

Working Paper in Economics No. 836

Risk management of stock portfolios with jumps at exogenous default events

Alexander Herbertsson

Department of Economics, September 2023

ISSN 1403-2473 (Print)
ISSN 1403-2465 (Online)



UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

RISK MANAGEMENT OF STOCK PORTFOLIOS WITH JUMPS AT EXOGENOUS DEFAULT EVENTS

ALEXANDER HERBERTSSON

ABSTRACT. In this paper we study equity risk management of stock portfolios where the individual stock prices have downward jumps at the defaults of an exogenous group of defaultable entities. The default times can come from any type of credit portfolio model. In this setting we derive computational tractable formulas for several stock-related quantiles, such as loss distributions of equity portfolios and apply it to Value-at-Risk computations. We start with individual stock prices and then extend the setting to a portfolio framework. In the portfolio case our studies considers both small-time expansions of the loss-distribution for a heterogeneous portfolio via a linearization of the loss, but also for general time points when the stock portfolio is large and homogeneous and where we use a conditional version of the law of large numbers. Most of the derived formulas will heavily rely on the ability to efficiently compute the number of defaults distribution of the entities in the exogenous group of corporates negative affecting the stock prices in our equity portfolio. If the stock prices are unaffected by the exogenous defaults then our framework collapses into the traditional Black-Scholes model under the real probability measure. Finally, we give several numerical applications. For example, in a setting where the jumps in the stock prices are at default times which are generated by a one-factor Gaussian copula model, we study the time evolution of Value-at-Risk (i.e. VaR as function of time) for stock portfolios, both for a 20-day period and for a two-year period. We also perform similar numerical VaR-studies in a setting where the individual default intensities follow a CIR process. Our results are compared with the corresponding VaR-values in the Black-Scholes case with same drift and volatilities as in the jump models. Not surprisingly, we show that the VaR-values in stock portfolios with downward jumps at defaults of external entities, will have substantially higher VaR-values compared to the corresponding Black-Scholes cases. The numerical computations of the number of default distribution will in all our studies use fast and efficient saddlepoint methods.

Keywords: equity portfolio risk; stock price modelling; credit portfolio risk; risk management; Value-at-Risk, intensity-based models; credit copula models, numerical methods.

JEL Classification: G33; G13; C02; C63; G32 .

1. INTRODUCTION

Simultaneous downward jumps in multiple stock prices at defaults of large companies is a very realistic feature. For example, at the default of Lehman Brothers in September 2008 there were downward jumps in most of the stock prices traded on financial markets all over the world.

In this paper we study equity risk management of stock portfolios where the individual stock prices have simultaneous downward jumps at the defaults of an exogenous group of defaultable entities, for example corporates or sovereign states. By "exogenous" we here mean that the entities, for example companies, will not be represented in the stock portfolio, that is stocks issued by the defaultable corporates are not present in the stock portfolio in our studies. The default times can come from any type of credit portfolio model. In this setting we derive computational tractable formulas to several stock-related quantiles, for example the loss distributions of equity portfolios and apply it to risk management computations such as Value-at-Risk of portfolios. We start with modelling an individual stock price and derive expressions for the expected value, conditional expected value, density and distribution for the stock. In the stock portfolio case our studies considers both small-time expansions of the loss-distribution to a heterogeneous portfolio via a linearization of the loss, but also for general time points when the stock portfolio is large and homogeneous, where we utilize a conditional version of the law of large numbers for a homogeneous stock portfolio. Most of the formulas in this paper will heavily rely on the ability to efficiently compute the number of defaults distribution of the entities in the exogenous group which are negative affecting the stock prices in our equity portfolio. In the case when the stock prices are unaffected by the exogenous defaults our stock price model collapses into the traditional Black-Scholes model under the real probability

Date: September 1, 2023.

The research was supported by Nasdaq Nordic Foundation, Vinnova, and Jan Wallanders och Tom Hedelius stiftelse. We are grateful for the comments received from participants at the 11th General AMaMeF conference 2023.

measure. Finally, we give several numerical applications. For example, in a setting where the jumps in the stock prices are at default times which are generated by one-factor Gaussian copula model, we study the time evolution of Value-at-Risk (i.e. VaR as function of time) for stock portfolios, both for a 20-day period with one-day steps and for a two-year period with one-month steps. In the 20-day period we use the linear approximation for the loss-process, while for the two-year period we utilize a large portfolio approximation formula for the loss-process to a large homogeneous stock portfolio. Furthermore, in a setting where the jumps in the stock price are at default times which have CIR-intensities, we also study the time evolution of Value-at-Risk for one stock over a two-year period. In all our numerical computations we also compare our results with the corresponding VaR-values in the Black-Scholes case with same drift and volatilities as in the jump models. Not surprisingly, we show that the VaR-values in stock portfolios with stock that jump downward at defaults of external companies will have substantially higher VaR-values compared to the corresponding Black-Scholes cases. The numerical computations of the number of default distribution will in all our VaR-studies use fast and efficient saddlepoint methods developed in Herbertsson (2022).

There exists a huge amount of academic papers that models stock prices with jumps and a vast majority of these articles which contains numerical/practical examples consider the case where jump times are driven by some sort of Poisson process. Furthermore, most of the jump-related equity papers model the stock price directly under the risk neutral probability measure and then apply the model for option pricing, such as e.g. the original paper Merton (1976). An example of an article that actually models the stock price under the real (physical) probability measure is the seminal paper Kou (2002) where the stock price jump either up or down at random times driven by a Poisson process with constant intensity. Kou (2002) mainly studies option pricing directly under the real probability measure by using asset pricing theory, consumption utilizes and the Euler equation where both the endowment process and the stock price follows the type of jump diffusion as defined in Kou (2002). More about option pricing models for stocks with jumps driven by Poisson processes (such as Levy processes) can be found in e.g. the books Schoutens (2003) or Cont & Tankov (2004).

In this paper all jumps in the stock prices are downward jumps occurring at the defaults in an exogenous group of defaultable entities. Hence, in this paper we have explicitly inserted "external" credit risk (from the external group of defaultable entities) into the equity dynamics or our stock price, effectively creating a type of *hybrid risk model*. Thus, the stock price model in this paper involves *both equity and credit risk*, although the credit risk comes from an external group of defaultable entities which can be corporate or sovereign states. Furthermore, we work under the real (physical) probability measure and focus on risk management such as VaR-computations of e.g. stock portfolios. To the best of our knowledge, this is the first paper that numerically computes VaR and related risk management quantities for stock portfolios where all the stock prices have simultaneous jumps at defaults in an external group of arbitrary many defaultable entities. Numerical studies of densities to the stock price, log-asset returns and related quantities in a model as in this paper are done in Herbertsson (2023).

Assuming only negative jumps in the stock prices will lead to a more conservative or prudent equity portfolio model which implies larger Value-at-Risk losses compared to a model which also includes positive jumps. Including only negative jumps in stock prices for e.g. VaR-models should therefore be more favourable among financial regulators (such as e.g. SEC, FCA, BaFin etc.) compared with frameworks that also contains positive jumps in equity prices. In our model it is possible to add another jump process in the dynamics of the stock price, for example a Poisson process with constant intensity and with positive jumps, e.g. as in Kou (2002). However, in this paper we are *only* interested in studying the effect of *external credit risk* on stock prices, coming from the external group of defaultable entities and therefore our jump-part in the dynamics of the stock will only include negative jumps occurring at the external default times. Furthermore, if the defaultable entities used in our stock price model have issued bonds (or stocks) which are publicly traded on major financial markets, then typically their default times are exogenously observed, as for example the default of Lehmann Brothers in 2008. On the other hand, if a Poisson process drives the times when the stock price jumps then these jump times can be difficult to observe exogenously and also difficult to assign to a specific financial event.

The rest of the paper is organized as follows. First, in Section 2 we consider one stock where the stock price can jump at default times belonging to an exogenous group of defaultable entities and then derive all relevant quantities such as the expected value, conditional expected value, density and distribution both for the stock and its loss process. Next, in Section 3 we generalize the single-stock dynamics in Section 2 to a heterogeneous portfolio of stocks and define the loss process for the stock portfolio. Furthermore, for small time points we make a linearization of the portfolio loss process and then derive a

computationally tractable expression for distribution of the linearized loss. For larger time points t , the linear approximations to the stock portfolio in Section 2 will fail but in Section 4 we outline a method that will work for arbitrary time points for large homogeneous stock portfolios, and derive a convenient expression for the distribution of the portfolio loss in such settings by using large portfolio approximations. In the numerical part of the paper covered in Section 5-6, we give several practical applications of our developed stock price model. First, in Section 5 we study Value-at-Risk over a two-year period for the loss of one single stock when the stock price is defined as in Section 2 in a model where the default times are exchangeable, conditional independent and have CIR-intensities. Finally, in Section 6 we repeat similar VaR-studies as in Section 5 but now for a portfolio of stocks in a setting with jumps in all stock prices occurring at default times driven by a one-factor copula model, and by using the small-time expansion formulas for the loss process derived in Section 3. In the Gaussian copula model we also in Section 6 do VaR-computations for large stock portfolios by using the large portfolio approximation formulas derived in Section 4, both for a 20-day period in time steps of one trading day, but also over a two-year period in time steps of one month. All computations done in Section 5 and Section 6 heavily rely on efficient numerical methods developed in Herbertsson (2022) for computing the distribution of number of defaults among the defaultable entities creating the jumps in the stock prices.

2. THE ONE-DIMENSIONAL CASE

In this subsection we consider one stock where the stock price can jump at default times belonging to an exogenous group of defaultable entities. We first define the dynamics of the stock price under the real (physical) probability measure \mathbb{P} that will be used throughout the first sections of the paper. Furthermore, we also derive all relevant quantities for the single stock, such as the expected value, conditional expected value, density and distribution both for the stock and its loss process. We start with the following definition of the stock price.

Definition 2.1. Consider a group of m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ with individual default times $\tau_1, \tau_2, \dots, \tau_m$ and let $\tilde{V}_1, \dots, \tilde{V}_m$ be random variables which have bounded expected values and satisfy $\tilde{V}_i \geq -1$, and are independent of $\tau_1, \tau_2, \dots, \tau_m$. Let company \mathbf{A} be an entity which does **not belong** to the group $\mathbf{C}_1, \dots, \mathbf{C}_m$ and let S_t denote the price of the stock to company \mathbf{A} at time t . The dynamics of S_t under the real probability measure \mathbb{P} is defined as

$$dS_t = S_{t-} dY_{t-} \quad (2.1)$$

where Y_t is given by

$$Y_t = \mu t + \sigma W_t + \sum_{i=1}^m \tilde{V}_i 1_{\{\tau_i \leq t\}} \quad (2.2)$$

and W_t is Brownian motion independent of the default times $\tau_1, \tau_2, \dots, \tau_m$ and $\tilde{V}_1, \dots, \tilde{V}_m$. Finally, $\sigma \geq 0$ is the so called volatility and μ is denoted as the drift of the stock price S_t .

Remark 2.2. We remark that the default times $\tau_1, \tau_2, \dots, \tau_m$ in Definition 2.1 can come from any credit portfolio model as long as the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ in the stock prices at the default times $\tau_1, \tau_2, \dots, \tau_m$ are independent of these defaults and also independent of the Brownian motion. We can for example work with heterogeneous or homogeneous copula based models studied in e.g. Li (2000), Gregory & Laurent (2005), Gregory & Laurent (2003), Andersen & Sidenius (2004), Crépey, Jeanblanc & Wu (2013), Burtschell, Gregory & Laurent (2009), Hofert & Scherer (2011) or heterogeneous or homogeneous conditional independent intensity based models such as in Bielecki, Cousin, Crépey & Herbertsson (2014b), Bielecki, Cousin, Crépey & Herbertsson (2014c) and Bielecki, Cousin, Crépey & Herbertsson (2014a) as well as heterogeneous or homogeneous contagion models studied in e.g. Herbertsson (2005), Herbertsson (2007), Herbertsson & Rootzén (2008), Herbertsson (2008b), Herbertsson (2008a), Herbertsson (2011), Cont, Deguest & Kan (2010), Cont & Kan (2011), Laurent, Cousin & Fermanian (2011), Frey & Backhaus (2008) and Frey & Backhaus (2010).

Remark 2.3. Relation to the model Kou (2002). Note that the stock price S_t in Definition 2.1 is related to the seminal paper Kou (2002). The main difference between Kou (2002) and Definition 2.1 is that Kou (2002) considers jumps coming from a Poisson process with constant intensity implying possible infinity many jumps, while the jumps in Definition 2.1 are due to the default times $\tau_1, \tau_2, \dots, \tau_m$ which comes from a finite group of m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$. Hence, Definition 2.1 implies that at each default time τ_i among the m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$, the stock price S_t will jump so that $\Delta S_{\tau_i} \neq 0$ and the

jump-times of S_t therefore have a direct financial interpretation, namely the default times τ_i among the firms $\mathbf{C}_1, \dots, \mathbf{C}_m$. Hence, the major difference between S_t in Definition 2.1 in this paper and the model by Kou (2002), is that we in Definition 2.1 have explicitly inserted "external" credit risk (from the external group $\mathbf{C}_1, \dots, \mathbf{C}_m$) into the equity dynamics for S_t , effectively creating a type of *hybrid risk model*, that is the stock price model S_t involves *both equity and credit risk*, although the credit risk comes from an external group of m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$. In Definition 2.1 it is possible to add another jump process in the dynamics of S_t , for example a Poisson process with constant intensity which jumps just as in Kou (2002). However, in this paper we are *only* interested in studying the effect of *external credit risk* on S_t coming from the external group of defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ and therefore our jump-part in the dynamics of S_t will only include the jumps coming from the default times $\tau_1, \tau_2, \dots, \tau_m$ of $\mathbf{C}_1, \dots, \mathbf{C}_m$. Another remark is that Kou (2002) mainly studies option pricing directly under the real measure \mathbb{P} , by using the Euler equation where both the endowment process and the stock price follows the type of jump diffusion as given in Section 2 of Kou (2002) and where the utility function has the special form $U(c, t) = e^{-\theta t} \frac{c^\alpha}{\alpha}$ for $0 < \alpha < 1$ or $U(c, t) = e^{-\theta t} \ln c$ for $\alpha = 0$. In this paper we will focus on equity risk management of stock portfolios (such as Value-at-Risk) where the individual stock prices have downward jumps down at the defaults of an exogenous group of defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$, as given in Definition 2.1, and we will consider both univariate and multivariate stock portfolios, as well as the case where the number of stocks in the portfolio is large. In our Value-at-Risk studies of the stock portfolios we are in particular interested in studying the effect of *external credit risk* coming from the external defaultable group of entities.

Finally we remark that if the defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ have issued bonds and/or stocks which are publicly traded on major financial markets, then typically the default times $\tau_1, \tau_2, \dots, \tau_m$ are *directly observable* on the market and the observations are exogenously observed regardless if the stock price model for S_t includes these defaults or not. This has to be compared with if a Poisson process drives the jumps which can be difficult to observe exogenously and also difficult to assign to specific financial events.

Remark 2.4. On the possibility to include company A in the group $\mathbf{C}_1, \dots, \mathbf{C}_m$. We remark that in Definition 2.1 it is possible to let company **A** be one of the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$, for example $\mathbf{A} = \mathbf{C}_m$ where we then set $\tilde{V}_m = -1$ so that $S_t = 0$ for $\tau_m \leq t$ where τ_m is the default time of **A**. Including **A** in the group $\mathbf{C}_1, \dots, \mathbf{C}_m$ where e.g. $\mathbf{A} = \mathbf{C}_m$ will create an extra complexity in the stock-related formulas, in particular if the default time of **A** will be correlated with the default times of $\mathbf{C}_1, \dots, \mathbf{C}_{m-1}$. However, in this paper we are *only* interested in studying the effect of *external credit risk* coming from the external defaultable group of entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ (for example when studying how the external credit risk affect Value-at-Risk for S_t), and we will therefore in this paper always assume that company **A** will *not belong* to the defaultable group $\mathbf{C}_1, \dots, \mathbf{C}_m$.

We now state the following useful proposition.

Proposition 2.5. *Let S_t be a stock price given by Definition 2.1 under the real probability measure \mathbb{P} . Then, with notation as above, we have*

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}} \right). \quad (2.3)$$

Proof. Let $\mathcal{F}_t^W = \sigma(W_s; s \leq t)$ be the filtration generated by the Brownian motion W_t and let $\mathcal{H}_t^i = \sigma(1_{\{\tau_i \leq s\}}; s \leq t)$ be the filtration generated by each default time τ_i and define the sigma-algebra \mathcal{V} as $\mathcal{V} = \sigma(\tilde{V}_1, \dots, \tilde{V}_m)$. Next we define the full filtration \mathcal{F}_t as

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \bigvee_{i=1}^m \mathcal{H}_t^i \vee \mathcal{V}. \quad (2.4)$$

Then Y_t in Definition 2.1 is a semimartingale with respect to the filtration \mathcal{F}_t . To see this, first note that since W_t is a Brownian motion, it is a martingale with respect to its own filtration \mathcal{F}_t^W . But due to Definition 2.1 the process W_t is independent of $\tau_1, \tau_2, \dots, \tau_m$ and $\tilde{V}_1, \dots, \tilde{V}_m$ so W_t will also be a martingale with respect to the full filtration \mathcal{F}_t given by (2.4). Hence, from (2.2) we see that Y_t can be written as a sum of local martingale with respect to \mathcal{F}_t , that is σW_t and a finite variation process, i.e. $\mu t + \sum_{i=1}^m \tilde{V}_i 1_{\{\tau_i \leq t\}}$ since \tilde{V}_i have bounded expected values. From Theorem 1 on p.102 in Protter (2003) we therefore conclude that Y_t is a semimartingale with respect to the filtration \mathcal{F}_t defined as in (2.4).

Next, we note that the differential form (2.1) can be rewritten as

$$S_t = S_0 + \int_0^t S_{s-} dY_{s-} \quad (2.5)$$

and letting S_t be given by $S_t = S_0 \tilde{S}_t$ then (2.5) can be rewritten as

$$S_0 \tilde{S}_t = S_0 \left(1 + \int_0^t \tilde{S}_{s-} dY_{s-} \right)$$

that is

$$\tilde{S}_t = 1 + \int_0^t \tilde{S}_{s-} dY_{s-}. \quad (2.6)$$

Hence, if we can find a solution to \tilde{S}_t in the SDE (2.6) then a solution to S_t in (2.5) is obtained from the relation $S_t = S_0 \tilde{S}_t$. Thus, for notational convenience we will without loss of generality assume that $S_0 = 1$ and drop the tilde notation in (2.6) so that we have

$$S_t = 1 + \int_0^t S_{s-} dY_{s-}. \quad (2.7)$$

Next, from Theorem 37 on p.84 in Protter (2003) we conclude that S_t in (2.7) is a semimartingale given by

$$S_t = \exp \left(Y_t - \frac{1}{2} [Y, Y]_t^c \right) \prod_{0 < s \leq t} (1 + \Delta Y_s) \exp(-\Delta Y_s) \quad (2.8)$$

where as usual $[Y, Y]_t^c$ denotes the path-by-path continuous part of the quadratic variation $[Y, Y]_t$, see p.70 in Protter (2003). Since $\mu t + \sigma W_t$ is a continuous process, we have

$$Y_t = \mu t + \sigma W_t + \sum_{i=1}^m \tilde{V}_i 1_{\{\tau_i \leq t\}} = \mu t + \sigma W_t + \sum_{0 < s \leq t} \Delta Y_s \quad (2.9)$$

so that

$$\exp \left(Y_t - \frac{1}{2} [Y, Y]_t^c \right) = \exp \left(\mu t + \sigma W_t - \frac{1}{2} [Y, Y]_t^c + \sum_{0 < s \leq t} \Delta Y_s \right). \quad (2.10)$$

Furthermore, note that

$$\prod_{0 < s \leq t} (1 + \Delta Y_s) \exp(-\Delta Y_s) = \exp \left(- \sum_{0 < s \leq t} \Delta Y_s \right) \prod_{0 < s \leq t} (1 + \Delta Y_s) \quad (2.11)$$

and from the definition of a Brownian motion and since $[Y, Y]_t^c$ is the continuous part of the quadratic variation $[Y, Y]_t$, we get

$$[Y, Y]_t^c = \sigma^2 t. \quad (2.12)$$

So (2.10)-(2.12) in (2.8) then gives

$$S_t = \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \prod_{0 < s \leq t} (1 + \Delta Y_s) \quad (2.13)$$

and in view of (2.2) we have

$$\prod_{0 < s \leq t} (1 + \Delta Y_s) = \prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}} \right)$$

so that (2.13) can be rewritten as

$$S_t = \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}} \right). \quad (2.14)$$

Recall that we set $S_0 = 1$ but from the arguments leading to (2.6) we can let S_0 be an arbitrary positive constant and this in (2.14) finally gives

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}} \right)$$

which proves (2.3) and this concludes the proposition. \square

Let \tilde{V}_i be the non-negative random variable in Definition 2.1 connected to default of company \mathbf{C}_i at the random default time τ_i in Definition 2.1. Then, Proposition 2.5 implies that for any default time τ_i among the m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ we have that

$$S_{\tau_i} = S_{\tau_i-} \left(1 + \tilde{V}_i\right) \quad \text{or equivalently} \quad \frac{S_{\tau_i} - S_{\tau_i-}}{S_{\tau_i-}} = \tilde{V}_i$$

i.e., there is a relative jump of random size \tilde{V}_i of the stock price S_t to company \mathbf{A} at the default time τ_i of entity \mathbf{C}_i where we remind that $\tilde{V}_i \geq -1$.

Note that if there are no jumps at the defaults of $\mathbf{C}_1, \dots, \mathbf{C}_m$, that is, if $\tilde{V}_n = 0$ for all k in Definition 2.1, then (2.3) in Proposition 2.5 implies that we are back in the classical Black-Scholes model under the real (physical) probability measure \mathbb{P} , with drift μ and volatility σ , that is

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right). \quad (2.15)$$

In the paper Kou (2002), the jumps \tilde{V}_i can be both positive or negative, where the jumps occur at the arrivals of a Poisson process, implying that the stock price can jump both up and down. In this paper we will consider all defaults among the m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ in Definition 2.1 as negative news for company \mathbf{A} , implying that the relative jumps \tilde{V}_i of the stock price S_t to company \mathbf{A} at each default of $\mathbf{C}_1, \dots, \mathbf{C}_m$ will be negative. Hence, in this paper the stock price S_t will jump downwards at the default times $\tau_1, \tau_2, \dots, \tau_m$. Furthermore, we define \tilde{V}_i as follows.

Definition 2.6. Let $\tilde{U}_1, \dots, \tilde{U}_m$ be arbitrary non-negative random variables which have bounded expected values and are independent of the default times $\tau_1, \tau_2, \dots, \tau_m$ and also independent of W_t in Definition 2.6. Then, we define the non-negative random variables $\tilde{V}_1, \dots, \tilde{V}_m$ as

$$\tilde{V}_i = e^{-\tilde{U}_i} - 1 \quad (2.16)$$

for each defaultable entity $\mathbf{C}_1, \dots, \mathbf{C}_m$.

From (2.16) it is easy to see that

$$\tilde{V}_i 1_{\{\tau_i \leq t\}} = \exp \left(-\tilde{U}_i 1_{\{\tau_i \leq t\}} \right) - 1 \quad \text{for all } t \geq 0$$

so that

$$\prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}} \right) = \exp \left(-\sum_{i=1}^m \tilde{U}_i 1_{\{\tau_i \leq t\}} \right). \quad (2.17)$$

Hence, in view of Definition 2.9 and Equation (2.17) we state the following corollary to Proposition 2.5.

Corollary 2.7. Let S_t be a stock price given by Definition 2.1 under the real probability measure \mathbb{P} and where the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ are distributed as in Definition 2.6 via the arbitrary non-negative random variables $\tilde{U}_1, \dots, \tilde{U}_m \in L_1$. Then, with notation as above, we have

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t - \sum_{i=1}^m \tilde{U}_i 1_{\{\tau_i \leq t\}} \right). \quad (2.18)$$

In this paper we are primarily interested in finding computationally tractable expressions for the distribution of the stock price S_t and use this distribution in various risk management applications under the real probability measure \mathbb{P} . Under Definition 2.1 and Definition 2.6 with heterogeneous distributions for $\tilde{U}_1, \dots, \tilde{U}_m$, then it is clear from Corollary 2.7 that the distribution of the stock price S_t will be a sum containing up to 2^m different terms. Furthermore, to find $\mathbb{P}[S_t \leq x]$ we need, for each set of defaultable entities $\mathbf{i}_k = (i_1, \dots, i_k)$, $\mathbf{i}_k \subseteq \{1, \dots, m\}$ among the group $\mathbf{C}_1, \dots, \mathbf{C}_m$, be able to find expressions for the distribution of $\sum_{n=1}^k \tilde{U}_{i_n}$. Note that there are $\binom{m}{k}$ different ways to pick out a subset $\mathbf{i}_k \subseteq \{1, \dots, m\}$ such that $\mathbf{i}_k = (i_1, \dots, i_k)$ which represents the defaults of the k entities $\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_k}$ among the m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ and where the ordering of i_1, \dots, i_k is ignored. The ordering of how the group $\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_k}$ defaults is not important, explaining the term $\binom{m}{k}$ compared to the case where ordering matters, which leads to $k! \binom{m}{k}$ different ways to pick out \mathbf{i}_k . The reason why we can ignore the ordering of the defaults follows from the structure of the jumps in (2.18) in Corollary 2.7 where we only need to keep track of *if*

an entity \mathbf{C}_i have defaulted or not. Thus, the total number of possible distinct terms in the expression for $\mathbb{P}[S_t \leq x]$ will be

$$\sum_{k=0}^m \binom{m}{k} = 2^m.$$

For example, if $m = 15$ with m different distributions of $\tilde{U}_1, \dots, \tilde{U}_m$ will then lead up to possibly $2^{15} = 32768$ different terms in the distribution $\mathbb{P}[S_t \leq x]$. These observations makes the definition of the stock price S_t in Definition 2.1 and Definition 2.6 with heterogeneous distributions for $\tilde{U}_1, \dots, \tilde{U}_m$, unusable from a practical point of view, even for moderate sizes m of the group of entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ that affect the stock price.

However, if $\tilde{U}_1, \dots, \tilde{U}_m$ are exchangeable, for example if $\tilde{U}_1, \dots, \tilde{U}_m$ is an i.i.d sequence and thus are homogeneous in their distributions, then the number of terms in the sums for $\mathbb{P}[S_t \leq x]$ will reduce from 2^m to just m terms, which will be practically to handle also for very large m -values, such as e.g. $m > 100$ entities in the group $\mathbf{C}_1, \dots, \mathbf{C}_m$. To see why the terms reduce from 2^m to m , let $N_t^{(m)}$ be a point process that counts the number of defaults among the m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ with default times $\tau_1, \tau_2, \dots, \tau_m$, that is

$$N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}. \quad (2.19)$$

Furthermore, if $\tilde{U}_1, \dots, \tilde{U}_m$ is an i.i.d sequence and if U_1, \dots, U_m is another i.i.d sequence with same distribution as $\tilde{U}_1, \dots, \tilde{U}_m$ then we have that

$$\sum_{i=1}^m \tilde{U}_i 1_{\{\tau_i \leq t\}} \stackrel{d}{=} \sum_{n=1}^{N_t^{(m)}} U_n \quad (2.20)$$

where $N_t^{(m)}$ is defined as in (2.19), so Corollary 2.7 and (2.20) therefore implies that

$$S_t \stackrel{d}{=} S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n \right) \quad (2.21)$$

where we remind that for two random variables X and Y , the notation $X \stackrel{d}{=} Y$ means that X and Y have same distribution. In view of Equation (2.20)-(2.21) we will sometimes use the notation U_1, \dots, U_m and V_1, \dots, V_m instead of $\tilde{U}_1, \dots, \tilde{U}_m$ and $\tilde{V}_1, \dots, \tilde{V}_m$ and sometimes write $S_t = \dots$ instead of $S_t \stackrel{d}{=} \dots$ in Equation (2.21).

Remark 2.8. The reason why the exchangeability of the jumps $\tilde{U}_1, \dots, \tilde{U}_m$ are important is that if this is not true, we have to keep track of *which* of the companies $\mathbf{C}_1, \dots, \mathbf{C}_m$ that have defaulted up to time t , while in the exchangeability case for $\tilde{U}_1, \dots, \tilde{U}_m$ we only need to keep track of *how many* of $\mathbf{C}_1, \dots, \mathbf{C}_m$ that have defaulted up to time t , i.e. we only need to model $N_t^{(m)}$ defined as in (2.19).

Next, we make following assumption on U_1, \dots, U_m and V_1, \dots, V_m .

Definition 2.9. Let U_1, \dots, U_m be an i.i.d sequence of exponentially distributed random variables which are independent of W_t and also independent of the default times $\tau_1, \tau_2, \dots, \tau_m$. Then, we define the i.i.d sequence V_1, \dots, V_m as

$$V_n = e^{-U_n} - 1 \quad \text{where} \quad U_n \stackrel{d}{=} \text{Exp}(\eta) \quad \text{with} \quad \mathbb{E}[U_n] = \frac{1}{\eta}. \quad (2.22)$$

From (2.22) in Definition 2.1 we see that U_n is exponentially distributed with density $\eta e^{-\eta u}$ for $u \geq 0$ and that $V_n \geq -1$ for each n .

Remark 2.10. Note that if $\eta \rightarrow \infty$ then $U_n \rightarrow 0$ almost surely under \mathbb{P} , so with a slight abuse of notation, we can identify $U_n = 0$ with " $\eta = \infty$ ".

The definition in (2.22) is similar to the one on p.1087 in Kou (2002), but where we here restrict ourselves to only negative jumps in stock price while Kou (2002) allows for both positive and negative stock price jumps. Assuming only negative jumps as in our model will lead to a more conservative or prudent stock price model which in particular will lead to larger Value-at-Risk losses, and should therefore

be more favourable among financial regulators (such as e.g. SEC, FCA, BaFin etc.) compared with models that also includes positive jumps in stock prices.

In view of Definition 2.9 and Equation (2.21) we state the following corollary to Proposition 2.5.

Corollary 2.11. *Let S_t be a stock price given by Definition 2.1 under the real probability measure \mathbb{P} and where the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ are distributed as V_1, \dots, V_m in Definition 2.9 with $\eta > 0$. Then, with notation as above,*

$$S_t \stackrel{d}{=} S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n \right). \quad (2.23)$$

Next, define the loss process $L_t^{(S)}$ for the stock S_t at time t with reference to the starting time 0, as

$$L_t^{(S)} = - (S_t - S_0) \quad (2.24)$$

where we note that a gain implies that the loss $L_t^{(S)}$ is negative. We are interested to compute Value-at-Risk for $L_t^{(S)}$ in our model for a stock price with jumps at defaults, that is, we want to compute

$$\text{VaR}_\alpha \left(L_t^{(S)} \right) = \inf \left\{ y \in \mathbb{R} : \mathbb{P} \left[L_t^{(S)} > y \right] \leq 1 - \alpha \right\} = \inf \left\{ y \in \mathbb{R} : F_{L_t^{(S)}}(y) \geq \alpha \right\} \quad (2.25)$$

where $F_{L_t^{(S)}}(x)$ is the distribution of $L_t^{(S)}$ and α is the confidence level, typically given by 95%, 99 or 99.9%, that is $\alpha = 0.95, \alpha = 0.99$ or $\alpha = 0.999$. So, if S_t is given as in Definition 2.1 with jumps as in Definition 2.9, then in view of Corollary 2.11 the loss $L_t^{(S)}$ in (3.9) can be reformulated as

$$L_t^{(S)} \stackrel{d}{=} S_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n \right) \right) \quad (2.26)$$

where for any $t > 0$ we have $\sup L_t^{(S)} = S_0$, since $S_t \geq 0$ almost surely. Next, we state the following useful theorem.

Theorem 2.12. *Let S_t be a stock price under the real probability measure \mathbb{P} defined as in Corollary 2.11. Then, with notation as above, we have that*

$$\mathbb{E} \left[S_t \mid N_t^{(m)} \right] = S_0 e^{\mu t} \left(\frac{\eta}{\eta + 1} \right)^{N_t^{(m)}} \quad \text{where} \quad \mathbb{E} \left[S_t \mid N_t^{(m)} = k \right] = S_0 e^{\mu t} \left(\frac{\eta}{\eta + 1} \right)^k \quad (2.27)$$

for $k = 0, 1, 2, \dots, m$ and

$$\mathbb{E} [S_t] = S_0 e^{\mu t} \mathbb{E} \left[\left(\frac{\eta}{\eta + 1} \right)^{N_t^{(m)}} \right] = S_0 e^{\mu t} \sum_{k=0}^m \left(\frac{\eta}{\eta + 1} \right)^k \mathbb{P} \left[N_t^{(m)} = k \right]. \quad (2.28)$$

Furthermore,

$$\mathbb{P} [S_t \leq x] = \sum_{k=0}^m \Psi_k(x, t, \mu, \sigma, S_0, \eta) \mathbb{P} \left[N_t^{(m)} = k \right] \quad (2.29)$$

where the mappings $\Psi_k(x, t, \mu, \sigma, u, \eta)$ for $u > 0$ are defined as

$$\Psi_k(x, t, \mu, \sigma, u, \eta) = \int_0^\infty \Phi \left(\frac{\ln \frac{x}{u} - \left(\mu - \frac{1}{2} \sigma^2 \right) t + y}{\sigma \sqrt{t}} \right) \frac{\eta e^{-\eta y} (\eta y)^{k-1}}{(k-1)!} dy \quad \text{for } 0 < k \leq m \quad (2.30)$$

and $\Psi_0(x, t, \mu, \sigma, u, \eta)$ for $u > 0$ is given by

$$\Psi_0(x, t, \mu, \sigma, u, \eta) = \Phi \left(\frac{\ln \frac{x}{u} - \left(\mu - \frac{1}{2} \sigma^2 \right) t}{\sigma \sqrt{t}} \right) \quad (2.31)$$

where $\Phi(x)$ is the distribution function to a standard normal random variable. Furthermore,

$$F_{L_t^{(S)}}(x) = \mathbb{P} \left[L_t^{(S)} \leq x \right] = 1 - \sum_{k=0}^m \Psi_k \left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta \right) \mathbb{P} \left[N_t^{(m)} = k \right] \quad (2.32)$$

where $x \leq S_0$ and for any $t > 0$ we have $\sup L_t^{(S)} = S_0$. The density $f_{S_t}(x)$ to S_t is given by

$$f_{S_t}(x) = \sum_{k=0}^m \psi_k(x, t, \mu, \sigma, S_0, \eta) \mathbb{P} \left[N_t^{(m)} = k \right] \quad \text{for } x > 0, t > 0 \quad (2.33)$$

where the mappings $\psi_k(x, t, \mu, \sigma, S_0, \eta)$ for $S_0 > 0, x > 0, t > 0$ are defined as

$$\psi_k(x, t, \mu, \sigma, S_0, \eta) = \frac{1}{x\sigma\sqrt{t}} \int_0^\infty \varphi \left(\frac{\ln \frac{x}{S_0} - (\mu - \frac{1}{2}\sigma^2)t + y}{\sigma\sqrt{t}} \right) \frac{\eta e^{-\eta y} (\eta y)^{k-1}}{(k-1)!} dy \quad \text{for } 0 < k \leq m \quad (2.34)$$

and $\psi_0(x, t, \mu, \sigma, S_0, \eta)$ for $S_0 > 0$ is given by

$$\psi_0(x, t, \mu, \sigma, S_0, \eta) = \frac{1}{x\sigma\sqrt{t}} \varphi \left(\frac{\ln \frac{x}{S_0} - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right) \quad (2.35)$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the density to a standard normal random variable.

Proof. We start with $\mathbb{P}[S_t \leq x]$ and note that

$$\mathbb{P}[S_t \leq x] = \sum_{k=0}^m \mathbb{P} \left[S_t \leq x \mid N_t^{(m)} = k \right] \mathbb{P} \left[N_t^{(m)} = k \right] \quad (2.36)$$

where Corollary 2.11 implies that

$$\mathbb{P} \left[S_t \leq x \mid N_t^{(m)} = k \right] = \mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t - \sum_{n=1}^k U_n \right) \mid N_t^{(m)} = k \right]. \quad (2.37)$$

From Definition 2.1 we know that W_t is independent of the default times $\tau_1, \tau_2, \dots, \tau_m$ and from Definition 2.9 we also know that the sequence U_1, \dots, U_m is independent of $\tau_1, \tau_2, \dots, \tau_m$. Thus, the process $N_t^{(m)}$ is independent of both W_t and U_1, \dots, U_m which in (2.37) gives

$$\begin{aligned} \mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t - \sum_{n=1}^k U_n \right) \leq x \mid N_t^{(m)} = k \right] \\ = \mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t - \sum_{n=1}^k U_n \right) \leq x \right] \end{aligned} \quad (2.38)$$

and the right hand side of (2.38) can be simplified to

$$\mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t - \sum_{n=1}^k U_n \right) \leq x \right] = \mathbb{P} \left[\sigma W_t - \sum_{n=1}^k U_n \leq \ln \frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2 \right) t \right]. \quad (2.39)$$

Let X and G_k be independent random variables, where X is a standard normal random variable and G_k is a gamma-distributed random variable so that $G_k \stackrel{d}{=} \text{Gamma}(k, \eta)$ where $k \geq 1$ is an integer. Then we note that

$$\sigma W_t \stackrel{d}{=} \sigma\sqrt{t}X \quad \text{and} \quad \sum_{n=1}^k U_n \stackrel{d}{=} G_k \stackrel{d}{=} \text{Gamma}(k, \eta) \quad (2.40)$$

where the last equality follows from the fact that a sum of k independent exponentially distributed random variables all with parameter η has distribution $\text{Gamma}(k, \eta)$. From Definition 2.9 we know that U_1, \dots, U_m are independent of W_t , which motivates why X and G_k in (2.40) are independent random variables. Next, let U and V be independent random variables with distributions $F_U(u)$ and $F_V(v)$. From standard probability theory we know that

$$\mathbb{P}[U + V \leq z] = \int F_U(z - v) dF_V(v) \quad (2.41)$$

see e.g. Theorem 2.1.1 on p.47 in Durrett (2010). If we define U and V as

$$U = \sigma\sqrt{t}X \quad \text{and} \quad V = -G_k \quad (2.42)$$

where X and G_k are same as in (2.40) then we have that

$$F_U(u) = \Phi \left(\frac{u}{\sigma\sqrt{t}} \right) \quad \text{and} \quad F_V(v) = 1 - F_{G_k}(-v) \quad \text{where } v \in (-\infty, 0] \quad (2.43)$$

so that

$$dF_V(v) = f_{G_k}(-v)dv \quad \text{for } v \in (-\infty, 0] \quad (2.44)$$

where $F_{G_k}(x)$ and $f_{G_k}(x)$ are the distribution function and density function to $G_k \stackrel{d}{=} \text{Gamma}(k, \eta)$ and as usual $\Phi(x)$ is the distribution function to a standard normal random variable. Now, (2.42), (2.43) and (2.44) in (2.41) then renders

$$\mathbb{P} \left[\sigma\sqrt{t}X - G_k \leq z \right] = \int_{-\infty}^0 \Phi \left(\frac{z-v}{\sigma\sqrt{t}} \right) f_{G_k}(-v)dv \quad (2.45)$$

and by making the change of variables $y = -v$ in (2.45) the integral in the right hand side of (2.45) can be rewritten as

$$\mathbb{P} \left[\sigma\sqrt{t}X - G_k \leq z \right] = \int_0^{\infty} \Phi \left(\frac{z+y}{\sigma\sqrt{t}} \right) f_{G_k}(y)dy. \quad (2.46)$$

By letting $z = \ln \frac{x}{S_0} - (\mu - \frac{1}{2}\sigma^2)t$ and $f_{G_k}(y) = \frac{\eta e^{-\eta y} (\eta y)^{k-1}}{(k-1)!}$ in (2.46), together with the relation (2.40), we get for any integer $k \geq 1$ that the right hand side of (2.39) can be written as

$$\mathbb{P} \left[\sigma W_t - \sum_{n=1}^k U_n \leq \ln \frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2 \right) t \right] = \int_0^{\infty} \Phi \left(\frac{\ln \frac{x}{S_0} - (\mu - \frac{1}{2}\sigma^2)t + y}{\sigma\sqrt{t}} \right) \frac{\eta e^{-\eta y} (\eta y)^{k-1}}{(k-1)!} dy. \quad (2.47)$$

By combining (2.38), (2.39) with (2.47) we get for for any integer $k \geq 1$ that

$$\mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t - \sum_{n=1}^k U_n \right) \leq x \mid N_t^{(m)} = k \right] = \Psi_k(x, t, \mu, \sigma, S_0, \eta) \quad (2.48)$$

where the mappings $\Psi_k(x, t, \mu, \sigma, u, \eta)$ for $u > 0$ are defined as

$$\Psi_k(x, t, \mu, \sigma, u, \eta) = \int_0^{\infty} \Phi \left(\frac{\ln \frac{x}{u} - (\mu - \frac{1}{2}\sigma^2)t + y}{\sigma\sqrt{t}} \right) \frac{\eta e^{-\eta y} (\eta y)^{k-1}}{(k-1)!} dy \quad \text{for integers } k \geq 1. \quad (2.49)$$

When $k = 0$ we have no defaults so and thus no jump-terms in the exponential expression of (2.38) implying that (2.38) reduces to

$$\mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right) \leq x \mid N_t^{(m)} = 0 \right] = \Psi_0(x, t, \mu, \sigma, S_0, \eta) \quad (2.50)$$

where $\Psi_0(x, t, \mu, \sigma, u, \eta)$ for $u > 0$ is defined as

$$\Psi_0(x, t, \mu, \sigma, u, \eta) = \mathbb{P} \left[u \cdot \exp \left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right) \leq x \right] = \Phi \left(\frac{\ln \frac{x}{u} - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right). \quad (2.51)$$

Thus, (2.48) - (2.51) together with (2.36) and (2.37) implies that

$$\mathbb{P} [S_t \leq x] = \sum_{k=0}^m \Psi_k(x, t, \mu, \sigma, S_0, \eta) \mathbb{P} [N_t^{(m)} = k]$$

which proves (2.29), (2.30) and (2.31). Next, consider the loss distribution $F_{L_t^{(S)}}(x) = \mathbb{P} [L_t^{(S)} \leq x]$. From the definition of $L_t^{(S)}$ in (3.9) we get after some trivial computations that

$$F_{L_t^{(S)}}(x) = \mathbb{P} [L_t^{(S)} \leq x] = 1 - \mathbb{P} \left[\frac{S_t}{S_0} \leq 1 - \frac{x}{S_0} \right] \quad (2.52)$$

and we can therefore reuse the formula for $\mathbb{P} [S_t \leq x]$ in (2.29), by letting $S_0 = 1$ in (2.29) and replace x in (2.29) with $1 - \frac{x}{S_0}$, rendering that

$$\mathbb{P} [L_t^{(S)} \leq x] = 1 - \sum_{k=0}^m \Psi_k \left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta \right) \mathbb{P} [N_t^{(m)} = k]$$

which proves (2.29). Next we prove the expressions for the density $f_{S_t}(x)$ to S_t and first note that $f_{S_t}(x) = \frac{d}{dx} \mathbb{P}[S_t \leq x]$ so (2.29) then implies that

$$f_{S_t}(x) = \sum_{k=0}^m \frac{\partial}{\partial x} \Psi_k(x, t, \mu, \sigma, S_0, \eta) \mathbb{P}[N_t^{(m)} = k]. \quad (2.53)$$

Next, we define $\psi_k(x, t, \mu, \sigma, S_0, \eta)$ as

$$\psi_k(x, t, \mu, \sigma, S_0, \eta) = \frac{\partial}{\partial x} \Psi_k(x, t, \mu, \sigma, S_0, \eta) \quad (2.54)$$

and for $k \geq 1$ with $x > 0, t > 0$ we then get from (2.49) and (2.54) and some elementary computations that

$$\psi_k(x, t, \mu, \sigma, S_0, \eta) = \frac{1}{x\sigma\sqrt{t}} \int_0^\infty \varphi\left(\frac{\ln \frac{x}{S_0} - (\mu - \frac{1}{2}\sigma^2)t + y}{\sigma\sqrt{t}}\right) \frac{\eta e^{-\eta y} (\eta y)^{k-1}}{(k-1)!} dy \quad \text{for } 0 < k \leq m \quad (2.55)$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the density to a standard normal random variable. In the same way, (2.51) and (2.54) implies that $\psi_0(x, t, \mu, \sigma, S_0, \eta)$ for $S_0 > 0, x > 0, t > 0$ is given by

$$\psi_0(x, t, \mu, \sigma, S_0, \eta) = \frac{1}{x\sigma\sqrt{t}} \varphi\left(\frac{\ln \frac{x}{S_0} - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right). \quad (2.56)$$

Hence, (2.54) with (2.55)-(2.56) inserted into (2.53) proves (2.33) - (2.35). Finally, we note that

$$\mathbb{E}[S_t] = \sum_{k=0}^m \mathbb{E}[S_t | N_t^{(m)} = k] \mathbb{P}[N_t^{(m)} = k] \quad (2.57)$$

where Corollary 2.11 implies that

$$\mathbb{E}[S_t | N_t^{(m)} = k] = \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^k U_n\right) \middle| N_t^{(m)} = k\right]. \quad (2.58)$$

By using exactly the same arguments which led to the right hand side in (2.38) we have that

$$\mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^k U_n\right) \middle| N_t^{(m)} = k\right] = \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^k U_n\right)\right]. \quad (2.59)$$

Furthermore, since W_t are independent of the jump terms U_1, \dots, U_m we get that the right hand side of (2.58) can be rewritten as

$$\mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^k U_n\right)\right] = \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)\right] \mathbb{E}\left[e^{-\sum_{n=1}^k U_n}\right]. \quad (2.60)$$

From standard Black-Scholes theory we have

$$\mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)\right] = S_0 e^{\mu t} \quad (2.61)$$

and from (2.40) we also note that

$$\mathbb{E}\left[e^{-\sum_{n=1}^k U_n}\right] = \mathbb{E}[e^{-G_K}] = \mathcal{L}_{G_K}(1) = \left(\frac{\eta}{\eta + 1}\right)^k \quad (2.62)$$

where $\mathcal{L}_{G_K}(s)$ is the Laplace transform to $G_k \stackrel{d}{=} \text{Gamma}(k, \eta)$ with $k \geq 1$ and $\eta > 0$, obtained from the moment generating function $\mathcal{M}_{G_K}(s)$ via the relation $\mathcal{L}_{G_K}(s) = \mathcal{M}_{G_K}(-s)$ and where standard probability theory gives us that $\mathcal{L}_{G_K}(s) = \left(\frac{\eta}{\eta + s}\right)^k$ for $s > -\eta$. Hence, combining (2.59)-(2.62) and inserting these relations in (2.58) for integers $k = 1, 2, \dots, m$ we get

$$\mathbb{E}[S_t | N_t^{(m)} = k] = S_0 e^{\mu t} \left(\frac{\eta}{\eta + 1}\right)^k \quad (2.63)$$

and since $\left(\frac{\eta}{\eta+1}\right)^0 = 1$ then (2.63) will also hold for $k = 0$. Thus, (2.63) for $k = 0, 1, \dots, m$ in (2.57) implies that

$$\mathbb{E}[S_t] = S_0 e^{\mu t} \sum_{k=0}^m \left(\frac{\eta}{\eta+1}\right)^k \mathbb{P}\left[N_t^{(m)} = k\right] = S_0 e^{\mu t} \mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right]$$

which proves (2.28). Finally, by using (2.63), we have

$$\mathbb{E}\left[S_t | N_t^{(m)}\right] = \sum_{k=0}^m \mathbb{E}\left[S_t | N_t^{(m)} = k\right] 1_{\{N_t^{(m)}=k\}} = \sum_{k=0}^m S_0 e^{\mu t} \left(\frac{\eta}{\eta+1}\right)^k 1_{\{N_t^{(m)}=k\}} = S_0 e^{\mu t} \left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}$$

which together with (2.63) proves (2.27) and this concludes the theorem. \square

We now make some remarks connected to Theorem 2.12.

Remark 2.13. As pointed out in Remark 2.3, if the defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ have issued bonds and/or stocks which are publicly traded on major financial markets, then typically the default times $\tau_1, \tau_2, \dots, \tau_m$ are *directly observable* on the market at the defaults, and these observations are done exogenously without the knowledge of S_t , that is, regardless if the stock price model for S_t includes the defaults or not. Hence, the point process $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ is in practice always observable making the quantities $\mathbb{E}\left[S_t | N_t^{(m)}\right]$ and $\mathbb{E}\left[S_t | N_t^{(m)} = k\right]$ given by (2.27) in Theorem 2.12 realistic to compute under the real probability measure \mathbb{P} . If the default times $\tau_1, \tau_2, \dots, \tau_m$ would be *unobservable on the market*, and therefore also making the counting process $N_t^{(m)}$ unobservable, or if the jumps would come from a Poisson process with arrival times that lack financial interpretation and therefore could not be observed directly, then it is less clear how to compute e.g. the quantity $\mathbb{E}\left[S_t | N_t^{(m)}\right]$ in practice, since $N_t^{(m)}$ would not be known to us. Note however that $\mathbb{E}[S_t]$ in (2.28) will always be possible to compute, regardless if $N_t^{(m)}$ is observable or not, since to find $\mathbb{E}[S_t]$ we do not need the exact value of $N_t^{(m)}$, but only its distribution.

Note that the η -parameter in the mapping $\Psi_0(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ in (2.31) for $k = 0$ will have no impact, and is only present for notational convenience given the sum in the expression of (2.29) which runs from $k = 0$ to $k = m$.

Some remarks on the expected stock price. Let $S_t^{(\text{BS})}$ be the stock price in the Black-Scholes model under the real probability measure \mathbb{P} given by (2.15), that is

$$S_t^{(\text{BS})} = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \quad (2.64)$$

so that

$$\mathbb{E}\left[S_t^{(\text{BS})}\right] = S_0 e^{\mu t}. \quad (2.65)$$

Let S_t be a stock price given by Definition 2.1 under the real probability measure \mathbb{P} and where the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ are distributed as V_1, \dots, V_m in Definition 2.9 with $\eta > 0$. Then, Equation (2.28) in Theorem 2.12 together with (2.65) implies that

$$\mathbb{E}[S_t] = \mathbb{E}\left[S_t^{(\text{BS})}\right] \mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right]. \quad (2.66)$$

We clearly see that if $0 < \eta < \infty$ then $\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right] < 1$ and therefore (2.66) implies the relationship

$$\mathbb{E}[S_t] < \mathbb{E}\left[S_t^{(\text{BS})}\right] \quad \text{when } 0 < \eta < \infty \quad (2.67)$$

which is intuitive clear since the S_t will always have a negative relative jumps at any default time τ_i where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, that is, for the same W_t in S_t given by Corollary (2.11) as in $S_t^{(\text{BS})}$ in (2.64) then Corollary (2.11) implies that $S_t \leq S_t^{(\text{BS})}$ almost surely under \mathbb{P} . If $U_n = 0$ for all n (that is if " $\eta = \infty$ ", see in Remark 2.10) this means that there will be no jumps at the defaults τ_i and S_t will coincide with

the Black-Scholes price $S_t^{(\text{BS})}$, that is $S_t = S_t^{(\text{BS})}$ as stated in Equation (2.64), so $\mathbb{E}[S_t] = \mathbb{E}[S_t^{(\text{BS})}]$. We note that

$$\frac{\partial}{\partial \eta} \mathbb{E} \left[\left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \right] = \mathbb{E} \left[\frac{N_t^{(m)}}{\eta(\eta+1)} \right] > 0 \quad (2.68)$$

so $\mathbb{E} \left[\left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \right]$ is strictly increasing in $\eta > 0$. Therefore, for a fixed time point t , and for any

$0 < \beta < 1$ the equation $\mathbb{E} \left[\left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \right] = \beta$ will have a unique solution in $\eta = \eta(\beta, t)$. This can be used

when calibrating η . For example, if we assume that the default counting process $N_t^{(m)}$ will make the expected value of the stock price S_t to be e.g. $\beta = 90\%$ of the corresponding expected value of the Black stock price $S_t^{(\text{BS})}$, up to time, say T , that is

$$\mathbb{E}[S_T] = \beta \mathbb{E}[S_T^{(\text{BS})}] \quad (2.69)$$

then, (2.66) and (2.69) implies for any $0 < \beta < 1$ that

$$\mathbb{E} \left[\left(\frac{\eta}{\eta+1} \right)^{N_T^{(m)}} \right] = \beta \quad (2.70)$$

which thus have a unique solution in $\eta^* = \eta(\beta, T) > 0$ and for most credit portfolio models this solution η^* has to be found numerically. Finally, we will often consider the equation $\mathbb{E}[S_T] = S_0$ so from (2.65), (2.69) and (2.70) we see that

$$\mathbb{E}[S_T] = S_0 \quad \text{if and only if} \quad \mathbb{E} \left[\left(\frac{\eta}{\eta+1} \right)^{N_T^{(m)}} \right] = e^{-\mu T} \quad (2.71)$$

where we note that the condition $\mathbb{E}[S_T] = S_0$ implies that the defaults among the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ "wipes" out the expected log-growth for a corresponding Black-Scholes model with drift μ up to time T . We will use condition (2.71) when calibrating η in our numerical studies presented in Section 5 and 6.

VaR-expressions and related quantities. Given the formula (2.32) for the distribution of the stock price loss process $F_{L_t^{(S)}}(x)$ in Theorem 2.12 we will be able to find Value-at-Risk for $L_t^{(S)}$ with confidence level α , denoted by $\text{VaR}_\alpha(L_t^{(S)})$, since from (2.25) and the fact that S_t is a continuous random variable, then

$$\text{VaR}_\alpha(L_t^{(S)}) = F_{L_t^{(S)}}^{-1}(\alpha) \quad \text{so that} \quad F_{L_t^{(S)}}(\text{VaR}_\alpha(L_t^{(S)})) = \alpha \quad (2.72)$$

where the second equation in (2.72) will be solved numerically to find $\text{VaR}_\alpha(L_t^{(S)})$. In the case when there are now jump at the defaults, i.e when $U_n = 0$ for all n , or equivalently, in view of Remark 2.10, if " $\eta = \infty$ ", then $S_t = S_t^{(\text{BS})}$ with $S_t^{(\text{BS})}$ given by Equation (2.15), and the expression for $\text{VaR}_\alpha(L_t^{(S)})$ in (2.72) can then be solved analytically, denoted by $\text{VaR}_\alpha^{\text{BS}}(L_t^{(S)})$ and given as

$$\text{VaR}_\alpha^{\text{BS}}(L_t^{(S)}) = S_0 \left(1 - \exp \left(\sigma \sqrt{t} \Phi^{-1}(1 - \alpha) + \left(\mu - \frac{1}{2} \sigma^2 \right) t \right) \right). \quad (2.73)$$

We will later in the numerical section use $\text{VaR}_\alpha^{\text{BS}}(L_t^{(S)})$ in (2.73) for the Black-Scholes model when comparing with $\text{VaR}_\alpha(L_t^{(S)})$ coming from a stock price S_t with jumps at the default arrivals in $N_t^{(m)}$ and where $\eta > 0$.

We finally remark that almost all formulas in Theorem 2.12 requires efficient and quick methods of computing the number of default distribution $\mathbb{P}[N_t^{(m)} = k]$.

3. THE MULTIDIMENSIONAL CASE: SMALL TIME APPROXIMATIONS TO LOSS DISTRIBUTIONS FOR HETEROGENEOUS STOCK PORTFOLIOS WITH JUMPS AT EXOGENOUS DEFAULTS.

In this section we generalize the single-stock dynamics in Section 2 to a heterogeneous portfolio of stocks. Furthermore, we also define the loss process for the stock portfolio. For small time points we make a linearization of the portfolio loss process and derive a computationally tractable expression for distribution of the linearized loss. We also consider the portfolio loss process and its linear approximation for small time points in the classical Black-Scholes portfolio case, i.e. without any jumps in the stock prices. In our numerical studies in Section 5 - 6 we will use the distribution of the linearized Black-Scholes portfolio loss as benchmark to the distribution of the linearized loss when the stock prices have jumps at defaults of some external defaultable entities.

Inspired by dynamics of a single-stock price S_t discussed in Section 2, and in particular Corollary 2.11 we now make the following definition.

Definition 3.1. Consider a group of m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ with individual default times $\tau_1, \tau_2, \dots, \tau_m$ and let $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Let the companies $\mathbf{A}_1, \dots, \mathbf{A}_J$ be J different exchangeable entities which do not belong to the group $\mathbf{C}_1, \dots, \mathbf{C}_m$ and let $S_{t,1}, \dots, S_{t,J}$ denote the stock prices of the companies $\mathbf{A}_1, \dots, \mathbf{A}_J$ at time t under the real probability measure \mathbb{P} . Then, for each entity \mathbf{A}_j we define the stock price $S_{t,j}$ as

$$S_{t,j} = S_{0,j} \exp \left(\left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t + \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1 - \rho_{S,j}^2} W_{t,j} \right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \quad (3.1)$$

where $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ are $J+1$ independent Brownian motions and $\rho_{S,j} \in [-1, 1]$ are constants. Furthermore, for each $j = 1, 2, \dots, J$ the m random variables $U_{1,j}, \dots, U_{m,j}$ are an i.i.d sequence distributed as

$$U_{n,j} \stackrel{d}{=} \text{Exp}(\eta) \quad \text{with} \quad \mathbb{E}[U_{n,j}] = \frac{1}{\eta} \quad (3.2)$$

where $U_{1,j}, \dots, U_{m,j}$ are independent of the processes $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ and also independent of the default times $\tau_1, \tau_2, \dots, \tau_m$. Furthermore, for each company \mathbf{A}_j the parameters $\sigma_j > 0$ and μ_j are the volatility and drift, same as in the one-dimensional case given in Definition 2.1 and Corollary 2.11.

We next make some remarks connected to Definition 3.1.

Remark 3.2. If we let $\tilde{U}_{1,j}, \dots, \tilde{U}_{m,j}$ be an i.i.d sequence with same distribution as $U_{1,j}, \dots, U_{m,j}$ then the jump term $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in (3.1) can be replaced by the more intuitive expression $\sum_{i=1}^m \tilde{U}_{n,j} 1_{\{\tau_i \leq t\}}$, just as in the single-stock case in Section 2, since $\sum_{n=1}^{N_t^{(m)}} U_{n,j} \stackrel{d}{=} \sum_{i=1}^m \tilde{U}_{n,j} 1_{\{\tau_i \leq t\}}$. However, in the derivations in our proofs etc. it will be more convenient from a notational point of view to use the first version, that is the term $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in (3.1).

Remark 3.3. Note that in Definition 3.1, all firms \mathbf{A}_j have stock prices $S_{t,j}$ with i.i.d jumps $U_{1,j}, \dots, U_{m,j}$, with same parameter η defined as in (3.2). We can of course also let the distributions for $U_{1,j}, \dots, U_{m,j}$, be different among different entities \mathbf{A}_j , for example by letting

$$U_{n,j} \stackrel{d}{=} \text{Exp}(\eta_j) \quad \text{with} \quad \mathbb{E}[U_{n,j}] = \frac{1}{\eta_j} \quad \text{where} \quad \eta_j \neq \eta_i \quad \text{for} \quad \mathbf{A}_j \neq \mathbf{A}_i. \quad (3.3)$$

However, allowing for heterogeneous jump parameters η_j among different firms \mathbf{A}_j , as in (3.3), will unfortunately make it difficult to find analytical formulas for the distribution of our stock portfolio losses. Therefore we will in this paper always assume homogeneous jump parameters, that is $\eta = \eta_1 = \eta_2 = \dots = \eta_J$, which will lead to analytical formulas for our portfolio related quantities.

Remark 3.4. Note that $\rho_{S,j} \in [-1, 1]$ and unless explicitly stated, we will throughout this paper always assume that at least one company \mathbf{A}_j has a correlation such that $\rho_{S,j} \neq -1, 1$ so that $\rho_{S,j} \in (-1, 1)$.

Remark 3.5. Since the collection of i.i.d sequences $U_{1,j}, \dots, U_{m,j}$ are exchangeable for all companies \mathbf{A}_j in Definition 3.1, that is $U_{k,j}$ and $U_{k',j'}$ have the same distribution for any pairs (k, j) and (k', j') , then, just as in Remark 2.2 we note that the default times $\tau_1, \tau_2, \dots, \tau_m$ in Definition 3.1 can come from any type credit portfolio model.

Remark 3.6. The stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ are correlated and have simultaneous jumps. Since $W_{t,0}$ and $W_{t,j}$ are independent Brownian motions for each j and $\rho_{S,j} \in [-1, 1]$, then from standard probability theory we know that $\rho_{S,j}W_{t,0} + \sqrt{1 - \rho_{S,j}^2}W_{t,j}$ used in (3.1) is also a Brownian motion. Hence, in view of Definition 2.1, Definition 2.9 and Corollary 2.11, it is clear that the dynamics of the stock price $S_{t,j}$ for each firm \mathbf{A}_j satisfies

$$dS_{t,j} = S_{t-,j}dY_{t-,j} \quad (3.4)$$

where $Y_{t,j}$ is given by

$$Y_{t,j} = \mu_j t + \sigma_j \left(\rho_{S,j}W_{t,0} + \sqrt{1 - \rho_{S,j}^2}W_{t,j} \right) + \sum_{n=1}^{N_t^{(m)}} (e^{-U_{n,j}} - 1). \quad (3.5)$$

Further, from the construction of $S_{t,j}$ in (3.1) and $U_{n,j}$ in (3.2), stated in Definition 3.1, the stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ will be "correlated" via the factor process $W_{t,0}$ when $\rho_{S,j} \neq 0$, and also "correlated" via the default counting process $N_t^{(m)}$ for the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$. In particular, all stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ will have a jump at the default times $\tau_1, \tau_2, \dots, \tau_m$, where the relative jumps of $S_{t,j}$ will be different almost surely under \mathbb{P} , although have same distribution, given by (3.2). Finally, each stock price $S_{t,j}$ will satisfy the results in Theorem 2.12.

Next, consider a weighted stock portfolio consisting of w_1, w_2, \dots, w_J stocks chosen for our portfolio at time $t = 0$, where the stocks are issued by the J companies $\mathbf{A}_1, \dots, \mathbf{A}_J$ with stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ that satisfy Definition 3.1. Then we define the portfolio value V_t as

$$V_t = \sum_{j=1}^J w_j S_{t,j}. \quad (3.6)$$

We will in this paper define an equally value-weighted portfolio V_t as follows.

Definition 3.7. Equally value-weighted portfolio. Let S_0 be a positive constant. We say that the portfolio V_t in (3.6) is an *equally value-weighted portfolio* if the weights w_j are chosen so that

$$w_j S_{0,j} = S_0 \quad \text{for } j = 1, 2, \dots, J \quad (3.7)$$

and thus

$$V_0 = \sum_{j=1}^J w_j S_{0,j} = \sum_{j=1}^J S_0 = JS_0. \quad (3.8)$$

The intuitive idea behind Definition 3.7 is that the portfolio weights w_j are chosen so that the value for the stock position in firm \mathbf{A}_j at time $t = 0$ will have the same amount given by S_0 for all companies $\mathbf{A}_1, \dots, \mathbf{A}_J$ that are contained in the portfolio V_t .

Next, we, define the portfolio loss process $L_t^{(V)}$ for a general portfolio V_t at time t with reference to the starting time 0, as

$$L_t^{(V)} = -(V_t - V_0) \quad (3.9)$$

where we note that a gain implies that the loss $L_t^{(V)}$ is negative. We are interested to compute Value-at-Risk for $L_t^{(V)}$ in our model given by Definition 3.1, that is, we want to compute

$$\text{VaR}_\alpha \left(L_t^{(V)} \right) = \inf \left\{ y \in \mathbb{R} : \mathbb{P} \left[L_t^{(V)} > y \right] \leq 1 - \alpha \right\} = \inf \left\{ y \in \mathbb{R} : F_{L_t^{(V)}}(y) \geq \alpha \right\} \quad (3.10)$$

where $F_{L_t^{(V)}}(x)$ is the distribution of $L_t^{(V)}$ and α is the confidence level, just as in (2.25). Define $X_{t,j}$ as

$$X_{t,j} = \left(\mu_j - \frac{1}{2}\sigma_j^2 \right) t + \sigma_j \left(\rho_{S,j}W_{t,0} + \sqrt{1 - \rho_{S,j}^2}W_{t,j} \right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \quad (3.11)$$

where the right hand side of (3.11) is same as in (3.1) in Definition 3.1 which then implies that

$$S_{t,j} = S_{0,j}e^{X_{t,j}}. \quad (3.12)$$

Then, for an equally value-weighted portfolio V_t as in Definition 3.7, the portfolio loss $L_t^{(V)}$ in (3.9) can be restated as

$$L_t^{(V)} = S_0 \left(J - \sum_{j=1}^J e^{X_{t,j}} \right). \quad (3.13)$$

We want to find $F_{L_t^{(V)}}(x) = \mathbb{P} \left[L_t^{(V)} \leq x \right]$ so that we for example can compute $\text{VaR}_\alpha \left(L_t^{(V)} \right)$ given by (3.10). Unfortunately, finding analytical or semi-analytical expressions to $F_{L_t^{(V)}}(x)$ is a challenging task. However, assuming that $|X_{t,j}|$ will be small for small t , then we can use a first order Taylor expansion of the term $e^{X_{t,j}}$, that is

$$e^{X_{t,j}} \approx 1 + X_{t,j} \quad \text{when } |X_{t,j}| \text{ is small} \quad (3.14)$$

which typically will hold for small t . So using (3.14) in (3.13) then implies that the loss $L_t^{(V)}$ for an equally value-weighted portfolio V_t as in Definition 3.7 is approximated by

$$L_t^{(V)} \approx -S_0 \sum_{j=1}^J X_{t,j} \quad \text{when } |X_{t,j}| \text{ is small for all } j. \quad (3.15)$$

For $X_{t,j}$ given by (3.11), we therefore define the linearized loss $L_t^{\Delta V}$ to the portfolio loss $L_t^{(V)}$ in an equally value-weighted portfolio, as

$$L_t^{\Delta V} = -S_0 \sum_{j=1}^J X_{t,j} \quad (3.16)$$

so that (3.15) then implies that

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx \mathbb{P} \left[L_t^{\Delta V} \leq x \right] \quad \text{when } |X_{t,j}| \text{ is small for all } j \quad (3.17)$$

which typically will hold for small t . Next, we state a theorem which provides computationally tractable semi-analytical expression to the distribution $\mathbb{P} \left[L_t^{\Delta V} \leq x \right]$ for the linearized loss $L_t^{\Delta V}$ defined as in (3.16), which is equivalently of finding the distribution of $\sum_{j=1}^J X_{t,j}$.

Theorem 3.8. *Consider an equally value-weighted portfolio as in Definition 3.7 where the J stock prices $S_{t,1}, \dots, S_{t,J}$ are defined as in Definition 3.1 under the real probability measure \mathbb{P} . Then, with notation as above, we have that*

$$\mathbb{P} \left[L_t^{\Delta V} \leq x \right] = 1 - \sum_{k=0}^m \Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) \mathbb{P} \left[N_t^{(m)} = k \right] \quad (3.18)$$

where the mappings $\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ for $k \geq 1$ are defined as

$$\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) = \int_0^\infty \Phi \left(\frac{y - \frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t}{\sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 \left(1 - \rho_{S,j}^2 \right) \right)}} \right) \frac{\eta e^{-\eta y} (\eta y)^{Jk-1}}{(Jk-1)!} dy \quad (3.19)$$

and for $k = 0$ the mapping $\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ is defined by

$$\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) = \Phi \left(\frac{-\frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t}{\sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 \left(1 - \rho_{S,j}^2 \right) \right)}} \right) \quad (3.20)$$

where $\Phi(x)$ and $\varphi(x)$ are the distribution function and density to a standard normal random variable.

Proof. First, since $S_0 > 0$, and in view of (3.16) we get after some rearranging

$$\mathbb{P} \left[L_t^{\Delta V} \leq x \right] = 1 - \mathbb{P} \left[\sum_{j=1}^J X_{t,j} \leq -\frac{x}{S_0} \right] \quad (3.21)$$

and we therefore seek the distribution of $\sum_{j=1}^J X_{t,j}$. From Definition 3.1 and (3.11) we can rewrite $X_{t,j}$ as

$$X_{t,j} = Z_{t,j} + \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \quad (3.22)$$

where $Z_{t,j}$ is defined by

$$Z_{t,j} = \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1 - \rho_{S,j}^2} W_{t,j} \right) \quad (3.23)$$

and the terms in the right hand side of (3.23) is same as in Equation (3.1) in Definition 3.1. Then,

$$\mathbb{P} \left[\sum_{j=1}^J X_{t,j} \leq -\frac{x}{S_0} \right] = \mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^{N_t^{(m)}} U_{n,j} \leq -\frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t \right] \quad (3.24)$$

For notational convenience we define $a(x)$ as

$$a(x) = -\frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t \quad (3.25)$$

so that (3.24) can be rewritten as

$$\mathbb{P} \left[\sum_{j=1}^J X_{t,j} \leq -\frac{x}{S_0} \right] = \mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^{N_t^{(m)}} U_{n,j} \leq a(x) \right]. \quad (3.26)$$

Next we note that

$$\mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^{N_t^{(m)}} U_{n,j} \leq a(x) \right] = \sum_{k=0}^m \mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^k U_{n,j} \leq a(x) \mid N_t^{(m)} = k \right] \mathbb{P} \left[N_t^{(m)} = k \right] \quad (3.27)$$

and since $W_{t,j}$ and $U_{n,j}$ are independent of $N_t^{(m)}$ for all j and n , then by using the same arguments which led to the right hand side in (2.38) in Theorem 2.12 we get

$$\mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^k U_{n,j} \leq a(x) \mid N_t^{(m)} = k \right] = \mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^k U_{n,j} \leq a(x) \right]. \quad (3.28)$$

From the definition of $Z_{t,j}$ in (3.23) we have that

$$\sum_{j=1}^J Z_{t,j} = \sum_{j=1}^J \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1 - \rho_{S,j}^2} W_{t,j} \right) = W_{t,0} \sum_{j=1}^J \sigma_j \rho_{S,j} + \sum_{j=1}^J \sigma_j \sqrt{1 - \rho_{S,j}^2} W_{t,j} \quad (3.29)$$

and since $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ are $J+1$ independent Brownian motions then (3.29) and standard results from probability theory together with some computations gives that

$$\sum_{j=1}^J Z_{t,j} \stackrel{d}{=} \sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2) \right)} X \quad (3.30)$$

where X is a standard normal random variable. Let G_{Jk} be random variables independent of X where G_{Jk} is a gamma-distributed random variable such that $G_{Jk} \stackrel{d}{=} \text{Gamma}(Jk, \eta)$ where $k \geq 1$ is an integer. Then, in view of Definition 3.1 and from standard probability theory and using the same arguments that led to (2.40) in Theorem 2.12 we have that

$$\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^k U_{n,j} \stackrel{d}{=} \sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2) \right)} X - G_{Jk}. \quad (3.31)$$

Next, by using (3.31) in a version of Equation (2.46) in Theorem 2.12, we obtain

$$\mathbb{P} \left[\sum_{j=1}^J Z_{t,j} - \sum_{j=1}^J \sum_{n=1}^k U_{n,j} \leq a(x) \right] = \int_0^\infty \Phi \left(\frac{a(x) + y}{\sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2) \right)}} \right) f_{G_{Jk}}(y) dy \quad (3.32)$$

where $f_{G_{Jk}}(y) = \frac{\eta e^{-\eta y} (\eta y)^{Jk-1}}{(Jk-1)!}$ is the density to G_{Jk} and $\Phi(x)$ is the distribution function to a standard normal random variable. If $k = 0$ there are no jump-terms so the right hand side of (3.28) reduces to

$$\mathbb{P} \left[\sum_{j=1}^J Z_{t,j} \leq a(x) \right] = \Phi \left(\frac{a(x)}{\sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2) \right)}} \right) \quad (3.33)$$

where we also used (3.30) for the distribution of $\sum_{j=1}^J Z_{t,j}$. Hence, using (3.32) for $k \geq 1$ and (3.33) for $k = 0$ in the right hand side of (3.28), and then use (3.27) and (3.26) together with the definition of $a(x)$ in (3.25) finally imply that (3.21) can be rewritten as

$$\mathbb{P} [L_t^{\Delta V} \leq x] = 1 - \sum_{k=0}^m \Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) \mathbb{P} [N_t^{(m)} = k]$$

where the mappings $\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ for $k > 1$ are defined by

$$\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) = \int_0^\infty \Phi \left(\frac{y - \frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t}{\sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2) \right)}} \right) \frac{\eta e^{-\eta y} (\eta y)^{Jk-1}}{(Jk-1)!} dy$$

and for $k = 0$ the mapping $\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ is defined as

$$\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) = \Phi \left(\frac{-\frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2} \sigma_j^2 \right) t}{\sqrt{t \left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j} \right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2) \right)}} \right)$$

and this proves (3.18), (3.19) and (3.20) which concludes the theorem. \square

We note that the η -parameter in the mapping $\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ in (3.20) for $k = 0$ will have no impact, and is only present for notational convenience given the sum in the expression of (3.18) which runs from $k = 0$ to $k = m$.

Remark 3.9. Note that Theorem 3.8 is stated for a heterogeneous stock portfolio so that the parameters $\mu_j, \sigma_j, \rho_{S,j}$ and $S_{0,j}$ can have different values for different firms \mathbf{A}_j but where the weights w_j in the portfolio V_t are chosen so that $w_j S_{0,j} = S_0$ for all companies where S_0 is a positive constant. Sometimes we want to study the case where the parameters for $S_{t,j}$ are identical for all firms \mathbf{A}_j , that is, when

$$S_{0,j} = S_0, \quad \mu_j = \mu, \quad \sigma_j = \sigma, \quad \text{and} \quad \rho_{S,j} = \rho_S \quad \text{for all firms } \mathbf{A}_1, \dots, \mathbf{A}_J \quad (3.34)$$

so that the stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ become exchangeable. Furthermore, by letting $w_j = 1$ for all companies we get an equally value-weighted portfolio as in Definition 3.7 and (3.34) together with Theorem 3.8 then implies that the mappings $\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ in the loss distribution $\mathbb{P} [L_t^{\Delta V} \leq x]$ given by (3.18) will simplify a bit, where we for $k \geq 1$ under (3.34) get

$$\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) = \int_0^\infty \Phi \left(\frac{y - \frac{x}{S_0} - J \left(\mu - \frac{1}{2} \sigma^2 \right) t}{\sigma \sqrt{t J (1 + (J-1) \rho_S^2)}} \right) \frac{\eta e^{-\eta y} (\eta y)^{Jk-1}}{(Jk-1)!} dy \quad (3.35)$$

and for $k = 0$ with condition (3.34), the mapping $\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ is simplified to

$$\Psi_0^V(x, t, \mu, \sigma, S_0, \rho_S, \eta) = \Phi\left(\frac{-\frac{x}{S_0} - J\left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{tJ(1 + (J-1)\rho_S^2)}}\right) \quad (3.36)$$

where the rest of the notation is same as in Theorem 3.8.

Given the formulas (3.18)-(3.20) in Theorem 3.8 for the distribution $F_{L_t^{\Delta V}}(x) = \mathbb{P}[L_t^{\Delta V} \leq x]$ where $L_t^{\Delta V}$ is the linear approximation to the portfolio loss $L_t^{(V)}$, we can find Value-at-Risk for $L_t^{\Delta V}$ with confidence level α , denoted by $\text{VaR}_\alpha(L_t^{\Delta V})$ and given as

$$\text{VaR}_\alpha(L_t^{\Delta V}) = F_{L_t^{\Delta V}}^{-1}(\alpha) \quad \text{so that} \quad F_{L_t^{\Delta V}}(\text{VaR}_\alpha(L_t^{\Delta V})) = \alpha \quad (3.37)$$

since $L_t^{\Delta V}$ is a continuous random variable. The Equation (3.37) can for most credit portfolio models only be solved numerically. Also, note that $\text{VaR}_\alpha(L_t^{\Delta V})$ will for small time points t , be an approximation to $\text{VaR}_\alpha(L_t^{(V)})$ defined in (3.10), that is

$$\text{VaR}_\alpha(L_t^{\Delta V}) \approx \text{VaR}_\alpha(L_t^{(V)}) \quad \text{so} \quad F_{L_t^{\Delta V}}^{-1}(\alpha) \approx F_{L_t^{(V)}}^{-1}(\alpha) \quad \text{for small time points } t. \quad (3.38)$$

Just as in Theorem 2.12, we again remark that the formulas in Theorem 3.8 and related computations as in (3.37) requires efficient and quick methods of computing the number of default distribution $\mathbb{P}[N_t^{(m)} = k]$.

In our numerical studies in Section 5 - 6 we will use the results in Theorem 3.8 together with efficient numerical methods for computing $\mathbb{P}[N_t^{(m)} = k]$ in an intensity based CIR model and also in a one-factor Gaussian copula model.

Remark 3.10. In the case when there are no jump at the defaults in Definition 3.1, i.e when $U_n = 0$ for all n , then $S_{t,j} = S_{t,j}^{(BS)}$ for all companies \mathbf{A}_j with $S_{t,j}^{(BS)}$ given by

$$S_{t,j}^{(BS)} = S_{0,j} \exp\left(\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t + \sigma_j\left(\rho_{S,j}W_{t,0} + \sqrt{1 - \rho_{S,j}^2}W_{t,j}\right)\right) \quad (3.39)$$

where $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ are $J + 1$ independent Brownian motions and the rest of the notation is same as in Definition 3.1. Note that $S_{t,1}^{(BS)}, \dots, S_{t,J}^{(BS)}$ will under (3.39) still be correlated via the factor process $W_{t,0}$ and recall that $\rho_S W_{t,0} + \sqrt{1 - \rho_S^2} W_{t,j}$ is a Brownian motion for each stock price $S_{t,j}^{(BS)}$.

In view of Remark 3.10, we now state the following corollary to Theorem 3.8 in the case where there are no jumps among the stock prices $S_{t,j}$.

Corollary 3.11. *Consider an equally value-weighted portfolio as in Definition 3.7 where the J stock prices $S_{t,1}^{(BS)}, \dots, S_{t,J}^{(BS)}$ are defined as in (3.39) under the real probability measure \mathbb{P} . Then, with notation as above,*

$$\mathbb{P}[L_t^{\Delta V} \leq x] = \Phi\left(\frac{\frac{x}{S_0} + \sum_{j=1}^J\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t}{\sqrt{t\left(\left(\sum_{j=1}^J\sigma_j\rho_{S,j}\right)^2 + \sum_{j=1}^J\sigma_j^2\left(1 - \rho_{S,j}^2\right)\right)}}\right) \quad (3.40)$$

and

$$\text{VaR}_\alpha(L_t^{\Delta V}) = S_0\left(\sqrt{t\left(\left(\sum_{j=1}^J\sigma_j\rho_{S,j}\right)^2 + \sum_{j=1}^J\sigma_j^2\left(1 - \rho_{S,j}^2\right)\right)}\Phi^{-1}(\alpha) - \sum_{j=1}^J\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t\right) \quad (3.41)$$

where $\Phi(x)$ is the distribution function to a standard normal random variable. Furthermore, if the stock prices $S_{t,1}^{(BS)}, \dots, S_{t,J}^{(BS)}$ also satisfy (3.34) in Remark 3.9 then (3.40)-(3.41) simplify to

$$\mathbb{P}[L_t^{\Delta V} \leq x] = \Phi\left(\frac{\frac{x}{S_0} + J\left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{tJ(1 + (J-1)\rho_S^2)}}\right) \quad (3.42)$$

and

$$\text{VaR}_\alpha(L_t^{\Delta V}) = S_0 \left(\sigma \sqrt{tJ(1 + (J-1)\rho_S^2)} \Phi^{-1}(\alpha) - J \left(\mu - \frac{1}{2}\sigma^2 \right) t \right). \quad (3.43)$$

Proof. From (3.21) in Theorem 3.8 we have

$$F_{L_t^{\Delta V}}(x) = \mathbb{P}[L_t^{\Delta V} \leq x] = 1 - \mathbb{P}\left[\sum_{j=1}^J X_{t,j} \leq -\frac{x}{S_0}\right] = 1 - \mathbb{P}\left[\sum_{j=1}^J Z_{t,j} \leq -\frac{x}{S_0}\right] \quad (3.44)$$

since there are now jumps and where $Z_{t,j}$ is defined as in (3.23) in Theorem 3.8. Now, (3.33) and (3.25) in Theorem 3.8 with some elementary computations together with the standard normal symmetry property $1 - \Phi(-y) = \Phi(y)$ then imply that

$$\mathbb{P}[L_t^{\Delta V} \leq x] = \Phi\left(\frac{\frac{x}{S_0} + \sum_{j=1}^J \left(\mu_j - \frac{1}{2}\sigma_j^2\right)t}{\sqrt{t\left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j}\right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2)\right)}}\right)$$

where $\Phi(x)$ is the distribution function to a standard normal random variable and this proves (3.40). Furthermore, from the definition in (3.37) we know that $\text{VaR}_\alpha(L_t^{\Delta V}) = F_{L_t^{\Delta V}}^{-1}(\alpha)$ so this with the distribution of $F_{L_t^{\Delta V}}(x)$ in (3.40) will then after some trivial computations yield that

$$\text{VaR}_\alpha(L_t^{\Delta V}) = S_0 \left(\sqrt{t\left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j}\right)^2 + \sum_{j=1}^J \sigma_j^2 (1 - \rho_{S,j}^2)\right)} \Phi^{-1}(\alpha) - \sum_{j=1}^J \left(\mu_j - \frac{1}{2}\sigma_j^2\right)t \right)$$

which proves (3.41). Finally, if we set the portfolio weights to $w_j = 1$ for all companies $\mathbf{A}_1, \dots, \mathbf{A}_J$ and if their stock prices $S_{t,1}^{(\text{BS})}, \dots, S_{t,J}^{(\text{BS})}$ also satisfy (3.34) in Remark 3.9, we get an equally value-weighted portfolio where $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$ and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \dots, \mathbf{A}_J$ and using this in (3.40)-(3.41) with some computations gives us the expressions (3.42)-(3.43) which concludes the corollary. \square

In our numerical studies in Section 5 and 6 we will use the "Black-Scholes" linear portfolio formulas in Corollary 3.11 as benchmark to expressions for the stock prices with jumps at defaults given in Theorem 3.8.

The results in Theorem 3.8 and Corollary 3.11 holds for heterogeneous stock portfolios which are equally value-weighted and have arbitrary size J , that is the number of stocks J in the portfolio can e.g. be small or large. The main drawback with the formulas in Theorem 3.8 and Corollary 3.11 is that these expressions for the linearized loss $L_t^{\Delta V}$ only work somewhat accurately as an approximation to the true loss $L_t^{(V)}$ when the time t is small, and the expressions will fail as time t starts to increase. For example, the linearized loss $L_t^{\Delta V}$ may produce VaR-values that are bigger than V_0 which is impossible since by construction it will hold that $L_t^{(V)} \leq V_0$ almost surely for all $t \geq 0$ under the real probability measure \mathbb{P} . However, in certain cases we can still find highly analytical approximation formulas for the loss distribution $\mathbb{P}[L_t^{(V)} \leq x]$ at any time point t and where the loss will never exceed V_0 , as will be seen in the next section.

4. THE MULTIDIMENSIONAL CASE: APPROXIMATION FORMULAS TO LOSS DISTRIBUTIONS FOR LARGE HOMOGENEOUS STOCK PORTFOLIOS WITH JUMPS AT EXOGENOUS DEFAULTS.

For larger time points t , the linear approximations to the stock portfolio in Theorem 3.8 and Corollary 3.11 will fail. If we however assume that the stock prices $S_{t,j}$ satisfy (3.34) in Remark 3.9, that is $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$ and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \dots, \mathbf{A}_J$ so that the $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ are exchangeable and the portfolio becomes homogeneous (given same weights), and if the number of stocks J in the portfolio are "large", then we will in this section derive approximation formulas for the loss distribution $\mathbb{P}[L_t^{(V)} \leq x]$ which will work for arbitrary time points t , that is both for large and small time points t and which will also guarantee that portfolio loss always will be smaller than V_0 almost surely for all $t \geq 0$ under the measure \mathbb{P} . Hence, in this section we will make two assumptions. First we assume that (3.34) holds together with Definition 3.1 under the real probability measure \mathbb{P} , with equal portfolio

weights w_j for all companies $\mathbf{A}_1, \dots, \mathbf{A}_J$ in the portfolio V_t . Our second assumption is that the number of stocks J in the portfolio are "large". Since the stock portfolio is equally weighted, and we are primarily interested in Value-at-Risk calculation of the portfolio, then due to linearity of VaR we can without loss of generality let $w_j = 1$ for each stock in the portfolio and thus define the portfolio value as $V_t = \sum_{j=1}^J S_{t,j}$. Due to the condition (3.34) the portfolio V_t will then be equally value-weighted portfolio as in Definition 3.7.

Remark 4.1. Homogenization of a heterogeneous stock portfolio: Assuming a completely homogeneous stock portfolio so that the parameters for each stock are the same is of course an unrealistic feature. Consider a heterogeneous stock portfolio with stocks defined as in Definition 3.1 and portfolio value \hat{V}_t and define S_0, μ, σ and ρ_S as the corresponding sample means of the parameters in this portfolio, that is

$$S_0 = \frac{1}{J} \sum_{j=1}^J S_{0,j} \quad \mu = \frac{1}{J} \sum_{j=1}^J \mu_j \quad \sigma = \frac{1}{J} \sum_{j=1}^J \sigma_j \quad \text{and} \quad \rho_S = \frac{1}{J} \sum_{j=1}^J \rho_{S,j}. \quad (4.1)$$

Next create a homogeneous stock portfolio as in Section 3 with parameters S_0, μ, σ and ρ_S as in (4.1) with portfolio value V_t and where $W_{t,0}, W_{t,1}, \dots, W_{t,J}, N_t^{(m)}$ and $U_{i,j}$ are the same as in the heterogeneous portfolio. For such homogeneous portfolios Hofmann & Platen (2000) as well as Guan, Xiaoqing & Chong (2003) proves that the value process \hat{V}_t for a large heterogeneous stock portfolio can be approximated arbitrary well by V_t in L_1 -sense as $J \rightarrow \infty$. Hofmann & Platen (2000) proves the result for portfolios with only diffusions while Guan et al. (2003) extends the proof to the case where the stocks also can jump due to Poisson processes. In view of the results of e.g. Hofmann & Platen (2000) and Guan et al. (2003) it is therefore still relevant to consider homogeneous stock portfolios in particular if these portfolios comes from doing a homogenization of a heterogeneous stock portfolio as in (4.1).

Given the assumption that (3.34) is satisfied, we now state the following theorem.

Theorem 4.2. *Let $S_{t,1}, \dots, S_{t,J}$ be stock prices defined as in Definition 3.1 which satisfies (3.34) under the real probability measure \mathbb{P} . Then, with notation as above, we have*

$$\lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J S_{t,j} = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta + 1} \right)^{N_t^{(m)}} \quad \text{a.s. under } \mathbb{P} \left[\cdot \mid W_{t,0}, N_t^{(m)} \right] \quad (4.2)$$

and

$$\lim_{J \rightarrow \infty} \mathbb{P} \left[\frac{1}{J} \sum_{j=1}^J S_{t,j} \leq x \right] = \mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta + 1} \right)^{N_t^{(m)}} \leq x \right]. \quad (4.3)$$

Furthermore, for large J we have

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta + 1} \right)^{N_t^{(m)}} \right) \leq x \right] \quad \text{for large } J \quad (4.4)$$

and if $\rho_S \neq 0$ then for $x \leq JS_0 = V_0$ it holds that

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx 1 - \sum_{k=0}^m \Phi \left(\frac{\ln \left(\left(1 - \frac{x}{JS_0} \right) \left(\frac{\eta+1}{\eta} \right)^k \right) - \left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t}{\sigma \rho_S \sqrt{t}} \right) \mathbb{P} \left[N_t^{(m)} = k \right] \quad \text{for large } J \quad (4.5)$$

where $\Phi(x)$ is the distribution function to a standard normal random variable.

Proof. From the construction in Definition 3.1 we know that $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ are $J + 1$ independent Brownian motions and for each j the jump terms where $U_{1,j}, \dots, U_{m,j}$ are also independent of the processes $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ and the default counting process $N_t^{(m)}$. Hence, for a fixed t , and conditional on the pair $W_{t,0}, N_t^{(m)}$, then $S_{t,1}, \dots, S_{t,J}$ will be an i.i.d sequence and therefore a conditional version of the law of large numbers implies that

$$\lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J S_{t,j} = \mathbb{E} \left[S_{t,j} \mid W_{t,0}, N_t^{(m)} \right] \quad \text{a.s. under the random measure } \mathbb{P} \left[\cdot \mid W_{t,0}, N_t^{(m)} \right] \quad (4.6)$$

where the subindex j in $\mathbb{E} \left[S_{t,j} \mid W_{t,0}, N_t^{(m)} \right]$ in the right hand side of (4.6) could be any positive integer due to the exchangeability of $S_{t,1}, \dots, S_{t,J}$. Next, by Definition 3.1 together with (3.34), we have that

$$\begin{aligned} & \mathbb{E} \left[S_{t,j} \mid W_{t,0}, N_t^{(m)} \right] \\ &= \mathbb{E} \left[S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma \left(\rho_S W_{t,0} + \sqrt{1 - \rho_S^2} W_{t,j} \right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\ &= S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma \rho_S W_{t,0} \right) \mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right]. \end{aligned} \quad (4.7)$$

Furthermore,

$$\mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] = \exp \left(\frac{\sigma^2 (1 - \rho_S^2) t}{2} \right) \mathbb{E} \left[\exp \left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| N_t^{(m)} \right] \quad (4.8)$$

since

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)}, \{U_{n,j}\}_{n=1}^m \right] \middle| W_{t,0}, N_t^{(m)} \right] \\ &= \mathbb{E} \left[\exp \left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} \right) \middle| W_{t,0}, N_t^{(m)}, \{U_{n,j}\}_{n=1}^m \right] \middle| W_{t,0}, N_t^{(m)} \right] \\ &= \mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} \right) \right] \mathbb{E} \left[\exp \left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\ &= \exp \left(\frac{\sigma^2 (1 - \rho_S^2) t}{2} \right) \mathbb{E} \left[\exp \left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| N_t^{(m)} \right] \end{aligned} \quad (4.9)$$

where the third equality in (4.9) follows from the fact that $W_{t,j}$ is independent of $W_{t,0}, N_t^{(m)}, \{U_{n,j}\}_{n=1}^m$ and the fourth equality in (4.9) is due to that $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ is independent of $W_{t,0}$, see e.g. 9.7(k) on p.88 in Williams (2000), and due to standard computations of $\mathbb{E} \left[\exp \left(\sigma \sqrt{1 - \rho_S^2} W_{t,j} \right) \right]$, which proves (4.8). Next, note that

$$\mathbb{E} \left[\exp \left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| N_t^{(m)} \right] = \left(\frac{\eta}{\eta + 1} \right)^{N_t^{(m)}} \quad (4.10)$$

since

$$\begin{aligned}
\mathbb{E} \left[\exp \left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| N_t^{(m)} \right] &= \sum_{k=0}^m \mathbb{E} \left[\exp \left(- \sum_{n=1}^k U_{n,j} \right) \middle| N_t^{(m)} = k \right] \mathbf{1}_{\{N_t^{(m)}=k\}} \\
&= \sum_{k=0}^m \mathbb{E} \left[\exp \left(- \sum_{n=1}^k U_{n,j} \right) \right] \mathbf{1}_{\{N_t^{(m)}=k\}} \\
&= \sum_{k=0}^m \left(\frac{\eta}{\eta+1} \right)^k \mathbf{1}_{\{N_t^{(m)}=k\}} \\
&= \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}}
\end{aligned} \tag{4.11}$$

where the second equality in (4.11) is due to that $\{U_{n,j}\}_{n=1}^m$ are independent of $N_t^{(m)}$ and the third equality in (4.11) follows from (2.62) in Theorem 2.12. So combining (4.7), (4.8) and (4.10) together with some computations then renders that

$$\mathbb{E} \left[S_{t,j} \middle| W_{t,0}, N_t^{(m)} \right] = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \tag{4.12}$$

and (4.12) in (4.6) finally implies

$$\lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J S_{t,j} = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \text{ a.s. under } \mathbb{P} \left[\cdot \middle| W_{t,0}, N_t^{(m)} \right] \tag{4.13}$$

which proves (4.2). The random measure $\mathbb{P} \left[\cdot \middle| W_{t,0}, N_t^{(m)} \right]$ is constructed from the probability measure \mathbb{P} used in this paper, and in particular Definition 3.1, so (4.13) then implies that $\frac{1}{J} \sum_{j=1}^J S_{t,j}$ converges weakly (i.e in distribution) to $S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right)$ under the probability measure \mathbb{P} when $J \rightarrow \infty$. To see this, note that

$$\mathbb{P} \left[\frac{1}{J} \sum_{j=1}^J S_{t,j} \leq x \right] = \mathbb{E} \left[\mathbb{P} \left[\frac{1}{J} \sum_{j=1}^J S_{t,j} \leq x \middle| W_{t,0}, N_t^{(m)} \right] \right] \tag{4.14}$$

and (4.13) implies that

$$\mathbb{P} \left[\frac{1}{J} \sum_{j=1}^J S_{t,j} \leq x \middle| W_{t,0}, N_t^{(m)} \right] \rightarrow \mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \leq x \middle| W_{t,0}, N_t^{(m)} \right] \tag{4.15}$$

as $J \rightarrow \infty$. Hence, (4.14)-(4.15) together with the law of iterated expectations then renders

$$\mathbb{P} \left[\frac{1}{J} \sum_{j=1}^J S_{t,j} \leq x \right] \rightarrow \mathbb{P} \left[S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \leq x \right] \text{ as } J \rightarrow \infty \tag{4.16}$$

which proves (4.3). Thus, if J is large then (4.16) implies that

$$\sum_{j=1}^J S_{t,j} \stackrel{d}{\approx} \Big|_{\mathbb{P}} JS_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \text{ for large } J \tag{4.17}$$

where $\stackrel{d}{\approx} \Big|_{\mathbb{P}}$ means "approximately equal in distribution under the probability measure \mathbb{P} ". Next, from the definition of the portfolio value V_t in (3.6) and the portfolio loss process $L_t^{(V)}$ in (3.9) together with the fact that $S_{t,0} = S_0$ for all stocks due to condition (3.34), we get that

$$L_t^{(V)} = V_0 - V_t = \sum_{j=1}^J S_{t,0} - \sum_{j=1}^J S_{t,j} = JS_0 - \sum_{j=1}^J S_{t,j} \tag{4.18}$$

so (4.17) and (4.18) with some simple calculations then imply that

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \right) \leq x \right] \quad \text{for large } J \quad (4.19)$$

which proves (4.4). We next want to find an more explicit expression of the right hand side in (4.19). First, we note that

$$\begin{aligned} & \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \right) \leq x \right] \\ &= \sum_{k=0}^m \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^k \right) \leq x \mid N_t^{(m)} = k \right] \mathbb{P} \left[N_t^{(m)} = k \right]. \end{aligned} \quad (4.20)$$

Since $W_{t,0}$ is independent of $N_t^{(m)}$ we get

$$\begin{aligned} & \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^k \right) \leq x \mid N_t^{(m)} = k \right] \\ &= \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^k \right) \leq x \right] \end{aligned} \quad (4.21)$$

and assuming $\rho_S \neq 0$, some calculations then renders that

$$\begin{aligned} & \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \left(\frac{\eta}{\eta+1} \right)^k \right) \leq x \right] \\ &= 1 - \Phi \left(\frac{\ln \left(\left(1 - \frac{x}{JS_0} \right) \left(\frac{\eta+1}{\eta} \right)^k \right) - \left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t}{\sigma \rho_S \sqrt{t}} \right) \end{aligned} \quad (4.22)$$

where $\Phi(x)$ is the distribution function to a standard normal random variable. So combining (4.20)-(4.22) and inserting these expression into (4.19) finally yields

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx 1 - \sum_{k=0}^m \Phi \left(\frac{\ln \left(\left(1 - \frac{x}{JS_0} \right) \left(\frac{\eta+1}{\eta} \right)^k \right) - \left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t}{\sigma \rho_S \sqrt{t}} \right) \mathbb{P} \left[N_t^{(m)} = k \right] \quad \text{for large } J$$

which proves (4.5) and this concludes the theorem. \square

We next make some remarks to the results in Theorem 4.2.

Remark 4.3. First, we note from (4.2) in Theorem 4.2, that when conditioning on $W_{t,0}, N_t^{(m)}$ and then studying the limit of $\frac{1}{J} \sum_{j=1}^J S_{t,j}$ when $J \rightarrow \infty$ we see that the individual diffusions $W_{t,j}$ as well as the individual jump terms $U_{n,j}$ vanish. Only the effect of $W_{t,0}$ and $N_t^{(m)}$ remains in the limit of $\frac{1}{J} \sum_{j=1}^J S_{t,j}$ on a simply form as stated in Equation (4.2). Secondly, if $\rho_S = 0$, meaning that there is no correlation through the factor process $W_{t,0}$ in the diffusion part among the stocks, then (4.2) collapses into

$$\lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J S_{t,j} = S_0 e^{\mu t} \left(\frac{\eta}{\eta+1} \right)^{N_t^{(m)}} \quad \text{a.s. under } \mathbb{P} \left[\cdot \mid N_t^{(m)} \right] \quad (4.23)$$

where the right hand side of (4.23) is a piecewise deterministic process with jumps at the default times τ_1, \dots, τ_m . If " $\eta = \infty$ " so that $U_{n,j} = 0$ for all pairs n, j (see also Remark 2.10) and if $\rho_S = 0$, then (4.23) reduces to the "standard" law of large numbers under the measure \mathbb{P} since from Remark 3.10 with $\rho_S = 0$ then implies that $S_{t,j} = S_{t,j}^{(\text{BS})}$ for all companies \mathbf{A}_j and $S_{t,1}, \dots, S_{t,J}$ will be an i.i.d sequence.

This observation together with Equation (2.65) gives $\mathbb{E} \left[S_{t,j}^{(\text{BS})} \right] = S_0 e^{\mu t}$ which is the right hand side of (4.23) without the point process $N_t^{(m)}$, since $U_{n,j} = 0$ for all n and j , that is,

$$\lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J S_{t,j} = S_0 e^{\mu t} \quad \mathbb{P} \text{-a.s.}$$

and this is just the (strong) law of large numbers under the measure \mathbb{P} , since $S_{t,1}, \dots, S_{t,J}$ is an i.i.d sequence.

For $\rho_S \neq 0$, define $F_{L_t^{(V)}}^{\text{LPA}}(x)$ as

$$F_{L_t^{(V)}}^{\text{LPA}}(x) = 1 - \sum_{k=0}^m \Phi \left(\frac{\ln \left(\left(1 - \frac{x}{JS_0} \right) \left(\frac{\eta+1}{\eta} \right)^k \right) - \left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t}{\sigma \rho_S \sqrt{t}} \right) \mathbb{P} \left[N_t^{(m)} = k \right]. \quad (4.24)$$

Then, if $\rho_S \neq 0$, the large portfolio approximation formula (4.5) in Theorem 4.2 implies that

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx F_{L_t^{(V)}}^{\text{LPA}}(x) \quad \text{for large } J. \quad (4.25)$$

Note that $F_{L_t^{(V)}}^{\text{LPA}}(x)$ in (4.24) is exactly equal to the right hand side of (4.4). From the probability in the right hand side of (4.4) it is clear that this probability will be one for $x > V_0 = JS_0$ and then $F_{L_t^{(V)}}^{\text{LPA}}(x) = 1$ for $x > V_0 = JS_0$. To see this, note that for each k we have that

$$\ln \left(\left(1 - \frac{x}{JS_0} \right) \left(\frac{\eta+1}{\eta} \right)^k \right) \rightarrow -\infty \quad \text{as } x \uparrow JS_0 = V_0.$$

so for each k we get

$$\lim_{x \uparrow JS_0} \Phi \left(\frac{\ln \left(\left(1 - \frac{x}{JS_0} \right) \left(\frac{\eta+1}{\eta} \right)^k \right) - \left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t}{\sigma \rho_S \sqrt{t}} \right) = 0$$

and this observation in (4.24) implies that

$$\lim_{x \uparrow JS_0} F_{L_t^{(V)}}^{\text{LPA}}(x) = 1. \quad (4.26)$$

Hence, in view of (4.24) and (4.26) the distribution $F_{L_t^{(V)}}^{\text{LPA}}(x)$ is only defined for $x \leq V_0 = JS_0$. Consequently, our LPA approximation formula in (4.25) imply that $F_{L_t^{(V)}}^{\text{LPA}}(x) = 1$ for $x > V_0 = JS_0$, that is for any time point t the loss will never be bigger than V_0 which is financially correct given our model setup, while the distribution for the linearized portfolio loss $L_t^{\Delta V}$ discussed in Section 3 can produce losses bigger than $V_0 = JS_0$ when t increases.

We here note that the distribution function $F_{L_t^{(V)}}^{\text{LPA}}(x)$ defined in (4.24) and used in the the right hand side of (4.5) in Theorem 4.2, will be much easier to evaluate than the corresponding distribution for the "small time" linear approximation $L_t^{\Delta V}$ to $L_t^{(V)}$, where $\mathbb{P} [L_t^{\Delta V} \leq x]$ is given by (3.18) in Theorem 3.8. More specific, the expression for $\mathbb{P} [L_t^{\Delta V} \leq x]$ in (3.18) will for each $k \geq 1$ in the sum involve computations of an integral given by (3.19) in Theorem 3.8, while the corresponding terms in the sum for $F_{L_t^{(V)}}^{\text{LPA}}(x)$ in (4.24) simply involves an evaluation of the distribution function to a standard normal random variable for each k in the sum. However, we remind that $\mathbb{P} [L_t^{\Delta V} \leq x]$ works for heterogeneous stock portfolios with arbitrary number of stocks J , in particular smaller J , while the approximation of $\mathbb{P} [L_t^{(V)} \leq x]$ via $F_{L_t^{(V)}}^{\text{LPA}}(x)$ in (4.25) is only feasible for large stock portfolio sizes J . On the other hand, $F_{L_t^{(V)}}^{\text{LPA}}(x)$ works for arbitrary time points t while $\mathbb{P} [L_t^{\Delta V} \leq x]$ is only a good approximation to $\mathbb{P} [L_t^{(V)} \leq x]$ for small time points t .

Let $\text{VaR}_\alpha \left(L_t^{(V)} \right)$ defined as in (3.10) be the Value-at-Risk for the stock portfolio loss $L_t^{(V)}$ with confidence level α . By using the large portfolio approximation formula (4.5) in Theorem 4.2, that is, relation

(4.25), we can for large J find an approximation to $\text{VaR}_\alpha \left(L_t^{(V)} \right)$ which then is given as the unique solution x^* to the equation $F_{L_t^{(V)}}^{\text{LPA}}(x^*) = \alpha$, that is

$$\text{VaR}_\alpha \left(L_t^{(V)} \right) \approx (F^{-1})_{L_t^{(V)}}^{\text{LPA}}(\alpha) \quad \text{for large } J \quad (4.27)$$

where $(F^{-1})_{L_t^{(V)}}^{\text{LPA}}(x)$ denotes the inverse function to the function $F_{L_t^{(V)}}^{\text{LPA}}(x)$ defined in (4.24). Since $F_{L_t^{(V)}}^{\text{LPA}}(x) = 1$ for $x > V_0 = JS_0$ we see that (4.27) can never produce a VaR value bigger than V_0 , contrary to the linearized portfolio loss VaR-values.

Just as in Theorem 2.12 and Theorem 3.8, we once again remark that formula in (4.5) in Theorem 4.2 and computations as in (4.27) requires efficient and quick methods of computing the number of default distribution $\mathbb{P} \left[N_t^{(m)} = k \right]$.

In the case when there are no jumps in the stock prices at the defaults of the exogenous group of defaultable entities in Definition 3.1, i.e when " $\eta = \infty$ " so that $U_{n,j} = 0$ for all pairs n, j (see also Remark 2.10), and thus $S_{t,j} = S_{t,j}^{(\text{BS})}$ for all companies \mathbf{A}_j where $S_{t,j}^{(\text{BS})}$ is given by (3.39) in Remark 3.10 and if $\rho_S \neq 0$, then (4.2) in Theorem 4.2 will reduce to

$$\lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J S_{t,j} = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \quad \text{a.s. under the random measure } \mathbb{P}[\cdot | W_{t,0}]. \quad (4.28)$$

Hence, from (4.28) and using the same arguments as in Theorem 4.2 we then have that

$$\mathbb{P} \left[L_t^{(V)} \leq x \right] \approx \mathbb{P} \left[JS_0 \left(1 - \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t + \sigma \rho_S W_{t,0} \right) \right) \leq x \right] \quad \text{for large } J. \quad (4.29)$$

We also note that the right hand side in (4.28) is on the exact same form as the stock price $S_t^{(\text{BS})}$ in the Black-Scholes model for a single stock, under the real probability measure \mathbb{P} given in (2.64), but now with the volatility $\sigma \rho_S$ instead of σ as in (2.64). Hence, for large J , the loss process $L_t^{(V)}$ will for the case when $U_{n,j} = 0$ for all n, j behave as the loss process for one single stock which follows the Black-Scholes dynamics with volatility $\sigma \rho_S$, drift μ and initial value JS_0 . From Equation (2.73) in Section 2 together with the large portfolio approximation in (4.29) we therefore in the case with no jumps in the stock price get that

$$\text{VaR}_\alpha \left(L_t^{(V)} \right) \approx JS_0 \left(1 - \exp \left(\sigma \rho_S \sqrt{t} \Phi^{-1}(1 - \alpha) + \left(\mu - \frac{1}{2} \sigma^2 \rho_S^2 \right) t \right) \right) \quad \text{for large } J. \quad (4.30)$$

In our numerical studies in Section 5 and 6 we will use the "Black-Scholes" LPA VaR formula in (4.30) as benchmark for the VaR-formulas obtained when using the LPA loss distribution (4.5) in Theorem 4.2 when the stock prices have jumps and are exchangeable.

5. NUMERICAL EXAMPLES WHEN THE DEFAULT TIMES HAVE CIR INTENSITIES

In this section we will study Value-at-Risk for the loss $L_t^{(S)} = -(S_t - S_0)$ for one single stock when the stock price S_t is given by Definition 2.1 under the real probability measure \mathbb{P} . Throughout this section we assume that the default times $\tau_1, \tau_2, \dots, \tau_m$ to the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ are exchangeable, conditional independent and have default intensities $\lambda_{t,i} = \lambda_t$ same for all entities where λ_t is a CIR-process. Furthermore, the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ in S_t at the defaults $\tau_1, \tau_2, \dots, \tau_m$ are distributed as V_1, \dots, V_m in Definition 2.9. In Subsection 5.1 we first discuss the parameters and other related quantizes. Then, in Subsection 5.2 we study Value-at-Risk for the loss of one individual stock with price under the real probability measure \mathbb{P} in a credit portfolio model with parameters as discussed in Subsection 5.1. Finally, in Subsection 5.3 we give some very important and useful remarks on the numerical computation of the loss distribution. The observations done in Subsection 5.3 will also hold for the loss distributions derived in Section 3 and Section 4 and for the credit portfolio model studies in Section 6.

5.1. The parameters and related quantities. In this subsection we discuss the modeling setup and its parameters that will hold in the rest of section and present some related quantities such as e.g. the distribution of the number of defaults $\left(\mathbb{P} \left[N_t^{(m)} = k \right] \right)_{k=0}^m$ for our model.

In the rest of this section we assume that the default times $\tau_1, \tau_2, \dots, \tau_m$ to the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ are exchangeable, conditional independent and have default intensities $\lambda_{t,i} = \lambda_t$ same for all entities. We set $\lambda_t = \lambda_{t,i}$ to be a Cox-Ingersoll-Ross process (CIR-process), that is

$$d\lambda_t = a_c (\mu_c - \lambda_t) dt + \sigma_c \sqrt{\lambda_t} dW_t^{(c)} \quad (5.1.1)$$

where $W_t^{(c)}$ is a Brownian motion under the physical probability measure \mathbb{P} , independent of the other random variables in S_t . Then the default times τ_1, \dots, τ_m are constructed as

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_s ds \geq E_i \right\} \quad (5.1.2)$$

where E_1, \dots, E_m is an i.i.d sequence of exponentially distributed random variables all with parameter one which are independent of $W_t^{(c)}$. From the construction (5.1.2) one can show that τ_1, \dots, τ_m are conditional independent given the trajectory of $(W_t^{(c)})_{t \geq 0}$. Furthermore, the marginal default distribution $F(t) = \mathbb{P}[\tau_i \leq t]$ is expressed as

$$F(t) = \mathbb{P}[\tau_i \leq t] = 1 - \mathbb{E} \left[e^{-\int_0^t \lambda_s ds} \right] \quad (5.1.3)$$

and is same for all entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ due to the exchangeability, where the quantity $\mathbb{E} \left[e^{-\int_0^t \lambda_s ds} \right]$ has closed formulas, see e.g. in Bielecki et al. (2014c) or Herbertsson (2022). The construction in (5.1.2)-(5.1.3) can be applied to arbitrary intensities λ_t , and thus not only to a CIR-process. From a practical point of view we want to have analytical expressions of the default distribution $F(t)$ in (5.1.3). Another example of intensity which gives analytical formulas for $F(t)$ is a shot-noise model as presented in e.g. Herbertsson, Jang & Schmidt (2011). The construction in (5.1.2)-(5.1.3) will also work for heterogeneous credit portfolios, that is when the intensities $\lambda_{i,t}$ are different among the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$.

Going back to our stock price model for S_t , we let the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ in S_t at the defaults $\tau_1, \tau_2, \dots, \tau_m$ be distributed as V_1, \dots, V_m in Definition 2.9 so $\tilde{V}_i = e^{-\tilde{U}_i} - 1$ where $\tilde{U}_1, \dots, \tilde{U}_m$ are i.i.d and exponentially distributed with parameter $\eta > 0$. Hence, given the above assumptions, the dynamics of the stock price S_t is same as in Corollary 2.11 and Theorem 2.12 where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and $\tau_1, \tau_2, \dots, \tau_m$ are exchangeable, conditionally independent, and have intensities $\lambda_{t,i} = \lambda_t$ as in (5.1.1). In our numerical examples we choose a CIR-process in (5.1.1) with the parameters $a_c = 0.6, \mu_c = 0.056, \sigma_c = 0.18$ and $\lambda_0 = 0.0262$ so that the individual one-year default probability is 0.0329 computed via well-known explicit expressions for the default probability $\mathbb{P}[\tau_i \leq t]$ when τ_i has a CIR-default intensity. Furthermore, we let the number of defaultable entities be $m = 125$, see also in Table 1. In Table 2 we show the expected number of defaults $\mathbb{E} \left[N_t^{(m)} \right]$ for $t = 1, 3, 6, 12, 18, 24$ months when individual default times have CIR-intensities as in Table 1 and where $m = 125$. So from Table 2 we see that our CIR-intensities implies that we expect for example around 2 defaults in six months, 4 defaults in one year, and 6 to 7 defaults in one and a half year. Consequently this is also the number of jumps that we expect to occur in our stock price up to each of these time points where each jump has the expected size of $\mathbb{E}[U] = \frac{1}{\eta}$. By our assumption of exchangeability we have that $\mathbb{E} \left[N_t^{(m)} \right] = m \mathbb{P}[\tau_i \leq t]$ so the individual default probabilities at $t = 1, 3, 6, 12, 18, 24$ months are obtained from Table 2 by dividing the numbers for $\mathbb{E} \left[N_t^{(m)} \right]$ with m . In Table 2 we see that after e.g. 6 months, there is a 0.1% probability of having 25 defaults or more among $\mathbf{C}_1, \dots, \mathbf{C}_m$, and after 24 months there is a 0.1% probability of 32 defaults or more among $\mathbf{C}_1, \dots, \mathbf{C}_m$.

Table 1. The parameters and related quantities for the CIR-process λ_t and the stock price S_t where we let $m = 125$.

λ_t	$\lambda_0 = 0.0262$	$a_c = 0.6$	$\mu_c = 0.056$	$\sigma_c = 0.18$	$\mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$
S_t	$S_0 = 50$	$\mu = 0.15 = 15\%$	$\sigma = 0.2 = 20\%$	$\eta = 26.71$	$\mathbb{E}[U] = \frac{1}{\eta} = 0.0374 = 3.74\%$

Next we turn to the parameters for the stock price model. We set $S_0 = 50, \mu = 0.15 = 15\%$, and $\sigma = 0.2 = 20\%$, see in Table 1. The jump parameter η is calibrated so that for $T = 1$ year we have

$$\mathbb{E}[S_T] = S_0 \quad \text{or equivalently} \quad \mathbb{E} \left[\left(\frac{\eta}{\eta + 1} \right)^{N_T^{(m)}} \right] = e^{-\mu T} \quad \text{for } T = 1 \quad (5.1.4)$$

see also Equation (2.71) in Section 2. Hence, η is calibrated so that the defaults from the CIR-model, "wipes" out the expected one-year log-growth for a corresponding Black-Scholes model with drift $\mu = 15\%$ and where $m = 125$. With the above parameters, we get that $\eta = 26.71$ via a numerical solver so that $\mathbb{E}[U_i] = \frac{1}{\eta} = 0.0374$, see also in Table 1.

Table 2. The expected number of defaults $\mathbb{E}[N_t^{(m)}]$ and $\text{VaR}_{99.9\%}(N_t^{(m)})$ for $t = 1, 3, 6, 12, 18, 24$ months when individual default times have CIR-intensities as in Table 1 and where $m = 125$.

t (in months)	1	3	6	12	18	24
$\mathbb{E}[N_t^{(m)}]$	0.2802	0.8818	1.875	4.116	6.596	9.222
$\text{VaR}_{99.9\%}(N_t^{(m)})$	20	25	25	25	27	32

From Theorem 2.12 and the definition of VaR we know that in order to compute $\text{VaR}_\alpha(L_t^{(S)})$ we need to compute the distribution of the number of defaults $(\mathbb{P}[N_t^{(m)} = k])_{k=0}^m$. Finding efficient numerical methods for $\mathbb{P}[N_t^{(m)} = k]$ is a non-trivial problem. We will in this paper use the method developed in Herbertsson (2022) to find $\mathbb{P}[N_t^{(m)} = k]$ which are based on saddlepoint methods for exchangeable, conditionally independent credit portfolio models and works both for intensity based frameworks as well as in factor copula settings. To find $\mathbb{P}[N_t^{(m)} = k]$ in the intensity based case we need the density $f_{Z_t}(z)$ to the random variable $Z_t = \int_0^t \lambda_u du$ where λ_t is a CIR-process defined as in (5.1.1). Details of how to find $f_{Z_t}(z)$ as well as numerical graphs of $f_{Z_t}(z)$ are found in e.g. Herbertsson (2022).

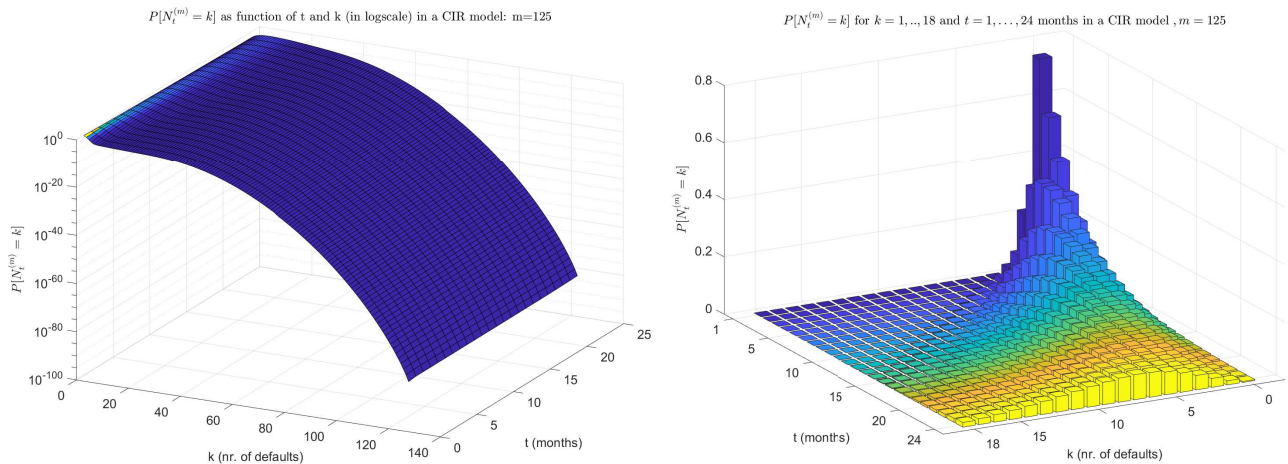


Figure 1. The time evolution of the distribution $\mathbb{P}[N_t^{(m)} = k]$ for $t = 1, 2, \dots, 24$ months when individual default times have CIR-intensities as in Table 1 where $m = 125$. **Left panel:** in log-scale for $k = 0, \dots, 125$. **Right panel:** for $k = 0, \dots, 18$. The plots in the two panels are viewed from different angles.

With the CIR-parameters parameters in Table 1 we compute $\mathbb{P}[N_t^{(m)} = k]$ with the saddlepoint method mentioned above and the left panel in Figure 1 plots, for $m = 125$, the time evolution of the distribution $\mathbb{P}[N_t^{(m)} = k]$ in log-scale where $k = 0, \dots, 125$ and $t = 1, 2, \dots, 24$ months. Furthermore, the right panel in Figure 1 displays the time evolution of the number of distribution $\mathbb{P}[N_t^{(m)} = k]$ in normal scale where

$k = 0, 1, \dots, 18$ when $m = 125$ and $t = 1, \dots, 24$ months when individual default times have CIR-intensities with parameters same as in the left panel in Figure 1. The plots in Figure 1 were generated with the saddlepoint algorithms found in Herbertsson (2022) and in these figures we write t in months, but the actual computations of $\mathbb{P} \left[N_t^{(m)} = k \right]$ are done with t measured in units of years. So for example two, six and 24 months mean that t is given by $t = \frac{2}{12}$, $\frac{6}{12}$ and $t = \frac{24}{12} = 2$ in our formulas for the computation of $\mathbb{P} \left[N_t^{(m)} = k \right]$. The same also hold for the results in Table 2.

5.2. VaR over a 2-year period for one stock when the jumps in the stock price are due to default times with CIR-intensities. In this subsection we will study Value-at-Risk for the loss $L_t^{(S)} = -(S_t - S_0)$ of one individual stock with price S_t given by Definition 2.1 under the real probability measure \mathbb{P} in a credit portfolio model as discussed in Subsection 5.1. Hence, the stock price S_t has jumps at the default times $\tau_1, \tau_2, \dots, \tau_m$ which are exchangeable and where the individual default times have CIR-intensities with parameters same as in Table 1. Furthermore, the jump parameter η is calibrated so that condition (5.1.4) holds and the rest of the parameters for S_t are displayed in Table 1.

In Figure 2-3 we study the time evolution of Value-at-Risk (in % of S_0) of a single stock for $t = 1, 2, \dots, 24$ months, computed with same stock parameters as in Table 1. More specific, for $m = 125$, the left panel in Figure 2 displays the time evolution of Value-at-Risk in % of S_0 for $t = 1, 2, \dots, 24$ months in the case when S_t has jumps coming from default times which have CIR-intensities with parameters same as in Table 1. The right panel in Figure 2 displays the Black-Scholes case for the stock price, i.e. with no jumps in S_t which has same drift and volatility parameters as in the left panel.

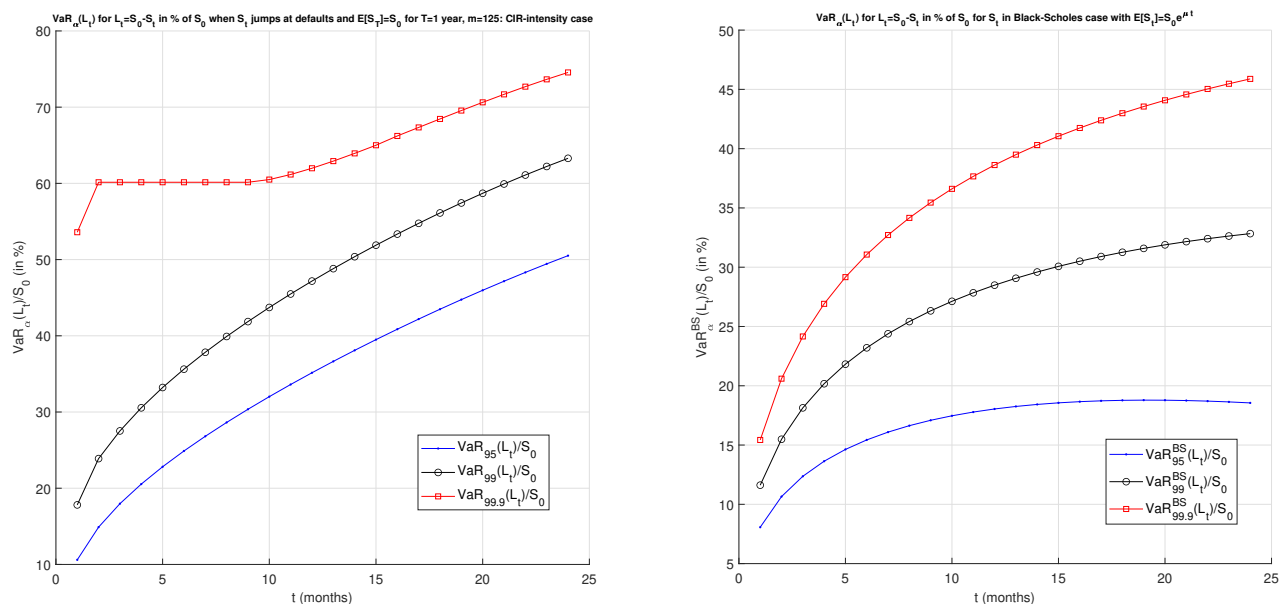


Figure 2. $m = 125$: The time evolution of Value-at-Risk (in % of S_0) of a single stock for $t = 1, 2, \dots, 24$ months. **Left panel:** In the case with jumps in S_t coming from default times which have CIR-intensities with parameters same as in Table 1. **Right panel:** In the Black-Scholes case, i.e. without jumps, where drift and volatility are same as in Table 1.

The interpretation of the results in Figure 2 is done as follows. For example, in the left panel of Figure 2, looking at the black line (99%-VaR), we see that for $t = \frac{14}{12}$, that is after 14 months, then there is a 1% probability of having a loss in the stock which is 50% or bigger, of the initial stock price S_0 at time $t = 0$. Similarly, for the red line (99.9%-VaR) in the left panel of Figure 2, at $t = \frac{20}{12}$, that is 20 months after the starting point $t = 0$, there is 0.1% probability of having a stock loss which is 70% (or bigger) of the starting value S_0 at time $t = 0$. The interpretation of the graphs in the right panel of Figure 2, i.e. the Black-Scholes case, should be done in the same way as in the left panel of Figure 2. Furthermore, in Figure 3 we plot the time evolution of the relative difference of Value-at-Risk (in %) between the case

with jumps in the stock price S_t coming from default times which have CIR-intensities with parameters same as in Table 1, and the standard Black-Scholes case, i.e. without jumps. The rest of the parameters for S_t are same as in Table 1.

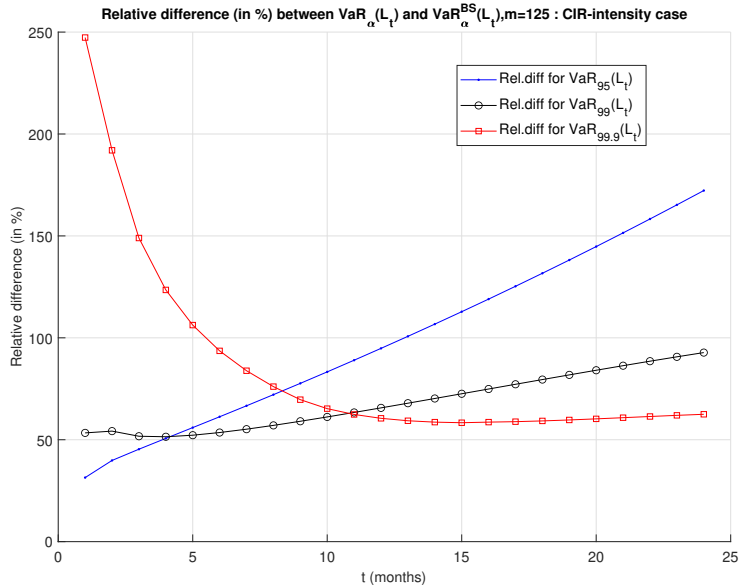


Figure 3. The time evolution of the relative difference of Value-at-Risk (in %) between the case of with jumps in the stock price S_t coming from default times which have CIR-intensities with parameters same as in Table 1, and the standard Black-Scholes case, i.e. without jumps. The relative difference is measured with respect to the Black-Scholes case. The rest of the parameters for S_t are same as in Table 1.

As can be seen in Figure 3, introducing downward jumps in S_t at the default times $\tau_1, \tau_2, \dots, \tau_m$ which are exchangeable and where the individual default times have CIR-intensities as in Subsection 5.1, will in general increase the Value-at-Risk up to around 50% and much more at some few time points (up to 250%) compared to the Black-Scholes model, and this holds for all three confidence levels $\alpha = 95\%$, 99% and $\alpha = 99.9\%$. For $\alpha = 95\%$, 99% the relative difference (jump-stock model vs. Black-Scholes) is almost linearly increasing in time t . Of course, that the relative VaR difference between the jump vs. non-jump case will increase as shown in Figure 3 is not surprising, but knowing exactly how big the difference actually is as function of different parameters as well as time t , requires the use of somewhat analytical formulas and efficient numerical methods.

5.3. Some remarks on the numerical computation of the loss distributions. In this subsection we give some important remarks on the computation of the loss distribution $F_{L_t^{(S)}}(x)$. The observations done in this subsection will also hold for the loss distributions derived in Section 3 and Section 4 and for the credit portfolio model studies in Section 6.

The computations in the left panel of Figure 2 are done by numerically solve Equation (2.72). From Theorem 2.12 we know that $F_{L_t^{(S)}}(x) = \mathbb{P} \left[L_t^{(S)} \leq x \right]$ is given by

$$F_{L_t^{(S)}}(x) = 1 - \sum_{k=0}^m \Psi_k \left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta \right) \mathbb{P} \left[N_t^{(m)} = k \right] \quad (5.3.1)$$

where the mappings $\Psi_k(x, t, \mu, \sigma, u, \eta)$ satisfy $0 \leq \Psi_k(x, t, \mu, \sigma, u, \eta) \leq 1$ and are defined in (2.30)-(2.31). By looking at e.g. the left panel in Figure 1 but also in the left panels of Figure 4, 11 and 13, we see that the probabilities $\mathbb{P} \left[N_t^{(m)} = k \right]$ are extremely small for moderate and large integers k for most time points t . For example, in the left panels of Figure 1 we have that $\mathbb{P} \left[N_t^{(m)} = k \right] < 10^{-14}$ for $k \geq 65$ at all time points t , and $\mathbb{P} \left[N_t^{(m)} = k \right] < 10^{-28}$ for $k \geq 85$ at all t . These observations means that we do not have to compute all the terms in the sum for $F_{L_t^{(S)}}(x)$ given by (5.3.1), but still have a very accurate

approximation to $F_{L_t^{(S)}}(x)$ in the truncated sum. For example, let ε be a very small positive constant, e.g. $\varepsilon \leq 10^{-9}$. Then, for each fixed t , there exists a subsequence $k_0, k_1, k_2, \dots, k_{m_t(\varepsilon)}$ of the integers $0, 1, 2, \dots, m$ such that

$$\sum_{j=0}^{m_t(\varepsilon)} \mathbb{P} \left[N_t^{(m)} = k_j \right] \geq 1 - \varepsilon. \quad (5.3.2)$$

In the credit portfolio models used in this paper the subsequence $k_0, k_1, k_2, \dots, k_{m_t(\varepsilon)}$ can always be chosen on the form $0, 1, \dots, m_t(\varepsilon)$, that is $k_j = j$ for $j = 0, 1, \dots, m_t(\varepsilon)$ so that (5.3.2) can be rewritten as

$$\sum_{k=0}^{m_t(\varepsilon)} \mathbb{P} \left[N_t^{(m)} = k \right] \geq 1 - \varepsilon \quad \text{and thus} \quad \sum_{k=m_t(\varepsilon)+1}^m \mathbb{P} \left[N_t^{(m)} = k \right] < \varepsilon \quad (5.3.3)$$

where it obviously holds that $m_t(\varepsilon) \leq m$ for any $0 < \varepsilon < 1$ and at all time points t . Typically, for the credit portfolio models studied in this paper it will often (but not always) hold that $m_t(\varepsilon) \ll m$ for most time points t . Given an arbitrary number $0 < \varepsilon < 1$, and for a fixed t , we can in view of the above observations define the function $F_{L_t^{(S)}}^\varepsilon(x)$ as

$$F_{L_t^{(S)}}^\varepsilon(x) = 1 - \sum_{k=0}^{m_t(\varepsilon)} \Psi_k \left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta \right) \mathbb{P} \left[N_t^{(m)} = k \right] \quad (5.3.4)$$

where the rest of the parameters and mappings are defined as in (5.3.1). Then, (5.3.1), (5.3.3) and (5.3.4) together with the triangle inequality implies that

$$\left| F_{L_t^{(S)}}(x) - F_{L_t^{(S)}}^\varepsilon(x) \right| \leq \varepsilon \quad \text{for all } x \in \mathbb{R} \quad (5.3.5)$$

where we in (5.3.5) also used that $0 \leq \Psi_k(x, t, \mu, \sigma, u, \eta) \leq 1$ for all k . Hence, for small ε , then (5.3.5) implies that $F_{L_t^{(S)}}^\varepsilon(x)$ will be a very sharp approximation to the loss distribution $F_{L_t^{(S)}}(x)$ in (5.3.1). Since it will often hold that $m_t(\varepsilon) \ll m$, then computing $F_{L_t^{(S)}}^\varepsilon(x)$ will be much faster than computing the exact distribution $F_{L_t^{(S)}}(x)$ while simultaneously have an accuracy of $F_{L_t^{(S)}}^\varepsilon(x)$ compared to $F_{L_t^{(S)}}(x)$ that is smaller than ε given the relation (5.3.5). Table 3 displays $m_t(\varepsilon)$ for $t = 1, 3, 6, 12, 18, 24$ months where $m = 125$ and $\varepsilon = 10^{-9}$ when the individual default times have CIR-intensities as in Table 1.

Table 3. The upper truncation level $m_t(\varepsilon)$ defined as in (5.3.3) for $t = 1, 3, 6, 12, 18, 24$ months where $m = 125$ and $\varepsilon = 10^{-9}$, when the individual default times have CIR-intensities as in Table 1.

t (in months)	1	3	6	12	18	24
$m_t(\varepsilon)$	53	54	54	54	54	55

Hence, from Table 3 we see that in order to have an accuracy of order $\varepsilon = 10^{-9}$ in our approximation $F_{L_t^{(S)}}^\varepsilon(x)$ to the exact distribution $F_{L_t^{(S)}}(x)$ at the time points $t = 1, 3, 6, 12, 18, 24$, we never need to have more than 56 terms in the sum of $F_{L_t^{(S)}}^\varepsilon(x)$ compared with 126 terms in $F_{L_t^{(S)}}(x)$ (recall that we start counting from 0, so e.g. $m_t(\varepsilon) = 55$ means 56 terms in the sum for $F_{L_t^{(S)}}^\varepsilon(x)$ etc.). Also note that for e.g. 99.9%-VaR computations, we will in our numerical solution of Equation (2.72) work with x^* -values so that $F_{L_t^{(S)}}(x^*) = 0.999$. Since we choose $\varepsilon = 10^{-9}$ and since both $F_{L_t^{(S)}}(x)$ and $F_{L_t^{(S)}}^\varepsilon(x)$ are continuous mappings in x , and the error-bound in (5.3.5) holds uniformly for all $x \in \mathbb{R}$, then the solution x_ε^* of the equation $F_{L_t^{(S)}}^\varepsilon(x_\varepsilon^*) = 0.999$ should therefore be extremely close to the exact VaR-solution x^* satisfying $F_{L_t^{(S)}}(x^*) = 0.999$. More specific, from (5.3.5) we have

$$10^{-9} \geq \left| F_{L_t^{(S)}}(x_\varepsilon^*) - F_{L_t^{(S)}}^\varepsilon(x_\varepsilon^*) \right| = \left| F_{L_t^{(S)}}(x_\varepsilon^*) - 0.999 \right|$$

so that the solution x_ε^* of the equation $F_{L_t^{(S)}}^\varepsilon(x_\varepsilon^*) = 0.999$ will give a value of $F_{L_t^{(S)}}(x_\varepsilon^*)$ that deviates at most 10^{-9} from $\alpha = 0.999 = 99.9\%$ which is very accurate. Hence, we can therefore approximate the

exact 99.9%-VaR value x^* with x_ε^* obtained from solving $F_{L_t^{(S)}}^\varepsilon(x_\varepsilon^*) = 0.999$ where the function $F_{L_t^{(S)}}^\varepsilon(x)$ is defined as in (5.3.4). Similar arguments obviously also hold for the 99%-VaR and 95%-VaR computations.

Furthermore, note that e.g 56 terms (i.e. $m_t(\varepsilon) + 1$) versus 126 terms when $m = 125$ (i.e $m + 1 = 126$) will mean a running time of VaR-computations with $F_{L_t^{(S)}}^\varepsilon(x)$ which is more than twice as fast compared with VaR-computations for the exact distribution $F_{L_t^{(S)}}(x)$.

Finally we again remark that the same type of truncation techniques done in this subsection will also hold for the loss distributions derived in Section 3 and Section 4, and will be applied in all of the computations done Section 6.

6. NUMERICAL EXAMPLES WHEN THE DEFAULT TIMES ARE DRIVEN BY A ONE-FACTOR GAUSSIAN COPULA MODEL

In the previous section we studied the time-evolution of Value-at-Risk for a single stock over a two-year period in time steps of one month where the stock has jumps at default times driven by a CIR-process.

In this section we will amongst others study the time-evolution of Value-at-Risk for a portfolio of stocks over a 20 day period in time steps of one trading day, with jumps in all stock prices occurring at default times of an external group of defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$. Throughout this section we assume that the default times $\tau_1, \tau_2, \dots, \tau_m$ to the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ are exchangeable, conditional independent and are driven by a one-factor Gaussian copula model. First, in Subsection 6.1 we briefly discuss the model for the default times and present the parameters used in this framework. We then also display related quantities such as e.g. the distribution of the number of defaults $\mathbb{P}[N_t^{(m)} = k]$ etc. Next, in Subsection 6.2 we study VaR for a portfolio consisting of $J = 70$ stocks by using the linear approximation formulas in Theorem 3.8. In Subsection 6.3 we consider a large portfolio with $J = 150$ stocks and then use the LPA (large portfolio approximation) formulas in Theorem 4.2 to compute VaR for this equity portfolio. Finally, in Subsection 6.4 we repeat similar studies as in Subsection 6.2 but now for a two-year period in steps of one month.

6.1. The parameters and related quantities. In this section we assume that the default times $\tau_1, \tau_2, \dots, \tau_m$ to the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ are exchangeable, conditional independent and are driven by a one-factor copula model as discussed in Subsection 3.2.1. Hence, the conditional default probability is same for all entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ and given by

$$\mathbb{P}[\tau_i \leq t | Z] = \Phi\left(\frac{\Phi^{-1}(F(t)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \quad (6.1.1)$$

where Z is standard normal random variable, ρ is the so-called default-correlation parameter, $\Phi(x)$ is distribution function to a standard normal random variable. Furthermore, $F(t) = \mathbb{P}[\tau_i \leq t]$ is the marginal default distribution same for all entities due to the exchangeability. For more about factor copula models, see e.g. in Nelsen (1999), McNeil, Frey & Embrechts (2005), Schönbucher (2003) or Hofert & Scherer (2011).

The jumps $\tilde{V}_1, \dots, \tilde{V}_m$ in S_t at the defaults $\tau_1, \tau_2, \dots, \tau_m$ are distributed as in the numerical studies in Section 5, that is $\tilde{V}_i = e^{-\tilde{U}_i} - 1$ where $\tilde{U}_1, \dots, \tilde{U}_m$ are i.i.d and exponentially distributed with parameter $\eta > 0$. Hence, given the above assumptions, the dynamics of the stock price $S_{t,j}$ is same as in Corollary 2.11 and Theorem 2.12 where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and $\tau_1, \tau_2, \dots, \tau_m$ are exchangeable, conditionally independent, and come from a one-factor Gaussian model as in (6.1.1). In our numerical examples we set $F(t) = \mathbb{P}[\tau_i \leq t] = 1 - e^{-\lambda t}$ and calibrate λ so that the one-year default probability is same as in the CIR-model in Section 5, that is $0.0329 = 3.29\%$ and this gives $\lambda = 0.0335$, see in Table 4. Furthermore, we set the "default-correlation" ρ to $\rho = 0.3$ and let the number of defaultable entities be $m = 125$, see in Table 4.

Table 4. The parameters and related quantities for the one-factor Gaussian copula model and the stock prices $S_{t,j}$ where we let $m = 125$.

Gauss copula	$m = 125$	$\rho = 0.3$	$F(t) = 1 - e^{-\lambda t}$	$\lambda = 0.0335$	$\mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$
$S_{t,j}$	$S_0 = 50$	$\mu = 0.15$	$\sigma = 0.2$	$\rho_S = 0.25$	$\eta = 21.98$
					$\mathbb{E}[U_{n,j}] = \frac{1}{\eta} = 0.0455 = 4.55\%$

In Table 5 we show the expected number of defaults $\mathbb{E} \left[N_t^{(m)} \right]$ and $\text{VaR}_{99.9\%} \left(N_t^{(m)} \right)$ for $t = 1, 5, 10, 15, 20$ days when the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and where $m = 125$. From Table 5 we see that the expected number of defaults the first 20 days will never exceed one default, and consequently the expected number of jumps in the stock prices the first 20 days will also be less than one. By our assumption of exchangeability we have that $\mathbb{E} \left[N_t^{(m)} \right] = m\mathbb{P} [\tau_i \leq t] = m(1 - e^{-\lambda t})$ so the individual default probabilities at $t = 1, 5, 10, 15, 20$ days are obtained from Table 5 by dividing the numbers for $\mathbb{E} \left[N_t^{(m)} \right]$ with m . Also note from Table 5 we e.g. see that after 10 days, there is a 0.1% probability of having 8 defaults or more among the entities in the exogenous group which are negative affecting the stock prices in our equity portfolio, and after 15 days there is a 0.1% probability of 11 defaults or more among the entities in the same exogenous group.

Table 5. The expected number of defaults $\mathbb{E} \left[N_t^{(m)} \right]$ and $\text{VaR}_{99.9\%} \left(N_t^{(m)} \right)$ for $t = 1, 5, 10, 15, 20$ days when individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and where $m = 125$.

t (in days)	1	5	10	15	20
$\mathbb{E} \left[N_t^{(m)} \right]$	0.0166	0.0829	0.1658	0.2487	0.3314
$\text{VaR}_{99.9\%} \left(N_t^{(m)} \right)$	2	5	8	11	13

Next we turn to the parameters for the stock price model. First, note that the linearized loss distribution given in Theorem 3.8 will work for heterogeneous portfolios of arbitrary size J . However, for simplicity we will consider the homogeneous case, that is the stock prices $S_{t,1}, \dots, S_{t,J}$ satisfy (3.34) in Remark 3.9, so that $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$ and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \dots, \mathbf{A}_J$ in the stock portfolio. Furthermore, we let the parameters μ and σ be same as in the CIR-model case studied in Section 5 so that $S_0 = 50, \mu = 0.15 = 15\%$, and $\sigma = 0.2 = 20\%$ and we let the stock correlation parameter ρ_S be $\rho_S = 0.25$, see in Table 4. The jump parameter η is calibrated so that condition (5.1.4) will hold, that is η is calibrated so that the defaults from the one-factor copula models, "wipes" out the expected one-year log-growth for a corresponding Black-Scholes model with drift $\mu = 15\%$ and where $m = 125$. With the default and stock parameters as in Table 4, we then get that $\eta = 21.98$ via a numerical solver so $\mathbb{E} [U_{n,j}] = \frac{1}{\eta} = 0.0455$, see also in Table 4.

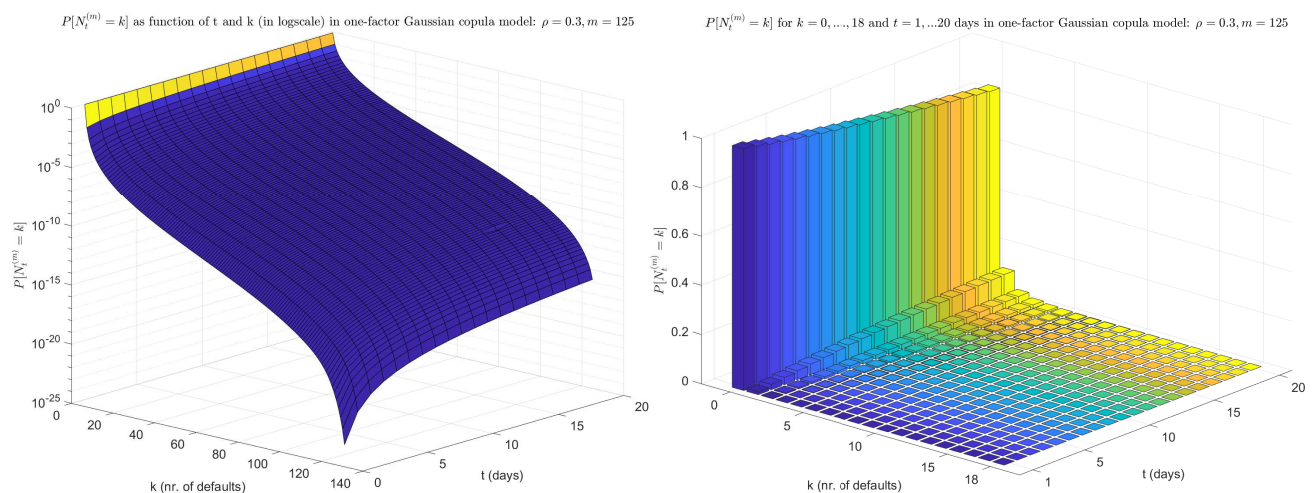


Figure 4. The time evolution of the distribution $\mathbb{P} \left[N_t^{(m)} = k \right]$ for $t = 1, 2, \dots, 20$ days in a one-factor Gaussian copula model with parameters as in Table 4 where $m = 125$ and $\rho = 0.3$. **Left panel:** in log-scale for $k = 0, \dots, 125$. **Right panel:** for $k = 0, \dots, 18$.

With the one-factor Gaussian copula parameters in Table 6.1 we compute $\mathbb{P}\left[N_t^{(m)} = k\right]$ as described above, and the left panel in Figure 4 plots, for $m = 125$, the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale where $k = 0, \dots, 125$ and $t = 1, 2, \dots, 20$ days. Furthermore, the right panel in Figure 4 displays the time evolution of the number of distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in normal scale where $k = 0, 1, \dots, 18$ when $m = 125$ and $t = 1, 2, \dots, 20$ days where the default times have same distribution as in Figure 4. The plots in Figure 4 were generated with the algorithms developed in Herbertsson (2022) and in these figures we write t in days, but the actual computations of $\mathbb{P}\left[N_t^{(m)} = k\right]$ are done with t measured in units of years. So for example two, six and 20 days mean that t is given by $t = \frac{2}{252}$, $\frac{6}{252}$ and $t = \frac{20}{252}$ in the formulas used for the computations of $\mathbb{P}\left[N_t^{(m)} = k\right]$, where we remind that 252 is the average number of trading days on e.g. the US-stock market.

6.2. VaR over a 20-day period for a linearized portfolio of stocks when the jumps are due to default times driven by a one-factor Gaussian copula model. In this subsection we study Value-at-Risk for a portfolio of stocks as function of time over a 20-day period in time steps of one trading day, with jumps in all stock prices occurring at default times $\tau_1, \tau_2, \dots, \tau_m$ which are exchangeable, conditional independent and are driven by a one-factor copula model as discussed in Subsection 6.1 and with parameters as in Table 4. We study VaR for a portfolio of $J = 70$ stocks by using the linear approximation formulas in Theorem 3.8.

In Figure 5-6 we study the time evolution of Value-at-Risk (in % of V_0) for a portfolio of $J = 70$ stocks discussed in Subsection 6.1 where $t = 1, 2, \dots, 20$ days, computed with same stock parameters as in Table 4. For $m = 125$, the left panel in Figure 5 displays the time evolution of Value-at-Risk in % of V_0 for $t = 1, 2, \dots, 20$ days in the case when S_t has jumps coming from default times in a one-factor Gaussian copula model with parameters as in Table 1. The right panel in Figure 5 displays the Black-Scholes case for the stock price, i.e. with no jumps in S_t which has same drift and volatility parameters as in the left panel. From the left panel of Figure 5, looking at the red line (99.9%-VaR), we see that for $t = \frac{12}{252}$, that is after 12 days, then there is a 0.1% probability of having a loss in the stock portfolio which is 42% or bigger, of the initial portfolio value V_0 at time $t = 0$. Furthermore, in Figure 6 we plot the time evolution of the relative difference of Value-at-Risk (in %) between the case with jumps in the stock prices $S_{t,j}$ coming from default times in a one-factor Gaussian copula model with parameters as in Table 1, and the standard Black-Scholes case, i.e. without jumps. The relative difference is measured with respect to the Black-Scholes case. The rest of the parameters for $S_{t,j}$ are same as in Table 4. As can be seen in Figure 6, introducing downward jumps in $S_{t,j}$ at the default times $\tau_1, \tau_2, \dots, \tau_m$ which comes from a one-factor Gaussian copula model, will for example increase the 99.9%-VaR up to around 1450% compared to the Black-Scholes model, and for the 99%-VaR up to 765%.

Furthermore, we also note the curves in the left panel of Figure 5 are not as smooth as in the left panel of Figure 2. The main reason for the somewhat non-smooth curves in Figure 5 is that the computations are done over a very short time period of 20 days, in steps of one trading day leading to a quite degenerated distribution for $\mathbb{P}\left[N_t^{(m)} = k\right]$ over k as seen e.g. the right panel in Figure 4. More specific, for $t = 1, 2, \dots, 20$ days, the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ will have a very high probability for $k = 0$ ("no defaults"), almost equal to one, while $\mathbb{P}\left[N_t^{(m)} = k\right]$ will be very small for $k \geq 1$. These degenerated distributions to $N_t^{(m)}$ follows from the small time points and will for the loss distributions $F_{L_t^{\Delta V}}(x) = \mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ given in (3.18)-(3.20) create more numerical challenges for the solver used in the numerical solutions of the VaR-equation $F_{L_t^{\Delta V}}(x) = \alpha$ compared to a less "degenerated" distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$. If $\mathbb{P}\left[N_t^{(m)} = k\right]$ is computed over a long period such as e.g. two years, then $\mathbb{P}\left[N_t^{(m)} = k\right]$ will have a quite large probabilities also for $k \geq 1$, see e.g. the right panel in Figure 13, and this will lead to very smooth VaR-curves in the Gaussian one-factor case, with same parameters as in Table 4, see the left panel in Figure 12 in Subsection 6.4. All the VaR-computations in the left panel of Figure 5 are done by numerically solving the equation $F_{L_t^{\Delta V}}(x) = \alpha$ where $F_{L_t^{\Delta V}}(x)$ is computed using Theorem 3.8 under condition (3.34) in Remark 3.9 so that the mappings $\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ in $F_{L_t^{\Delta V}}(x)$ are given by (3.35)-(3.36). Furthermore, in our computations of $F_{L_t^{\Delta V}}(x)$ we use the same truncation techniques as discussed in Subsection 5.3. Finally,

the VaR-computations in the right panel of Figure 5 are done by using Equation (3.43) in Corollary 3.11 for the "Black-Scholes" linear portfolio case.

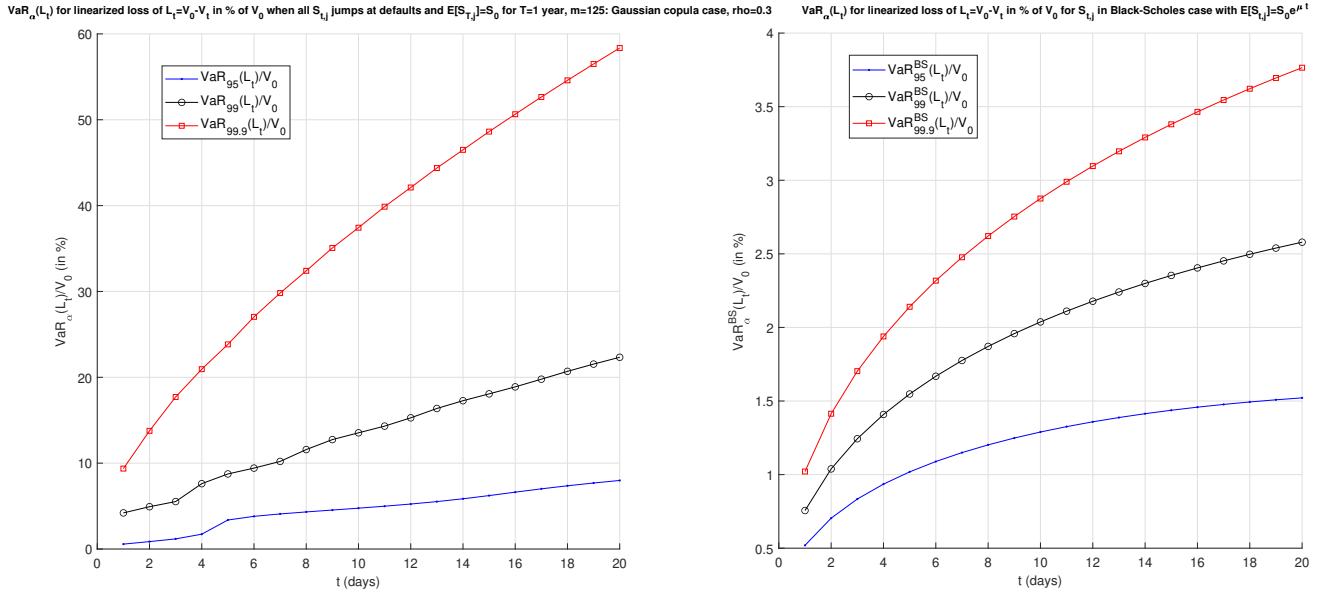


Figure 5. $m = 125$: The time evolution of Value-at-Risk (in % of V_0) of a linearized stock portfolio with $J = 70$ stocks for $t = 1, 2, \dots, 20$ days. **Left panel:** In the case with jumps in the stock price where the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4. **Right panel:** In the Black-Scholes case, i.e. without jumps, and where the drift and volatility are same as in the left panel.

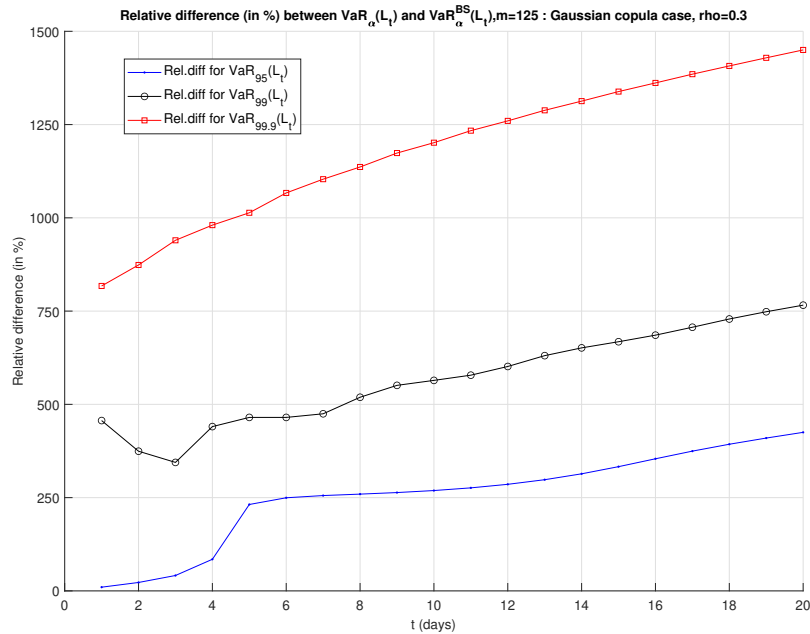


Figure 6. The time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \dots, 20$ days between the case of linearized stock portfolio with $J = 70$ stocks with jumps in the stock price where the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4, and the linearized Black-Scholes case, i.e. without jumps, where drift and volatility is same as in the jump case. The relative difference is measured with respect to the Black-Scholes case.

As discussed in Section 4, the linearized loss $L_t^{\Delta V}$ will only work somewhat accurately as an approximation to the true loss $L_t^{(V)}$ when the time t is small, that is if $|X_{t,j}|$ is small for all j when t is small. Recall that $X_{t,j}$ is defined as in (3.11) and from the expression in (3.11) it is clear that the more potential number of jump terms $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in the expression for $X_{t,j}$, that is the more defaultable entities m , the less likely it will be that $|X_{t,j}|$ is "small". Thus, $|X_{t,j}|$ should in general grow in the number of defaultable entities m . So it is therefore of interest to study $S_{t,j}$ and its linear approximation $S_0(1 + X_{t,j})$ as function of number of defaulted entities m for different time points t . Hence, Figure 7 displays the expected value of $S_{t,j}$ and its linear approximation $S_0(1 + X_{t,j})$, that is $\mathbb{E}[S_{t,j}]$ and $S_0\mathbb{E}[(1 + X_{t,j})]$ as function of number of defaulted entities m , for $t = 5, 10, 20$ and $t = 252$ days where $X_{t,j}$ is defined as in Equation (3.11) with parameters as in Table 4. The jumps in the stock price occur at default times driven by a one-factor Gaussian copula model with parameters as in Table 4. The number of defaultable entities m runs from 5 up to 135 in Figure 7.

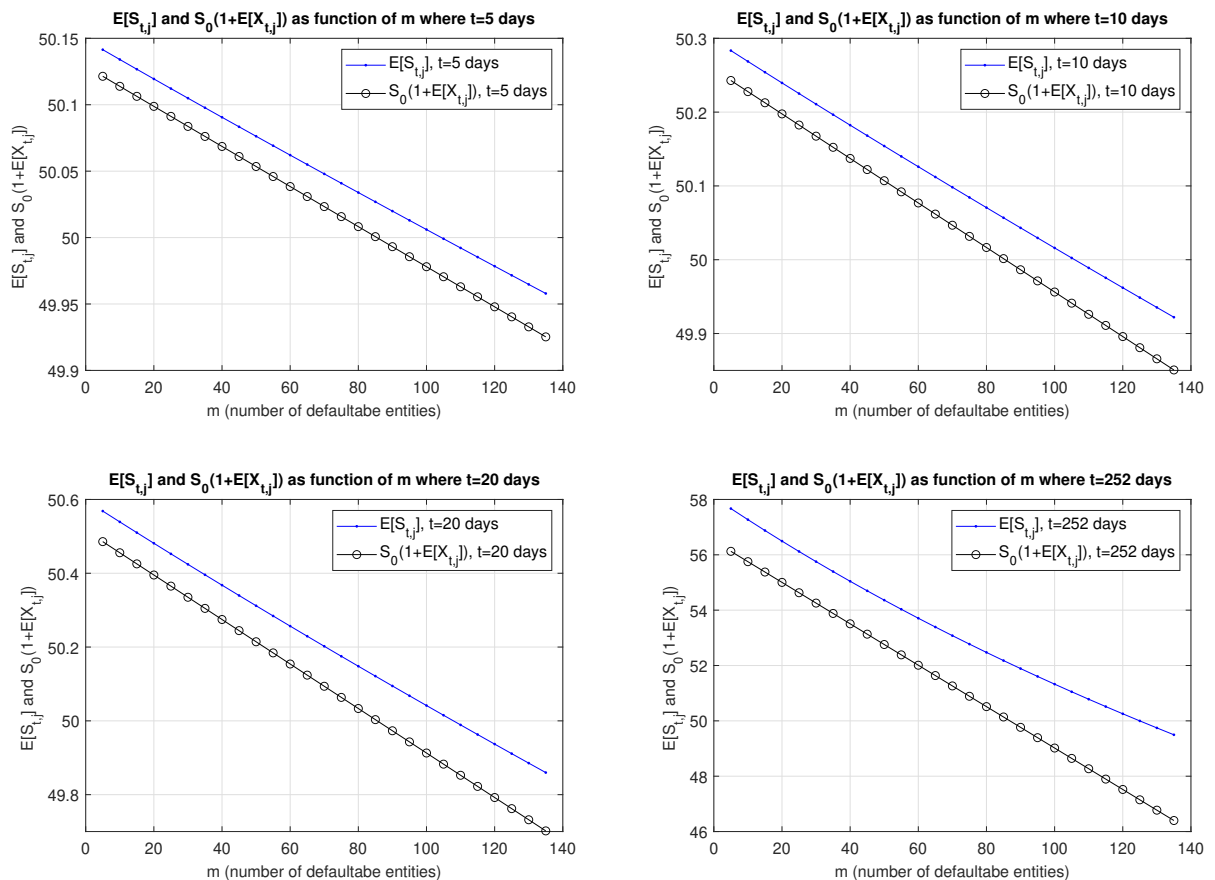


Figure 7. Expected value of $S_{t,j}$ and its linear approximation $S_0(1 + X_{t,j})$, that is $\mathbb{E}[S_{t,j}]$ and $S_0\mathbb{E}[(1 + X_{t,j})]$ as function of number of defaulted entities m , for $t = 5, 10, 20$ and $t = 252$ days where $X_{t,j}$ is defined as in Equation (3.11) with parameters as in Table 4. The jumps in the stock price occur at default times driven by a one-factor Gaussian copula model with parameters as in Table 4.

Furthermore, Figure 8 shows the relative difference between $\mathbb{E}[S_{t,j}]$ and $S_0\mathbb{E}[(1 + X_{t,j})]$ in percent, as function of number of defaulted entities m for $t = 5, 10, 20$ days in the left panel and for $t = 252$ days in the right panel, where $X_{t,j}$ is defined as in Equation (3.11) with model and parameters same as in Figure 7. The relative difference is measured with respect to $\mathbb{E}[S_{t,j}]$. From Figure 8 we e.g. see that the relative error, or difference for $t = 5$ days, never exceeds 0.07% when $m \leq 135$. Also, when $t = 1$ year, that is $t = 252$ days, then the relative error is always smaller than 6%. In all plots in Figure 8 the relative difference is increasing when $m \geq 20$.

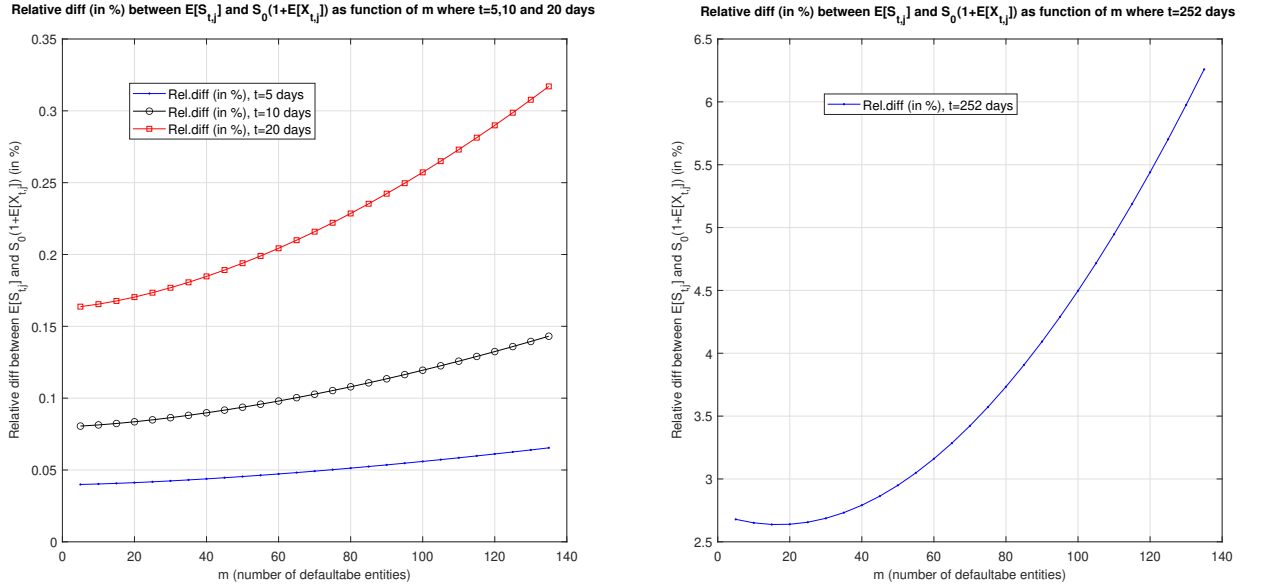


Figure 8. Relative difference between $E[S_{t,j}]$ and $S_0 E[(1+X_{t,j})]$ in percent as function of number of defaulted entities m for different time points t where $X_{t,j}$ is defined as in Equation (3.11) with model and parameters same as in Figure 7. The relative difference is measured with respect $E[S_{t,j}]$. **Left panel:** for $t = 5, 10, 20$ days. **Right panel:** for $t = 252$ days.

6.3. VaR over a 20-day period for a large homogeneous stock portfolio where jumps in stocks are due to default times driven by a one-factor Gaussian copula model. In this subsection we study Value-at-Risk for a large homogeneous portfolio of stocks as function of time over a 20-day period in time steps of one trading day. The stock prices in the portfolio have jumps occurring at default times $\tau_1, \tau_2, \dots, \tau_m$ which are exchangeable, conditional independent and are driven by a one-factor Gaussian copula model as discussed in Subsection 6.1 and with parameters as in Table 4. We study VaR for a portfolio of $J = 150$ stocks by using the LPA approximation formulas in Theorem 4.2 and we do our VaR studies for two different levels of the default correlation parameter ρ in the one-factor Gaussian copula model. First, in the left panel of Figure 9 we display the time evolution of Value-at-Risk in % of V_0 for $t = 1, 2, \dots, 20$ days in the case when $S_{t,j}$ has jumps coming from default times in a one-factor Gaussian copula model with parameters as in Table 1 so the default-correlation ρ is set to $\rho = 0.3$. The right panel in Figure 9 displays the same quantities as in the left panel but now with the default-correlation parameter $\rho = 0.6$ and $\eta = 13.92$ so that condition (5.1.4) holds, just as in the left panel of Figure 9. Comparing the VaR-curves in the left and right panel in Figure 9 we see that e.g. the 99% and 99.9%-VaR plots for $\rho = 0.6$ in the right panel are much higher than the corresponding curves for $\rho = 0.3$ in the left panel where $\eta = 13.92$, with the rest of the parameters same as in the left panel. For example, looking at the red line (99.9%-VaR) in the right panel with $\rho = 0.6$, we see that after 12 days there is a 0.1% probability of having a loss in the portfolio which is 80% or bigger, of the initial portfolio value V_0 at time $t = 0$. However, when $\rho = 0.3$ in the left panel, there is for the same time, that is 12 days, a 0.1% probability of having a loss in the portfolio which is 33% or bigger, of the initial portfolio value V_0 at time $t = 0$. The big differences between the curves for same α -levels in the two panels are due to the fact that a default-correlation of $\rho = 0.6$ will create probabilities $\mathbb{P}[N_t^{(m)} = k]$ that are substantially larger for lower k -values compared to the corresponding probabilities in the case when $\rho = 0.3$. Looking at the left panel in Figure 11 which displays the time evolution of the distribution $\mathbb{P}[N_t^{(m)} = k]$ in log-scale for $k = 0, \dots, 125$ and $t = 1, 2, \dots, 20$ days in a one-factor Gaussian copula model where $\rho = 0.6$ and comparing these probabilities with the corresponding values for $\mathbb{P}[N_t^{(m)} = k]$ in the left panel in Figure 4 where $\rho = 0.3$, we see that the levels of $\mathbb{P}[N_t^{(m)} = k]$ when $\rho = 0.6$ for some k are a factor 10^5 higher compared with the probabilities $\mathbb{P}[N_t^{(m)} = k]$ when $\rho = 0.3$ for the same k -values. The curves in the left panel of Figure 9 are not as smooth as in e.g. the left panel of Figure 2 and the reason for this is the same

as discussed in Subsection 6.2. As will be seen in Subsection 6.4, longer periods such as two years, will lead to very smooth VaR-curves in the Gaussian one-factor case, with same parameters as in Table 4.

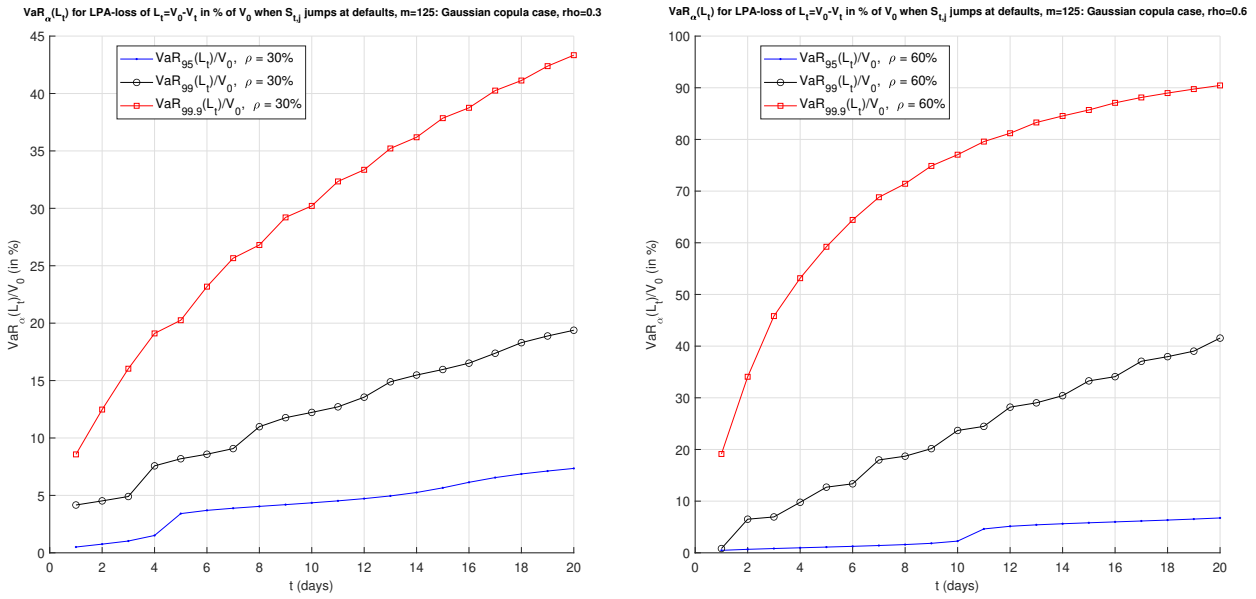


Figure 9. The time evolution of Value-at-Risk (in % of V_0) computed with the LPA-formula in Theorem 4.2 for $t = 1, 2, \dots, 20$ days of a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 4. **Left panel:** Default-correlation parameter $\rho = 30\%$ and $\eta = 21.98$. **Right panel:** Default-correlation parameter $\rho = 60\%$ and $\eta = 13.92$. In both panels condition (5.1.4) holds.

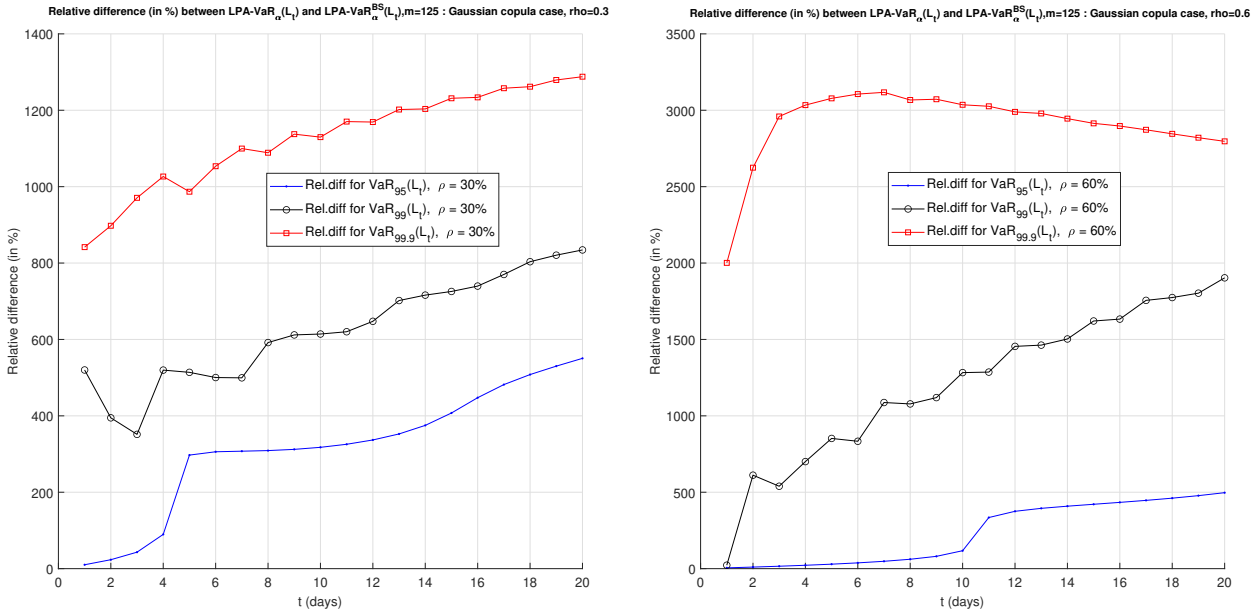


Figure 10. The time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \dots, 20$ days between a stock portfolio with jumps as in Figure 9 using the LPA formula in Theorem 4.2 and the standard Black-Scholes case, i.e. without jumps given by right panel in Figure 11. The relative difference is measured with respect to the Black-Scholes case. All parameters for the jump-model are as in Figure 9. **Left panel:** With default-correlation parameter $\rho = 30\%$ and $\eta = 21.98$. **Right panel:** With default-correlation parameter $\rho = 60\%$ and $\eta = 13.92$.

Next, in the two panels in Figure 10 we display the time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \dots, 20$ days between a stock portfolio with jumps as in Figure 9 using the LPA-formula (4.5) in Theorem 4.2 and the standard Black-Scholes case, i.e. without jumps given by in the right panel of Figure 11 computed with the Black-Scholes LPA-formula in Equation (4.30) with parameters as in Table 4. As can be seen in Figure 10 the differences between the jump vs non-jump VaR-cases are huge. For example, the 99.9%-VaR for $\rho = 0.6$ in the right panel is for some time points around 3000% higher than the corresponding 99.9%-VaR values in the Black-Scholes portfolio case. In our VaR-computations in Figure 9 we use the same truncation techniques for the LPA-portfolio loss distributions as discussed in Subsection 5.3.

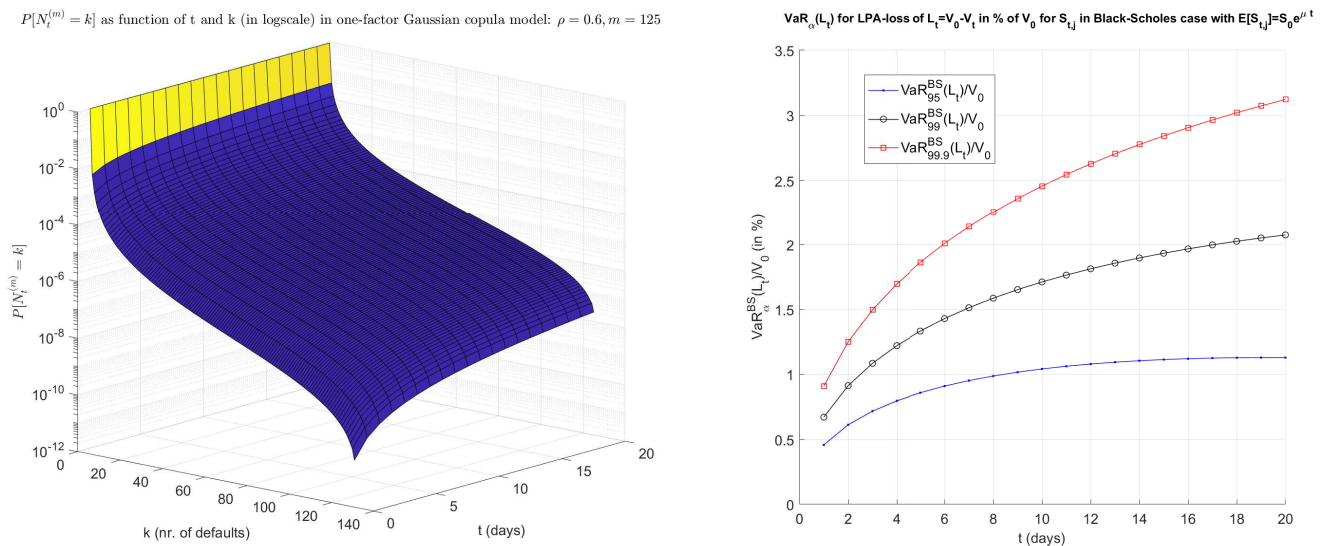


Figure 11. **Left panel:** The time evolution of the distribution $\mathbb{P}[N_t^{(m)} = k]$ in log-scale for $k = 0, \dots, 125$ and $t = 1, 2, \dots, 20$ days in a one-factor Gaussian copula model where $m = 125, \rho = 0.6, \eta = 13.92$ and rest of the parameters are same as in Table 4. **Right panel:** The time evolution of Value-at-Risk (in % of V_0) for $t = 1, 2, \dots, 20$ days of a homogeneous portfolio with $J = 150$ stocks in the Black-Scholes case computed with the LPA-formula in Equation (4.30) and with parameters as in Table 4.

6.4. VaR over a 2-year period for a large homogeneous stock portfolio where jumps in stocks are due to default times driven by a one-factor Gaussian copula model. In this subsection we repeat similar studies and for the same model and same parameters as in Subsection 6.2 but now for a two-year period in steps of one month. The obtained VaR-curves in this subsection will be smooth and continuous, just as in the CIR-case where we also studies VaR over a two-year period. Hence, Figure 12 shows the same type of VaR-curves as in Figure 5, but for a two-year period, and all parameters in Figure 12 are same as in Figure 5, and given by Table 4. By comparing the curves in the left panel of Figure 12 with the graphs in left panel in Figure 5, we clearly see that the VaR-values over a two-year period are very smooth and continuous. Not surprisingly, the VaR-values for the two-year period are also much higher than for the 20-day period. For example, looking at the red line (99.9%-VaR) in Figure 12 we see that after 12 months, then there is a 0.1% probability of having a loss in the portfolio which is 90% or bigger, of the initial portfolio value V_0 at time $t = 0$.

In Figure 13 we display the time evolution of the distribution $\mathbb{P}[N_t^{(m)} = k]$ for $t = 1, 2, \dots, 24$ months in a one-factor Gaussian copula model with parameters as in Table 4 where $m = 125$ and $\rho = 0.3$. Comparing the probabilities in Figure 13 over a two-year period with those in Figure 4 over a 20-day period, we see that there are huge differences. Furthermore, in the two-year case our probabilities are now much less degenerated, i.e. not centered around $k = 0$ as in the 20-day period, and this fact also explains the much more smooth curves in in the left panel of Figure 12 compared with those in left panel in Figure 5. All computations in Figure 12 are done as in Subsection 6.3 and with the same parameters, and the only difference is that we now consider a two-year period in steps of one month. Furthermore,

just as in previous subsections, we will in our VaR-computations in Figure 12 use the same truncation techniques for the LPA-portfolio loss distributions as discussed in Subsection 5.3.

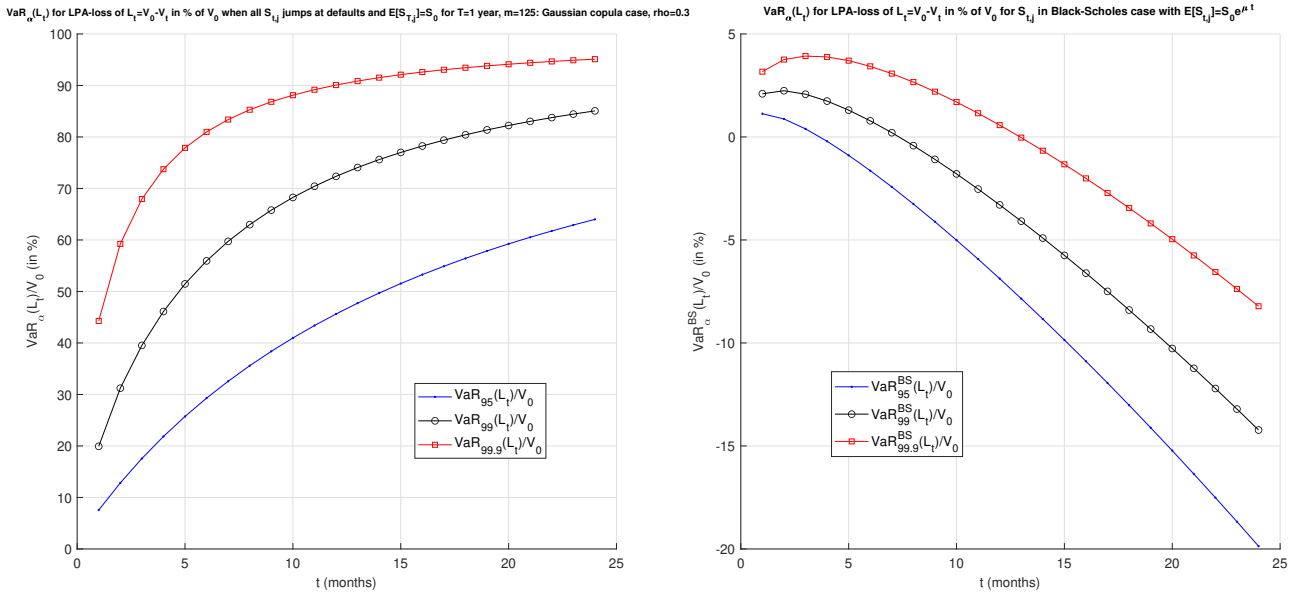


Figure 12. The time evolution of Value-at-Risk (in % of V_0) computed with the LPA-formulas in a homogeneous portfolio with $J = 150$ stocks for $t = 1, 2, \dots, 24$ months. **Left panel:** In the case with jumps in the stock price where the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and using the LPA-formula in Theorem 4.2. **Right panel:** In the Black-Scholes case, i.e. without jumps, using the LPA-formula in Equation (4.30) and with parameters as in Table 4.

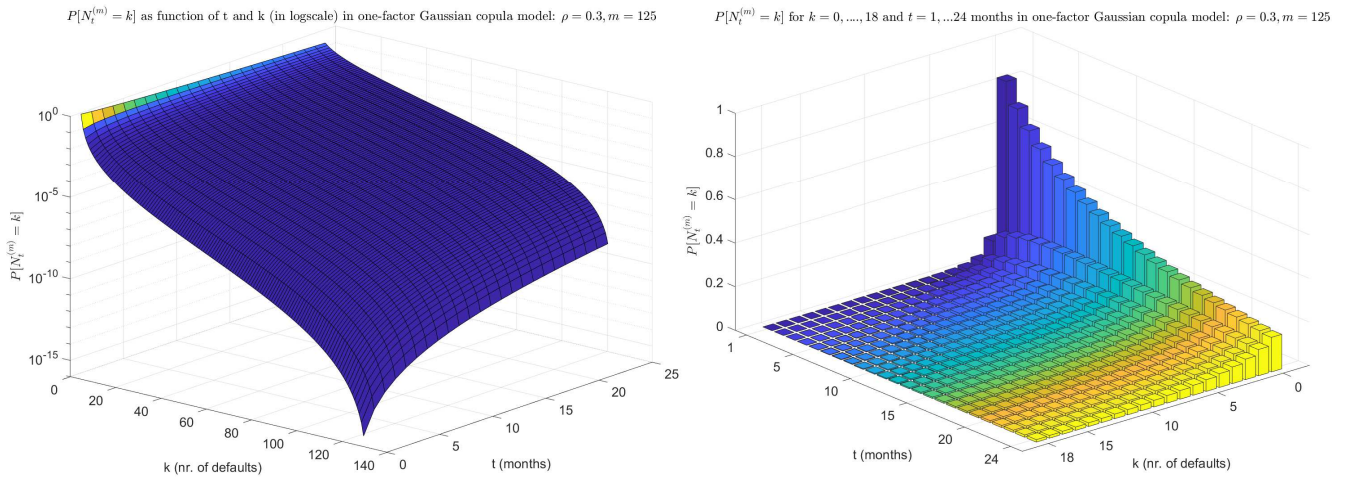


Figure 13. The time evolution of the distribution $\mathbb{P}[N_t^{(m)} = k]$ for $t = 1, 2, \dots, 24$ months in a one-factor Gaussian copula model with parameters as in Table 4 where $m = 125$ and $\rho = 0.3$. **Left panel:** in log-scale for $k = 0, \dots, 125$. **Right panel:** for $k = 0, \dots, 18$. The plots in the panels are viewed from different angles.

Note that the right panel in Figure 12 shows the VaR-values for the Black-Scholes case, i.e. without jumps, using the LPA-formula in Equation (4.30) and with same drift, stock-correlation and volatility parameters as in the left panel, see also in Table 4. From the right panel in Figure 12 we see that in

the Black-Scholes LPA portfolio model it is extremely difficult to obtain losses over a two-year period, where we remind that a negative loss is a gain. For example, we see that after 20 months there is a 95% probability of having a gain which is 15.2% or more of the initial portfolio value V_0 . Similarly, after 20 months there is a 99% probability of having a gain which is 10.3% or more of the initial portfolio value V_0 and 99.9% probability of having a gain which is 4.96% or more of the initial portfolio value V_0 . The intuitive explanation of these VaR-results in the Black-Scholes LPA portfolio setting is that the growth rate will for longer time periods beat the downside risk given by the volatility term, while such positive stock-prognoses are newer possible in the corresponding stock price model with jumps at external defaults over the same time period of 20 months, as clearly seen in the left panel of Figure 12.

Table 6. The expected number of defaults $\mathbb{E} [N_t^{(m)}]$ and $\text{VaR}_{99.9\%} (N_t^{(m)})$ for $t = 1, 6, 12, 18, 24$ months when individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and where $m = 125$.

t (in months)	1	6	12	18	24
$\mathbb{E} [N_t^{(m)}]$	0.3480	2.073	4.113	6.118	8.090
$\text{VaR}_{99.9\%} (N_t^{(m)})$	13	39	55	66	74

In Table 6 we show the expected number of defaults $\mathbb{E} [N_t^{(m)}]$ for $t = 1, 3, 6, 12, 18, 24$ months when the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and where $m = 125$. So from Table 6 we see that our one-factor Gaussian copula model implies that we expect for example around 2 defaults in six months, 4 defaults in one year, and 8 defaults in two-years. Consequently this is also the number of jumps that we expect to occur in our stock price up to each of these time points where each jump has the expected size of $\mathbb{E} [U] = \frac{1}{\eta}$. By our assumption of exchangeability we have that $\mathbb{E} [N_t^{(m)}] = m\mathbb{P}[\tau_i \leq t]$ so the individual default probabilities at $t = 1, 3, 6, 12, 18, 24$ months are obtained from Table 6 by dividing the numbers for $\mathbb{E} [N_t^{(m)}]$ with m . From Table 6 we also see that after e.g. 6 months, there is a 0.1% probability of having 39 defaults or more among the entities in the exogenous group which are negative affecting the stock prices in our equity portfolio, and after 24 months (i.e. 2 years) there is a 0.1% probability of 74 defaults or more among the entities in the exogenous group negative affecting the stock prices in our equity portfolio, when using the parameters in Table 4.

REFERENCES

- Andersen, L. & Sidenius, J. (2004), ‘Extensions to the Gaussian copula: random recovery and random factor loadings’, *Journal of Credit Risk* **1**(1), 29–70.
- Bielecki, T. R., Cousin, A., Crépey, S. & Herbertsson, A. (2014a), A bottom-up dynamic model of portfolio credit risk - part ii: Common-shock interpretation, calibration and hedging issues. in *Recent Advances in Financial Engineering 2012*, eds. Takahashi, A., Muromachi, Y. and Shibata, T. World Scientific.
- Bielecki, T. R., Cousin, A., Crépey, S. & Herbertsson, A. (2014b), ‘Dynamic hedging of portfolio credit risk in a Markov copula model’, *Journal of Optimization Theory and Applications* **161**(1), 90–102.
- Bielecki, T. R., Cousin, A., Crépey, S. & Herbertsson, A. (2014c), ‘A Markov copula model of portfolio credit risk with stochastic intensities and random recoveries’, *Communications in Statistics - Theory and Methods* **43**(7), 1362–1389.
- Burtschell, X., Gregory, J. & Laurent, J.-P. (2009), ‘A comparative analysis of CDO pricing models’, *Journal of derivatives* **16**(4), 9–37.
- Cont, R., Deguest, R. & Kan, Y. H. (2010), ‘Recovering default intensity from cdo spreads: Inversion formula and model calibration’, *SIAM Journal on Financial Mathematics* **1**, 555–585.
- Cont, R. & Kan, Y. H. (2011), ‘Dynamic hedging of portfolio credit derivatives’, *SIAM Journal on Financial Mathematics* **2**, 112–140.
- Cont, R. & Tankov, P. (2004), *Financial Modelling with Jump Processes*, Chapman & Hall, London.
- Crépey, S., Jeanblanc, M. & Wu, D. (2013), ‘Informationally dynamized Gaussian copula’, *International Journal of Theoretical and Applied Finance* **16**(2).
- Durrett, R. (2010), *Probability. Theory and Examples. Fourth Edition*, Cambridge University Press, Cambridge. UK.
- Frey, R. & Backhaus, J. (2008), ‘Pricing and hedging of portfolio credit derivatives with interacting default intensities’, *International Journal of Theoretical and Applied Finance* **11**(6), 611–634.
- Frey, R. & Backhaus, J. (2010), ‘Dynamic hedging of synthetic CDO tranches with spread and contagion risk’, *Journal of Economic Dynamics and Control* **34**(4), 710–724.
- Gregory, J. & Laurent, J.-P. (2003), ‘I will survive’, *RISK* **16**, 103–107.
- Gregory, J. & Laurent, J.-P. (2005), ‘Basket default swaps, CDO’s and factor copulas’, *Journal of Risk* **7**(4), 103–122.

- Guan, L. K., Xiaoping, L. & Chong, T. K. (2003), ‘Asymptotic dynamics and value-at-risk of large diversified portfolios in a jump-diffusion market’, *Quantitative Finance* **4**(2), 129.
- Herbertsson, A. (2005), Dynamic dependence modelling in credit risk. Licentiate thesis. Department of Mathematics. Chalmers University of Technology. Defended 2005-05-11. Opponent: Prof. Dr Rüdiger Frey, Universität Leipzig.
- Herbertsson, A. (2007), Pricing portfolio credit derivatives, PhD thesis, University of Gothenburg.
- Herbertsson, A. (2008a), Default contagion in large homogeneous portfolios. Chapter 14 in “The Credit Derivatives Handbook Global Perspectives, Innovations, and Market Drivers”, Gregoriou, G.N and Ali, P.U, (eds), McGraw-Hill.
- Herbertsson, A. (2008b), ‘Pricing synthetic CDO tranches in a model with Default Contagion using the Matrix-Analytic approach’, *Journal of Credit Risk* **4**(4), 3–35.
- Herbertsson, A. (2011), ‘Modelling default contagion using Multivariate Phase-Type distributions’, *Review of Derivatives Research* **14**(1), 1–36.
- Herbertsson, A. (2022), Saddlepoint approximations for credit portfolios with stochastic recoveries. Working paper. Centre for Finance, University of Gothenburg.
- Herbertsson, A. (2023), Saddlepoint approximations for credit portfolio distributions with applications in equity and credit risk. In preparation.
- Herbertsson, A., Jang, J. & Schmidt, T. (2011), ‘Pricing basket default swaps in a tractable shot-noise model’, *Statistics and Probability Letters* **81**(8), 1196–1207.
- Herbertsson, A. & Rootzén, H. (2008), ‘Pricing k^{th} -to-default swaps under default contagion: the matrix-analytic approach’, *Journal of Computational Finance* **12**(1), 49–78.
- Hofert, M. & Scherer, M. (2011), ‘CDO pricing with nested archimedean copulas’, *Quantitative Finance* **11**(5), 775–787.
- Hofmann, N. & Platen, E. (2000), ‘Approximating large diversified portfolios’, *Mathematical Finance* **10**(1), 77–88.
- Kou, S. (2002), ‘A jump-diffusion model for option pricing’, *Management Science* **48**(8), 1086–1101.
- Laurent, J.-P., Cousin, A. & Fermanian, J.-D. (2011), ‘Hedging default risks of CDOs in Markovian contagion models’, **2**, 1773–1791. *Quantitative Finance*.
- Li, D. X. (2000), ‘On default correlation: A copula function approach’, *The Journal of Fixed Income* **9**, 43–54.
- McNeil, A. J., Frey, R. & Embrechts, P. (2005), *Quantitative Risk Management*, Princeton University Press, Oxford.
- Merton, R. C. (1976), ‘Option pricing when underlying stock returns are discontinuous’, *Journal of Financial Economics* **3**(1-2), 125–144.
- Nelsen, R. B. (1999), *An Introduction to Copulas*, Springer, New York.
- Protter, P. E. (2003), *Stochastic Integration and Differential Equations. Second Edition.*, Springer, New York.
- Schönbucher, P. J. (2003), *Credit Derivatives Pricing Models. Models, Pricing and Implementation*, Wiley, UK.
- Schoutens, W. (2003), *Levy Processes in Finance. Pricing Financial Derivatives*, Wiley, Chichester.
- Williams, D. (2000), *Probability with Martingales.*, Cambridge mathematical textbooks, Cambridge.

(ALEXANDER HERBERTSSON), CENTRE FOR FINANCE, DEPARTMENT OF ECONOMICS, SCHOOL OF BUSINESS, ECONOMICS AND LAW, UNIVERSITY OF GOTHENBURG, P.O BOX 640, SE-405 30 GÖTEBORG, SWEDEN
 Email address: alexander.herbertsson@cff.gu.se