Thesis for the degree of licentitate of philosophy

Sharp bounds on the height of some arithmetic Fano varieties

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Göteborg, Sweden 2023

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Typeset with LaTeX
Printed by Stema
Borås, Sweden 2023

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Abstract

In the framework of Arakelov geometry one can define the height of a polarized arithmetic variety equipped with an hermitian metric over its complexification. When the arithmetic variety is Fano, the complexification is K-semistable and the metrics are normalized in a natural way, we find in this thesis a universal upper bound on the height in a number of cases. For example for the canonical integral model of toric varieties of low dimension (in paper 1) and for general diagonal hypersurfaces (in paper 2). The bound is sharp with equality for the projective space over the integers equipped with a Fubini-Study metric. These results provide positive cases of a conjectural general bound that we introduce, which can be seen as an arithmetic analog of Fujita's sharp upper bound on the anti-canonical degree of an n-dimensional K-semistable Fano variety in [11]. An extension of the toric result to arbitrary dimension hinges on a conjectural sharp bound for the second largest anti-canonical degree of a toric K-semistable Fano variety in a given dimension. A version of the conjecture for log-Fano pairs is also introduced (in paper 2), which is settled in low dimensions for toric log-pairs and for simple normal crossings hyperplane divisors in projective space. Along the way we define a canonical height of a K-semistable arithmetic (log) Fano variety, making a connection with positively curved (log) Kähler-Einstein metrics.

Keywords: Arakelov geometry, Kähler-Einstein metrics, toric geometry, K-stability, Fano varieties, height bounds

Contents

List of publications

This thesis consists of an extended summary and the following appended manuscripts.

Paper 1: R. Andreasson. R. J. Berman. Sharp bounds on the height of K-semistable toric Fano varieties I. Preprint. Arxiv:2205.00730, (2023).

Paper 2: R. Andreasson. R. J. Berman. Sharp bounds on the height of K-semistable toric Fano varieties II, the log case. In preparation, (2023).

Author contribution:

Paper 1: The original idea for the conjectural height bound was due to the supervisor. The authors both contributed substantially to the proof of the main theorem.

Paper 2: A continuation and generalization of the ideas in paper 1. The authors both contributed substantially for the proofs of the main theorems with the exception of Theorem 1.5 that was due to the supervisor.

Acknowledgements

I would first and foremost like to thank my supervisor Robert for his contagious enthusiasm in answering my questions, our projects, and for mathematics in general. I would also like to thank my co-supervisor Jakob for his sound advice, interest in my ideas and our fruitful collaborations. Also thanks to my second co-supervisor Klas for explaining many things to me outside of my field. Thanks Håkan for being generally helpful and a good examiner. A special thanks to Dennis and Bo for their patience with my countless ill-formed questions and for many nice discussions. Furthermore I would like to thank my other colleagues at the department, equally for their answers as for their questions; Ludvig, Douglas, Vincent, Rahim, Mattias, Robin, Jan, Mingchen, Antonio, Remi, Lars-Martin, Simon, Mykola, Daniel, David, Richard, Mats, Elizabeth and Hossein. At last I would like to thank my family for their support; and for bearing with me, Louise.

Rolf Andreasson
Göteborg, 2023

Part 1

Introduction and summary

1.1 Elementary introduction

This thesis is about an invariant of certain arithmetic varieties that we will refer to as the canonical height. From another perspective, it is about universal height bounds on these arithmetic varieties. Arguably the easiest example of height is that of the *naive height* of a rational number, given by

$$h\left(\frac{a}{b}\right) = \log \max(a, b) \tag{1.1}$$

where a and b are required to be relatively prime. The height can be thought of as the arithmetic complexity of the rational number $\frac{a}{b}$. Indeed, when storing the rational number a/b in computer memory, $h(a/b)$ is approximately the number of bits needed.

Note that one could also first define the *very naive height* by just letting $h(a/b) = \log \max(a, b)$, regardless of the relative primeness of a or b. This is no longer a well defined notion for rational numbers, rather for their actual symbolic representation as *a over b*. Then we could define a *canonical height* by considering the infimum over such representations and this would only depend on the actual rational number. This will in fact coincide with the naive height, and indeed representing a rational number by a/b for relatively prime a and b is one canonical representation of it. This is not, in spirit, unlike how we will define a canonical height of an arithmetic Fano variety.

A more algebraic way of presenting the above is to consider instead of the rational number the linear equation that defines it. Namely, the rational number a/b is a solution to the equation $a = bx$. Since a and b are only defined up to multiplying both of them with an integer, it would be nice if the same was true for x. One can *homogenize* the equation and consider the equation of two variables $ay = bx$. Any integer solution x, y still represents the rational number a/b but now there are many solutions, since we can multiply an old solution x, y by an integer and get a new one. So the symbolic representation of the rational number a/b just means an integer solution to $ay = bx$.

Remark. *An elementary example where the ideas of heights appear, can be found in a proof of the irrationality of $\sqrt{2}$. Lets work with the definition of the canonical height of a symbolic representation of a rational number. Assume, in pursuit of contradiction, that*

$$\sqrt{2} = a/b \tag{1.2}$$

for some integers a and b, so that $\sqrt{2}$ is rational. Then the canonical height of $\sqrt{2}$ is at most $\log \max(a, b)$. It is a small exercise to realize that by squaring both sides of (1.2), a and b must be divisible by two. Thus the canonical height of $\sqrt{2}$ is at most $\log(\max(a, b)/2)$. But now we can repeat the above argument again and again, and infer eventually that the canonical height of $\sqrt{2}$ must be $-\infty$, which can be shown is not the case for a rational number (by arguing via a relative primeness).

Looping back to the discussion on representing rational numbers by equations, we have shown above that the homogeneous equation $2b^2 = a^2$ does not have an integer solution.

The concept of height has been used for various problems in Diophantine geometry. To show the flavour of this, consider an algebraic equation of two variables, such as

$$x^2 = y^3 + 17. \tag{1.3}$$

If we are interested in rational solutions, a simple exercise shows that we can just as well consider integral solutions to the homogenization

$$zx^2 - y^3 - 17z^3 = 0 \tag{1.4}$$

in projective three-space. In modern language, these solutions are called rational points. The sought after results in Diophantine geometry are to understand the properties of the set of solutions. For example, if there are a finite or infinite number of solutions, or if there are any solutions at all. In any case, it has been useful to introduce a size of a solution, the height. We can introduce a notion of height of any solution (a, b, c) (where a, b, c are chosen relatively prime) by letting

$$h([a, b, c]) = \log \max(a, b, c), \tag{1.5}$$

and consider the set of solutions of height bounded by B. This set will always be finite. Thus a statement about the finiteness of the set of solutions is equivalent to an upper bound on the height of the solutions. In the case that there are infinitely many solutions, one can ask refined questions about how the number of solutions of height smaller than some number B grows as B becomes large.

Conjecturally these basic properties of the set of solutions should be dictated by the geometry of the underlying analytic space consisting of all solutions in the complex numbers. For algebraic curves, both of these notions are in addition tightly connected to the degree of the defining polynomial. This is particularly well understood in the case of non-singular algebraic curves, like the example (1.4).

Firstly, for the arithmetic of algebraic curves, lets restrict for simplicity to the case that there is at least one rational point (for more on this, see the remark of section 1.6).

- Curves of degree one are in a very strong sense equivalent to the projective line (the two-dimensional sphere). Curves of degree two were understood by Euclid and can be related to the projective line again by stereographic projection, this operation preserves both the geometry and the study of rational points.

The projective line over $\mathbb{Z}$ has infinitely many rational points given by primitive pairs of integers (a, b).

- Curves of degree three can have both finitely and infinitely many rational points, depending on the exact curve. In any case, it is a finitely generated abelian group. This is the content of Mordell's theorem [22].

- Higher degree curves has finitely many rational points. This is the content of Falting's theorem [9].

Secondly, the degree is intertwined with the geometry of the space of complex solutions, namely,

- if the degree is either one or two, the space is a sphere and the geometry is positively curved in the sense that there exists a Riemannian metric of constant positive curvature.

- If the degree is precisely three, the space is a torus (the surface of a donut) and the geometry is flat in the sense that there exists a flat Riemannian metric.

- If the degree is larger than three, the space is negatively curved in the sense that there exists a Riemannian metric of constant negative curvature. The topology becomes increasingly more complicated with increasing degree in the sense that the space has more and more 'holes' (here, the sphere has zero 'holes' while the torus has one).

Going back to the arithmetics, the theorems of Mordell and Falting both uses crucially some notion of height. In this thesis we will be concerned with the case when the underlying analytic space is of any dimension but positively curved in a certain sense, more precisely it is Fano. Conjecturally according to part of the Manin conjecture [10], the rational points are infinitely many in this case (if there are any at all). Here heights enters when one tries to understand how the number of solutions grows according to their height. How the questions in this thesis enters more precisely, was described recently in [3].

In addition to the height of points, which can be seen as zero-dimensional varieties, it is useful to define the height of higher dimensional varieties. Naively, for hypersurfaces, the height of a variety could be defined as the height of the coefficient vector of the equation that defines it. This fits nicely with the naive height of points of $\mathbb{P}^1$ which coincides with the height in this sense of the linear equation that defines them.

One central point about heights whether of points or entire varieties, is that there are many more sensible ways to define them than the naive height. For some applications, not much would change by defining for example

$$h([a:b:c]) = \log(|a|^2 + |b^2| + |c^2|)^{1/2}, \tag{1.6}$$

since this new definition only differs up to a uniformly bounded term from the old one. More generally, we could use any norm on, in this case, $\mathbb{R}^3$. For higher-dimensional varieties, if we define the height as the naive height of the coefficient vector, then we could still realize the underlying abstract variety in different ways as a subscheme of projective space.

In the generality of Arakelov theory, one can use any positively curved hermitian metric on the complexification of an arithmetic line bundle to define a notion of height in an intrinsic way. One benefit of this framework is that it will be possible to find a type of canonical height on an arithmetic Fano variety in a way that would not be possible otherwise (see section 1.9).

As explained, the notion of height depends on a choice of metric. The idea put forth in this thesis is to define a canonical height by maximizing the height over all metrics, and in the meantime get a universal bound for the height with respect to any metric. It turns out that this invariant will only be finite under a subtle algebraic condition on the underlying complex variety called K-semistability. Additionally, under the slightly stronger condition of K-polystability, there exists a maximizing metric which is a Kähler-Einstein metric. A metric is Einstein if it is a solution of the vacuum Einstein field equations from the theory of General Relativity. The potential role played by Einstein metrics in Arakelov geometry was perhaps first pointed out by Yuri Manin in his *New dimensions in geometry* [21] from -85. This thesis pursues that largely unexplored direction.

Organization

In sections 1.1 and 1.2 we introduce the analytic theory of complex projective varieties upon which much of the sequel is built. In section 1.3 we introduce the algebro-geometric counterpart of height, degree. In section 1.4 we briefly introduce arithmetic varieties. In section 1.5 we are finally ready to introduce the theory underlying the thesis, Arakelov theory, built on both arithmetic and complex geometry. In section 1.7 we introduce toric varieties, the class of varieties that large parts of the papers studies. In section 1.8 we summarize the contents of paper 1 about universal height bounds on arithmetic toric Fano varieties. In section 1.9 we briefly describe logarithmic pairs. In section 1.10 we summarize paper 2, about a logarithmic generalization of paper 1, and some applications to height bounds on arithmetic diagonal Fano hypersurfaces.

1.2 Complex projective varieties

Later on we will be concerned with varieties over $\mathbb{Z}$, which are objects of a quite algebraic nature. However, one of the main points of Arakelov geometry, the framework on which much of the thesis is based, is that it is useful when studying algebraic equations with coefficients in $\mathbb{Z}$ to also consider the complex space cut-out by the polynomials. This brings us to *complex projective varieties.*

A *complex projective variety* is the zero set of one or more homogeneous polynomials in complex projective space. If the zero set of the Jacobian of this polynomial never intersects the variety itself, it is non-singular and we have a manifold. Many of the constructions in this thesis are easier to understand in the non-singular setting, so this is what we will focus on. They can often be extended to the singular setting, with some restrictions on the severity of the singularities.

One important example is complex projective space, $\mathbb{P}^n$, itself, given by the space of complex lines through the origin in $\mathbb{C}^{n+1}$. In the introduction we saw another, given by the vanishing of the single homogeneous polynomial (1.4), a hypersurface in embedded in $\mathbb{P}^3$. Another important class of examples in this thesis is that of toric varieties. Due to their highly symmetric nature they have their whole own theory, described in section 1.8. For now, we can think of them as the projective varieties defined by the vanishing of one or more binomials.

1.3 Kähler geometry

Complex projective manifolds are special among differentiable manifolds and have many remarkable properties. For one, they are complex manifolds, meaning that after identifying their charts with subsets of $\mathbb{C}^n$ the transitions maps can be chosen to be holomorphic. This allows to define what it means for a function, differential form or general tensor field to be holomorphic. One can also introduce a $\bar{\partial}$-operator, whose kernel are precisely the holomorphic functions.

Furthermore, even among complex manifolds projective manifolds are special. Partly because they are *Kähler*, meaning that they admit Kähler metrics. A *Kähler metric* is a Riemannian metric on a complex manifold such that there exists holomorphic coordinates where the metric looks like the standard metric on $\mathbb{C}^n$ to order one (not just to order zero which one can always achieve). This turns out to be an extremely useful condition and restricts the geometry and topology of Kähler manifolds, and thus also of complex projective varieties, in a variety of ways. Another way to define Kähler manifolds, is to say they are Riemannian, symplectic and complex at the same time, together with a relation relating the three structures. In terms of tensor, we have by definition a Riemannian metric g, and symplectic form ω, and a complex structure J, with the compatibility condition that $g(J\cdot,\cdot) = \omega$. A natural setting in complex algebraic geometry is to fix the complex structure, and then to consider different Kähler structures, by varying ω. As long as one varies ω within one cohomology class $[\omega_0]$, it turns out that they are all of the form

$$\omega = \partial\bar{\partial}u + \omega_0,$$

a statement referred to as the $\partial\bar{\partial}$-lemma. This is convenient as it turns equations for defining special Kähler metrics into equations for a real valued function u.

Line bundles

On a compact Kähler manifold there are no non-constant holomorphic functions. Instead, one has line bundles, whose sections are locally holomorphic functions. For example, on projective space, on example is the line bundle given at a point by the line naturally defined by it. It is usually denoted by $\mathcal{O}(-1)$. Duals and tensor products in of this line bundle are written in additive notation $\mathcal{O}(k)$ for $k \in \mathbb{Z}$. The fact that the space of sections, denoted $H^0(X, \mathcal{O}(k))$, are given by the degree k homogeneous polynomials showcases the strong connection to algebra. On any complex projective manifold, a realization of it as a complex submanifold of projective space defines multiple line bundles on it by pullback of the $\mathcal{O}(k)$'s. A more intrinsic approach is to consider the variety in a more abstract language, equipped with a line bundle whose space of sections embeds the variety in projective space, and identifies the sections with homogeneous polynomials. These line bundles are called ample, geometrically defined by them admitting positively curved metrics.

Metrics on line bundles

A metric on a line bundle L is a pointwise norm $||\cdot||$ on each fiber such that $||s||^2$ is a smooth function for any holomorphic section s. Given a local trivialization e of the line bundle, i.e. locally $s = fe$ for a holomorphic function f, we can write

$$||s||^2 = |f|^2 \exp(-\phi) \tag{1.7}$$

for a function ϕ. Even though ϕ is only defined locally, it is useful to write ϕ for the metric, and then we mean a collection of locally defined functions, which glues together to a globally defined metric. The (Chern) curvature of the metric is given by $i\partial\bar{\partial}\phi$, and is a globally defined, closed, $2-$form. The class of the curvature form is called the first Chern class $c_1(L)$ of the line bundle L. An alternative to representing the metric with a locally defined function ϕ, is to choose some reference metric ϕ_0, then $u + \phi_0$ is a new metric for any smooth function u. The $\partial\bar{\partial}-$lemma from the section on Kähler geometry now implies that any Kähler metric ω in the first Chern class of a line bundle is the curvature form of some metric on the line bundle. I.e.

$$\omega = i\partial\bar{\partial}\phi. \tag{1.8}$$

This metric is by definition positively curved. Up to metric scaling this is a one-to-one correspondence.

Looping back to the introductory discussion, on $n-$dimensional projective space, any norm on $\mathbb{C}^{n+1}$ defines a positively curved metric on $\mathcal{O}(1)$. The Euclidean norm gives the *Fubini-Study metric*. Additionally, for a line bundle on a general projective manifold, if one can embed it with the space of sections into projective space, one obtains a positively curved metric by pullback of the Fubini-Study metric. More precisely the embedding identifies the line bundle with the pullback of some $\mathcal{O}(k)$. It should be noted that not every positively curved metric on a line bundle is of this sort. However, the existence of a positively curved metric implies that the line bundle is ample (by the Kodaira embedding theorem [15]), so that the line bundle can be identified with the pullback of some $\mathcal{O}(k)$ on projective space.

Fano varieties

On a projective manifold X there is a special line bundle, the *canonical line bundle*, K_X. The sections of it are locally of the form $f(z)\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$ for f a holomorphic function and are thus sometimes called holomorphic top-forms. One can also define the dual of this line bundle, dubbed the anti-canonical line bundle, or $-K_X$. The positivity properties of K_X and $-K_X$ play a major role for the geometry, and mostly conjecturally, the arithmetic of the underlying arithmetic variety (when there is one). The ones with K_X ample are in a certain sense negatively curved, while ones with $-K_X$ ample are positively curved. The questions in this thesis concern the case when $-K_X$ is positive. These projective manifolds are also called *Fano manifolds*, or *Fano varieties* when possibly singular.

For metrics on $-K_X$, the locally defined volume form

$$\exp(-\phi)\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$$

glues together perfectly to become a global volume form and will be denoted $\exp(-\phi)$.

1.4 Degree and volume

If we have a polynomial, an obvious affine invariant is the degree. Algebraically, the degree is one possible measure of the complexity of the polynomial. Furthermore, recall from the introduction that the degree is also a measure on the geometric complexity of the underlying analytic space. For a polynomial in one complex variable, the degree can be recovered geometrically by the number of zeros. For a hypersurface X in complex projective space $\mathbb{P}^{n+1}$, there is a similar procedure where the degree of the defining polynomial can be recovered by intersecting the variety with n hyperplanes, chosen generally so that the intersections are transverse. This notion turns out to be homological, i.e. only depending on an appropriate topological class of the hyperplanes and the hypersurface. For the intersection number, computing the degree in this case, we write

$$\mathcal{O}(1)^n \cdot X \tag{1.9}$$

where $\mathcal{O}(1)$ is the line bundle whose zero sets of its sections are the hyperplanes. Since it it homological, one can now compute this in a variety of ways. Either in the above intersection theoretic way, or cohomologically as an actual integral of a representative differential form on X. Since any line bundle on $\mathbb{P}^{n+1}$ is of the form $\mathcal{O}(k)$ for some k, X sitting inside $\mathbb{P}^{n+1}$ is cohomologically precisely $\mathcal{O}(d)$. Somewhere along the line, it became usual to call the number L^n of a polarized variety (X, L) the *degree* of L, comparing it to the elementary degree of a hypersurface. However, note that these notions are distinct. The first notion is relative to an embedding, while the second is intrinsic. From now on, lets focus on the latter notion. We can also compute the degree via differential forms on X. In particular, given any Kähler metric ω in the first Chern class $c_1(L)$ of L, we have

$$L^n = \int_X \omega^n. \tag{1.10}$$

Thus the algebraic invariant $L^n/n!$ of the polarized manifold (X, L) also coincides with the symplectic, and also the Riemannian volume of the Kähler manifold X, equipped with the Kähler metric ω. For an ample line bundle L, the number $L^n/n!$ is called the volume of L.

There is yet another way of calculating the volume of an ample line bundle, called the Hilbert-Samuel formula, namely through the asymptotics of the dimension of the space of holomorphic sections $H^0(X, kL)$ of higher and higher tensor powers kL (written here additively) of the line bundle L. More precisely, by the Riemann-Roch theorem,

$$\mathrm{Vol}(L) = \lim_{k\to\infty} \frac{1}{n!k^n} \mathrm{dim} H^0(X, kL). \tag{1.11}$$

This result may seem like the most obscure way of computing the degree or volume at the moment but in the arithmetic setting this is the closest to how we will define the arithmetic volume or degree. This arithmetic version of the degree will be the notion of height that we work with.

Fujita's universal sharp bound

In general, there cannot be any universal upper bound on the degree of any line bundle over an n-dimensional projective variety without further restrictions. For example, just take the repeated tensor power of an ample line bundle. Thus the degree in the sense of L^n does not really measure the complexity of the underlying variety X, rather of the of the pair (X, L).

In the Fano case, there is a canonical choice of ample line bundle, $-K_X$. One can ask if there is a universal bound on the degree of the anti-canonical line bundle for varieties of a fixed dimension n. If singularities are allowed, there are no lower bounds other than zero. There is however always an upper bound. For some time it was believed in addition that projective space maximized the anti-canonical degree among all Fano varieties of a given dimension. There are however toric counterexamples already in dimension 4. What was eventually established, first in [11] in the non-singular case, and then extended in [18], is that projective space is the maximizer as long as one restrict to varieties that satisfy a certain condition called K-semistability (see section 5.1).

Theorem 1 (Fujita, Liu). *Let X be a K-semistable Fano variety of dimension n. Then*

$$(-K_X)^n \leq (-K_{\mathbb{P}^n})^n \tag{1.12}$$

with equality if and only if $X = \mathbb{P}^n$

So put differently, one can differentiate between any K-semistable Fano variety and $\mathbb{P}^n$ just by looking at the degree of the anti-canonical line bundle. The same is true for a number of other invariants such as Seshadri constants, Fano index and alpha-invariants, see [20], [14] and [24], respectively.

1.5 Kähler-Einstein metrics

A *Kähler-Einstein metric* is a Kähler metric (see section 1.3), which is also a solution to the Einstein field equations, meaning that the Ricci curvature is proportional to the metric. This condition restricts the projective manifold to one of three cases depending on the sign of the curvature. The case of positive curvatures corresponds exactly to Fano manifolds. In terms of the symplectic form ω defining the Kähler metric, the Kähler-Einstein equation in this case reads

$$\mathrm{Ric}(\omega) = \omega,$$

after possible rescaling the metric.

As noted earlier, the height that we will consider will depend on a choice of metric on a line bundle, specifically, on the anti-canonical line bundle $-K_X$. We will argue that a natural metric to use, when it is available, is a Kähler-Einstein metric. Regardless, they will play a role in finding metric-universal bounds on the height. Anyhow, we define them and briefly mention the relevant theory. By a Kähler-Einstein metric on $-K_X$ we mean a metric whose curvature form defines the sympletic form of a Kähler metric which is Einstein. In fact, any Einstein metric which is also Kähler on a Fano manifold essentially arises as the curvature form of a metric on $-K_X$. The equation that should be satisfies with a metric ϕ on $-K_X$ for the curvature form of ϕ to be Einstein is the following complex Monge-Ampère equation.

$$(i\partial\bar{\partial}\phi)^n = C\exp(-\phi(z)) \tag{1.13}$$

for an arbitrary constant C (related to scaling the metric). The operator in the left hand side is usually called the *complex Monge-Ampère operator*, a fully non-linear, second order differential operator which locally takes the form

$$\det\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\phi\right)_{i,j}.$$

If equation (1.13) holds then the curvature form $\omega = i\partial\bar{\partial}\phi$ will be Einstein due to the expression for the Ricci curvature

$$\mathrm{Ric}(\omega) = -i\partial\bar{\partial}\log(\omega^n/\mathrm{d}z\wedge\mathrm{d}\bar{z}) \tag{1.14}$$

on Kähler manifolds

Variational theory

At this point, we have not explained why Kähler-Einstein metrics might be natural to use to define canonical heights. This will become clear in Section 7, where we will see that they appear as certain height-maximizing metrics, after a normalization is introduced. Understanding this turns out to be closely related to the variational theory of Kähler-Einstein metrics.

The starting point of the variational theory is the complex Monge-Ampère operator admits a functional primitive, i.e. there exists a functional on the space of positively curved metrics on $-K_X$, whose functional derivative is the complex Monge-Ampère operator. This functional is the *Monge-Ampère energy*, given by

$$\mathcal{E}_{\phi_0}(\phi) := \frac{1}{n+1}\sum_{j=0}^{n}\int(\phi-\phi_0)(\mathrm{dd}^c\phi)^j \wedge (\mathrm{dd}^c\phi_0)^{n-j}. \tag{1.15}$$

Here ϕ_0 is another metric on $-K_X$, which is needed to define the functional but for the sake of the variational theory it is completely arbitrary which one to use. The right hand side of the Kähler-Einstein equation also admits a functional primitive and one can define the *Ding functional* [8],

$$\mathcal{D}_{\phi_0}(\phi) := \mathcal{E}_{\phi_0} + \log\int e^{-\phi}, \tag{1.16}$$

whose Euler-Lagrange equation is the Kähler-Einstein equation (1.13). It turns out that this functional is concave in an appropriate sense, and through a vast body of work, the existence of Kähler-Einstein metrics on Fano varieties is reduced to a coercivity notion of the Ding functional. The coercivity condition is not always satisfied, and so Kähler-Einstein metrics does not exist on all Fano varieties. This is in contrast to the cases when K_X is positive or trivial, and the Ricci curvature is negative or zero, respectively. In these cases Kähler-Einstein metric always exists.

Remarkably, the coercivity of the Ding functional can be characterized by an algebraic condition on the underlying projective variety. Although it should be noted that this condition can be difficult to check in practice. This algebraic notion is called *K-stability*. A slightly weaker notion, *K-polystability*, precisely characterizes existence of Kähler-Einstein metrics, allowing for continuous automorphisms that leave the Ding functional non-coercive in certain directions. This massive achievement is referred to as the, now settled, Yau-Tian-Donaldsson conjecture [6][23]. The later proof in [4][5] was based on the variational method and it was eventually extended to the singular setting in [17][19]. An even weaker stability notion is called *K-semistability*, and it precisely characterizes the case when the Ding functional is bounded from above, see [16] and paper 1. Note that even in the K-semistable case, the supremum of the Ding functional itself is not a very interesting invariant since it depends on the choice of reference metric ϕ_0.

1.6 Arithmetic varieties

An *arithmetic variety* is a certain *scheme* over $\mathbb{Z}$ with some additional properties. Every arithmetic variety (here assumed projective) can always be realized as "something" cut-out by a tuple of homogeneous polynomials with coefficients in $\mathbb{Z}$. So far, when discussing varieties over $\mathbb{C}$ we have not made distinction between the abstract variety as an algebraic object and the associated analytic space. For the case of arithmetic varieties this is too sloppy. For example, the arithmetic varieties $X^2 + Y^2 = Z^2$ and $X^2 + Y^2 = -Z^2$ clearly are quite different from an arithmetic perspective. The first equation admits infinitely many integral points, called the Pythagorean triples, given by $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$ for arbitrary choices of integers a and b. The second equation does not admit a single integral solution, indeed, it does not even admit a real solution. But the analytic spaces they give rise to are exactly the same. A biholomorphism between them is given by sending $Z \mapsto iZ$. The correct notion which captures the arithmetic features of "things" cut-out by homogeneous polynomials with integer coefficients yet is flexible enough to not differentiate between realizations that really give rise to the same algebraic object is that of a *projective scheme* over $\mathbb{Z}$.

Remark. *Note that the above example seems to be a counterexample to the statement that arithmetic Fano varieties admits many rational points. And it is, if one is not cautious and reformulates the conjecture to be that there are many rational points if there are any at all. Alternatively, one leaves the world of ordinary arithmetic and lets* $\mathbb{Z}$ *be exchanged for another number field, in the above examples the Gaussian integers,* $\mathbb{Z}[i]$.

Just as for complex algebraic varieties, to an arithmetic variety $\mathcal{X}$ we will consider *arithmetic line bundles* $\mathcal{L}$ over $\mathcal{X}$, and occasionally *arithmetic sections* of $\mathcal{L}$. These notions are easiest understood when $\mathcal{X}$ is a hypersurface of projective space $\mathcal{P}^n_{\mathbb{Z}}$ cut-out by an integer coefficient degree d homogeneous polynomial. This polynomial is a section of $\mathcal{O}(d)$ over $\mathcal{X}$, the lattice of integer coefficient polynomials of degree d. We get more arithmetic sections of arithmetic line bundles on $\mathcal{X}$ by taking restrictions of integer coefficient homogeneous polynomials. The space of arithmetic sections of an arithmetic line bundle is always a lattice.

Lastly, some terminology that needs explanation is that of integral models. An arithmetic variety $\mathcal{X}$ is clearly also a variety over $\mathbb{Q}$. But to one variety X over $\mathbb{Q}$ one can associated several arithmetic varieties. Then $\mathcal{X}$ is called an *integral model* of X. For example, consider the quadratic hypersurface $\mathcal{X}$ defined by

$$x^2 + y^2 = 0.$$

Consider the map F taking $x = a - b$ and $y = a + b$. Pulling back the hypersurface under this map results in the new hypersurface $\mathcal{Y}$ defined by

$$2a^2 + 2b^2 = 0.$$

Since the map is defined over $\mathbb{Z}$ this counts as a morphism of arithmetic varieties. However, note that the inverse of F is given by $b = \frac{x+y}{2}$ and $a = \frac{y-x}{2}$. Thus $\mathcal{X}$ and $\mathcal{Y}$ are not equivalent as arithmetic varieties. However they are equivalent over $\mathbb{Q}$, and we say that $\mathcal{X}$ and $\mathcal{Y}$ are different integral models X, the hypersurface defined over $\mathbb{Q}$.

1.7 Heights and Arakelov geometry

Accepting the analogy between heights and degree as different measures of complexity, one could try to obtain a theory of heights by some intersection theory on arithmetic varieties. But it turns out that it is quite difficult to construct a well-behaved intersection theory for schemes over $\mathbb{Z}$. A picture to have in mind is that schemes over

$\mathbb{Z}$ have an extra scheme-theoretical 'dimension' compared to the dimensions related to the accompanying analytic space. Moving in this dimension is related to looking at the scheme from the perspective of a larger and larger prime. More precisely, the arithmetic variety fibers over this dimension and the fiber over a prime is precisely the arithmetic variety over $\mathbb{Z}_p$ that one obtains by reducing the coefficients modulo p. This extra dimension poses some non-compactness (or non-properness) issues for a well behaved intersection theory, analogous to how two intersecting lines in the affine plane intersects exactly once, only if they are not parallel, but always intersect exactly once in the projective plane. The solution proposed by Arakelov in [1] (and extended to higher dimensions in [13]) is to compactify the situation by, loosely speaking, add a (so to say) "prime at infinity". This turns out to correspond to adding, in an appropriate sense, the corresponding complex analytic space, together with an hermitian metric on an accompanying line bundle.

With this data one can define a kind of intersection theory of *metrized arithmetic divisors* that can be used to define a reasonable theory of heights, that for zero-dimensional subvarieties, i.e. points, agrees with the naive height if the correct metric is chosen. The theory is called an intersection theory because it has many features that resembles the ordinary algebro-geometric intersection theory. It should be noted that intersection numbers in this theory can not as clearly be understood as the actual number of elements in some set of intersection points. But this is also good because we want the intersection theory to describe heights, which even in the naive setting does not seem to be related to the number of intersection points of anything.

The height, in the Arakelov sense, is associated with the data consisting of an arithmetic variety $\mathcal{X}$ together with an arithmetic line bundle $\mathcal{L}$, where one in addition has fixed an hermitian metric ϕ on the complexification L over X, the complexification of $\mathcal{X}$. One defines the height, or *arithmetic degree*, in parallel with the degree of a line bundle over a complex variety, by the top *arithmetic intersection number*

$$(\mathcal{L})_{\phi}^{n+1}. \tag{1.17}$$

The fact that it is an $n+1$:th self-intersection is related to the extra dimension from the Spec($\mathbb{Z}$)-direction.

Before we properly define the height in general, we showcase a formula for it for a degree d hypersurface $\mathcal{X}$ of projective space. If $\mathcal{X}$ is cut-out by a section s of $\mathcal{O}(d)$, then we can restrict $\mathcal{O}(k)$ to $\mathcal{X}$ to get an arithmetic line bundle on $\mathcal{X}$ and then (up to some numerical constants that we omit)

$$(\mathcal{O}(k))_{\phi}^{n+1} = \frac{1}{2}\mathcal{E}(\phi')_{\phi_0} + \int_{\mathbb{P}^{n+1}} \log ||s||_{\phi'}(\partial\bar{\partial}\phi')^{n+1}. \tag{1.18}$$

Here ϕ is a positively curved metric on $\mathcal{O}(k)|_{\mathcal{X}}$, and ϕ' is an arbitrary positively curved extension of ϕ to $\mathcal{O}(k)$. Here the reference metric ϕ_0 is the metric on $\mathbb{P}^n$ associated to the max-norm in the naive definition of the height. The required extension ϕ' always exists, but the result is independent of choice of the extension.

It is instructive to compare this to the naive height of a hypersurface. Recall that the naive height in this case would be the logarithm of the maximum of the coefficients of s, when the coefficients are chosen relatively prime. From this it follows that if the coefficients are all distinct, large, prime numbers, then the naive height is large as well. From the above formula, one can guess that a similar behaviour is true for the Arakelov height, noting that $(\partial\bar{\partial}\phi)^{n+1}$ is a positive measure.

In general we define the height via an analog of the Hilbert-Samuel formula (1.11). First, recall the notion of an arithmetic section of an arithmetic line bundle. The set of arithmetic sections $H^0(\mathcal{X}, \mathcal{L})$ sits inside $H^0(X, L)$ as a lattice, i.e. a free $\mathbb{Z}$-module. Now, the analog of the classical Hilbert-Samuel formula (1.11), is a sort of asymptotic formula the number of points belonging to the lattice $H^0(\mathcal{X}, k\mathcal{L})$ inside the L^∞-ball of $H^0(X, kL)$. More precisely, according to the arithmetic Hilbert-Samuel formula (see [13]),

$$\frac{(\mathcal{L})_{\phi}^{n+1}}{(n+1)!} = \lim_{k\to\infty} \frac{1}{k^{n+1}} \log \operatorname{vol}\{s \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} : \sup_X ||s||_{k\phi} \leq 1\} \tag{1.19}$$

where $H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R}$ can be identified with the subspace of real sections in $H^0(X, kL)$. Here the volume should be computed with respect to a Lebesgue measure normalized such that a fundamental domain of the lattice $H^0(\mathcal{X}, k\mathcal{L})$ gets unit volume. This normalization is essentially where the arithmetic model enters.

Importantly, the definition leads to the following formula for the difference of the heights with respect to two metrics ϕ and ϕ_0.

$$(\mathcal{L})_{\phi}^{n+1} - (\mathcal{L})_{\phi_0}^{n+1} = \frac{(n+1)}{2}\mathcal{E}_{\phi_0}(\phi), \tag{1.20}$$

revealing a connection between heights and Kähler-Einstein metrics, which we will come back to in section 1.9.

1.8 Toric geometry

Toric varieties is a class of varieties that enjoy a particularly simple and large group of symmetries that often makes their study considerable simpler than the case of a general variety. Many constructions in algebraic geometry become combinatorial and convex geometric, while their theory as Kähler manifolds becomes deeply intertwined with convex analysis.

A *projective toric variety* X of dimension n admits a faithful action of the complex torus $\mathbb{C}^{*n} := (\mathbb{C} \setminus \{0\})^n$, (i.e., the action is as big as possible). Additionally, there is a dense orbit. In other words, the entirety of X can be though of as a compactification of $\mathbb{C}^{*n}$. Given an ample line bundle L on X, one can choose a basis of $H^0(X, kL)$ which is equivariant with respect to the torus action. Over the dense subset $\mathbb{C}^{*n}$, these must be monomials. But not all monomials can become holomorphic sections in the compactification, since the space of sections of a line bundle over a projective variety is always finite-dimensional. In fact, to each toric variety equipped with an ample line bundle there is a (non-unique) polytope, called the *moment polytope*, such that a monomial $z_1^{a_1} z_2^{a_2} ... z_n^{a_n}$ is part of $H^0(X, L^k)$ if and only if $(a_1, ..., a_n)/k \in P$. Conversely, given any polytope with rational vertices, one gets a toric variety by (for a sufficiently large $k \in \mathbb{N}$) considering the closure of the image of the map $(\mathbb{C}^*)^n \to \mathbb{P}^{\mathbb{N}}$, $z \to [z^{p_1}, z^{p_2}, ..., z^{p_{N_k}}]$ where we have used the notation $z^p = z_1^{p_1} ... z_{N_k}^{p_{N_k}}$ and where $p_1, ..., p_{N_k}$ are the lattice (or integer) points of the scaled polytope kP.

Once this correspondence between toric varieties equipped with ample line bundles, (X, L), and polytopes, P, is set up, there is a vast and growing dictionary translating between notions in algebraic and Kähler geometry, to notions in convex geometry and analysis. The degree of an ample line bundle is given by the volume of the corresponding polytope. Toric Fano varieties equipped with the anti-canonical line bundle correspond to (duals) of polytopes of a certain type called *Fano polytopes.* $(S^1)^n$ invariant metrics on L with positive curvature correspond to convex functions u on $\mathbb{R}^n$ with a certain growth condition encoded by P. The Kähler-Einstein equation for $(S^1)^n$-invariant metrics becomes a certain real Monge-Ampère equation, and it turns out that all Kähler-Einstein metrics on toric Fano varieties has this symmetry, up to biholomorphisms.

A neat fact about toric varieties is that they are naturally defined over the integers. This can be seen directly from their occurrence as closures of images of monomial maps into projective space. A monomial is canonically defined over $\mathbb{Z}$ simply because we can choose the coefficient in front of it to be one. Thus all toric varieties have a *canonical integral model* (that we will just refer to as the canonical integral model). With respect to this model, the height of the canonical integral model of a toric Fano variety takes the following form

$$(-\mathcal{K}_{\mathcal{X}})_u^{n+1} = 2\int_P -u^* \mathrm{d}p \tag{1.21}$$

where $u^*(p) := \sup_x \langle x, y\rangle - u(x)$ is the Legendre transform of u.

1.9 Canonical heights and sharp height bounds on toric Fano varieties (Paper 1)

There is a philosophy that virtually any statement in algebraic geometry should have a counterpart in Arakelov geometry. Recall the Fujita theorem 1 about the sharp upper bound on the anticanonical degree of a K-semistable Fano variety. In the first paper we introduce and investigate a possible arithmetic or Arakelov version of this, i.e. a sharp upper bound on the arithmetic anti-canonical degree of an arithmetic Fano variety. As is evident by now, the notion of arithmetic degree also depends on a choice of metric on $-K_X$. One immediate problem with a bound of the type asked of above is that if one scales the metric on $-K_X$, i.e. $\phi \mapsto \phi + c$, then the height changes additively as $(-\mathcal{K}_{\mathcal{X}})_{\phi+c}^{n+1} = (-\mathcal{K}_{\mathcal{X}})_{\phi}^{n+1} + (-K_X)^n c$. Thus we need to normalize the metrics somehow. One seemingly natural normalization is to require that the volume form defined by ϕ, denoted $\exp(-\phi)$ has total volume 1, that is $\int_X e^{-\phi} = 1$. We call such metrics normalized and one can wonder whether there is an upper bound on the possible anti-canonical arithmetic degrees of arithmetic Fano varieties metrized with normalized metrics. The first paper attached to the thesis introduces the following conjecture regarding the above question.

Conjecture 1. *Let $\mathcal{X}$ be an arithmetic Fano variety such that X is K-semistable and ϕ a normalized, continuous, positively curved hermitian metric on $-K_X$, then*

$$(-\mathcal{K}_{\mathcal{X}})_{\phi}^{n+1} \leq (-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}})_{\phi_{\mathrm{FS}}}^{n+1} \tag{1.22}$$

where $\mathbb{P}^n_{\mathbb{Z}}$ is the projective space over the integers and ϕ_{FS} is up to a biholomorphism the normalized Fubini-Study metric.

Let us stress that the right hand side above is explicitly known,

$$(-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}})^{n+1}_{\phi_{\mathrm{FS}}} = \frac{1}{2}(n+1)^{n+1}((n+1)\sum_{k=1}^{n} k^{-1} - n + \log(\frac{\pi^n}{n!})). \quad (1.23)$$

Let us for clarity also state the conjecture in convex-theoretic language in the toric case.

Conjecture. *Let P be the moment polytope of a toric Fano variety with barycenter in the origin so that the variety is K-semistable. Let u be any convex function on $\mathbb{R}^n$ with Legendre transformation with domain P (finite on precisely P). Then*

$$2\int_P u^*(p)\mathrm{d}p - 2V(P)\log\left(\pi^n \int_{\mathbb{R}^n} \exp(-u)\right) \leq (-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}})^{n+1}_{\phi_{\mathrm{FS}}} \quad (1.24)$$

where $V(P)$ is the volume of P.

In general, we define the *canonical height* of an arithmetic Fano variety to be the supremum of the height of a fixed arithmetic variety with respect to continuous positively curved normalized metrics on $-K_X$.

That the Fubini-Study metric, which is Kähler-Einstein, appears as the maximizer of the height on projective space - and thus as a metric whose height corresponds to the canonical heights - is part of a general pattern. In fact we show the following series of statements relating Kähler-Einstein metrics and K-stability with height maximization.

Proposition 1. *If the supremum of $(-\mathcal{K}_{\mathcal{X}})^{n+1}_{\phi}$ over normalized metrics ϕ is attained, it is attained at a Kähler-Einstein metric. Conversely if X has a Kähler-Einstein metric, the normalization of it attains the supremum. The supremum is finite if and only if X is K-semistable.*

The idea is quite simple, given the large existing body of work concerning the variational theory of Kähler-Einstein metrics [4], and relies on the fact that taking the supremum of the height over all normalized metrics is the same as taking the supremum over all metrics, if one adds to the height the term $(-K_X)^n \log \int_X \exp(-\phi)$. In the latter case, the resulting functional differs from the Ding-functional by just an additive metric independent term.

Remark. *Let us note that the supremum in the definition could be taken, without changing it, over all embeddings into projective space and the metrics taken to be pullbacks of Fubini-Study metrics (this follows from Demailly approximations [7]). In this way, one would stay closer to the naive theory of heights. One advantage of fully utilizing the generality of Arakelov theory is that the supremum is a maximum whenever a Kähler-Einstein metric exists, this would likely only happen on extremely rare occasions otherwise.*

Remark. *At least naively, to compute the canonical height, when it is finite, one needs an explicit Kähler-Einstein metric. But this is almost never the case. However upper bounds on the canonical height are by definition also upper bounds for the height in general for normalized metrics.*

The main result of the paper is a resolution of the conjecture for toric Fano varieties, equipped with their canonical integral model, for low dimensions, or alternatively, if a conjecture that we dubbed the *gap hypothesis* holds for the algebro-geometric degree.

Theorem 2. *Conjecture 1 is true for the canonical integral model $\mathcal{X}$ of any toric Fano manifold X as long as the dimension of X is at most 6. It is also true in any dimension for certain classes of singular toric Fano varieties (see paper 1 for the precise statement).*

The proof is based on a metric independent bound for the toric Ding-functional for K-semistable toric Fano varieties, utilizing the functional Santalo inequality from [2]. More precisely, we can prove in this case that

$$(-\mathcal{K}_{\mathcal{X}})_{\phi}^{n+1} \leq \frac{n+1}{2}(-K_X)^n \log\left(\frac{(2\pi^2)^n n!}{(-K_X)^n}\right). \tag{1.25}$$

The bound only depends on the degree $(-K_X)^n$ and the dimension, but is not optimal. However given that the degree is at most the degree of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, one can deduce the conjectural sharp bound. This last step is what we call the *gap hypothesis*, or more precisely,

Conjecture 2. *Let X be a K-semistable toric Fano variety which is not $\mathbb{P}^n$, then $(-K_X)^n \leq (-K_{\mathbb{P}^{n-1}\times\mathbb{P}^1})^n$.*

For toric Fano varieties with certain singularities, for example quotient singularities, the above conjecture can be proven by convex geometric techniques. For non-singular toric Fano varieties there is a complete classification up to dimension 6 for which Conjecture 2 can be checked by going through the classification. This completes the proof of Conjecture 1 in these cases. Recall that since the volume of an ample line bundle on a toric variety is related to the volume of the corresponding polytope, the above conjecture can be formulated in the language of volumes of a certain class of polytopes. Let us also note that, for non-singular but possibly non-toric Fano varieties, there is no counter-example produced by the existing classifications in dimension 2 and 3.

1.10 Logarithmic pairs

In the second paper of the thesis, summarized in the next section, logarithmic pairs play an important role. A *logarithmic pair*, is just a pair (X, Δ), consisting of a variety X, and a *divisor* Δ, i.e. a formal $\mathbb{Z}$-linear combination of codimension one subvarieties. By definition divisors are locally cut-out by holomorphic functions and in fact one can always associated a line bundle with an associated section which cuts out the divisor. For a divisor Δ we will denote the line bundle by Δ as well. For a logarithmic pair one defines the *log-canonical line bundle* by $K_{(X,\Delta)} = K_X + \Delta$. In several ways, this line bundle will play the role of the ordinary canonical line bundle in the logarithmic setting. The main merit of these definitions is that they naturally appear when doing various birational operations on an ordinary variety, for example blow-ups.

Another situation where log pairs pop-up naturally are branched covers, of which a prototypical example is $z \mapsto z^n$ on $\mathbb{C}$. Since this map has degenerate Jacobian at any coordinate hyperplane if $n \geq 2$, a holomorphic differential form on the image will not pull back to a holomorphic differential form on the domain but will have a certain singularity at 0. More precisely, if $w = z^n$, then $\mathrm{d}z$ corresponds to $\frac{1}{n}w^{1-\frac{1}{n}}\mathrm{d}w$ outside of 0. The (affine) logarithmic pair $(\mathbb{C}, (1-1/n)[0])$ contains the information of the singularity of a differential forms that arises from pullback under the z^n map. In general, from a branched cover, one can construct an associated log-pair in the sense that the canonical line bundle pulls back to the log-canonical line bundle of the log-pair.

Once log pairs have been introduced, many objects we have so far mentioned generalize to logarithmic versions. We have log Fano pairs for which $-(K_X + \Delta)$ admits a positively curved metric, and the corresponding algebraic degree-like invariant $-(K_X + \Delta)^n$. We have the notion of *log Kähler-Einstein metrics* ω which satisfy in, a certain precise sense, the singular Kähler-Einstein type equation

$$\mathrm{Ric}(\omega) = \omega + [\Delta] \tag{1.26}$$

where $[\Delta]$ is the current of integration along Δ. This is also the equation satisfied by the pullback of a smooth Kähler-Einstein metric under a blow-up, or what should be solved on the base of a branched cover for the pullback metric to be an ordinary Kähler-Einstein metric. The theory of heights also admits a straightforward logarithmic version if Δ is a divisor cut-out by an arithmetic section.

1.11 Canonical heights and sharp height bounds on some logarithmic pairs and diagonal hypersurfaces (Paper 2)

In the second paper, we extend conjecture 1 regarding the sharp upper bound on the height of metrized arithmetic Fano varieties to logarithmic pairs. An *arithmetic log Fano pair* is an arithmetic log-pair where the complexified log-anti-canonical bundle $-(K_X + \Delta)$ is ample.

If we have an arithmetic variety $\mathcal{X}$ and Δ is cut-out by an arithmetic section we have the notion of an arithmetic log-pair. We can consider the Arakelov invariant, $-(\mathcal{K}_{\mathcal{X}} + \Delta)^{n+1}_{\phi}$, for a metric ϕ on $-K_{(X,\Delta)}$. In this setting, ϕ no longer defines a volume form but due to the arithmetic structure on Δ, we have a preferred choice of section s_Δ cutting out Δ. Recall that the set of arithmetic sections is a lattice, so that given any section cutting out a divisor, we can divide by an integer to obtain a section corresponding to a primitive point of the lattice. Clearly this section cuts-out the same complex analytic variety. Notice that a similar procedure is not possible without the arithmetic information. Thus, from the data of an arithmetic log Fano pair, we obtain a measure on X by considering locally the expression $\exp(-\phi)|s_\Delta|^2$, which glues into a global measure on X. We say that ϕ is normalized when this integrates to 1. Thus we can still define a logarithmic version of our canonical height on an arithmetic log Fano variety by considering

$$\sup_{\phi} -(\mathcal{K}_{\mathcal{X}} + \Delta)^{n+1}_{\phi}, \tag{1.27}$$

where the supremum is taken over normalized metrics. Analogous to the non-logarithmic setting, whenever maximizers exists, they are log Kähler-Einstein. The log-version of conjecture 1 reads

Conjecture 3. *Let $(\mathcal{X}, \mathcal{D})$ be an arithmetic Fano variety such that $(X, \mathcal{D})$ is K-semistable and ϕ a normalized, continuous, positively* *curved hermitian metric on* $-(K_X + \Delta)$. *Then*

$$(-\mathcal{K}_{\mathcal{X}} + \mathcal{D})_\phi \leq (-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}})_{\phi_{\mathrm{FS}}}, \tag{1.28}$$

where $\mathbb{P}^n_{\mathbb{Z}}$ *is the projective space over the integers and* ϕ_{FS} *is the normalized Fubini-Study metric. Additionally, equality happens only for this case, up to a biholomorphism.*

One of the main results is the resolution of the conjecture in the toric case in low dimension.

Theorem 3. *Conjecture 3 holds for the canonical integral model of non-singular toric log-pairs of dimension at most 3.*

The proof follows the same strategy as Theorem 1 from Paper 1 and follows from a logarithmic version of the gap hypothesis, conjecture 2. The added difficulty that restricts the dimension to at most 3 is that a logarithmic version of the 'gap hypothesis' must be established, while there are uncountable many K-semistable non-singular toric log Fano pairs, in any dimension.

A second result, that goes beyond the toric setting, is that in the particular case where $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$, and $\mathcal{D}$ is supported on hyperplanes that are in simple normal crossing arrangement (an snc hyperplane arrangement for short), then the main conjecture is true.

Theorem 4. *If* $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$, *and* $\mathcal{D}$ *is supported on a simple normal crossings hyperplane arrangement (equipped with the canonical integral model), then Conjecture 3 holds.*

Although the result goes beyond the toric setting, the proof uses an elementary convexity property of the height in the logarithmic case, together with an explicit characterization of the K-semistable log Fano snc hyperplane arrangements from [12] to reduce to the toric case. For toric hyperplane arrangements, the canonical height can be explicitly calculated.

As explained in section 10, logarithmic pairs appear naturally in certain constructions in algebraic geometry. For example in the case of branched covers. This is used in another result of paper 2, where we consider conjecture 1 for the arithmetic Fano diagonal hypersurfaces. These are the hypersurfaces of the form,

$$\sum_{k=0}^{n+1} a_k X_k^d = 0. \tag{1.29}$$

Here a_k are integers and $d \leq n+1$ so that the above equation defines an arithmetic Fano variety. A computation shows that the canonical height is minimized when $a_k = 1 \forall k$, the so called Fermat hypersurface of degree d. The map $X_k \mapsto X_k^d$ on $\mathbb{P}^{n+1}$ expresses the Fermat hypersurface as a branched cover over $\mathbb{P}^n$. The branching happens precisely over a simple normal crossings hyperplane arrangement and the techniques from the proof of 4 we can prove the following.

Theorem 5. *Conjecture 1 is true for diagonal Fano hypersurfaces.*

Remark. *The above statement is formulated entirely in terms of heights and arithmetic. But our proof, while quite simple, requires deep input from the theory of K-stability and Kähler-Einstein metrics. For example the explicit characterization of K-semistable simple normal crossing hyperplane arrangements, or that a Kähler-Einstein metric on a toric log-pair, when it exists, is invariant under the real torus action (up to a biholomorphism).*

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Part 2

Appended papers

Paper 1

SHARP BOUNDS ON THE HEIGHT OF K-SEMISTABLE TORIC FANO VARIETIES, I.

ROLF ANDREASSON, ROBERT J. BERMAN

Abstract. Inspired by Fujita's algebro-geometric result that complex projective space has maximal degree among all K-semistable complex Fano varieties, we conjecture that the height of a K-semistable metrized arithmetic Fano variety $\mathcal{X}$ of relative dimension n is maximal when $\mathcal{X}$ is the projective space over the integers, endowed with the Fubini-Study metric. Our main result establishes the conjecture for the canonical integral model of a toric Fano variety when $n \leq 6$ (the extension to higher dimensions is conditioned on a conjectural "gap hypothesis" for the degree). Translated into toric Kähler geometry this result yields a sharp lower bound on a toric invariant introduced by Donaldson, defined as the minimum of the toric Mabuchi functional. We furthermore reformulate our conjecture as an optimal lower bound on Odaka's modular height. In any dimension n it is shown how to control the height of the canonical toric model $\mathcal{X}$, with respect to the Kähler-Einstein metric, by the degree of $\mathcal{X}$. In a sequel to this paper our height conjecture is established for any projective diagonal Fano hypersurface, by exploiting a more general logarithmic setup.

1. Introduction

1.1. The height of K-semistable Fano varieties. Let $(\mathcal{X}, \mathcal{L})$ be a projective flat scheme $\mathcal{X}$ over $\mathbb{Z}$ of relative dimension n, endowed with a relatively ample line bundle $\mathcal{L}$. The complexification of $(\mathcal{X}, \mathcal{L})$ will be denoted by (X, L). In other other words, X is the complex projective variety consisting of the complex points of $\mathcal{X}$ and L is the corresponding ample line bundle over X.

A central role in arithmetic and Diophantine geometry is played by the *height* of $(\mathcal{X}, \mathcal{L})$, which is defined with respect to a continuous metric $\|\cdot\|$ on L. This is an arithmetic analog of the algebro-geometric degree of (X, L), i.e., of the top intersection number L^n on X. The height of $(\mathcal{X}, \mathcal{L}, \|\cdot\|)$ - also known as Faltings' height - is defined as the $(n+1)-$fold arithmetic intersection number of the metrized line bundle $(\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$, introduced by Gillet-Soulé in the context of Arakelov geometry [43, 19] (see Section 1.1). We recall that in Arakelov geometry the metric $\|\cdot\|$ on L plays the role of a "compactification" of $\mathcal{X}$. Accordingly, a metrized line bundle $(\mathcal{L}, \|\cdot\|)$ is usually denoted by $\overline{\mathcal{L}}$. The definition of height naturally extends to any $\mathbb{Q}-$line bundle $\mathcal{L}$, using homogeneity.

In contrast to the algebro-geometric degree of L the height of $\overline{\mathcal{L}}$ can rarely be computed explicitly and all one can hope for, in general, is explicit bounds on the height. When $\mathcal{L}$ is the relative canonical line bundle, that we shall denote by $\mathcal{K}_{\mathcal{X}}$ and $n = 1$, such conjectural upper bounds are motivated by the Bogolomov-Miyaoka-Yau inequality on X and imply, in particular, the effective Mordell conjecture, concerning explicit upper bounds on the number of rational points on $X_{\mathbb{Q}}$ and the abc-conjecture [79, 95, 84]. Here we shall be concerned with the opposite situation where $\mathcal{X}$ is an *arithmetic Fano variety*, in the sense that the relative anti-canonical line bundle is defined as a relative ample $\mathbb{Q}-$line bundle that we denote by $-\mathcal{K}_{\mathcal{X}}$, using additive notation for tensor products (see Section 2.2.1). In particular, X is a complex *Fano variety*;

a variety whose canonical line bundle $-K_X$ defines an ample $\mathbb{Q}-$line bundle. We will also, for simplicity, assume that X is normal. As shown in [10] in the toric case and then [46] in general, for any complex Fano variety X

$$(-K_X)^n \leq (-K_{\mathbb{P}^n_{\mathbb{C}}})^n \tag{1.1}$$

under the assumption that X is *K-semistable.* Moreover, equality holds iff $X = \mathbb{P}^n_{\mathbb{C}}$ [61]. In contrast, when X is not K-semistable the degree $(-K_X)^n$ gets arbitrarily large in any given dimension n, for singular X (see [34, Ex 4.2] for simple two-dimensional toric examples). The notion of K-stability first arose in the context of the Yau-Tian-Donaldson conjecture for Fano manifolds, saying that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable [92, 38]. The conjecture was settled in [28] and very recently also established for singular Fano varieties [57, 63]. From a purely algebro-geometric perspective K-stability can be viewed as a limiting form of Chow and Hilbert-Mumford stability [82], that enables a good theory of moduli spaces (see the survey [98]).

Is there an arithmetic analog of the inequality 1.1? More precisely, it seems natural to ask if, under appropriate assumptions, the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is bounded from above by the height $(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ of the relative anti-canonical line bundle on the projective space $\mathbb{P}^n_{\mathbb{Z}}$ over the integers, endowed with its standard Kähler-Einstein metric (the Fubini-Study metric)? This would yield an explicit bound on the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$, since the height of Fubini-Study metric on projective space was explicitly calculated in [52, §5.4], giving, after volume-normalization,

$$(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1} = \frac{1}{2}(n+1)^{n+1}\left((n+1)\sum_{k=1}^{n} k^{-1} - n + \log(\frac{\pi^n}{n!})\right) \tag{1.2}$$

If such a universal bound is to hold one needs, however, to impose a normalization condition on the metric on $-K_X$. Indeed, $\overline{\mathcal{L}}^{n+1}$ is additively equivariant with respect to scalings of the metric. Accordingly, the metric $\|\cdot\|$ on $-K_X$ will henceforth be assumed to be *volume-normalized* in the sense that the corresponding volume form on X has total unit volume. As it turns out, the supremum of the height $\overline{-\mathcal{K}_{\mathcal{X}}}^{n+1}$ over all volume-normalized metrics on $-K_X$ with positive curvature current is finite if and only if X is K-semistable (Theorem 2.4). It seems thus natural to make the following conjecture:

Conjecture 1.1. *Let $\mathcal{X}$ be an arithmetic Fano variety of relative dimension n over $\mathbb{Z}$. If the complexification X of $\mathcal{X}$ is K-semistable, then the following height inequality holds for any volume-normalized continuous metric on $-K_X$ with positive curvature current:*

$$(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} \leq (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1},$$

where $-K_{\mathbb{P}^n_{\mathbb{C}}}$ is endowed with the volume normalized Fubini-Study metric. Moreover, if $\mathcal{X}$ is normal equality holds if and only if $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and the metric is Kähler-Einstein, i.e. coincides with the Fubini-Study metric, modulo the action of an automorphism.

More generally, when $\mathbb{Z}$ is replaced by the ring of integers of a number field F, i.e. a finite field extension F of $\mathbb{Q}$, the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ should be divided by the degree $[F:\mathbb{Q}]$. But, for simplicity, we will focus on the case when $F = \mathbb{Q}$ (see Section 6.2 for a generalization of the previous conjecture). The converse "only if" statement to the previous conjecture does hold (as a consequence of Theorem 2.4). Moreover, the conjecture is compatible with taking products (Prop 2.10). The inequality in the previous conjecture is equivalent to the following inequality for any continuous metric on $-K_X$ with positive curvature current, as follows from a simple scaling argument,

$$\frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}}{(n+1)} + \frac{(-K_X)^n}{2}\log\mu(X) \leq c_n \tag{1.3}$$

where $\mu(X)$ denotes the volume of X with respect to the measure μ on X corresponding to the metric $\|\cdot\|$ on $-K_X$ and c_n denotes the constant in the right hand side of formula 1.2. Some intruiging relations between the conjectural bound 1.3 and the Manin-Peyre conjecture, concerning the density of rational points on Fano varieties, are discussed in [7].

Our main result concerns the case when X is toric and $\mathcal{X}$ is its canonical toric integral model (see [68, Section 2] and [22, Def 3.5.6]).

Theorem 1.2. *Let X be an $n-$dimensional K-semistable toric Fano variety and denote by $\mathcal{X}$ its canonical model over $\mathbb{Z}$. Then the previous conjecture holds under anyone of the following conditions:*

- *$n \leq 6$ and X is $\mathbb{Q}-$factorial (equivalently, X is non-singular or has abelian quotient singularities)*
- *X is not Gorenstein or has some abelian quotient singularity*

Note that when $n = 2$ any toric variety is, in fact, $\mathbb{Q}-$factorial. More generally, we will show that the curvature assumption may be dispensed with if the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is replaced by the $\chi-$arithmetic volume $\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right)$ of $\overline{-\mathcal{K}_{\mathcal{X}}}$ (whose definition is recalled in Section 2.2.2). We expect that the maximum of $\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right)$ over all integral models $(\mathcal{X}, -\mathcal{K}_{\mathcal{X}})$ of a given toric Fano variety $(X, -K_X)$ is attained at the canonical integral model $\mathcal{X}$ featuring in the previous theorem. This expectation is inspired by a conjecture of Odaka discussed in Section 1.4 below.

The key ingredient in the proof of Theorem 1.2 is the following bound estimating the arithmetic volume $\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right)$ of any volume-normalized metric on $-K_X$ in terms of the algebro-geometric volume $\mathrm{vol}(X)$ (Prop 3.7):

$$\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right) \leq -\frac{1}{2}\mathrm{vol}(X)\log\left(\frac{\mathrm{vol}(X)}{(2\pi^2)^n}\right) \quad \mathrm{vol}(X) := (-K_X)^n/n! \tag{1.4}$$

Since $\mathrm{vol}(X)$ is maximal for $X = \mathbb{P}^n$ the right hand side above is bounded by a constant C_n only depending on the dimension n. Under the "gap hypothesis" that $\mathbb{P}^{n-1} \times \mathbb{P}^1$ has the second largest volume among all $n-$dimensional K-semistable X we show that the bound 1.4 implies Conjecture 1.1 for the canonical integral model $\mathcal{X}$ of a toric Fano variety X. The proof of Theorem 1.2 is concluded by verifying the gap hypothesis under the conditions in Theorem 1.2. But we do expect that the gap hypothesis above holds for any toric Fano variety (see Section 3.2.1).

In a sequel [1] to the present paper Conjecture 1.1 is established for any diagonal Fano hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ (i.e. $\mathcal{X}$ is the subscheme cut out by a homogeneous polynomial of the form $a_0x_0^d + ... + a_{n+1}x_{n+1}^d$ for any given integers a_i, with no common divisors, and $d \leq n+1$). Although $\mathcal{X}$ is not toric the proof, somewhat suprisingly, is reduced to a simple toric logarithmic case.

1.2. **The height of toric Kähler-Einstein metrics.** In the toric case, X is K-semistable if and only if it is K-polystable and thus admits a toric Kähler-Einstein metric [96, 8], i.e. a toric continuous metric on $-K_X$ whose curvature form defines a Kähler metric with constant positive Ricci curvature on the regular locus of X. Moreover, in general, any volume-normalized Kähler-Einstein metric maximizes $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$. This means that the inequality in the previous theorem is equivalent to the corresponding inequality for the volume-normalized toric Kähler-Einstein metric on $-K_X$. The special role of the Kähler-Einstein condition in arithmetic (Arakelov) geometry - as an analog of minimality of $\mathcal{X}$ over Spec $\mathbb{Z}$ - was emphasized already in the early days of Arakelov geometry by Manin [69]. It is, however, rare that the Kähler-Einstein metric and the corresponding height, can be explicitly computed. In fact, in the Fano case this seems to only have been achieved when X is homogeneous [67, 25, 54, 88, 89, 90]. The following result, complementing the general upper bound 1.4, yields a rather precise control on its height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ in the toric case:

Theorem 1.3. *Let X be an $n-$dimensional toric Fano variety and denote by $\mathcal{X}$ its canonical model over $\mathbb{Z}$. Then the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ of any volume-normalized Kähler-Einstein metric satisfies*

$$\frac{(n+1)!}{2}\mathrm{vol}(X)\log\left(\frac{n!m_n\pi^n}{\mathrm{vol}(X)}\right) \leq (\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} \leq \frac{(n+1)!}{2}\mathrm{vol}(X)\log\left(\frac{(2\pi)^n\pi^n}{\mathrm{vol}(X)}\right)$$

where m_n denotes the largest lower bound on the Mahler volume of a convex body. In particular, $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} > 0$.

We also provide an infinite family of toric varities X for which the height of the corresponding Kähler-Einstein can be explicitely computed as a function $f(v)$ of vol(X) of the same form as in the previous theorem; $f(v) = v\log(av^{-1})$ for some constant a. The constant m_n in the previous theorem is the largest constant satisfying

$$m_n \leq \mathrm{vol}(P)\mathrm{vol}(P^*),$$

where P^* denotes the polar dual of any given convex body P containing the origin in its interior (the role of P in the present setting is played by the moment polytope of X). According to Mahler's conjecture, the constant m_n is equal to $(n+1)^{n+1}/(n!)^2$ (which is realized for a simplex P). The case $n = 2$ was settled in [66], but for our purposes the following general bound from [51] will be enough:

$$m_n \geq (\frac{\pi}{2e})^{n-1}(n+1)^{n+1}/(n!)^2,$$

which implies the strict positivity of $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$. Combining the previous theorem with the upper bound 1.1 thus yields the following universal bounds:

Corollary 1.4. *Let X be an $n-$dimensional toric Fano variety and denote by $\mathcal{X}$ its canonical model over $\mathbb{Z}$. Then the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ of any volume-normalized Kähler-Einstein metric satisfies the following universal bounds*

$$0 < (\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} \leq \frac{n(n+1)^{n+1}}{2}\log\left(\frac{2\pi^2 n!}{n+1}\right)$$

Incidentally, the upper bound above is related to a question posed in [73], asking whether $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is bounded from above by a universal constant C_n, under the assumption that X be non-singular and $\overline{-\mathcal{K}_{\mathcal{X}}}$ be relatively ample. This is a stronger condition than having positive curvature, as we assume. We also allow singularities, but our results concern only the toric case. Under the conditions in Theorem 1.2 our upper bound may be improved to the sharp bound $(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ (given by formula 1.2). As for the lower bound it is sharp in any dimension n. Indeed, there are $n-$dimensional K-semistable ($\mathbb{Q}$-factorial) Fano varieties X such that vol(X) and thus (by Theorem 1.3) $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is arbitrarily close to 0; see Example 3.1.

1.3. **Donaldson's toric invariant.** Let now (X, L) be a polarized complex projective manifold. A prominent role in Kähler geometry is played by Mabuchi's K-energy functional $\mathcal{M}$ [65], defined on the space $\mathcal{H}(X, L)$ of all smooth metrics $\|\cdot\|$ on L with positive curvature. Its critical points are the metrics whose curvature form ω define a Kähler metric on X with constant scalar curvature. The precise definition of $\mathcal{M}$ is recalled in Section 4.1. Since the definition of $\mathcal{M}$ only involves its differential, the functional $\mathcal{M}$ is only defined up to addition by a real constant. However, when (X, L) is toric Donaldson [38] exploited the toric structure to define the Mabuchi functional $\mathcal{M}$ as a canonical functional on toric metrics:

$$\mathcal{M}_L := \int_{\partial P} u d\sigma - a \int_P u dx - \int_P \log\det(\nabla^2 u) dx, \quad a := \int_{\partial P} d\sigma / \int_P dx \tag{1.5}$$

where P is the moment polytope in $\mathbb{R}^n$ corresponding to the polarized toric manifold (X, L), whose boundary ∂P comes with a measure $d\sigma$ induced by Lebesgue measure dx on $\mathbb{R}^n$ and the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ and u is the smooth bounded convex function on P corresponding to a toric metric on L under Legendre transformation (see Section 3.1.2). In particular, in the last section of [38] Donaldson introduced an invariant of a polarized toric manifold (X, L), defined as the infimum of the toric Mabuchi functional $\mathcal{M}_L$ defined by formula 1.5. Here we show that Theorem 1.2 implies that when X is a Fano variety and $L = -K_X$ a slight perturbation of Donaldson's invariant is minimal when X is complex projective space, under the conditions on X appearing in Theorem 1.2:

Theorem 1.5. *Let X be a K-semistable toric Fano variety of dimension n, satisfying the conditions in Theorem 1.2. Then the invariant*

$$X \mapsto \inf_{\mathcal{H}(X, -K_X)} \mathcal{M}_{-K_X} - \frac{(-K_X)^n}{n!} \log\left(\frac{(-K_X)^n}{n!}\right)$$

is minimal for $X = \mathbb{P}^n$ (and only then), where the inf is attained at the metric on $-K_{\mathbb{P}^n}$ induced by the Fubini-Study metric.

In the previous theorem the Fano variety X is allowed to be singular. The Mabuchi functional for singular general Fano varieties was introduced in [37, 9] and Donaldson's formula 1.5 was extended to singular toric Fano varieties in [8]. In general, for Fano varieties the Mabuchi functional $\mathcal{M}$ is bounded from below iff X is K-semistable [56] (see the discussion following Theorem 2.4).

1.4. **The arithmetic Mabuchi functional and Odaka's modular height.** For a general polarized manifold (X, L) the infimum of the Mabuchi functional $\mathcal{M}$ is not canonically defined (since $\mathcal{M}$ is only defined up to addition by a constant). But to any given integral model $(\mathcal{X}, \mathcal{L})$ of a polarized complex variety (X, L) one may, as shown by Odaka [77], attach a particular Mabuchi functional $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ which (up to a multiplicative normalization) is given as the following sum of arithmetic intersection numbers:

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\overline{\mathcal{L}}) := \frac{a}{(n+1)!}\overline{\mathcal{L}}^{n+1} - \frac{1}{n!}(-\overline{\mathcal{K}}_{\mathcal{X}}) \cdot \overline{\mathcal{L}}^n, \quad a = -n(K_X \cdot L^{n-1})/L^n \tag{1.6}$$

where, as in the previous section, $\overline{\mathcal{L}}$ denotes the metrized line bundle $(\mathcal{L}, \|\cdot\|)$. In the definition of the second arithmetic intersection number above one also needs to endow $-K_X$ with a metric and one is confronted with two different natural choices: either the metric induced by the volume form $\omega^n/n!$ of the Kähler metric ω defined by the curvature form of $(\mathcal{L}, \|\cdot\|)$ or the *normalized* volume form ω^n/L^n (which has unit total volume). The first choice is the one adopted in [77] and we show that when X is a toric Fano variety and $(\mathcal{X}, \mathcal{L})$ is the canonical integral model of

(X,L) this choice coincides with Donaldson's one (formula 1.5). However, for our purposes the second volume-normalized choice turns out to be the appropriate one. It yields, in particular, the shift by the logarithm of $(-K_X)^n$ appearing in Theorem 1.5:

$$2\mathcal{M}_{(\mathcal{X},-\mathcal{K}_X)} = \mathcal{M}_{-K_X} - \frac{(-K_X)^n}{n!}\log\left(\frac{(-K_X)^n}{n!}\right)$$

(Prop 5.2). The point is that with this choice the following formula holds in the arithmetic setting:

$$\sup \frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}}{(n+1)!} = -\inf_{\mathcal{H}(X,-K_X)} \mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})} \tag{1.7}$$

where the sup ranges over all volume-normalized metrics in $\mathcal{H}(X,-K_X)$ (see Prop 5.3). As a consequence, Conjecture 1.1 is equivalent to the inequality

$$\inf_{\mathcal{H}(X,-K_X)} \mathcal{M}_{(\mathcal{X},-K_{\mathcal{X}})} \geq \inf_{\mathcal{H}(\mathbb{P}^n,-K_{\mathbb{P}^n})} \mathcal{M}_{(\mathbb{P}^n_{\mathbb{Z}},\dots)}. \tag{1.8}$$

Theorem 1.5 thus follows from Theorem 1.2.

1.4.1. *Odaka's modular height.* Let (X_F,L_F) be an $n-$dimensional polarized variety defined over a number field F. In [77] Odaka introduced the following invariant of (X_F,L_F), dubbed the *intrinsic K-modular height of* (X_F,L_F) :

$$h(X_F,L_F) = \inf_{(\mathcal{X},\mathcal{L})} \inf_{\mathcal{H}(X,L)} \mathcal{M}_{(\mathcal{X},\mathcal{L})}, \tag{1.9}$$

where $(\mathcal{X},\mathcal{L})$ is a model of (X_F,L_F) over the rings of integers $\mathcal{O}_{F'}$ of a finite field extension F' of F and $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ now denotes the arithmetic K-energy 1.6, divided by the degree $[F':\mathbb{Q}]$. In contrast to [77], we will employ the volume-normalized metric on $-K_X$ in the definition of $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$, discussed in the previous section. As shown in 1.6, for a polarized abelian variety (X_F,L_F), Odaka's modular height $h(X_F,L_F)$ essentially coincides with Faltings' stable modular height of (X_K,L_K) [41] (see Section 6.4). Furthermore, as explained in [77], $h(X_F,L_F)$ can be viewed as a "large rank limit" of Bost's and Zhang's intrinsic heights appearing in [17, 18, 101], where the role of K-semistability is played by Chow semistability (see formula 6.7). We propose the following

Conjecture 1.6. *Let $X_{\mathbb{Q}}$ be a Fano variety defined over $\mathbb{Q}$. Then Odaka's modular invariant $h(X_{\mathbb{Q}},-K_{X_{\mathbb{Q}}})$, normalized as above, is minimal when $X_{\mathbb{Q}}=\mathbb{P}^n_{\mathbb{Q}}$.*

According to a conjecture of Odaka [78] any globally K-semistable integral model $(\mathcal{X},-\mathcal{K}_{\mathcal{X}})$ of $(X,-K_X)$ minimizes $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ over all models $(\mathcal{X},\mathcal{L})$ (the function field analog of this minimization property is established in [16]; see also [98, Remark 7.9]). Global K-semistability means that all the fibers of $\mathcal{X}\to$ Spec $\mathcal{O}_F$ are K-semistable. In other words, in addition to the K-semistability of the generic fiber X_F this means that the variety $X_{\mathbb{F}_p}$ over the finite field $\mathbb{F}_p$, corresponding to the integral model $\mathcal{X}$, is K-semistable for any prime ideal p. For example, as pointed out to us by Odaka the canonical model $\mathcal{X}$ of a K-semistable toric Fano variety $X_{\mathbb{Q}}$ appearing in Theorem 1.2 is globally K-semistable. Thus if Odaka's minimization conjecture holds, then Theorem 1.2 implies Conjecture 1.6 for any toric Fano variety $X_{\mathbb{Q}}$ satisfying the conditions in Theorem 1.2.[1] Anyhow, the positivity statement in Theorem 1.3 implies that the modular invariant $h(X_{\mathbb{Q}},-K_{X_{\mathbb{Q}}})$ is negative for any K-semistable toric Fano variety $X_{\mathbb{Q}}$.

[1]During the revision of the first preprint version of the present paper Odaka's minimization conjecture was settled in [49] under slightly stronger assumptions than global K-semistability.

1.5. **Organization.** In Section 2 we start by recalling the complex-geometric and arithmetic setup before proving Theorem 2.4, relating upper bounds on the height of Fano varieties to K-semistability. The proof leverages an arithmetic analog of the Ding functional. In Section 3 we specialize to the toric situation and prove the sharp height inequality in Theorem 1.2, stated in the introduction and the height bounds for Kähler-Einstein metrics in Theorem 1.3. We also show that Conjecture 1.1 is compatible with taking products. We then go on, in Section 4, to deduce Theorem 1.5 concerning the sharp lower bound on Donaldson's toric Mabuchi functional. In Section 5 Donaldson's functional is related to Odaka's arithmetic Mabuchi functional, which, in turn is related to the arithmetic Ding functional. In the last section we make a comparison with the function field case, formulate a generalized version of Conjecture 1.1 and compare with previous work of Bost and Zhang, Odaka and Faltings.

We have made an effort to make the paper readable for the reader with a background in arithmetic geometry, as well as for the complex geometers, by including most of the background material needed for the proofs of the main results.

1.6. **Acknowledgements.** We are grateful to Bo Berndtsson, Dennis Eriksson, Gerard Freixas i Montplet, Benjamin Nill, Yuji Odaka, Per Salberger, Chenyang Xu and Ziquan Zhuang for illuminating discussions/comments and, in particular, to Alexander Kasprzyk for updating the database [72]. This work was supported by grants from the Knut and Alice Wallenberg foundation, the Göran Gustafsson foundation and the Swedish Research Council.

2. Heights, arithmetic volumes and K-stability of Fano varieties

In this section we show, in particular, that the height of a polarized integral model $(\mathcal{X},\mathcal{L})$ of a Fano manifold $(X,-K_X)$ is bounded from above - as the metric on $\mathcal{L}$ ranges over all volume-normalized metrics with positive curvature current - if and only if $(X,-K_X)$ is K-semistable (Theorem 2.4). See also [77] for further connections between K-stability of polarized varieties (X,L) and arithmetic geometry. The main new feature here, compared to [77], is that we leverage an arithmetic version of the Ding functional in Kähler geometry, while [77] considers an arithmetic version of the Mabuchi functional (the two functionals are compared in Section 5).

2.1. **Complex geometric setup.** Throughout the paper X will denote a compact connected complex normal variety, assumed to be $\mathbb{Q}-$Gorenstein. This means that the canonical divisor K_X on X is defined as a $\mathbb{Q}-$line bundle: there exists some positive integer m and a line bundle on X whose restriction to the regular locus X_{reg} of X coincides with the m:th tensor power of $K_{X_{\mathrm{reg}}}$, i.e. the top exterior power of the cotangent bundle of X_{reg}. We will use additive notation for tensor powers of line bundles.

2.1.1. *Metrics on line bundles.* Let (X,L) be a polarized complex projective variety i.e. a complex normal variety X endowed with an ample line bundle L. We will use additive notation for metrics on L. This means that we identify a continuous Hermitian metric $\|\cdot\|$ on L with a collection of continuous local functions ϕ_U associated to a given covering of X by open subsets U and trivializing holomorphic sections e_U of $L\to U$:

$$\phi_U := -\log(\|e_U\|^2), \tag{2.1}$$

which defines a function on U. Of course, the functions ϕ_U on U do not glue to define a global function on X, but the current

$$dd^c\phi_U := \frac{i}{2\pi}\partial\bar{\partial}\phi_U$$

is globally well-defined and coincides with the normalized curvature current of $\|\cdot\|$ (the normalization ensures that the corresponding cohomology class represents the first Chern class $c_1(L)$ of L in the integral lattice of $H^2(X,\mathbb{R})$). Accordingly, as is customary, we will symbolically denote by ϕ a given continuous Hermitian metric on L and by $dd^c\phi$ its curvature current. The space of all continuous metrics ϕ on L will be denoted by $\mathcal{C}^0(L)$. We will denote by $\mathcal{C}^0(L)\cap \mathrm{PSH}(L)$ the space of all continuous metrics on L whose curvature current is positive, $dd^c\phi \geq 0$ (which means that ϕ_U is plurisubharmonic, or psh, for short). Then the exterior powers of $dd^c\phi$ are defined using the local pluripotential theory of Bedford-Taylor [11]. The *volume* of an ample line bundle L may be defined by

$$\mathrm{vol}(L) := \lim_{k\to\infty} k^{-n} \dim H^0(X, L^{\otimes k}) = \frac{1}{n!}L^n = \frac{1}{n!}\int_X (dd^c\phi)^n \tag{2.2}$$

using in the second equality the Hilbert-Samuel theorem and where ϕ denotes any element in $\mathcal{C}^0(L)\cap \mathrm{PSH}(L)$.

More generally, metrics ϕ are defined for a $\mathbb{Q}-$line bundle L : if mL is a bona fide line bundle, for $m\in\mathbb{Z}_+$, then $m\phi$ is a bona fide metric on mL.

Remark 2.1. The normalization of ϕ_U used here coincides with the one in [6, 8], but it is twice the one employed in [11].

2.1.2. *Metrics on $-K_X$ vs volume forms on X.* First consider the case when X is smooth. Then any smooth metric $\|\cdot\|$ on $-K_X$ corresponds to a volume form on X, defined as follows. Given local holomorphic coordinates z on $U\subset X$ denote by e_U the corresponding trivialization of $-K_X$, i.e. $e_U = \partial/\partial z_1\wedge\cdots\wedge\partial/\partial z_n$. The metric on $-K_X$ induces, as in the previous section, a function ϕ_U on U and the volume form in question is locally defined by

$$e^{-\phi_U}(\frac{i}{2})^{n^2} dz\wedge d\bar{z}, \quad dz := dz_1\wedge\cdots\wedge dz_n, \tag{2.3}$$

on U, which glues to define a global volume form on X. In other words, $e^{-\phi_U}$ is the density of the volume form with respect to the local Euclidean volume form. Accordingly, we will simply denote the volume form in question by $e^{-\phi}$, abusing notation slightly. When X is singular any continuous metric ϕ on $-K_X$ induces a measure on X, symbolically denoted by $e^{-\phi}$, defined as before on the regular locus X_{reg} of X and then extended by zero to all of X. We will say that a measure dV on X is a continuous volume form it it corresponds to a continuous metric on $-K_X$. A Fano variety has log terminal singularities iff it admits a continuous volume form dV with finite total volume [9, Section 3.1].

2.1.3. *K-semistability.* We briefly recall the notion of K-semistability (see [38, 82, 97, 75] for more background). A polarized complex projective variety (X,L) is said to be *K-semistable* if the Donaldson-Futaki invariant $\mathrm{DF}(\mathscr{X},\mathscr{L})$ of any test configuration $(\mathscr{X},\mathscr{L})$ for (X,L) is non-negative. A test configuration $(\mathscr{X},\mathscr{L})$ is defined as a $\mathbb{C}^*-$equivariant normal model for (X,L) over the complex affine line $\mathbb{C}$. More precisely, $\mathscr{X}$ is a normal complex variety endowed with a $\mathbb{C}^*-$action ρ, a $\mathbb{C}^*-$equivariant holomorphic projection π to $\mathbb{C}$ and a relatively ample $\mathbb{C}^*-$equivariant $\mathbb{Q}-$line bundle $\mathscr{L}$ (endowed with a lift of ρ) :

$$\pi : \mathscr{X}\to\mathbb{C}, \quad \mathscr{L}\to\mathscr{X}, \quad \rho:\ \mathscr{X}\times\mathbb{C}^*\to\mathscr{X} \tag{2.4}$$

such that the fiber of $\mathscr{X}$ over $1 \in \mathbb{C}$ is equal to (X, L). Its *Donaldson-Futaki invariant* $\mathrm{DF}(\mathscr{X}, \mathscr{L}) \in \mathbb{R}$ may be defined as a normalized limit, as $k \to \infty$, of Chow weights of a sequence of one-parameter subgroups of $GL\left(H^0(X, kL)\right)$ induced by $(\mathscr{X}, \mathscr{L})$ (in the sense of Geometric Invariant Theory). As a consequence, (X, L) is K-semistable if, for example, (X, kL) is Chow semi-stable, for k sufficiently large [82]. However, for the purpose of the present paper it will be more convenient to employ the intersection-theoretic formula for $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ established in [97, 75]:

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{a}{(n+1)!}\overline{\mathscr{L}}^{n+1} + \frac{1}{n!}\mathscr{K}_{\overline{\mathscr{X}}/\mathbb{P}^1} \cdot \overline{\mathscr{L}}^n, \quad a = -n(K_X \cdot L^{n-1})/L^n$$

where $\overline{\mathscr{L}}$ denotes the $\mathbb{C}^*-$equivariant extension of $\mathscr{L}$ to the $\mathbb{C}^*-$equivariant compactification $\overline{\mathscr{X}}$ of $\mathscr{X}$ over $\mathbb{P}^1$ and $\mathscr{K}_{\overline{\mathscr{X}}/\mathbb{P}^1}$ denotes the relative canonical divisor.

Remark 2.2. Usually the definition of $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ involves a factor of $1/L^n$, but the present definition will be more convenient here (since the factor L^n is positive it does not alter the definition of K-stability). It is made so that $\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \overline{\mathscr{L}}^{n+1}$ when $\mathscr{L} = -\mathscr{K}_{\overline{\mathscr{X}}/\mathbb{P}^1}$.

2.2. **Arithmetic setup.** We will say that $\mathcal{X}$ is an *arithmetic variety,* if $\mathcal{X}$ is a a projective flat scheme $\mathcal{X} \to \mathrm{Spec}\ \mathbb{Z}$ of relative dimension n and $\mathcal{X}$ is reduced and satisfies Serre's conditions S_2 (this is, for example, the case if $\mathcal{X}$ is normal). A *polarized arithmetic variety* $(\mathcal{X}, \mathcal{L})$ is an arithmetic variety endowed with a relatively ample $\mathbb{Q}-$line bundle $\mathcal{L}$. We will denote by X the $n-$dimensional complex projective algebraic variety consisting of the complex points of $\mathcal{X}$, which is assumed to be normal and by L the ample line bundle over X induced by $\mathcal{L}$. This means that L is the restriction to the complexification X of the generic fiber $X_{\mathbb{Q}}$ of the structure morphism $\mathcal{X} \to \mathrm{Spec}\ \mathbb{Z}$. We will then say that $(\mathcal{X}, \mathcal{L})$ is a *model* for (X, L) *over* $\mathbb{Z}$ (or an *integral model* for (X, L))). For any positive integer k we may identify the free $\mathbb{Z}-$module $H^0(\mathcal{X}, k\mathcal{L})$ with a lattice in $H^0(X, kL)$:

$$H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{C} = H^0(X, kL).$$

By definition a *metrized line bundle* $\overline{\mathcal{L}}$ is a line bundle $\mathcal{L} \to \mathcal{X}$ such that the corresponding line bundle $L \to X$ is endowed with a metric $\|\cdot\|$. We will use the additive notation ϕ for metrics $\|\cdot\|$ on L discussed in the previous section:

$$\overline{\mathcal{L}} := (\mathcal{L}, \phi).$$

2.2.1. *Arithmetic Fano varieties.* We will say that the relative canonical line bundle of an arithmetic variety $\mathcal{X}$ is defined as a $\mathbb{Q}-$line bundle, denoted by $\mathcal{K}$, if there exists a positive integer m such that the mth reflexive power $\omega_{X/\mathrm{Spec}\mathbb{Z}}^{[m]}$ of the dualizing sheaf $\omega_{X/\mathrm{Spec}\mathbb{Z}}$ of $\mathcal{X}$ is locally free. Then the line bundle $m\mathcal{K}$ over $\mathcal{X}$ may be identified with $\omega_{X/\mathrm{Spec}\mathbb{Z}}^{[m]}$ (see [55, Section 1.1] for a more general setup of canonical line bundles attached to schemes over regular excellent rings). An arithmetic variety $\mathcal{X} \to \mathrm{Spec}\ \mathbb{Z}$ will be called an *arithmetic Fano variety* if

- the canonical line bundle $\mathcal{K}$ of $\mathcal{X}$ is well-defined as a $\mathbb{Q}-$line bundle and its dual $-\mathcal{K}$ is relatively ample
- the complexification X of $\mathcal{X}$ is normal and thus defines a complex Fano variety (i.e. $-K_X$ is ample)

Example 2.3. If $\mathcal{X}$ is, locally, a complete intersection, then $\mathcal{K}$ is defined as a line bundle (i.e. $m = 1$) [55, Section 1.1]. In particular, if $\mathcal{X}$ is the subscheme of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by an irreducible homogeneous polynomial of degree d with integer coefficents, then $\mathcal{K}$ is well-defined as a line bundle and $\mathcal{X}$ is an arithmetic Fano variety iff $d \leq n + 1$.

2.2.2. *The $\chi-$arithmetic volume, heights and arithmetic intersection numbers.* In the arithmetic setup there are different analogs of the volume $\mathrm{vol}(L)$ of an ample line bundle L. Here we shall focus on the one defined by the following asymptotic arithmetic Euler characteristic originating in [42] (called the $\chi-$*arithmetic volume* [22, 23] and the *sectional capacity* in [83]):

$$\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right) := \lim_{k\to\infty} k^{-(n+1)} \log \mathrm{Vol}\left\{ s_k \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} : \sup_X \|s_k\|_{\phi} \leq 1 \right\}, \tag{2.5}$$

where $H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R}$ may be identified with the subspace of real sections in $H^0(X, kL)$. If the metric on L has positive curvature current, then, by the arithmetic Hilbert-Samuel theorem [53, 99],

$$\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right) = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}, \tag{2.6}$$

where $\overline{\mathcal{L}}^{n+1}$ denotes the top arithmetic intersection number in the sense of Gillet-Soulé [52], which, defines the *height* of $\mathcal{X}$ with respect to $\overline{\mathcal{L}}$ [43, 19]. For the purpose of the present paper formula 2.5 may be taken as the definition of $\overline{\mathcal{L}}^{n+1}$ (arithmetic intersections between general $n+1$ metrized line bundles could then be defined by polarization). More generally, $\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)$ is naturally defined for $\mathbb{Q}-$line bundles, since it is homogeneous with respect to tensor products of $\overline{\mathcal{L}}$:

$$\widehat{\mathrm{vol}}_{\chi}\left(m\overline{\mathcal{L}}\right) = m^{n+1}\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right), \text{ if } m \in \mathbb{Z}_+ \tag{2.7}$$

Moreover, $\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)$ is additively equivariant with respect to scalings of the metric:

$$\widehat{\mathrm{vol}}_{\chi}\left(\mathcal{L}, \phi + \lambda\right) = \widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right) + \frac{\lambda}{2}\mathrm{vol}(L), \text{ if } \lambda \in \mathbb{R}, \tag{2.8}$$

as follows directly from the definition.

2.3. **Upper bounds on the $\chi-$arithmetic volume vs K-semistability of Fano varieties.** We are now ready to prove the following theorem, relating upper bounds on the $\chi-$arithmetic volume of a metrized integral model of $(X, -K_X)$ to K-semistability:

Theorem 2.4. *Let $(\mathcal{X}, \mathcal{L})$ be a polarized arithmetic variety such that X is a Fano variety and $L = -K_X$. Then the following is equivalent:*

1. *$(X, -K_X)$ is K-semistable*
2. *The supremum of $\widehat{\mathrm{vol}}_{\chi}(\mathcal{L}, \phi)$ over all continuous volume-normalized metrics ϕ on $-K_X$ is finite.*
3. *The supremum of $\widehat{\mathrm{vol}}_{\chi}(\mathcal{L}, \phi)$ over all continuous volume-normalized metrics ϕ on $-K_X$, which are invariant under complex conjugation, is finite.*

Recall that on any complex projective variety X which is defined over $\mathbb{R}$ there is a globally defined complex conjugation map (whose orbits on X correspond to the maximal ideals of the scheme $X_{\mathbb{R}}$) and in Arakelov geometry it is often assumed that the metrics are invariant under complex conjugation [86].

Before embarking on the proof we recall the definition of the *Ding functional* on $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$, introduced in [36], which depends on the choice of a reference metric ψ_0 in $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$:

$$\mathcal{D}_{\psi_0}(\psi) := -\frac{1}{\mathrm{vol}(-K_X)}\mathcal{E}_{\psi_0}(\psi) - \log \int_X e^{-\psi}, \tag{2.9}$$

where the functional $\mathcal{E}_{\psi_0}$ is a primitive of $(dd^c\psi)^n/n!$ (see formula 2.12). More generally, as shown in [9] $\mathcal{D}_{\psi_0}(\psi)$ can be extended to the space $\mathcal{E}^1(-K_X)$ of all metrics in $\mathrm{PSH}(-K_X)$ with finite energy and a finite energy metric ψ minimizes $\mathcal{D}_{\psi_0}(\psi)$ iff ψ is a Kähler-Einstein metric, i.e. $dd^c\psi$ defines a Kähler metric on the regular locus of X with constant positive Ricci curvature. When ψ is volume-normalized this equivalently means that

$$\frac{(dd^c\psi)^n}{\mathrm{vol}(-K_X)n!} = e^{-\psi}$$

on the regular locus of X. The identity 2.6 was extended to finite energy metrics in [12]. But for our purposes it will be enough to work with continuous metrics.

Remark 2.5. In general, any Kähler-Einstein metric ψ in $\mathcal{E}^1(-K_X)$ is locally bounded [9]. In the toric case this implies that ψ is, in fact, continuous [32, Prop 4.1].

By introducing an arithmetic version of the Ding functional we show that item 2 in the previous theorem is equivalent to the Ding functional $\mathcal{D}_{\psi_0}$ being bounded from below on $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ (which is equivalent to lower boundedness of the Mabuchi functional; see 4.7). By [56] this is equivalent to K-semistability when X is non-singular. In the proof of Theorem 2.4 we explain how to extend this result to general Fano varieties, leveraging the very recent solution of the Yau-Tian-Donaldson conjecture for singular Fano varieties [57, 63]. The equivalence with item 3 leverages the recent result [102].

Remark 2.6. The proof will also reveal that $(X, -K_X)$ is K-polystable iff the supremum in item 2 above is attained at some locally bounded metric ψ in $PSH(-K_X)$. In general, any such a maximizer is a Kähler-Einstein metric.

2.3.1. *Proof of Theorem 2.4.* We start with two lemmas. First, to a given continuous metric ϕ on L we associate, following [11], a continuous psh metric ψ on L defined as the following point-wise envelope:

$$P\phi := \sup\{\psi : \ \psi\,\mathrm{psh},\ \psi \leq \phi\}. \tag{2.10}$$

Remark 2.7. More generally, when L is big the envelope above has to be replaced by its upper semi-continuous regularization in order to obtain a psh metric. However, when L is an ample line bundle over a normal variety X, as we assume here, the envelope $P\phi$ is already continuous (see [21, Lemma 7.9]).

Lemma 2.8. *Assume that $\mathcal{L}$ is relatively ample and let ϕ be a continuous metric on L with positive curvature current. Then the arithmetic $\chi-$volume may be expressed as the following top arithmetic intersection number:*

$$\widehat{vol}_\chi(\mathcal{L}, \phi) = \frac{(\mathcal{L}, P\phi)^{n+1}}{(n+1)!}.$$

Proof. When ϕ is psh the lemma follows directly from [99, Thm 1.4] (the latter proof reduces to the original arithmetic Hilbert-Samuel theorem in [53], where X is assumed non-singular, using a perturbation argument on a resolution of X). In fact, the result [99, Thm 1.4] applies more generally when $\mathcal{L}$ is merely assumed to be relatively nef over the closed points of Spec $\mathbb{Z}$. Next, the general case follows from the case when ϕ is psh (applied to $P\phi$) by the following simple observation:

$$\sup_X \|s\|_\phi = \sup_X \|s\|_{P\phi}, \quad \text{if } s \in H^0(X, kL),$$

as follows directly from the definition 2.10 of $P\phi$ (see [11, Prop 1.8]). □

In order to state the next lemma consider the following functional on $\mathcal{C}^0(L) \cap \mathrm{PSH}\,(L)$, defined with respect to a given reference $\psi_0 \in \mathcal{C}^0(L) \cap \mathrm{PSH}(L)$:

$$\mathcal{E}_{\psi_0}(\psi) := \frac{1}{(n+1)!} \int_X (\psi - \psi_0) \sum_{j=0}^{n} (dd^c\psi)^j \wedge (dd^c\psi_0)^{n-j} \tag{2.11}$$

Alternatively, the functional $\mathcal{E}_{\psi_0}$ may be characterized as the primitive of the one-form on $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ defined by the measure $(dd^c\psi)^n/n!$:

$$d\mathcal{E}_{\psi_0}(\psi) = \frac{1}{n!}(dd^c\psi)^n, \quad \mathcal{E}_{\psi_0}(\psi_0) = 0. \tag{2.12}$$

It follows directly from the definition of $\mathcal{E}_{\psi_0}(\psi)$ and the classical Hilbert-Samuel formula 2.2 that

$$\mathcal{E}_{\psi_0}(\psi + c) = \mathcal{E}_{\psi_0}(\psi) + c\mathrm{vol}(L), \ \ \forall c \in \mathbb{R} \ . \tag{2.13}$$

The following lemma is an arithmetic refinement of the previous formula:

Lemma 2.9. *(change of metrics formula). For any two continuous metrics on L, which are invariant under complex conjugation,*

$$\widehat{\mathrm{vol}}_{\mathcal{X}}\,(\mathcal{L}, \phi_1) - \widehat{\mathrm{vol}}_{\mathcal{X}}\,(\mathcal{L}, \phi_2) = \frac{1}{2}\left(\mathcal{E}_{\psi_0}(P\phi_1) - \mathcal{E}_{\psi_0}(P\phi_2)\right) \tag{2.14}$$

Proof. When ϕ_i are psh this is well-known and follows from basic properties of arithmetic intersection numbers; see formula 5.1 or [77, Prop 2.2]). Alternatively, the result follows from the previous lemma combined with [11, Thm A]. In order to check that the multiplicative normalizations adopted here are compatible note that the scaling relations 2.8 and 2.13 are indeed compatible. □

2.3.2. *Conclusion of the proof of Theorem 2.4.* Consider the following functional on the space $\mathcal{C}^0(-K_X)$ of continuous metrics on $-K_X$

$$\mathcal{D}_{\mathbb{Z}}(\phi) := -2\frac{\widehat{\mathrm{vol}}_{\mathcal{X}}\,(\mathcal{L}, \phi)}{\mathrm{vol}(-K_X)} - \log \int_X e^{-\phi}. \tag{2.15}$$

Since this functional is invariant under scalings of the metric, $\phi \mapsto \phi + c$, the finiteness statement in the second point of the proposition amounts to showing that the infimum of $\mathcal{D}_{\mathbb{Z}}(\phi)$ over $\mathcal{C}^0(-K_X)$ is finite. Now fix a continuous psh metric ψ_0 on $-K_X$ and consider the following extension of the Ding functional 2.9 to all of $\mathcal{C}^0(-K_X)$:

$$\mathcal{D}_{\psi_0}(\phi) := -\frac{1}{\mathrm{vol}(-K_X)}\mathcal{E}_{\psi_0}(P\phi) - \log \int_X e^{-\phi}. \tag{2.16}$$

Combining the previous two lemmas reveals that

$$\mathcal{D}_{\mathbb{Z}}(\phi) = \mathcal{D}_{\psi_0}(\phi) + C_0, \ \ C_0 := -\frac{2\,(\mathcal{L}, \psi_0)^{n+1}}{\mathrm{vol}(-K_X)(n+1)!} \tag{2.17}$$

Next, observe that

$$\inf_{\mathcal{C}^0(-K_X)} \mathcal{D}_{\psi_0} = \inf_{\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)} \mathcal{D}_{\psi_0} \tag{2.18}$$

Indeed, this follows directly from the fact that the operator $\phi \mapsto P\phi$ from $\mathcal{C}^0(L)$ to $\mathcal{C}^0(L) \Cap \mathrm{PSH}(L)$ is increasing and satisfies $P^2 = P$.

"3" $\implies$ "1". Let us first recall how Item 2 implies Item 1. First Item 2 implies, thanks to the identities 2.17 and 2.18, that the infimum of $\mathcal{D}_{\psi_0}$ over $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ is finite. Thus it follows from results in [6] that $(X, -K_X)$ is K-semistable. Let us next show how to refine the proof in [6] to show the stronger statement "3" $\implies$ "1". More generally, we will show that when X is defined over the real field $\mathbb{R}$ X is K-semistable if the infimum of $\mathcal{D}_{\psi_0}$ over the space $\overline{\mathcal{C}^0(-K_X)} \Cap \mathrm{PSH}(-K_X)$ is finite, where $\overline{\mathcal{C}^0(L)}$ denotes the subspace of $\mathcal{C}^0(L)$ consisting of metrics which are invariant under complex conjugation. To this end let us first summarize the main steps in the proof in [6]. First, a test configuration $(\mathscr{X}, \mathscr{L})$ for $(X, -K_X)$ and a given metric ϕ for $-K_X$ in $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ determines a ray ϕ_t in $\mathrm{PSH}(-K_X)$ emanating from ϕ parametrized by $t \in [0, \infty[$ (i.e. $\phi_0 = \phi$). Using the notation in formula 2.4 the ray ϕ_t is defined by

$$\phi_{-\log|\tau|} = \rho(\tau)^*(\Phi_{|\mathscr{X}_\tau}), \quad \tau \in \mathbb{C}^*$$

where Φ is the S^1-invariant metric on the restriction of $\mathcal{L}$ to the inverse image $\pi^{-1}(\mathbb{D})$ in $\mathcal{X}$ of the unit-disc $\mathbb{D} \subset \mathbb{C}$ defined by

$$\Phi := \sup\left\{\Psi : \ \Psi_{|\pi^{-1}(\partial\mathbb{D})} = \phi, \quad \Psi \in \mathcal{C}^0(\mathcal{L}) \Cap \mathrm{PSH}(\mathcal{L}_{|\pi^{-1}(\mathbb{D})})\right\}, \tag{2.19}$$

where we have used the $\mathbb{C}^*-$action ρ to identify X with X_τ for any τ in the unit-circle $\partial\mathbb{D}$. By [6, Thm 1.3]

$$\mathrm{vol}\ (-K_X)^{-1}\mathrm{DF}\ (\mathscr{X}, \mathscr{L}) \geq \lim_{t\to\infty}\left(t^{-1}\mathcal{D}_{\phi_0}(\phi_t)\right).$$

When $\mathcal{D}_{\phi_0}(\phi_t)$ is bounded from below this means that $\mathrm{DF}(\mathscr{X}, \mathscr{L}) \geq 0$, showing that X is K-semistable. Now assume that X is defined over the real field $\mathbb{R}$. Then it follows from [102, Thm 1.1] that in order to check K-semistability of $(X, -K_X)$ it is enough to consider test configurations $(\mathscr{X}, \mathscr{L})$ defined over $\mathbb{R}$. Thus, we just have to verify that for such test configurations, if the given metric ϕ is taken to be in $\overline{\mathcal{C}^0(-K_X)} \Cap \mathrm{PSH}(-K_X)$, then the ray ϕ_t remains in $\overline{\mathcal{C}^0(-K_X)} \Cap \mathrm{PSH}(-K_X)$, for all $t > 0$. Since $(\mathscr{X}, \mathscr{L})$ is defined over $\mathbb{R}$ there is a complex conjugation map F from $\mathscr{X}$ to $\mathscr{X}$ (that lifts to $\mathscr{L}$) and thus it is enough to show that $F^*\phi = \phi$ implies that $F^*\Phi = \Phi$. But this follows from the definition 2.19 of Φ only using that F^* preserves the psh property of a metric (as follows from a direct local calculation that reduces to the fact that the Laplacian $i\partial_z\partial_{\bar{z}}$ in $\mathbb{C}$ is invariant under $z \mapsto \bar{z}$).

"1" $\implies$ "2". First recall that any K-semistable normal Fano variety (i.e. such that $(X, -K_X)$ is K-semistable) has log terminal singularities [74, Thm 1.3]. In the case that X is non-singular it was shown in [56] that if X is K-semistable, then the infimum of the Ding functional $\mathcal{D}_{\psi_0}$ over $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ is finite. Thus, by formula 2.18, so is the infimum of $\mathcal{D}_{\psi_0}$ over $\mathcal{C}^0(-K_X)$. The proof in [56] relied, in particular, on the resolution of the Yau-Tian-Donaldson conjecture in [28] for Fano manifolds. But thanks to the recent resolution of the Yau-Tian-Donaldson conjecture for singular Fano varieties the proof in [56] can be extended to singular Fano varieties, mutatis mutandis. We briefly summarize the argument, using Deligne pairings as in [6] (rather than the Bott-Chern classes used in [56]). The starting point is the result [60, Thm 1.3], saying that if X is K-semistable then there exists a test configuration $(\mathscr{X}, \mathscr{L})$ for $(X, -K_X)$ whose central fiber X_0 is given by a K-polystable Fano variety. More precisely, the test configuration is *special* in the sense that $\mathscr{L}$ is the relative anti-canonical line bundle. Since the central fiber X_0 of $\mathscr{X}$ is K-polystable it admits, by the solution of the Yau-Tian-Donaldson conjecture for singular Fano varieties [63] (building on [57]) a Kähler-Einstein metric ϕ_{KE}. It thus follows from [9, Thm 4.8] that the Ding functional is bounded from below on

$\mathcal{C}^0(-K_{X_0}) \Cap \mathrm{PSH}(-K_{X_0})$. More precisely, its infimum is attained at the Kähler-Einstein metric ϕ_{KE} :

$$\inf_{\mathcal{C}^0(-K_{X_0})\Cap\mathrm{PSH}(-K_{X_0})} \mathcal{D} = \mathcal{D}(\phi_{KE}) > -\infty. \tag{2.20}$$

Now, given a metric ϕ in $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ let Φ be the corresponding metric on $\mathscr{L} \to \pi^{-1}(\mathbb{D})$ defined by formula 2.19. It induces a metric on the $(n+1)-$fold Deligne pairing $\langle \mathscr{L}, \mathscr{L}, ..., \mathscr{L} \rangle \to \mathbb{D}$ that we denote by $\langle \Phi \rangle$ (see [6, Section 2.3]). Consider the corresponding twisted metric on $-\langle \mathscr{L}, \mathscr{L}, ..., \mathscr{L} \rangle \to \mathbb{D}$ defined by

$$-\langle \Phi \rangle - \log \int_{X_\tau} e^{-\Phi_{|X_\tau}},$$

dubbed the *Ding metric* in [6]. Fixing a trivialization $S(\tau)$ of $\langle \mathscr{L}, \mathscr{L}, ..., \mathscr{L} \rangle \to \mathbb{D}$ we may identify this metric with a function $\psi(\tau)$ on $\mathbb{D}$:

$$\psi(\tau) := \log \left(\|S(\tau)\|^2_{\langle \Phi \rangle} \right) - \log \int_{X_\tau} e^{-\Phi_{|X_\tau}},$$

For a fixed τ this metric coincides with the Ding functional $\mathcal{D}(\phi_\tau)$ up to an additive constant depending on τ (by the "change of metrics formula" for Deligne pairing; see [6, Section 2.3]). In particular, there exists $a \in \mathbb{R}$ such that

$$\psi(1) := \mathcal{D}_{\psi_0}(\phi) + a, \quad \psi(0) \geq b := \log \left(\|S(0)\|^2_{\langle \phi_{KE} \rangle} \right) - \log \int_{X_0} e^{-\phi_{KE}}, \tag{2.21}$$

using 2.20 in the inequality. As shown in [6, Prop 3.5] $\psi(\tau)$ is subharmonic on $\mathbb{D}$ and the first term $\langle \Phi \rangle$ is continuous on $\mathbb{D}$ (as follows from [71, Thm A]; see the proof of [6, Prop 3.6]). Moreover, the second term is also continuous on $\mathbb{D}$, as shown when X is non-singular in [56, Lemma 1.9] and in general in [58, Lemma 7.1]. As a consequence,

$$\psi(0) \leq \int_{\partial D} \psi d\theta = \psi(1),$$

using that $\psi(\tau)$ is S^1-invariant in the last equality. Finally, invoking formula 2.21 shows that $\mathcal{D}_{\psi_0}(\phi)$ is uniformly bounded from below, as desired.

2.4. **Compatibility of Conjecture 1.1 with taking products.** The previous theorem shows, in particular, that the K-semistability assumption in Conjecture 1.1 is necessary. We next show that the conjecture is compatible with taking products:

Proposition 2.10. *Let $\mathcal{X}_1, ..., \mathcal{X}_m$ be arithmetic Fano varieties which are K-semistable over $\mathbb{C}$. Assume that the inequality in Conjecture 1.1 holds for all $\mathcal{X}_i$ (for any volume-normalized metrics on $-K_{X_i}$ with positive curvature current). Then the inequality holds for $\mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ with* strict *inequality (for any volume-metric normalized metric on $-K_{X_1 \times \cdots \times X_m}$ with positive curvature current). More precisely,*

$$\widehat{vol}_\chi(\overline{-\mathcal{K}_{\mathcal{X}_1 \times \cdots \times \mathcal{X}_m}}) < \widehat{vol}_\chi(\overline{-\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}).$$

Proof. By a simple induction argument it is enough to consider the case when $m = 2$. First note that, in general, given two polarized metrized arithmetic varieties $(\mathcal{X}_i, \overline{\mathcal{L}_i})$ of relative dimension n_i

$$\frac{\widehat{\mathrm{vol}}_\chi \left(\rho_1^* \overline{\mathcal{L}_1} \otimes \rho_2^* \overline{\mathcal{L}_2} \right)}{\mathrm{vol} \left(\rho_1^* \overline{\mathcal{L}_1} \otimes \rho_2^* \overline{\mathcal{L}_2} \right)} = \frac{\widehat{\mathrm{vol}}_\chi \left(\overline{\mathcal{L}_1} \right)}{\mathrm{vol} \left(\overline{\mathcal{L}_1} \right)} + \frac{\widehat{\mathrm{vol}}_\chi \left(\overline{\mathcal{L}_2} \right)}{\mathrm{vol} \left(\overline{\mathcal{L}_2} \right)} \tag{2.22}$$

where ρ_1 and ρ_2 denote the natural morphisms from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively (as follows readily from formula 2.5).

Assume now that the inequality in Conjecture 1.1 holds for $\overline{-\mathcal{K}_{\mathcal{X}_1}}$ and $\overline{-\mathcal{K}_{\mathcal{X}_2}}$. Endow $-K_{X_1 \times X_2}$ with the induced product metric (which is volume-normalized, since the metrics on $-K_{X_i}$ are assumed to be volume-normalized). The identity 2.22 yields

$$\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}}\right) = \widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}_1}}\right)\mathrm{vol}\left(-K_{X_2}\right) + \widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}_1}}\right)\mathrm{vol}\left(-K_{X_1}\right).$$

Accordingly, by assumption,

$$\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}}\right) \leq \widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}}\right)\mathrm{vol}\left(-K_{X_2}\right) + \widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}}\right)\mathrm{vol}\left(-K_{X_1}\right),$$

where the projective spaces have been induced by the volume-normalized Fubini-Study metric and we have used that $\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\right)$ is positive for any n (as shown in 3.6). Hence, applying Fujita's inequality 1.1, yields

$$\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}}\right) \leq \widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}}\right)\mathrm{vol}\left(-K_{\mathbb{P}^{n_1}_{\mathbb{C}}}\right) + \widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}}\right)\mathrm{vol}\left(-K_{\mathbb{P}^{n_2}_{\mathbb{C}}}\right).$$

But, the rhs above equals $\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^{n_1}_{\mathbb{Z}}\times\mathbb{P}^{n_2}_{\mathbb{Z}}}}\right)$ (by the identity 2.22), which is strictly smaller then $\widehat{\mathrm{vol}}_\chi\left(\overline{-\mathcal{K}_{\mathbb{P}^{n_1+n_2}_{\mathbb{Z}}}}\right)$, by the toric case, considered in Section 3.2.2.

All that remains is thus to show that the sup of $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}})$ over all continuous volume-normalized metrics coincides with the sup restricted to the ones which have positive curvature current and are product metrics. First, as shown in the proof of Theorem 2.4 we may restrict to those with positive curvature current. To prove that we may restrict to product metrics first consider the case when $(X_i, -K_{X_i})$ are both K-polystable. They thus admit Kähler-Einstein metrics and the corresponding product metric is Kähler-Einstein on $X_1 \times X_2$ and, as a consequence, realizes the sup of $(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}})^{n+1}$, as pointed out in Remark 2.6 (strictly speaking, in the singular case the Kähler-Einstein metric is merely known to be locally bounded, but it can, in a standard way, be approximated by continuous ones). Finally, in the case when $(X_i, -K_{X_i})$ are merely K-semistable we will use the following general observation. If X_1 and X_2 are K-semistable Fano varieties over $\mathbb{C}$, then the inf of the Ding functional (formula 2.9) corresponding to $X_1 \times X_2$ coincides with the inf over product metrics. To prove this first recall the definition of the twisted Ding functional $\mathcal{D}_{\psi_0,\gamma}$ corresponding to a given locally bounded psh metric ψ_0 and $\gamma \in]0,1]$:

$$\mathcal{D}_{\psi_0,\gamma}(\psi) = -\frac{1}{\mathrm{vol}(-K_X)}\mathcal{E}_{\psi_0}(\psi) - \log\int_X e^{-(\gamma\psi+(1-\gamma)\psi_0)}$$

By Hölder's inequality $\mathcal{D}_{\psi_0,\gamma}(\psi)$ is decreasing in γ. Since, as shown in the proof of Theorem 2.4, $\mathcal{D}_{\psi_0,1}$ is bounded from below when X is K-semistable, so is $\mathcal{D}_{\psi_0,\gamma}(\psi)$ for any $\gamma \in]0,1[$. More precisely, $\mathcal{D}_{\psi_0,\gamma}(\psi)$ is coercive for any given $\gamma \in]0,1[$ (see the proof of [5, Cor 3.6]) and thus $\mathcal{D}_{\psi_0,\gamma}$ admits a minimizer ψ_γ and the minimizers are precisely the solutions to the twisted Kähler-Einstein equation

$$\frac{(dd^c\psi)^n/n!}{\mathrm{vol}(-K_X)} = \frac{e^{-(\gamma\psi+(1-\gamma)\psi_0)}}{\int_X e^{-(\gamma\psi+(1-\gamma)\psi_0)}}$$

(see [9]). Accepting this claim for the moment we may, given two K-semistable Fano varieties X_1 and X_2 and $\gamma \in]0,1[$ thus take two twisted KE-metrics $\psi^{(1)}_\gamma$ and $\psi^{(2)}_\gamma$ on $-K_{X_1}$ and $-K_{X_2}$, respectively. The corresponding product metric ψ_γ on $-K_{X_1\times X_2}$ is a twisted KE-metric and thus minimizes the twisted Ding function $\mathcal{D}_{\psi_0,\gamma}$ on $X_1\times X_2$. Moreover, as $\gamma\to 1$

$$\mathcal{D}_{\psi_0}(\psi_\gamma)\to\inf\mathcal{D}_{\psi_0}. \tag{2.23}$$

Indeeed, in general, the inf of $\mathcal{D}_{\psi_0,\gamma}$ converges towards the inf of $\mathcal{D}_{\psi_0,1}$ (as follows from a general simple convexity/continuity argument, using that $\mathcal{D}_{\psi_0,\gamma}(\psi)$ is concave in γ, detailed in [1]). Since $\mathcal{D}_{\psi_0,\gamma}(\psi)$ is decreasing in γ this proves the convergence 2.23. Finally, since in our setup ψ_γ is a product metric it follows that the inf of $\mathcal{D}_{\psi_0}$ coincides with the inf restricted to product metrics, as desired. □

3. Sharp height inequalities in the toric case

We now specialize to the case when X is toric Fano variety.

3.1. **The toric setup.**

We start by recalling the notation for toric metrics employed in [8] and the relation to the canonical toric integral model.

3.1.1. *The moment polytope $P(L)$.* Let X be an $n-$dimensional complex projective toric variety, i.e. a complex projective variety endowed with the action of the $n-$dimensional complex torus $\mathbb{C}^{*n}$ with an open dense orbit. We shall denote by T_c the complex torus and by T the real maximal compact subtorus of T_c, i.e. $T=(S^1)^n$. Let L be a toric ample line, i.e. an ample line bundle over X endowed with a T_c-action covering the action of T_c on X. It induces a bounded convex polytope $P(L)$ in $\mathbb{R}^d$ with non-empty interior, defined as follows. Consider the induced action of the group T_c on the space $H^0(X,kL)$ of global holomorphic sections of $kL\to X$ (for k a given positive integer). Decomposing the action of T_c according to the corresponding one-dimensional representations e^m, labeled by $m\in\mathbb{Z}^n$:

$$H^0(X,kL)=\oplus_{m\in B_k}\mathbb{C}e^\alpha \tag{3.1}$$

the lattice polytope $P_{(X,L)}$ may be defined as the convex hull of $k^{-1}B_k$ in $\mathbb{R}^n$. More generally, by homogeneity, $P_{(X,L)}$ is defined for any ample $\mathbb{Q}-$line bundle.

In particular, if X is Fano, then the polytope $P(-K_X)$ has vertices in $\mathbb{Q}^n$ and may be represented as follows:

$$P(-K_X)=\{p\in\mathbb{R}^n:\ \langle l_F,p\rangle\geq -1,\ \forall F\}, \tag{3.2}$$

where F ranges over all facets of $P(-K_X)$ and l_F denotes the unique primitive element in $\mathbb{Z}^n$ which is an interior normal to the facet F (i.e. $P(-K_X)$ is the dual of the polytope with primitive vertices l_F). Conversely, any such polytope corresponds to a Fano variety X [33, 8].

Example 3.1. When $X=\mathbb{P}^n$ the polytope $P(-K_X)$ is $(n+1)\left(\Sigma_n-(1,....,1)\right)$ where Σ_n denotes the $n-$dimensional unit-simplex. An infinite family of two-dimensional toric Fano varieties $X_{p,q}$, parametrized by two prime numbers p and q, is obtained by letting $P(-K_{X_{p,q}})$ be the polytope which is dual to the polytope with the four primitive vertices $(\pm p,\pm q)$. In particular, $\mathrm{vol}(-K_{X_{p,q}})=2/(pq)$ tends to zero when pq tends to infinity.

Remark 3.2. From an invariant point of view, the real vector space $\mathbb{R}^n$ above arises as $M\otimes_{\mathbb{Z}}\mathbb{R}$, where M is the lattice $\mathrm{Hom}(T_c,\mathbb{C}^*)$ of characters of the group T_c (cf. [33]).

3.1.2. *Logarithmic coordinates and the Legendre transform ϕ^* of a metric ϕ on L.* Since X is toric we can identify T_c with its open orbit in X. Let Log be the map from T_c to $\mathbb{R}^n$ defined by

$$\mathrm{Log}\colon T_c \to \mathbb{R}^n,\ \ \mathrm{Log}(z) := x := (\log(|z_1|^2), ..., \log(|z_n|^2)).$$

The real compact torus T acts transitively on its fibers. We will refer to x as the (real) *logarithmic coordinates* on T_c. Let L be a toric ample line bundle over X and assume that P contains the origin, $0 \in P$, and denote by e^0 the corresponding $T-$invariant element in $H^0(X, kL)$. Any continuous $T-$invariant metric $\|\cdot\|$ on L induces a continuous function on $\mathbb{R}^n$ that we shall denote by $\phi(x)$, defined as

$$\phi(x) := -\log\left(\left\|e^0\right\|^2(z)\right),\ \ z \in T_c \Subset X,\ \ x := \mathrm{Log}\ z.$$

Thus, in the present additive notation ϕ for metrics we have $\phi(x) = \phi_U(z)$, when $U = T_c$, abusing notation slightly. The Legendre transform of $\phi(x)$, which defines a lower-semicontinuous convex function on $\mathbb{R}^n$ (taking values in $]-\infty, \infty]$) will be denoted by ϕ^* :

$$\phi^*(p) := \sup_{x \in \mathbb{R}^n} \langle p, x\rangle - \phi(x).$$

A $T-$invariant continuous metric ψ on L is psh iff the corresponding function $\psi(x)$ on $\mathbb{R}^n$ is convex (iff $\psi(x) = \psi^{**}(x)$). We will denote by $\psi_{P(L)}$ the unique continuous convex function on $\mathbb{R}^n$ whose Legendre transform is equal to 0 on $P(L)$ and equal to ∞ on the complement of $P(L)$:

$$\psi_{P(L)}(x) := \sup_{p \in P(L)} \langle p, x\rangle \qquad (\psi^*_{P(L)} = 0 \text{ on } P,\ \psi^*_{P(L)} = \infty \text{ on } P(L)^c) \tag{3.3}$$

It corresponds to a continuous psh metric on L (see the proof of [8, Prop 3.3]) and it will be used as a canonical reference metric in the present toric setup. It follows that for any other continuous metric ϕ on L

$$\phi - \psi_{P(L)} \in L^\infty(\mathbb{R}^n),\ \ P(L) = \overline{\{\phi^* < \infty\}}. \tag{3.4}$$

Remark 3.3. From an invariant point of view the logarithm coordinates take value in $N \otimes \mathbb{R}$, where N is the lattice $\mathrm{Hom}(\mathbb{C}^*, T_c)$ of one-parameter subgroups of T_c, i.e. the dual of the lattice $\mathrm{Hom}(T_c, \mathbb{C}^*)$ of characters of T_c.

3.1.3. *Pushing forward measures from X to $\mathbb{R}^n$.* For any $T-$invariant continuous psh metric ψ on L the push-forward of the measure $(dd^c\psi)^n/n!$ on L under the map Log is given by

$$\mathrm{Log}\left(\frac{(dd^c\psi)^n}{n!}\right) = \det(\nabla^2\phi)dx,$$

(since the integral along the T^n-fibers equals $(2\pi)^n$). The measure in the right hand side is defined in the weak sense of Alexandrov. Since the closure of the image of $\mathbb{R}^n$ under the sub-gradient map of $\phi(x)$ equals P it follows that

$$\mathrm{vol}(L) = \int_P dy := \mathrm{Vol}(P).$$

Next consider the case when $L = -K_X$. Then

$$e_0 := z_1\frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n\frac{\partial}{\partial z_n} \tag{3.5}$$

defines a T_c-invariant global holomorphic section of $-K_X$, trivializing $-K_X$ over $U := \mathbb{C}^{*n}$. We can thus identify a continuous metric ϕ on $-K_X$ with the corresponding function ϕ_U on $\mathbb{C}^{*n}$

(formula 2.1) and volume form on X (formula 2.3) expressed as follows on $\mathbb{C}^{*n}$, with respect to the local holomorphic coordinate $\log z$:

$$e^{-\phi_U}(\frac{i}{2})^n d(\log z_1)\wedge d(\log \overline{z}_1)\wedge\cdots\wedge d(\log z_n)\wedge d(\log \overline{z}_n)$$

symbolically denoted by $e^{-\phi}$. Using again that the integral along the T^n-fibers equals $(2\pi)^n$ yields

$$\int_X e^{-\phi}=\pi^n\int_{\mathbb{R}^n}e^{-\phi(x)}dx. \tag{3.6}$$

3.1.4. *K-semistability and toric Kähler-Einstein metrics.* We recall the following result, which is a combination of the results [8, Thm 1.2] and [6, Cor 1.2] (which are formulated in terms of T_c-equivariant K-polystability and K-polystability, respectively).

Proposition 3.4. *Let X be a toric Fano variety. The following is equivalent:*

- *X is K-semistable*
- *X is K-polystable*
- *X admits a $T-$invariant Kähler-Einstein metric*
- *The barycenter of $P(-K_X)$ coincides with the origin* 0.

3.1.5. *The arithmetic $\chi-$volume of a toric metric.* Any toric ample line bundle $L\to X$ admits a canonical integral model $\mathcal{L}\to\mathcal{X}$ over $\mathbb{Z}$ with $\mathcal{X}$ normal (see [68, Section 2] and [22, Def 3.5.6]).

The following result is a special case of the main result of [22, Thm 3] (combined with Lemma 2.8):

Proposition 3.5. *Let $L\to X$ be an ample toric line bundle and denote by $(\mathcal{X},\mathcal{L})$ its canonical toric model over $\mathbb{Z}$. Assume that ϕ is a continuous $T-$invariant metric on L. Then*

$$2\widehat{vol}_\chi(\mathcal{L},\phi)=-\int_{P(L)}\phi^* d\lambda$$

An alternative analytic proof of this formula can also be given, using that the integral lattice $H^0(\mathcal{X},k\mathcal{L})$ in $H^0(X,kL)$ is generated by the T_c-equivariant bases e^m appearing in the decomposition 3.1 [68]. Since this basis is ortonormal wrt the L^2-norm on $H^0(X,kL)$ induced by the metric $\psi_{P(L)}$ on L, defined by formula 3.3 and the Haar measure on the unit-torus $T\Subset X$, applying [11, Thm A] yields

$$\widehat{2\mathrm{vol}}(\mathcal{L},\phi)=\mathcal{E}_{\psi_{P(L)}}(\phi). \tag{3.7}$$

When ϕ is toric the right hand side above coincides, by [8, Prop 2.9], with the right hand side of the formula in the previous proposition.

3.1.6. *Arithmetic toric Fano varieties.* Now assume that X is a toric Fano varity, so that $-K_X$ defines an ample $\mathbb{Q}-$line bundle. Then the canonical integral model $\mathcal{X}$ of X over $\mathbb{Z}$ is a normal arithmetic Fano variety, i.e. the relative anti-canonical divisor $-\mathcal{K}$ on $\mathcal{X}$ defines a relatively ample $\mathbb{Q}-$line bundle on $\mathcal{X}$. Indeed, $-\mathcal{K}$ coincides with the canonical integral model $\mathcal{L}$ of $-K_X$. This follows (just as in the function field case considered in [6, Lemma 2.2]) from the fact that the fibers of the structure morphism $\mathcal{X}\to$Spec $\mathbb{Z}$ are reduced and irreducible.

3.2. **Proof of Theorem 1.2.** Given a Fano variety X, let ϕ be a continous metric on $-K_X$ which is volume-normalized. We will prove the following more general formulation of the inequality in Theorem 1.2 (where the psh assumption on ϕ has been dispensed with):

$$\widehat{\mathrm{vol}}_{\chi}\left(-\mathcal{K},\phi\right) \leq \widehat{\mathrm{vol}}_{\chi}\left(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\right)$$

where the metric on $-K_{\mathbb{P}^n}$ is the one induced by the volume-normalized Fubini-Study metric.

A $T-$invariant continuous metric ϕ will, as above, be identified with a function $\phi(x)$ on $\mathbb{R}^n$. If ϕ is moreover volume-normalized Prop 3.5 gives

(3.8)

$$2\widehat{\mathrm{vol}}_{\chi}\left(-\mathcal{K},\phi\right)/\mathrm{vol}(-K_X) = -\mathcal{D}_{\mathbb{Z}}(\phi) = -\mathcal{D}_{\psi_P}(\phi) = -\int_P \phi^* dy/V + \log\int_{\mathbb{R}^n} e^{-\phi(x)}dx + n\log\pi,$$

where $\mathcal{D}_{\mathbb{Z}}(\phi)$ and $\mathcal{D}_{\psi_P}(\phi)$ are the Ding type functionals defined by formula 2.15 and formula 2.16, respectively, and we have used formula 3.6.

We start by recording the following explicit formula for the arithmetic volume of projective space $\mathbb{P}^n$, endowed with a volume normalized Kähler-Einstein metric (which may be assumed to be the metric induced by the Fubini-Study metric).

Lemma 3.6. *The following formulas holds for the metrics ϕ_{KE} on the anti-canonical line bundles of $\mathbb{P}^n_{\mathbb{C}}$ induced by a volume normalized toric Kähler-Einstein metric:*

$$X = \mathbb{P}^n_{\mathbb{C}} \implies 2\widehat{\mathrm{vol}}_{\chi}\left(-\mathcal{K},\phi_{KE}\right) = \frac{(n+1)^n}{n!}\left((n+1)\sum_{k=1}^n k^{-1} - n + \log(\frac{\pi^n}{n!})\right) > 0$$

Proof. First consider the case when $X = \mathbb{P}^n_{\mathbb{C}}$, whose canonical integral model is given by $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$. The canonical model of the anti-canonical line bundle of $\mathbb{P}^n_{\mathbb{C}}$ is given by $\mathcal{O}(1)^{\otimes n+1} \to \mathbb{P}^n_{\mathbb{Z}}$. As shown in [52, §5.4] (using the induction formula for the height; see also [85, Prop 3.10]) the height h_{FS} of $\mathcal{O}(1) \to \mathbb{P}^n_{\mathbb{Z}}$ endowed with the Fubini-Study metric ϕ_{FS} is given by

$$h_{FS} = \frac{1}{2}\sum_{k=1}^n\sum_{m=1}^k m^{-1}.$$

Since $(n+1)\phi_{FS}$ defines a Kähler-Einstein metric on $-K_{\mathbb{P}^n}$ and $\pi^{-n}\int_{\mathbb{P}^n} e^{-(n+1)\phi_{FS}} = 1/n!$ this gives

$$2\widehat{\mathrm{vol}}_{\chi}\left(-\mathcal{K},\phi_{KE}\right) - n\log\pi = (n+1)^{n+1}\frac{h_{FS}}{(n+1)!} + \frac{(n+1)^n}{n!}\log(\frac{1}{n!}) = \frac{(n+1)^n}{n!}\left(h_{FS} + \log(\frac{1}{n!})\right),$$

using formula 2.6 in the first term, combined with the homogeneity property 2.7 and, in the second term, the scaling property 2.8. Rewriting the formula for h_{FS} above as a triangle sum and changing the order of summation then concludes the proof of the formula of the lemma. The last positivity statement will be shown in the course of the proof of Lemma 3.8. □

The key ingredient in the proof of Theorem 1.2 is the following universal bound on the arithmetic volume, in terms of the ordinary volume:

Proposition 3.7. *For any $n-$dimensional toric Fano variety X which is K-semistable, the following bound holds for any volume-normalized continuous metric ϕ on $-K_X$,*

$$2\widehat{vol}_{\chi}\left(-\mathcal{K},\phi\right) \leq -\mathrm{vol}(X)\log\left(\frac{\mathrm{vol}(X)}{(2\pi^2)^n}\right), \quad \mathrm{vol}(X) := \mathrm{vol}(-K_X).$$

Proof. First recall that, as shown in the beginning of the proof of Theorem 2.4, it is equivalent to establish the upper bound for $-\mathcal{D}_{\psi_P}(\phi)$ when ϕ is a continuous psh metric on L. Since X is assumed K-semistable it follows from Prop 3.4 that X admits a $T-$invariant Kähler-Einstein metric. In general, a Kähler-Einstein metric ϕ on $-K_X$ minimizes the Ding functional $\mathcal{D}_{\psi_0}$ [9]. Thus in the toric case the infimum of $\mathcal{D}_{\psi_0}$ coincides with the infimum over all continuous $T-$invariant psh metrics. As explained in Section 3.1.2 such a metric may be identified with a convex function $\phi(x)$ on $\mathbb{R}^n$ satisfying $\phi - \psi_P \in L^\infty(\mathbb{R}^n)$. By formula 3.8 it will be enough to show that for such convex functions

$$-\int_P \phi^* dy/V + \log \int_{\mathbb{R}^n} e^{-\phi(x)} dx \leq -\log V + n\log(2\pi), \quad V := \mathrm{vol}(-K_X). \tag{3.9}$$

Since 0 is contained in the interior of P the measure $e^{-\phi}dx$ on $\mathbb{R}^n$ has finite moments. Recall that, by Prop 3.4 the barycenter of P coincides with $0 \in \mathbb{R}^n$ and, as a consequence, the left hand side in inequality 3.9 is invariant under translations of ϕ, $\phi(x) \mapsto \phi(x+a)$ for any given $a \in \mathbb{R}^n$ [8, Lemma 2.14]. As a consequence, in order to prove the inequality 3.9 we may as well assume that

$$\int_{\mathbb{R}^d} x e^{-\phi} dx = 0.$$

By the functional form of Santaló's inequality [3, Lemma 2.14] this implies that

$$\int_{\mathbb{R}^n} e^{-\phi^*(y)} dy \cdot \int e^{-\phi(x)} dx \leq (2\pi)^n$$

(where equality holds if $\phi = \phi^*$ i.e. if $\phi(x) = |x|^2/2$). Moreover, by Jensen's inequality

$$-\int_P \phi^* d\lambda/V \leq \log\left(\int_P e^{-\phi^*(y)} dy/V\right) = \log\left(\int_{\mathbb{R}^n} e^{-\phi^*(y)} dy/V\right),$$

using in the last equality that $\phi^* = \infty$ on the complement of P (see formula 3.4). Combining the latter two inequalities yields the desired inequality 3.9. □

Recall that $\mathbb{P}^n$ has maximal volume among all K-semistable $n-$dimensional Fano varieties (as shown in [10] in the toric case and in [46] in general). We next show that it will be enough to prove that, in the toric case, the next to largest volume is attained by $\mathbb{P}^{n-1} \times \mathbb{P}^1$:

Lemma 3.8. *For any $n-$dimensional toric Fano variety X which is K-semistable*

$$\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \implies \widehat{\mathrm{vol}}_\chi(-\mathcal{K}, \phi) < \widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\right)$$

where $-K_{\mathbb{P}^n}$ is endowed with the volume-normalized Fubini-Study metric.

Proof. First observe that the function of $\mathrm{vol}(X)$ appearing in the rhs of the inequality in the previous proposition is increasing when $\mathrm{vol}(X) \leq (2\pi^2)^n/e$. This bound is, in fact, satisfied for any K-semistable X. Indeed, by [10],

$$\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^n) = \frac{(n+1)^n}{n!} < (2\pi^2)^n/e. \tag{3.10}$$

(using, in the last inequality a simple induction argument). Thus, by the previous proposition, it will be enough to show that

$$-\mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)/(2\pi^2)^n) < 2\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\right). \tag{3.11}$$

for any $n \geq 2$. To this end first note that

$$-\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)/(2\pi^2)^n) = -\frac{2n^{n-1}}{(n-1)!}(\log(\frac{2n^{n-1}}{(n-1)!}) - n\log(2\pi^2))$$

We check that the inequality holds for $n=2$ and with induction in mind we simplify the right hand side of 3.11 with $n+1$ for n and get

$$\begin{aligned}
2\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^{n+1}_\mathbb{Z}}}\right) =& -\frac{(n+2)^{n+1}}{(n+1)!}(n+1-(n+2)\sum_{k=1}^{n+1}\frac{1}{k} + \log((n+1)!) - (n+1)\log(\pi))\\
=& -(\frac{n+2}{n+1})^{n+1}\frac{(n+1)^n}{n!}((n-(n+1)\sum_{k=1}^{n}\frac{1}{k} + \log(n!) - n\log(\pi))+\\
&(1-(n+2)\sum_{k=1}^{n+1}\frac{1}{k} + (n+1)\sum_{k=1}^{n}\frac{1}{k} + \log(n+1) - \log(\pi)))\\
=&(\frac{n+2}{n+1})^{n+1}2\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}\right) - (\frac{n+2}{n+1})^{n+1}(1-\log(\pi)+\log(n+1) - \frac{n+2}{n+1} - \sum_{k=1}^{n}\frac{1}{k})
\end{aligned}$$

Here we observe for later use that $\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}\right) > 0 \forall n \geq 1$ by evaluating it at $n=1$ and then using the above to perform induction and noting that

$$-(1-\log(\pi)+\log(n+1) - \frac{n+2}{n+1} - \sum_{k=1}^{n}\frac{1}{k}) > -(-\log(\pi)+\log(2)) = \log(\frac{\pi}{2}) > 0$$

for $n \geq 1$. We have used that $-\log(n+1) + \sum_{k=1}^n \frac{1}{k}$ is increasing and can thus be estimated from below by putting $n=1$. We also simplify the left hand side of 3.11,

$$\begin{aligned}
-\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)/(2\pi^2)^{n+1}) =& -\frac{2(n+1)^n}{n!}(\log(\frac{2(n+1)^n}{n!}) - (n+1)\log(2\pi^2))\\
=& -(\frac{n+1}{n})^n\frac{2n^n}{n!}((\log(\frac{2n^n}{n!}) - n\log(2\pi^2))+\\
&(\log((\frac{n+1}{n})^n - \log(2\pi^2))\\
=& -(\frac{n+1}{n})^n\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)/(2\pi^2)^n)\\
&-2\frac{(n+1)^n}{n!}(-\log((\frac{n+1}{n})^n) - \log(2\pi^2)).
\end{aligned}$$

Fix $n \geq 2$ and assume $-\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)/(2\pi^2)^n) \leq 2\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}\right)$. Define for brevity $e_n = (1+\frac{1}{n})^n$ and estimate

$$
\begin{aligned}
&2\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^{n+1}_{\mathbb{Z}}}}\right) - (-\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)/(2\pi^2)^{n+1}))\\
&\quad = e_{n+1}\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\right) - (-e_n\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)/(2\pi^2)^n))\\
&\qquad + 2\frac{(n+1)^n}{n!}(\log(\frac{(n+1)^n}{n}) - \log(2\pi^2))\\
&\qquad - \frac{(n+2)^{n+1}}{(n+1)!}(1-\log(\pi)+\log(n+1) - \frac{n+2}{n+1} - \sum_{k=1}^n \frac{1}{k})\\
&\quad > 2\frac{(n+1)^n}{n!}(\log(\frac{(n+1)^n}{n}) - \log(2\pi^2))\\
&\qquad - \frac{(n+2)^{n+1}}{(n+1)!}(1-\log(\pi)+\log(n+1) - \frac{n+2}{n+1} - \sum_{k=1}^n \frac{1}{k})\\
&\quad = \frac{(n+2)^{n+1}}{(n+1)!}(\frac{(n+1)^n}{n!}/\frac{(n+2)^{n+1}}{(n+1)!}2(\log((\frac{n+1}{n})^n) - \log(2\pi^2))\\
&\qquad - 1 + \log(\pi) - \log(n+1) + \frac{n+2}{n+1} + \sum_{k=1}^n \frac{1}{k})\\
&\quad = \frac{(n+2)^{n+1}}{(n+1)!}[\frac{2}{e_n}(\log(e_n) - \log(2\pi^2)) + \log(\pi) + \sum_{k=1}^{n+1}\frac{1}{k} - \log(n+1)].
\end{aligned}
$$

In the inequality above we have used $\widehat{\mathrm{vol}}_\chi\left(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\right) > 0 \forall n \geq 1$ and $e_n < e_{n+1}$ and the induction hypothesis. Next check numerically that this last expression is positive for $n = 2, 3$. For $n \geq 4$ we have

$$
\begin{aligned}
&\frac{2}{e_n}(\log(e_n) - \log(2\pi^2)) + \log(\pi) + \sum_{k=1}^{n+1}\frac{1}{k} - \log(n+1)\\
&\qquad > \frac{2}{e_4}(\log(e_4) - \log(2\pi^2)) + \log(\pi) + \gamma > 0.
\end{aligned}
$$

We used again that $e_n < e_{n+1}$ and the fact that $\sum_{k=1}^{n+1}\frac{1}{k} - \log(n+1) > \gamma$ [93], where γ is the Euler-Mascheroni constant. The last inequality is checked numerically. □

We expect that any K-semistable toric Fano variety X, not equal to $\mathbb{P}^n$, satisfies the volume bound in the previous lemma (see the following section). Here we will show that this is the case under the conditions of Theorem 1.2. First, the singular cases are handled using the following bound.

Lemma 3.9. *Let X be a singular K-semistable toric Fano variety. Then*

$$
\mathrm{vol}(-K_X) \leq \frac{1}{2}(n+1)^n/n!
$$

if anyone of the following conditions hold:

- *X is $\mathbb{Q}-$factorial (or equivalently, X has abelian quotient singularities).*
- *X is not Gorenstein*

In particular, by the first point, when $n = 2$ this inequality holds for any singular K-semistable toric Fano variety X.

Proof. The result concerning the first point is the toric case of [61, Thm 3] concerning quotient singularities, but in the toric case it also follows from the proof of [10, Thm 1.2]. For future reference we recall the argument in [10]. Let P be a given polytope with rational vertices and represent P as the intersection of hyperplanes $\{p \in \mathbb{R}^n : \langle l_F, p\rangle \geq -a_F\}$, where the index F ranges over the facets of P, l_F is a primitive vector in $\mathbb{Z}^n$ and a_F is a non-zero positive numbers. In the present Fano case $a_F = 1$. Moreover, since X is assumed to be $\mathbb{Q}-$factorial for any vertex p_0 of P there are precisely n facets $F_1, ..., F_n$ of P intersecting p_0, numbered so that the corresponding normals define a positively oriented bases in $\mathbb{R}^n$ [33]. Fixing a vertex p_0 of P we denote by P' the image of P under the map

$$p \mapsto \left(\frac{\langle l_{F_1}, p\rangle + a_{F_1}}{a_{F_1}}, ..., \frac{\langle l_{F_n}, p\rangle + a_{F_n}}{a_{F_n}}\right), \tag{3.12}$$

which is a polytope in $[0, \infty[^n$. Moreover, assuming that 0 is the barycenter of P the barycenter of P' is $(1, ..., 1)$. By [10, Thm 1.5] the volume $\mathrm{Vol}(P')$ of any such polytope is maximal when P' is $(n+1)$ times the unit-simplex in $[0, \infty[^n$ with vertex at $(0, ..., 0)$. Hence,

$$\mathrm{Vol(P')} \leq \mathrm{(n+1)^n/n!}, \;\; \mathrm{Vol}(P') = \frac{\delta}{a_{F_1} \cdots a_{F_n}} \mathrm{Vol}(P) \tag{3.13}$$

where δ is the determinant of the map $p \mapsto (\langle l_{F_1}, p\rangle, ..., \langle l_{F_n}, p\rangle)$. Thus δ is a positive integer and $\delta = 1$ iff the map is invertible, i.e. if and only if $l_{F_1}, ..., l_{F_n}$ generate $\mathbb{Z}^n$, which is equivalent to the T_c-invariant neighbourhood U_0 corresponding to the vertex p_0 being biholomorphic to $\mathbb{C}^n$ [33]. Hence, if X is singular (i.e. X is not non-singular), then there must be some vertex p_0 with $\delta \geq 2$. Since $a_{F_i} = 1$ this concludes the proof.

To prove the second point we employ a similar argument. This time, for X possibly not $\mathbb{Q}$-factorial, there might be more than n facets intersecting a vertex p_0. Still, there are at least n facets intersecting at p_0, and we can construct the map 3.12 by choosing any n of them. Next note that if $\delta = 1$, the map and its inverse have integer coefficients (since $a_{F_i} = 1$ when X is Fano) and since p_0 is mapped to 0, $p_0 \in \mathbb{Z}^n$. Since p_0 was arbitrary, it follows that P is a lattice polytope and hence X is Gorenstein. Thus $\delta \geq 2$ and we are done. ☐

The volume bound in the previous lemma implies the volume bound in Lemma 3.8 is satisfied:

$$\frac{(n+1)^n}{2n!} \leq \frac{2n^{n-1}}{(n-1)!} \iff (1 + 1/n)^n \leq 4. \tag{3.14}$$

The lhs in the latter inequality increases to e, which is, indeed, smaller than 4. This proves Theorem 1.2 in the singular cases. Finally, in the case that X is non-singular there are, for any given dimension n only a finite number of cases to check in order to verify the volume bound in Lemma 3.8. When $n \leq 6$ we may apply the database [72] of all non-singular Fano varieties of dimension n. The condition that the barycenter of P vanishes, corresponds in the data base to the condition "zero dual barycentre". Adding the condition $(-K_X)^n \geq n!\mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$ the database only furnishes $\mathbb{P}^n$ and $\mathbb{P}^{n-1} \times \mathbb{P}^1$, as desired.

3.2.1. *Remarks on the "gap hypothesis".* In order to extend the proof of Theorem 1.2 to a any dimension n one would need to establish the following conjecture (established above under the conditions in Theorem 1.2):

Conjecture 3.10. *(the "gap hypothesis"). For any $n-$dimensional toric K-semistable Fano manifold X different from* $\mathbb{P}^n$, $\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$.

This conjecture might even hold without the toric assumptions in any dimension (as pointed out to us by Ziquan Zhuang this appears to be a folklore conjecture). For example, when $n = 3$ and X is non-singular it follows from the well-known classification of three dimensional Fano manifolds (see the "big table" in [2, Section 6]) that the only Fano manifolds X, different from $\mathbb{P}^3$, which do not satisfy the inequality in question are $\mathbb{P}^3$ blown-up in one point and $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$. But both of these are K-unstable, i.e. they are not K-semistable. Indeed, these two Fano manifolds are toric and if they were K-semistable they would satisfy the gap hypothesis, by the toric case ($n \leq 6$) applied to $n = 3$. Let us also point out that in the toric case it is only $\mathbb{P}^{n-1} \times \mathbb{P}^1$ that saturates the inequality in the "gap hypothesis" when $n \leq 6$ and it seems thus natural to ask if this is also the case when $n > 6$? However, in the general non-toric case the inequality is also saturated by the non-singular quadratic hypersurface X_2 in $\mathbb{P}^{n+1}$, i.e. the base of the Ordinary Double Point (ODP). Moreover, as pointed out to us by Yuji Odaka, in the general case our "gap hypothesis" is reminiscent of the ODP-conjecture in [87], very recently settled in the toric case [70]. More precisely, in our setup, the ODP-conjecture implies that

$$\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)(n/I(X)), \tag{3.15}$$

where $I(X)$ denotes the Fano index of X (i.e. largest positive integer such that $K_X/I(X)$ is a line bundle). However, $I(X) \leq n$ when $X \neq \mathbb{P}^n$ (with equality iff $X = X_2$) and hence the inequality 3.15 is weaker than our "gap hypothesis".

3.2.2. *The case of products in any dimension.*

Lemma 3.11. *The "gap hypothesis" holds when X is the product of K-semistable Fano varieties $X_1, ..., X_M$ (not necesseraily assumed toric).*

Proof. By a simple induction argument we may as well assume that $M = 2$. Let, without loss of generality, $n := \dim(X_1) \geq \dim(X_2) =: m > 1$. Note that if $m = 1$ we are done since then, $\mathrm{vol}(X) = \mathrm{vol}(X_1)\mathrm{vol}(X_2) \leq \mathrm{vol}(\mathbb{P}^{N-1})\mathrm{vol}(\mathbb{P}^1) = \mathrm{vol}(\mathbb{P}^{N-1} \times \mathbb{P}^1)$ using that, by Fujita's inequality 1.1, complex projective space maximizes the volume among K-semistable Fano varieties in each dimension. Using again that complex projective space maximizes the volume in each given dimension and defining for brevity $e_k := (1 + \frac{1}{k})^k$ we get $\mathrm{vol}(X) = \mathrm{vol}(X_1)\mathrm{vol}(X_2) \leq \mathrm{vol}(\mathbb{P}^n)\mathrm{vol}(\mathbb{P}^m)$

$$= \frac{(n+1)^n}{n!}\frac{(m+1)^m}{m!} = \frac{(n+2)^{n+1}}{(n+1)!}\frac{m^{m-1}}{(m-1)!}(\frac{n+1}{n+2})^{n+1}(\frac{m+1}{m})^m$$

$$= \frac{(n+2)^{n+1}}{(n+1)!}\frac{m^{m-1}}{(m-1)!}\frac{e_m}{e_{n+1}} < \frac{(n+2)^{n+1}}{(n+1)!}\frac{m^{m-1}}{(m-1)!} = \mathrm{vol}(\mathbb{P}^{n+1})\mathrm{vol}(\mathbb{P}^{m-1})$$

where in the last inequality we have used that e_k is increasing in k. We may continue in similar manner until we have $\mathrm{vol}(\mathbb{P}^{N-1} \times \mathbb{P}^1)$ in the right hand side and we are done. □

As explained in the previous section, it follows from the previous lemma that Conjecture 1.1 holds when $\mathcal{X}$ is a product of toric arithmetic Fano varieties, i.e. $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_M$, where $\mathcal{X}_i$ is endowed with its canonical integral structure.

3.3. **The height of toric Kähler-Einstein metrics; proof of Theorem 1.3.** By Prop 3.7 it only remains to prove the lower bound. Using the notation in the proof of Prop 3.7 we have that, for any continuous convex function ψ on $\mathbb{R}^n$ such that $\psi - \psi_P$ is bounded,

$$2(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}/\mathrm{vol}(-K_X) \geq -\int_P \psi^* dy/\mathrm{Vol(P)} + \log \int_{\mathbb{R}^n} e^{-\psi} dx + n \log \pi$$

In particular, taking $\psi = \psi_P$ the first term in the right hand side vanishes. Moreover,

$$I := \int_{\mathbb{R}^n} e^{-\psi_P} dx = n!\mathrm{Vol}(P^*),$$

where P^* denotes the polar dual of P, i.e. P^* consists of all $x \in \mathbb{R}^n$ such that $x \cdot p \leq 1$ for all $p \in P$. Indeed,

$$I = \int_{[0,\infty[} e^{-t}(\psi_P)_* dx = \int e^{-t}\frac{dV(t)}{dt} dt = \int e^{-t} V(t) dt = \int_0^\infty e^{-t} t^n dt \mathrm{Vol}(P^*),$$

where $V(t)$ is the Lebesgue volume of $\{\psi_P < t\}$ i.e. of tP^*. Hence,

$$2(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} \geq \mathrm{Vol}(P)\left(\log\left(n!\mathrm{Vol}(P^*)\right) + n \log \pi\right).$$

Since, by definition, $\mathrm{Vol}(P^*)\mathrm{Vol}(P) \geq m_n$ this concludes the proof of the lower bound in the theorem. Next, by [51, Cor 1.8] (see also [14])

$$m_n \geq (\frac{\pi}{2e})^{n-1}(n+1)^{n+1}/(n!)^2 = (\frac{\pi}{2e})^{n-1}\frac{(n+1)}{n!}\sigma_n,$$

where $\sigma_n = \mathrm{vol}(\mathbb{P}^n)$. Since $\mathrm{Vol}(P) \leq \sigma_n$ (by 3.10) this means that

$$n!\pi^n m_n \mathrm{Vol}(P)^{-1} \geq n!\pi^n m_n \sigma_n^{-1} = \pi(\frac{\pi^2}{2e})^{n-1}(n+1) > 1$$

proving the positivity in the theorem.

3.4. **Examples.** We next provide examples of families of toric varities X for which the height of the corresponding Kähler-Einstein can be explicitely computed as a function of $\mathrm{vol}(X)$ of the same form as in Theorem 1.3. The examples are based on the following

Proposition 3.12. *Let X_1 and X_2 be two K-semistable toric Fano varieties of dimension n with moment polytopes P_1 and P_2 such that $P_2 = AP_1$ for an invertible linear transformation A (the polytopes are linearly equivalent). Denote the canonical integral models of X_1 and X_2 by $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. Then, with heights taken with respect to the volume-normalized Kähler-Einstein metrics,*

$$\frac{(\overline{-\mathcal{K}_{\mathcal{X}_2}})^{n+1}/(n+1)!}{(-K_{X_2})^n/n!} = \frac{(\overline{-\mathcal{K}_{\mathcal{X}_1}})^{n+1}/(n+1)!}{(-K_{X_1})^n/n!} - \frac{1}{2}\log \det A.$$

As a consequence, for X a K-semistable toric Fano variety of dimension n,

$$(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} = \frac{(\mathrm{n}+1)!}{2}\mathrm{vol}(X)\log\left(\frac{a}{\mathrm{vol}(X)}\right) \tag{3.16}$$

where a is a constant independent of the choice of X within a class of toric varieties with linearly equivalent moment polytopes. More precisely,

$$a = \mathrm{vol}(X)\exp(\frac{2(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}/(n+1)!}{\mathrm{vol}(X)}) \tag{3.17}$$

and Proposition 3.12 ensures the claimed independence.

Proof. (of Proposition 3.12) Recall that, with heights taken with respect to Kähler-Einstein metrics,

$$\frac{(\overline{-\mathcal{K}_{\mathcal{X}_2}})^{n+1}/(n+1)!}{(-K_{X_2})^n/n!} = -\frac{1}{2}\sup_{\phi} - \frac{1}{\mathrm{vol}(P_2)}\int_{P_2}\phi^*(p)\mathrm{d}p + \log\int_{\mathbb{R}^n}\exp(-\phi(x))\mathrm{d}x.$$

Changing variables in the integrals, $p \mapsto A^t p'$ and $x \mapsto Ax'$ we get

$$\frac{(\overline{-\mathcal{K}_{\mathcal{X}_2}})^{n+1}/(n+1)!}{(-K_{X_2})^n/n!} = -\frac{1}{2}(\sup_{\phi(A\cdot)} - \frac{1}{\mathrm{vol}(P_1)}\int_{P_1}\phi^*(A^t p')\mathrm{d}p' + \log\int_{\mathbb{R}^n}\exp(-\phi(Ax'))\mathrm{d}x' + \log\det\mathrm{A}).$$

Next we rename $\phi' = \phi(A\cdot)$ and use that then $\phi'^* = \phi^*(A^t\cdot)$ to get the result. □

Example 3.13. Recall the K-semistable toric Fano varieties $X_{q,p}$ parametrized with two prime numbers from Example 3.1. The corresponding polytope $P(-K_{X_{p,q}})$ is the image of the polytope $P(-K_{\mathbb{P}^1\times\mathbb{P}^1}) = \mathrm{conv}\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ under the linear map A given in matrix form by $\begin{bmatrix} \frac{1}{2p} & \frac{1}{2p} \\ \frac{-1}{2q} & \frac{1}{2q} \end{bmatrix}$. Thus the family $\mathcal{F} = \{\mathbb{P}^1\times\mathbb{P}^1, X_{p,q} : p, q \text{ prime}\}$ comprise an example of a family of K-semistable toric Fano varieties with linearly equivalent moment polytopes. Thus by 3.16, for $X \in \mathcal{F}$,

$$(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} = \frac{(\mathrm{n}+1)!}{2}\mathrm{vol}(X)\log\left(\frac{a}{\mathrm{vol}(X)}\right)$$

with, by 3.17, 3.6 and a simple computation,

$$a = \mathrm{vol}(\mathbb{P}^1\times\mathbb{P}^1)\exp\left(\frac{2(\overline{-\mathcal{K}_{\mathbb{P}^1\times\mathbb{P}^1}})^{n+1}/(n+1)!}{\mathrm{vol}(\mathbb{P}^1\times\mathbb{P}^1)}\right) = 4\exp(2-\log\pi^2).$$

Recall also that vol $(-K_{X_{p,q}}) = 2/(pq)$ so that in this family the heights with respect to the Kähler-Einstein metrics are explicitely computed by the previous formula.

4. Sharp bounds on Donaldson's toric Mabuchi functional

Let (X, L) be a polarized complex manifold and denote by $\mathcal{H}(X, L)$ the space of all smooth metrics ψ on L whose curvature form $dd^c\psi$ is positive, $dd^c\psi > 0$.

4.1. **The Mabuchi functional (recap).** The Mabuchi functional $\mathcal{M}$ on $\mathcal{H}(X, L)$ is defined, up to addition by a constant, by declaring that its differential on $\mathcal{H}(X, L)$ at a given point ψ is represented by the following measure on X :

$$d\mathcal{M}_{|\psi} := (-S(\psi)+a)\frac{(dd^c\psi)^n}{n!}, \quad a := n(-K_X)\cdot L^{n-1}/L^n, \tag{4.1}$$

where $S(\psi)$ denotes the scalar curvature of the Kähler form $(dd^c\psi)$, i.e. the trace of the Ricci curvature:

$$S(\psi)\frac{(dd^c\psi)^n}{n!} := \mathrm{Ric}\ (dd^c\psi)\wedge\frac{(dd^c\psi)^{n-1}}{(n-1)!}.$$

Recall that the Ricci curvature $\mathrm{Ric}(dd^c\psi)$ of the Kähler form $dd^c\psi$ is the $(1,1)-$form defined as the curvature of the metric on $-K_X$ induced by the volume form of $dd^c\psi$. We have followed Donaldson's multiplicative normalizations in [38, formula 3.2.1], which differ from the original definition in [65], where the measure $\frac{(dd^c\psi)^n}{n!}$ on X is volume-normalized. At any rate, formula 4.1 only determines the Mabuchi functional $\mathcal{M}$ up to an additive constant.

4.1.1. *The case when X is a Fano manifold and $L = -K_X$.* We now specialize to the case when $L = -K_X$ and note that a choice of reference metric ψ_0 in $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ induces a particular choice of Mabuchi functional, i.e. a functional whose differential satisfies formula 4.1, that we shall denote by $\mathcal{M}_{\psi_0}$. This is a consequence of the thermodynamical formalism introduced in [5], which expresses

$$\mathcal{M}_{\psi_0}(\psi) := \mathrm{vol}(-K_X)F_{\psi_0}(MA(\psi)), \tag{4.2}$$

where $MA(\psi)$ is the probability measure on X defined by the normalized volume form of the Kähler metric $dd^c\psi$:

$$MA(\psi) := \frac{1}{n!}(dd^c\psi)^n/\mathrm{vol}(L) \tag{4.3}$$

and $F_{\psi_0}(\mu)$ denotes the *free energy functional* on the space $\mathcal{P}(X)$ of all probability measures on X,defined as follows:

$$F_{\psi_0}(\mu) := -E_{\psi_0}(\mu) + \mathrm{Ent}_{dV_0}(\mu) \in]-\infty, \infty] \tag{4.4}$$

Here $\mathrm{Ent}_{dV_0}(\mu)$ denotes the *entropy* of μ relative to the volume form dV_0 on X induced by ψ_0 (i.e. $dV_0 = e^{-\psi_0}$ in the notation of Section 2.1.2) defined by

$$\mathrm{Ent}_{dV_0}(\mu) := \int \log \frac{\mu}{dV_0}\mu$$

when $\mu \in L^1(X, dV_0)$ and otherwise $\mathrm{Ent}_{dV_0}(\mu) := \infty$. Furthermore, $E_{\psi_0}(\mu)$ is the *pluricomplex energy* of μ, relative to ψ_0, introduced in [4], which may be defined as a Legendre-Fenchel transform of the functional $\mathcal{E}_{\psi_0}/\mathrm{vol}(L)$ (defined by formula 2.11). For our purposes it will be enough to define $E_{\psi_0}(\mu)$ when μ is of the form $\mu = MA(\psi)$ for ψ in $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$:

$$E_{\psi_0}(MA(\psi)) = \frac{\mathcal{E}_{\psi_0}(\psi)}{\mathrm{vol}(L)} - \int_X (\psi - \psi_0)MA(\psi). \tag{4.5}$$

We recall that formula 4.2 follows readily from the fact that on the subspace of all volume forms μ in $\mathcal{P}(X)$ the differential of E_{ψ_0} at $\mu \in \mathcal{P}(X)$ is represented by the function $\psi_0 - \psi_\mu$:

$$dE_{\psi_0|\mu} = -(\psi_\mu - \psi_0)$$

(this formula is dual to formula 2.12 in the sense of Legendre transforms; see [5]).

Remark 4.1. Formula 4.2 defines $\mathcal{M}_{\psi_0}(\psi)$ on the space $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ as a function taking values in $]-\infty, \infty]$. More generally, the functional $\mathcal{M}_{\psi_0}(\psi)$ is well-defined as soon as $E(MA(\psi)) < \infty$ (see [5, 9]). For ψ smooth formula 4.2 is essentially equivalent to a formula for the Mabuchi functional appearing in [91] and [27].

4.1.2. *The case when X is a singular Fano variety.* In the case when X is a singular Fano variety we will denote by $\mathcal{H}(X, -K_X)$ the space of all continuous metrics ψ on L such that ψ is smooth on the regular locus X_{reg} of X and $dd^c\psi > 0$ on X_{reg}.

4.2. **Proof of Theorem 1.5.** First recall the following basic inequality that holds on any Fano variety [9, Lemma 4.4]:

$$F_{\psi_0}(MA(\psi)) \geq \mathcal{D}_{\psi_0}(\psi) \tag{4.6}$$

as follows from the non-negativity of the relative entropy between two probability measures (or Jensen's inequality). In fact, the following identity holds [9, Lemma 4.4]:

$$\inf_{\mathcal{C}^0(L)\cap\mathrm{PSH}(L)} F_{\psi_0}(MA(\psi)) = \inf_{\mathcal{C}^0(L)\cap\mathrm{PSH}(L)} \mathcal{D}_{\psi_0}(\psi), \tag{4.7}$$

(the two infima above may, equivalently, be restricted to $\mathcal{H}(X,L)$; see the regularization result in [13]).

Combining Theorem 1.2 with the inequality 4.6 the proof is concluded by invoking the following formula relating $\mathcal{M}_{\psi_P}$(where ψ_P is the canonical toric reference defined by formula 3.3) to Donaldson's toric Mabuchi functional

$$\mathcal{M}_{-K_X}(\psi) := \int_{\partial P} \psi^* d\sigma - n\int_P \psi^* dx - \int_P \log\det(\nabla^2\psi^*)dx, \tag{4.8}$$

where ψ^* denotes the Legendre transform of the $T-$invariant metric $\psi \in \mathcal{H}(X,-K_X)$ and $d\sigma$ is the measure on ∂P, absolutely continuous wrt the $(n-1)-$dimensional Lebesgue measure $d\lambda_{\partial P}$, defined by $d\sigma = d\lambda_{\partial P}/\,\|l_F\|$ on a facet F of ∂P, where $\|l_F\|$ denotes the Euclidean norm of a primitive normal vector to F.

Lemma 4.2. *Let X be an $n-$dimensional toric Fano variety. The following identity holds on the space of all $T-$invariant metrics in $\mathcal{H}(X,-K_X)$:*

$$\mathcal{M}_{\psi_P} = \mathcal{M}_{-K_X} - \mathrm{vol}(-K_X)\log\mathrm{vol}(-K_X)$$

Proof. This formula is essentially the content of [8, Prop 4.6], but since the normalizations are a bit different we recall the proof. First identifying a toric metric ψ with a convex function on $\mathbb{R}^n$ (as in Section 3.1.2) formula 4.2, combined with formula 4.5, yields

$$\mathcal{M}_{\psi_P}(\psi) = -\mathcal{E}_{\psi_P}(\psi) + \int_{\mathbb{R}^n} (\psi-\psi_P)(dd^c\psi)^n/n! + \int_{\mathbb{R}^n} \log\left(\frac{MA(\psi)}{e^{-\psi_P}dx}\right)\mathrm{vol}(-K_X)MA(\psi) =$$

$$= \int_P \psi^* d\lambda + \int_{\mathbb{R}^n} \psi(dd^c\psi)^n/n! + \int_{\mathbb{R}^n} \log\det(\nabla^2\psi)\det(\nabla^2\psi) - \mathrm{vol}(-K_X)\log\mathrm{vol}(-K_X).$$

By [8, Lemma 4.7] making the change of variables $y = \nabla\psi$ the second term above may be expressed

$$\int_{\mathbb{R}^n} \psi(dd^c\psi)^n/n! = \int_{\partial P} \psi^* d\sigma - (n+1)\int u dp, \tag{4.9}$$

giving

$$\mathcal{M}_{\psi_P}(\psi) = \int_{\partial P} \psi^* d\sigma - n\int_P \psi^* d\lambda + \int_{\mathbb{R}^n} \log\det(\nabla^2\psi)\det(\nabla^2\psi) - \mathrm{vol}(-K_X)\log\mathrm{vol}(-K_X).$$

Again making the change of variables $y = \nabla\psi$ in the remaining integral over $\mathbb{R}^n$ concludes the proof, using the standard relation $\det(\nabla^2\psi)(x)\det(\nabla^2\psi^*)(\nabla\psi(x)) = 1$ (which follows from the fact that the map $y \mapsto \nabla\psi^*(y)$ is the inverse of $x \mapsto \nabla\psi(x)$). □

5. Relations to the arithmetic Mabuchi functional

Given an integral model $(\mathcal{X},\mathcal{L})$ of a polarized variety (X,L) consider the *arithmetic Mabuchi functional* $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ on $\mathcal{H}(X,L)$ defined by

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) := \frac{a}{(n+1)!}\overline{\mathcal{L}}^{n+1} + \frac{1}{n!}\overline{\mathcal{K}}_{\mathcal{X}}\cdot\overline{\mathcal{L}}^n, \quad a = -n(K_X\cdot L^{n-1})/L^n$$

where $\overline{\mathcal{L}} = (\mathcal{L}, \psi)$ and $\overline{\mathcal{K}}_{\mathcal{X}}$ is endowed with the metric induced by the measure $MA(\psi)$ on X, i.e. the normalized volume form of the Kähler form $dd^c\psi$. As discussed in Section 1.4 this functional coincides, up to additive and multiplicative normalizations, with the arithmetic Mabuchi functional introduced in [77].

Lemma 5.1. *The differential of the functional $\psi \mapsto 2\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\mathcal{L}, \psi)$ on $\mathcal{H}(X, L)$ satisfies the defining formula 4.1 of the Mabuchi functional.*

Proof. As pointed out in [77] this formula can be deduced from the formula for the Mabuchi functional in [91, 27]. But for completeness and to check the normalizations we provide a simple direct proof. First recall the following property of arithmetic intersection numbers which holds if $\mathcal{L}_0 \to \mathcal{X}$ is the trivial line bundle (which is a consequence of the restriction formula [19, Prop 2.3.1]):

$$(\mathcal{L}_0, \phi_0) \cdot (\mathcal{L}_1, \phi_0) \cdot ... \cdot (\mathcal{L}_n, \phi_n) = \frac{1}{2} \int_X \phi_0 dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n, \tag{5.1}$$

where ϕ_0 is the globally well-defined function on X defined by formula 2.1 when e_U is the standard global trivialization 1 of the trivial line bundle over X, i.e. $\phi_0/2 = -\log \|s\|_{\phi_0}$, where s is a global trivialization of $\mathcal{L}$. In particular, differentiating along a curve $t \mapsto \psi_t$ in $\mathcal{H}(X, L)$ and using the symmetry of arithmetic intersection numbers gives

$$\frac{d}{dt}\left((\mathcal{L}, \psi_t)^{n+1}\right) = (n+1)\left(\mathcal{L}_0, \frac{d\psi_t}{dt}\right) \cdot (\mathcal{L}, \psi_t)^n = \frac{1}{2}\int_X \frac{d\psi_t}{dt}(dd^c\psi)^n$$

where $\frac{d\psi_t}{dt}$ is a globally well-defined function on X and can thus be identified with a metric on the trivial line bundle that we denote by $\mathcal{L}_0$. Likewise, denoting by ρ_t a local density for $MA(\psi_t)$ with respect the Euclidean measure defined by local holomorphic coordinates,

(5.2)

$$\frac{d}{dt}\left((\mathcal{K}_{\mathcal{X}}, \log \rho_t)(\mathcal{L}, \psi_t)^n\right) = (\mathcal{K}_{\mathcal{X}}, \log \rho_t)\, n\left(\mathcal{L}, \frac{d\psi_t}{dt}\right) \cdot (\mathcal{L}, \psi_t)^{n-1} + \left(\left(\mathcal{L}_0, \frac{d}{dt}\log \rho_t\right) \cdot (\mathcal{L}, \psi_t)^n\right)$$

where we have used Leibniz rule. Applying formula 5.1, the second term above may, after multiplication by 2, be expressed as

$$= \int_X \frac{d}{dt}\log \rho_t (dd^c\psi_t)^n = n!\mathrm{vol}(L)\int_X \frac{d}{dt}\log \rho_t \rho_t = n!\mathrm{vol}(L)\frac{d}{dt}\int_X \rho_t = 0,$$

using in the last equality that $\int_X \rho_t = \mathrm{vol}(L)$ for any t. Likewise, applying formula 5.1 to the first term in formula 5.2 yields

$$2\,(\mathcal{K}_{\mathcal{X}}, \log \rho_t)\left(\mathcal{L}, \frac{d\psi_t}{dt}\right)/n = \int_X \frac{d\psi_t}{dt} dd^c\,(\log \rho_t) \wedge (dd\psi_t)^{n-1} = -\int_X \frac{d\psi_t}{dt}\mathrm{Ric}(dd^c\psi_t) \wedge (dd\psi_t)^{n-1}.$$

All in all, this concludes the proof. □

The following proposition relates the arithmetic Mabuchi functional $\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}$ to Donaldson's toric Mabuchi functional $\mathcal{M}_{-K_X}$ (formula 4.8):

Proposition 5.2. *Given a toric Fano variety X denote by $\mathcal{X}$ its canonical integral model. Then the following formula holds for any $T-$invariant metric in $\mathcal{H}(X, -K_X)$:*

$$2\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})} = \mathcal{M}_{-K_X} - \mathrm{vol}(-K_X)\log \mathrm{vol}(-K_X)$$

Proof. In this case $a = n$ and we can thus decompose $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ as

$$(5.3) \qquad \frac{1}{(n+1)!}\overline{\mathcal{L}}^{n+1} + \frac{1}{n!}(\overline{\mathcal{L}} + \overline{\mathcal{K}}_{\mathcal{X}}) \cdot \overline{\mathcal{L}}^{n} = -\frac{1}{(n+1)!}\overline{\mathcal{L}}^{n+1} + \frac{1}{2}\int \log(\frac{MA(\psi)}{e^{-\psi}})(dd^c\psi)^n/n!,$$

where, in the last equality, we have exploited that $\mathcal{L} + \mathcal{K}_{\mathcal{X}}$ is trivial so that formula 5.1 applies. Applying formula 3.7 to the first term in the rhs above thus gives

$$2\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) := -\mathcal{E}_{\psi_P}(\psi) + \int \log(\frac{MA(\psi)}{e^{-\psi}})(dd^c\psi)^n/n! =$$

$$= \mathrm{vol}(-K_X)\left(-\frac{1}{V(X)}\mathcal{E}_{\psi_P}(\psi) + \langle \psi - \psi_P, MA(\psi)\rangle + \int \log(\frac{MA(\psi)}{e^{-\psi_P}})MA(\psi)\right).$$

The rhs in the last equation above equals $\mathcal{M}_{\psi_P}(\psi)$ (by definition 4.2). Invoking Lemma 4.2 thus concludes the proof. □

Next, consider an arithmetic Fano variety $\mathcal{X}$ (defined in Section 2.2.1). Denote by $\mathcal{D}_{\mathbb{Z}}(\psi)$ the functional defined by formula 2.15, corresponding to the integral model $\mathcal{L} = -\mathcal{K}_{\mathcal{X}}$. In this arithmetic setup the following variants of the inequality 4.6 and the identity 4.7 hold.

Proposition 5.3. *When $\mathcal{L} = -\mathcal{K}_{\mathcal{X}}$ the following relations hold:*

$$2\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})} \geq \mathrm{vol}(-K_X)\mathcal{D}_{\mathbb{Z}}$$

and

$$\inf_{\mathcal{C}^0(L)\cap PSH(L)} 2\mathcal{M}_{(\mathcal{X},\mathcal{L})} = \mathrm{vol}(-K_X) \inf_{\mathcal{C}^0(L)\cap PSH(L)} \mathcal{D}_{\mathbb{Z}}.$$

Proof. First note that the second term in the decomposition 5.3 of $\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)$ is precisely the entropy of $(dd^c\psi)^n/n!$ relative to $e^{-\psi}$:

$$\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi) = -\frac{(\mathcal{L},\psi)^{n+1}}{(n+1)!} + \mathrm{Ent}_{e^{-\psi}}\left((dd^c\psi)^n/n!\right).$$

Since the entropy between two probability measure is non-negative (by Jensen's inequality) this proves the inequality in the proposition when the measure $e^{-\psi}$ has unit total volume. Then general case then follows from a simple scaling argument. Next, to prove the identity in the proposition fix a reference metric ψ_0 in $\mathcal{H}(X, -K_X)$ and rewrite the previous formula as

$$(5.4) \qquad \frac{\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)}{\mathrm{vol}(-K_X)} = -\left(\frac{(\mathcal{L},\psi)^{n+1}}{(n+1)!\mathrm{vol}(-K_X)} + \langle \psi - \psi_0, MA(\psi)\rangle\right) + \frac{1}{2}\mathrm{Ent}_{e^{-\psi_0}}\left(MA(\psi)\right).$$

Accordingly, expressing $(\mathcal{L},\psi)^{n+1} = (\mathcal{L},\psi_0)^{n+1} + (n+1)!\mathcal{E}_{\psi_0}(\psi)/2$, using Lemma 2.9, gives

$$\frac{\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)}{\mathrm{vol}(-K_X)} = -\frac{1}{2}F_{\psi_0}\left(MA(\psi)\right) - \frac{1}{(n+1)!}(\mathcal{L},\psi_0)^{n+1},$$

where $F_{\psi_0}(\mu)$ is the free energy functional 4.4. The proof is thus concluded by invoking the identity 4.7 and using Lemma 2.9 again. □

Remark 5.4. When $-K_X$ admits a Kähler-Einstein metric ϕ_{KE} both infima in the previous proposition are attained at ϕ_{KE} [9]. The identity then follows directly from the Kähler-Einstein equation, giving $MA(\phi_{KE}) = e^{-\phi_{KE}}$, when ϕ_{KE} is volume-normalized.

In Section 6.2 the inequality in the previous proposition will be generalized to any model $(\mathcal{X},\mathcal{L})$ of $(X, -K_X)$, by introducing an arithmetic Ding functional $\mathcal{D}_{(\mathcal{X},\mathcal{L})}$, coinciding with the functional $\mathcal{D}_{\mathbb{Z}}$ under the conditions in the previous proposition.

6. Discussion and outlook

6.1. **The function field analog.** Recall that, according to the philosophy of Arakelov geometry, the function field analog of a metrized arithmetic variety $\mathcal{X} \to \text{Spec } \mathbb{Z}$ is a flat projective morphism

$$\mathscr{X} \to \mathscr{B}$$

from a normal complex projective variety $\mathscr{X}$ to a fixed regular complex projective curve $\mathscr{B}$. In particular, the analog of the setup of arithmetic Fano varieties in Conjecture 1.1 appears when $\mathscr{X}$ is normal, the relative anti-canonical divisor $-\mathscr{K}_{\mathscr{X}/\mathscr{B}}$ defines a relatively ample $\mathbb{Q}-$line bundle and the generic fiber is K-semistable. The analog of the inequality in Conjecture 1.1 does hold in this situation, but not the uniqueness statement. More precisely, if $(X, -K_X)$ is assumed K-semistable then it follows from [31] (see the beginning of [31, Section 1.7.1]) that

$$(-\mathscr{K}_{\mathscr{X}/\mathscr{B}})^{n+1} \leq 0. \tag{6.1}$$

Equality holds for the trivial fibrations $\mathscr{X} = X \times \mathscr{B}$ for any K-semistable X. In particular,

$$(-\mathscr{K}_{\mathscr{X}/\mathscr{B}})^{n+1} \leq (-\mathscr{K}_{\mathbb{P}^n \times \mathscr{B}/\mathscr{B}})^{n+1} (= 0) \tag{6.2}$$

which is the function field analog of the inequality in Conjecture 1.1. Note that when $\mathscr{B} = \mathbb{P}^1$ and the standard $\mathbb{C}^*-$action on $\mathbb{P}^1$ lifts to $\mathscr{X}$, the inequality 6.1 follows directly from the definition of K-semistability.

Remark 6.1. The analog of the volume-normalization (appearing in Conjecture 1.1) is automatically satisfied in the function field case. Indeed, the second term in the corresponding Ding functional $\mathcal{D}_{(\mathcal{X}_{\mathscr{X}/\mathscr{B}}, -\mathscr{K}_{\mathscr{X}/\mathscr{B}})}$, discussed in the following section, then vanishes.

In contrast to Conjecture 1.1 projective space thus plays no special role in the function field case (since equality holds in the inequality 6.2 for *any* product $\mathscr{X} = X \times \mathscr{B}$). Conversely, it should be stressed that the analog of the inequality 6.1 *fails* in the arithmetic situation (by the strict positivity in Lemma 3.6). Hence, the function field analogy is somewhat deceptive. Our general motivation for Conjecture 1.1 is rather the analogy with the corresponding result over $\mathbb{C}$ (corresponding to the trivial morphism $X \to \text{Spec } \mathbb{C}$) and the fact that projective space maximizes the degree of $-K_X$ [46], among K-semistable X of a given dimension (as well as a range of other positivity properties of $-K_X$; see, for example, the discussion and references in the introduction of [62]).

6.2. **A generalization of Conjecture 1.1.** Consider a Fano variety X_F defined over a number field F, i.e. a field extension F of $\mathbb{Q}$ of finite degree $[F : \mathbb{Q}]$. Let $(\mathcal{X}, \mathcal{L})$ be a normal irreducible model of $(X_F, -K_{X_F})$ over the ring of integers $\mathcal{O}_F$ of F such that $\mathcal{K}_{\mathcal{X}/\text{Spec}\mathcal{O}_F}$ is defined as a $\mathbb{Q}-$line bundle. We will denote by ψ a collection of continuous psh ψ_σ metrics on $-K_{X_\sigma}$ as σ ranges over all embeddings of the field F into $\mathbb{C}$, where X_σ denotes the corresponding complex projective varieties. To the model $(\mathcal{X}, \mathcal{L})$ we attach an *arithmetic Ding functional,* defined as follows. First consider a model $(\mathcal{X}, \mathcal{L})$ of $(X_F, -K_{X_F})$ such that $\mathcal{L} + \mathcal{K}_{\mathcal{X}/\text{Spec}\mathcal{O}_F}$ defines a bona fide line bundle. Then

$$\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi) := -\frac{(\mathcal{L}, \psi)^{n+1}}{[F : \mathbb{Q}](n+1)(-K_X)^n} + \frac{1}{[F : \mathbb{Q}]}\widehat{\deg}\pi_*(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\text{Spec}\mathcal{O}_F}),$$

where the second term above denotes the arithmetic (Arakelov) degree of the line bundle $\pi_*(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\text{Spec}\mathcal{O}_F}) \to \text{Spec}\mathcal{O}_F$, endowed with the L^2-metric induced by the metric ψ on $\mathcal{L}$ (i.e. on $-K_X$). More generally, when $\mathcal{K}_{\mathcal{X}/\text{Spec}\mathcal{O}_F}$ is merely defined as a $\mathbb{Q}-$line bundle we fix a positive integer r such that $r(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\text{Spec}\mathcal{O}_F})$ is defined as a line bundle and replace

$\widehat{\deg}\pi_* H^0(\mathcal{X}, (\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})$ with $r^{-1}\widehat{\deg}\pi_* H^0(\mathcal{X}, (r(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})),$ where now $\pi_*(r(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}))$ is endowed with the $L^{2/r}-$metric induced by ψ. Concretely, this means that

$$2r^{-1}\widehat{\deg}\pi_*(r(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})) = -\sum_\sigma \log \int_{X_\sigma} |s_r|^{2/r} e^{-\psi_\sigma}, \tag{6.3}$$

where s_r is a generator of the rank one $\mathcal{O}_F-$module $H^0(\mathcal{X}, r(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}))$ and $|s_r|^{2/r}e^{-\psi_\sigma}$ denotes corresponding measure on X_σ.

The functional $\mathcal{D}_{(\mathcal{X},\mathcal{L})}$ coincides with the functional $\mathcal{D}_\mathbb{Z}$, defined in formula 2.15, up to an additive constant and a factor of two. Indeed, replacing s_r with $1 \in H^0(X_\sigma, \mathbb{C})$ in formula 6.3 and applying the product formula in $\mathcal{O}_F$ gives

$$[F : \mathbb{Q}]\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi) := \mathcal{D}_\mathbb{Z}(\psi)/2 + \frac{1}{r}\log|p|\sum_p \mathrm{ord}_p(1), \tag{6.4}$$

where $\mathrm{ord}_p(1)$ denotes the order of vanishing at the closed point p in $\mathrm{Spec}\mathcal{O}_F$ of the rational section "1" of the line bundle $\pi_*(r(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})) \to \mathrm{Spec}\mathcal{O}_F$ coinciding with $1 \in H^0(X_\mathbb{Q})$ on the generic fiber and $|p|$ denotes the norm of the ideal in $\mathcal{O}_F$ defined by p (i.e., the cardinality of the corresponding residue field; in particular, $\log|p| \geq 0$).

Remark 6.2. The functional $\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi)$ is the arithmetic analog of the degree of the Ding line bundle of a test configuration $(\mathscr{X}, \mathscr{L})$ for $(X, -K_X)$ introduced in [6]. As shown in [47] a Fano variety X is K-semistable iff the degree of the Ding line bundle is non-negative for any test configuration $(\mathscr{X}, \mathscr{L})$.

Now consider the following invariant of the Fano variety X_F :

$$\mathcal{D}(X_F) := (-K_X)^n \inf \mathcal{D}_{(\mathcal{X},\mathcal{L})},$$

where the inf runs over all integral models $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ and metrics ψ as above.

Conjecture 6.3. *Let X_F be a K-semistable Fano variety defined over a number field F. Then*

$$\mathcal{D}(X_F) \geq (-K_{\mathbb{P}^n})^n \mathcal{D}_{(\mathbb{P}^n_\mathbb{Z}, -\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}})}(\psi_{FS}),$$

where ψ_{FS} denotes the volume-normalized Fubini-Study metric ψ_{FS} on $-K_{\mathbb{P}^n}$. Equivalently, for any model $(\mathcal{X}, \mathcal{L})$ and continuous psh metric ψ, normalized so that $\widehat{\deg}\pi_* H^0(\mathcal{X}, (\mathcal{L}+\mathcal{K}_{\mathcal{X}/Spec\mathcal{O}_F}) = 0$,

$$\frac{1}{[F : \mathbb{Q}]}(\mathcal{L}, \psi)^{n+1} \leq (-\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}, \psi_{FS})^{n+1}.$$

Moreover, equality holds if and only if $(\mathcal{X}, \mathcal{L}) = (\mathbb{P}^n_\mathbb{Z}, -\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}})$ and ψ coincides with ψ_{FS}, up to the action of an automorphism of $\mathbb{P}^n$.

For example, by formula 6.4, when $F = \mathbb{Q}$ and $\mathcal{L}$ equals $-\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$ the second inequality in the previous conjecture specializes to Conjecture 1.1 if $\pi_*\mathcal{O}_\mathcal{X}$ coincides with $\mathcal{O}_F$ (which ensures that the integral lattice in $H^0(X, \mathcal{O}_X)$ corresponding to $H^0(\pi_*\mathcal{O}(\mathcal{X}))$ is generated by the constant function 1 on X). This condition can always be achieved after a base change, using Stein factorization (thanks to Dennis Eriksson and Gerard Freixas i Montplet for pointing this out). We expect that any integral model $(\mathcal{X}, \mathcal{L})$ which is globally K-semistable realizes the infimum defining the invariant $\mathcal{D}(X_F)$, inspired by Odaka's conjecture discussed in Section 1.4.

In general, the following inequality between the arithmetic Mabuchi functional and the arithmetic Ding functional holds, showing, in particular, that Conjecture 6.3 implies Conjecture 1.6 concerning Odaka's modular invariant.

Proposition 6.4. *Let $(\mathcal{X},\mathcal{L})$ be a normal irreducible model of $(X,-K_X)$ over Spec $\mathcal{O}_F$ which is $\mathbb{Q}-$Gorenstein. Then*

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) \geq \mathrm{vol}(\text{-}K_X)\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi)$$

with equality iff ψ is a Kähler-Einstein metric and $\mathcal{L} = -\mathcal{K}_{\mathcal{X}/Spec\mathcal{O}_F}$.

Proof. To simplify the notation we assume that $r=1$ and $F=\mathbb{Q}$ (but the proof in the general case is essentially the same). Let s be a generator of the rank one module $H^0(\mathcal{X},\mathcal{L}+\mathcal{K})$, where $\mathcal{K} := \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$. It follows directly from the definitions that we need to prove that

$$\frac{1}{n!L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n - \widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}) \geq 0$$

with equality iff the conditions in the proposition hold. After scaling the metric we may as well assume that $\widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}) = 0$, i.e. that $|s|^2e^{-\psi}$ is a probability measure on X. Then

$$\frac{2}{n!L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n \geq \int_X \log(\frac{MA(\psi)}{|s|^2e^{-\psi}})MA(\psi) =: \mathrm{Ent}_{|s|^2e^{-\psi}}(MA(\psi))$$

Indeed, by the restriction formula for arithmetic intersection numbers [19, Prop 2.3.1]

$$\frac{2}{n!L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n = \int_X \log\left(\frac{MA(\psi)}{|s|^2e^{-\psi}}\right)MA(\psi) + (s=0)\cdot\overline{\mathcal{L}}^n, \tag{6.5}$$

where $(s=0)$ denotes the subscheme of $\mathcal{X}$ cut out by s. The second term above is non-negative since $(s=0)$ is supported on the closed fibers (by assumption s is non-vanishing over the generic fiber). Moreover, since $\mathcal{L}$ is relatively ample the term vanishes iff s is globally non-vanishing, i.e. $\mathcal{L}+\mathcal{K}$ is trivial. Finally, the first term above is proportional to the relative entropy $\mathrm{Ent}_{|s|^2e^{-\psi}}(MA(\psi))$ between the probability measures $MA(\psi)$ and $|s|^2e^{-\psi}$ and thus non-negative and vanishes iff the two probability measures coincide, i.e. ψ is Kähler-Einstein. □

6.3. **Comparison with bounds on Bost-Zhang's normalized heights.** The arithmetic Ding functional $\mathcal{D}_{(\mathcal{X},\mathcal{L})}$ is reminiscent of Bost's normalized height h_{norm}, introduced in [18] in the general setup of polarized variety (X_F,L_F) defined over a number field F :

$$h_{\mathrm{norm}}(\mathcal{L},\psi) := \frac{(\mathcal{L},\psi)^{n+1}}{[F:\mathbb{Q}](n+1)(L_F)^n} - \frac{1}{[F:\mathbb{Q}]N}\widehat{\deg}\pi_*(\mathcal{X},\mathcal{L}),$$

assuming that the rank N of the vector bundle $\pi_*(\mathcal{X},\mathcal{L}) \to \mathrm{Spec}\mathcal{O}_F$ is non-zero and $\pi_*(\mathcal{X},\mathcal{L})$ is endowed with the L^2-norm induced by the continuous psh metrics ψ_σ on L_σ and the volume forms $MA(\psi_\sigma)$ on X_σ (defined by formula 4.3). When L_F is very ample it is shown in [18] that the functional $h_{\mathrm{norm}}(\mathcal{L},\cdot)$ is bounded from below iff the Chow point of (X_F,L_F) is semistable wrt the action of the group $GL(N,F)$ on the Chow variety (in the sense of Geometric Invariant Theory). More precisely, it it shown in [18] that the semi-stability in question is equivalent to a lower bound on Bost's intrinsic normalized height of (X_F,L_F) :

$$\inf h_{\mathrm{norm}} > -\infty$$

where the infimum runs over all models $(\mathcal{X},\mathcal{L})$ and metrics ψ as above. In fact, by [18, Prop 2.1] and [101, Thm 4.4] the Chow-semistability in question is equivalent to the following explicit lower bound:

$$h_{\mathrm{norm}}(\mathcal{L},\psi) \geq -\frac{1}{2}\sum_{n=1}^{N+1}\sum_{m=1}^{n}\frac{1}{m} - \frac{1}{2}\log N \tag{6.6}$$

(it is moreover conjectured in [101] that the first term in the right hand side above may be replaced by 0).

In this setup the role of the normalization $\widehat{\deg}\pi_* H^0(\mathcal{X}, (\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}) = 0$ in Conjecture 6.3 is thus played by the normalization $\widehat{\deg}\pi_* H^0(\mathcal{X}, \mathcal{L}) = 0$. However, in contrast to Conjecture 6.3 the lower bound 6.6 on $h_{\mathrm{norm}}(\mathcal{L}, \psi)$ corresponds to a *lower* bound on $(\mathcal{L}, \psi)^{n+1}$ for any normalized metric. Note also that one virtue of the normalization condition in Conjecture 6.3 is that it is comparatively explicit, since $\pi_*(\mathcal{L} + \mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})$ has rank one (so that formula 6.3 applies, showing that it is enough to assume that the volume forms $|s_r|^{2/r} e^{-\psi_\sigma}$ on X_σ are normalized). Another advantage of this normalization condition is that it applies to any continuous metric ψ (at the price of replacing $(\mathcal{L}, \psi)^{n+1}$ with the $\chi-$arithmetic volume of $\mathcal{L}$, as in Theorem 2.4).

Finally, we recall that when $\mathcal{L}$ is replaced by $k\mathcal{L}$ for a large positive integer k it follows from [77, Thm 3.7] that there exists constants a and b (depending only on (X_F, L_F)) such that $a > 0$

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)/L^n = h_{\mathrm{norm}}(k\mathcal{L}, \psi) - a \log N_k + b + o(1), \tag{6.7}$$

as $k \to \infty$, where N_k denotes the rank of $\pi_* H^0(\mathcal{X}, k\mathcal{L})$ which diverges as $k \to \infty$. Unfortunately, the diverging term $a \log N_k$ makes it impossible to infer lower bounds on $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ from lower bounds on $h_{\mathrm{norm}}(k\mathcal{L})$. Since $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ coincides with $\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi)$ when $\mathcal{L}$ equals $-\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$ this means that Conjecture 6.3 can not be deduced from bounds of the form 6.6 by letting k (and hence N) tend to infinity.

6.4. **Comparison with Odaka's and Faltings' modular heights.** Finally, let us compare our normalizations of the arithmetic Mabuchi functional with those of Odaka [78] and Faltings [44]. First of all our multiplicative normalization for the arithmetic Mabuchi functional $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ (formula 1.6) are made so that $\pm\mathcal{M}_{(\mathcal{X},\pm K_{\mathcal{X}})} = (\pm\mathcal{K}_{\mathcal{X}})^{n+1}/(n+1)!$, assuming that $\mathbb{F} = \mathbb{Q}$ (in the general case $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ is divided by $[\mathbb{F} : \mathbb{Q}]$). Moreover, as discussed in Section 1.4.1, we are employing the metric on $-K_X$ induced by the *normalized* volume form ω^n/L^n of the Kähler form ω defined by a given metric ψ on $\mathcal{L}$ with positive curvature (i.e. $\omega = dd^c\psi$). Comparing with Odaka's arithmetic Mabuchi functional, that we shall denote by $\mathcal{M}^{(O)}_{(\mathcal{X},\mathcal{L})}(\psi)$, thus yields

$$\frac{1}{(n+1)!L^n}\mathcal{M}^{(O)}_{(\mathcal{X},\mathcal{L})} = \mathcal{M}_{(\mathcal{X},\mathcal{L})} + \frac{1}{2}\frac{L^n}{n!}\log(L^n/n!). \tag{6.8}$$

In the case that $\mathcal{X}$ is an abelian variety it was shown in [78] that the infimum of Odaka's arithmetic Mabuchi functional over all metrics on $\mathcal{L}$ with positive curvature coincides with Faltings' (modular) height [44], up to a multiplicative and additive constants depending on L^n. Here we note that our additive normalizations are consistent with those of Faltings:

Proposition 6.5. *Let $\mathcal{X}$ be a projective and flat scheme over $\mathbb{Z}$ and assume that $\mathcal{K}_{\mathcal{X}}$ is trivial. For any relatively ample line bundle $\mathcal{L}$ over $\mathcal{X}$*

$$\inf_{\psi} \frac{1}{V}\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) = -\frac{1}{2[\mathbb{F} : \mathbb{Q}]}\log\frac{1}{2^n}\left|\int_{X(\mathbb{C})}\Omega \wedge \bar{\Omega}\right|, \tag{6.9}$$

where Ω is a generator of $H^0(\mathcal{X}, \mathcal{K}_{\mathcal{X}})$ and the inf ranges over all psh metrics ψ on $\mathcal{L}$ and $V := L^n/n!$.

Proof. This is essentially equivalent to [78, Thm 2.11], using the relation 6.8. Anyhow, in order to verify that all normalizations are consistent we provide a simple direct proof. Assume, to simplify the notation, that $\mathbb{F}=\mathbb{Q}$. Recall that Faltings' modular height [44] is defined as the arithmetic degree of $\pi_*(\mathcal{X},K_{\mathcal{X}})$, with respect to the L^2-metric on $H^0(X,K_X)$ defined by $\|\Omega\|^2:=\frac{1}{2^n}\left|\int_{X(\mathbb{C})}\Omega\wedge\bar{\Omega}\right|$. This is precisely the right hand side in formula 6.9. As for the the the left hans side it is is given by

$$\int_X \log\left(\frac{(dd^c\psi)^n/Vn!}{\frac{i^{n^2/2}}{2^n}\Omega\wedge\bar{\Omega}/\|\Omega\|^2}\right)\frac{(dd^c\psi)^n}{Vn!}=\int_X \log\left(\frac{(dd^c\psi)^n/Vn!}{\frac{i^{n^2/2}}{2^n}\Omega\wedge\bar{\Omega}/\|\Omega\|^2}\right)\frac{(dd^c\psi)^n}{Vn!}-\log\|\Omega\|^2.$$

(as follows readily from the definitions, just as in formula 6.5). Now, by Jensen's inequality this expression is minimal precisely when the two probability measures $(dd^c\psi)^n/Vn!$ and $2^{-n}i^{n^2/2}\Omega\wedge\bar{\Omega}/\|\Omega\|^2$ coincide, which, equivalently, means that $dd^c\psi$ is a Kähler-Einstein metric. By the Calabi-Yau theorem such a metric exists for any given ample L, which concludes the proof. $\square$

The previous proposition has the following consequence, when combined with well-known properties of Faltings' modular height of abelian varieties (cf. the discussion in relation to [78, Thm 2.11] and [78, Section 2.3.2]). Consider a polarized abelian variety $(X_{\mathbb{F}_0},L_{\mathbb{F}_0})$ defined over a given number field $\mathbb{F}_0$. Then the infimum of $\mathrm{vol}(L)^{-1}\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ over all metrics, finite field extensions $\mathbb{F}$, models over $\mathcal{O}_{\mathbb{F}}$ and positively curved metrics on $L\to X_{\mathbb{F}}(\mathbb{C})$ is attained at any semi-stable reduction of the Néron model $\mathcal{X}$ of $X_{\mathbb{F}}$, when L is endowed with a Kähler-Einstein metric. Moreover, in the particular case of elliptic curves it was observed in [35, page 29] that the minimal value of the aforementioned infimum over all $X_{\mathbb{F}}$ is attained at the semistable reduction of the Néron model $\mathcal{X}_0$ of any elliptic curve with vanishing $j-$invariant ($\mathcal{X}_0$ is uniquely determined for any sufficently large field extension). Thus the role of $\mathcal{X}_0$ among all models of elliptic curves, is somewhat analogous to the role of $\mathbb{P}^n_{\mathbb{Z}}$ in Conjectures 1.1, 1.6. However, it should be stressed that in the setup of Fano varieties the choice of multiplicative normalization is crucial. Indeed, while $\mathbb{P}^n_{\mathbb{Z}}$ minimizes $\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi_{\mathrm{KE}})$ over the canonical toric integral models of all K-semistable toric Fano varieties X (assuming that $n\leq 6$) it does not minimize $\mathrm{vol}(-K_X)^{-1}\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi_{\mathrm{KE}})$. In fact, for all we know it could actually be the case that $\mathrm{vol}(-K_X)^{-1}\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi_{\mathrm{KE}})$ is *maximal* on $\mathbb{P}^n_{\mathbb{Z}}$. For example, this turns out to be the case in the more general setup of Fano orbifolds (not assumed toric) when X has relative dimension one (a proof will appear in a separate publication).

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Paper 2

SHARP BOUNDS ON THE HEIGHT OF K-SEMISTABLE TORIC FANO VARIETIES II, THE LOG CASE

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Abstract. In our previous work we conjectured - inspired by an algebro-geometric result of Fujita - that the height of an arithmetic Fano variety $\mathcal{X}$ of relative dimension n is maximal when $\mathcal{X}$ is the projective space $\mathbb{P}^n_{\mathbb{Z}}$ over the integers, endowed with the Fubini-Study metric, if the the corresponding complex Fano variety is K-semistable. In this work the conjecture is settled for diagonal hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{Z}}$. The proof is based on a logarithmic extension of our previous conjecture, of independent interest, which is established for toric log Fano varieties of relative dimension at most three, hyperplane arrangements on $\mathbb{P}^n_{\mathbb{Z}}$, as well as for general logarithmic arithmetic Fano surfaces.

1. Introduction

This is a sequel to [2], where a conjectural arithmetic analog of Fujita's sharp bound for the degree (volume) of K-semistable Fano varieties over $\mathbb{C}$ was proposed, concerning arithmetic Fano varieties $\mathcal{X}$. In [2] the case when $\mathcal{X}$ is the canonical integral model of a toric Fano variety X was settled when the relative dimension n is at most six (the extension to any n is conditioned on a conjectural gap hypothesis for the algebro-geometric degree). Here we will, in particular, show that the conjecture introduced in [2] holds for any diagonal Fano hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ (see Section 1.1.3 below). The proof is based on the following extension of the conjecture in [2] to the logarithmic setting, which is the main focus of the present work:

Conjecture 1.1. *Let $(\mathcal{X},\mathcal{D})$ be an arithmetic log Fano variety. Then the following inequality of arithmetic intersection numbers holds for any volume-normalized continuous metric on $-(K_X+\Delta)$ with positive curvature current if (X,Δ) is K-semistable:*

$$(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1} \leq (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1},$$

where $-K_{\mathbb{P}^n_{\mathbb{C}}}$ is endowed with the volume-normalized Fubini-Study metric. Moreover, if $\mathcal{X}$ is normal equality holds if and only if $(\mathcal{X},\mathcal{D}) = (\mathbb{P}^n_{\mathbb{Z}},0)$ and the metric is Kähler-Einstein, i.e. coincides with the Fubini-Study metric up to the action of an automorphism of $\mathbb{P}^n_{\mathbb{C}}$.

By definition, an *arithmetic log Fano variety* $(\mathcal{X},\mathcal{D})$ is a projective flat scheme $\mathcal{X}$ over $\mathbb{Z}$ together with an effective $\mathbb{Q}-$divisor $\mathcal{D}$ on $\mathcal{X}$ such that

$$-\mathcal{K}_{(\mathcal{X},\mathcal{D})} := -(\mathcal{K}_{\mathcal{X}} + \mathcal{D})$$

defines a relatively ample $\mathbb{Q}-$line bundle, where $\mathcal{K}_{\mathcal{X}}$ denotes the relative canonical divisor on $\mathcal{X}$. We also assume that the corresponding complex variety X is normal and thus defines a complex Fano variety. Following standard procedure we denote by $\overline{\mathcal{L}}$ a metrized line bundle, i.e. a line bundle $\mathcal{L}$ on $\mathcal{X}$ endowed with an Hermitian metric over the complex points X of $\mathcal{X}$. Arithmetic intersection numbers of metrized line bundles were introduced by Gillet-Soulé in the context of Arakelov geometry [7]. The top arithmetic intersection number of $\overline{\mathcal{L}}$ is called the *height* of $\overline{\mathcal{L}}$. The height of $\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}$ with respect to the volume-normalized Fubini-Study metric, appearing in the previous conjecture, is explicitly given by the following formula [1, Lemma 3.6], which, essentially, goes back to [20, §5.4]:

$$(1.1) \qquad (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1} = \frac{1}{2}(n+1)^{n+1}\left((n+1)\sum_{k=1}^{n} k^{-1} - n + \log(\frac{\pi^n}{n!})\right).$$

As for the notion of K-stability it originally appeared in the context of the Yau-Tian-Donaldson conjecture for Fano manifolds X (see the survey [49] for recent developments, including connections to moduli spaces and the minimal model program in birational geometry). By [30] and [32, Thm 1.6] a log Fano variety (X,Δ) over $\mathbb{C}$ is *K-polystable* (which is a slightly stronger condition than K-semistability) if and only if it admits a *log Kähler-Einstein metric,* i.e. a locally bounded metric on X whose curvature current ω induces a Kähler metric with constant Ricci curvature on the complement of Δ in the regular locus of X. After volume-normalization such a metric maximizes the height $(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}$ among all volume-normalized locally bounded metrics on $-(K_X+\Delta)$ with positive curvature (as shown precisely as in the case that $\mathcal{D}=0$ considered in [2, Section 2.3]).

The K-semistability of (X,Δ) implies that (X,Δ) is Kawamata Log Terminal (klt) in the usual sense of birational geometry (see Remark 2.1). An important class of klt log Fano varieties (X,Δ) is provided by (smooth) *Fano orbifolds*, where the coefficients of Δ are of the form $(1-1/m_i)$ for positive integers m_i. Diophantine aspects of Fano orbifolds have recently been explored in a number of works, building on Campana's program [13] and its developments by Abramovich [1] (see [46] for a very recent survey). In particular, a logarithmic generalization of the Manin-Peyre conjecture for the density of rational points of bounded height on Fano varieties is proposed in [41], which, for example, is addressed for log Fano hyperplane arrangements and toric varieties in [11] and [40], respectively. See [4] for relations between height bounds, K-stability and the Manin-Peyre conjecture.

1.1. **Main results.**

1.1.1. *Toric log Fano varieties.* We first consider the case when $(\mathcal{X},\mathcal{D})$ is the canonical integral model of a toric log Fano variety (X,Δ) (see [34, Section 2] and [12, Def 3.5.6]). One advantage of the logarithmic setup is that on any given toric Fano variety X there exist an infinite number of toric $\mathbb{Q}-$divisors D such that $-(K_X+\Delta)$ is a K-semistable log Fano variety. Building on [1], where the case when $D=0$ was considered, we show

Theorem 1.2. *Let $(\mathcal{X},\mathcal{D})$ be the canonical integral model of a K-semistable toric log Fano variety (X,Δ). Conjecture 1.1 holds for $(\mathcal{X},\mathcal{D})$ under anyone of the following conditions:*

- *$n\leq 3$ and X is $\mathbb{Q}-$factorial (equivalently, X has at worst abelian quotient singularities)*
- *X is not Gorenstein or has some abelian quotient singularity*

The starting point of the proof is the bound

$$(1.2) \qquad \frac{(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}}{(n+1)!} \leq \frac{1}{2}\mathrm{vol}(X,\Delta)\log\left(\frac{(2\pi^2)^n}{\mathrm{vol}(X,\Delta)}\right) \quad \mathrm{vol}(X,\Delta) := \frac{-(K_X+\Delta)^n}{n!},$$

shown precisely as in the case when $\Delta=0$, considered in [1]. For $X=\mathbb{P}^n$ the previous theorem is verified by an explicit calculation. In the remaining case, $X\neq\mathbb{P}^n$, the bound in Conjecture 1.1 follows, just as in [1], from combining the bound 1.2 with the following logarithmic analog of the "gap hypothesis" introduced in [1]:

$$(1.3) \qquad \mathrm{vol}(X,\Delta) \leq \mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)$$

for any K-semistable $n-$dimensional Fano variety (X,Δ) such that $X \neq \mathbb{P}^n$. In the case that X is singular the logarithmic gap hypothesis does hold in any dimension, just as in [1]. In the non-singular case there is, for any dimension, only a finite number of toric Fano varieties X. For $n \leq 6$ these appear in the database [52], which, as observed in [1], settles the gap hypothesis for $n \leq 6$, when $\Delta = 0$. However, in the present case there is for any given toric variety X an infinite number of toric divisors Δ on X such that (X,Δ) is a K-semistable Fano variety. In order to establish the logarithmic gap-hypothesis 1.3 we thus introduce the following invariant of a Fano manifold X :

$$S(X) := \sup_{\Delta} \{\mathrm{vol}(X,\Delta) : \, (X,\Delta) \, \text{K-semistable} \log \text{Fano}\}$$

and show, by solving the corresponding optimization problem, that $S(X) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$ when $X \neq \mathbb{P}^n$ and $n \leq 3$.

The invariant $S(X)$ and the corresponding maximizers Δ appear to be of independent interest in Kähler geometry. This is illustrated by some examples in Section 3.1, where we make contact with a rigidity property of the corresponding log Kähler-Einstein metric, first exhibited in [43].

1.1.2. *Hyperplane arrangements.* We next turn to the case when $\mathcal{X}$ is the projective space over the integers, $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and $\mathcal{D}$ is a hyperplane arrangement, i.e. its irreducible components are hyperplanes.

Theorem 1.3. *Conjecture 1.1 holds when $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and $\mathcal{D}$ is a hyperplane arrangement with simple normal crossings.*

The proof employs a convexity argument to reduce the problem to the case when $\mathcal{D}$ is toric, which is covered by Theorem 3.1. The argument leverages the explicit characterization of K-semistable hyperplane arrangements established in [24] and yields the following explicit bound:

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}}{(n+1)!} \leq \frac{1}{2}\mathrm{vol}(X,\Delta)\log\left(\frac{(n+1)^n e^{2a_n}}{(n+1)!\mathrm{vol}(X,\Delta)}\right), \quad a_n = \frac{(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}}{(n+1)^{n+1}} \tag{1.4}$$

with equality iff $\mathcal{D}$ is toric.

1.1.3. *Application to diagonal hypersurfaces.* Given a positive integer d and integers a_i, consider the diagonal hypersurface $\mathcal{X}_a$ of degree d in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by the homogeneous polynomial

$$\sum_{i=0}^{n+1} a_i x_0^d,$$

The corresponding complex variety X_a is Fano if and only if $d \leq (n+1)$ and is always K-polystable (and, in particular, K-semistable); see, for example, [51] for an algebraic proof. Using the results stated in the previous two sections we will establish Conjecture 1.1 for $\mathcal{X}_a$, endowed with the trivial divisor 0 :

Theorem 1.4. *Conjecture 1.1 holds for any diagonal hypersurface $\mathcal{X}_a$ which is Fano (i.e. $d \leq n+1$) when the divisor $\mathcal{D}$ is trivial. More precisely,*

$$(\overline{-\mathcal{K}_{\mathcal{X}_a}})^{n+1} \leq (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1} + (1-d)(n+2-d)^n \sum_{i=0}^{n+1} \log|a_i|.$$

and the inequality is strict if $d \geq 2$.

Note that the schemes $\mathcal{X}_a$ are mutually non-isomorphic over $\mathbb{Z}$, for any given degree d of at least two. In fact, in general, they are not even isomorphic over $\mathbb{Q}$. The proof of the previous theorem is first reduced to the case of a Fermat hypersurface, i.e. the case when $a_i = 1$. Expressing $\mathcal{X}_a$ as a Galois cover of $\mathbb{P}^n_{\mathbb{Z}}$ the estimate 1.4 can then be applied with $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and Δ the corresponding branching divisor, which reduces the problem to a simple toric case.

We recall that the Manin-Peyre conjecture has been settled for Fano hypersurfaces of sufficiently small degree [39]. In particular, there is an extensive literature on the diagonal case (see, in particular, [9] for Diophantine result with respect to random coefficients a_i).

1.1.4. *Arithmetic log surfaces.* Consider a polarized arithmetic log surface $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, i.e. an arithmetic log surface $(\mathcal{X}, \mathcal{D})$ endowed with a relatively ample line bundle $\mathcal{L}$. Assume that the complexification $\mathcal{L} \otimes \mathbb{C}$ is isomorphic to $-K_{(X,\Delta)}$ (where (X, Δ) denotes, as before, the complexification of $(\mathcal{X}, \mathcal{D})$). The *arithmetic log Mabuchi functional* of $(\mathcal{X}, \mathcal{D}; \overline{\mathcal{L}})$ is defined by

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}}) := \frac{1}{2}\overline{\mathcal{L}}^2 + \overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})} \cdot \overline{\mathcal{L}}, \tag{1.5}$$

where $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$ is endowed with the volume-normalized metric induced by the curvature current ω of $\overline{\mathcal{L}}$ (when $\mathcal{D} = 0$ this coincides, up to normalization, with the arithmetic Mabuchi functional introduced in [38]). For a given integral model $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ is minimized on a log Kähler-Einstein metric, if such a metric exists, and then

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}}) = -(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^2/2,$$

if the metric is volume-normalized and $\mathcal{L} = -\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$.

Theorem 1.5. *Let $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ be a polarized arithmetic log surface $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ such that the complexification (X, Δ) of $(\mathcal{X}, \mathcal{D})$ is a K-semistable Fano variety and $\mathcal{L} \otimes \mathbb{C} = -K_{(X,\Delta)}$. Then*

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}}) \geq \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},0)}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}}) \quad (= -1 - \log \pi)$$

where $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ is endowed with the Fubini-Study metric. Moreover, equality holds iff $(\mathcal{X}, \mathcal{D})$ is isomorphic to $(\mathbb{P}^1_{\mathbb{Z}}, 0)$ and $\mathcal{L}$ is isomorphic to $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$, endowed with a metric coinciding with the Fubini-Study metric, up to the application of an automorphism of $\mathbb{P}^1_{\mathbb{Z}}$ and a scaling of the metric.

Corollary 1.6. *Conjecture 1.1 holds for arithmetic log Fano surfaces.*

In the setup of the previous theorem the corresponding complex variety X is always equal to $\mathbb{P}^1$ and thus (X, Δ) is a hyperplane arrangement. Accordingly, applying Theorem 1.3, the proof of Theorem 1.5 is reduced to showing that the canonical integral model $(\mathcal{X}_c, \mathcal{D}_c; -\mathcal{K}_{(\mathcal{X}_c,\mathcal{D}_c)})$ of $(X, \Delta; -K_{(X,\Delta)})$ obtained by setting $\mathcal{X}_c = \mathbb{P}^1_{\mathbb{Z}}$ and taking $\mathcal{D}_c$ to be the Zariski closure of $\Delta_{\mathbb{Q}}$ in $\mathcal{X}_c$ minimizes $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ over all integral models $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of $(X, \Delta; -K_{(X,\Delta)})$, for any fixed metric on $-K_{(X,\Delta)}$. This minimization property can be viewed as logarithmic version of Odaka's minimization conjecture (proposed in dimension in [38]). Our proof builds on [38], leveraging log canonical thresholds. We also show that the minimum is uniquely attained for $(\mathbb{P}^1_{\mathbb{Z}}, 0)$. See also [22] for very recent progress on Odaka's minimization conjecture in another direction.

1.2. **Outlook on the case of $\mathbb{P}^1_{\mathbb{Z}}$ endowed with a divisor with three components.** Consider now the special case of $(\mathbb{P}^1, \mathcal{D}_c)$, where $\mathcal{D}_c$ is the canonical model of a divisor $\Delta_{\mathbb{Q}}$ on $\mathbb{P}^1$ with three components and $-K_{(\mathbb{P}^1,\Delta_{\mathbb{Q}})}$ is endowed with the volume-normalized log Kähler-Einstein metric. In this case an explicit formula for the canonical height $\overline{(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D}_c)})}^2$ is established in the sequel [3] to the present work. More precisely, expressing

$$\Delta_{\mathbb{Q}} = \sum_{i=1}^{3} w_i p_i, \;\; V = 2 - \sum_{i=1}^{3} w_i > 0$$

(where V is the algebraic degree of $-K_{(\mathbb{P}^1,\Delta_{\mathbb{Q}})}$) it is shown in [3] that, when $(\mathbb{P}_1, \Delta_{\mathbb{Q}})$ is K-polystable,

$$\frac{\overline{(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D}_c)})}^2}{2V} = \frac{1}{2}(1 + \log\pi - \log\frac{V}{2}) - \frac{\gamma(0, \frac{V}{2}) + \sum_{i=1}^{3}\gamma(w_i, w_i + \frac{V}{2})}{V}, \tag{1.6}$$

where $\gamma(a,b)$ may be expressed in terms of Hurwitz zeta function $\zeta(s,x)$ and its derivative $\zeta'(s,x)$ wrt s :

$$\gamma(a,b) = F(b) + F(1-b) - F(a) - F(1-a), \;\; F(x) := \zeta(-1,x) + \zeta'(-1,x).$$

Moreover, the right hand side of formula 1.6 is shown to extend real-analytically, wrt the coefficients w_i, to the case $K_{(\mathbb{P}^1,\Delta_{\mathbb{Q}})} > 0$ (i.e. $V < 0$), as long as $w_i \in]0,1]$. In this case formula 1.6 computes $\overline{(\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D}_c)})}^2/2V$ and can be related to Kudla's program [29] and the Maillot-Rössler conjectures [35, 36], expressing the height of Shimura varieties $(\mathcal{X},\mathcal{D})$, wrt canonical metrics, in terms of the Dedekind zeta function $\zeta_{\mathbb{F}}(s)$ of the number field $\mathbb{F}$ attached to $(\mathcal{X},\mathcal{D})$ and its derivative $\zeta'_{\mathbb{F}}(s)$ at $s = -1$. More precisely, this connection concerns the 18 Shimura varieties that are isomorphic to $(\mathbb{P}^1, \Delta)$ over $\mathbb{F}$, for a particular orbifold divisor Δ. This leads, in particular, to some intriguing connections to the recent work [50] on quaternionic Shimura curves.

1.3. **Acknowledgments.** Thanks to Dennis Eriksson, Mattias Jonsson and Yuji Odaka for discussions. This work was supported by a Wallenberg Scholar grant from the Knut and Alice Wallenberg foundation.

2. General setup

2.1. **Log Fano varieties over $\mathbb{C}$ and volume-normalized metrics on $-(K_X + \Delta)$.** A *log pair* (X,Δ) over $\mathbb{C}$ is a normal complex projective variety X together with an effective $\mathbb{Q}-$divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}-$Cartier, i.e. defines a $\mathbb{Q}-$line bundle, where K_X denotes the canonical divisor on X [27]. In the logarithmic setting this bundle plays the role of the canonical line bundle and is thus called the *log canonical line bundle* and is denoted by $K_{(X,\Delta)}$. A log pair (X,Δ) is said to be a *log Fano pair* if Δ is effective and $-(K_X+\Delta) > 0$. Any continuous metric $\|\cdot\|$ on $-(K_X+\Delta)$ induces a measure μ on X in a standard fashion. Indeed, when X is regular and $\Delta = 0$ this follows directly from the definition of metrics on $-K_X$ (see [2, Section 2.1.2]). In general, denoting by X_{reg} the regular locus of X, this construction yields a measure on $X_{reg} - \operatorname{supp}(\Delta)$ whose push-forward to X, under the inclusion map, yields a measure on X (see also [5, Section 3.1] for a slightly different representation of this measure). This measure has finite mass iff the log pair (X,Δ) is *klt* in the standard sense of birational algebraic geometry (see [5, Section 3.1] and Remark 2.1 below). A continuous metric on $-(K_X+\Delta)$ will be said to be *volume-normalized* if the corresponding measure is a probability measure.

2.1.1. *Local representations of metrics and measures.* As in [1] we will use additive notation for metrics on holomorphic line bundles $L \to X$. This means that we identify a continuous Hermitian metric $\|\cdot\|$ on L with a collection of continuous local functions ϕ_U associated to a given covering of X by open subsets U and trivializing holomorphic sections e_U of $L \to U$:

$$\phi_U := -\log(\|e_U\|^2). \tag{2.1}$$

The curvature current of the metric may then, locally, be expressed as

$$dd^c\phi_U := \frac{i}{2\pi}\partial\bar{\partial}\phi_U$$

Accordingly, as is customary, we will symbolically denote by ϕ a given continuous Hermitian metric on L and by $dd^c\phi$ its curvature current. We will denote by $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ the space of all continuous metrics on L whose curvature current is positive, $dd^c\phi \geq 0$ (which means that ϕ_U is plurisubharmonic, or psh, for short).

Given a log Fano pair (X, Δ) the measure corresponding to a given continuous metric ϕ on $-K_{(X,\Delta)}$ may be locally on X_{reg} be expressed as

$$\mu_\phi = e^{-\phi_U} |s_U|^{-2} (\frac{i}{2})^{n^2} dz \wedge d\bar{z}, \ \ dz := dz_1 \wedge \cdots \wedge dz_n$$

by taking $e_U = \partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n \otimes e_\Delta$ where e_Δ is a local trivialization of the $\mathbb{Q}-$line bundle over X_{reg} corresponding to the divisor Δ and $s_U e_\Delta$ is the (multi-valued) holomorphic section cutting out Δ.

2.1.2. *Log Kähler-Einstein metrics.* Given a log Fano pair (X, Δ) a metric ϕ on (X, Δ) is said to be a *log Kähler-Einstein metric,* if ϕ is a locally bounded metric and its curvature current $dd^c\phi$ induces a Kähler metric with constant positive Ricci curvature on the complement of Δ in X_{reg} [5]. When X is smooth any log Kähler-Einstein metric is, in fact, continuous (see [26, 21] for more general higher order regularity results).

2.1.3. *K-semistability.* We next recall the definition of K-semistability in terms of intersection numbers (see the survey [49] for more background). A *test configuration* for a log Fano pair (X, Δ) is a $\mathbb{C}^*-$equivariant normal model $(\mathscr{X}, \mathscr{L})$ for $(X, -(K_{(X,\Delta)})$ over the complex affine line $\mathbb{C}$. More precisely, $\mathscr{X}$ is a normal complex variety endowed with a $\mathbb{C}^*-$action ρ, a $\mathbb{C}^*-$equivariant holomorphic projection π to $\mathbb{C}$ and a relatively ample $\mathbb{C}^*-$equivariant $\mathbb{Q}-$line bundle $\mathscr{L}$ (endowed with a lift of ρ) :

$$\pi : \mathscr{X} \to \mathbb{C}, \quad \mathscr{L} \to \mathscr{X}, \quad \rho : \ \mathscr{X} \times \mathbb{C}^* \to \mathscr{X} \tag{2.2}$$

such that the fiber of $\mathscr{X}$ over $1 \in \mathbb{C}$ is equal to $(X, -(K_{(X,\Delta)})$. A log Fano pair (X, Δ) is said to be *K-semistable* if the Donaldson-Futaki invariants $\mathrm{DF}_\Delta(\mathscr{X}, \mathscr{L})$ are non-negative for any test configuration $(\mathscr{X}, \mathscr{L})$ of (X, Δ) :

$$n!\mathrm{DF}_\Delta(\mathscr{X}, \mathscr{L}) = \frac{n}{(n+1)}\overline{\mathscr{L}}^{n+1} + \mathscr{K}_{(\overline{\mathscr{X}},\mathscr{D})/\mathbb{P}^1} \cdot \overline{\mathscr{L}}^n,$$

where $\overline{\mathscr{L}}$ denotes the $\mathbb{C}^*-$equivariant extension of $\mathscr{L}$ to the $\mathbb{C}^*-$equivariant compactification $\overline{\mathscr{X}}$ of $\mathscr{X}$ over $\mathbb{P}^1$ and $\mathscr{K}_{(\overline{\mathscr{X}},\mathscr{D})/\mathbb{P}^1}$ denotes the relative log canonical divisor of the pair $(\overline{\mathscr{X}},\mathscr{D})$ with $\mathscr{D}$ denoting the flat closure in $\overline{\mathscr{X}}$ of the $\mathbb{C}^*-$orbit of the divisor Δ.

Remark 2.1. If a log Fano variety (X, Δ) is K-semistable, then (X, Δ) is klt [10, Cor 9.6]. When X is non-singular and Δ has simple normal crossings this means that all the coefficients of Δ along its irreducible components are strictly smaller than 1.

2.2. **Arithmetic log Fano varieties.** As explained in the book [28] the notion of log pairs can be extended to schemes over excellent rings. Here we will consider the case when the ring in question is $\mathbb{Z}$. Henceforth, $\mathcal{X}$ will denote an *arithmetic variety,* i.e. a projective flat scheme $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ of relative dimension n such that $\mathcal{X}$ is reduced and satisfies Serre's conditions S_2 (this is, for example, the case if $\mathcal{X}$ is normal). We will denote by π the corresponding structure morphism to Spec $\mathbb{Z}$,

$$\pi : \mathcal{X} \to \operatorname{Spec} \mathbb{Z}.$$

A *log pair* $(\mathcal{X}, \mathcal{D})$ over $\mathbb{Z}$ (also called an *arithmetic log variety*) of relative dimension n is an arithmetic variety $\mathcal{X}$ endowed with an effective $\mathbb{Q}-$divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{K}_{\mathcal{X}} + \mathcal{D}$ is $\mathbb{Q}-$Cartier, i.e. defines a $\mathbb{Q}-$line bundle, where $\mathcal{K}_{\mathcal{X}}$ denotes the relative canonical divisor on $\mathcal{X}$ (see [28, Section 1.1]). The complexification of $(\mathcal{X}, \mathcal{D})$ will be denoted by (X, Δ) and $(\mathcal{X}, \mathcal{D})$ will be called an *integral model of* (X, Δ) (although, strictly speaking, $(\mathcal{X}, \mathcal{D})$ is an integral model of the corresponding log pair over $\mathbb{Q}$). A log pair $(\mathcal{X}, \mathcal{D})$ over $\mathbb{Z}$ will be called an *arithmetic log Fano variety* if $-(\mathcal{K}_{\mathcal{X}} + \mathcal{D})$ is relatively ample and the corresponding complex variety X is normal. In particular, (X, Δ) is a log Fano variety over $\mathbb{C}$.

More generally, an arithmetic variety $\mathcal{X}$ endowed with a relatively ample line bundle $\mathcal{L}$ will be said to be *polarized.*

2.3. **Arithmetic intersection numbers and heights.** We recall some well-known facts about heights (see[2] for more background and references). A *metrized line bundle* $\overline{\mathcal{L}}$ is a line bundle $\mathcal{L} \to \mathcal{X}$ such that the corresponding line bundle $L \to X$ is endowed with a metric, that we shall denote by ϕ (as in Section 2.1.1); $\overline{\mathcal{L}} := (\mathcal{L}, \phi)$. The $\chi-$*arithmetic volume* of a polarized arithmetic variety $(\mathcal{X}, \mathcal{L})$ is defined by

$$\widehat{\operatorname{vol}}_{\chi}\left(\overline{\mathcal{L}}\right) := \lim_{k\to\infty} k^{-(n+1)} \log \operatorname{Vol}\left\{ s_k \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} : \sup_X \|s_k\|_{\phi} \leq 1 \right\}, \tag{2.3}$$

where $H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R}$ may be identified with the subspace of real sections in $H^0(X, kL)$. More generally, $\widehat{\operatorname{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)$ is naturally defined for $\mathbb{Q}-$line bundles, since it is homogeneous with respect to tensor products of $\overline{\mathcal{L}}$:

$$\widehat{\operatorname{vol}}_{\chi}\left(m\overline{\mathcal{L}}\right) = m^{n+1}\widehat{\operatorname{vol}}_{\chi}\left(\overline{\mathcal{L}}\right), \text{ if } m \in \mathbb{Z}_+ \tag{2.4}$$

Moreover, $\widehat{\operatorname{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)$ is additively equivariant with respect to scalings of the metric:

$$\widehat{\operatorname{vol}}_{\chi}\left(\mathcal{L}, \phi + \lambda\right) = \widehat{\operatorname{vol}}_{\chi}\left(\overline{\mathcal{L}}\right) + \frac{\lambda}{2}\operatorname{vol}(L), \text{ if } \lambda \in \mathbb{R}. \tag{2.5}$$

If the metric on L has positive curvature current (i.e. if ϕ is psh), then, by the arithmetic Hilbert-Samuel theorem,

$$\widehat{\operatorname{vol}}_{\chi}\left(\overline{\mathcal{L}}\right) = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}, \tag{2.6}$$

where $\overline{\mathcal{L}}^{n+1}$ denotes the top arithmetic intersection number in the sense of Gillet-Soulé [20], which, defines the *height* of $\mathcal{X}$ with respect to $\overline{\mathcal{L}}$ [19, 7]. For the purpose of the present paper formula 2.6 may be taken as the definition of $\overline{\mathcal{L}}^{n+1}$ (arithmetic intersections between general $n+1$ metrized line bundles could then be defined by polarization, i.e. using multilinearity).

Following standard practice we will use the shorthand $h_\phi(\mathcal{X},\mathcal{L})$ for the height $(\mathcal{L},\phi)^{n+1}$ and $\hat{h}_\phi(\mathcal{X},\mathcal{L})$ for the normalized height:

$$h_\phi(\mathcal{X},\mathcal{L}) := (\mathcal{L},\phi)^{n+1},\ \hat{h}_\phi(\mathcal{X},\mathcal{L}) := \frac{(\mathcal{L},\phi)^{n+1}}{(n+1)L^n}.$$

The definition of $\hat{h}_\phi(\mathcal{X},\mathcal{L})$ is made so that

$$\hat{h}_{\phi+\lambda}(\mathcal{X},\mathcal{L}) = \hat{h}_\phi(\mathcal{X},\mathcal{L}) + \lambda/2,\ \text{ if } \lambda \in \mathbb{R}$$

We also recall that, given two continuous psh metrics ϕ and ϕ_0 on the complexification $L \to X$ of $\mathcal{L} \to \mathcal{X}$, we have that

$$2h(\mathcal{L},\phi) - 2h(\mathcal{L},\phi_0) = \mathcal{E}(\phi,\phi_0) := \frac{1}{(n+1)!}\int_X (\phi-\phi_0)\sum_{j=0}^{n}(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}. \tag{2.7}$$

2.4. **The canonical height of an arithmetic log Fano variety.** We define the *canonical height* $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ of an arithmetic log Fano variety $(\mathcal{X},\mathcal{D})$ by

$$h_{\mathrm{can}}(\mathcal{X},\mathcal{D}) := \sup\left\{h_\phi\left(-\mathcal{K}_{(\mathcal{X},\mathcal{D})}\right) : \phi \text{ cont. psh}, \int_X \mu_\phi = 1\right\}$$

(when $\mathcal{D} = 0$ we shall use the short hand $h_{\mathrm{can}}(\mathcal{X})$ for $h_{\mathrm{can}}(\mathcal{X},0)$). As shown precisely as in the case $\mathcal{D}=0$, considered in [2], $h_{\mathrm{can}}(\mathcal{X},\mathcal{D}) < \infty$ iff the corresponding log Fano variety (X,Δ) over $\mathbb{C}$ is K-semistable. Moreover, (X,Δ) is K-polystable iff the sup defining $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ is attained at some continuous metric ϕ, namely a log Kähler-Einstein metric. Hence, if (X,Δ) is K-polystable, then the canonical height $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ is computed by any volume-normalized log Kähler-Einstein metric.

3. Toric log Fano varieties

A log pair (X,D) over $\mathbb{C}$ is said to be *toric* if X and D are toric, i.e. if X is toric and the $\mathbb{Q}-$divisor D is invariant under the torus action on X. Any toric log Fano variety admits a canonical integral model $(\mathcal{X},\mathcal{D})$ which is log Fano (see [34, Section 2] and [12, Def 3.5.6]). In this section we will prove the following

Theorem 3.1. *Let $(\mathcal{X},\mathcal{D})$ be the canonical integral model of a K-semistable toric log Fano variety (X,D). Conjecture 1.1 holds for $(\mathcal{X},\mathcal{D})$ under anyone of the following conditions:*

- *$n \leq 3$ and X is $\mathbb{Q}-$factorial (equivalently, X has at worst abelian quotient singularities)*
- *X is not Gorenstein or has some abelian quotient singularity*

We start by introducing some notation, following [6]. Given a toric log Fano variety (X,D) set $L = -(K_X+\Delta)$ and denote by P the corresponding moment polytope in $\mathbb{R}^n$. Then

$$P = \{p \in \mathbb{R}^n : \langle l_F, p\rangle \geq -a_F,\ \forall F\}, \tag{3.1}$$

where $a_F \in]0,1]$ (generalizing the Fano case when $a_F = 1\forall F$; see [6]) and l_F is a primitive integer vector. As shown in [6] (X,Δ) is K-semistable iff 0 is the barycenter of P iff the log Ding functional $\mathcal{D}_{\psi_P}$ is bounded from below. Moreover, the infimum of $\mathcal{D}_{\psi_P}$ is attained at a $T-$invariant psh metric ϕ on L. We will identify the metric ϕ with a continuous convex function on $\mathbb{R}^n$ as in [2]. More precisely, on $(\mathbb{C}^*)^n \hookrightarrow X$, let $x_i = \log(|z_i|^2)$. Trivializing $-(K_X+\Delta)$ with $\frac{\mathrm{d}z_1}{z_1} \wedge ... \wedge \frac{\mathrm{d}z_n}{z_n} \otimes s_U e_\Delta$ over $U = (\mathbb{C}^*)^n$, and abusing notation slightly, we let $\phi(x) := \phi_U(z)$

in the chosen trivialization over $U = (\mathbb{C}^*)^n$. Then ϕ as a function of x is a continuous convex function on $\mathbb{R}^n$ and as in formula 3.8 in [2], we still have that

$$\mathcal{D}_{\psi_P}(\phi) = \int_P \phi^* dy/V - \log \int_{\mathbb{R}^n} e^{-\phi(x)} dx - n \log \pi, \ \ V := \mathrm{vol}(P)$$

(since the support of $\mathcal{D}$ is contained in the complement of $(\mathbb{C}^*)^n$ in X). Thus the inequality in Proposition 3.7 in [2] generalizes to the canonical toric model $\mathcal{L}$ of L (which coincides with $-\mathcal{K}_{(\mathcal{X},\mathcal{D})}$) :

$$2\widehat{\mathrm{vol}}_{\chi}\left(-\mathcal{K}_{(\mathcal{X},\mathcal{D})}, \phi\right) \leq -\mathrm{vol}(X, \Delta) \log\left(\frac{\mathrm{vol}(X,\Delta)}{(2\pi^2)^n}\right) \ \ \mathrm{vol}(X,\Delta) := \mathrm{vol}(-K_{(X,\Delta)}). \tag{3.2}$$

We will first prove Theorem 3.1 in the case that $X = \mathbb{P}^n$, using the following lemma, formulated in terms of the divisor D_0 cut out by the T_c-invariant element of $H^0(X, -K_X)$ (given by $\frac{\mathrm{d}z_1}{z_1} \wedge ... \wedge \frac{\mathrm{d}z_n}{z_n}$ over $(\mathbb{C}^*)^n$). In other words,

$$D_0 = \sum_F D_F,$$

where D_F is the irreducible divisor corresponding to the facet F of the moment polytope corresponding to X (see [6]). The lemma is a special case of formula Proposition 3.12 from [2].

Lemma 3.2. *Let $\mathcal{X}$ be the canonical integral model of an $n-$dimensional K-semistable toric Fano variety X and denote by D_0 the standard anti-canonical divisor on X. Then*

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},(1-t)\mathcal{D}_0)}})^{n+1}/(n+1)!}{(-(K_X+(1-t)D_0))^n/n!} = \frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}/(n+1)!}{(-K_X)^n/n!} - \frac{1}{2}\log(t^n) \quad t^n = \left(\frac{(-(K_X+(1-t)D_0))^n}{(-K_X)^n}\right)$$

with respect to the volume-normalized Kähler-Einstein metrics.

We next deduce the following

Lemma 3.3. *Let $(\mathcal{X}, \mathcal{D})$ be a toric K-semistable log Fano variety such that $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$. Then $(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1} \leq (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ with equality iff $\mathcal{D} = 0$.*

Proof. First observe that there exists $t \in [0,1]$ such that $\mathcal{D} = (1-t)\mathcal{D}_0 =: \mathcal{D}_t$. This is a special case of [24, Cor 1.6], which applies to $\mathbb{P}^n$, in any dimension n, using that toric log Fano varieties are never uniformly K-stable. It will thus be enough to show that $t \mapsto (\overline{-\mathcal{K}_{(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t)}})^{n+1}$ is increasing on $[0,1]$ (and thus its maximum is attained at $t = 1$). By the previous lemma

$$\frac{2(\overline{-\mathcal{K}_{(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t)}})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^n/n!} = t^n 2\frac{(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^n/n!} - t^n \log(t^n).$$

Differentiating wrt (t^n) reveals that the right hand side above is increasing with respect to t iff $2\frac{(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^n/n!} \geq 1$. The latter inequality is indeed satisfied, as follows from the explicit formula 1.1. $\square$

Combining the universal bound 3.2 with Lemma 3.8 from [2], all that remains to prove Theorem 3.1 is to establish the "logarithmic gap hypothesis"

$$\mathrm{vol}(X,\Delta) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1). \tag{3.3}$$

Proposition 3.4. *The logarithmic gap hypothesis holds for all toric K-semistable log Fano varieties (manifolds)* (X, Δ) *such that* $X \neq \mathbb{P}^n$ *iff the following bound holds for all Fano varieties (manifolds)* $X \neq \mathbb{P}^n$

$$S(X) \leq \operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \quad S(X) := \sup \{\operatorname{vol}(-(K_X + \Delta)) : (X, \Delta) \text{ K-semistable}\}. \tag{3.4}$$

The "logarithmic gap hypothesis" holds for all log Fano varieties (X, Δ) *such that* X *is* $\mathbb{Q}-$*factorial and of dimension* $n \leq 3$ *and for any dimensions* n *if* X *has some abelian quotient singularity or if* X *is not Gorenstein.*

Proof. Since, trivially, $\operatorname{vol}(X, \Delta) \leq S(X)$ the first equivalence follows directly from the definitions. Next, let us show the last statement of the proposition, first assuming that X is singular, which means that the moment polytope P of (X, Δ) is "singular" in the sense that there exists a vertex of ∂P such that the corresponding primitive vectors $l_{F_1}, ..., l_{F_n}$ do not generate $\mathbb{Z}^n$. It follows from the proof of Lemma 3.9 from [2] that

$$\operatorname{vol}(P) \leq \frac{1}{2}(n+1)^n/n! \leq \operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1))$$

Indeed, since $a_F \leq 1$ the first inequality follows from the inequality (3.13) from [2], using that $\delta \geq 2$, according to the singularity assumption on P (for the the second inequality see formula (3.14) from [2]). All that remains is thus to show the bound 3.4 for $S(X)$ when $n \leq 3$ and X is non-singular. First assume that $n = 2$. This means, by classical classification results, that X is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up $X^{(m)}$ of $\mathbb{P}^2$ in m points for $m \leq 3$. But $(-K_{X^{(m)}})^2 = (-K_{\mathbb{P}^2})^2 - m$ and thus $\operatorname{vol}(X_1) \leq 4 = \operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$, proving the bound 3.4. Finally, consider the case when $n = 3$. Starting with the trivial bound $\operatorname{vol}(X, \Delta) \leq \operatorname{vol}(X)$ it follows the classification [52] of all non-singular toric Fano varieties of dimension 3 that it is enough to show that the bound 3.4 holds when X is $\mathbb{P}^3$ blown-up in one point or $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2))$ (whose degrees are 56 and 62, respectively). According to the following proposition the corresponding invariants $S(X)n!$ are, approximately, given by 41.8 and 30.3, respectively, which are well below the degree 54 of $\mathbb{P}^2 \times \mathbb{P}^1$, as desired. □

3.1. **The invariant** $S(X)$ **for** $n \leq 3$. In the proof above we used the following result.

Proposition 3.5. *After rounding to the nearest decimal place the invariant* $n!S(X)$ *(formula 3.4) is given by* 41.8 *and* 30.3 *when* X *equals* $\mathbb{P}^3$ *blown up in one point and* $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$, *respectively.*

Proof. Given a convex subset P of $\mathbb{R}^n$ let

$$s(P) := \sup \{\operatorname{vol}(P_0) : P_0 \subset P, b_{P_0} = 0\},$$

where P_0 is a closed subset of P with barycenter b_{P_0} at the origin. We will compute $s(P)$ when P is the moment polytope of the manifolds X appearing in the proposition, showing at the same time that $s(P) = S(X)$. The moment polytopes P of both $\mathbb{P}^3$ blown up in one point and $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ are of the form a simplex, with a simplex subset removed, by chopping off a vertex (see ID 20 and ID7 in the database [52])). After a general linear transformation, they are of the form $(a\Delta_3 - \mathbf{1}) - (b\Delta_3 - \mathbf{1})$ where Δ_3 is the standard unit simplex in dimension three, $\mathbf{1}$ is the vector with all ones and a and b are positive real numbers. For $\mathbb{P}^3$ blown up in one point we can transform the moment polytope to $(4\Delta_3 - \mathbf{1}) \backslash (2\Delta_3 - \mathbf{1})$ and for $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ we get $(5\Delta_3 - \mathbf{1}) \backslash (\Delta_3 - \mathbf{1})$. In the first case, the linear transformation is unimodular, but in the second case the transformation has determinant 2. This will not matter when computing $s(P)$ as long as we correct for the non-unit determinant. Next we compute the barycenter b_P

of these polytopes, a simple task using the explicit barycenter of the standard unit simplex, $b_{\Delta_n} = \mathbf{1}/(n+1)$, and then scaling and linearity properties of the volume times the barycenter. The barycenter of $(a\Delta_3 - \mathbf{1})\backslash(b\Delta_3 - \mathbf{1})$ is given by $\frac{a^3/3!(a/4-1)-b^3/3!(b/4-1)}{a^3/3!-b^3/3!}\mathbf{1}$. Next we use a general fact, to be proved in the lemma below, stating that the closed subset P' of P which maximizes volume, with the relaxed constraint

$$b_{P'} \cdot \mathbf{1} = 0 \tag{3.5}$$

is the one given by $P \cap H$ where H is a half-space with normal $\mathbf{1}$. In our case, by symmetry, this P' automatically satisfies the stronger constraint $b_{P'} = 0$. Moreover, since the boundary of $P \cap H$ is parallel to a facet of P it corresponds to a divisor Δ on X defining a log Fano pair (X, Δ). Thus (X, Δ) is also the K-semistable log Fano pair realizing the sup in the definition of $S(X)$, showing that $s(P) = S(X)$. We can find H by imposing the constraint. We introduce the weight w such that

$$P \cap H = ((a - w)\Delta_3 - \mathbf{1})\backslash(b\Delta_3 - \mathbf{1}).$$

From here it is clear that if $b_{P'} \cdot \mathbf{1} = 0$, then, in fact, the entire barycenter will vanish and the condition $b_{P'} \cdot \mathbf{1} = 0$ turns into the following fourth order polynomial equation for w :

$$(a - w)^3/3!((a - w)/4 - 1) - b^3/3!(b/4 - 1) = 0.$$

The solution w and the corresponding value $s(P)$ for $\mathbb{P}^3$ blown up in one point, is given by $w = \frac{2}{3}(5 - \frac{4}{\sqrt[3]{19-3\sqrt{33}}} - \sqrt[3]{19 - 3\sqrt{33}})$ and $\mathrm{n!s(P)} = \mathrm{n!vol}(P') = ((4 - w)^3 - 2^3) \approx 41.8$ and for $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$, $w = (4 - \sqrt[3]{\frac{4}{2-\sqrt{2}}} - \sqrt[3]{2(2 - \sqrt{2})})$ and $\mathrm{n!S(P)} = \frac{1}{2}\mathrm{n!vol}(P') = \frac{1}{2}((5-w)^3 - 1^3) \approx 30.3$, where we have corrected for the non-unimodular transformation used in the second case. □

In the above proof we used the following

Lemma 3.6. *Let P be a closed subset of $\mathbb{R}^n$ with the origin as an interior point. Given $v \in \mathbb{R}^n$ assume that $\int_P x \cdot v > 0$. Then the maximum*

$$\max_{Q \subset P: \int_Q v \cdot x \mathrm{d}\lambda(\mathrm{x}) = 0} \int_Q \mathrm{d}\lambda$$

is attained at $Q = P \cap H$ with H a closed half-space with outward pointing normal v. Here $\mathrm{d}\lambda$ is Lebesgue measure.

Proof. Without loss of generality we can assume that $v = (0, ..., 0, 1)$. Denote by $(x_1, x_2, ..., x_{n-1}, y)$ the coordinates on $\mathbb{R}^n$. Since the origin is an interior point of P and $\int_P x \cdot v > 0$ there is a closed half-space H as in the lemma satisfying $\int_{P \cap H} y \mathrm{d}\lambda = 0$. Hence, any candidate Q for the maximum in question satisfies $\int_{P \cap H} y \mathrm{d}\lambda = \int_Q \mathrm{y d}\lambda$. Subtracting the left hand side from the right hand side and vice versa yields $\int_{P \cap H \backslash Q} y \mathrm{d}\lambda = \int_{\mathrm{Q} \backslash \mathrm{P} \cap \mathrm{H}} \mathrm{y d}\lambda$. Since $\sup_{P \cap H \backslash Q} y \leq \inf_{Q \backslash (P \cap H)} y$ it follows that $\mathrm{vol}(P \cap H \backslash Q) \geq \mathrm{vol}(Q \backslash P \cap H)$ which, in turn, implies that $\mathrm{vol}(P \cap H) \geq \mathrm{vol}(Q)$, as desired. □

In fact, with just a slight variation of the argument above, any maximizer must be of the special form above and, in addition, assuming connectedness of P, the maximizer is unique. The proof of the previous proposition thus reveals that the unique toric divisor Δ on X realizing the sup defining the invariant $S(X)$ is a multiple of the prime divisor D_F defined by the zero-section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbb{P}^2$ and hyperplane "at infinity" in $\mathbb{P}^3$ blown up at the origin in $\mathbb{C}^3 \subset \mathbb{P}^3$, respectively (i.e the zero-section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^2$). A similar argument also applies when X is the blow-up of $\mathbb{P}^2$ at the origin in $\mathbb{C}^2$ (i.e. the first Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^2$).

The unique maximizer for the invariant $S(X)$ is then a log Fano pair (X, Δ) for a multiple of the hyperplane D "at infinity" (i.e. the zero-section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^2$). Interestingly, this K-polystable log pair (X, Δ) was also singled out in [43, Cor 1.5] by the following rigidity property (answering a question of Cheltsov): it admits a rigid Kähler-Einstein metric in the sense that for any other multiple cD the log pair (X, cD) does not admit a Kähler-Einstein metric. The same rigidity property holds for the two three-dimensional log pairs discussed above (since there is a unique half-space H satisfying the constraint in formula 3.5).

3.2. **Estimates on the canonical height.** Theorem 1.3 from [2] (and its corollary) generalizes directly to the case of log Fano pairs and their Kähler-Einstein metrics in any relative dimension n (with the same proof, by letting P be the moment polytope corresponding to (X, Δ)) :

$$\frac{1}{2}\mathrm{vol}(X, \Delta) \log\left(\frac{n! m_n \pi^n}{\mathrm{vol}(X, \Delta)}\right) \leq \frac{h_{\mathrm{can}}(\mathcal{X}, \mathcal{D})}{(n+1)!} \leq \frac{1}{2}\mathrm{vol}(X, \Delta) \log\left(\frac{(2\pi)^n \pi^n}{\mathrm{vol}(X, \Delta)}\right)$$

Interestingly, Lemma 3.2 reveals that the family of log Fano pairs $(\mathcal{X}, \mathcal{D})$ appearing in the lemma may be explicitly expressed in terms of the algebro-geometric volume $\mathrm{vol}(X, \Delta)$ in the same functional form as the one appearing in the previous upper and lower bounds:

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}}{(n+1)!} = \frac{1}{2}\mathrm{vol}(X, \Delta) \log\left(\frac{be^{2a}}{\mathrm{vol}(X, \Delta)}\right)$$

with $a := \frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}/(n+1)!}{(-K_X)^n/n!}$ and $b = \mathrm{vol}(X)$.

4. Hyperplane arrangements

In this Section we prove Theorem 1.3 concerning hyperplane arrangements. Recall that a log Fano pair (X, Δ) is called a *log Fano hyperplane arrangement* if $X = \mathbb{P}^n$ and $\Delta = \sum_{i=1}^m w_i H_i$ where $w_i \in \mathbb{Q}_{>0}$ and the H_i are distinct hyperplanes. Furthermore we will call (X, Δ) *simple normal crossing*, abbreviated snc, if the support of Δ has simple normal crossings.

For an snc log Fano hyperplane arrangement, if $m = n + 1$ and all the weights w_i are equal, then (X, Δ) is a toric log-pair (see Lemma 3.2). The following lemma shows that for given hyperplanes $H_1, ..., H_m$ and a fixed volume $\mathrm{vol}(X, \Delta)$, the "toric" weights form the vertices of the convex polytope of all weights w_i corresponding to K-semistable (X, Δ).

Lemma 4.1. *Fix $m \geq 1$ and a real number $0 < D \leq (-K_{\mathbb{P}^n})^n = (n+1)^n$. Let as before for a real m-tuple w, $\Delta = \sum_{i=1}^m w_i H_i$ for distinct hyperplanes H_i. Then the set of weights*

$$S = \{w \in \mathbb{R}^n : (-(K_{\mathbb{P}^n} + \Delta))^n = D \text{ and } (\mathbb{P}^n, \Delta) \text{ is } \mathrm{K-semistable}\}$$

is either empty or $m \geq n+1$ and S is a polytope with $\binom{n}{m}$ vertices given by any reordering of the tuple $w_1 = w_2 = ... = w_{n+1} = \frac{1}{m}(n + 1 - D^{1/n})$, $w_l = 0$ $\forall l > n + 1$.

Proof. By [24], for $w \in \mathbb{R}^n$ and $\Delta = \sum_{i=1}^m w_i H_i$, $(\mathbb{P}^n, \Delta)$ is K-semistable and log Fano if and only if w is in the convex set C defined by the following inequalities:

$$0 \leq w_i < 1 \ \forall i = 1, ..., m$$

$$k \sum_{i=1}^m w_i \geq (n+1) \sum_{j=1}^k w_{i_j} \ \forall 1 \leq k \leq n \ \forall 1 \leq i_1 < ... < i_k \leq m. \tag{4.1}$$

Here it should be noted that in fact, it suffices to consider the second inequality for index combinations i_j of length 1. The other inequalities for larger index combinations follows. Hence, K-semistability of $(\mathbb{P}^n, \Delta)$ is equivalent to

$$\begin{aligned} & 0 \le w_i < 1 \ \forall i = 1, ..., m \\ & w_i \le \frac{1}{n+1} \sum_{j=1}^m w_j \ \forall i = 1, ..., m. \end{aligned} \tag{4.2}$$

Fix m and D as in the statement of the theorem. The goal is to understand the intersection of the above set with the set $\{w : -(K_{\mathbb{P}^n} + \Delta) = D\}$. Note first that

$$(-(K_{\mathbb{P}^n} + \sum_{i=1}^m w_i H_i))^n = (n+1 - \sum_{i=1}^m w_i)^n.$$

Let $C := n+1 - D^{1/n}$, so that $\{w : -(K_{\mathbb{P}^n} + \Delta) = D\} = \{w : \sum_{i=1}^m w_i = C\}$. Thus with S defined as in 4.1

$$\begin{aligned} S = \{w : & \sum_{i=1}^m w_i = C \\ & 0 \le w_i < 1 \ \forall i = 1, ..., m \\ & w_i \le \frac{C}{n+1} \ \forall i = 1, ..., m\} \end{aligned}$$

Observe that since $0 \le C < n+1$, the inequality $w_i < 1$ is superfluous. After a convenient rescaling we get

$$\begin{aligned} \frac{n+1}{C} S = \{w : & \sum_{i=1}^m w_i = n+1 \\ & 0 \le w_i \le 1 \ \forall i = 1, ..., m\}. \end{aligned}$$

Clearly if $m < n+1$, $\frac{n+1}{C} S$ is empty. For $m \ge n+1$, any vertex of $\frac{n+1}{C} S$ is given by the intersection of $\frac{n+1}{C} S$ with some collection of the inequalities put to equality. But clearly all such points must be of the form of having $n+1$ ones and $m-(n+1)$ zeros. And on the other hand any such point is a vertex. □

Fixing the volume $\mathrm{vol}(X, \Delta)$ is, when $X = \mathbb{P}^n$, tantamount to fixing the isomorphism class of the $\mathbb{Q}-$line bundle $-(K_X + \Delta)$ (since the rank of the Picard group of $\mathbb{P}^n$ is one). The following lemma shows that, in this case, the maximal height is convex with respect to the weights of Δ.

Lemma 4.2. *Consider an arithmetic Fano variety $\mathcal{X}$ and a curve $t \mapsto (\mathcal{X}, \mathcal{D}_t)$ of arithmetic log Fano varieties where $\mathcal{D}_t = \sum_{i=1}^m w_i(t)\mathcal{D}_i$ for some $m \ge 1$, irreducible divisors $\mathcal{D}_i$ over $\mathbb{Z}$ and $w : [0,1] \to \mathbb{R}^m$ an affine function. Additionally assume that all the $\mathcal{D}_t$ are linearly equivalent, which equivalently means that $-(\mathcal{K} + \mathcal{D}_t)$ isomorphic to $\mathcal{L}$ for a line bundle $\mathcal{L} \to \mathcal{X}$ independent of t. Then the function $h : [0,1] \to]-\infty, \infty]$ defined as*

$$t \mapsto h_{can}(\mathcal{X}, \mathcal{D}_t) \tag{4.3}$$

is strictly convex. Equivalently the function $t \mapsto \hat{h}_{can}(\mathcal{X}, \mathcal{D}_t)$ is strictly convex.

Proof. By assumption we can identify $-(\mathcal{K}+\mathcal{D}_t)$ with $\mathcal{L}$ for a line bundle $\mathcal{L}$ independent of ϕ. Thus the height $h_\phi(\mathcal{X},\mathcal{D}_t)$ for a fixed metric on $\mathcal{L}$ is independent of t. Likewise, $h_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)$ coincides with $\hat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)$ up to multiplication by a constant independent of t. Next, express

$$\hat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)=\sup_\phi \hat{h}(\mathcal{X},\mathcal{D}_t)+\frac{1}{2}\log\int_X \mu_{(\phi,\mathcal{D}_t)},$$

where the sup ranges over all continuous psh metrics on $\mathcal{L}$. Introducing an arbitrary volume form $\mathrm{d}V$ on X we can rewrite

$$\int_X \mu_{(\phi,\mathcal{D}_t)}=\int_X \exp(-\phi-\sum_{i=1}^m w_i(t)\psi_{D_i}-\log \mathrm{d}V)dV.$$

By Hölder's inequality this expression is convex in t, since $w_i(t)$ is affine. It is even strictly convex since the D_i are distinct. This means that $\hat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)$ is the supremum over a set independent of t, of a collection of strictly convex functions and thus is itself, strictly convex. □

4.1. **Conclusion of the proof of Theorem 1.3.** In the following we will use the notation $\mathcal{D}_w=\sum_{i=1}^m w_iH_i$ for $w\in\mathbb{R}^m$ for fixed hyperplanes H_i defined over $\mathbb{Z}$. Let $(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_{w'})$ be a K-semistable snc log Fano hyperplane arrangement. Define for brevity $d:=(-(K_{\mathbb{P}^n}+\Delta(w')))^n$. Set, as in Lemma 4.1, $S=\{w\in\mathbb{R}^n:(-(K_{\mathbb{P}^n}+\Delta_w))^n=d \text{ and } (\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_w) \text{ is } \mathrm{K}-\mathrm{semistable}\}$. Consider the function $h(w)$ defined by

$$h(w)=h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_w)$$

Restricted to the convex set S, $h|_S$ is convex by Lemma 4.2. Next by Lemma 4.1, S is the convex hull of weight vectors $(w^k)_{k=1,\ldots,\binom{m}{n+1}}$, each corresponding to toric log Fano pairs, all equivalent to $(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D}_0)$. Here $\mathcal{D}_0$ is the toric standard anti-canonical divisor and t is the unique number such that $(-(K_{\mathbb{P}^n}+(1-t)\mathcal{D}_0)^n=d$. By Jensens inequality it follows directly from expressing w as a convex combination of the w^k, i.e. $w=\sum_{k=1}^{\binom{m}{n+1}}\lambda_k w^k$, that

$$h(w)\leq\sum_{k=1}^{\binom{m}{n+1}}\lambda_k h(w^k)=h(w^1)=h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D}_0)$$

with equality iff Δ_w is toric. We have thus reduced to a toric case, which we have already handled. Specifically, the bound 1.4 follows directly from Lemma 3.2. For Theorem 1.3, recall that it was observed in the proof of Lemma 3.3 that the volume dependent bound in 1.4 is increasing with volume, so that a universal bound is given for maximal volume, i.e. when $\Delta=0$, yielding the result.

5. Diagonal hypersurfaces

In this section we will deduce Theorem 1.4 from the results in the previous sections. The starting point of the proof is the following analytic representation of the height:

Lemma 5.1. *(Restriction formula) Let $\mathcal{X}$ be the subscheme of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by a homogeneous polynomial s of degree d with integer coefficients and ϕ a continuous psh metric on $\mathcal{O}(d)\to\mathbb{P}^{n+1}_{\mathbb{C}}$. Then the height $h_\phi(\mathcal{X}_d,\mathcal{O}(d))$ of the restriction of $(\mathcal{O}(d),\phi)$ to $\mathcal{X}$ may be expressed as*

$$\frac{2h_\phi(\mathcal{X}_d,\mathcal{O}(d))}{(n+1)!}=(n+2)\mathcal{E}(\phi,d\phi_0)+\int_{\mathbb{P}^{n+1}}\log\left(\|s\|^2_\phi\right)\frac{(dd^c\phi)^{n+1}}{(n+1)!},$$

where ϕ_0 is the Weil metric on $\mathcal{O}(1)$ and $\mathcal{E}$ is the functional defined by formula 2.7, corresponding to $\mathcal{O}(d) \to \mathbb{P}^{n+1}$.

Proof. This is well-known, but for completeness we provide a proof. Consider first the general situation where $\mathcal{X}$ is a subscheme (of relative dimension n) of a regular projective flat scheme $\mathcal{Y}$ cut out by a section s of a relatively ample line bundle $\mathcal{L} \to \mathcal{Y}$. Then, given a metric ϕ on the complexification L of $\mathcal{L} \to \mathcal{Y}$, the restriction formula for arithmetic intersection numbers [7, Prop 2.3.1] gives

$$(\mathcal{L}, \phi)^{n+2} \cdot \mathcal{Y} = (\mathcal{L}, \phi)^{n+1} \cdot \mathcal{X} - \int_Y \log ||s||_\phi (dd^c\phi)^{n+1} \tag{5.1}$$

In particular, setting $\mathcal{Y} = \mathbb{P}^{n+1}_{\mathbb{Z}}$ and $\mathcal{L} = \mathcal{O}(d)$ gives and

$$\frac{2h_\phi(\mathcal{X}_d, \mathcal{O}(d))}{(n+1)!} = (n+2)\frac{h_\phi(\mathbb{P}^{n+1}_{\mathbb{Z}}, \mathcal{O}(d))}{(n+2)!} + \int \log ||s||^2_\phi \frac{(dd^c\phi)^{n+1}}{(n+1)!}.$$

The proof is thus concluded by invoking the well-known fact that $h_\phi(\mathbb{P}^{n+1}_{\mathbb{Z}}, \mathcal{O}(d))/(n+2)! = \mathcal{E}_{\mathbb{P}^{n+1}}(\phi, \phi_0)$. For example, this is a special case of the toric formula in [1, formula 3.7]. □

In general, if $\mathcal{X}$ is subscheme of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ of codimension one, then $\mathcal{K}_{\mathcal{X}}$ is well-defined as line bundle over $\mathcal{X}$. More precisely, by the adjunction formula, there is an isomorphism of line bundles over $\mathbb{Z}$,

$$\mathcal{K}_{\mathcal{X}} \simeq \left(\mathcal{K}_{\mathbb{P}^{n+1}_{\mathbb{Z}}} - \mathcal{O}(\mathcal{I}/\mathcal{I}^2)\right)_{\mathcal{X}},$$

where $\mathcal{I}$ is the ideal sheaf cutting out $\mathcal{X}$ [28, formula 1.6.2, page 8]. In particular, if $\mathcal{X}$ is cut out by a homogeneous polynomial s of degree d, then

$$-\mathcal{K}_{\mathcal{X}} \simeq -\mathcal{K}_{\mathbb{P}^{n+1}_{\mathbb{Z}}} - \mathcal{O}(d) \simeq \mathcal{O}(n+2-d). \tag{5.2}$$

Hence, $-\mathcal{K}_{\mathcal{X}}$ is relatively ample iff $d \leq n+1$. Now assume that the complex variety X defined by the complex points of $\mathcal{X}$ is non-singular. Then, by the adjunction isomorphism 5.2, a metric ϕ_0 on $\mathcal{O}(n+2-d)|_X$ may be identified with a metric on $-K_X$.

5.1. **Reduction to Fermat hypersurfaces.** Given integers a_i consider the subscheme $\mathcal{X}_a$ of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by the homogeneous polynomial

$$s_a := \sum_{i=0}^{n+1} a_i x_i^d.$$

Denote by X_a the corresponding complex variety, which is non-singular and consider the map

$$F_a(x) := (a_0^{-1/d} x_0, ..., a_{n+1}^{-1/d} x_{n+1}) : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}. \tag{5.3}$$

We will identify the map F_a with an automorphism of $\mathbb{P}^{n+1}$, admitting a standard lift to the total space of $\mathcal{O}(1) \to \mathbb{P}^{n+1}$. We can then express

$$X_a = F_a(X_1).$$

Proposition 5.2. *Let k be a positive integer and ϕ a metric on $\mathcal{O}(n+2-d)|_{X_a}$. Then*

$$2\hat{h}_\phi(\mathcal{X}_a, \mathcal{O}(n+2-d)) + \log \int_{X_a} e^{-\phi} =$$

$$= 2\hat{h}_{F_a^*\phi}(\mathcal{X}_1, \mathcal{O}(n+2-d)) + \log \int_{X_1} e^{-F_a^*\phi} + (\frac{n+2-d}{(n+1)} - 1)d^{-1} \sum_i \log(|a_i|^2).$$

The proof of the proposition follows from combining the following two lemmas:

Lemma 5.3. *Let k be a positive integer and ϕ a metric on $\mathcal{O}(k)$. Then*

$$2\widehat{h}_\phi(\mathcal{X}_a,\mathcal{O}(k))=2\widehat{h}_{F_a^*\phi}(\mathcal{X}_1,\mathcal{O}(k))+\frac{1}{d}\sum_i\log(|a_i|^2)\frac{k}{(n+1)}$$

Proof. We will use that any continuous psh metric ϕ on $\mathcal{O}(k)|_X$ is the restriction of a continuous psh metric on $\mathcal{O}(k)\to\mathbb{P}^{n+1}$, that we shall denote by the same symbol ϕ [15]. First consider the case when $k=d$ and denote by s_a the section of $\mathcal{O}(d)$ cutting out the scheme $\mathcal{X}_a$. By the restriction formula (Lemma 5.1)

$$\frac{2}{(n+1)!}h_\phi(\mathcal{X}_a,\mathcal{O}(k))=(n+2)\mathcal{E}_{\mathbb{P}^{n+1}}(\phi,d\phi_0)+\int_{\mathbb{P}^{n+1}}\log|s_a|^2_\phi MA(\phi).$$

Rewriting

$$\int_{\mathbb{P}^{n+1}}\log|s_a|^2_\phi MA(\phi)=\int_{\mathbb{P}^{n+1}}\log|(F_a^{-1})^*s_1|^2_\phi MA(\phi)=\int_{\mathbb{P}^{n+1}}\log|s_1|^2_{F_a^*\phi}MA(F_a^*\phi),$$

thus reveals that

$$\frac{2}{(n+1)!}h_{F_a^*\phi}(X_1,\mathcal{O}(k))-\frac{2}{(n+1)!}h_\phi(X_a,\mathcal{O}(k))=(n+2)\left(\mathcal{E}_{\mathbb{P}^{n+1}}(F_a^*\phi,d\phi_0)-\mathcal{E}_{\mathbb{P}^{n+1}}(\phi,d\phi_0)\right)$$

Now, denote by G_a the standard lift of F_a from X to $-K_X$ and its tensor powers. We can then express

$$G_a^*\phi=F_a^*\phi+c_a,\ \ c_a:=\frac{k}{(n+2)}\frac{1}{d}\sum_i\log(|a_i|^2)$$

Indeed,

$$G_a^*(e^{-\frac{n+2}{k}\phi}dz d\bar{z})=(e^{-\frac{n+2}{k}F_a^*\phi})F_a^*(dzd\bar{z})=(e^{-\frac{n+2}{k}F_a^*\phi})\prod_i|a_i|^{-2/d}(dzd\bar{z}). \tag{5.4}$$

Hence,

$$2h_{F_a^*\phi}(\mathbb{P}^{n+1},\mathcal{O}(k))=2h_{G_a^*\phi-c_a}(\mathbb{P}^{n+1},\mathcal{O}(k))=2h_{G_a^*\phi}(\mathbb{P}^{n+1},\mathcal{O}(k))-c_a\frac{k^{n+1}}{(n+1)!}$$

But, by Lemma 7.1 in the appendix

$$\mathcal{E}_{\mathbb{P}^{n+1}}(G_a^*\phi,d\phi_0)=\mathcal{E}_{\mathbb{P}^{n+1}}(\phi,d\phi_0)$$

Hence, $\frac{2}{(n+1)!}h_{F_a^*\phi}(X_1,\mathcal{O}(k))-\frac{2}{(n+1)!}h_\phi(X_a,\mathcal{O}(k))=$

$$=(n+2)\left(\mathcal{E}_{\mathbb{P}^{n+1}}(F_a^*\phi,d\phi_0)-\mathcal{E}_{\mathbb{P}^{n+1}}(\phi,d\phi_0)\right)=-(n+2)c_a\frac{k^{(n+1)}}{(n+1)!}$$

As a consequence,

$$2\hat{h}_{F_a^*\phi}(X_1,\mathcal{O}(k))-2\hat{h}_\phi(X_a,\mathcal{O}(k))=-(n+2)c_a\frac{k^{n+1}}{(n+1)!}.\frac{1}{dk^n/n!}=-(n+2)c_a\frac{k}{d(n+1)}=$$

$$=-\frac{1}{d}\sum_i\log(|a_i|^2)(n+2)\frac{k}{(n+2)}\frac{k}{d(n+1)}=-\frac{1}{d}\sum_i\log(|a_i|^2)\frac{k^2}{d(n+1)}$$

Since we have assumed that $k=d$ this means that

$$2\widehat{h}_\phi(X_a,\mathcal{O}(k))-2\widehat{h}_{F_a^*\phi}(X_1,\mathcal{O}(k))=\frac{1}{d}\sum_i\log(|a_i|^2)\frac{d}{(n+1)}$$

Finally, for any given integer k we can express $k = d\lambda$ for $\lambda = k/d$ and use the basic scaling property

$$\widehat{h}_{\lambda\phi}(\mathcal{X}, \lambda\mathcal{L}) = \lambda\widehat{h}_{\phi}(\mathcal{X}, \mathcal{L}),$$

to get

$$2\widehat{h}_{\phi}(X_a, \mathcal{O}(k)) - 2\widehat{h}_{F_a^*\phi}(X_1, \mathcal{O}(k)) = \frac{1}{d}\sum_i \log(|a_i|^2)\frac{k}{(n+1)}$$

which concludes the proof. □

Lemma 5.4. *Given a metric ϕ on $\mathcal{O}(n+2-d)$ we have that*

$$\log\int_{X_a} e^{-\phi} = \log\int_{X_1} e^{-F_a^*\phi} - d^{-1}\sum_i \log(|a_i|^2).$$

Proof. It will be convenient to lift the integrals to the corresponding affine cones. To this end let X be the non-singular hypersurface of $\mathbb{P}^{n+1}$, cut out by a given homogeneous polynomial s in $\mathbb{C}^{n+2}$. Let A_X be the affine cone over X :

$$A_X = \{s = 0\} \subset \mathbb{C}^{n+2}$$

Let Ω_s be the holomorphic top form on $A_X - \{0\}$ defined by the relation

$$\Omega_s \wedge ds = dz \text{ on } Y, \quad dz := dz_0 \wedge \cdots \wedge dz_{n+1}.$$

Denote by δ interior multiplication with the generator of the standard $\mathbb{R}_+-$action, acting by scalings, on $\mathbb{C}^{n+2}$. Assume now that X is Fano. A given metric ϕ on $-K_X$ then induces a one-homogeneous function r on A (using the adjunction isomorphism 5.2 and by identifying $\mathbb{C}^{n+2} - \{0\}$ with the complement of the zero-section in $\mathcal{O}(1)^* \to X$). Moreover, lifting the adjunction isomorphism 5.2 to A_X yields the following well-known formula (which applies in the general setup of Fano varieties over local fields; cf. [39, Lemma 4.2.2]):

$$\int_X e^{-\phi} = c\int_{\{r=1\}\cap A_X} \delta(\Omega_s \wedge \overline{\Omega_s})$$

for a non-zero constant c only depending on n and d. Hence, recalling that F denotes the scaling map in formula 5.3 on $\mathbb{C}^{n+2}$,

$$\int_X e^{-\phi} = c\int_{\{F^*r=1\}\cap F^*A_X} F^*(\delta(\Omega_s \wedge \overline{\Omega_s}).$$

Using that the $\mathbb{R}_+-$action commutes with F thus gives

$$\int_X e^{-\phi} = c\int_{\{F^*r=1\}\cap A_{F^*X}} \delta(F^*(\Omega_s) \wedge \overline{F^*\Omega_s}).$$

Applying F^* to the defining relation for Ω_s yields

$$F^*\Omega_s \wedge d(F^*f) = F^*dz \text{ on } Y.$$

Since $F^*dz = a_0^{-1/d}\cdots a_{n+1}^{-1/d}dz$ this shows that $F^*\Omega_s = \Omega_{F^*s}a_0^{-1/d}\cdots a_{n+1}^{-1/d}$, which concludes the proof. □

It follows directly from Proposition 5.2 that

$$h_{\text{can}}(\mathcal{X}_a) = h_{\text{can}}(\mathcal{X}_1) + (n+1)(n+2-d)^n d\left(\frac{n+2-d}{(n+1)} - 1\right)d^{-1}\sum_i \log(|a_i|^2). \tag{5.5}$$

In particular, since $n+2-d \leq n+1$, this means that $h_{\text{can}}\left(\mathcal{X}_a\right) \leq h_{\text{can}}\left(\mathcal{X}_1\right)$ and thus the proof of Theorem 1.4 is reduced to the case of the Fermat hypersurface $\mathcal{X}_1$.

Remark 5.5. More generally, let $\mathcal{X}$ be a hypersurface in $\mathbb{P}_{\mathbb{Z}}^{n+1}$ cut-out by a homogeneous polynomial s of degree d of the form T^*s_1 where $T \in GL(n+2, \mathbb{C})$. Then formula 5.5 can be generalized as follows (as shown in essentially the same manner as before):

$$h_{\text{can}}\left(\mathcal{X}\right) = h_{\text{can}}\left(\mathcal{X}_1\right) + (n+1)(n+2-d)^n d\left(\frac{n+2-d}{(n+1)} - 1\right)\log(|\det T|^2).$$

5.2. **Reduction to log hyperplane arrangements.** Fix a degree $d(\leq n+1)$ and denote by $\mathcal{X}$ the corresponding Fermat hypersurface. The Fermat hypersurface of degree one will be denoted by $\mathcal{Y}$. We will next express the canonical height $h_{\text{can}}\left(\mathcal{X}\right)$ in terms of the canonical height $h_{\text{can}}\left(\mathcal{Y}, \mathcal{D}\right)$ where $\mathcal{D}$ is the divisor on $\mathcal{Y}$ defined by

$$\mathcal{D} = (1-1/d)\left[x_0 = 0\right] + ... + (1-1/d)\left[x_{n+1} = 0\right],$$

where x_i denotes the homogenous coordinates on $\mathbb{P}_{\mathbb{Z}}^{n+1}$ restricted to $\mathcal{Y}$.

Proposition 5.6. *Denote by $\mathcal{X}$ the Fermat hypersurface of a given degree $m(\leq n+1)$ and by $\mathcal{Y}$ the Fermat hypersurface of degree one, endowed with the divisor $\mathcal{D}$. Then*

$$\hat{h}_{can}\left(\mathcal{X}\right) = \hat{h}(\mathcal{Y}, \mathcal{D}) - \frac{1}{2}\log\frac{V(X)}{V(Y, \Delta)} \tag{5.6}$$

Proof. By the adjunction formula we have isomorphisms $-K_{\mathcal{X}} \simeq (n+2-m)\mathcal{O}(1)|_{\mathcal{X}}$ and $-\mathcal{K}_{(\mathcal{Y},\Delta)} \simeq (n+2-m)(1/m)\mathcal{O}(1)|_{\mathcal{Y}}$. Consider the following morphism:

$$F: \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}, \ \ (x_0, ..., x_{n+1}) \mapsto (x_0^m, ..., x_{n+1}^m),$$

which induces a map $\mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ and a lift to $\mathcal{O}(1)$, which is naturally defined over $\mathbb{Z}$ and satisfies $F^*\mathcal{O}(1) \simeq m\mathcal{O}(1)$. In particular, it induces a morphism

$$F: \mathcal{X} \to \mathcal{Y}, \ \ F^*(-\mathcal{K}_{(\mathcal{Y},\Delta)}) \simeq -K_{\mathcal{X}}$$

under the adjunction isomorphisms. We recall that, by basic functorial properties of heights,

$$\hat{h}(\mathcal{X}, F^*\mathcal{L}) = \hat{h}(\mathcal{Y}, \mathcal{L}). \tag{5.7}$$

In fact, in this case this formula follows directly from the analytic representation of the height in Lemma 5.1, using that F preserves the Weil metric ϕ_0. In particular, setting $\mathcal{L} := (n+2-m)(1/m)\mathcal{O}(1)_{|\mathcal{Y}}$ and using the adjunction isomorphisms yields $\hat{h}(\mathcal{X}, -K_{\mathcal{X}}, F^*\phi) = \hat{h}(\mathcal{Y}, -\mathcal{K}_{(\mathcal{Y},\Delta)}, \phi)$. Thus, all that remains is to show that

$$\int_X \mu_{F^*\phi} = m^{-(n+1)}\int_Y \mu_\phi,$$

where we have used that $F^*\phi$ induces a metric on $-K_X$ and ϕ induces a metric on $-K_{(Y,\Delta)}$. Since F has topological degree $m^{(n+1)}$ we have $F_*[X_m] = m^{(n+1)}[Y]$ as homology classes and thus it will be enough to show that

$$F^*\mu_\phi = m^{2(n+1)}\mu_\phi.$$

To this end consider the affine piece $\mathbb{C}^{n+1}$ of $\mathbb{P}^{n+1}$ where $x_0 \neq 0$. Setting $z_i = x_i/x_0$ for $i = 1, ..., n+1)$ we can, locally, parametrize X by the coordinates $z_1, ..., z_n$. In these coordinates a metric ψ on the restriction of $\mathcal{O}(1)$ to X induces, by the adjunction isomorphism, a metric on $-K_X$ and thus a measure on X locally expressed as

$$\mu_\psi = \frac{e^{-(n+2-m)\psi}}{(m|z_{n+1}|^{m-1})^2}\frac{i}{2}\mathrm{d}z_1\wedge \mathrm{d}\bar{z}_1\cdots\frac{i}{2}\mathrm{d}z_n\wedge \mathrm{d}\bar{z}_n. \tag{5.8}$$

To see this, recall that, by definition,

$$\mu_\psi := \|\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\|_\psi^{-2}\frac{i}{2}\mathrm{d}z_1\wedge \mathrm{d}\bar{z}_1\cdots\frac{i}{2}\mathrm{d}z_n\wedge \mathrm{d}\bar{z}_n.$$

In the affine piece $\mathbb{C}^{n+1}$ of $\mathbb{P}^{n+1}$ we can express

$$s = fx_0^{\otimes 2},\quad f = 1+\sum_{i=1}^n z_i^m + z_{n+1}^m$$

By the adjunction isomorphism 5.2 we have

$$\|\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\| := \|\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\wedge \mathrm{d}s\| = \|\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\wedge \mathrm{d}f\|\left\|x_0^{\otimes m}\right\|.$$

Since $\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\wedge \mathrm{d}f = \mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\wedge \mathrm{d}z_{n+1}\partial f/\partial z_{n+1}$ this means that

$$\|\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\|^2 := \left|\frac{\partial f}{\partial z_{n+1}}\right|^2\|\mathrm{d}z_1\wedge\cdots\wedge \mathrm{d}z_n\wedge \mathrm{d}z_{n+1}\|\left\|x_0^{\otimes m}\right\| = e^{(n+2)\psi}e^{-m\psi},$$

giving

$$\mu_\phi = \frac{e^{-(n+2-m)\psi}}{|\partial f/\partial z_{n+1}|^2}\frac{i}{2}\mathrm{d}z_1\wedge \mathrm{d}\bar{z}_1\cdots\frac{i}{2}\mathrm{d}z_n\wedge \mathrm{d}\bar{z}_n, \tag{5.9}$$

which proves 5.8. Likewise, we can parametrize the affine piece of Y by the coordinates $z_1,...,z_n$. A given metric ϕ on the restriction of $\mathcal{O}(1)$ to Y induces, by the adjunction isomorphism a measure (defined with respect to the divisor $\mathcal{D}$)

$$\mu_\phi = e^{-(n+2-m)m^{-1}\phi}\left|z_{n+1}\right|^{-2(1-1/m)}\left|z_1\right|^{-2(1-1/m)}\cdots\left|z_n\right|^{-2(1-1/m)}\frac{i}{2}\mathrm{d}z_1\wedge \mathrm{d}\bar{z}_1\cdots\frac{i}{2}\mathrm{d}z_n\wedge \mathrm{d}\bar{z}_n.$$

Since $F^*z_i = z_i^m$ this means that

$$F^*\mu_\phi = e^{-(n+2-m)f^*(m^{-1}\phi)}\left|z_{n+1}\right|^{-2(m-1)}\left|z_1\right|^{-2(m-1)}\cdots\left|z_n\right|^{-2(m-1)}\frac{i}{2}\mathrm{d}(z_1^m)\wedge \mathrm{d}(\bar{z}_1^m)\cdots\frac{i}{2}\mathrm{d}(z_n^m)\wedge \mathrm{d}(\bar{z}_n^m).$$

Finally, since $\mathrm{d}(z^m) = mz^{m-1}$ this proves the desired identity 5.6, using the representation 5.9 with $\psi = F^*(m^{-1}\phi)$. □

5.3. **Conclusion of the proof of Theorem 1.4.** The affine projection $(x_0,..,x_{n+1})\mapsto(x_0,...,x_n)$ induces an isomorphism from $\mathcal{Y}$ to $\mathbb{P}^n_{\mathbb{Z}}$, identifying $(\mathcal{Y},\mathcal{D})$ with a hyperplane arrangement $(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D})$ with simple normal crossings. It follows readily from the definition of $\mathcal{D}$ and the criterion 4.1 that $(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D})$ is K-semistable. Hence, combining Proposition 5.6 with refined bound following the statement of Theorem 1.3 yields

$$\hat{h}_{\mathrm{can}}\left(\mathcal{X}\right)\leq \hat{h}_{\mathrm{can}}\left(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t\right)-\frac{1}{2}\log\frac{V(X)}{V(\mathbb{P}^n,\Delta_t)},$$

where $\mathcal{D}_t$ is the toric divisor on $\mathbb{P}^n_{\mathbb{Z}}$ such that $(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t)$ is K-semistable and $V(\mathbb{P}^n,\Delta_t) = V(\mathbb{P}^n,\Delta)$. The explicit formula for $\hat{h}_{\mathrm{can}}\left(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t\right)$ thus yields

$$\hat{h}_{\mathrm{can}}\left(\mathcal{X}\right)\leq \hat{h}_{\mathrm{can}}\left(\mathbb{P}^n_{\mathbb{Z}}\right)-\frac{1}{2}\log\frac{V(X)}{V(\mathbb{P}^n)}.$$

Multiplying both sides with $V(X)$ reveals that

$$h_{\mathrm{can}}(\mathcal{X}) \leq \lambda h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}) - \frac{1}{2}V(X)\log\lambda, \quad \lambda := V(X)/V(\mathbb{P}^n).$$

Since $\lambda \in]0,1[$ it thus follows from Lemma 3.2 that the right hand side above is increasing with respect to λ and thus maximal when $\lambda = 1$, giving $h_{\mathrm{can}}(\mathcal{X},\mathcal{D}) \leq h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}})$. Moreover, the equality is strict if $d \geq 2$ since then $\lambda < 1$.

6. Arithmetic log surfaces

In this section we will prove Theorem 1.5. As explained in Section 1.1.4, by Theorem 1.3, the proof is reduced to showing that for any fixed metric on $-K_{(X,\Delta)}$:

- The canonical integral model $(\mathcal{X}_c,\mathcal{D}_c;-\mathcal{K}_{(\mathcal{X}_c,\mathcal{D}_c)})$ of $(X,\Delta;-K_{(X,\Delta)})$ obtained by setting $\mathcal{X}_c = \mathbb{P}^1_{\mathbb{Z}}$ and taking $\mathcal{D}_c$ to be the Zariski closure of $\Delta_{\mathbb{Q}}$ in $\mathcal{X}_c$ minimizes $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ over all integral models $(\mathcal{X},\mathcal{D};\mathcal{L})$ of $(X,\Delta;-K_{(X,\Delta)})$
- When $\mathcal{D} = 0$ the minimum is uniquely attained for $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$, up to isomorphisms over $\mathbb{Z}$.

6.1. **Preliminaries on log canonical thresholds.** Following [28, 45] a log pair $(\mathcal{X},\mathcal{D})$ is said to be *log canonical (lc)* if for any normal blow-up morphism $p:\mathcal{Y}\to\mathcal{X}$

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - p^*\mathcal{D} = \sum_i a_iE_i, \;\; a_i \geq -1, \;\; \mathcal{K}_{\mathcal{Y}/\mathcal{X}} := \mathcal{K}_{\mathcal{Y}} - p^*\mathcal{K}_{\mathcal{X}}$$

where the prime divisor E_i is either an exceptional divisor of p or the proper transform of a component of $\mathcal{D}$. The *log canonical threshold* of a $\mathbb{Q}-$divisor F on $\mathcal{X}$ with respect to the log pair $(\mathcal{X},\mathcal{D})$ is defined by

$$\mathrm{lct}\,(\mathcal{X},\mathcal{D};F) := \sup_{t>0}\{t:\ (\mathcal{X},tF+\mathcal{D})\,\text{is lc}\}$$

The following lemma follows readily from the definition:

Lemma 6.1. *For any normal blow-up morphism $p:\mathcal{Y}\to\mathcal{X}$*

$$lct\,(\mathcal{X},\mathcal{D};F) \leq \inf_i \frac{a_i - b_i + 1}{c_i}$$

where a_i, b_i and c_i denote the order of vanishing along the $p-$exceptional prime divisor E_i of $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$, $p^*\mathcal{D}$ and p^*F, respectively and i ranges over all $p-$exceptional prime divisors.

6.2. **Preparations for the proof of Theorem 1.5.** The following result is a logarithmic generalization of [38, Thm 2.14 (3)] (in the case of arithmetic surfaces).

Lemma 6.2. *Let (X,D) be a log Fano curve over $\mathbb{C}$ and $(\mathcal{X},\mathcal{D})$ an arithmetic log Fano model for (X,D) such the fibers $\mathcal{X}_b$ of $\mathcal{X}$ are reduced and irreducible and the divisor $\mathcal{D}$ is horizontal (i.e. $\mathcal{D}$ is the Zariski closure of D). Assume that*

$$\alpha(\mathcal{X},\mathcal{D}) := \inf_{b,F} lct\,(\mathcal{X},\mathcal{D}+\mathcal{X}_b;\,F) \geq 1/2$$

where the inf runs over all effective $\mathbb{Q}-$divisors F on $\mathcal{X}$ linearly equivalent to $-\mathcal{K}_{(\mathcal{X},\mathcal{D})}$ and closed points b in the base $\mathcal{B} := \operatorname{Spec}\mathbb{Z}$ such that F does not contain the support of $\mathcal{X}_b$. Then

$$\frac{1}{2}\overline{\mathcal{L}}^2 + \overline{\mathcal{K}}_{(\mathcal{X}',\mathcal{D})}\cdot\overline{\mathcal{L}} \geq \frac{1}{2}\overline{\mathcal{L}}^2 + \overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}\cdot\overline{\mathcal{L}}$$

for any relatively ample model $(\mathcal{X}',\mathcal{D}';\mathcal{L}')$ of $(X,D;-K_{(X,D)})$ and given metrics on $-K_{(X,D)}$ and L.

In the proof we fix once and for all metrics on $-K_{(X,D)}$ and L and set

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}) := \frac{1}{2}\overline{\mathcal{L}}^2 + \overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})} \cdot \overline{\mathcal{L}} \tag{6.1}$$

for the corresponding metrized lines bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$. Thus $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L})$ specializes to the arithmetic log Mabuchi functional $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ (formula 1.5) precisely when the metric on $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$ is the one induced from the curvature form ω of $\overline{\mathcal{L}}$. But here it will be convenient to consider the present more general setup.

Proof when D is trivial. To fix ideas we first consider the case when D is trivial. Set $\mathcal{B} :=$ Spec $\mathbb{Z}$ and $\mathcal{L} := -\mathcal{K}_{\mathcal{X}}$. To simplify the notation we will remove the bar indicating the metric in the notation for the arithmetic intersection numbers. Anyhow, all the arithmetic intersections will be computed over the closed points b in the base $\mathcal{B}$ and are thus independent of the choice of metric (since they are proportional to the algebraic intersections on the scheme $\pi^{-1}(b)$ over the residue field of b). If F_1 and F_2 are $\mathbb{Q}-$divisors we will write $F_1 \geq F_2$ if $F_1 - F_2$ is effective.

Step 1: It is enough to consider the case of a relatively semi-ample model of the form $(\mathcal{X}', \mathcal{L}') = (\mathcal{Y}, p^*\mathcal{L} - E)$ where $p : \mathcal{Y} \to \mathcal{X}$ is the blow-up along a closed subscheme $\mathcal{Z}$ of $\mathcal{X}$ and E is an effective $p-$exceptional divisor on $\mathcal{Y}$ whose support contains all the $p-$exceptional prime divisors and such that for any $b \in \pi(\mathcal{Z})$ $p^*\mathcal{L} - E$ admits a global section s_b not vanishing identically along $\mathcal{Y}_b$.

This is shown precisely as in the proof of [37, Prop 3.10] - for completeness a proof is given in Step 1 in Section 6.3 below.

Step2: The inequality holds in the case of Step 1.

First observe that

$$\mathcal{M}_{\mathcal{Y}}(\mathcal{L}') - \mathcal{M}_{\mathcal{X}}(\mathcal{L}) = \mathcal{L}' \cdot \left(\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E\right). \tag{6.2}$$

Indeed, rewriting

$$\mathcal{M}_{\mathcal{X}}(\mathcal{L}) = -\frac{\mathcal{L}^2}{2} + \cdot \mathcal{L} \cdot (\mathcal{L} + \mathcal{K}_{\mathcal{X}})$$

(and likewise for $(\mathcal{Y}, \mathcal{L}')$) the left hand side in formula 6.2 may be expressed as

$$\frac{p^*\mathcal{L}^2 - \mathcal{L}'^2}{2} + \mathcal{L}' \cdot (\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - E) = \frac{E \cdot (p^*\mathcal{L} + \mathcal{L}')}{2} + \mathcal{L}' \cdot (\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - E).$$

Since $E \cdot p^*\mathcal{L} = 0$ this proves formula 6.2.

Since $\mathcal{L}'$ is relatively semi-ample it will thus be enough to show that the vertical exceptional divisor $\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E$ is effective. This means, by the assumption on $\alpha(\mathcal{X})(:= \alpha(\mathcal{X}, 0))$, that it is enough to show that

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E \geq 0 \tag{6.3}$$

To fix ideas first assume that $\pi(\mathcal{Z})$ is supported on a single point that we denote by b. By Step 1, we can express $s_b = p^*s$ for a global section s of $\mathcal{L} \to \mathcal{X}$ whose zero-divisor F does not vanish identically on $\mathcal{X}_b$ and such that $p^*F - E$ is effective. Since F is a contender for the inf defining $\alpha(\mathcal{X})$ we have

$$\alpha(\mathcal{X}) \leq \text{lct}(\mathcal{X}, \mathcal{X}_b;\, F),$$

Next, since $p^*F \geq E$ it follows from Lemma 6.1 that

$$\text{lct}(\mathcal{X}, \mathcal{X}_b;\, F) \leq \inf_i \frac{a_i + 1 - b_i}{c_i}$$

where i runs over the $p-$exceptional irreducible prime divisors E_i of $\mathcal{Y}$ and a_i, b_i and c_i denote the order of vanishing along E_i of $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$, $\mathcal{Y}_b$ and E_{ex} respectively. Note that $c_i > 0$ (since the support of E contains the support of all $p-$exceptional divisors) and $b_i \geq 1$ (since $\mathcal{Z}$ is assumed to be supported in $\mathcal{X}_b$). Thus

$$\alpha(\mathcal{X}) \leq \inf_i \frac{a_i + 1 - b_i}{c_i} \leq \inf_i \frac{a_i}{c_i},$$

giving

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E \geq \sum_i a_i E_i - \left(\min_j \frac{a_j}{c_j}\right) E_i = \sum_i \left(\frac{a_i}{c_i} - \left(\min_j \frac{a_j}{c_j}\right)\right) c_i E_i \geq 0,$$

which proves 6.3. Finally, consider the general case when the support of $\pi(\mathcal{Z})$ consists of a finite number of points b_m in $\mathcal{B}$. We then split the vertical divisors $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$ and E into the components $\mathcal{K}^{(m)}_{\mathcal{Y}/\mathcal{X}}$ and $E^{(m)}$ over b_m :

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E = \sum_m \mathcal{K}^{(m)}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E^{(m)}$$

and apply the previous bound for each fixed m (with b replaced by b_m) to get that $\mathcal{K}^{(m)}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E^{(m)} \geq 0$ and thus $\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E \geq 0$, as desired.

Proof for log pairs. Just as in the previous case it is enough to consider the special case of Step 2. In this case formula 6.2 readily generalizes to

$$\mathcal{M}_{(\mathcal{Y},q^*\mathcal{D}')}(\mathcal{L}') - \mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}) = \mathcal{L}' \cdot \left(\mathcal{D}' - p^*\mathcal{D} + \mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E\right).$$

As before it will thus be enough to show that

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} + q^*\mathcal{D}' - p^*\mathcal{D} - \alpha(\mathcal{X},\mathcal{D})E \geq 0. \tag{6.4}$$

To simplify the exposition we will assume that $\pi(\mathcal{Z})$ is a single closed point in $\mathcal{B}$, denoted by b (the general case is shown in a similar way by decomposing wrt the components of $\pi(\mathcal{Z})$ as above). By the definition of $\alpha(\mathcal{X},\mathcal{D})$

$$\alpha(\mathcal{X},\mathcal{D}) \leq \mathrm{lct}(\mathcal{X},\mathcal{D}+\mathcal{X}_b;\, F), \tag{6.5}$$

Next, since $p^*F - E$ is effective, i.e. $p^*F \geq E$ Lemma 6.1 yields

$$\mathrm{lct}(\mathcal{X},\mathcal{D}+\mathcal{X}_b;\, F) \leq \inf_i \frac{a_i + 1 - d_i - b_i}{c_i},$$

where a_i, b_i, c_i and d_i are the order of vanishing along E_i of $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$, $\mathcal{Y}_b$, E_{ex} and $p^*\mathcal{D}$ respectively. In particular, $b_i \geq 1$ since $\mathcal{Z}$ is supported in $\mathcal{X}_b$, Hence,

$$\alpha(\mathcal{X},\mathcal{D}) \leq \inf_i \frac{a_i + 1 - d_i - b_i}{c_i}.$$

Next, we may decompose

$$p^*\mathcal{D} = (p^*\mathcal{D})_{\mathrm{hor}} + (p^*\mathcal{D})_{\mathrm{ex}},$$

where $(p^*\mathcal{D})_{\mathrm{hor}}$ is the horizontal divisor obtained as the proper transform of the horizontal divisor $\mathcal{D}$ and $(p^*\mathcal{D})_{\mathrm{ex}}$ is $p-$exceptional. By 6.5 $\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - (p^*\mathcal{D})_{\mathrm{ex}} - \alpha E_{\mathrm{ex}} \geq$

$$\geq \sum_i (a_i - d_i)E_i - \left(\min_j \frac{a_j - b_j + 1 - d_j}{c_j}\right) c_i E_i \geq \sum_i \left(\frac{(a_i - d_i)}{c_i} E_i - \left(\min_j \frac{a_j - d_j}{c_j}\right)\right) c_i E_i \geq 0$$

using that $b_j \geq 1$. Hence,

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} + \mathcal{D}' - p^*\mathcal{D} - \alpha(\mathcal{X},\mathcal{D})E_{\text{ex}} \geq \mathcal{D}' - (p^*\mathcal{D})_{\text{hor}}.$$

But, since both $(\mathcal{X}',\mathcal{D}')$ and $(\mathcal{X},\mathcal{D})$ are models for (X,D) and $\mathcal{D}$ is assumed horizontal it follows that $q^*\mathcal{D}' - (p^*\mathcal{D})_{\text{hor}}$ is an effective vertical divisor and hence

$$q^*\mathcal{D}' - (p^*\mathcal{D})_{\text{hor}} \geq 0,$$

which concludes the proof of the inequality 6.4.

Lemma 6.3. *Assume that (X,D) is a K-semistable log Fano curve over $\mathbb{C}$. For the canonical model $(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})$ of (X,D)*

$$\alpha(\mathcal{X},\mathcal{D}) \geq 1/2.$$

Proof. By inversion of adjunction on surfaces over excellent schemes [45]

$$\text{lct }(\mathcal{X},\mathcal{D}+\mathcal{X}_b;\, F) = \text{lct }(\mathcal{X},\mathcal{D}_{|\mathcal{X}_b}; F_{|\mathcal{X}_b}),$$

if F does not contain the support of the divisor $\mathcal{X}_b$. In the present case $\mathcal{X}_b = \mathbb{P}^1_{\mathbb{F}_b}$, where b has been identified with a prime number and $\mathbb{F}_b$ denotes the field with b elements. Decomposing

$$D = \sum w_i D_i$$

the K-semistability assumption is, by 4.2, equivalent to the condition

$$w_j \leq \frac{1}{2}\sum_i w_i, \ \ \forall j. \tag{6.6}$$

We recall that for any curve C over a perfect field (here taken to be $\mathbb{P}^1_{\mathbb{F}_b}$) an effective $\mathbb{Q}-$divisor F on C is lc iff all its coefficients are less then are equal to one [28, 45]. Since $-K_{\mathbb{P}^1_{\mathbb{F}_b}}$ is linearly equivalent to $\mathcal{O}(2)$ and $D_{i|\mathcal{X}_b}$ is linearly equivalent to $\mathcal{O}(1)$ it thus follows from the weight condition 6.6 that $\text{lct}(\mathcal{X},\mathcal{D}_{|\mathcal{X}_b}; F_{|\mathcal{X}_b}) \geq 1/2$. Indeed, it is enough to consider the case when $F = (2-\sum_i w_i)[x]$, where $[x]$ denotes the prime divisor on $\mathbb{P}^1_{\mathbb{F}_b}$ corresponding to a closed point x in $\mathbb{P}^1_{\mathbb{F}_b}$. Then

$$\frac{1}{2}F + \mathcal{D}_{|\mathcal{X}_b} = (1-\frac{1}{2}\sum_i w_i)[x] + \sum w_i[x_i],$$

In the case that $x \neq x_i$ for any i the coefficients of $F/2 + \mathcal{D}_{|\mathcal{X}_b}$ are indeed less than or equal to 1, since $w_i \in [0,1]$. Moreover, if $x = x_j$ then the coefficient of index j equals $(1-\frac{1}{2}\sum_i w_i) + w_j$ which is less than are equal to 1, by the weight condition 6.6.

We will also need the following lemma, shown precisely as in the case when $\mathcal{D} = 0$ considered in [1, Prop 5.3]. □

Lemma 6.4. *Let $(\mathcal{X},\mathcal{D};\mathcal{L})$ be a polarized arithmetic log surfaces $(\mathcal{X},\mathcal{D};\mathcal{L})$ such that the complexification (X,Δ) of $(\mathcal{X},\mathcal{D})$ is a Fano variety and $\mathcal{L}\otimes\mathbb{C} = -K_{(X,\Delta)}$. A metric realizes the infimum*

$$\inf_{\|\cdot\|} \mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L},\|\cdot\|)$$

over all locally bounded metrics on $-(K_{(X,D)})$ with positive curvature current iff the metric is a log Kähler-Einstein metric. In particular, in the case when $D = 0$ any minimizer coincides with the Fubini-Study metric up to the application of an automorphism of X and a scaling of the metric. Moreover,

$$\inf_{\|\cdot\|} \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}\left(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})},\|\cdot\|\right) = -\sup_{\|\cdot\|}\frac{1}{2}\left(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})},\|\cdot\|\right)^2$$

where the sup in the right hand side is restricted to volume-normalized metrics.

6.3. **Conclusion of the proof of Theorem 1.2 and Corollary 1.6.** Combining the previous first two lemmas immediately yields

$$\mathcal{M}_{(\mathcal{X}',\mathcal{D}')}(\overline{\mathcal{L}'}) \geq \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}(\overline{-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}}).$$

Applying the third lemma above thus gives

$$\mathcal{M}_{(\mathcal{X}',\mathcal{D}')}(\overline{\mathcal{L}'}) \geq -\sup_{\|\cdot\|}\left(\frac{1}{2}\left(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})},\|\cdot\|\right)^2\right)$$

where the infimum in the left hand side is restricted to volume-normalized metrics. Invoking Theorem 1.3 and using that the Fubini-Study metric is a minimizer when $\mathcal{D}=0$ (by Lemma 6.4) thus proves the inequality in Theorem 1.5 and it corollary. Moreover, Theorem 1.3 implies that the inequality is strict, as soon as D is non-trivial.

6.3.1. *The equality case.* Consider now the case of equality in Theorem 1.5 (and, as a consequence, $D=0$) :

$$\mathcal{M}_{(\mathcal{X}',\mathcal{D}')}(\overline{\mathcal{L}'}) = \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},0)}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}}) \tag{6.7}$$

where, in the rhs, $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ is endowed with the Fubini-Study metric. By the minimizing property in Lemma 6.1, when $D=0$, the metric on $\mathcal{L}$ coincides with the Fubini-Study metric up to the application of an automorphism of X and scaling of the metric. All that remains is to show is thus that $(\mathcal{X}',\mathcal{D}')$ is isomorphic to $(\mathbb{P}^1_{\mathbb{Z}},0)$. To this end first note that since $\mathcal{X}(=\mathbb{P}^1_{\mathbb{Z}})$ and $\mathcal{X}'$ have the same generic fiber they are birationally equivalent. Thus, there exists a normal variety $\mathcal{Y}$, which is flat and projective over $\mathcal{B}$, dominating both $\mathcal{X}$ and $\mathcal{X}'$, with birational morphisms

$$p:\mathcal{Y}\to\mathcal{X},\;\; q:\mathcal{Y}\to\mathcal{X}' \tag{6.8}$$

which are the identity over the generic point in $\mathcal{B}$ (a concrete construction is given in Step 1 below). It will thus be enough to show that the equality 6.7 implies that p can be taken to be an isomorphism. Indeed, if p is an isomorphism we get a birational morphism q from $\mathbb{P}^1_{\mathbb{Z}}$ to $\mathcal{X}'$ and any such morphism is an isomorphism (since the fibers of $\mathbb{P}^1_{\mathbb{Z}}$ over $\mathcal{B}$ are all reduced and irreducible). Moreover, when $\mathcal{X}'$ is equal to $\mathbb{P}^1_{\mathbb{Z}}$ any $\mathcal{L}$ whose complexification equals $-K_{\mathbb{P}^1}$ is isomorphic to $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ and the components of any divisor $\mathcal{D}'$ on $\mathcal{X}'$ whose complexification is trivial are fibers $\mathcal{X}_{b_i}$ of $\mathbb{P}^1_{\mathbb{Z}}$ (using again that the fibers of $\mathbb{P}^1_{\mathbb{Z}}$ over $\mathcal{B}$ are all reduced and irreducible). Hence, the assumed equality 6.7 implies, since $\mathcal{X}_{b_i}^2=0$ and $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ is relatively ample that $\mathcal{D}'$ is trivial, i.e. $\mathcal{D}'=0$.

Thus all that remains is to show that the assumed equality in formula 6.7 implies that the morphism $p:\mathcal{Y}\to\mathcal{X}$ (in formula 6.8) can be taken to be an isomorphism.

Step 1: In the case of arithmetic surfaces $p:\mathcal{Y}\to\mathcal{X}$ can be taken as the successive blow-ups of $\mathcal{X}$ along a finite number of closed points x_i in regular surfaces $\mathcal{X}_i$ and there exists a $p-$exceptional and $p-$ample divisor E on $\mathcal{Y}$ and a morphism q from $\mathcal{Y}$ to $\mathcal{X}$ *such that $q^*\mathcal{L}'=p^*\mathcal{L}-E$. In particular, $\mathcal{M}_{\mathcal{X}'}(\mathcal{L}')=\mathcal{M}_{\mathcal{Y}}(p^*\mathcal{L}-E)$.*

This is shown as in the beginning of the proof of [37, Prop 3.10], as next explained. First note that since $\mathcal{X}$ and $\mathcal{X}'$ have the same generic fiber they are birationally equivalent. Since $\mathcal{X}$ is normal this means that there exists a morphism $h:U\to\mathcal{X}'$ from a Zariski open subset U in $\mathcal{X}$ of codimension two. As a consequence, $h^*\mathcal{L}'$ extends to a $\mathbb{Q}-$line bundle $\mathcal{L}''$ on $\mathcal{X}$ coinciding with $-\mathcal{K}_{\mathcal{X}}$ on the generic fiber. Since $\mathcal{X}=\mathbb{P}^1_{\mathbb{Z}}$ this implies that $\mathcal{L}''$ is isomorphic to $-\mathcal{K}_{\mathcal{X}}$ (using, for example, that $\pi:\mathcal{X}\to \operatorname{Spec}\mathbb{Z}$ has reduced irreducible fibers). Now fix a positive integer k such that $k\mathcal{L}'$ is a relatively very ample line bundle and take a basis s_i' in the free $\mathbb{Z}-$module $H^0(\mathcal{X}',k\mathcal{L}')$. Then $s_i:=h^*s_i$ extends, since $\mathcal{X}$ is normal, to a unique element in $H^0(\mathcal{X},k\mathcal{L})$. Denote by $\mathcal{J}$ the ideal sheaf on $\mathcal{X}$ generated by the sections s_i. Since $\mathcal{X}$ is a regular surface we get after successive blow-ups (as stated in Step 1) a morphism $p:\mathcal{Y}\to\mathcal{X}$ from a regular surface $\mathcal{Y}$ to $\mathcal{X}$ with the property that $p^*\mathcal{J}$ defines an effective $p-$exceptional divisor E_k on $\mathcal{Y}$ (using that $\mathbb{Z}$ is an excellent ring) [53]. Set

$$E:=k^{-1}E_k \quad (E_k:=p^*\mathcal{J})$$

By construction, E_k is $p-$ample,

$$H^0(\mathcal{Y},kp^*\mathcal{L}-E_k)\cong H^0(\mathcal{X},k\mathcal{L}\otimes\mathcal{J})\cong H^0(\mathcal{X}',k\mathcal{L}') \tag{6.9}$$

and the global sections of $kp^*\mathcal{L}-E_k$ induce a morphism q to $\mathcal{X}'$ such that $q^*\mathcal{L}'=p^*\mathcal{L}-E$. Finally, note that

$$\mathcal{M}_{\mathcal{X}'}(\mathcal{L}')=\mathcal{M}_{\mathcal{Y}}(q^*\mathcal{L}'),$$

as follows directly from the fact that p is an isomorphism between Zariski open subsets of $\mathcal{X}'$ and $\mathcal{Y}$ and, as a consequence, the $\mathbb{Q}-$line bundle $q^*\mathcal{L}'$ is trivial on the support of the divisor $q^*\mathcal{K}_{\mathcal{X}}-\mathcal{K}_{\mathcal{Y}}$.

Step 2: $\mathcal{M}_{\mathcal{Y}}(p^*\mathcal{L}-E)=\mathcal{M}_{\mathcal{X}}(\mathcal{L})\implies$ *p is an isomorphism, when* $\mathcal{X}=\mathbb{P}^1_{\mathbb{Z}}$

Replacing $q^*\mathcal{L}'$ with $p^*\mathcal{L}-E$ in formula 6.2 yields, since $\mathcal{M}_{\mathcal{Y}}(p^*\mathcal{L}-E)=\mathcal{M}_{\mathcal{X}}(\mathcal{L})$,

$$(p^*\mathcal{L}-E)\cdot\left(\mathcal{K}_{\mathcal{Y}/\mathcal{X}}-\frac{1}{2}E\right)=0.$$

It follows, since, by construction, $p^*\mathcal{L}-E$ is $p-$ample, that

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}}=\frac{1}{2}E. \tag{6.10}$$

Now, since $p:\mathcal{Y}\to\mathcal{X}$ is the blow-up along a finite number of closed points x_i in regular surfaces $\mathcal{X}_i$,

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}}=\sum c_iE_i,\ \ c_i\geq 1 \tag{6.11}$$

where the sum runs over all prime $p-$exceptional divisors E_i. Hence,

$$E=\sum_i 2c_iE_i\geq\sum_i 2E_i.$$

But this contradicts the isomorphisms 6.9, if the number of points x_i is non-zero. Indeed, denote by E_1 the strict transform of the $p-$exceptional divisor on $\mathcal{Y}$ induced from the exceptional divisor on the first point x_1 blown-up on $\mathcal{X}(=\mathbb{P}^1_{\mathbb{Z}})$. Then it it follows from the previous inequality and the construction of E that the restriction of the ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ to a neighbourhood of x_1 in the fiber $\mathcal{X}_{\pi(x_1)}$ is contained in the $2k$th power $\mathfrak{m}_{x_1}^{2k}$ of the maximal ideal $\mathfrak{m}_{x_1}$ on $\mathcal{X}_{\pi(x_1)}$ defined by the point x_1. But, in general, for $\mathcal{X}=\mathbb{P}^1_{\mathbb{Z}}$, the line bundle $k\mathcal{L}_{|\mathcal{X}_{\pi(x)}}\otimes\mathfrak{m}_x^{2k}$ on $\mathcal{X}_{\pi(x)}$ is trivial for any closed point x on $\mathcal{X}$ (since $\mathcal{L}_{|\mathcal{X}_b}:=-K_{\mathcal{X}_b}=\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_b}}(2)$). But this contradicts the isomorphism 6.9, since $\mathcal{L}'$ is relatively ample. Hence, the number of points x_i must be zero, as desired.

Combing these two steps thus concludes, as discussed above, the proof of Theorem 1.5. Finally, Corollary 1.6 can be deduced from Theorem 1.5 using a generalization of Lemma 6.4. But here we instead proceeds as follows. Given an arithmetic log Fano surface $(\mathcal{X},\mathcal{D})$ set

$\mathcal{L} := -\mathcal{K}_{\mathcal{X}}$ and endow $\mathcal{L}$ and $-\mathcal{K}_{\mathcal{X}}$ with the same metric induced from a volume-normalized metric on $-K_X$ with positive curvature current. Then, by definition 6.1,

$$-\frac{1}{2}\overline{\mathcal{K}}^2_{(\mathcal{X},\mathcal{D})} = \mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}).$$

Hence, combining Step one and Step two above yields

$$-\frac{1}{2}\overline{\mathcal{K}}^2_{(\mathcal{X},\mathcal{D})} \geq \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},0)}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}}) = -\frac{1}{2}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}})^2$$

and the equality case is deduced precisely as before.

7. Appendix

In the proof of Lemma 5.3 we used the following result (applied to $X = \mathbb{P}^n_{\mathbb{C}}$).

Lemma 7.1. *Let X be a Fano manifold and V a holomorphic vector field on X. Denote by F_t the flow of the real part of V on X at time t and by $(F_t^V)^*\phi$ its action on a given continuous metric ϕ on $-K_X$ with positive curvature current. If X admits a Kähler-Einstein metric, then*

$$\frac{d}{dt}\mathcal{E}(F_t^*\phi, \psi_0) = 0$$

for any fixed metric ψ_0 on $-K_X$.

Proof. This is well-known and essentially goes back to [25], but for the convenience of the reader we provide a proof in the spirit of the present paper and its precursor [2]. Consider the *Ding functional* on the space of all continuous metrics on $-K_X$ with positive curvature current, defined by

$$\mathcal{D}(\phi) := -\frac{n!}{(-K_X)^n}\mathcal{E}(\phi) - \log \int_X \mu_\phi,$$

where μ_ϕ is the measure on X corresponding to the metric ϕ (see Section 2.1, with $\Delta = 0$) and $\mathcal{E}(\phi)$ is a shorthand for the functional $\mathcal{E}(\phi, \psi_0)$ defined in formula 2.7. Since, $\mu_{F^*\phi} = F^*\mu_\phi$ for any biholomorphism F of X it follows that $\mathcal{E}(F_t^*\phi)$ and $\mathcal{D}(F_t{}^*\phi)$ have the same derivative. Moreover, in general, the function $t \mapsto \mathcal{D}(F_t^*\phi)$ is linear. Indeed, its derivative is the Futaki invariant of V (see the claim on page 73 in [47], where $\mathcal{D}$ is denoted by F_ω). Hence, all that remains is to verify that $\mathcal{D}$ is bounded from below (since then $t \mapsto \mathcal{D}(F_t^*\phi)$ must be constant). But this follows from the existence of a Kähler-Einstein metric, since such a metric minimizes $\mathcal{D}$, as recalled in [2, Section 2.3]. □

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