\pi iu\log N/\mathcal{L}}du.$$

The integral I_1 may be extended to the full contour $\mathcal{C}$ at the cost of an error term of size

$$\int_{C_0} \left| \frac{\phi(u)}{u} \right| du \ll \int_{\mathcal{L}^\varepsilon}^{\infty} \frac{1}{u^{2/\varepsilon^2+1}} du \ll \frac{1}{\mathcal{L}^{2/\varepsilon}}.$$

We split the extended integral into

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u)}{u} \cos\left(2\pi u \frac{\log N}{\mathcal{L}}\right) du - \frac{1}{2\pi} \int_{\mathcal{C}} \frac{\phi(u)}{u} \sin\left(2\pi u \frac{\log N}{\mathcal{L}}\right) du.$$

The first integral has odd integrand, so the integral over $C_0 \cup C_1$ vanish. The remaining integral

$$\frac{1}{2\pi i} \int_{C_2} \frac{\phi(u)}{u} \cos\left(2\pi u \frac{\log N}{\mathcal{L}}\right) du$$

converges to $\phi(0)/2$ as η tends to 0, by the Cauchy residue theorem. The integrand in the second integral is entire when $N > 1$, so as η tends to 0 we obtain

$$\begin{aligned} -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi(u)}{u} \sin\left(2\pi u \frac{\log N}{\mathcal{L}}\right) du &= -\frac{\log N}{\mathcal{L}} \int_{-\infty}^{\infty} \phi(u) \frac{\sin(2\pi u \log N/\mathcal{L})}{2\pi u \log N/\mathcal{L}} du \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} \widehat{\phi}(u) \chi_{[-\log N/\mathcal{L}, \log N/\mathcal{L}]}(u) du = -\frac{1}{2} \int_{-\log N/\mathcal{L}}^{\log N/\mathcal{L}} \widehat{\phi}(u) du, \end{aligned}$$

where we used Plancherel's theorem in the second step.

Similarly to I_1, we may also extend the second integral I_2 to $C_0 \cup C_1 \cup C_2$ at the cost of an error of size $\mathcal{L}^{-2/\varepsilon}$, due to the rapid decay of ϕ. Letting η tend to 0, we see that the extended integral is equal to

$$\frac{2\gamma + X_L'(1/2) - 4\zeta'(2)/\zeta(2)}{\mathcal{L}} \widehat{\phi}\left(\frac{\log N}{\mathcal{L}}\right).$$

All in all, we have that (6.29) is equal to

$$\frac{i^k \mu(N)}{N} \left(\frac{\phi(0)}{2} - \frac{1}{2} \int_{-\log N/\mathcal{L}}^{\log N/\mathcal{L}} \widehat{\phi}(u) du + \frac{2\gamma + X_L'(1/2) - 4\zeta'(2)/\zeta(2)}{\mathcal{L}} \widehat{\phi}\left(\frac{\log N}{\mathcal{L}}\right) \right) + O\left(\frac{1}{N\mathcal{L}^{2-\varepsilon}}\right).$$

The term inside the parenthesis is bounded as N tends to infinity. Hence, the whole expression is of order $O(1/N)$. This agrees with the corresponding assertion in [Mil09, Lemma 3.4], although we obtained the result through different means. Collecting the term (6.28) and the various error terms and inserting into (6.21), we have the following result.

Proposition 6.5. *The Ratios conjecture predicts that*

$$\begin{aligned} \mathcal{D}_{H_k^*(N)}(\phi) = \frac{1}{\mathcal{L}} \int_{-\infty}^{\infty} \left[2 \log\left(\frac{\sqrt{N}}{\pi}\right) + \psi\left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{\pi i u}{\mathcal{L}}\right) \right] \phi(u) du \\ + \sum_p \sum_{j=1}^{\infty} \frac{2(p-1)\log p}{p^{2j} \mathcal{L}} \widehat{\phi}\left(\frac{2j \log p}{\mathcal{L}}\right) + O\left(N^{-1/2+\varepsilon}\right), \end{aligned}$$

where k is fixed and $N \to \infty$ through the primes.

Remark 6.6. We have observed that the Ratios Conjecture does not place any assumptions on the support of $\widehat{\phi}$, other than compactness. We also observe that the error term, which stems from the Ratios Conjecture recipe, has smaller size than in Proposition 5.4. Thus, the Ratios Conjecture is a very strong assumption, and yield correspondingly strong results.

Remark 6.7. We already observed that the main terms in Propositions 5.4 and 6.5 have identical shape. Thus, the Ratios Conjecture correctly predicts the 1-level density down to an error of size $O\left(N^{-(3-2\sigma')/6+\varepsilon} + N^{-1/3+\varepsilon}\right)$ for any $\sigma' < \sigma$, provided that $\sigma = 3/2$.

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A

The Riemann-Stieltjes integral

We recall the construction of the Riemann-Stieltjes integral along with some of its central properties. Most of the material, as well as the notation and the exposition, is from [Rud76, Ch. 6].

The Riemann integral can be constructed as follows. First, we divide the interval $[a, b]$ into N subintervals $[x_i, x_{i+1}]$, where

$$a = x_0 < x_1 < ... < x_N = b.$$

We call the set of points $\{x_i\}$ a *partition* of $[a, b]$, and denote it by P. For a bounded function $f : [a, b] \to \mathbb{R}$, we let

$$\begin{aligned}
M_i &:= \sup\{f(x) : x \in [x_{i-1}, x_i]\}, \\
m_i &:= \inf\{f(x) : x \in [x_{i-1}, x_i]\}, \\
\Delta_i &:= x_i - x_{i-1},
\end{aligned}$$

for $i = 1, ..., N$. Now, we consider the *upper* and *lower Riemann sums*

$$\begin{aligned}
U(P, f) &:= \sum_{i=0}^{N} M_i \Delta_i, \\
L(P, f) &:= \sum_{i=0}^{N} m_i \Delta_i.
\end{aligned}$$

The function f is called *Riemann integrable* if the values

$$\inf_P U(P, f) \quad \text{and} \quad \sup_P L(P, f)$$

both exist and are equal; the integral

$$\int_a^b f(x)dx$$

is then defined as this value.

The Riemann-Stieltjes integral is a slight generalisation of this concept. Namely, instead of considering the differences Δ_i, we let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function on $[a, b]$ and consider the differences $\Delta g_i = g(x_{i-1}) - g(x_i)$. With M_i and m_i as above, we define

$$U(P, f, g) := \sum_{i=0}^{N} M_i \Delta g_i,$$
$$L(P, f, g) := \sum_{i=0}^{N} m_i \Delta g_i,$$

and say that f is *Riemann integrable with respect to* g if the values

$$\inf_P U(P, f, g) \quad \text{and} \quad \sup_P L(P, f, g)$$

both exist and are equal. The *Riemann integral of* f *against* g is denoted by

$$\int_a^b g(x) dg(x)$$

and is defined as this value.

By choosing $g(x) = x$, we recover the usual Riemann integral. However, g can be chosen quite freely; in partiicular, it need not be continuous. For instance, if $g(x) = \lfloor x \rfloor$ is the floor function, then g has a discontinuity at each integer, where it "jumps" a step of length 1. Hence, one can show that

$$\int_a^b f(x) d\lfloor x \rfloor = \sum_{a \leq n \leq b} f(n).$$

Slightly more generally, if

$$g(x) = \sum_{n \leq x} a_n$$

for some nonnegative real numbers a_n, then

$$\int_a^b f(x) dg(x) = \sum_{a \leq n \leq b} a_n f(n).$$

If g is differentiable, then we have the equality

$$\int_a^b f(x) dg(x) = \int_a^b f(x) g'(x) dx;$$

in other words, the Riemann-Stieltjes integral reduces to an ordinary Riemann integral [Rud76, Theorem 6.17]. If f is Riemann integrable with respect to g, then g is Riemann integrable with respect to f and we have the equality

$$\int_a^b f(x) dg(x) = f(a)g(a) - f(b)g(b) - \int_a^b g(x) df(x),$$

referred to as integration by parts (see e.g. [HP57, Theorem 3.3.1]).