ON THE CVA OF CREDIT DEFAULT SWAPS: THE IMPLICATION OF DEPENDENCE USING A COPULA APPROACH

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Abstract

This study examines the nature and background to the Credit Value Adjustment(CVA), a concept that has gained focus due the it's heightened importance for financial institutions subsequent to the 2008 financial crisis. CVA can be defined as the the price that should be added to the bilateral defaultable contract to adjust for the existing Counterparty Credit Risk (CCR) so that the contract will have the same value as a corresponding risk-free contract. This thesis aims to derive and implement a CVA measure of a Credit Default Swap (CDS) under the presence of Wrong Way Risk (WWR). The Credit Default Swap is roughly an insurance against potential losses suffered from a default of an obligor, typically a company or a sovereign state , often denoted by the reference entity. The CDS contains of the buyer and seller of the CDS, and the reference entity. The buyer of protection has to pay a quarterly payment to the seller of protection, which in turn has to pay a nominal amount to the buyer in the case of default of the reference entity. In this setting, WWR can be defined as the risk of a negative relationship between the reference entity and the sellers credit quality. We are using the semi-analytical expression derived in Herbertsson (2023) to examine CVA under different values parameters correlation and default intensity, which is the default rate for a certain time period conditional on no earlier default. In line with Arismendi-Zambrano et al. (2022), the results show that CVA is increasing with both parameters. An additional two studies are made on the time-series CVA. The first one exhibited an increase in CVA losses during the 08' crisis and the Europen Debt Crisis. Furthermore, we examine the time-series CVA under different values of correlation which confirmed a positive relationship between them.

Keywords*: Credit Value Adjustment, Counterparty Credit Risk, Wrong Way Risk, Credit Default Swap, Semi-Analytical Model, Interest Rate Swap*

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1 Introduction

Despite the prolific focus on the financial crisis of 07'-09', both in academia and the media, the majority of the coverage has been on the credit losses derived from defaults. However, in the context of increased securitization in the financial markets and the subsequent growing markets of derivatives, an increasingly important element in the modeling of credit risk are the losses stemming from the deterioration of credit quality. Naturally, these can cause considerable devaluations in financial institutions' portfolios. In fact, a substantial portion of the losses, and as much as two thirds of the losses endured during the crisis did not stem from actual defaults, but rather from losses derived from the write-downs of outstanding derivatives. This was in turn caused by the increase in probability of default of counterparties.

According to the Basel Committee, Credit Value Adjustment ("CVA") risk, is defined as "the risk of losses arising from changing CVA values in response to changes in counterparty credit spreads and market risk factors that drive prices of derivative transactions" (BCBS 2019b). It can also be described as the difference between the value of a portfolio when the counterparty is default-free and the value of the portfolio having adjusted for the counterparties risk of default. Counterparty Credit Risk ("CCR") is shortly defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows (BCBS 2019a).

A credit default swap is a contract which gives protection against the losses suffered from the default of a certain obligor, sometimes denoted as the reference name. The CDS involves the seller of the swap, the buyer of the CDS and the reference entity. The buyer of the CDS has to pay a quarterly payment to the seller of the swap, which in turn has to pay a nominal amount to the buyer in the case of default of the reference entity before the maturity. This is central to the thesis as we use a semi-analytical model derived by Herbertsson (2023) for CVA on a Credit Default Swap in the presence of Wrong way Risk by performing three different numerical studies.

Wrong Way Risk is another concept which will play a vital role in this thesis going forward. Much like CVA risk, WWR has gained significant attention following the 2008 financial crisis. Wrong Way risk refers to the potential negative correlation between credit exposure and counterparty credit quality. If WWR is not taken into account when valuing derivatives, it can lead to an underestimation of the CVA measure. Several studies, including Brigo and Capponi (2008), Brigo, Morini, and Pallavicini (2013), Arismendi-Zambrano et al. (2022) have demonstrated that incorporating WWR results in an increase in CVA. Černý and Witzany (2014) also show that ignoring WWR in the CVA calculation of an interest rate swap can lead to a nearly three-fold underestimation of the true CVA. Thus, ignoring wrong way risk in the CVA calculation should be avoided. This is supported by our results which are indicating that WWR and default correlation has a substantial impact on CVA values.

Previous literature on the modelling of CVA on CDS contracts are scarce. The characteristics of the derivatives contract inhibits the use of relatively straightforward CVA models such as the ones used on interest rate swaps. However, Brigo and Chourdakis (2009) develops a copula model on a CDS which follows a stochastic default intensity process. The default intensity is the default rate at any point in time assuming that the obligor has survived up to that point. By allowing the default intensity to follow a stochastic process, Brigo and Chourdakis (2009) introduce spread risk into the model in addition to the default risk. This thesis is different in the way that the default intensity is kept constant, and our focus is on the modelling of correlation of default times and its effect on CVA.

Brigo and Chourdakis (2009) correlates the default times between the counterparties, but does so through correlating the associated uniforms of two exponential random variables using a copula function. A copula is a function that connects multiple marginal distributions, which creates a cumulative multivariate distribution function, allowing for the modelling of dependence. There is advantages of doing it in the way of Brigo and Chourdakis (2009), however, one drawback is that the model loses a big part of their economic intuition. Therefore, we model the default times directly through our copula, hence making the interpretation more explicit. Crépey, Jeanblanc, and Wu (2013) also develops a Gaussian Copula model that can be used for dynamic valuation and hedging of counterparty risk on credit derivatives. The Gaussian copula model allows for the assessment of related model risk by comparing the CVA in their Copula model with a common shocks model that accounts for wrong way risk by allowing for the possibility of a common default of the reference entity and of the counterparty. As opposed to Crépey, Jeanblanc, and Wu (2013) we do not benchmark our model but sensitise it for different values of our correlation and default intensity parameter.

Our contribution to the literature is twofold. Firstly, we study the time evolution of CVA inspired by Arismendi-Zambrano et al. (2022), but on a Credit Default Swap rather than on an Interest Rate Swap which is done in Arismendi-Zambrano et al. (2022). The CDS adds on complexity as a third party is introduced into the contract. Secondly, we use the semi-analytical CVA expression for a Gaussian copula derived in Herbertsson (2023) to study the effects of WWR using different levels of correlation. These contributions are achieved by answering the following set of research questions:

- *How will Wrong Way Risk affect the value of a single CVA measure by changing the correlation and default intensity parameters?*

- *How did the time evolution of CVA progress throughout the 08' crisis and the European Debt Crisis?*
- *Would changes in the correlation parameter have had substantial impact on the historic time evolution of CVA?*

The thesis is organized as follows: In Section 2, central concepts that are necessary for the comprehension of the rest of this paper such as credit risk, counterparty risk, wrong way risk and credit default swaps are explained on a relative non-technical basis. Section 3 outlines and explains different ways of modelling credit risk, such as intensity based models. Section 4 pinpoints a general CVA valuation model, applicable to any type of contract. In Subsection 5, the models derived from Herbertsson (2023) are discussed. Subsequently, the numerical results will be presented and analyzed in Section 6 divided into three main sections. Lastly, in Section 7 we will discuss and conclude this thesis.

2 Central Concepts

In this section we aim at presenting and describing some central concepts that are vital for the understanding of the topics that this thesis contains. Firstly, in Subsection 2.1 the concept of credit risk is explained. Moving on, Subsection 2.2 sets out to describe the way in which counterparty risk exists within the context of a Credit Default Swap. In the next section we also discuss the concept of wrong way risk, which was shortly introduced previously. Lastly, Subsection 2.4 describes the nature of a Credit Default Swap. The definitions that we use in this section is mainly taken from the sources of Schmid (2002), Schönbucher (2003) and Brigo, Morini, and Pallavicini (2013).

2.1 Credit risk

Credit risk, although sometimes referred to as default risk, can be categorized into default risk and spread risk. Default risk is intrinsically linked to the obligation of a payment for which the obligor has to honor, i.e.,default risk can be defined as the overall risk of loss that can be derived from the nonpayments from the obligor. In extension, default risk can be divided into several sub-sections: Arrival risk, which is a term for uncertainty whether a default will occur or not; timing risk, reffering to the uncertainty about the precise time of default; recovery risk, describing the uncertainty of the severity of losses if a default has happened; market risk, meaning the risk of changes in the market price of assets that can default; default correlation risk, describing the risk that several obligors default together during some specific time period. In this thesis, an added focus will be on the default correlation risk, as it´s being modelled by calculating the correlation between default times using copulas.

2.2 Counterparty Credit Risk

" Counterparty credit risk (CCR) is the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default...CCR creates a bilateral risk of loss: the market value of the transaction can be positive or negative to either counterparty to the transaction. The market value is uncertain and can vary over time with the movement of underlying market factors."

(BCBS 2019a)

Counterparty credit risk can be thought of as a subset of credit risk that gained focus both from managers and regulators, since many of the losses incurred during the 2007/2008 crisis resulted from events associated with counterparty credit risk. Today, counterparty risk is considered by the majority of market participants as the most essential financial risk. The concept of counterparty credit risk is vital to understand as CVA is the main metric used to estimate marking losses on the market due to exposure to CCR (International Settlements 2011).

Counterparty credit risk is the risk that one or both of the counterparties in a bilateral financial contract will not meet their contractual obligations. There are two aspects which are specific for counterparty credit risk compared to traditional "loan-based" credit risk: (1) The value of a derivatives contract in the future is uncertain, in most cases significantly so. The value of a derivative at a potential default date will be the net value of all future cashflows to be made under that contract which can be positive or negative and is in models often seen as a stochastic process derived form the random underlying cashflows; (2) Since the value of a derivatives contract can be positive or negative, counterparty risk is typically bilateral. In other words, in a derivatives transaction, each counterparty has credit risk to the other. This bilateral nature of counterparty risk has been a particularly important feature of the recent credit crisis.

As the previous discussion indicates, an important distinction to make within the context of CCR is the difference between the modelling of unilateral and bilateral counterparty risk. In the modelling of bilateral risk it is assumed that both of the parties of the contract carry the risk of default. However, as in the case of this thesis, unilateral counterparty risk only assumes that it is the counterparty that can default. Hence, the other party is treated as being risk-free and cannot default. While there is no risk-free counterparty in reality, often, one of the counterparties are generally much safer than its counterparty, making the assumption of unilateral counterparty risk more realistic.

2.3 Wrong Way Risk

According to (Gregory 2010) the price of counterparty risk can generally be derived from taking the probability of default of the counterparty times the expected exposure and loss given default, with the crucial assumption that the terms are independent of each other. If the default time, the exposure and loss given default are not statistically independent, which is the more realistic scenario then the analysis becomes increasingly complex. If there is a dependence between any of the terms, either between expected exposure and probability of default or loss given default, it will create a form of wrong way risk ("WWR"). The dependence means that it´s no longer possible to multiply the expected value of the terms with each other. In the context of the financial contract studied in this paper, i.e., a CDS, there would exist WWR if there existed a strong relationship between the credit quality of the reference entity and the counterparty.

Quite evidently, if we have a US treasury bond as a reference entity in the CDS, with the seller of protection being a US-based bank, the deterioration in credit quality of the treasury bond would most likely be strongly correlated with the credit quality of the US investment bank selling protection on the bond. Note that the correlation can go both ways. A weakening of the economy, and in extension a weakening of its bonds, could indicate a less profitable counterparty. The opposite is also true. The default of a counterparty, especially a large one, could precipitate a weakening of the economy. The WWR-correlation is even more true in the times of the "too big to fail" where some banks are considered cornerstones in the financial markets due to their extensive balance sheets. The direction of the dependence does not necessarily matter, but it has to be understood and modelled properly in order for institutions to not mismanage the risk of their credit portfolios. In practice, if wrong way risk is not modelled correctly, the institution buying protection in the CDS, could end up with a substantially smaller CVA than its true value, which in the case of a default of the seller of CDS-protection will imply higher losses for the buyer of CDS-protection.

2.4 CDS description

In this subsection we explain the concept of Credit Default Swaps ("CDS") in more detail. The notation and framework in this section closely follows that of Herbertsson (2022) and Herbertsson (2023). The CDS is frequently traded and very liquid contract where the underlying asset is typically bonds issued by the so called reference entity. Due to the increased securitization of the financial markets there exists more complex derivatives on portfolios of CDSs such as the CDS index, credit default swap option (CDS swaptions), basket default swaps etc. However, this thesis only deals with the traditional single name credit default swap. The CDS involves three parties which we here denote **A**, **B** and **C**, as seen in Figure 1 where the structure of a CDS is clearly illustrated. Counterparty **A**, also called the protection buyer, enters into the CDS contract with the protection seller **B** in order to buy protection against the potential credit losses on bonds issued by the reference entity **C**, within the coming T years. The seller of the CDS, counterparty **B**, promises to cover credit losses suffered by counterparty **A** at a possible default of **C** up to time T. As a compensation for this hedge, **A** pays **B** a fee of $\frac{R(T)N}{4}$ quarterly until maturity, T, or τ_C, depending on whichever takes place first, i.e. if default occurs before maturity or not.

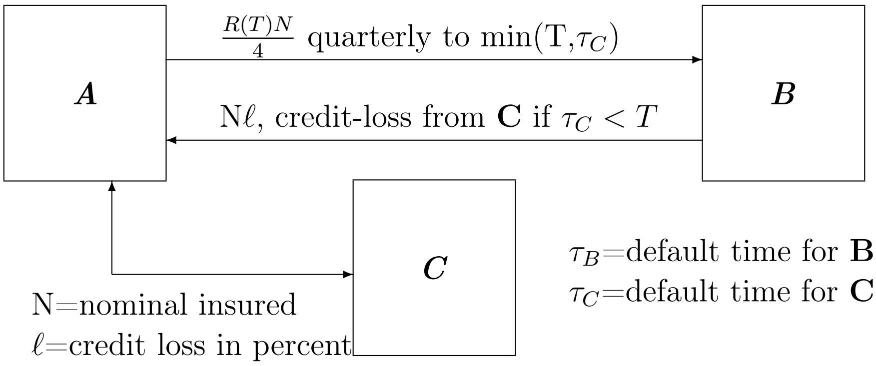

Figure 1: The structure of a Credit Default Swap, where the seller of protection, **B**, can default but the buyer of protection, **A** is risk-free, i.e., can not default.

Furthermore, we denote τ_C to be the default time of the underlying reference entity of the CDS and τ_B the default time of the Counterparty **B**. In this thesis we make the assumption that the protection buyer **A**, also called the investor, is default-free and hence bear no credit risk. The settlement of the CDS spread $R(T)$ is done so that, at inception, the expected present value of the cash flows between **A** and **B** is equal under the risk neutral measure. Hence, $R(T)$ is then expressed as (see also

Herbertsson (2022) and Herbertsson (2023))

$$R(T) = \frac{\mathbb{E}[1_{\{\tau_C \leq T\}} D(\tau)(1-\phi)]}{\sum_{n=1}^{4T} \mathbb{E}[D(t_n)\frac{1}{4}1_{\{\tau > t_n\}} + D(\tau_C)(T - t_{n-1})1_{\{t_{n-1} < \tau_C < t_n\}}]} \tag{2.4.1}$$

where $1_{\{\tau_C \leq T\}}$ is an indicator function which takes on the value of 1 if the default time τ_C takes place before or at the maturity time T and zero otherwise. Furthermore, $D(t)$ is the discount factor given by $D(t) = e^{-\int_0^t r_s ds}$ where r_t is the risk-free rate at time t. The numerator in (2.4.1) is referred to as the default leg, which represents the anticipated cash flow from **B** to **A**. Meanwhile, the denominator in (2.4.1) known as the premium leg, which represents the expected cash flow from **A** to **B** in the form of premiums. The part of the denominator $D(\tau_C)(T - t_{n-1})1_{\{t_{n-1} < \tau < t_n\}}$ in (2.4.1) is often called the accrued premium which is payed to **B** in the case of a default before T. By assuming that the recovery rate is constant and that the short term risk-free rate, r_t, is a deterministic function of time such that $r_t = r(t)$, it is possible to derive $R(T)$ as (see e.g. in Herbertsson (2022), Herbertsson (2023))

$$R(T) = \frac{(1-\phi)\int_0^T D(t) f_{\tau_C}(t)dt}{\sum_{n=1}^{4T}\left(D(t_n)\frac{1}{4}(1 - F(t_n)) + \int_{t_{n-1}}^{t_n} D(s)(s - t_{n-1}) f_{\tau_C}(s)ds\right)} \tag{2.4.2}$$

where $F(t) = \mathbb{P}[\tau_C \leq t_n]$, represents the default probability at time t_n for the reference entity **C**, while $f_{\tau_C}(t)$ is the density of default time τ_C, and $\int_{t_{n-1}}^{t_n} D(s)(s-t_{n-1}) f_{\tau_C}(s)ds$ is the accrued premium. Note that assuming that $r_t = r(t)$ is deterministic, then it follows that the interest rate is not dependent of the default time τ_C (Herbertsson 2022). Under the assumption that: (1) the accrued premium is ignored, (2) If the default is in the period $[\frac{n-1}{4}, \frac{n}{4}]$, the loss is paid at time $t_n = \frac{n}{4}$, which is at the end of the quarter rather than immediately at τ_C, then Herbertsson (2023) obtains a semi-closed formula for $R(T)$ from (2.4.2) given by

$$R(T) = \frac{(1-\phi)\sum_{n=1}^{4T} D(t_n)\big(F(t_n) - F(t_{n-1})\big)}{\sum_{n=1}^{4T}(D(t_n)(1 - F(t_n))\frac{1}{4}}. \tag{2.4.3}$$

Hence, from (2.4.2) and (2.4.3) we clearly see that in order to compute $R(T)$ more explicit, it is necessary to have a more specific model to compute the default time, τ_C, and this will be discussed in the following section.

3 Credit Risk Modelling

From previous sections we concluded that we need more explicit models for default times. This sections adds on to the previous sections about credit risk and it's subcategories by introducing credit risk models that are traditionally used in the credit risk industry. First, in Subsections 3.1., the firm-value model is explained, albeit briefly. Secondly, Subsection 3.2, describes the concept of intensity-based models. After having discussed the models of credit risk, Subsection 3.3 introduces Copulas as a way of modelling dependence. Lastly the specifics of the Gaussian copula is discussed.

According to McNeil, Frey, and Embrechts (2005) there exists two main methodologies to model credit risk. The first one being a firm-value Model, also referred to as a structural model. The other being a so called reduced form model, which contains an intensity based approach. The firm-value model stems from the Merton model but has since then been revised and developed upon but at its core it relies on assumptions regarding the firm´s capital structure and assets. The default event is then determined by a threshold based on the assets value. The reduced-form models relies on exogenous and stochastic models to determine when a default event will occur.

3.1 Structural Models - Firm-Value Modelling

As previously mentioned, structural models are used to calculate the probability of default for a firm using the balance sheet and the value of the assets and liabilities.

The structural model is built on the assumption that a firm defaults when the value of its debt exceeds those of its assets. The model uses the market value of the assets, which is problematic on its own since one does not observe the market value of a firm´s assets on the balance sheet, but rather the accounting value of it. However, this is not the case for publicly traded companies for which both the market value of the equity and liabilities are observable (Merton 1974).

The Merton model is set up in a similar way as the Black-Scholes model. The assets in a firm has a value V_t and can be computed under risk neutral measure as

$$V_t = V_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

where σ is the the annual volatility for the asset, r is the risk-free interest rate and W_t is a random variable that follows a standard Brownian motion (Lando 2004).

In the Merton model, it is assumed that a company has limited liability and has issued two types of claims: debt and equity where the debt is a zero-coupon bond with face value D and maturity T. We let the value of the bond at time t be denoted as B_t and let S_t denote the value of the equity where $0 \leq t \leq T$. Hence, the companies corporate structure is

$$\text{Assets} = \text{equity} + \text{debt} \quad \text{or} \quad V_t = S_t + B_t$$

If the companies bond reaches maturity and the assets is larger than its debt, i.e. $V_T > D$, the owner of the company pays the holders of the bond. If the assets are smaller than the debt at maturity T, i.e. $V_t < D$, the entity that owns the companies equity will declare bankruptcy and hand over to the debt holders who recovers the remaining assets instead of D. Note that in Merton Model, a default of the company can only happen at maturity T when the company needs to pay back its debt (Lando 2004).

There is however extensions to the Merton Model, such as the Black-Cox, (see Black and Cox (1976) and Chapter 2 in Lando (2004)), where the company can default at any time t when the assets of the company falls below its debt i.e. $V_t < D(t)$. The default time for a extended Merton model can than be defined as:

$$\tau = \inf\{t > 0 : V_t \leq D(t)\} \tag{3.1.1}$$

where Equation (3.1.1) implies that τ is equal to the first time t when $V_t < D(t)$. The Merton Model and its extension is one way to model default time. However, in this thesis, we model default times using a so called intensity-based model.

3.2 Intensity-Based Modelling

There are several types of reduced-form models such as the intensity-based approach or credit migrations models. The focus of this section will be on the intensity-based approach and is built on the outlines presented in e.g. Herbertsson (2022), McNeil, Frey, and Embrechts (2005) and Lando (2004). The expectations are computed under a risk neutral probability measure. As will be seen from the subsequent sections of this thesis, an absolute vital role in evaluating the respective conditional expectations is had by the *default intensity*, which is also the starting point of the intensity based approach.

In the intensity based model the probability of default is considered an unexpected event and is modelled using a stochastic process. The modeling is based on a default intensity λ_t which follows a stochastic process where for all t it holds that $\lambda_t \geq 0$. If we consider a single obligor with default time τ than the default intensity is the arrival intensity of τ given market filtration $\mathcal{F}_t$. Intuitively the market filtration $\mathcal{F}_t$ is the flow of information in the market at time t which generates a default time τ within a certain time period. So, conditional on the market information in $\mathcal{F}_t$ and time period $[t, t+\Delta t)$, where Δt is "small", the default time τ will arrive within the time period with approximately the probability $\lambda_t \Delta t$ (Herbertsson 2022). The next step is to give a rigorous mathematical definition for τ that satisfies the above structure. We let $(X_t)_{t \leq 0}$ be built on a d-dimensional stochastic process meaning that $X_t = (X_{t,1}, X_{t,2}, ..., X_{t,d})$ where $\mathcal{G}_t^X$ is the filtration which is generated by X_t i.e. $\mathcal{G}_t^X = \sigma(X_s; s \leq t)$, where d is a integer (Lando 2004).

If we let λ be a positive function $\lambda : \mathbb{R}^d \to [0, \infty)$ we define λ_t as a function of X_t meaning that $\lambda_t = \lambda(X_t)$. Lastly we consider a random threshold E_1 to be a random variable with a exponential distribution with parameter 1 which means that $\mathbb{E}[E_1] = 1$. We let E_1 be independent of $(X_t)_{t \leq 0}$, so the default time can then be defined as

$$\tau = \inf\left\{ t \geq 0 : \int_0^t \lambda(X_s)\, ds \geq E_1 \right\} \tag{3.2.1}$$

where τ will be equal to the first time the increasing default intensity $\lambda(X_s)$ reaches the random variable E_1 (Lando 2004). This is the definition for the deafult time τ using the intensity based model which is quite different compared with the definition of the default time in the firm-value model in (3.1.1). The construction (3.2.1) which generates random variable τ is visualized in Figure 2.

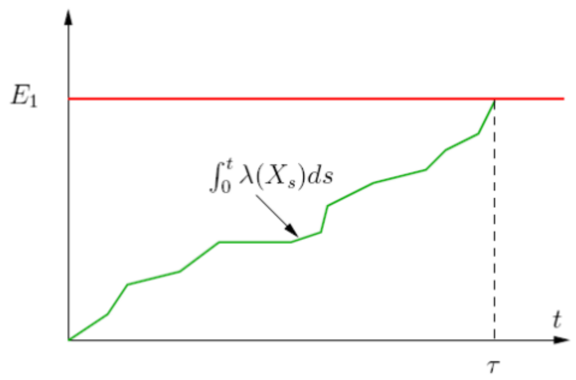

Figure 2: The construction of τ with default threshold E_1
(Herbertsson 2022, p.47)

The definition of τ gives us the tools to construct the survival probability which is essential when estimating CVA. Given (3.2.1) and by using rules for conditional expectations we have for $u \geq t$ that

$$\begin{aligned}\mathbb{P}\big[\tau > t\big] &= \mathbb{E}\big[1_{\tau > t}\big] = \mathbb{E}\big[\mathbb{P}\big[\tau > t \mid \mathcal{G}_u^X\big]\big] \\ &= \mathbb{E}\left[\mathbb{P}\left[\int_0^t \lambda(X_s)ds < E_1 \mid \mathcal{G}_u^X\right]\right] \\ &= \mathbb{E}\left[1 - \mathbb{P}\left[\int_0^t \lambda(X_s)ds < E_1 \mid \mathcal{G}_u^X\right]\right] \\ &= \mathbb{E}\left[e^{-\int_0^t \lambda(X_s)}\right]\end{aligned}$$

where the filtration $\mathcal{G}_u^X$ is independent of E_1 so

$$\mathbb{P}[\tau > t] = \mathbb{E}\big[e^{-\int_0^t \lambda(X_s)ds}\big]. \tag{3.2.2}$$

In this thesis the default intensity is assumed to be constant, meaning that the integral in the exponent in (3.2.2) is equal to $\int_0^t \lambda(X_s)ds = \lambda t$ resulting in the following expression for the default distribution $F(t)$ to τ,

$$F(t) = \mathbb{P}[\tau \leq t] = 1 - \mathbb{P}[\tau > t] = 1 - e^{-\lambda t} \tag{3.2.3}$$

which will be used to construct the probability of default in the following sections.

3.2.1 The Default Intensity and CDS Spread

As it was mentioned in Section 2.4 on the description of a CDS contract, they are highly liquid instrument. Another important aspect of the CDS instrument and it's liquidity is that it is possible to calibrate the risk-neutral default probability from it. This makes for an easy calibration of the intensity. Using that assumption, Herbertsson (2022) derives and simplifies (2.4.3) to

$$R(T) = 4(1-\phi)(e^{\frac{\lambda}{4}} - 1) \tag{3.2.1.1}$$

Furthermore, as shown in Herbertsson (2022) in the case that λ is small, and then by using a first-order Taylor-approximation, $R(T)$ can be simplified further to

$$R(T) \approx (1-\phi)\lambda. \tag{3.2.1.2}$$

If we assume that the CDS spread of the counterparty is observed on the market, given by $S(T)$ than S is the market CDS-spread. Furthermore, if we want to calibrate our models so that $R(T) = S$ and by making the assumptions stated above. Then (3.2.1.2) implies that we can calibrate the default intensity λ from the market CDS-spread S as follows

$$\lambda \approx \frac{S}{1-\phi} \tag{3.2.1.3}$$

where S is the market CDS spread and ϕ being the estimated recovery rate which in turn implies that the expression in the denominator is the loss given default. This is commonly referred to as the credit triangle which will be used in this thesis to compute the default intensity. It is worth mentioning that there exist a term structure for the market CDS spread where $S(T_i)$ for $i = 1, 4, 5, 7$, (i.e. there exist different CDS spreads for different maturities for the same asset in the market which creates a CDS-term structure). The most common way to adjust the model for the term structure is to use piecewise constant default intensities using e.g. bootstrapping which would calibrate the CDS spreads perfectly. However, under the assumption of constant default intensities, the CDS spread is constant for all maturities meaning that the existence of a term structure is ignored and only one market price for a CDS is considered regardless of the maturity (Herbertsson 2022). In this thesis, the term structure is ignored because of the assumption of constant default intensity which makes it easier to compute the CVA for a CDS contract, however it is not very realistic as the prices for a CDS is different for each maturity.

3.3 Dependency Models using Copulas

The copula approach has emerged as one of the most popular ways to model dependence. In the following section we discuss copula as a concept and subsequently we introduce a Gaussian copula which is used in this thesis. We can use these copulas in combination with the default intensity model to produce a dependent CVA model.

3.3.1 Copula

Copulas provide a means of modeling dependence between multiple random variables. They are functions that connect multiple marginal distributions, each of which is uniform on an interval [0,1], to create a cumulative multivariate distribution function. Sklar's theorem, see Appendix (A.1), demonstrates that any multivariate distribution function can be expressed as a copula, and that the copula representation is exclusive when the marginal distributions are uninterrupted (McNeil, Frey, and Embrechts 2005).

Instead of specifying the multivariate distribution, we can use the copula function to model dependence by indicating both the marginal function and the copula. According to O'Kane (2008), when it comes to credit modeling, $F_i(t_i)$ represents the likelihood of obligor i defaulting prior to time t_i, that is $F_i(t_i) = \mathbb{P}[\tau_i < t_i]$. In this context, the m-dimensional copula function C, which has m uniform marginals, and due to Sklar's theorem, can be characterized as follows:

$$C(F_1(t_1), F_2(t_2), ..., F_m(t_m)) = \mathbb{P}[\tau_1 \leq t_1, \tau_2 \leq t_2, ..., \tau_m \leq t_m].$$

When analyzing portfolio credit risk, the concept of a copula is useful because it allows for the direct calibration of an obligor's default probability curve based on market credit default swap (CDS) data, without having to take into account the dependence structure (O'Kane 2008).

To summarize, Sklar's theorem allows for the separation of the dependence structure of the random variable $\mathbf{X} = (X_1, ..., X_m)$ into two parts where the first part is the marginal distributions F_i. The second part is the joint distribution of $\mathbf{X}$ which is specified by the copula C. The copula is mainly used in applied multivariate analysis because it completely describes the joint distribution of $\mathbf{X} = (X_1, X_2, ..., X_m)$ via the marginal distributions $F_i(x) = \mathbb{P}[X_i \leq x]$ for $i = 1, 2, ..., m$. (Brigo, Morini, and Pallavicini 2013), see also McNeil, Frey, and Embrechts (2005). Therefore, the copula ensure that we do not have to consider the dependency between the marginal distributions and instead create the dependency using a copula.

3.3.2 Gaussian Copula model

Copulas can be classified into different types based on their parameters. One way to broadly categorize them is into one-parameter and two-parameter copulas. The former includes the Archimedean copula family and the well-known Gaussian copula, which is the most commonly used copula in finance (Meissner 2013).

The Gaussian copula model assumes that the data follows a normal distribution, i.e., Gaussian distribution, with a mean of zero and a standard deviation of one. This model can be described for $i = 1, 2, ..., m$ as

$$C^{GC}(F_1(t_1), F_2(t_2), ..., F_m(t_m)) = \Phi^m(\Phi^{-1}(F_1(t)), \Phi^{-1}(F_2(t)), ..., \Phi^{-1}(F_m(t))) \quad (3.3.2.1)$$

where $\Phi^m(\cdot)$ is the m-dimensional cumulative normal distribution (CDF) for a m dimensional normally distributed random variable and its inverse $\Phi^{-1}(\cdot)$ where $\Phi^{-1}(F_i(t))$ is the threshold of default (O'Kane 2008).

In O'Kane (2008) the relation (3.3.2.1) is obtained by first defining a default indicator X_i for obligor i as a function of a common factor Z, which may represent the economic environment, and an idiosyncratic factor Y_i which is unique for each obligor. We have a sequence of m random variables $(Y_1, Y_2, ..., Y_m)$ that all come from the same distribution, which is a standard normal distribution. The variable Z also comes from a standard normal distribution, but is independent of Y variables. Consequently, X_i can be expressed as

$$X_i = \sqrt{\rho} Z + \sqrt{1 - \rho} Y_i \quad (3.3.2.2)$$

where we note that X will be a standard normal variable. The modelling of X can be extended by assuming that the correlation parameter ρ can be correlated with the economic environment Z differently for obligor $i = 1, 2, ..., m$, (see for example Gregory and Laurent (2004)). The correlation parameter in (3.3.2.2) can therefore be replaced with ρ_i and rewritten into

$$X_i = \sqrt{\rho_i} Z + \sqrt{1 - \rho_i} Y_i. \quad (3.3.2.3)$$

The next step is to find a definition for the default time τ_i in the intensity based model relying on the works of O'Kane (2008) and Herbertsson (2022). If we define $\Phi^{-1}\big(F_i(t)\big)$ as the threshold for defaults where $F_i(t)$ is the default distribution for obligor i and combine it with the default indicator from (3.3.2.3), then the default time for each obligor i is defined as

$$\tau_i = \inf\{t > 0 : X_i \leq \Phi^{-1}\big(F_i(t)\big)\}. \tag{3.3.2.4}$$

This is the construction of default time τ_i using the intensity based approach and Gaussian copula which tells us that default time τ_i equals the first time the random level X is reached by the increasing threshold $\Phi^{-1}\big(F_i(t)\big)$. Based on the definition for default intensity in (3.3.2.4) we know that the default time τ_i will not occur if the stochastic process X_i is below or equal to the default threshold $\Phi^{-1}\big(F_i(t)\big)$. This means that the probability of default i.e. the probability that the default time τ_i is smaller or equal to t, can be expressed as

$$\mathbb{P}\big[\tau_i \leq t\big] = \mathbb{P}\big[X_i \leq \Phi^{-1}\big(F_i(t)\big)\big] = \Phi(\Phi^{-1}(F_i(t))) = F_i(t).$$

Given the default indicator in equation (3.3.2.3) and the definition of default time τ_i in (3.3.2.4) we note that

$$\tau_i \leq t \quad \text{if and only if} \quad \sqrt{\rho_i}Z + \sqrt{1-\rho_i}Y_i \leq \Phi^{-1}\big(F_i(t)\big). \tag{3.3.2.5}$$

The default time τ_i is therefore constructed through a dependency by the variable Z, and the size of this dependency, i.e. correlation, is set by the correlation parameter ρ_i which as previously mentioned is set individually for each obligor. Conditional on Z the default times are independent meaning that the probability of default conditional on Z can be written as

$$\mathbb{P}[\tau_1 \leq t, \tau_2 \leq t, ..., \tau_m \leq t \mid Z] = \prod_{i=1}^{m} \mathbb{P}[\tau_i \leq t \mid Z].$$

The next step is to find the conditional default probability $\mathbb{P}[\tau_i \leq t \mid Z]$ and first we note from (3.3.2.5) that

$$\tau_i \leq t \quad \text{if and only if} \quad Y_i \leq \frac{D_i(t) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}$$

whch gives us that

$$\mathbb{P}[\tau_i \leq t|Z] = \mathbb{P}\left[Y_i \leq \frac{\Phi^{-1}\big(F_i(t)\big) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\Big|Z\right].$$

Since Y_i is a normally distributed random variable with a mean of zero and standard deviation of one, i.e. $Y_i \sim N(0,1)$ and independent of Z, the default probability conditional on Z $\mathbb{P} = [\tau_i \leq t \mid Z]$ can therefore be rewritten as

$$\mathbb{P}[\tau_i \leq t|Z] = \Phi\left(\frac{\Phi^{-1}\big(F_i(t)\big) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right) \tag{3.3.2.6}$$

which holds for $\tau_1, ..., \tau_m$ where $m \geq 2$ and can be applied to multiple obligors. In this thesis, two obligors, (i.e. $m = 2$), are being considered, however, recall that counterparty **A** is risk free so only counterparty **B** and **C** has a probability of default so $i = \mathbf{B}, \mathbf{C}$.

4 A General Credit Value Adjustment Formula for Univariate Counterparty Credit Risk

The aim of this section is firstly to describe a CVA model in a more detailed way compared to the non-technical intuitive introduction in Section 1 and 2. Additionally, Subsection 4.1.1 and Subsection 4.1.2 derives an independent CVA model where the correlation between default times are not modelled. It should be noted that both Subsection 4.1 and 4.1.1 are general, or arbitrary, i.e., they can be applicable to any type of derivative or portfolio of derivatives. Subsection 4.1.2 on the other hand is strictly applicable to the valuation of a CDS. It is largely based on the mathematical framework presented by Brigo and Chourdakis (2009) and Herbertsson (2023).

4.1 The Valuation of Univariate Counterparty Risk

In this section we will closely follow the notation and setup presented in Herbertsson (2022), as well in the papers of Brigo and Masetti (2005), Brigo and Capponi (2008), Brigo and Chourdakis (2009) and Brigo, Morini, and Pallavicini (2013). Consider any bilateral derivatives contract where it is assumed that one counterparty in the bilateral contract is risk-free, meaning that the counterparty cannot default. In this case the risk free party is denoted by **A**, the buyer of the derivatives contract, also called the investor. However, the second party, **B**, which is the seller of the derivatives contract, also called the *counterparty* is not risk-free and can subsequently default. In this context it is only part **A** that faces counterparty credit risk. This is called *unilateral counterparty risk.*

The default time of the counterparty **B** is denoted by τ_B , and let ϕ_B be the percent recovery at default for the counterparty. Naturally, loss given default in percent can be calculated as $\ell_B = 1 - \phi_B$. From the investor **A**'s point of view, $\Pi^D(t,T)$ denotes the discounted value from the derivatives contract between the counterparty **B** and the investor **A**. The discounted value is derived from taking the sum of all discounted cash flows payed from the counterparty to the investor in the period t to T. Moreover, we denote the "default free" version of $\Pi^D(t,T)$ as $\Pi(t,T)$, which is the sum of all discounted cash flows when assuming $\tau_B = \infty$, i.e., the counterparty **B** will never default. Moving on, we can define the net present value NPV(t,T) of the risk-free cash flow as:

$$NPV(t,T) = \mathbb{E}\big[\Pi(t,T)|\mathcal{F}_t\big] \tag{4.1.1}$$

where $\mathcal{F}_t$ is the full information available at time t. The expectation is taken under the risk neutral martingale measure, i.e. under the pricing measure. For $s \leq t$ let $D(s,t)$ denote the discount factor at time s for the maturity t. It is now possible to give a detailed definition of $\Pi^D(t,T)$ in the context of the notation used above. Let

$(x)^+$ denote the part of x that is positive, that is $(x)^+ = \max(x, 0)$. Note that in the purpose of notational convenience, $NPV(\tau_B, T)$ is written as $NPV(\tau_B)$. In the view of the above notation and arguments, we then get that

$$\Pi^D(t,T) = 1_{\{\tau_B > T\}}\Pi(t,T) + \\ 1_{\{t \leq \tau_B \leq T\}}\big[\Pi(t,\tau_B) + D(t,\tau_B)(\phi(NPV(\tau_B))^+ - (-NPV(\tau_B))^+)\big]. \tag{4.1.2}$$

Next we give a more detailed motivation of (4.1.2) and explain each term for themselves. The first part of Equation (4.1.2) tells us that if the default time is greater than the time of maturity, that is, the counterparty does not default before time T and $\tau_B > T$ then $\Pi^D(t,T) = \Pi(t,T)$. In the other case, if the default time is less than the time of maturity, i.e. if $t \leq \tau_B \leq T$ there is two scenarios, either $NPV(\tau_B) > 0$ or $NPV(\tau_B) \leq 0$. In the first case, if $NPV(\tau_B) > 0$, the investor have a positive exposure against the counterparty at the time of defualt, meaning that the counterparty is owing $NPV(\tau_B) > 0$ to the investor. Since $NPV(\tau_B) > 0$ we know that $NPV(\tau_B) = (NPV(\tau_B))^+$, which in turn means that $-NPV(\tau_B) < 0$ so that $(-NPV(\tau_B))^+ = 0$. Moreover, as the counterparty defaults the investor will only be able to recover a part of the value of the CDS, which is $\phi NPV(\tau_B)$.

Considering the second case, if $NPV(\tau_B) \leq 0$, it is instead the counterparty that has a positive exposure to the investor at the time of default, meaning that the investor instead owes the counterparty the amount of $-NPV(\tau_B) > 0$ which has to be paid to the counterparty at the time of default. It also follows that $(-NPV(\tau_B))^+ > 0$ and $(NPV(\tau_B))^+ = 0$. As a result the counterparty has to pay an amount of $(-NPV(\tau_B))^+ > 0$ to the investor. This explains the term $-(-NPV(\tau_B))^+ > 0$ in (4.1.2) above. Taking the different terms into account it clearly proves the relation in (4.1.2).

4.1.1 The General Unilateral CVA Formula

Again, following the framework of Herbertsson (2022) and recalling (4.1.1) from previous section, with the notation as above we have that

$$\mathbb{E}\big[\Pi^D(t,T)|\mathcal{F}_t\big] = \mathbb{E}\big[\Pi(t,T)|\mathcal{F}_t\big] - (1-\phi)\mathbb{E}\big[1_{\{t\leq\tau_B\leq T\}}D(t,\tau_B)(NPV(\tau_B))^+|\mathcal{F}_t\big] \tag{4.1.1.1}$$

where the second term on the right hand side of (4.1.1.1)is non-negative:

$$\mathbb{E}\big[1_{\{t\leq\tau_B\leq T\}}D(t,\tau_B)(NPV(\tau_B))^+|\mathcal{F}_t\big] > 0 \tag{4.1.1.2}$$

Furthermore, (4.1.1.1) and (4.1.1.2) implies that

$$\mathbb{E}\big[\Pi^D(t,T)|\mathcal{F}_t] \leq \mathbb{E}[\Pi(t,T)|\mathcal{F}_t\big].$$

The interpretation of last inequality is that the cash flows are smaller in the default-case than the corresponding case where there is no default, which is reasonable of course. Additionally, (4.1.1.1) tells us that the cash flows from the default-case is the value of the cash flows in the case of a no-default subtracting the value of an option of the quantity NPV(τ_B) with a strike of zero. Also, the term $\mathbb{E}\big[1_{\{t\leq\tau_B\leq T\}}D(t,\tau_B)(NPV(\tau_B))^+|\mathcal{F}_t\big]$ from the right hand side of Equation (4.1.1.1), essentially helps the investor decide how much they should add on to the part that is risky (default-case), $\mathbb{E}[\Pi^D(t,T)|\mathcal{F}_t]$ for them to make the value of the CDS to have the same value as a contract that is risk free, i.e., without the default risk of the counterparty. Letting (4.1.1.1) we can compute the quantitites $\mathbb{E}[\Pi^D(0,T)]$ for today, i.e., $t = 0$, as

$$\mathbb{E}[\Pi^D(0,T)] = \mathbb{E}\big[\Pi(0,T)\big] - (1-\phi)\mathbb{E}\big[1_{\{\tau_B\leq T\}}D(0,\tau_B)(NPV(\tau_B))^+\big]$$

which can be re-written as

$$\big(1-\phi\big)\mathbb{E}\big[1_{\{\tau_B\leq T\}}D(0,\tau_B)(NPV(\tau_B))^+\big] = \mathbb{E}\big[\Pi(0,T)\big] - \mathbb{E}\big[\Pi^D(0,T)\big] \tag{4.1.1.3}$$

The left hand side of (4.1.1.3) is the Credit Value Adjustment at time $t = 0$, which we now denote as CVA_0 so that

$$CVA_0 = \big(1-\phi\big)\mathbb{E}\big[1_{\{\tau_B \le T\}} D(0,\tau_B)(NPV(\tau_B))^+\big] \tag{4.1.1.4}$$

There are different ways to compute CVA_0 over a specified time period, in this thesis the bucketing approach is used. It's possible to split the interval [0, T] into a "grid" or buckets $0 = t_0 < t_1 < t_2 < ... < t_J$, giving us the following equation

$$\begin{aligned}
&\mathbb{E}\big[1_{\{\tau_B \le T\}} D(0,\tau_B)(NPV(\tau_B))^+\big] \\
&= \sum_{j=1}^{J} \mathbb{E}\big[1_{\{t_{j-1} \le \tau_B \le t_j\}} D(0,\tau_B)(NPV(\tau_B))^+\big] \\
&= \sum_{j=1}^{J} \mathbb{E}\big[1_{\{t_{j-1} \le \tau_B \le t_j\}} D(0,\tau_B)(\mathbb{E}[\Pi(\tau_B,T)|\mathcal{F}_{\tau_B}])^+\big]
\end{aligned} \tag{4.1.1.5}$$

where $\text{NPV}(\tau_B)$ is shorthand for $\text{NPV}(\tau_B, T)$ for notational convenience, and

$$NPV(t,T) = \mathbb{E}\big[\Pi(t,T)|\mathcal{F}_t\big].$$

Note that for any bilateral derivatives contract, if our bucket grid has small intervals $[t_{j-1}, t_j]$ we can assume that the default time τ_B is replaced with the nearest t_j bigger than τ_B in the grid, i.e. we can use the approximation, which is

$$\begin{aligned}
&\mathbb{E}[1_{\{t_{j-1} < \tau_{\le} t_j\}} D(0,\tau_B)(\mathbb{E}[\Pi(\tau_B,T) \mid \mathcal{F}_{\tau_B}])^+] \\
&\approx \mathbb{E}[1_{\{t_{j-1} < \tau \le t_j\}} D(0,t_j)(\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}])^+]
\end{aligned} \tag{4.1.1.6}$$

Inserting (4.1.1.6) into (4.1.1.5) and using (4.1.1.4) we get that

$$\begin{aligned}
&\mathbb{E}[\Pi^D(0,T)] \approx \mathbb{E}[\Pi(0,T)] \\
&-(1-\phi)\sum_{j=1}^{J} \mathbb{E}\big[1_{\{t_j-1 \le \tau_B \le T\}} D(0,t_j)(\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}])^+\big]
\end{aligned} \tag{4.1.1.7}$$

In general, one has to rely on Monte Carlo Simulations in order to evaluate (4.1.1.7). The exception to this case is if τ_B is independent of the random variable $\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}]$ as well as the interest rate. Since τ_B is independent of the random variable $\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}]$, it is possible to do further simplifications of the above formula since

$$\begin{aligned}&\mathbb{E}\big[1_{\{t_{j-1}<\tau_B\leq t_j\}}D(0,\tau_B)(\mathbb{E}[\Pi(\tau,T) \mid \mathcal{F}_T])^+\big] \\ &\quad = \mathbb{P}[t_{j-1}<\tau_B\leq t_j]\mathbb{E}\big[D(0,t_j)(\mathbb{E}[\Pi(t_j,T)|\mathcal{F}_{t_j}])^+\big].\end{aligned} \tag{4.1.1.8}$$

Moreover, we use the rule that $\mathbb{P}[t_{j-1}<\tau_B\leq t_j] = \mathbb{P}[\tau_B\leq t_j] - \mathbb{P}[\tau_B\leq t_{j-1}]$. Hence, in the case that τ_B is independent of the random variable $\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}]$ and the interest rate, then the implication from (4.1.1.7) and (4.1.1.8) is that

$$\begin{aligned}&\mathbb{E}[\Pi^D(0,T)] \approx \mathbb{E}[\Pi(0,T)] \\ &- (1-\phi)\sum_{j=1}^{J}\mathbb{P}[t_{j-1}\leq\tau_B\leq t_j]\mathbb{E}\Big[D(0,t_j)(\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}])^+\Big]\end{aligned} \tag{4.1.1.9}$$

On the assumption that the default time is independent of the exposure, it is possible to combine (4.1.1.4) and (4.1.1.9) in order to get

$$CVA_0 \approx (1-\phi)\sum_{j=1}^{J}\mathbb{P}[t_{j-1}\leq\tau_B\leq t_j]\mathbb{E}\big[D(0,t_j)(\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}])^+\big] \tag{4.1.1.10}$$

Lastly, assuming that the interest rate is independent random variable $\Pi(s,t)$ it is possible to rewrite (4.1.1.10) as

$$CVA_0 \approx (1-\phi)\sum_{j=1}^{J}\mathbb{P}[t_{j-1}\leq\tau_B\leq t_j]\mathbb{E}[D(0,t_j)]\mathbb{E}\big[(\mathbb{E}[\Pi(t_j,T) \mid \mathcal{F}_{t_j}])^+\big] \tag{4.1.1.11}$$

Let us again note that this chapter can be applicable to any derivatives contract. In the following section we derive the CVA model to take into account *Wrong Way Risk* by modelling the correlation between τ_C and τ_B.

5 A Semi-Analytical Model for CVA of a CDS under WWR

In this section we outline the model derived in Herbertsson (2023) as a way to compute a time-series CVA measure for equation (4.1.1.11) under wrong way risk, i.e., a model that allows for a correlation between τ_B, and τ_C. This is the model that we implement in Section 6. The semi-analytical model below is solely applicable to the derivative of a CDS. Herbertsson (2023) develops a contagion model to study CVA on a CDS, and then also uses a one-factor Gaussian copula model to compute the CVA of a CDS as a benchmark to CVA-values obtained in the contagion model. The CVA expression for the CDS and its derivations for the Gaussian copula model in this thesis is taken from Herbertsson (2023). An important benefit of the model is its semi-analytical nature, which eliminates the need for Monte Carlo simulations that can be both computationally demanding and time-consuming.

We start by looking more closely on $\mathbb{E}[\Pi(t,T)|\mathcal{F}_t]$ in equation (4.1.1.11). In the copula model derived in Section 3.3.2 we have a dependency between the default time τ_C and τ_B which results in the following definition for market filtration,

$$\mathcal{F}_t = \mathcal{H}_t^C \vee \mathcal{H}_t^B \vee \sigma(Z).$$

Recall that Z is a variable that drives the dependence between τ_B and τ_C. $\mathcal{H}_t^C$ is the filtration for counterparty **C** at time t, i.e. $\mathcal{H}_t^C = \sigma(\tau_C \leq s; s \leq t)$. $\mathcal{H}_t^B$ is the filtration for counterparty **B** at time t, i.e. $\mathcal{H}_t^B = \sigma(\tau_B \leq s; s \leq t)$, and $\sigma(Z)$ is the filtration for the economic environment Z. Hence, the dependence between τ_B and τ_C is driven through common factors in the economic environment that affects both of the counterparties. From Herbertsson (2023) we have

$$\mathbb{E}[\Pi(t,T)|\mathcal{F}_t] = \frac{-R(0,T)}{4}\sum_{n=n_t}^{n_T} D(t,t_n)\mathbb{P}[\tau_C > t_n|\mathcal{F}_t] + LGD^{(C)}\mathbb{E}[D(t,\tau_C)1_{\{t<\tau_C<T\}}|\mathcal{F}_t] \tag{5.1}$$

where the first step is to find an expression for $\mathbb{P}[\tau_C > t_n|\mathcal{F}_t]$. For $s > t$ and given market filtration $\mathcal{F}_t$ we have

$$\mathbb{P}[\tau_i > s|\mathcal{F}_t] = \mathbb{E}[1_{\{\tau_i>s\}}|\mathcal{F}_t]$$

and that

$$\mathbb{P}[\tau_i > s|\mathcal{F}_t] = 1_{\{\tau_i>t_n\}}\mathbb{P}[\tau_i > s|\mathcal{F}_t] = 1_{\{\tau_i>t_n\}}\mathbb{E}[1_{\{\tau_i>s\}}|\mathcal{F}_t]$$

By the use of Lemma 1, (see (A.2) Appendix), in combination with filtration $\mathcal{H}_t^j \vee \sigma(Z)$where $j \neq i$ we then get

$$\mathbb{P}[\tau_i > s|\mathcal{F}_t] = 1_{\{\tau_i>t_n\}}\frac{\mathbb{P}[\tau_i > s|\mathcal{H}_t^j \vee \sigma(Z)]}{\mathbb{P}[\tau_i > t|\mathcal{H}_t^j \vee \sigma(Z)]}. \tag{5.2}$$

However, note that conditional on the economic environment Z, default time τ_i and τ_j are independent of each other and as a result, we then get

$$\mathbb{P}[\tau_i > s|\mathcal{H}_t^j \vee \sigma(Z)] = \mathbb{P}[\tau_i > s|\ \sigma(Z)] = \mathbb{P}[\tau_i > s|\ Z] \tag{5.3}$$

and similarly, we have that

$$\mathbb{P}[\tau_i > t|\mathcal{H}_t^j \vee \sigma(Z)] = \mathbb{P}[\tau_i > t|\ \sigma(Z)] = \mathbb{P}[\tau_i > t|\ Z]. \tag{5.4}$$

Inserting (5.3) and (5.4) and insert into (5.2), we have

$$\mathbb{P}[\tau_i > s|\mathcal{F}_t] = 1_{\{\tau_i>t_n\}}\frac{\mathbb{P}[\tau_i > s|Z]}{\mathbb{P}[\tau_i > t|Z]}$$

which could also be rewritten as

$$\mathbb{P}[\tau_i > s|\mathcal{F}_t] = 1_{\{\tau_i>t_n\}}\frac{1-\mathbb{P}[\tau_i \leq s|Z]}{1-\mathbb{P}[\tau_i \leq t|Z]}. \tag{5.5}$$

Next we use (3.3.2.6) in (5.5) which finally renders

$$\mathbb{P}[\tau_i > s|\mathcal{F}_t] = 1_{\{\tau_i > t_n\}} \frac{1 - \Phi\left(\frac{\Phi^{-1}(F_i(s)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right)}{1 - \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right)}.$$

Moving on, we consider $\mathbb{E}[D(t, \tau_C)1_{\{t<\tau_C<T\}}|\mathcal{F}_t]$ in Equation (5.2). From Herbertsson (2023) we then get

$$\mathbb{E}[D(t, \tau_C)1_{\{t<\tau_C<T\}}|\mathcal{F}_t] = \mathbb{E}[D(t, \tau_C)1_{\{t<\tau_C<T\}}|\mathcal{H}_t^C \vee \mathcal{H}_t^B \vee \sigma(Z)]$$

where we replace filtration $\mathcal{F}_t$ with filtration $\mathcal{H}_t^C \vee \sigma(Z)$ resulting in

$$\mathbb{E}[D(t, \tau_C)1_{\{t<\tau_C<T\}}|\mathcal{H}_t^C \vee \sigma(Z)] = \int_0^\infty D(t, s)1_{\{t<s\leq T\}} f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z))ds$$

which can be rewritten as

$$\mathbb{E}[D(t, \tau_C)1_{\{t<\tau_C<T\}}|\mathcal{H}_t^C \vee \sigma(Z)] = \int_t^T D(t, s) f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z))ds \tag{5.6}$$

where $f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z))$ is the conditional density to τ_C for $s \geq t$ given the information $\mathcal{H}_t^C \vee \sigma(Z)$, that is

$$f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z)) = \frac{\partial}{\partial s} P[\tau_C \leq s|\mathcal{H}_t^C \vee \sigma(Z)]. \tag{5.7}$$

However, for $s \geq t$ we have that

$$\begin{aligned}
\mathbb{P}[\tau_C > s|\mathcal{H}_t^C \vee \sigma(Z)] &= 1_{\{\tau_C > t\}} \frac{\mathbb{P}[\tau_C > s|\sigma(Z)]}{\mathbb{P}[\tau_C > t|\sigma(Z)]} \\
&= 1_{\{\tau_C > t\}} \frac{\mathbb{P}[\tau_C > s|Z]}{\mathbb{P}[\tau_C > t|Z]}
\end{aligned}$$

that is,

$$\mathbb{P}[\tau_C > s|\mathcal{H}_t^C \vee \sigma(Z)] = 1_{\{\tau_C > t\}} \frac{\mathbb{P}[\tau_C > s|Z]}{\mathbb{P}[\tau_C > t|Z]}. \tag{5.8}$$

Furthermore, from (3.3.2.6) it is possible to rewrite (5.8) as (see also in Herbertsson (2023))

$$\mathbb{P}[\tau_C > s|\mathcal{H}_t^C \vee \sigma(Z)] = 1_{\{\tau_C > t\}} \frac{1 - \Phi(\frac{\Phi^{-1}(F_C(s)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}})}{1 - \Phi(\frac{\Phi^{-1}(F_C(t)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}})}. \tag{5.9}$$

and since we know that $\mathbb{P}[\tau_C \leq s|\mathcal{H}_t^C \vee \sigma(Z)] = 1 - \mathbb{P}[\tau_C > s|\mathcal{H}_t^C \vee \sigma(Z)]$ we use (5.7) to then get $f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z))$ as

$$\begin{aligned} & f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z)) = \\ & \quad 1_{\{\tau_C > t\}} \frac{1}{1 - \Phi(\frac{\Phi^{-1}(F_C(t)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}})} \cdot \frac{F_C^{'}(s)}{\sqrt{1-\rho_i}} \\ & \quad \cdot \frac{1}{\varphi(\Phi^{-1}(F_C(s))} \cdot \varphi\Big(\frac{\Phi^{-1}(F_C(s)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\Big) \end{aligned} \tag{5.10}$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and using the Inverse function rule we know that $\frac{d}{dx}\Phi^{-1}(x) = \frac{1}{\Phi'(\Phi^{-1}(x))} = \frac{1}{\varphi(\Phi^{-1}(x))}$, which can also be seen in Herbertsson (2023). Equation (5.10) gives us an explicit expression for $f_{\tau_C}(s|\mathcal{H}_t^C \vee \sigma(Z))$ that in turn can be plugged into (5.6) in order to find a expression for $\mathbb{E}[D(t,\tau_C)1_{\{t<\tau_C<T\}}|\mathcal{H}_t^C \vee \sigma(Z)]$. Thus, plugging (5.10) into (5.6) gives us

$$\begin{aligned} \mathbb{E}[D(t,\tau_C)1_{\{t<\tau_C<T\}}|\mathcal{F}_t] = 1_{\{\tau_C > t\}} & \frac{1}{1 - \Phi(\frac{\Phi^{-1}(F_C(t)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}})} \cdot \frac{1}{\sqrt{1-\rho_i}} \\ & \cdot \int_t^T D(t,s) \cdot \frac{F_C^{'}(s)}{\varphi(\Phi^{-1}(F_C(s))} \cdot \varphi\Big(\frac{\Phi^{-1}(F_C(s)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\Big) ds. \end{aligned}$$

Furthermore, if $F_C(s) = 1 - e^{-\lambda_C s}$ then $F_C^{'}(s) = \lambda_C e^{-\lambda_C s}$ and if we use constant interest rate, then $D(t,s) = e^{-r(s-t)}$ where $s \geq t$, which gives a rather explicit expression. Moving on we can plug (5.10) and (5.9) into (5.1) which for $0 \leq t \leq T$ yields

$$\begin{aligned} & \mathbb{E}[\Pi(t,T)|\mathcal{F}_t] = 1_{\{\tau_C > t\}} \frac{1}{1 - \Phi(\frac{\Phi^{-1}(F_C(t)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}})} \\ & \cdot \Big(\frac{-R(0,T)}{4} \sum_{n=n_t}^{n_T} D(t,t_n) \big(1 - \Phi\Big(\frac{\Phi^{-1}(F_C(t_n)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\Big) + (1-\phi) I_C(t,Z) \Big) \end{aligned} \tag{5.11}$$

where $I_C(t, Z)$ is given by

$$I_C(t,Z) = \int_t^T \frac{D(t,s)}{1-\rho_i} \cdot \frac{F_C^{'}(s)}{\varphi(\Phi^{-1}(F_C(s))} \cdot \varphi\left(\frac{\Phi^{-1}(F_C(s)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right) ds. \tag{5.12}$$

This expression for $\mathbb{E}[\Pi(t,T|\mathcal{F}_t]$ is also found in Herbertsson (2023). In view of (5.11) and (5.12) in the case of $0 \leq t \leq T$ it is possible to to rewrite $\mathbb{E}[\Pi(t,T)|\mathcal{F}_t]$ as

$$\mathbb{E}[\Pi(t,T)|\mathcal{F}_t] = 1_{\{\tau_C > t\}} H(t,T,Z) \tag{5.13}$$

where $H(t,T,Z)$ is given by the terms to the right of the indicator function $1_{\{\tau_C > t\}}$ in (5.11), that is

$$H(t,T,Z) = \frac{1}{1-\Phi\left(\frac{\Phi^{-1}(F_C(t)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right)} \tag{5.14}$$
$$\cdot\left(\frac{-R(0,T)}{4} \sum_{n=n_t}^{n_T} D(t,t_n)\left(1-\Phi\left(\frac{\Phi^{-1}(F_C(t_n)) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right)\right) + (1-\phi_C) I_C(t,Z)\right).$$

and where $I_C(t,Z)$ is same as in (5.12). We know from (4.1.1.4) and (4.1.1) that

$$CVA_0 = (1-\phi)\mathbb{E}[1_{\{\tau_B \leq T\}} D(0,\tau_B)(\mathbb{E}[\Pi(\tau_B,T) \mid \mathcal{F}_{\tau_B}])^+]. \tag{5.15}$$

If we let $0 = t_0^{(B)} < t_1^{(B)} ... < t_{J_B}^{(B)} = T$ be a bucketing-grid of the interval $[0,T]$ then we have

$$\mathbb{E}[1_{\{\tau_B \leq T\}} D(0,\tau_B)(\mathbb{E}[\Pi(\tau_B,T) \mid \mathcal{F}_{\tau_B}])^+] = \sum_{j=1}^{J_B} \mathbb{E}[1_{\{t_{j-1}^B < \tau_B \leq t_j^B\}} D(0,\tau_B)(\mathbb{E}[\Pi(\tau_B,T) \mid \mathcal{F}_{\tau_B}])^+] \tag{5.16}$$

Next, from Equation (10.2.11) in Herbertsson (2022) we use the approximation of

$$\mathbb{E}[1_{\{t_{j-1}^B < \tau_B \leq t_j^B\}} D(0,\tau_B)(\mathbb{E}[\Pi(\tau_B,T) \mid \mathcal{F}_{\tau_B}])^+] \tag{5.17}$$
$$\approx \mathbb{E}[1_{\{t_{j-1}^B < \tau_B \leq t_j^B\}} D(0,t_j^{(B)})(\mathbb{E}[\Pi(t_j^{(B)},T) \mid \mathcal{F}_{t_j^{(B)}}])^+]$$

that is, if $t^B_{j-1} < \tau_B \le t^B_j$ we approximate $D(0, \tau_B)$ with $D(0, t^B_j)$ and similarly, $\mathbb{E}\big[\Pi(\tau_B, T) \mid \mathcal{F}_{\tau_B}\big]$ with $\mathbb{E}\big[\Pi(t^B_j, T) \mid \mathcal{F}_{t^B_j}\big]$. Note that the approximation in (5.17) will be better and more accurate the smaller the "bucket-grid" $\left[t^{(B)}_{j-1},\, t^{(B)}_j\right]$ is. Hence, if $s < t$ we will next find a simplification of the quantity of

$$\mathbb{E}[1_{\{s<\tau_B\le t\}} D(0,t)(\mathbb{E}[\Pi(t,T) \mid \mathcal{F}_t])^+] \tag{5.18}$$

in the final CVA-computing step. In the view of (5.13) and the fact that $1_{\{\tau_C>t\}} \ge 0$, it is possible to rewrite (5.18) as the following equation

$$\begin{aligned}
&\mathbb{E}[1_{\{s<\tau_B\le t\}} D(0,t)(\mathbb{E}[\Pi(t,T) \mid \mathcal{F}_t])^+] \\
&\quad = \mathbb{E}[1_{\{s<\tau_B\le t\}} D(0,t) 1_{\{\tau_C>t\}} \big(H(t,T,Z)\big)^+]
\end{aligned} \tag{5.19}$$

where we recall that $H(t, T, Z)$ is given by (5.14). Furthermore, as in Herbertsson (2023), we make the same assumption that the interest is deterministic, so that $D(0, t)$ is a deterministic function of t, then (5.19) can be rewritten as

$$\mathbb{E}[1_{\{s<\tau_B\le t\}} D(0,t) 1_{\{\tau_C>t\}} \big(H(t,T,Z)\big)^+] = D(0,t)\mathbb{E}[1_{\{s<\tau_B\le t\}} 1_{\{\tau_C>t\}} \big(H(t,T,Z)\big)^+].$$

Next, following Herbertsson (2023), we note that

$$\mathbb{E}[1_{\{s<\tau_B\le t\}} 1_{\{\tau_C>t\}} \big(H(t,T,Z)\big)^+] = \mathbb{E}\big[\mathbb{E}[1_{\{s<\tau_B\le t\}} 1_{\{\tau_C>t\}} \big(H(t,T,Z)\big)^+ | Z]\big]$$

which can be rewritten using rules for conditional expectations, see (A.3) in Appendix, into

$$\mathbb{E}\big[\mathbb{E}[1_{\{s<\tau_B\le t\}} 1_{\{\tau_C>t\}} \big(H(t,T,Z)\big)^+ | Z]\big] = \mathbb{E}\big[\big(H(t,T,Z)\big)^+ \mathbb{E}[1_{\{s<\tau_B\le t\}} 1_{\{\tau_C>t\}} \big| Z]\big]$$

and we know that

$$\mathbb{E}[1_{\{s<\tau_B\le t\}} 1_{\{\tau_C>t\}} \big| Z] = \mathbb{E}[1_{\{s<\tau_B\le t\}} | Z]\mathbb{E}[1_{\{\tau_C>t\}} \big| Z]$$

which can be rewritten using the definition for conditional probabilities, see (A.4) in Appendix, as

$$\mathbb{E}[1_{\{s<\tau_B\leq t\}}|Z]\mathbb{E}[1_{\{\tau_C>t\}}|Z]] = \mathbb{P}[s<\tau_B\leq t|Z]\mathbb{P}[\tau_C>t\ |Z].$$

Hence, for $s<t$ we have that

$$\begin{aligned}&\mathbb{E}[1_{\{s<\tau_B\leq t\}}1_{\{\tau_C>t\}}\big(H(t,T,Z)\big)^+] = \\ &\quad \mathbb{E}\big[\mathbb{P}[s<\tau_B\leq t|Z]\cdot\mathbb{P}[\tau_C>t\ |Z]\cdot\big(H(t,T,Z)\big)^+\big].\end{aligned} \tag{5.20}$$

Using (3.3.2.6) we can rewrite $\mathbb{P}[s<\tau_B\leq t|Z]$ in (5.20) as

$$\begin{aligned}&\mathbb{P}[s<\tau_B\leq t|Z] = \\ &\quad \Phi\bigg(\frac{\Phi^{-1}(F_B(t))-\sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\bigg)-\Phi\bigg(\frac{\Phi^{-1}(F_C(s))-\sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\bigg).\end{aligned} \tag{5.21}$$

If we plug (3.3.2.6) and (5.21) into (5.20) we then get

$$\begin{aligned}&\mathbb{E}[1_{\{s<\tau_B\leq t\}}1_{\{\tau_C>t\}}\big(H(t,T,Z)\big)^+] = \\ &= \mathbb{E}\Bigg[\Bigg(\Phi\bigg(\frac{\Phi^{-1}(F_B(t))-\sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\bigg)-\Phi\bigg(\frac{\Phi^{-1}(F_B(s))-\sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\bigg)\Bigg) \\ &\cdot\Bigg(1-\Phi\bigg(\frac{\Phi^{-1}(F_C(t))-\sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\bigg)\Bigg)\cdot\big(H(t,T,Z)\big)^+\Bigg].\end{aligned} \tag{5.22}$$

We note that the the expectation in (5.22) is simply a univariate integral

$$\mathbb{E}\Big[1_{\{s<\tau_B\leq t\}}1_{\{\tau_C>t\}}\big(H(t,T,Z)\big)^+\Big] = \int_{-\infty}^{\infty} G(s,t,T,z)\varphi(z)dz \tag{5.23}$$

where $\varphi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$, and where $G(s,t,T,z)$ is given by

$$\begin{aligned}G(s,t,T,z) =&\Bigg(\Phi\bigg(\frac{\Phi^{-1}(F_B(t))-\sqrt{\rho_i}z}{\sqrt{1-\rho_i}}\bigg)-\Phi\bigg(\frac{\Phi^{-1}(F_B(s))-\sqrt{\rho_i}z}{\sqrt{1-\rho_i}}\bigg)\Bigg) \\ &\cdot\Bigg(1-\Phi\bigg(\frac{\Phi^{-1}(F_C(t))-\sqrt{\rho_i}z}{\sqrt{1-\rho_i}}\bigg)\cdot\big(H(t,T,z)\big)^+\Bigg)\end{aligned} \tag{5.24}$$

and $H(t, T, z)$ is defined as in (5.14), with Z replaced by z,

Next, by using (5.23) for $s = t_{j-1}^{(B)}$ and $t = t_j^{(B)}$ for each "bucket interval", i.e. $t_{j-1}^{(B)}$ to $t_j^{(B)}$ together with the approximation in (5.17) inserted into (5.16) we can finally find an approximation to CVA_0 in Equation (5.15) given by

$$CVA_0 \approx (1 - \phi) \sum_{j=1}^{J_B} D(0, t_j^B) \int_{-\infty}^{\infty} G\Big(t_{j-1}^{(B)}, t_j^{(B)}, T, z\Big)\varphi(z)dz \tag{5.25}$$

where the mapping $G(s, t, T, z)$ is given by (5.24). Equation (5.25) is the formula that will be used to numerically compute CVA_0 in Section 6.

6 Numerical Results

This section sets out to answer the three research questions presented in the introduction by conducting three different numerical studies using the Gaussian CVA model derived in Herbertsson (2023):

- *How will Wrong Way Risk affect the value of a single CVA measure by changing the correlation and default intensity parameters?*

- *How did the time evolution of CVA progress throughout the 08' crisis and the European Debt Crisis?*

- *Would changes in the correlation parameter have substantial impact on the historic time evolution of CVA?*

Herbertsson (2023) develops a contagion model to study CVA on a CDS, and then also uses a one-factor copula Gaussian copula model to compute the CVA ofr a CDS as a benchmark to CVA-values obtained in the contagion model. The CVA expression for the CDS and it's derivation in Section 5 of this thesis for the Gaussian copula model is taken from Herbertsson (2023). Note that in this thesis our focus is on the CVA measure under WWR for a one-factor Gaussian copula model, as it is explained in Subsection 2.3. Naturally, a focal point of this section will relate to the correlation parameter in the Gaussian copula model, measured by ρ, which can only take on positive values between 0 and 1. Another central parameter in the model is the parameter is the CDS spreads for both parties. Note that given the constant default intensity, which is derived from the credit triangle in (3.2.1.3), the only determining factor of the default intensity will be the CDS spread. Given that this thesis deals with the instrument of a CDS, there is one CDS spread and ρ each for **B** and **C**.

As mentioned in previous sections, this the main focus in this thesis is the risk of default, and not the spread risk, which is why we have chosen to keep the default intensity constant. Another choice would have been to let the default intensity follow a stochastic process, or for it to be piece-wise constant. A default intensity that is piece-wise constant means that it is piecewise constant between fixed time points $T_1, T_2, ..., T_j$. The advantage of piecewise constant is that it is possible to perfectly calibrate the intensity against the CDS-term structure. Despite its advantages, it's easier to model the CVA using a constant default intensity and this thesis only focuses on the default risk. It should also be noted that the Gaussian copula model works for an arbitrary choice of the marginal distributions $F_i(t) = \mathbb{P}[\tau_i \leq t]$.

The computations using the CVA formula in Equation (5.25) are extremely time consuming, evaluating (5.25) for one setup $(\rho_B, \rho_C, \lambda_B, \lambda_C, \phi_B, \phi_C)$ takes approximately 80 seconds to compute using matlab on a standard laptop. To put this in perspective, when calculating the time-series of the CVA from 2007 to 2014 using all available trading days on four credit default swaps, 7772 calculations had to be computed separately meaning that is would take approximately 172 hours to compute them in MATLAB. However, the computations were done in MATLAB using the parfor command in order to decrease the time it takes to run each calculation. The parfor command computes each iteration separately using every processor core available. In this thesis the number of processor cores used in the computations of the CVA time series were 24, which was possible by connecting several computers plus a server to our Matlab-computation. By using the parfor command, MATLAB could calculate 24 calculations simultaneously which decreased the time needed considerably for all CVA computations. The described technique of utilizing all cores is commonly referred to as parallel computing, (see MathWorks (2022) for full description).

The first numerical study conducted is a sensitivity analysis and relates to the first research question stated on the previous page where we look at how a single CVA value is changing depending on the value of CDS_{B}, CDS_{C}, ρ_B and ρ_C. Essentially, our first numerical study evaluates the CVA of a CDS as a function of WWR and the change of the $\text{CDS}_{\text{B,C}}$ which is presented in Subsection 6.1

For the second study we present a time evolution of the CVA measure in an attempt to answer the second research question. The focus of the second study will mainly be set on the development of the CVA during the 08' crisis and especially the European Debt Crisis that followed a few years later. The time-series of the CVA is presented in Subsection 6.2.

The last numerical study attempts to answer the third research question, that is to study the time evolution of the CVA for different values of ρ. The study will allow us to draw conclusions on how the modelling of correlation would have affected CVA losses historically and is presented in Subsection 6.3.

Note that when the CVA value is calculated in this thesis, it is done for one monetary unit, e.g., one euro, yen or dollar. If the CDS on $\mathbf{C}$ is for a value of $N = 10$ million monetary units then CVA_0^N is given by $CVA_0^N = N \cdot CVA_0$ due to the linearity of

CVA. Also note that CVA_0 is calculated at time $t = 0$ where CVA_0 can be interpreted as the sum of monetary units that has to be added on the CDS-contract between **A** and **B** so that **B** can be regarded as risk-free.

6.1 CVA under Change of Parameters

Subsection 6.1 aims to answer the first research question:*How will Wrong Way Risk affect the value of a single CVA measure by changing the correlation and default intensity parameters?*. We perform a sensitivity analysis and evaluate the CVA of a CDS as a function of WWR and the change of CDS_B and CDS_C. In the one-factor Gaussian copula model, it is difficult to know the true value of the correlation between the default times of **B** and **C**. We therefore want to investigate the way that CVA behaves under different values of ρ, the correlation parameter. Moreover, given that we have a constant default intensity, it is interesting to investigate how the model would have behaved based on different value of the CDS spread. This is interesting to study as we compute the default default intensity using the CDS spread through the credit triangle, (see equation (3.2.1.3)).

The recovery rate, ϕ_B and ϕ_C are set to 0.4 for both **B** and **C**, i.e. $\phi_B = \phi_C = 0.4$, so that 60 percent is lost in the event of a default for both **B** and **C**. It should be clarified that given that we have a heterogeneous model, i.e., we allow for different parameter values for **B** and **C**, the recovery rate does not have to be the same value for the counterparties. In this thesis, the reference entity, **C**, is government bonds issued by sovereign states. It would be interesting to study the actual recovery rates of countries, however, that is not within the scope of this thesis. The risk-free interest rate is assumed to be constant and set equal to 0.03 and we use continuous compounding.

Figure 3 displays the CVA computed using (5.25) as a function of the CDS spread for party **B** and **C** which are ranging from 50 to 500 bps while keeping the correlation constant at $\rho_B = \rho_C = \rho = 0.5$. Note that we also keep the risk-free rate, r, constant at $r = 0.03$ in numerical study 1. The CVA value is increasing with both $\mathrm{CDS_B}$ and $\mathrm{CDS_C}$. The increase of $\mathrm{CDS_B}$ is more close to a linear increase than $\mathrm{CDS_C}$. In the case of a CDS spread of zero for **B**, i.e., $\mathrm{CDS_B}$=0 and $\mathrm{CDS_C} = 0.8$ for **C**, the CVA losses amount to approximately 15 bps. If we keep $\mathrm{CDS_C} = 0$ and instead increasing the CDS spread for **B** so that $\mathrm{CDS_B} = 0.8$, the CVA losses amount to approximately 50 bps. Hence, the model is much more sensitive to a change of $\mathrm{CDS_B}$ than $\mathrm{CDS_C}$. Note that we use the CDS spreads to find λ_B and λ_C via the credit triangle in equation (3.2.1.3). Hence, Figure 3 displays the change in CVA by the change of parameters λ_C and λ_B, and as mentioned above, the CVA is increasing in default intensity λ_C and λ_B.

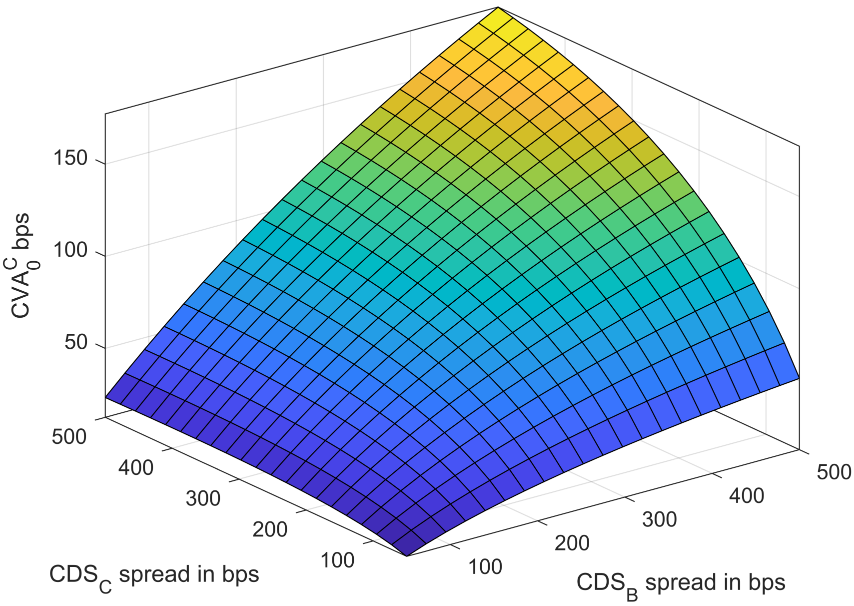

Figure 3: CVA with respect to the change of CDS_B and CDS_C. The CDS spreads are ranging from 0-500, where $\rho_B = \rho_C = 0.5$, and $\phi_B = \phi_C = 0.4$ are kept constant. Furthermore the interest rate is kept constat at $r = 0.03$

Next, Figure 4 shows the sensitivity to the correlation factor ρ_B and ρ_C where these variables range from 0 to 1 with an interval of 0.05 steps while keeping the spread constant at 100 bps for both CDS_B and CDS_C. As expected the CVA losses in bps increase with an increase in both ρ_B and ρ_C. As illustrated by Figure 4 below, the velocity of the increase in CVA seems to be marginally higher for ρ_C than for ρ_B. By keeping ρ_B constant at zero it's clear that an increase in ρ_C has a noticeable effect on CVA. However, when keeping $\rho_C = 0$, the same can not be said for ρ_B, which when increased yield close to zero increase in CVA losses, as shown by the ρ_B axis.

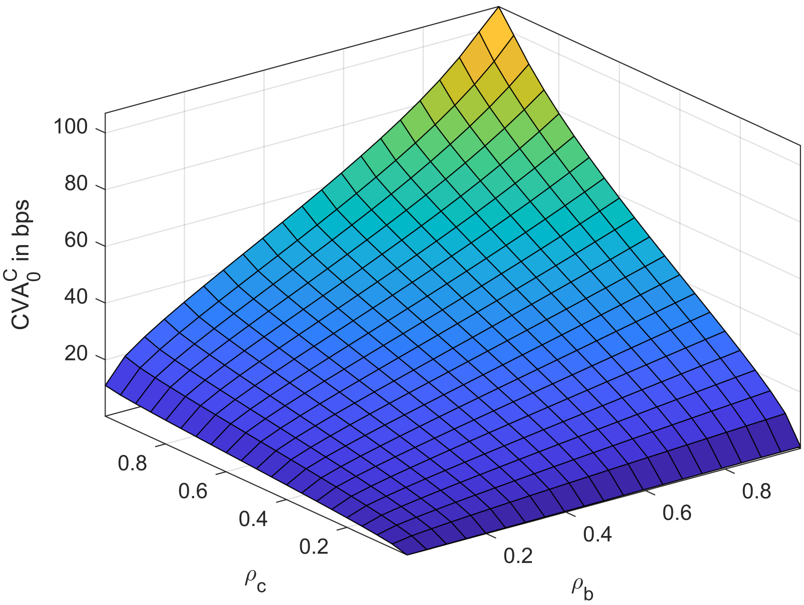

Figure 4: CVA with respect to the change of parameters ρ_C and ρ_B. The recovery rate is kept the same as in Figure 3, i.e., $\phi_B = \phi_C = 0.4$. The spread is kept constant so that $CDS_B = CDS_C = 100$. From the credit triangle we can derive that $\lambda_B = \lambda_C = 1.67\%$. Note that the risk-free interest rate is kept constant at $r = 0.03$ as in Figure 3.

Figure 5 gives a two-dimensional version of the plot in Figure 4 for different fixed

ρ_B-values, respective ρ_C-values. In the right panel of Figure 5, to the right, we see that for higher levels of ρ_B, the level of CVA is increasing. It is also evident that for higher levels of both ρ_B and ρ_C the rate of increase in CVA is increasing, Figure 5 confirms this as well.

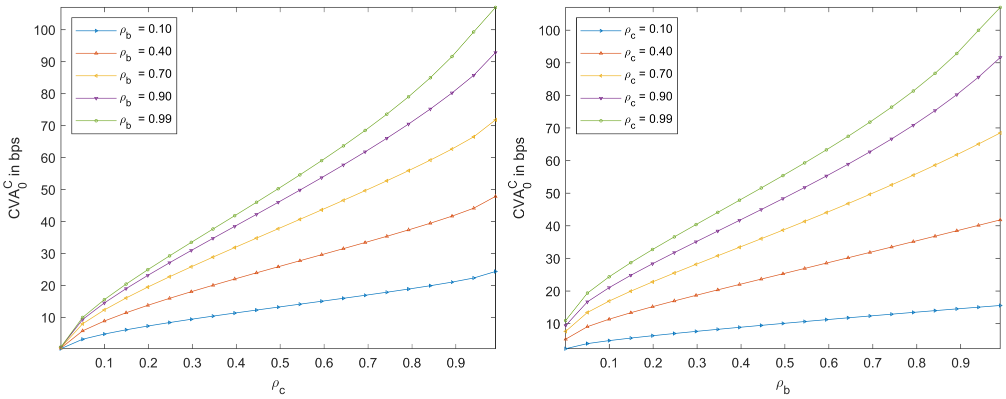

Figure 5: CVA sensitised for different levels of ρ_B and ρ_C. The recovery rate and CDS parameters are kept the same as in Figure 4. The left panel illustrates CVA as a function of ρ_C for $\rho_B = 0.10, 0.40, 0.70, 0.90, 0.99$. The right panel illustrates CVA as a function of ρ_B for $\rho_C = 0.10, 0.40, 0.70, 0.90, 0.99$.

Inarguably, there is a correlation between arbitrary obligors in a Gaussian copula model. The correlation between obligors became painfully evident when several states had to bail out banks due to their status as *too big to fail* during the 08' crisis. As a result, it is crucial to take WWR into account. Still, the true level of dependence between sovereign states, that is **C** and banks, that is **B**, is very difficult to measure. Table 1 below is exhibiting different levels of CVA sensitised for different levels of ρ. Table 1 illustrates that ignoring WWR and setting ρ close to 1, always results in the underestimation of CVA, assuming that the true value of ρ is not below 0.1. As an example, when $\rho_C = 0.4$ the CVA is underestimated by 27.13 bps if the true level of correlation is $\rho_B = 0.9$. Instead, if the true level of correlation was $\rho_B = 0.4$ the model underestimates the CVA by 10.66 bps.

Table 1: CVA in bps of a 5y CDS as a function of $\rho_c = 0.10, 0.40, 0.70, 0.90, 0.99$ and $\rho_b = 0.10, 0.40, 0.70, 0.90, 0.99$. We keep the same setup as in the Figures of this section, i.e., $\phi_B = \phi_C = 0.4$, a constant spread so that $CDS_B = CDS_C = 100$. Hence, from the credit triangle we can derive that $\lambda_B = \lambda_C = 1.67\%$. The risk-free interest rate is also kept constant at $r = 0.03$

		ρ_c				
		0.10	0.40	0.70	0.90	0.99
	0.10	4.79	11.35	16.91	21.03	24.36
	0.40	8.86	22.01	33.42	41.67	47.84
ρ_b	0.70	12.34	31.84	49.64	62.68	71.79
	0.90	14.52	38.48	61.79	80.22	92.84
	0.99	15.56	41.81	68.48	91.62	106.97

6.2 Time Evolution of CVA for an Average Counterparty

In this subsection we study the time evolution of CVA on a Credit Default Swap where the counterparty, **B** has a CDS spread equal to the CDS of 20 major European and US banks. The time-series data for the CDS spreads of both **B** and **C** have been downloaded from the S&P Capital IQ platform. We explore the time evolution of CVA where the reference enitity, **C** is a Credit Default Swaps on either Spanish, Italian, French or German bonds as the reference entity running with a 5-year maturity. The study in this subsection presents how the CVA varies during the financial crisis of 08' as well as the European Debt Crisis that erupted late 2009.

The time evolution, or, time-series CVA is obtained by re-calculating the CVA_0 using (5.25) for each day in the time-interval. This means that we for each time point recalibrate λ_B and λ_B through the credit triangle using the CDS_B and CDS_C. For each day the constant risk-free rate, r, is also updated with the overnight 1-year Libor rate. However, for the calculations in this subsection the correlation factor ρ is fixed at $\rho_B = \rho_C = 0.5$. Furthermore, the recovery rate is also kept constant in this study at the same level as for study 1 in Subsection 6.1 where $\phi = 0.4$.

The 1-year Libor rate is used as the proxy for the risk-free rate as illustrated in the right panel of Figure 5 and is downloaded from the SP Capital IQ platform. The counterparty of the contract in all of our studies is an average comprised of 20 banks, which namely are: BNP Paribas Cardif, Deutsche Bank, Crédit Agricole, Barclays, Banco Santander, Sociétè Generale, Intesa Sanpaolo, UniCredit, Credit Suisse, JP Morgan Chase, Bank of America, Citigroup, Wells Fargo, Goldman Sachs, Allianz, Morgan Stanley, Standard Chartered, Capital One, American Express and BBVA. The average spread of the 20 banks named above is presented in the left panel of Figure 6. We can view this CDS average as the CDS spread for a likely counterparty, **B** given that it is highly likely that a buyers of a CDS does business with one of the counterparties included in our group of 20 international banks. Furthermore, if we assume that the portfolio of banks is homogenous, which is a reasonable assumption for the above banks, it is proven in Lemma 9.2.1 in Herbertsson (2022) that the average CDS spread is equal to the individual CDS spread.

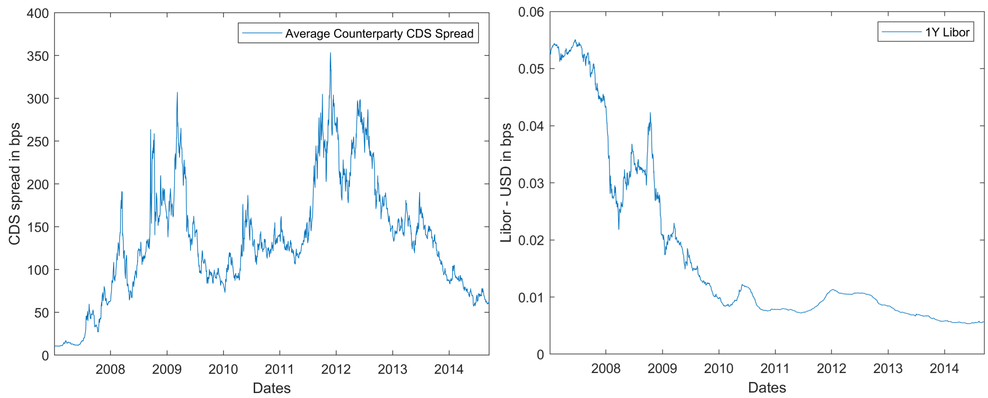

Figure 6: Average Counterparty CDS Spread and 1-year overnight Libor Rate. In the left panel the average Counterparty CDS Spread of 20 major US and European banks is presented. In the right panel the 1-year overnight Libor Rate used as a proxy for the risk-free rate is presented. Source: SP Capital IQ platform.

Figure 8 displays the time evolution of univariate CVA_0 for a CDS between January 2007 and December 2014, where the counterparty **B** has the CDS spread given in the left panel of Figure 6 for the same time period. For four different reference entities (Italy, Spain, Germany and France) Figure 7 displays the CDS spread, i.e., the CDS spreads for the reference entities, **C**. The time period presented in and Figure 7 runs from the beginning of 2008 until the end of 2014 as in Figure 8. Figure 7 shows a slight increase in CDS-spreads for all reference entities (Italy, Spain, Germany and France) in the first quarter of 2008. Subsequently, the CDS spreads rise sharply towards the end of 2008 and throughout the first quarter of 2009. From Figure 7 and Figure 8 it is evident that as the CDS spreads increase, an increase in CVA losses follow. From the buyer's perspective, having a position against the banks contained within the average Counterparty CDS spread, would have resulted in substantial CVA losses.

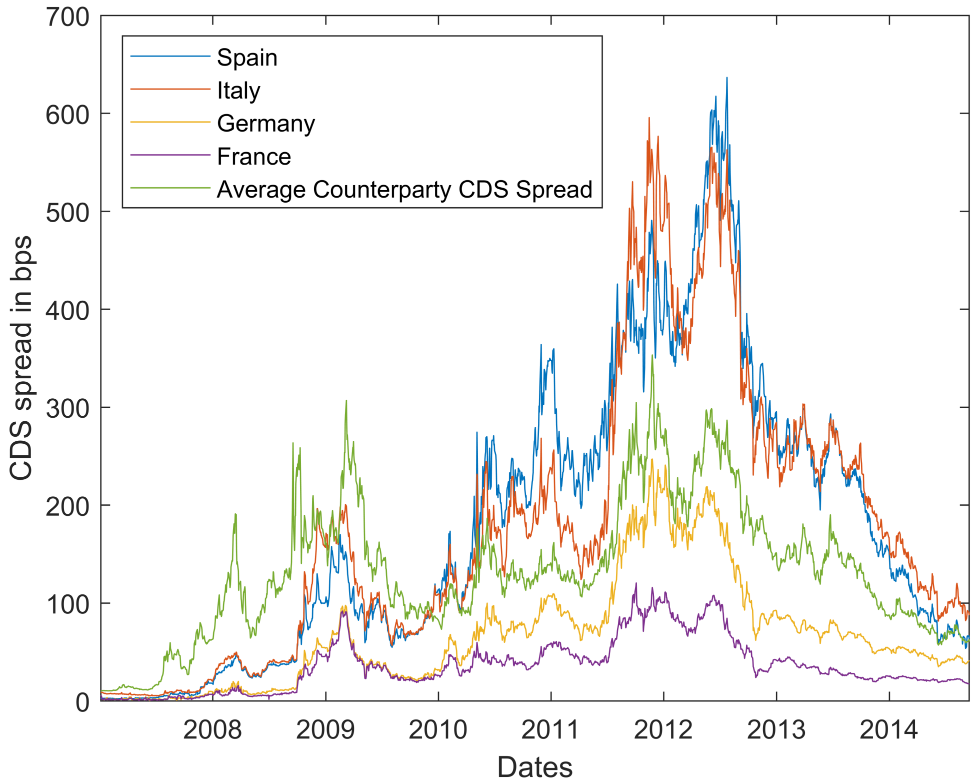

Figure 7: 5y CDS Spreads for Italy, Spain, Germany, France and the average Counterparty CDS Spread. Source: S&P Capital IQ platform.

The increase in CVA_0 losses is also true for the period between 2011-2013 where we can clearly see the implications of the European Debt Crisis. Not surprisingly, given the extensive debt positions of the reference entities, the CDS spreads of Spain and Italy tops the graph and consequently incur the greatest CVA losses during the period. As seen in Figure 7, The CDS spread of the bond issued by Germany performs relatively well compared to it's European peers but also relative to its own history. During the European Debt Crisis, the German spread is consistently trending below or marginally above the spread of the financial crisis, rendering CVA losses that are modest in this context.

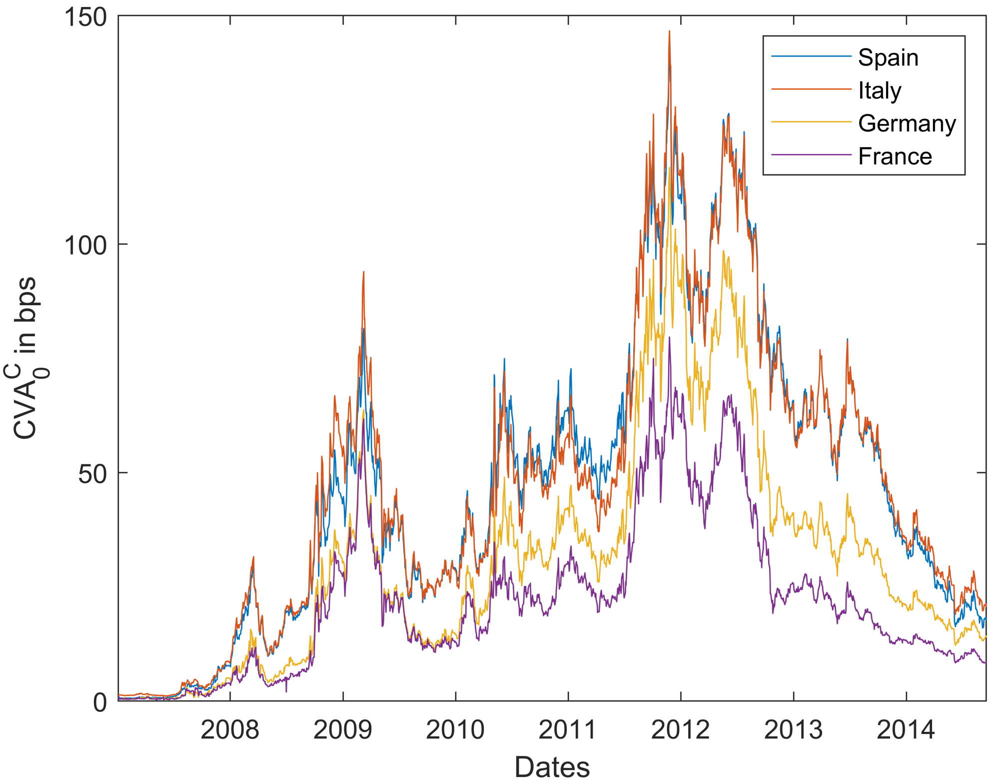

Figure 8: Time evolution for univariate CVA_0 on a CDS where CVA_0 is computed by (5.25) for sovereign bonds issued by Italy, Spain, Germany and France where the counterparty, **B** for the CDS's is our average counterparty CDS spread given in the left panel of Figure 6. Furthermore, as previously $\phi_B = \phi_c = 0.4$.

As an example, in the beginning of 2012, if a CDS buyers, **A**, enters into a contract to hedge against the default of an Italian bond, a counterparty, **B**, having a CDS spread of approximately 350 bps, risk to lose almost 150 bps, 1.5%, or 1.5 million monetary units assuming that the nominal is 100 million monetary units, at the peak of the European Debt Crisis, if the counterparty A would default.

6.3 Time evaluation of CVA as a function of default correlation

Our third numerical study examines the effect that ρ has on our previously obtained time evolution of CVA. Figure 9 plots the estimated time evolution of CVA on a CDS using Equation (5.25) for a set of values for the parameter ρ. In the numerical study in this section we chose the same values for ρ_B and ρ_C meaning that $\rho_B = \rho_C = \rho$. The three set of values are, $\rho = 0.3, 0.6$ and $\rho = 0.9$ where the CVA_0 for each ρ is plotted during the same time period as in Figure 7 and Figure 8, (i.e. January 2007 until December 2014). The results in Figure 9 illustrates that for higher levels of ρ, the level of CVA_0 is generally higher, however, during the European Debt Crisis the difference in CVA_0 between $\rho = 0.6$ and $\rho = 0.9$ is much smaller in the time period between 2011 July to 2012 August. To illustrate the differences more clearly we plot the relative differences in CVA_0 between the different set of values for ρ in Figure 10.

Relative differences are employed to assess and compare two quantities while considering the inherent sizes of the entities under examination. The relative difference has been computed by first subtracting between two CVA_0 values, one with a higher correlation and the other with a lower correlation. Next, the resulting difference is divided by the CVA_0 value corresponding to the lower level of correlation ρ. This step helps determine the relative change or ratio between the two values. To present the result as a percentage, the computed ratio is then multiplied by 100. The described process was used to compute the results in Figure 10. The Figure illustrates the relative difference between $\rho = 0.3$ and $\rho = 0.6$, which is represented by the blue line and the relative difference between $\rho = 0.3$ and $\rho = 0.9$ which is represented by the orange line. The difference is consistently above 0% for the entire time period between January 2007 and December 2014. Figure 10 clearly illustrates that there is a positive correlation between ρ and CVA_0.

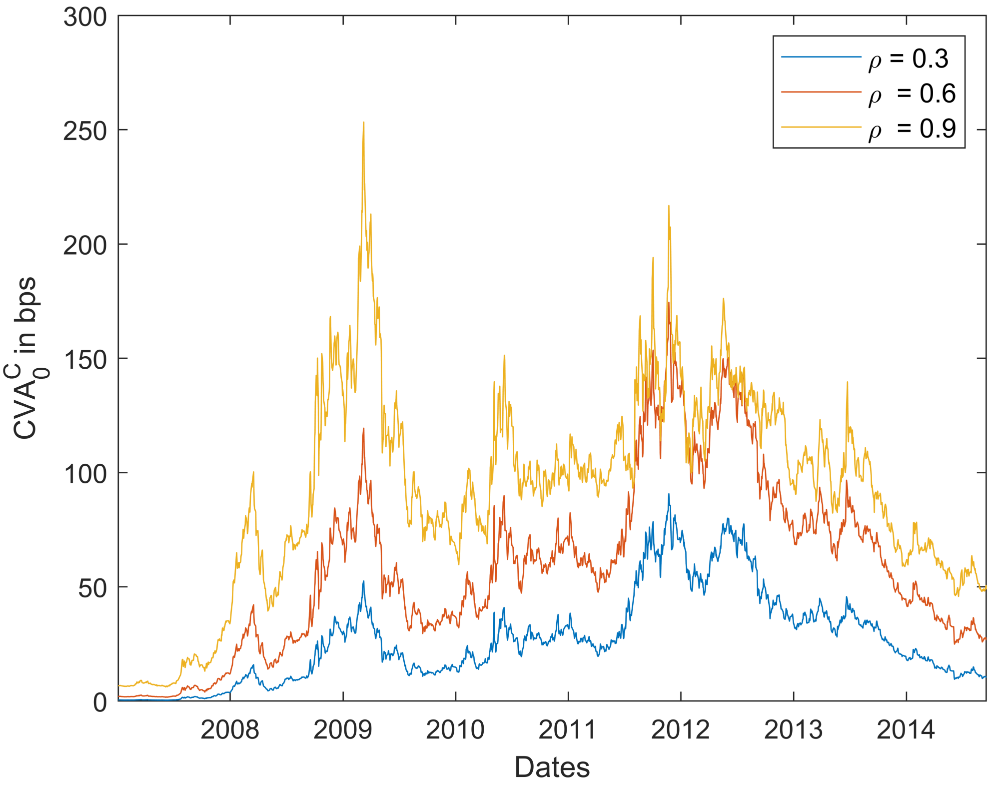

Figure 9: The time evolution for univariate CVA on a CDS computed by (5.25) for a CDS on an Italian government bond for $\rho = 0.3, 0.6, 0.9$. The parameters are $\phi_B = \phi_C = 0.4$ and where the counterparty **B** has CDS spread from the average counterparty CDS spread given in the left panel of Figure 6. The overnight 1y Libor rate is used as a proxy for the risk-free rate. The constant default intensity, λ_B and λ_C is derived by the credit triangle in (3.2.1.3).

Interestingly, the relative differences between $\rho = 0.3$ and $\rho = 0.9$ in Figure 10 is much more volatile compared with $\rho = 0.3$ and $\rho = 0.6$. During the Financial Crisis, the relative differences is very large, while during the European Debt Crisis the difference is almost negligible. The highest relative difference in CVA_0 is between $\rho_C = 0.3$ and

$\rho_C = 0.9$ which occurs on the 11 June 2007 and is equal to approximately 1544%. Conversely, it is true that the least difference of 71% is obtained on 8 November 2011.

The results in Figure 9 and Figure 10 supports the results found in Section 6.1 regarding the positive correlation between correlation parameter ρ and CVA_0. However, the effect seems to depend on the time period, especially for the relative difference between $\rho = 0.3$ and $\rho = 0.9$. As illustrated in Figure 8, during the Financial crisis, the CDS spread of the the average counterparty was greater than the CDS spread of the Italian Bond, yielding the highest relative difference, however during the European Debt Crisis, the CDS spread of the the average counterparty was smaller than the CDS spread of the Italian Bond, yielding the smallest relative difference.

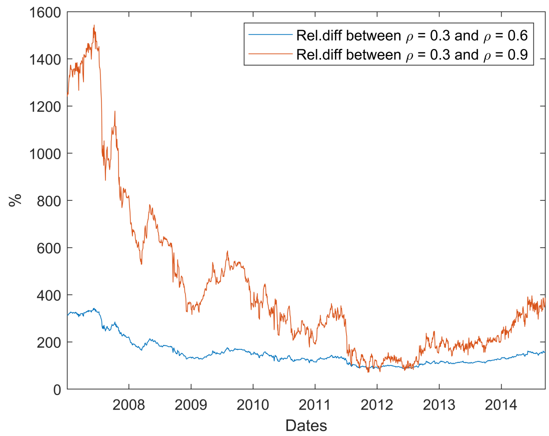

Figure 10: The time evolution of the relative difference in CVA_0 between the case when $\rho = 0.3$ and $\rho = 0.6$ as well as when $\rho = 0.3$ and $\rho = 0.9$ in the one factor Gaussian copula model with identical parameters as in Figure 9. The relative difference is measured in % between the beginning of 2008 and end of 2014 and is computed using the results from Figure 9. The relative differences has been computed by $\frac{CVA_0^{\rho=0.09} - CVA_0^{\rho=0.03}}{CVA_0^{\rho=0.03}}$ and $\frac{CVA_0^{\rho=0.06} - CVA_0^{\rho=0.03}}{CVA_0^{\rho=0.03}}$, the described computation is then multiplied with 100 in order to get the ratio as a percentage.

7 Conclusion

We implement the univariate CVA model for a CDS derived in Herbertsson (2023) for a one-factor Gaussian copula model. Using the univariate CVA model we were able to present three separate numerical analysis. Firstly, a sensitivity analysis with a focus on the correlation and default intensity parameters was conducted. Secondly, a time-series CVA measure enabled us to investigate how CVA behaved throughout the financial crisis of 2008 as well as the European Debt Crisis. Lastly, the time-series CVA model was tested using a set of ρ numbers, allowing for the analysis of time-series CVA under different levels of WWR.

Our implementation of Herbertsson (2023) shows that Wrong Way Risk affects the value of a single CVA measure by changing the correlation parameters, ρ_B and ρ_C. It was also found that CVA is increasing in the default intensity, λ_B and λ_C. From the results of the sensitivity analysis, we draw the conclusion that neglecting WWR has a detrimental effect on the modelling of CVA, as it could be severely underestimated. It is also clear that the default intensity for both **B** and **C** has a positive relationship with CVA. Hence, both of the parameters, λ_B and λ_C as well as ρ_B and ρ_C showed a positive relationship with the unilateral CVA value on the CDS with **C** as a reference entity and **B** as a counterparty. The positive correlation between the parameters and the unilateral CVA value is heavily in line with the results shown in Brigo and Capponi (2008), Brigo, Morini, and Pallavicini (2013) and Arismendi-Zambrano et al. (2022).

The results of the time evolution of CVA in our second numerical analysis illustrates the enormous losses that the financial institutions endured during the financial crisis in 2008 and 2009 by neglecting WWR. We can also draw the conclusion that the correlation between the CDS spread and CVA losses are significant as increases in

CDS spreads are generally followed by increases in CVA losses.

Lastly, the time-series CVA for a range of ρ values is presented in the last numerical analysis where the the results show that there is a positive relationship between ρ and CVA, i.e., when the correlation parameter increases, so does the CVA. The positive relationship between ρ and CVA verifies the result obtained in our sensitivity analysis in numerical study one which showed that CVA is increasing in both ρ_B and ρ_C.

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A Appendix

A.1 Sklar’s Theorem

In Sklar’s theorem we let $X_1, X_2, ..., X_m$ be a series of m real valued random variables with $F_i(x) = \mathbb{P}[X_i \leq x]$ as their distribution functions for $i = 1, 2, ..., m$ and denote F as their joint distribution. We then consider $\mathbf{X}$ to be a random m-dimensional vector $\mathbf{X} = (X_1, X_2, ..., X_m)$ with a joint distribution $F(\mathbf{x})$ which can be expressed as

$$F(\mathbf{x}) = \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, ..., X_m \leq x_m]$$

where $\mathbf{x} = (x_1, x_2, ..., x_m)$. In this setting, there exist an m-dimensional copula C which gives us

$$F(\mathbf{X}) = C(F_1(x_1), F_2(x_2), ..., F_m(x_m))) \tag{A.1}$$

Note that if F_i is continuous for all i then the copula is unique. If not, the copula is uniquely determined on $RanF_1 \times ... \times RanF_m$ where the range space for the function

F is denoted as F_i. This theorem lets us separate the dependence structure of random variable $\mathbf{X}$ into two parts where the first part is the marginal distributions F_i. The second part is the joint distribution of $\mathbf{X}$ which is specified by the copula C, (for full proof see Nelsen (1999))((Herbertsson 2022)).

A.2 Lemma 1

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ represents the full market information. Let $\mathcal{G}_t$ be the filtration at time t. Furthermore define the filtration $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ where we let $\mathcal{H}_t = \sigma(1_{\tau \leq s}; s \leq t)$. From Lemma 7.3.4.1 on p.420 in Janblanc, Yor, and Chesney (2009) we know that

$$1_{\{\tau > t\}} \mathbb{E}[Z \mid \mathcal{F}_t] = 1_{\{\tau > t\}} \frac{\mathbb{E}[1_{\{\tau > t\}} Z \mid \mathcal{G}_t]}{\mathbb{E}[1_{\{\tau > t\}} \mid \mathcal{G}_t]} \tag{A.2}$$

where the variable Z is a $\mathcal{F}$ measurable random variable. For a full proof we refer to Janblanc, Yor, and Chesney (2009).

A.3 Rules for conditional expectations

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which means that all variables are $\mathcal{F}$-measurable, additionally the sigma-algebras exist in $\mathcal{F}$. Consider the sigma algebras $\mathcal{G}$ and $\mathcal{H}$ where $\mathcal{H} \subseteq \mathcal{G}$, then

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}] = \mathbb{E}[X \mid \mathcal{H}] \tag{A.3}$$

where we let X be a random variable (Williams 1991).

A.4 Rules for conditional probabilities

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ and $\mathcal{F}$ be sigma-algebras, additionally let the indicator function 1_A be a random variable where A is a event in

$\mathcal{F}$ meaning that 1_A is $\mathcal{F}$-measurable. It is then possible to define the conditional probability $\mathbb{P}[A \mid \mathcal{G}]$ as

$$\mathbb{P}[A \mid \mathcal{G}] = \mathbb{E}[1_A \mid \mathcal{G}] \tag{A.4}$$

where we note that $\mathbb{P}[A \mid \mathcal{G}]$ is a random variable and $\mathcal{G}$-measurable (Williams 1991; Herbertsson 2022).