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The influence of numbers when students solve equations

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Abstract

Is it possible that some students' primary difficulty with equation-solving is neither handling the literal symbols nor the equality, but the numbers used as coefficients? It is well known that many students find algebra a difficult topic, and there is much research on how students experience this strand of mathematics, with indications of how it can be taught. Still, a perspective not often fronted in this research – that has been suggested as an area potentially important – is how numbers, other than natural numbers, in algebra, are perceived by students. Such kinds of numbers (negative numbers and decimal fractions) have been used in this thesis to explore how the numbers influence students' equation-solving. Two studies with a phenomenographic approach have explored how students (n_1=5, n_2=23) perceive linear equations of similar structure but with different kinds of numbers as coefficients, e.g., $819 = 39 \cdot x$ and $0.12 = 0.4 \cdot x$. In the second study, a test was also used to investigate the magnitude of the influence of a change of coefficients for 110 students while solving equations with a calculator. The findings show that equations with decimal fractions and negative numbers are less likely to be solved by these students, and decimal fractions as coefficients can even make a student unable to recognize a kind of equation they just solved with natural numbers. The interviews display that, depending on the number in a linear equation, some students focus on different aspects of the equation, and that the numbers influence what meaning the students see in the equation and how they can justify their solution. Following the phenomenographic approach, differences in the way that students experience the equations were specified, and critical aspects were formulated. This implies a wider use of different kinds of numbers in teaching algebra, as different kinds of numbers hold different challenges, thereby also varying learning potential, for students.

Sammanfattning

Är det möjligt att vissa elevers huvudsakliga svårighet med ekvationslösning är – varken att hantera bokstavssymboler eller likheten – utan talen? Det är väl känt att många elever tycker att algebra är ett svårt ämne, och därför finns det mycket forskning om hur elever erfar den här delen av matematikämnet, med indikationer om hur algebra bör undervisas. Trots det, är ett område som ofta inte lyfts fram i forskning – men som har föreslagits som potentiellt viktigt – hur tal, andra än de naturliga talen, upplevs av elever i den algebraiska kontexten. Negativa tal och tal i decimalform har använts i den här uppsatsen för att utforska hur talens egenskaper påverkar elever när de löser ekvationer. Två studier med fenomenografisk ansats har utforskat hur studenter ($n_1=5$, $n_2=23$) erfar linjära ekvationer av liknande struktur men med olika tal som koefficienter, ex. $819 = 39 \cdot x$ och $0.12 = 0.4 \cdot x$. I den andra studien användes även ett test för att undersöka omfattningen av påverkan på 110 studenters hantering av ekvationer (med miniräknare) då koefficienterna byts ut. Resultaten visar hur det är mindre sannolikt att elever löser ekvationer med decimaltal och negativa tal, samt att decimaltal som koefficienter kan göra elever oförmögna att känna igen en typ av ekvation som de precis har löst, men med naturliga tal. Intervjuerna visar att beroende på typ av tal i ekvationer så fokuserar elever på olika delar i ekvationen, och talen påverkar med vilken mening eleven förstår ekvationen och berättigar sin lösning. Genom den fenomenografiska ansatsen identifierades skillnader i hur eleverna erfar ekvationerna och kritiska aspekter formulerades. Det här implicerar en bredare användning av tal i algebraundervisning, eftersom olika typer av tal bär med sig olika utmaningar och därmed också olika potential för elevers lärande.

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List of publications

This thesis contains the following papers:

- **Holmlund, A.** (2022). Different ways of experiencing linear equations with a multiplicative structure. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, M. Hähkiöniemi, B. E. Jessen, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. R. i. Portaankorva-Koivisto, J., J. Ö. Sigurjónsson, O. Viirman, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future. Proceedings of Norma 20. The ninth Nordic Conference on Mathematics Education, Oslo, 2021* (Vol. 17, pp. 89–96). SMDF.
- **Holmlund, A.** (n.d.). The significance of numbers when learning to solve linear equations. *Manuscript, under review.*

Contents

Chapter 1 Introduction

An important part of being a teacher is, not just to be able to understand difficult equations and how to solve them, but also to be able to see the subject matter in the eyes of the students. If a teacher does not know what students need to learn, then the choice of content is based on chance and quite likely ineffective.

This is especially delicate in algebra, a subject matter that combine several areas of mathematics, like knowledge of numbers, algorithms, and mathematical structures of different kinds. It is a challenge to teach as the students need to use several areas of their prior knowledge when interpreting algebra. There is a lot of research on students' previous experiences concerning specific concepts like the variable and the equality sign (see section 2.2.1-2.2.2). However, it is important also to look at how students' experience equations in their complexity, the combination of numbers, variables and equalities, not just as isolated concepts.

Starting doctoral studies, I recalled that several of my students in vocational education did not have any trouble solving equations with natural numbers but found similar equations, about an electric problem with decimal fractions, as more difficult. Exploring the reason for this has led to this thesis about how different types of numbers in equations influence students' experiences of the equations (see Figure 1.1).

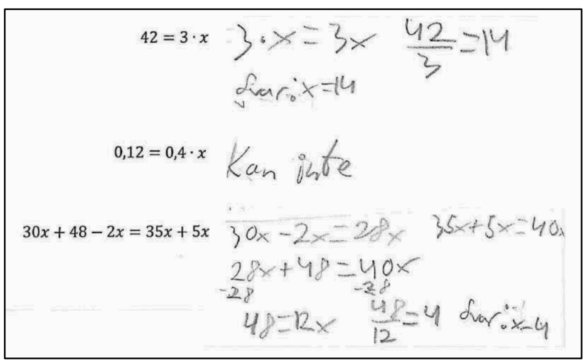


Figure 1.1: The relevance of numbers – one students' answers to three equations in consecutive order from a test preceding the first study in this thesis.

In contrast to many studies that elaborate on effective strategies for equation-solving (see section 2.1.2), this thesis focuses on the effort made by students to understand what they see *before* using a strategy.

If "you want to prepare learners to handle future, novel, situations in powerful ways, you have to help them to learn to see those situations in powerful ways"

(Marton, 2014, p. 68). Students do not need to see all possible structures and strategies of equation-solving, but they need help to guide their focus and direct their attention to the relevant aspects of equations and the solution process. When students learn to see equations in powerful ways, they will also handle them powerfully. This epistemological assumption from the phenomenographic approach has guided the research in this thesis.

This thesis adds knowledge on what aspects need to come into focus in teaching in upper secondary school regarding equation-solving so that students can develop as mathematical individuals and handle formulas (and equations) in powerful ways in their professional life.

1.1 Aim and specific object

The aim of this thesis is to contribute to research regarding students' experience of equations while solving them and how the nature of numbers influence the way students conceive equations. Therefore, this research explores the following question: How does the nature of numbers influence students' conceptions of equations while solving them?

Chapter 2 Previous research on solving linear equations

Equation solving is a key topic in educational research on algebra and there is a lot of research specifically concerning teaching and learning of linear equations (Blanton et al., 2015). This chapter will review some of this research; starting with overviewing the process and how the equation-solving competence usually is described, and then reviewing central concepts. Finally, a perspective that is not as often raised in research – the influence of coefficients, and especially from different number domains (Kieran, 2022; Vlassis & Demonty, 2022) – will be displayed.

2.1 Solving equations from a learner's perspective

This section will display different views of the process of solving equations. Starting with an overview of the concept linear equations, then the process of solving equations and research on students' strategies. Finally, the common description of algebraic thinking will be elaborated on in relation to the thesis.

2.1.1 The subject matter linear equations

There is no official definition of what an equality is, but the definitions suggested usually include a statement of mathematical equality (Tossavainen at al., 2011). There is therefore a wide range of equations, starting by simple mathematical sentences like 7=7 advancing to differential equations. The subject matter in focus in this thesis is linear equations, also called polynomial equations of degree one – where the exponent for any variable is zero or one. This allows for a wide range of different ways of writing the equation as a gap or a point can be used to indicate that there is a variable of degree one (Herscovics & Linchevski, 1994; Xie & Cai, 2002), wherefore $4 + \square = 9$ can be interpreted as a linear equation.

Linear equations are usually a part of students' first encounter with algebra as well as equations. Some equations are easier for beginners to interpret with their previous knowledge. A distinction, often referred to, is made between "arithmetic equations", that only include a variable on one side of the equality, e.g., $4 \cdot x + 6 = 30$, and "non–arithmetical equations", that contain variables on both sides, e.g., $4 \cdot x + 6 = 7 \cdot x - 12$, (Filloy & Rojano, 1989). The latter is also called an "algebraic" equation (Andrews & Sayers, 2012). Non-arithmetical equations are not possible to "undo" by simply using arithmetic skills and inverting the operations but requires handling the variable and seeing that the left and right side of the equality are of the same nature (Andrews & Sayers, 2012; Filloy & Rojano, 1989; Kieran, 1992). A remark here is that the first kind of equation, the arithmetical, is possible to regard as either arithmetical or algebraic from a student's perspective, depending on how the student *sees* the equation. Either a student sees solving $4 \cdot x + 6 = 30$ as an arithmetic procedure, e.g., $(30 - 6)/4 = x$ when "undoing" calculations, or as a structure where variables and coefficients are arranged, e.g., $4 \cdot x - 0 \cdot x = 30 - 6$ when collecting similar terms. With this idea in mind, it is very likely that students solving linear equations in the same classroom are practising ideas from different areas of mathematics – some are solving equations arithmetically and some algebraically. Therefore, practising algebra in the classroom does not necessarily evolve students' algebraic thinking. This makes it rather difficult for teachers to know the quality of students' learning of algebra.

Equations in this thesis concern the first kind of equations, arithmetic equations, with the unknown on one side of the equality. The research interest here is how students recognize similar structures from an algebraic perspective.

2.1.2 The process of equation-solving

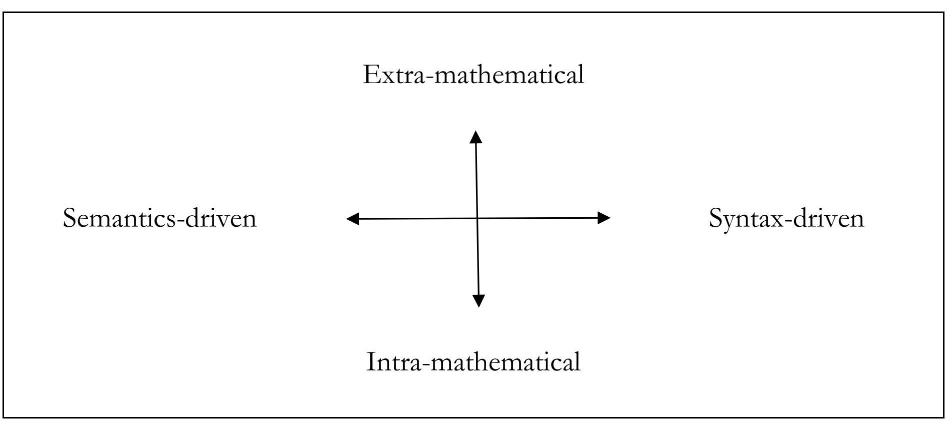
There are many different aspects covered in the literature regarding equation-solving. Pierce and Stacey (2004) suggest that the thinking process in algebra that students need to develop concerns: recognition of conventions and basic properties, identification of structure, and identification of key features (which includes linking form with solution type). In this section, research concerning these parts of equation-solving with special relevance for this thesis, will be reviewed: how students direct their attention, see meaning in equations and apply a strategy.

There is much literature, that we soon shall address, that concerns students' choice of strategy when solving equations. However, before applying a strategy student needs to notice that the specific task is suited for a certain strategy. Depending on where a student directs her attention, she can interpret the task in different ways. How the equation is presented is therefore of significance – even the spacing between the symbols has been registered as affecting students

(Kirshner, 1989). Another example is from 11–12-year-old students in China that had learnt to solve linear equations arithmetically with brackets for the unknown value (•) (Xie & Cai, 2022). After learning formal strategies, the students tended to favour a formal strategy over arithmetic strategies more for equations with a variable **x**, compared with the same equations with bracket (•) representing the unknown. Insecurity with the new strategy and an insecurity concerning the new symbol – not seeing how it is irrelevant – resulted in lower solution frequencies for the equations with x as variable. For example, 97.6% of the students (N=126) solved $9-(\bullet)=5$, but only 77.8% of the same students solved $9-x=5$ on the same test. Another example of the importance of what students notice is from a study by de Geer (1987). He gave similar word-problems to 12-13-year-old students but with different numbers (e.g., changing a number larger than one for another between zero and one). By this, he could show that the students changed their approach to the problem because of the change of numbers, even though they had calculators.

It is not only the parts of the equation, like x and the spacing, that influence a student's experience. The combination of all symbols needs to be interpreted using the student's prior experiences. One such aspect of experiencing equations often emphasized in research is whether the equation is seen as an object to handle or a process to perform (Tossavainen et al., 2011; Tuominen et al., 2018; Wettergren et al., 2021). Two other aspects where experiences of equations can differ are described by Carraher & Schliemann (2007) in Table 2.1; the dimension of focus and the dimension of control. The focus dimension concerns whether an algebraic

Table 1.1: Two underlying dimension when approaching algebra, by Carraher & Schliemann (2007,p. 677). The vertical "focus" dimension and the horizontal "control" dimension.



task can be interpreted as purely mathematical or with references from other domains, e.g., applying a context to the problem. The control dimension concerns how students overview the solution process – while solving – and meanwhile justify their solutions. Either inferences are drawn by refering to the solution process as transformations of something concrete, e.g., adding by thinking of several things put together, or by theoretical assumptions, e.g., relying on mathematical properties and rules of the representation form (Balacheff, 2001; Carraher & Schliemann, 2007; Kirshner, 2001). The terms *semantic* and *syntacitc* are used for this distinction. These concepts, known from the terminology of linguistic are also since long established in mathematics education for distinguishing between either regarding algebraic expressions as primarily representing a content or as displaying a certain form (Balacheff, 2001; Filloy & Rojano, 1989). To conclude this section, a student interprets what she has noticed in an equation, and depending on how aspects of the equation are in focus, the whole equation can be experienced in different ways.

Moving on, a student either approaches the task with a strategy or a sequence of choices that afterwards can be regarded as a strategy (Threllfall, 2002). Distinctions between strategies – in some texts referred to as methods – for solving an equation are commonly made in studies on equation-solving (e.g., Andrews, 2020; Andrews & Öhman, 2019; Filloy & Rojano, 1989; Vlassis, 2002; Xie & Cai, 2022). Kieran (1992, p. 400) has categorized students' equation-solving methods as follows:

a. use of number facts,
b. use of counting techniques,
c. cover-up,
d. undoing,
e. trial-and-error substitution,
f. transposing,
g. performing the same operation on both sides.

The two last strategies are considered as formal (Kieran, 1992), transposing being considered a shortened version of "doing the same thing on both sides". Advantages with these two strategies have been discussed under a long period of time, e.g., transposing terms is also called the "the Viète model" and operating on both sides with inverses "the Eulerian model" (Filloy & Rojano, 1989), but they are also discussed in more recent literature (Andrews, 2020; Andrews & Öhman, 2019). There are benefits to both formal methods. Transposition is a general solution method to linear equations with one unknown, but it does not connect to students' previous experiences (Vlassis, 2002). Some authors stress that there is a risk that the method of transposition can become a rout-learnt strategy, "swap side swap sign", without knowledge of the equation-solving process (Andrews & Öhman, 2019; Kieran, 1992). However, operating on both sides does not seem to be a guarantee for success in algebra (Andrews & Öhman, 2019).

There are many factors that influence the choice of strategy a student makes in solving equations. The potential context is one such example. Similar (simple) equations, one set in a context and one not, can have different solution frequency for students, where the equation set in a context tend to have the higher solution frequency (Linsell, 2009, p. 37). A possible reason is that a context can help students to see an equation from new perspectives. However, factors of a context can also impede the solution process when the order of the numbers in a word problem does not align with the order of the numbers in respective equation (Clement, 1982). A frequently occurring example in studies concerning research on teaching and learning algebra is when "Six times as many students as professors" should be formulated as $S = 6P$, but sometimes is mistaken for $6S = P$ as it follows the word-order. The choice of strategy can also be directed by what numbers are included in equations, e.g., large numbers prohibit the use of number facts (Herscovics & Linchevski, 1994). Moreover, the choice of strategy has also been registered as culturally dependent, e.g., students from different countries tend to expand parentheses or use the factorized form of equations to varying extent (Star et al., 2022).

2.1.3 Algebraic thinking

There have been many attempts to describe the ability to handle algebra. A term usually referred to is *algebraic thinking* that, depending on how it is described, puts emphasize on different aspects of algebra and equation-solving.

In 1980s researchers started to discuss what characterizes algebraic thinking (e.g., Filloy & Rojano, 1989). There is still no consensus as to what characterizes algebraic thinking in mathematics education, but two terms that often are used in discussing algebra are the use of symbols and making generalizations. Some emphasize making generalizations as what characterizes algebraic thinking (Mason, 2008) while others stress the symbolization process as most relevant (Kaput, 2008). Radford (2018) means that both views are too narrow as it is fully possible to perform algebraic reasoning with natural language (without symbolism) and generalizing is a central competence in other areas than mathematics and is therefore not exclusive to algebraic thinking. Radford (2018) means that algebraic thinking deals with "indeterminate quantities in an *analytic* manner" (p. 8) and with several different modes of representation. The word analytic is central in his definition, making deductions as a manner of reasoning.

Summarizing different views in research, Kieran (2022) has come to a threefold distinction of (early) algebraic thinking. The three dimensions of (early) algebraic thinking are: analytical thinking, structural thinking, and functional thinking. (As the last one is not relevant to this thesis, it will not be described further.) Analytic thinking is described as the "thinking that underpins the transformations and equivalence aspects of equations and equation-solving", while structural thinking "is more aligned with seeing and expressing structure and properties within numbers" (Kieran, 2022, p. 1134). An interesting notion is that Kieran (2022) regards these dimensions not as conflicting but complementary and overlapping. She explains that analytic and structural thinking could "be collapsed into the single, more general dimension of relational thinking" (p. 1134), but making the separation reveals thinking with two different focuses. These theoretical distinctions can be useful in order to specify what algebraic thinking is and thereby register and quantify its occurrences (Molina & Castro, 2021; Vlassis and Demonty, 2022).

2.2 Central concepts

Linear equations usually consist of variables, equality signs, operations, and numbers. The first two mentioned are commonly discussed in research concerning teaching and learning algebra, probably because they have a central role in the algebraic activity. In this section, some of this research will be reviewed, but also research on numbers in equations – that is a topic not often at center of research on teaching and learning algebra – will be summarized. Finally, the last section will address the concept structure, as it is of relevance to this thesis.

2.2.1 The equality sign

Many researchers emphasize the importance of an evolved understanding of equivalence for the learning of algebra (Kieran, 1981; Matthews, Rittle-Johnsson & Taylor, 2012). This is easy to understand as the idea of equality is part of what defines an equation (Tossavainen et al., 2011) and without knowledge on how to handle it, equation-solving is reduced to performing computations. This section will raise topics in research concerning how equality is perceived by students and how it is relevant to the ability to solve equations.

Mathematically the equal sign is used to state that two expressions represent the same thing (Kiselman & Mouwitz, 2008). Equality is an example of an equivalence relation, since it is reflexive $a = a$ (any expression is equal to itself), symmetric ($a = b \leftrightarrow b = a$), and transitive (if $a = b$ och $b = c \rightarrow a = c$). Studies have shown that difficulties to conceptualize these properties can lead to difficulties with identifying what an equation is (Tossavainen et al., 2011). Accepting these properties of the equality is the foundation for understanding an equation as a structure (Kieran, 1992). A distinction is usually made between a relational understanding of the equality sign and an operational understanding (Kiearan, 1981; Madej, 2022). The latter way of considering the equality sign implies that it is a sign for doing something and one side as representing "the answer" (Kieran, 1981; Tossavainen et al., 2011). This conception seems to be persistent and can be viewed even among students at the university level (Tossavainen et al., 2011). The description of students' understanding of equality has also been developed further, indicating that there are different levels of operational and relational understanding of equality (Rittle-Johnsson, Taylor, Matthews, 2011). For example, students who approach equations with an operational view can be more or less willing to accept operations on the right side of the equality, as in $37 = 5 \cdot x + 2$.

Giving a correct definition of equality as a relation does not necessarily imply that this understanding is displayed in students handling of the equality sign (Madej, 2022; Sumpter & Löwenhielm, 2022). However, reaching a relational understanding of equality has been seen to be a prerequisite for success in algebra (Knuth et al., 2006).

2.2.2 The variable

A central concept when solving equations is the variable – explicitly suggested as one of the "big ideas" within mathematics education focusing on algebra (Blanton et al., 2015). That the variable is central in teaching and research on algebra is not surprising as symbolization is usually considered one of the characteristics of algebra (Kaput, 2008). However, the variable has proven a versatile concept as it is presented in different contexts and therefore also inhibits different properties depending on the situation.

The term variable was originally introduced by Leibnitz (1646–1716) and then denoted symbols that could assume different values in a functional relationship (Philipp, 1992). Since then, the variable has become a term commonly used for all literal symbols in mathematics (though a variable does not have to be represented by a letter). Therefore, the meaning of a variable is now decided by its context (Schoenfeld & Arcavi, 1988), and the variable can have different uses in different contexts. A common distinction in literature is between a variable that is an unknown (e.g., in equations), a general number (e.g., in algebraic expressions), or a variable in a function (Ursini & Trigueros, 2001). However, there are other categorizations, and in some of them, coefficients are mentioned explicitly as a kind of variable – sometimes called a parameter (Partanen & Tolvanen, 2019; Schoenfeld & Arcavi, 1988; Usiskin, 1988).

Several studies have registered how students struggle to perceive the varying meaning of variables in different contexts (Küchemann, 1978; Usiskin, 1988; Ursini & Trigueros, 2001). Students can find it confusing that the same letter (usually x, y or z) can mean different things in different contexts (Rystedt et al., 2016). Studies have shown that the variable as an unknown number, in the setting of solving linear equations, is not the most difficult context for students to accept (Küchemann, 1978). Textbook analyses have shown that the variable as unknown (in equations) is usually well-represented in tasks, which gives students a lot of practise (Ursinin & Trgiueros, 2001; Partanen & Tolvanen, 2019), which might explain why this variable is easier for the students to accept. However, it is not only the mathematical meanings of variables that can confuse students. For example, the variable can be interpreted as a label for something e.g., D meaning Daniel's hight, as an abbreviation for a word, e.g., c being short for cat or corresponding to the letter's position in the alphabet (Küchemann, 1978; MacGregor & Stacey, 2007).

2.2.3 Numbers in equations

Coefficients are to some extent variables – as parameters in equations – but as they often occur in the form of numbers there are additional dimensions to consider of how students perceive them.

Numbers are relevant as a surface feature in equations, as a change of numbers can alter how an equation looks and direct the solver's attention to different aspects of the equation. Linchevski and Livneh (1999) identified that certain number combinations can trigger students to make incorrect assumptions, e.g.,

$50 - 10 + 10 + 10$ is seen as $50 - (3 \cdot 10)$, since students are used to seeing multiplication as repeated addition. They call the numbers that induce certain associations *biasing number combinations*. They mean that this is especially delicate in the transition from arithmetic to algebra, as students often are encouraged to use number combinations and replace them with more efficient ones. "Numbers are loaded entities, entities that children are encouraged to take into consideration, to refer to- not to ignore" (Linchevski and Livneh, 1999, p192).

Further on, numbers are also relevant as a prerequisite in algebra (Bush & Karp, 2013). There are many examples in research literature how students understanding of numbers can be insufficient or incorrect – which presumably affects how those numbers are handled in algebraic expressions. It is well documented that some students confer characteristics of whole numbers to numbers of other domains, called the "whole number bias" (Ni & Zhou, 2005). The similar term "natural number bias" (Alibali & Sidney, 2015; Vamvakoussi, Van Dooren & Verschaffel, 2012) makes it explicit that the bias is for *positive* whole numbers. Some of these biases are specific to the syntax of whole numbers, e.g., the reading rule for whole numbers always has one as the unit, but for decimal fractions, the unit varies and have to be specified (Resnick et al., 1989). This can make it difficult to know how "zero point fourteen" (which is the logic of the Swedish way to say 0.14) relates to the whole number fourteen. Other examples of natural number bias concern values when operating with natural numbers, e.g., that multiplication makes bigger, and division makes smaller – which is neither the case for decimal fractions less than one nor negative numbers (Christou, 2015; Ekenstam & Greger, 1983; Greer, 1987).

A lot of studies on algebra in mathematics education concern reasoning with whole numbers (Stephens et al., 2017). Negative numbers and fractions as coefficients have received little attention in research on algebraic thinking (Kieran, 2022; Vlassis & Demonty, 2022). However, there are exceptions. It has been registered that changing the numbers in an equation to negative numbers can make students that are otherwise able to solve equations, unable to isolate x as it is preceded by a negative coefficient (Vlassis, 2002). The difficulties associated with negative numbers in equations are explained both by the abstract nature of the numbers – how imagining a concrete representation is made more difficult – and by the tendency to detach the minus sign from negative numbers (Vlassis, 2002; Vlassis & Demonty, 2022). Vlassis & Demonty (2022) explicitly showed that algebraic ability concerns how numbers are viewed, as they found that students who considered numbers preceded by a minus sign as "subtractive numbers" also had an increased algebraic performance. Another example is from Hackenberg & Lee (2015) who explore the idea that fractional knowledge and algebraic thinking are related. They suggest that being able to see numbers as compositions with different units (e.g., seeing 35 as five units of seven and as seven units of five) can help students to represent problems of multiplicative character algebraically, not only with natural numbers as multipliers, e.g., $y = 5 \cdot x$, but also with fractions as multipliers, e.g., $x = 1/5 \cdot y$. In their study, they also register students' difficulty in giving a concrete meaning to equations when coefficients are fractions (Hackenberg & Lee, 2015). Another link between students' algebraic ability and their experiences of numbers from other domains is the well explored notion that the variable often is expected by students to represent a natural number, which then impedes their algebraic performance (Christou, 2015; Christou et al., 2022; Christou & Vosniadou, 2012).

Looking into textbooks on algebra, numbers are present in almost every task. It is, therefore, interesting to consider how students should relate to numbers in algebra. Kaput (2008) explains that arithmetic expressions should be viewed in a new way in algebra (in the strand generalized arithmetic), focusing on their forms instead of their values when calculated. In a popular scientific publication, the two reputable researchers Blanton and Kaput (2003, p. 71) use the term "an algebraic use of numbers" when students need to look for patterns in a task and see how number values depend on various factors, and from this generalize. Kieran (2018, p. 101) emphasizes that learning the structure of numbers and numerical operations is vital to (early) algebraic thinking. Concluding this, we can see that numbers are relevant in algebraic thinking as carrying structural properties, both internally – looking at the composition of the numbers – and externally – when numbers are part of an expression or a pattern.

2.2.4 Structure

Equations are usually defined by the relations between the elements, such as when quadratic equations refer to the structure $ax^2 + bx + c = 0$, where $a \neq 0$. In this thesis, students are exposed to similar linear equations, but with varying coefficients, wherefore "structure" is a central concept as it describes what unites the equations. However, looking into different definitions, there are several ways of interpreting the term structure.

The word structure is explained by the Cambridge dictionary (n.d.) as "the way in which the parts of a system or object are arranged or organized". This would imply that the mathematical term structure concerns the arrangement of mathematical objects. Examining the structure concept further, Kieran (2018) presents a rather paradoxical relation between the two concepts generalization and structure. On the one hand, some definitions claim that generalization in arithmetic involves seeing structure. According to Kaput (2008), arithmetic generalizations include seeing general relationships concerning arithmetic operations and their form (i.e., structure). On the other hand, structure is sometimes defined as seeing generality, building on basic arithmetic principles. Mason (2009, p. 10) defines mathematical structure "to mean the identification of general properties which are instantiated in the particular situations as relationships between elements". Kieran (2018) concludes that generalization and structure are two concepts closely intertwined but argues that the term structure does not only concern the most basic arithmetic properties but also includes many other mathematical properties such as prime decomposition, multiples, and powers of ten.

However, mathematical relationships can be noticed at different levels of generality. Venkat et al. (2019, p. 16) explain that seeing local relationships between elements lead to awareness of *emergent structure* while thinking of the domain of applicability. The same relation in different presentations leads to seeing *mathematical structure*. In a way, this clarifies the relation between structure and generality – because there are many mathematical aspects that can be generalized when we look at a particular expression as $4 \cdot x$. Still, we need several objects to see commonality and generalize, e.g., $4 \cdot x$ and $9 \cdot x$ can be seen as emergent structures of the sort "expressions where the coefficients are quadratic numbers" or simply "a discrete number of x:es". An additional object, preferably in a different form of representation, could help to see the mathematical structure that unites the particular objects (Venkat et al., 2019).

In separating the mathematical properties of a structure from specific examples, a student might perceive constraints in the variation allowed. Mason (2003) calls these possibly perceived constraints by a student the "range-of-permissible-change", e.g., Viète (who introduced symbolic algebra – being born in the 16th century – perceived the range of permissible change in roots of equations to only included positive solutions, as it was not common to acknowledge negative numbers in his time (Viète & Witmer, 2006, p. 8).

The aspects of "structure" that are of interest in this thesis refer to similar forms of algebraic expressions and equations, characterized by their mathematical relations. Hoch and Dreyfus (2004, p. 2) formulate this as:

> Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure.

When exemplifying a structure, Hoch and Dreyfus (2004) mention quadratic equations and how transforming them into standard form can reveal a similar internal order. When the equations are not in the same form, the similar structure is still there – the structure is just hidden. They exemplify that the expressions $30x^2 - 28x + 6$ and $(5x - 3)(6x - 2)$ are of similar structure but are interpreted in different ways. This difference in the two ways of interpreting the expressions could also be termed as having "different structures" as they also show two ways of ordering the symbolic expression as either expanded or factorized. However, in this thesis, the term structure refers to an internal order, both concerning the internal order between quantities and operations, and regarding how the symbolic expression is ordered.

Chapter 3 The phenomenographic approach

This chapter will first give a short overview how the phenomenographic research tradition emerged and has developed. Then some basic assumptions within the approach will be outlined, and finally, varying ways to describe conceptions will be elaborated on.

3.1 The phenomenographic research tradition

In the 1960s and 70s, new qualitative research methods emerged in pedagogical research in Sweden, which was otherwise dominated by quantitative methods (Englund, 2004). The research approach of phenomenography was developed from the work of a research group in Gothenburg in the 1970s, called the INOM group (named after the Swedish terms for learning and perception of the outside world, "INlärning och OMvärldsuppfattning") (Åkerlind, 2018; Marton, 2014). The term phenomenography was given by Marton (1981) as he proposed that research that describes people's experiences of phenomena belongs in this research approach.

The methodology was developed in the 1990s. Then, Marton and Booth (1997) made a publication that both elaborated further on phenomenography as methodology, investigating the structure of awareness, but also described learning as a change of awareness. This latter theorization gave a new direction for further research focusing on learning, initially called the "new phenomenography" which at the beginning of the 20th century was called Variation theory (Marton, 2014; Åkerlind, 2018). The variation theory came to embrace the exploration of critical aspects in teaching, e.g., how students' learning outcomes change when different patterns of variation elicit critical aspects of an object of learning (Marton & Booth, 1997; Åkerlind, 2018).

3.2 Basic principles of phenomenography

The only possible way to experience the world is through our senses. It is therefore impossible to separate the world from our experience of the world (Marton, 1981).

However, the aim in phenomenography is – rather than making statements about the world – to explore how it is experienced, by taking the perspective of the people experiencing it. This is called the second-order perspective. The experience of the world is non-dualistic, it is not mental or physical, but consists of an internal relationship between the observer and the phenomenon (Marton, 2015; Marton & Booth, 1997), e.g., the relation between a student and an equation in a textbook. Exploring the nature of these kinds of relations – ways of experiencing – is the focus of research in phenomenography (Marton, 1981).

An individual's experience is constantly changing. Some aspects of a phenomenon are experienced in the foreground, while others are seen in the background (Marton & Booth, 1997; Marton, 2014). However, as a person is aware of several aspects in the background, a change in situation can put other aspects in the foreground and change the way of seeing. Hence, someone's awareness can comprise of different ways of experiencing a phenomenon, but all of these are not activated at the same time (Marton & Booth, 1997).

The study object within a phenomenographic study is the collective awareness, the existing ways of experiencing a phenomenon within a group. A way of experiencing a phenomenon can be resembled to a snapshot of the awareness when focusing on an object (Marton & Booth, 1997). There is an exhaustible number of features to notice, but as we only can hold a limited number of aspects in focus at the same time, variation in the different ways of experiencing the phenomenon can be studied within a group (Marton, 2014; Marton & Booth, 1997). It is the *essential* variations in the varying experiences within a group that are researched in phenomenography – the more individual aspects of the experience, such as taste, etc., are left out. The assumption that there is a limited number of ways of experiencing a phenomenon within a group has also been empirically supported (Marton & Booth, 1997). So, in using a phenomenographic research approach, the goal is to describe the essential differences in the existing ways of experiencing within a group by specifying what aspects of a phenomenon are seen differently. The most common method used to do this is to perform interviews (Marton & Booth, 1997), but there are other possible methods – such as analyzing group work (Cederqvist, 2022).

An example of a study within the phenomenographic scope is a study on how 5-6-year-old children experience numbers (Björklund & Kempe, 2019). The simplest way of experiencing numbers found in the group of children is seeing numbers merely as "words" used randomly without a numerical meaning. A more refined way of experiencing found included seeing order in these words, which developed the words to become "names" of positions in a sequence. Consequently, the different ways of experiencing usually have a logical relation to each other as the aspects possible to discern concern the same phenomenon, e.g., the more refined ways of experiencing numbers include discerning that numbers are called different words. Marton & Booth (1997, p. 125) explain that "the hierarchical structure can be defined in terms of increasing complexity, in which the different ways of experiencing the phenomenon in question can be defined as subsets of the component parts and relationships within more inclusive or complex ways of seeing the phenomenon."

In his publications, Marton uses many different terms synonymously to *ways of experiencing*: ways of understanding, conceptions, and ways of seeing (Marton, 2014; Marton and Booth, 1997) – to mention a few. It might seem confusing to use all these different words to denote the same thing. However, Marton (2014) claims that several terms can be useful as no word is perfect and they direct attention to different aspects of what they all denote.

3.3 Describing ways of experiencing

As the phenomenographic research tradition span over more than four decades, there is variation in the terminology used and how ways of experiencing are described in different studies. This section will first look at how experiences of a phenomenon are described in Marton's earlier publication (e.g., Marton & Booth, 1997) and later (e.g., Marton, 2014). Thereafter, some examples of mathematical phenomenographic studies that use different terminology when presenting their results will be given.

Marton and Booth (1997) describe how a way of experiencing is composed of aspects of a different kind. *Referential aspect* is a term used for the meaning seen in the phenomenon, e.g., numbers can be seen as random "words". There are also *structural aspects* in an experience, these concern both how the internal parts of a phenomenon are related to each other and how these parts are related to the context. Exemplifying with ways of experiencing numbers, structural aspects both concerns how the ordering of numbers relate to seeing cardinality in the numbers, but also how numbers are separated from the context, e.g., how are numbers separated from the situation and the representations (cf. Marton & Booth, 1997)? Referential and structural aspects are usually presented in an outcome space, which is a description of all the different ways of experiencing a phenomenon in a group

(Marton & Booth, 1997). This is one possible terminology to use in phenomenography, but not the only one.

In a later publication Marton (2014) does not mention structural or referential aspects at all but rather prefers the terms *aspects* and *features*. These are terms commonly used in studies concerning variation theory. "Aspects" are dimensions of variation in a phenomenon (e.g., ordering or not ordering numbers), whereas features constitute the values in that dimension (Marton, 2014). When a situation gives the opportunity to discern new aspects in a phenomenon, this is called "opening up a dimension of variation". What aspects are called *critical aspects* are specific to groups (or individuals) and concern the aspects that need to come into focus for students to discern a phenomenon in a more refined way. Marton (2014, p. 24) explains that as long as "the learner has not already made a specific necessary aspect her own, it is a critical aspect for her. It is one thing she has to learn to meet the learning target".

Throughout the phenomenographic tradition we can see a variation in how phenomenographic studies are presented. Even within mathematics education there are several ways of presenting phenomenographic studies (cf. Björklund & Kempe; 2019; Neuman, 1999; Wong et al, 2002). Some studies explicitly position themselves within the phenomenographic tradition (Neuman, 1999), whereas other do not, even though the aim of their studies is within the phenomenographic scope (Wong et al., 2002; Björklund and Kempe, 2019). The choice of terminology also varies, e.g., some studies use the terminology from variation theory (Björklund and Kempe, 2019), and others a "freer" terminology, using terms such as "what" and "how" to describe the way aspects of the phenomenon are perceived (Neuman, 1999).

To conclude this comparison, even though a study is performed within the phenomenographic or variation theoretical tradition, there is the liberty to choose how to present the results and what terms to use. The two studies in this thesis are both within the phenomenographic tradition. Still, the first uses the terminology introduced by Marton & Booth (1997) concerning referential and structural aspects, while the latter uses more of the terminology from variation theory.

4. Method

In this chapter methodological issues, as design, data-collection, analysis, ethical considerations, and limitations will be described and reflected upon.

4.1 Choice of method

In order to learn more about the role of numbers for students when learning to solve equations, in particular in the form of decimal fractions and negative numbers, a phenomenographic approach was chosen as it aims at revealing students' understandings in a qualitative way.

There is no set method in phenomenographic research, even though semi-structured interview is the most common method to collect empirical data (Marton & Booth, 1997; Neuman, 1999). The semi-structured interview has the advantage that the interviewer can control the direction of the interview and lead the subject back to speaking about the phenomenon. There is also the possibility to ask further questions if something needs to be clarified. However, there are other methods that could be used to address the question asked in this thesis. An example would be to film the students while working in groups with the equations (cf. Cederqvist, 2022) and interpret their actions and discussions. This way of observing students in groups has the benefit of putting the students in a more natural environment. In an interview there is the risk of making the students nervous which could possibly change their natural way of experiencing the equations. However, as the research intended to capture students' experiences of equations while solving them on their own, not their collective ability, semi-structured interviews was chosen. With the knowledge that the interview setting might make the students nervous, specific attention was paid to making the situation as natural as possible, with informal chatting and without any stress.

Marton and Booth (1997, pp. 129-130) describe phenomenographic interviews as taking place at two different levels, two ways of relating to the interviewee's awareness. At the first level, the subject responds to a concrete phenomenon, whereas at the second level she reflects on her own experience of the phenomenon. Marton and Booth (1997) describe both levels of an interview as important in learning about ways to experience a phenomenon, but that the second level, the meta-cognitive level, is more difficult to handle as it is a new way of reflecting for the student and requires that the student distinguishes the phenomena from the situation. This part of the interview can be supported by alternative questions or offering interpretations of what the student have said earlier in the interview (Marton & Booth, 1997). In the two studies described in this thesis, the equations presented on paper are in focus, with questions that direct students' attention to the addressed phenomenon. Though, as the role of different numbers is an important part of the studies it has been relevant to ask the students to reflect on their experience and compare equations, in a meta-cognitive way. Both levels of reflection are analyzed as equally important.

In the second article, quantitative methods have been applied, which is not traditionally within the scope of phenomenographic studies. However, phenomenograpic investigation is suitable to use in mixed methods as it does not reject the physical nature of objects (Feldon & Tofel-Grehl, 2018), even though the phenomenograpic approach is focused on the subjects' experience. The quantitative methods were primarily used to motivate further inquiry, seeing how relevant the issue of different kinds of numbers is to a larger group of students.

4.2 Selection

In the first study, six 16-year-old students attaining vocational education, specializing in electricity, in an upper secondary school in Sweden participated (though only five of these were analyzed as one interviewee did not solve either equation). The school class that all six students were a part of was selected for this study as a choice of convenience, as the researcher previously had worked at the school – but not taught this class. As a part of a larger project, the class had participated in a test on learning algebra which revealed that several students had difficulties solving a multiplicative equation containing decimal fractions, despite calculator aid and having just solved a similar whole number equation. The six students were therefore selected for semi-structured interviews to clarify their reasoning or some other aspect of their answers. As the phenomenographic approach was selected as the scientific approach in a later stage, identifying a wide range of possible experiences in the group was not prioritized in the selection process– but rather searching for examples of ways of experiencing multiplicative equations that can be developed further. Of the six students selected, four had managed to solve the whole number equation $42 = 3 \cdot x$, but not the decimal equation $0.12 = 0.4 \cdot x$. The other two selected students had solved both equations on the test.

In the second study, seven different classes of 16-year old's, studying their first year in vocational education participated. All secondary schools with vocational education specializing in electricity in two large cities in the western part of Sweden were contacted through email. Contact was established with three schools, two municipal and one owned by a well-known school company in Sweden. One of the schools was the same as in the first study, but with another class that the researcher had not taught. In phenomenographic studies, the aim is to display variation in ways to see a phenomenon and therefore to have as wide a range of experiences as possible in the data collection (Marton & Booth, 1997, p129). To investigate the extent of the variation in the students' performances, a survey was initially performed with the students. This was used as a selection tool for the interviews. All students present in the schools on the days that the questionnaire was distributed participated (111 students). After the results from the questionnaire were reviewed, 23 students were gradually selected for interviews between one and six weeks later. The selection was guided by the ambition to get as wide a range of different experiences as possible. Phenomenographic studies usually include about 15–30 informants (cf. Larsson, 2010), which should be enough not to miss any ways of experiencing, but also sufficiently few not to greatly exceed the needed number of interviews to cover the variation in the data. This is based on the basic principle in phenomenographic research; that there is a limited number of ways to experience a phenomenon within a group (Marton & Booth, 1997, p32). To find these ways of experiencing there is the possibility to conduct interviews until saturation is reached and no new conceptions present themselves. Therefore, the data collection was concluded after 23 interviews.

4.3 Data collection

This thesis concern two separate data collections that will be reported in this section, one collection for each paper.

The first data collection – which took place at the beginning of 2020 – was initially considered a pilot study but was subsequently decided to constitute a sub-study on its own. The interviews were performed during the classes' scheduled mathematics lessons but in a separate classroom. Each interview had a duration of approximately 10 minutes.

During the interviews in the first data collection, the student received equations on paper, one at a time. The tasks and follow-up questions in the interview guide were developed in cooperation with two senior researchers. The guide consisted of six multiplicative equations and one startup task to assess their ability to use the calculator. In letting the students address the equations under similar conditions, the intention was to probe their understanding of the phenomenon of multiplicative equations on the form $a = b \cdot x$. The students were asked to think out loud concerning the equations and possible solutions, but also to reflect on their experienced degree of difficulty concerning the different equations and to compare them. They had a calculator, pen, and paper as tools during the interview. The students' reflections and the conversation with the researcher were recorded on a mobile phone and after the interviews transferred to a computer.

The second data collection – that took place in 2020–2021 – was considered as a continuation of the first data collection, in search for more information on how students perceive equations with numbers from other number domains, this time with different arithmetical structures (not just multiplicative equations). An initial test was planned to guide the selection of students for interviews. However, as this selection tool was distributed to a large number of students (n=111) it came to be regarded as containing important data that could display the extent of the issue that students solve equations with some kind of numbers, but not others. Therefore, this data was analysed as a part of its own in the second paper.

The questions on the test consist of ten equations with three different kinds of arithmetical structures and coefficients from different number domains; natural numbers, decimal fractions, one with negative numbers, and one without specified coefficients, displayed as $a = b \cdot x$ (the equations are specified in Paper 2). Some equations were borrowed from the first study described in this thesis and from other studies (Vlassis, 2002), whereas others were carefully designed in collaboration with two senior researchers to capture different aspects of the phenomena. Advice in the design of the questionnaire was also sought from a professor in statistics. One of the ten equations was added after the test had been performed in one class to add further information in the rest of the data collection. At the beginning of the test, three simple tasks were used to test students' abilities to use the calculator. The quality of the survey was piloted on three other students prior to the data collection, leading to some alterations. The degree of difficulty was not changed, but several formulations were adjusted.

From the survey, 23 students were selected to participate in interviews. As in the previous study, the interviews were made during the classes' scheduled mathematics lessons – but in separate rooms. The students were offered to use a pencil, paper, and calculator during the interview. The ten equations from the test (and one additional) were used as a base in the interview guide, though without the goal to discuss them all – but rather to choose the ones appropriate to probe the specific student's understanding. Some students developed their thoughts in a rich way addressing only three equations, whereas others approached seven equations during the interview. The interviews lasted between 15 and 40 minutes. Mostly the length of the interviews was decided in relation to whether there seemed to be more to know about their way of seeing. Unfortunately, the length of the interviews was sometimes affected by the length of the class's lessons.

The interviews in the second data collection were recorded by a video camera, capturing the conversation as audio and the students' writings as visual data (except for the last three interviews that only recorded audio as they were performed digitally – due to the pandemic – which meant that it would have been technically more demanding to only capture students' writing). A mobile phone was also used as a audio recording device, as a safety measure. The choice to film the students' equation solving was not obvious, as video cameras might make students nervous. However, a discreet camera was chosen, and the students were told that it was just their writing that was in focus, not themselves. With a video camera, it is easier to follow the conversation as the students make notes. A weakness is that the camera did not capture what the students typed on the calculator. Sometimes the researcher asked the students to describe their activity with the calculator (e.g., "What are you typing on the calculator?") or confirmed loudly what the students had typed, but in a few cases, this activity is not possible to extract from the recordings.

4.4 Analysis

This section will display the analysis of the collected data. First, some general phenomenographic guidelines that have been relevant will be presented, and thereafter concrete examples from the analysis of the first study will be given.

4.4.1 A phenomenographic analysis

As in the phenomenographic tradition, the formation of the outcome space was guided by the variation found in the empirical data (Larsson, 2010). The focus of the analysis was to construct categories of description from the data to reveal critical differences between the categories and not the conceptions in full. Marton and Booth (1997) emphasize that there is no complete description of any experience. The descriptions do not focus on the feelings of the observer like a psychological approach would. Neither is the physical nature of the object considered, as in a study from a first-order perspective. Instead, categories of descriptions intend to capture the variation in subjects' experiences of the phenomenon's character (Marton & Booth, 1997).

Marton & Booth (1997) outline three rules that constitute as criteria for a way of experiencing. The first is that each conception should relate to the phenomenon and show a distinct way of seeing it. The second is that the ways of experiencing must relate to each other in a logical way, which often is hierarchical. The third rule is that the number of categories should be limited and only include categories necessary to display the variation. These rules have been present in both analyses reported on in this thesis.

In a phenomenographic analysis, there are two different traditions of how to approach the collected data (Larsson, 2010). The first approach assumes that one student can display several ways to experience a phenomenon in a single data collection. The empirical material is then considered a *pool of meaning,* where the interviews are considered a great collection of statements. Each statement is situated in two contexts: the context of the collective and the context of the individual interview (Marton & Booth, 1997). By focusing on one dimension of variation in the phenomenon at a time while keeping other aspects frozen and viewing the statements in these two different contexts, there is an openness in the analysis to variation and to taking the second-order perspective. The other approach in a phenomenographic analysis is to consider the whole interview as representing a specific way of experiencing the phenomenon (Larsson, 2010). The analysis is then focused on finding similar conceptions of a phenomenon between individuals. An advantage of this tradition is that the resulting categories are more related to the individual's actual experiences, as it is not divided into different categories. Both studies presented in this thesis have applied the first approach based on the argument that there is a possibility that one person experiences a phenomenon in several ways during an interview. However, the first smaller study did not contain as much data, which made it easier to also take individual experiences into consideration. Therefore, the outcome space of the first study displays ways of experiencing multiplicative *equations* (where the phenomenon is two or more equations of a similar kind but with different numbers) and not ways of experiencing *one equation* independent of the kind of number used as coefficients (as in Paper 2).

4.4.2 Analysis Paper 1

When the material from the first data collection was finished, a sortation of what interview data that was considered relevant started. The interview guide included some questions that the students answered in the interviews but that later was disregarded, e.g., an equation with fractions that was judged as too difficult for most of the students. The analysis was therefore limited to addressing three equations with whole numbers and decimals ($0.12 = 0.4 \cdot x, 42 = 3 \cdot x$ and $0.25 = 5 \cdot x$), and a task where students' skills in performing division with decimals on the calculator were tested ($\frac{0,36}{0,6}$). The focus of the study was narrowed down to focus on how students can experience multiplicative equations with decimal fractions and whole numbers. As one student did not solve either equation with or without decimals, he was excluded from the analysis.

The interviews were transcribed. As a first part of the analysis the entire transcripts were read several times. By reading them repeatedly an idea started to form on three aspects of the phenomena that was discerned in different ways by the interviewed students; the role of the type of numbers used, the solution methods and their aim in handling the phenomenon. The parts of the transcript that concerned these dimensions of variation were marked in different colors to help make connections between different statements. The printed transcripts were also cut into chunks, depending on whether they concerned the equation with whole numbers or the equations with decimals. Two distinctly different ways of experiencing the phenomena emerged in the analysis, which seemed like a not too large number of conceptions for a group of five people.

An example from the analysis process concerns *the role of the type of numbers used*, which was an identified aspect that varied in students' ways of perceiving multiplicative equations. All students that addressed the whole number equation ($42 = 3 \cdot x$) connected division as a solution method to the equation. However, they all approached the equation with two decimal fractions ($0.12 = 0.4 \cdot x$) in another way;

> Student two: x should be three because three times 0.4 is 0.12 […] But I can't describe how I get it.
>
> Student five: Something times it will be 0.12…
>
> Student six: […] I thought it had something to do with these two, like times or division. But I'm not sure.

The two students making the first two statements could explain why they divided one or both sides of the equations with whole numbers for the first equation. Then, as we see in the quotes, as the equations include decimal fractions, they are no longer able to connect division as a solution method and explain why. As the students did not connect the same solution method to the two equations, they experienced them in different ways. The two first quotes are, therefore, examples of when numbers are *discerned as a part of the structure*, which is a part of experiencing equations as *Different kinds of equations* (conception B in Paper 1).

The third quote, on the other hand, instantly links multiplication and division to the equation with decimal fractions. However, an uncertainty is displayed of which of the two operations to apply to the new equation, which was not present when the equation only concerned natural numbers. The type of number used is now *discerned as affecting the solution method,* which is a part of experiencing equations as *Different tasks* (conception C in Paper 1).

As two categories were distinguished in the group, a third was constructed to display the most advanced way of experiencing multiplicative equations as the goal of learning, even though this understanding was not present in the interviews. The categories were then critically reviewed by seeing how the interviews relate to the conceptions presented, to see how well the conceptions display the interviews' all different ways of experiencing the phenomenon. The result was also discussed and revised in consultation with three senior researchers.

4.4.2 Analysis Paper 2

The analysis of the second study was two-fold as the test and the interview data were analyzed separately.

The quantitative data that, in the form of 111 test replies, was first corrected in two different ways. In one correction, the answer was marked as right if the correct answer was found anywhere in the solution, whereas the other correction categorized a solution as correct if the correct answer was marked in a way that could imply that the answer was given intentionally. Later in the analysis process, when the rules of correction could be ruled out as influencing the results, the second correction was adapted and reported on in the paper. One student was excluded from further analysis as he did not show proficiency in using the calculator in the initial "control tasks", as the calculator is an important tool in being able to apply similar solution methods to equations with decimal fractions or negative numbers, as with natural numbers.

Thereafter, solution frequencies were calculated for different equations and confidence intervals (using the formula for a normal distribution with 95% confidence). These values display how competent this group of students are in solving these equations. To receive a measure of how relevant these results are, a significance test was performed, using SPSS. The test used, called McNemar's test, compares paired nominal data. In this case the individual is the unit, and his solution to two different equations (e.g., $819 = 39 \cdot x$ and $0.12 = 0.4 \cdot x$) is the paired data that can have nominal results, either correct (1) or not correct (0). By comparing individuals that solve the first equation but not the other, to students that solve the second equation but not the first, we can evaluate the null hypothesis that the likelihood that a student solves the first equation p_1 is the same as the likelihood that she solves the second equation p_2.

The second part of the quantitative analysis was to gather solutions from students that had solved $819 = 39 \cdot x$ but not a similar equation ($0.12 = 0.4 \cdot x$, $-24 = 6 \cdot x$ or $0.657 = 0.045 \cdot x$). After a first review, a sketch was made of categories that could explain why students did not use the same solution method for the other equation as the equation with natural numbers. The answers were then again categorized and solutions that were hard to interpret were then categorized in consultation with another researcher (see Paper 2). The final number of categories was six:

1. *Natural number bias*, for answers that implied that a negative number or decimal fraction somehow was mistaken for a natural number.
2. *Wrong operation*, for solutions that wrote down or used the wrong operation.
3. *Error in calculation,* where there was an error that implied computational issues.
4. *Correct solution but does not provide the correct answer*, for solutions that gave a correct explanation but had not marked an answer.
5. *Other issues concerning numbers,* a category that did not meet the above criteria, but implied difficulties concerning the numbers in the equations.
6. *The answer does not reveal why,* all solutions that did not give a hint as to why the students only solved the equation with natural numbers, e.g., the student left a blank space.

The qualitative data (all 23 interviews) were initially transcribed verbatim. During the transcription, the researcher was acquainted with the data material, making small notes to get an overview of them. A few of the interviews were selected for initial analysis, where the researcher looked at how different aspects of a linear equation were focused on by different individuals while ignoring other aspects. The goal was to identify critical variation in how certain aspects was experienced. When having reviewed an aspect, such as numerical operations, both in the pool of data and in the individual contexts, a new aspect was chosen as focus and the process was repeated. Two aspects were then considered as containing interesting variation in the data; the aim that the students experienced in the equation, and how the numbers in the equations were considered as specific, belonging to a group of numbers, or in a general way. This categorization guided the coding of the entire data set. The qualitative data analysis tool NVivo was used in this process.

Analyzing the interviews further showed that the students experienced the equation solving as either a concrete transformation or as a theoretical action. This aspect of seeing different meanings in the equation was seen to relate to how the numbers were considered. Reviewing the dataset, this aspect gave explanations as to why the equations of similar structure were experienced in such different ways. Five different ways of experiencing equations were found that display both how students see different meanings in the equation, and how the structure and/or numbers are in the foreground or background of their attention.

Finally, the five ways of experiencing were compared, looking for critical differences between the conceptions – what the students must see in the phenomenon to gain more refined ways of experiencing. In this way, four critical aspects were discerned that can guide teaching to help students evolve in equation solving.

The main analysis was performed by one researcher. However, the analysis has at all stages (from initial analysis to final editing) been supported and discussed thoroughly with three senior researchers. Additionally, the process and results have been presented in several seminars and conferences.

4.5 Ethical considerations

The data collection in these studies has been performed in alignment with the ethical research guidelines from the Swedish research council (2002). All participants were 15 years or older, and they were thereby considered old enough to decide whether they wanted to participate or not. No sensitive data was handled in the studies.

All students, both those answering the questionnaire and those asked to participate in interviews, were made aware that participation was voluntary. A confirmation of this was given as a few students did not want to identify themselves and thereby did not make themselves available for interviews, and a few students declined the request to participate in interviews. As students accepted the invitation to be interviewed, they were also made aware of the possibility to withdraw their participation at any time. The aim of the study, and that the data would be analyzed and compiled in order to create scientific articles – where their names would not be mentioned – was also accounted for.

4.6 Limitations

The idea of phenomenography is to reveal qualitative differences in how people perceive a phenomenon (Marton, 1981; Marton & Booth, 1997). In both studies, there is a knowledge claim regarding how a certain group of students experiences the phenomena in a qualitative way. However, in a phenomenographic study, the purpose of the research is to describe possible ways to perceive a phenomenon in a certain group, but that can exist in other similar groups.

In the first study in this thesis, there is an apparent limitation of having few participants and short interviews. However, the main claim of this article is to be explorative rather than generalizing. In the second study, there is a relatively large (n=111) number of students considered in the selection for interviews, ending up with 23 completed interviews when saturation in the different ways of experiencing was considered reached. As the students in the present group showed a wide range of understandings, it is possible to find qualitatively similar experiences in other groups of students. The research is, therefore, relevant as the phenomena addressed in this study are something that students and teachers work with every year. The results give a qualitative description of how it is possible to experience the phenomena, and educationally critical differences between the understandings, i.e., the critical aspects (Marton & Booth, 1997).

The initial questions posed in Chapter 1.1 speak of "students" as a general term, which implies claims of general nature. This claim refers to students in a similar stage in their education but could also be relevant in lower secondary school when students start to learn to solve equations. The general claim of this thesis, how students are influenced by the nature of numbers when solving equations, is supported by the analysis of the test in Paper 2, as this test has collected information from all 111 students. However, in the quantitative analysis presented

– when using McNemar's test to compare the probabilities for students to solve two equations of similar structure but with different numbers – the number of observations used in the calculations is once again smaller and would have benefited from a larger sample (e.g., 26 students solved $819 = 39 \cdot x$ but not $0.12 = 0.4 \cdot x$, and four students that solved $0.12 = 0.4 \cdot x$ but not $819 = 39 \cdot x$). Another possible limitation regarding the significance testing concerns the case of the multiple-comparison problem (Rice, 2007) – that there is an increased likelihood of finding a significant result when making repeated comparisons with one variable. In Paper 2, repeated comparisons was made as the probabilities of solving multiplicative equations with negative numbers or decimal fractions, were compared to the probability of solving the same equation with natural numbers, $819 = 39 \cdot x$. However, as the results seemed credible, considering earlier research, this was not considered a threat anymore.

5. Summary of papers

5.1 Paper 1: Different ways of experiencing linear equations with a multiplicative structure

This study explores five students' experiences of multiplicative equations, differing only in the type of numbers used as coefficients, such as: $0.12 = 0.4 \cdot x$, $42 = 3 \cdot x$ and $0.25 = 5 \cdot x$. The students had calculator aid as a way of making the calculations equally simple independent on numbers.

The results display that changing coefficients in a multiplicative equation for decimal fractions can make students unable to discern a structure they are usually familiar to handle. This is seen when a student that has solved $42 = 3 \cdot x$ by dividing with three on both sides – explaining "then the three goes away, so it is x on this side" (p.5 in Paper 1), in the next moment is asked if there are similarities with $0.12 = 0.4 \cdot x$, but thinks a while and then concludes "I'm not sure".

Furthermore, the phenomenographic analysis show two ways of experiencing multiplicative equations (see table 5.1, where conception A was not found empirically, but was theoretically constructed). In this study we can see that both the experience of multiplicative equations as an equality to balance (conception B) and as tasks to solve (conception C) can be influenced by a new kind of numbers – but in different ways. In the former case, the new set of numbers makes it difficult to *recognize* that this is an equation of similar structure as the equation with whole numbers, that can be balanced in the same way. In the latter case, the focus is on finding the correct operation to solve the task – even with decimal fractions as coefficients – but the new kind of numbers are seen as possibly *affecting* the solution method.

Despite the small empirical basis – which limits the possibility of making generalizations concerning other students – these findings are relevant to the teaching of algebra. First, it shows that it is not possible to teach equation-solving in whole numbers and expect all students to transfer the solution method by themselves to other numerical contexts. Secondly, it supports research that claims that issues concerning whole number bias are relevant for students learning algebra (cf. Christou & Vosniadou, 2012).

Table 5.1: Outcome space horizontally displaying three ways of experiencing linear equations. In the columns variation in different aspects concerning the experiences are outlined (cf. Marton & Booth, 1997). Table from Holmlund (2022, p. 92).

	Referential aspect	*Structural aspects*	
Ways of experiencing	Multiplicative equations are discerned as…	The relation between multiplication in the equations and division as a solution method.	The role of the type of number used in multiplicative equations.
A. The same kind of equation	… a multiplicative structure independent of type of number.	Discerned as inverse operations.	Discerned as irrelevant for the structure.
B. Different kinds of equations	…equations to balance in different ways depending on type of number.	Discerned as inverse operations, but not next to decimal fractions.	Discerned as a part of the structure.
C. Different tasks	… tasks to find the value of the unknown.	Discerned as operations connected in some way.	Discerned as affecting the solution methods.

5.2 Paper 2: The significance of numbers when learning to solve linear equations

In this paper, students' attention to the nature of numbers while solving linear equations is investigated. The research results are divided in two parts. The first concerns answers, from 110 students in vocational education, to a test with equations. The second part address 23 student interviews.

The quantitative results, concerning a test with ten equations that the students (n=110) performed within 40-50 minutes with a calculator at hand, is displayed in Table 5.2. Reading the table horizontally displays that students' frequencies of correct answers are lower for linear equations with decimal fraction and negative numbers, than with natural numbers. The right-most column displays that the probability of solving an equation with natural numbers (column Equation 1) is significantly different for all structures, compared to the probability of solving equations with other rational numbers (column Equation 2). Further on, solutions were categorized according to putative reasons why a student did not answer the

Table 5.2: Frequencies of correct answers to different items on the test arranged horizontally according to their structure. To the left, p values showing the probability that there is equal probability for students to solve Equation 2 as Equation 1 (Holmlund, n.d.)

Structure	Equation 1 ($a, b, c \in \mathbb{Z}^+$)	Correct (%) (95% CI)	Equation 2 ($a, b, c \in \mathbb{Q}$)	Correct (%) (95% CI)	p [a] value
$\mathbf{a = b \cdot x}$	$819 = 39 \cdot x$	89.1 (83.2–95.0)	$0.12 = 0.4 \cdot x$	69.1 (60.4–77.8)	< 0.001
			$0.657 = 0.045 \cdot x$	78.7 (71.0–86.4)	0.006
			$-24 = 6 \cdot x$	74.5 (66.4–82.7)	< 0.001
$\mathbf{a = b + x}$	$352 = 63 + x$	90.9 (85.5–96.3)	$2.01 = 0.434 + x$	83.6 (76.7–90.6)	0.011
$\mathbf{a = b \cdot x + c}$	$37 = 7 \cdot x + 5$	89.1 (83.2–95.0)	$2.5 = 0.8 \cdot x + 2.1$	52.7 (43.4–62.1)	< 0.001

a. *p* values were determined with McNemar test, binominal distribution and exact significance (1-sided).

same way to equations of similar multiplicative structure (Figure 5.1). These results show that some students tend to treat equations with decimal fraction containing few digits as natural numbers (natural number bias).

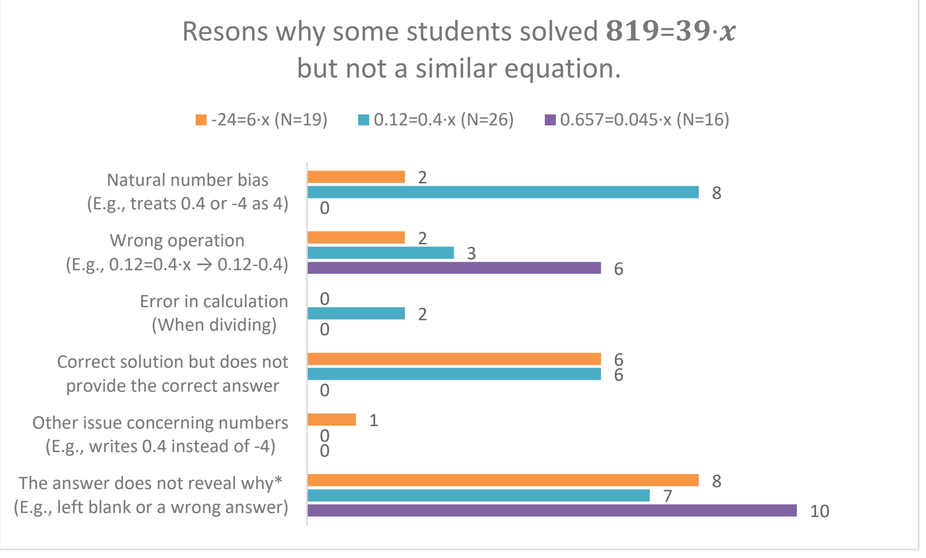


Figure 5.1: Indications given in answers why students solved linear equation with natural numbers, but not a similar one with other numbers. *Some student used division to solve other multiplicative equations on the test (three for $-24 = 6 \cdot x$, three for $0.12 = 0.4 \cdot x$ and four for $0.657 = 0.045 \cdot x$) (Holmlund, n.d).

Further insight as to why students do not treat equations of similar structure – but with different types of numbers – in the same powerful way, was given by the 23 student interviews. Using a phenomenographic approach, five ways of experiencing were found (Table 5.3). These ways of seeing the equations are not individual, but a student can change conception depending on what aspects of an equation that are focused upon. Two conceptions (E and D) describe when students do not see a structure in the equations, whereas conception C–A see structure but in different ways and with varying focus on the numbers. The descriptions outline that when experiencing equations semantically (C), with concrete inner representations, numbers can impede the possibility to recognize familiar equations. An example is given where the equations $-24 = 6 \cdot x$ and $0.12 = 0.4 \cdot x$ does not have a meaning to a student, as division is seen as "fitting a quantity into another", which is not possible if the product is lesser than the given factor. Also, when an equation is experienced syntactically, as a system of symbols, numbers can still be foregrounded and affect how the equation is experienced, e.g., concerning imagined constraints or rules for what numbers are allowed (B).

In comparing the conceptions, four critical aspects were formulated that describe the differences between the categories (Figure 5.2). These aspects can be beneficial to use in teaching to help students to a more refined way of experiencing equations.

Summary

The nature of the numbers in an equation directs students' attention – what they notice and how they interpret the equations – but the numbers are also given varying attention depending on how the equation is experienced and how well the number fits in this experience. This is summarized in a framework of five categories that shows that whether numbers are experienced in foreground or background, and whether the equation is seen as a semantic or syntactic construct, influences how much the nature of the numbers influence the solution process. This framework can be used for research, when analyzing students' algebraic thinking, but also in teaching. The results show the importance for teachers to incorporate how students perceive the numbers in their teaching, and to use a wide range of numbers.

Table 5.3: Displaying five conceptions of how linear equations can be experienced while solving with a decreasing influence from the numbers on the solution process as the structure comes more in focus (from E to A). Figure from Holmlund (n.d.).

The linear equation is experienced as…	Meaning discerned in the structure	Aspect in foreground
A. …a relation between operations and elements	Syntactic meaning	Structure
B. …a relation between numbers	Syntactic meaning	Numbers and structure
C. …a relation that must represent a situation	Semantic meaning	Numbers and structure
D. …numbers to compute	Searches for a meaning	Operation
E. …an unfamiliar equation	Searches for a meaning	Numbers

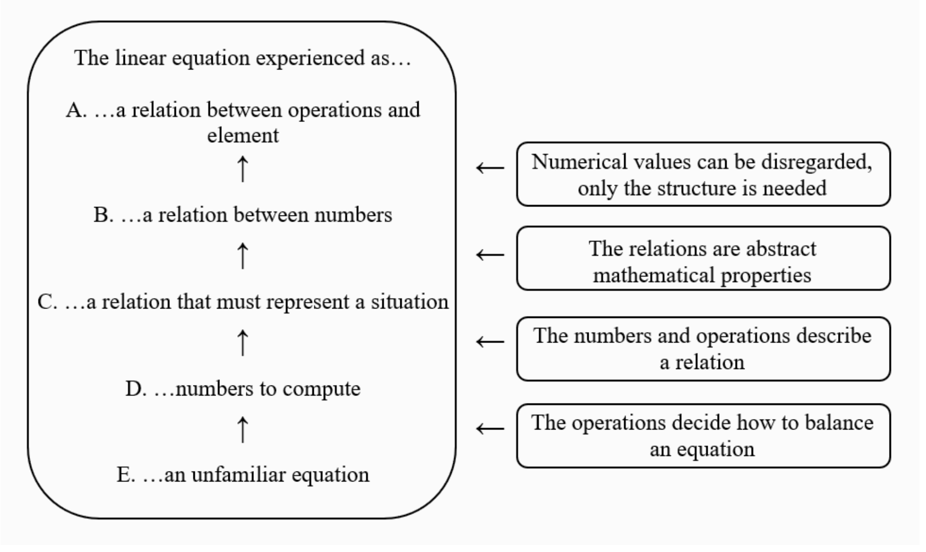


Figure 5.2: Schematic overview of five ways of experiencing linear equations, with critical aspects on the right. Figure from Holmlund (n.d.).

6. Conclusion and outlook

Finally, combining the current papers in this thesis gives a joint contribution to research that addresses the initial research question of how the nature of numbers influences students' conceptions of equations in the solution process. These contributions are summarized in the following bullet points:

- In both Paper 1 and Paper 2 there are examples where students could not see the similarities between a multiplicative equation with whole numbers that they just solved and an equation of similar structure with decimal fractions. Hence, decimal fractions in linear equations can cause what looks like a "temporary loss of previous abilities" (Filloy & Rojano, 1989, p. 21) – when students do not recognize an equation they were previously able to solve. In a somewhat similar way, Vlassis (2002) found that, despite previous equation solving, some students were unable to isolate x when there was a negative coefficient. Our results show that the presence of decimal numbers can be critical to students' equation-solving abilities, even though they have a calculator and are able to solve a corresponding equation with natural numbers.
- Looking at the papers, we can see that the influence of decimal fractions and negative numbers on students' equation-solving is not negligible. It is registered on two occasions and with a large group of students. We can also see from the results in Paper 2 that the influences are of statistical significance. This calls for increased attention to students' experiences of different kinds of numbers in research on learning algebra, which otherwise is scarce (Kieran, 2022; Vlassis & Demonty, 2022).
- The nature of numbers, e.g., if they are large, small, discrete, negative, or positive, can affect how students experience an equation, not only directing (or limiting) students' attention. It can also limit the possibility of experiencing the meaning of the equation in certain ways, e.g., the difficulty to experience the solution process of $-24 = 6 \cdot x$ as fitting 6 into –24 (see Paper 2). This displays the importance of the work that

students do before they apply a strategy – looking for what is relevant in the equation and trying to see meaning in it.

- The studies have provided two outcome spaces (Table 5.1 and Table 5.3) displaying how students perceive equations (*multiplicative equations* in Paper 1 and *a linear equation* in Paper 2). Comparing these, we can see that the most refined ways of seeing (conception A in both tables) corresponds to each other, where numbers do not influence the equation-solving. Conception B in the first study, experiencing multiplicative equations as "different kinds of equations" to balance, can corresponds to several conceptions in the second framework as it describes possible experiences where balancing the equation works for natural numbers, but not for other numbers. Conception C in the first study describes a focus on operations, but without a clear understanding of why it works – which can be compared to conception D in the second study, experiencing an equation as "numbers to compute". Overall, the conceptions found in the second study were formed from several interviews – longer than in the first study – and therefore gives more elaborate explanations on the influence of numbers and are more suited for using in research e.g., as a framework when analysing students ways of experiencing equation-solving or what possibilities to experience equations that are enabled in teaching or textbooks.
- Linear equations with small natural numbers are easily solved for most students in these studies and do not offer enough variation to fully display all dimensions of variation of the structures of equations, such as its abstract character and the relevance of numbers. Whereas equations with negative numbers and decimal fractions can be more difficult to visualize while solving and thereby encourages a more syntactic understanding of the equation-solving process. The presence of these numbers can also challenge students' possible natural number bias, e.g., multiplication makes bigger. In this way, varying the kinds of numbers used in equations – including negative numbers and decimal fractions less than one – opens other dimensions of variation in the structure of linear equations and thereby enables other learning opportunities.

Returning to the statement made by Marton (2014, p. 68) – that in order to prepare students to handle new situations in powerful ways, we need to "help them to learn to see those situations in powerful ways" – the results from this thesis can be a tool in helping students achieve a more powerful way of seeing linear equations. One possibility is to put focus on the critical aspects in Figure 5.2 (in section 5.2) in teaching. This can direct students' attention to a syntactic and algebraic view on equations, and with this – a focus on numbers' form and structure rather than their values (see Kaput, 2008; Kieran, 2018). Also, the five conceptions formulated in Paper 2 can be used to assess students' experiences of equations.

Another possibility for teaching, which is also an outlook at potential further research, is to investigate the learning potential in using different numbers in algebra teaching more systematically. A study with this kind of approach has been made by Zazkis (2001), who displays the learning potential of using large numbers to impede urges to compute and instead focus on structures. This might also hold for negative numbers or very small numbers – which is an interesting area of research. In line with these thoughts, Greer (2006) wonders why second-degree equations like this one, $2.67x^2 - 3.86x - 12.23 = 0$, are rare even though students should be able to solve them with a calculator. The learning potential in these numbers, with many decimals, would also be interesting to examine further. This could be done through designing teaching and measuring students' progress, either quantitively with pre- and posttests, or qualitatively, seeing how handling numbers in equations might be connected to the quality of their algebraic thinking (see Vlassis & Demonty, 2022). Another suggestion for future research is to look at how textbook supports students' understanding of equations as syntactic structures, comparing how theory and problems in textbooks suggest that equation-solving should be justified – relying on mathematical theory or concrete examples.

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