ACTA UNIVERSITATIS GOTHOBURGENSIS
GOTHENBURG STUDIES IN EDUCATIONAL SCIENCES 471

Teaching and learning mathematics with integrated small-group discussions

– A learning study about scaling geometric figures

Jenny Svanteson Wester

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Abstract

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The aim of this thesis is to contribute to a deeper understanding of the relationship between teaching and student learning in mathematics when small-group discussions are used in teaching. This thesis focuses on student learning of specific subject matter when small group discussions are used in whole-class teaching. The data analysed was generated in a learning study about enlarging and reducing two-dimensional geometric figures in Grade 8. The learning study involved four cycles, and five classes and three teachers participated. The data consist of 10 video-recorded lessons and 33 video-recorded small-group discussions. Variation theory was the theoretical framework used to analyse the data. Results show that the use of small-group discussions as a planned and integrated part of whole class teaching can contribute to widening the space of learning and increase students' opportunities to learn what was intended during the lesson. In the study it was found that the small-group discussions solely did not provide sufficient opportunities for students to learn what was intended. Instead, lessons with pre-planned tasks for small-group discussion integrated in whole-class discussions, seems to provide more powerful learning opportunities in relation to what was intended to be compared to lessons with a less systematic use of small-group discussions. In the small-group discussions different ways of experiencing the object of learning were made possible to explore and in the subsequent whole-class discussions those different ways of experiencing were further explored. The results show that teachers benefit from listening to small-group discussions and when students report on such discussions. It was shown that teachers' insights about the students' ways of experiencing the object of learning were vital for

enacting whole-class teaching with small-group discussions in a powerful way. The teachers changed their teaching in response to what they noticed about what could be critical for student learning about the object of learning. The result of this study suggests that it is not a matter of whether small-group discussions should be used or not, but *how* small-group discussions can be used during whole class teaching to support student learning of an intended object of learning in mathematics for the *whole* class.

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världen, som du sa. Vänner är världen. Där har du rätt. Så tack alla vänner. Det är en lyx att ha er. Ni har gjort allt mycket roligare.

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Göteborg i oktober 2022
Jenny

Contents

List of Tables

List of Figures

Chapter 1 Introduction

This thesis is about the relationship between teaching and students' learning when small-group discussions are used in whole-class teaching in mathematics. Classroom discussions are often seen as of the utmost importance in mathematics teaching (e.g. Ball, 1993; Hiebert & Waerne, 1993; Kosko, 2012; McCrone, 2005), but are used for different purposes. For instance, classrooms discussions, both in small groups and in whole class can be used as an opportunity for students to develop social and communicative skills (e.g. McCrone, 2005), to solve mathematical problems together and to explain or critically review their own and other students' reasoning (e.g. Francisco, 2013). However, what students learn about *a mathematical topic* from small-group discussions, or what learning opportunities are constituted when small-group discussions are used in whole-class teaching, and why, is more or less a black box. Several researchers emphasise that how and in under what conditions small-group discussions help promote students' learning in mathematics remains an open question (e.g. Francisco, 2013; McCrone, 2005; Weber et al., 2008). This thesis will provide insights on the use of small-group discussions in a whole-class teaching context, and how this relates to student learning in mathematics and what learning opportunities are offered.

It is well known that classroom discussions can promote students' learning of mathematics (e.g. Hiebert & Wearne, 1993; Kosko, 2012; Stein et al., 2008). Kosko (2012) found, for example, that students who regularly participated in classroom discussions about mathematics enhanced their understanding of mathematical ideas to a greater extent than students who never or hardly ever participated in classroom discussions. Nevertheless, research on small-group discussions have to a large extent focused on students' development of social and communicative capabilities, such as the ability to collaborate, engage or participate in discussions (e.g. Ayalon & Even, 2016; Lampert & Cobb, 2003; McCrone, 2005; Sjöblom, 2022), or students' mathematical reasoning when solving mathematical problems together (e.g. Francisco, 2013; Hunter, 2014; Larsson, 2015; Stein et al., 2008). Questions have also been raised about the organisation and implementation of small-group discussions in relation to how to improve mathematical discussions in the classroom (e.g. Cengiz et al., 2011; Kosko, 2012; McCrone, 2005; Stein et al., 2008). In these studies, the role of the teacher during the classroom discussions, and how this role affects students' learning when classroom discussions are used, have been examined (e.g. Cengiz et al., 2011; Forslund Frykedal & Hammar Chiriac, 2011; Lobato et al., 2005; McCrone, 2005; Webb et al., 2019). Findings indicate that it seems to be a challenge for teachers to lead problem-based mathematics discussions in the classroom (Shaughnessy et al., 2021), and implement classroom discussions that build on students' responses about the topic taught (e.g. Ball, 1993; Bray, 2011; Even, 2005; Forslund Frykedal & Hammar Chiriac, 2011; Mason & Davis, 2013). The challenge seems to be about paying attention to and using students' experiences of the topic taught, and simultaneously being aware of what students should learn about the topic. According to Forslund Frykedal & Hammar Chiriac (2011), this could be a reason that some teachers avoid using small-group discussions as a teaching arrangement.

Other studies argue that having students share their solution strategies will not necessarily generate learning about mathematics (e.g. Ball, 1991, 1993; Clarke, 2001; Mc Crone, 2005; Sfard & Kieran, 2001). In this thesis, students solve problems by the means of small-group discussions and share their solutions with the whole class, but student sharing of solution strategies is not in focus. Instead, the focus is on student learning about specific mathematical content, and learning opportunities constituted when small-group discussions are used in whole-class teaching.

Previous studies have mainly examined student learning in small-group discussions separately, and not as a part of a whole-class setting (Francisco, 2013; Weber et al., 2008; Cengiz et al., 2011). There is less known about students' learning of mathematical content afforded in lessons using small-group discussions as a part of a whole-class teaching, i.e. student learning and learning opportunities constituted from the lesson as a whole. Moreover, there seems to be a lack of knowledge that is directly useful for teachers. How can teaching with small-group discussions be designed to contribute to the enhancement of students' learning about the topic taught? The study presented in this thesis addresses this gap in research by exploring the relationship between teaching and student learning during lessons when small-group discussions are used for teaching about enlarging and reducing two-dimensional geometric figures.

1.1 Relating classroom teaching to student learning

There is an extensive amount of research focusing on student learning processes and learning outcomes. Studies that address the *relationship* between teaching and student learning are more rare (Svensson, 2016). The relationship between teaching and learning is about *what* is made visible in the learning situation, that is, what is focused upon, rather than how or by whom (Nuthall, 2004). In that way, there is a sharper focus on what enables learning, and on the relationship between the handling of the content and student opportunities to learn the same. The decisive factor for learning is what is focused upon in the teaching situation, regardless of the methods or arrangement of teaching (Xu, 2019). Studies that analyse the relationship between teaching and student learning are challenging to conduct, but studies of this kind are highly relevant. Hiebert and Grouws (2007) point out that the first step in making progress in establishing connections between teaching and learning is to understand why these studies are so difficult to carry out. Then there is a need to construct a useful theory of teaching and learning since this can direct the researcher's attention to particular relationships and suggest where to look when formulating the research question and providing a research design.

Furthermore, Hiebert and Grouws (2007) point out that instruments for measuring student learning and describing classroom teaching that are reliable and valid is a continuing methodological challenge. Describing teaching is in many ways more challenging than measuring student learning, partly due to the complexity and partly since it has received relatively little attention. Moreover, it has been suggested that teachers should be seen to a greater extent as co-researchers, and not only as learners in the studies, and should share the focus on improving teaching and establishing connections between teaching and learning (Carlgren, 2018). There is a real need for teachers and researchers to collaborate in the development of knowledge about teaching that is relevant and directly useful in teachers' practice (Carlgren, 2018; Svensson, 2016). These studies address research questions about classroom teaching that are worth exploring (Carlgren, 2018) and explore specific cases of teaching and learning rather than developing decontextualized and broadly generalizable results (Svensson, 2016). The notion of *learning opportunities*[1] has been a useful construct for studying relationships between teaching and learning (e.g. Ekdahl, 2019; Hiebert & Grouws, 2007; Kullberg et al., 2017; Maunula, 2018; Runesson, 2007). Cai et al. (2020) argue that the construct can help researchers and teachers to understand the relationship between teaching and learning and thus work towards maximizing the quality of learning opportunities for each individual student. In this thesis, the construct *learning opportunities* is used in relation to how the object of learning (e.g. Runesson, 1999), i.e. the specific content for the lesson, is handled during the lesson and how this relates to more or less powerful learning. Previous research shows that variation theory is a powerful analytical tool to analyse relationships between teaching and student learning opportunities about an object of learning in mathematics, in terms of how the object of learning is handled during the lesson (e.g. Häggström, 2008; Kullberg et al., 2017). Previously, variation theory has only been used for analysing students' learning opportunities in the whole-class part of lessons in mathematics (e.g. Kullberg et al., 2017; Maunula, 2018; Pang & Marton, 2005). Students' learning opportunities in small-group discussions have only been analysed in an isolated setting i.e. separate from a whole-class setting. For example, Berge and Ingerman (2017) and Ingerman et al. (2009) examined students' opportunities to develop conceptual understanding of force and friction in Newtonian mechanics during small-group discussions.

In this study, an intervention in the form of a learning study (e.g. Lo, Hung & Chik, 2007; Pang & Marton, 2003) conducted in collaboration with teachers makes it possible to empirically investigate the relationship between teaching and students' learning when teaching involves whole-class teaching with elements of planned small-group discussions. The analysis is directed towards how the object of learning is handled during the entire lesson, i.e. in the small-group discussions and in the whole-class part of the lessons as well as how they relate to each other and to the learning opportunities during the lessons.

1.2 Aim and research questions

The overall aim of this thesis is to contribute to a deeper understanding of the relationship between teaching and student learning in mathematics when small-group discussions are a part of whole-class teaching. The following research questions are posed.

[1] 'Opportunities to learn' is used in this study with a similar meaning to 'possibilities to learn' used within variation theory (Kullberg et al., 2017).

- What learning opportunities are constituted when small-group discussions are used in whole-class teaching?
- What is the significance of small-group discussions for student learning about scaling two-dimensional geometric figures?

To be able to answer the research questions, two sub-goals also needed to be resolved. The first sub-goal was to develop a lesson design on the mathematical topic that included small-group discussions in a whole-class setting. The second sub-goal was to find a way of analysing small-group discussions and the teaching in whole-class with variation theory. The thesis will thereby, besides answering the research questions, also contribute to the development of a lesson design including small-group discussions in a whole-class setting, and a way of analysing lessons with small-group discussions using variation theory as a theoretical framework.

1.3 Structure of the thesis

The thesis consists of seven chapters. The first chapter is an introduction including a problem formulation, aim and research questions. This is followed by Chapter 2 which presents a description of the theoretical framework of the thesis. The same chapter also describes learning study – the model used for generating the data. Chapter 3 describes previous research about small-group- and whole-class discussions and the mathematical topic taught that is of particular interest for the study. Chapter 4 is a summary of the author's licentiate thesis (Svanteson Wester, 2014) which serves as a background to the study. The design of the study is found in Chapter 5. Chapter 6 describes the results in terms of summaries of the three articles. In the final chapter, there is a discussion of the results in relation to previous research, and implications for practice and further research are also discussed.

Chapter 2 Variation theory and learning study

The study explores how the aspects of the object of learning were handled, during small-group discussions and during teaching in whole-class, and what learning opportunities were offered. Variation theory asserts that learning means discerning aspects of the object of learning that have not been previously discerned, and that these aspects can only be discerned if they are experienced as dimensions of variation (Marton & Tsui, 2004; Runesson, 2006). The theory enables analysis of the relationship between teaching in terms of how the object of learning is handled and students' opportunities to learn in the classroom (Marton & Tsui, 2004). Studies using variation theory as a framework for analysing students' opportunities to learn from teaching have primarily analysed whole-class discussions (e.g. Maunula, 2018; Kullberg et al., 2017; Pang & Marton, 2005; Runesson, 1999) or small-group discussions isolated from whole-class teaching (e.g. Berge & Ingerman, 2017).

In this thesis, as was outlined in the previous chapter, student learning opportunities for a specific object of learning are analysed in small-group discussions as well as in whole-class discussions and the lesson is thus analysed as a whole. Variation theory provides concepts and tools for analysis and offers the possibility of making detailed descriptions of the relationship between teaching and students' learning opportunities in commensurable terms. The concepts of discernment, object of learning and critical aspects are useful tools for investigating the lesson as a whole. Together, they allow a better understanding of students' learning opportunities regarding enlarging and reducing two-dimensional geometric figures, when small-group and whole-class discussions are used as a teaching arrangement. The students' answers in pre- and post-tests also make it possible to analyse the students' learning in relation to these learning opportunities. The data in this thesis has been generated from a learning study (Pang & Marton, 2003). As mentioned before, a learning study is a systematic cyclic process aimed at improving teaching and learning, and has been shown to promote learning for students, teachers and researchers. With variation theory as a theoretical framework, learning study is a cyclic research arrangement designed to discover aspects related to the object of learning that are critical for students' learning of the object of learning, and design teaching in order to make it possible for students to discern these aspects. In this chapter, the theoretical framework – variation theory – is presented (Marton, 2015). The presentation mainly focuses on the concepts used in the thesis for analysis. This is followed by a presentation of the model used in the intervention – learning study.

2.1 Variation theory

From a variation theory perspective, there is always something to be learned in a lesson: a skill, a phenomenon or a capability, a so-called object of learning. How a student experiences or sees an object of learning depends on what aspects are discerned by the student, whether the student relates the aspects to each other and whether the aspects are discerned simultaneously. Learning is the experiencing of a specific object of learning in a new and more differentiated way (Marton & Tsui, 2004), and is seen as a process of differentiation with regard to an object of learning rather than enrichment or accumulation (Marton, 2015). This view is supported by psychologists like Gibson and Gibson (Gibson & Gibson, 1955), who suggest that there is a constant differentiation of the experienced world. The philosopher Hirst (1974) describes learning in a similar way, in terms of parts of a landscape, where the subject as a whole corresponds to the landscape. Learning then involves an exploration of a landscape where different parts of the landscape gradually become more and more apparent. Learning can be compared to interacting, exploring and focusing on different parts of the landscape, and the learner changes his or her way of seeing or experiencing a phenomenon by discerning new aspects, a relation between aspects or a relation between aspects and the whole (Marton, 2015). Learning can also mean learning to do certain things, i.e., practical skills, by discerning particular aspects of these practical skills, for example, discerning aspects of actions such as being able to do a neat high jump or learning to knead dough. People have a tendency to notice things when they are changed or varied against a fixed background or when something remains unchanged against a changing background. For example, it is easier for us to notice the tomatoes on a green plant when they are red and ripe compared to when they are green and unripe. The tomatoes are discerned against the same background only when they change colour against the green leaves, i.e., they vary their colour in relation to the background, which is invariant. In Svanteson Wester (2014), the students were supposed to discern change in length and change in area when they were scaling two-dimensional geometric figures. A challenge arose because the aspects change in length and change in area co-varied, i.e., it is not possible to keep one of those aspects invariant and vary the other, as they both change at the same time when scaling. It turned out that it was *the relationship* between the two aspects that was decisive for students' ability to master enlarging and reducing two-dimensional geometric figures in a powerful way.

2.1.1 Ways of experiencing

Variation theory is rooted in the research approach called phenomenography (Marton, 1981; Marton & Booth, 1997). Phenomenography is a qualitive empirical research approach that was developed during the 1970s by the INOM group at the University of Gothenburg (Marton et al., 1977), and that aims to explore people's qualitatively different ways of experiencing the same thing. The ontological position in both phenomenography and variation theory is built on a non-dualistic view of the world. The world and the experiencing of the world cannot be separated. Experiencing a phenomenon is constituted in the relationship between the person and the phenomenon. The meaning emerges in the meeting and is a relationship between the person who experiencing and what is experienced. Hence, the human being is seen as an active meaning maker in relation to the world around her (Marton & Booth, 1997).

Results from phenomenographic research form the basis for the development of variation theory (Marton & Booth, 1997). The theory was developed from a need, not only to be able to describe different ways of experiencing (or seeing) a phenomenon, but also to be able to plan for and explain how teaching can enable a development towards more complex experiencing of a phenomenon (Marton, 2015; Runesson & Kullberg, 2017). Marton and Booth (1997) argued for the possibility of giving a theoretical description of learning from the phenomenographic perspective. The qualitatively different ways of experiencing a phenomenon are described by Marton and Booth (1997) in terms of a number of qualitatively different categories of description. These are often organized in an outcome space describing the variation in ways of experiencing the phenomenon. The differences between different ways of experiencing a phenomenon are critically important for teaching and learning. An example of students' different ways of experiencing the concept of scaling arose in Svanteson Wester (2014). When students were asked, in a pre-test, to enlarge a triangle by scale factor 4:1, some students interpreted the scale factor as though it meant that one triangle was going to become four triangles. Other students understood the scale factor as though it was the area of the triangle which was going to be four times bigger (see Figure 1)

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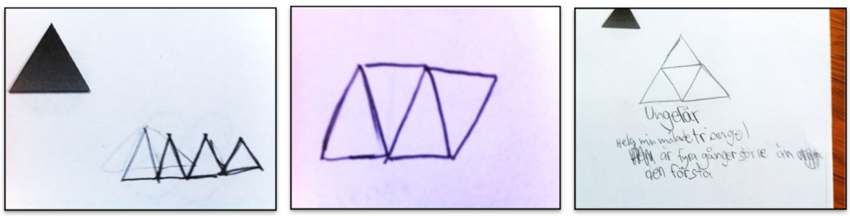

Figure 1. Students' answers in the pre-test when enlarging a triangle by scale factor 4:1.

Only a few students in the pre-test understood the scale factor as meaning that the triangle's sides were going to be four times longer. The purpose of the Svanteson Wester (2014) study, using variation theory as a guiding principle, was to explore the relationship between teaching and students' learning regarding enlarging and reducing of two-dimensional geometric figures. The results showed that certain aspects proved to be critical for the students to experiencing the concept of scaling when enlarging and reducing two-dimensional geometric figures, such as discerning length and change in length in the figures, and relating this to similar figures.

According to Marton (2015), learning is seen as a process in which the learner changes his or her way of experiencing a phenomenon by discerning new aspects or by discerning a relation between aspects or a relation between aspects and the whole. In Svanteson Wester (2014), the results showed that it was crucial for the students to discern the relationship between change in length and change in area when scaling two-dimensional geometric figures. A 'way of seeing' something refers to experiencing a phenomenon in a specific way, by discerning certain aspects of it simultaneously. In Svanteson Wester (2014), a simultaneous discernment of change in length and change in area arose as being crucial during teaching, since both of the aspects change when scaling the figure, but in different ways. Learning is a matter of a change in the qualitative way a phenomenon is experienced (Marton & Booth, 1997). Accordingly, students' learning difficulties in relation to specific content can be explained and specified in terms of aspects of that content that they have yet to discern. The difference between learning and not learning is thereby explained as discerning, or not discerning, certain aspects or relations between certain aspects of that which is to be learned.

2.1.2 Using variation theory to analyse the relationship between teaching and student learning opportunities

Variation theory provides tools to explore what it takes to learn a certain object of learning, and has been used in several studies to analyse classroom teaching (e.g. Ekdahl, 2019; Häggström, 2008; Kullberg, 2010; Marton & Tsui, 2004; Runesson, 1999) with a focus on what is made possible to learn and what is actually learned, based on how a specific content has been dealt with. The theory has also been used when analysing how a lesson design can be developed and improved to enable student learning of a specific object of learning (e.g. Runesson & Kullberg, 2017; Ryberg, 2020).

For a student to learn, according to Marton (2015), the student must direct his or her attention towards the object of learning in the lesson. The term critical aspect is used for aspects that are seen as critical for learning the object of learning in an intended way (Marton & Tsui, 2004), and learning is due to seeing the object of learning in a new way by discerning certain aspects that have not previously been discerned and discerning them simultaneously. The critical aspects are related to the individual student, and the same ones will not be critical for all students in the class. The object of learning is therefore dynamic and can change in the process of learning (Pang & Ki, 2016), since new critical aspects for the students can emerge during teaching (Runesson, 2007). When teachers examine the object of learning together with the students, the teachers are able to see the object of learning based on what the students communicate about their experiencing of it, and thus develop a deeper understanding of how students can experiencing what they are supposed to learn, but also, to use this knowledge to develop a more powerful way of teaching (Cheung & Wong, 2014; Huang & Shimizu, 2016). According to variation theory, discernment of critical aspects occurs during systematic interaction between the student and what is to be learnt, and variation is the agent that generates such interaction (Leung 2012). Thus, variation is a necessary condition for learning to take place and those aspects of the object of learning which are critical for understanding the object of learning in an intended way, need to be varied in order to be discernible by the student. Variation theory therefore makes it possible to address research questions concerning students' learning opportunities in relation to what was intended to be learnt, the object of learning constituted during the lesson and differences in students' learning opportunities.

A critical aspect of an object of learning that is enacted by dimensions of variation and invariance can be used as an analytical tool to compare learning opportunities for the object of learning. What is made possible to learn in the teaching can thus be understood by studying which critical aspects of the object of learning are simultaneously in focus and whether these are opened as dimensions of variation (Marton & Tsui, 2004; Marton, 2015). For example, if any two-dimensional geometric figure is to be enlarged (or reduced) in a lesson, different shapes of two-dimensional geometric figures are in the foreground and explored. A critical aspect to discern when scaling two-dimensional geometric figures could be to discern the lengths in different shapes (e.g. a rectangle, a triangle and a circle) of figures. The lengths in a two-dimensional geometric figure can be opened up as a dimension of variation, such as height, width, perimeter, radius, diameter and diagonal. Thus, the distinction between dimensionality and directionality may be crucial when discerning lengths, e.g. the perimeter of a rectangle is one-dimensional, but has two directions in the form of length and width, and also encloses the area of the figure.

There are several different ways of identifying potential critical aspects. According to Lo (2012), critical aspects are to be found from previous research, from teachers' experiencing of their previous teaching about the actual object of learning and from screening interviews and pre-tests where students express their experiencing of ways of seeing the object of learning. Even during teaching, additional critical aspects can be identified as the opportunity arises to listen carefully to the students' ways of experiencing the object of learning. Some critical aspects do not emerge through collaborative discussions between teachers around the object of learning before teaching, nor during pre-tests or interviews with students, but are only revealed when students interact with the object of learning during the lesson. Pang and Ki (2016) argue that critical aspects are relational in nature, in that they relate to qualitatively different ways of experiencing the same phenomenon. They cannot easily be derived from the content itself. The teacher should strive to consider students' different ways of seeing the object of learning being taught. To be aware of students' different ways of seeing the object of learning seems to be important when designing teaching. Teachers can arrange for individual differences to become apparent by focusing on students' ways of seeing the object of learning during the lesson and taking these as a starting point when teaching (Lo, 2012), which can contribute to an expanded common ground in the lesson and further enhance learning opportunities (Al-Murani & Watson, 2009).

One of the most specific tenets of variation theory is that seeing differences must precede seeing sameness (Marton & Pang, 2013) and this has certain implications. The opportunity to discern and focus on critical aspects is created according to a structured and well-planned pattern of variation. The patterns of variation are contrast, generalization and fusion. To structure a teaching sequence, Marton (2015) argues that the lesson should begin with a form of fusion. In this initial fusion, which Marton (2015) calls the undifferentiated whole, it is not possible for the students to discern critical aspects of what is to be learned, but rather it involves an experiencing of the undivided whole. However, contrast is needed in order for students to discern critical aspects. For instance, to understand the concept of a linear relationship when enlarging and reducing two-dimensional geometric figures, the students need to discern how it differs from a non-linear relationship: a linear relationship must be compared to a non-linear relationship to have a meaning of its own. To understand the change in length, when scaling two-dimensional geometric figures in relation to a given scale factor, the students also need to discern the change in area and how it differs from the change in length. Contrast can also be seen when a teacher enacts patterns of variation related to the students' previous knowledge or experiencing. Complementary to contrast is generalization. For instance, to understand the idea of a linear relationship when enlarging and reducing two-dimensional geometric figures, the student must be able to perceive sameness, such as the same aspect or set of aspects (e.g. change in length and change in area) within different geometric figures or within different scale factors. After having, in a way, taken the whole apart, you have to put it back together again. This final fusion is possible as the focus is on the simultaneous variation of two or more aspects. Different critical aspects of the object of learning vary simultaneously and are brought to the attention of the students at the same time; this provides an opportunity for the students to perceive the relationship between the aspects and thus the object of learning is changing as a whole. Lo (2012) argues that there must be a whole to which the parts belong, before the parts can be understood. We simply cannot learn more details without knowing what they are details of.

What students learn and understand in a learning situation depends on their ability to understand the enactment of patterns of variation. If the teacher can consciously systematize the presentation of the patterns of variation, making it easier for the students to discern the critical aspects and link them to the whole, it contributes to the students' learning more effectively (Marton & Tsui, 2004). Taking the critical aspects into account is, in turn, critical to enabling students to develop the intended way of experiencing the object of learning. As mentioned before, teachers should, according to Pang and Ki (2016), not simply consider the pattern of variation in those aspects that may be derived from examining the subject matter itself. They should also explore the aspects that the students focus upon. Pang and Ki (2016) pointed out that students' alternative ways of experiencing are often very difficult to understand and articulate, and that a teacher may be able to see what a student does incorrectly, but fail to see why, for example, that students may have unstable and fragmentary views of the object of learning. As a result, in some cases, according to Marton and Pang (2013), the teacher may miss opening up a dimension of variation that would have allowed the student to discern critical aspects. It is also possible that the students themselves will open up dimensions of variation that are critical to understanding the object of learning in the intended way.

Variation theory is to a great extent a theory about the object of learning. Describing how the object of learning is handled during the lessons implies describing a space of dimensions of variation that is opened up in the classroom interaction. This space of opened dimensions of variation is a space of learning and creates both constraints and opportunities to learn (Runesson, 2005). To study teaching from a variation theory perspective, as Runesson (1999, 2005) states, is to study teaching in terms of a potentially experienced space of variation and invariance, where students' and teachers' consciousness is directed to a specific object of learning. This space of variation is constituted differently depending on how the aspects of the object of learning are highlighted and problematized, and which dimensions of variation are opened. Thus, the nature of the space of variation that is constituted in the teaching can be considered critical for students' learning. Runesson (1999) points out that whichever teaching method is used, a range of variation is generated. Aspects are elaborated upon by opening various dimensions of variation, forming a space of variation. Based on the space of variation, a wider or a narrower space of learning can be constituted (Marton & Tsui, 2004; Runesson, 2005). A rich situation for learning is then when students, during the lesson, are offered an opportunity to discern critical aspects and relationships between these aspects and the whole. To do this, a lesson design with a strategic use of variation related to the critical aspects is needed. In order to learn something specific, certain specific critical aspects have to be discerned and students need an opportunity to experiencing a systematic variation in relation to these aspects (Leung, 2012; Runesson, 2005). The variation can be opened up by the teacher or by the students during the lesson (Maunula, 2018), and it can also be jointly constituted between teacher and students in whole-class discussions during teaching, as in Svanteson Wester (2014). A mathematics teaching that is rooted in variation is one that purposefully provides students with an opportunity to experiencing variation through, for example, carefully designed tasks, in order to create a rich opportunity for learning (Leung, 2012).

2.1.3 The dynamic object of learning and student learning opportunities

The object of learning is central to variation theory and also includes three analytical perspectives: the intended, the enacted and the lived object of learning (Häggström, 2008; Marton & Tsui, 2004; Al-Murani & Watson, 2009; Lo, 2012). The intended object of learning is defined by the teacher and describes the teacher's perspective on what the teacher wants the students to learn, i.e., the learning intentions with the lesson, and can be something that the teacher has reflected on to a greater or lesser extent, in relation to the content and the students' ways of thinking. The enacted object of learning is what the students meet in the actual teaching and describes the observer's perspective on the object of learning that emerges in the teaching situation in the classroom. It is a result of an analysis of how the object of learning is handled during the lesson. According to Runesson (2005), when the teacher and the students, the students together in small groups, or the students individually interact with the object of learning in the classroom, a space of variation and invariance is opened. This space of variation results in a space of learning that enables students to discern certain critical aspects. In other words, the space of variation, which is jointly constituted, can be described as an enacted object of learning. The enacted object may be seen as a potential for learning. It is the teacher and the students together who constitute the enacted object of learning. The aspects of an intended object of learning that are enacted in the lesson define the space of learning and constitute the limits for what is made possible to learn (Marton & Tsui, 2004). The lived object of learning describes what the students actually learnt from the teaching and can be identified through, for example, the use of pre- and post-tests.

The process of articulation, i.e., the variation generated by the students during the lesson, moves the students' experiencing of the intended object of learning from the private domain into the public domain, contributing to the potential development of the enacted object of learning by making it discernible for the students in the classroom. The enacted object of learning during teaching is influenced not only by the teacher's intended object of learning but also by what the students communicate about their experiencing of the intended object of learning. Depending on which aspects of the intended object of learning are explored in the lesson, different learning opportunities emerge. Teachers who address students' ways of seeing the intended object of learning and explore aspects related to the intended object of learning (new aspects or students' confusions about existing aspects) with a systematic variation, expand the shared common ground in the classroom (Al-Murani & Watson, 2009).

When a dimension of variation of an aspect has been opened up and systematically pursued in a whole-class discussion, a process of systematic variation has been enacted in the classroom, called an 'exchange systematicity' by Al-Murani and Watson (2009), i.e., a systematic exchange of variation by means of a dialogue or a discussion in the classroom. Al-Murani and Watson (2009) claim that attention to a systematic exchange of variation gives insights into how learning opportunities are jointly developed in public in the co-construction of the enacted object of learning. Teachers are able to shape the variation exchange so that through the process of co-constructing the enacted object of learning, the chances of the intended object of learning corresponding with the lived object of learning increase (Al-Murani & Watson, 2009; Pang & Ki, 2016). The way in which teachers and students handle the intended object of learning, in regard to which aspects students' awareness is directed towards and whether dimensions of variation are opened, appears to be crucial for students' learning opportunities during the lesson (Runesson, 1999). Lo (2012) acknowledges the dynamic character of the enacted object of learning, and argues for its unpredictability. By focusing on how the object of learning is handled in whole-class discussions and small-group discussions, and by acknowledging that students and teacher together constitute the learning opportunities in a lesson, this thesis draws on Al-Murani and Watson's (2009) conclusion about student learning opportunities and systematic exchange of variation in critical aspects.

This thesis pays attention to students' learning opportunities in relation to an intended object of learning, not only in the public part of the lesson but also learning opportunities when students in small groups, without any support from the teacher, jointly explore the intended object of learning. The research interest is directed towards the lesson as a whole and therefore the analysis focuses on the object of learning as it is enacted both in whole-class discussions and in groups.

As stated previously, studies using variation theory as their theoretical framework have primarily analysed the enacted object of learning in whole-class teaching (e.g. Kullberg et al., 2017) or enacted objects of learning in student small-group discussions as an isolated setting (Berge & Ingerman, 2017; Ingerman et al., 2009). In order to gain knowledge about learning opportunities in both whole-class and small-group discussions, the aspects of the object of learning that are made possible to discern in the different discussions have been identified.

2.2 Learning study

The data for this thesis has been generated from a learning study (Pang & Marton, 2003; Marton & Runesson, 2015). Learning study is a systematic process of inquiry which involves planning, implementing, evaluating and analysing a research lesson (Cheng & Lo. 2013). Learning study provides a model to examine the linkage of theory and practice through an investigation of how teaching designed by theory could improve student learning, and how improved teaching practices inform the development of theory (Lo & Marton, 2012). The aim of a learning study is to find out why certain objects of learning are difficult to teach and to explore ways for improving teaching and student learning of those objects of learning (Kullberg et al., 2019). Learning study has recently been used for more complex objects of learning (Pang & Runesson, 2019). The learning study reported on in this thesis, premised on variation theory, explores a complex object of learning: enlarging and reducing two-dimensional geometric figures and overcoming the illusion of linearity, a topic that several studies have shown is a challenging topic for students to learn (e.g. Ayan & Bostan, 2018; De Bock et al., 1998; Fernández et al., 2014) and for teachers to teach about (Svanteson Wester, 2014).

Learning study was developed in Hong Kong and Sweden, almost twenty years ago, inspired by the Japanese lesson study (Fernandez & Yoshida, 2004; Lewis, 2002; Stiegler & Hiebert, 1999) and design experiment (Brown, 1992; Cobb et al., 2003), framed by a learning theory (Marton, 2015). Initially, the purpose of the learning study was to test variation theory (Marton & Booth, 1997; Marton & Tsui, 2004) and how it can be used as a guiding principle in teaching to enhance student learning. Since then, learning study has increasingly developed into a research approach. With variation theory as a theoretical framework, learning study can be used as a systematic cyclic process and provides a platform to help teachers to develop an innovative learning environment that promotes student learning. Moreover, the learning study approach also helps to create a professional learning community that will support teacher learning within school (Cheng & Lo, 2013; Lo & Marton, 2012). Each learning study can be regarded as a case study (e.g. Lo et al., 2007; Lo et al., 2005) in which a team of teachers and a researcher plan, examine and refine a lesson collaboratively several times in an iterative process in order to develop a powerful lesson design and enhance student learning opportunities (Kullberg et al., 2019; Lo & Marton, 2012). The collaboration can create a sustainable and shareable knowledge-base that can support teachers in implementing learning opportunities in the classroom (Cai et al., 2017a; 2017c) for example, Kullberg et al., (2019) suggest that learning study with variation theory as a framework can offer teachers mechanisms to create such public knowledge.

There can be several reasons for choosing to use learning study as a research approach in a study. One reason can be that learning study provides an opportunity to investigate the relationship between teaching and student learning regarding a specific object of learning. Another reason can be that learning study can generate knowledge about the specific object of learning in the form of refined theoretical descriptions of the specific object of learning, i.e., how it is constituted in terms of critical aspects (Kullberg et al., 2019). Learning study aims to provide insights that can be used both as theoretical descriptions and practical guidance to improve teaching and students' learning (Carlgren et al., 2017).

The strength of learning study lies in the systematic analysis of learning in relation to a particular content, resulting in the establishment of knowledge in connection to problems raised in teachers' practice (e.g. Cai et al., 2017b) and the development of the design of lessons (e.g. Carlgren, 2012; Carlgren et al., 2017). Teachers play a crucial role in bringing the intended object of learning to life in the classroom and learning study provides a mechanism through which the intended, enacted and lived object of learning can be brought closer together (Huang & Shimizu, 2016). Based on theoretically grounded teaching and a cyclical research design, this study investigates the relationship between teaching and student learning of a particular mathematical content during lessons when small-group and whole-class discussions are used as teaching arrangements.

2.2.1 The learning study cycle

A learning study is carried out in an iterative process with cycles with an initial object of learning as the point of departure for the inquiry process (e.g. Cheng & Lo, 2013; Marton & Runesson, 2015; Pang & Marton, 2003). The different steps shown in Figure 2 are activities that help the teacher team and the researcher to define the intended object of learning. Three steps of variation guide the inquiry process. The first, variation in the students' way of seeing the object of learning (V1), the second, the variation in the teachers' own ways of seeing and handle this object of learning in the past (V2) and the third, variation as a guiding principle of pedagogical design (V3). During the process, the teacher team and the researcher together explore the teaching about an intended object of learning that is challenging to teach or hard for students to learn. The process gives the teacher team and the researcher an opportunity to explore the object of learning and what aspects might be critical for students to learn the object of learning. In order to explore and develop teaching of the intended object of learning, the teacher team and the researcher plan, implement, analyse and revise the lesson, guided by variation theory (Cheng & Lo, 2013). The plan is rather detailed, especially concerning what aspects of the object of learning should be made possible to discern. In the first cycle, the lesson design is planned based on critical aspects identified from teachers' previous experiencing and previous research results, and critical aspects identified by means of students' pre-tests. The aim with the students' pre-tests is to capture what the students in the particular class are struggling with in relation to what they are supposed to learn during the lesson. The planned lesson design is implemented in one of the classes by one of the participating teachers, while the other teachers and the researcher observe the lesson and/or video record it. The lesson is then analysed and revised jointly by the teacher team and the researcher, based on lesson observations, video recording of the lesson and students' post-tests. The aim with the students' post-tests is to capture and describe what the students have actually learnt, i.e. the lived object of learning, and what students still have to learn. The results of the post-test in combination with observations and video recordings are the basis for the next revision of the lesson design for the next class. The lesson is analysed with a focus on how the object of learning was handled during the lesson. The revised lesson design is usually carried out by another teacher in the teacher team.

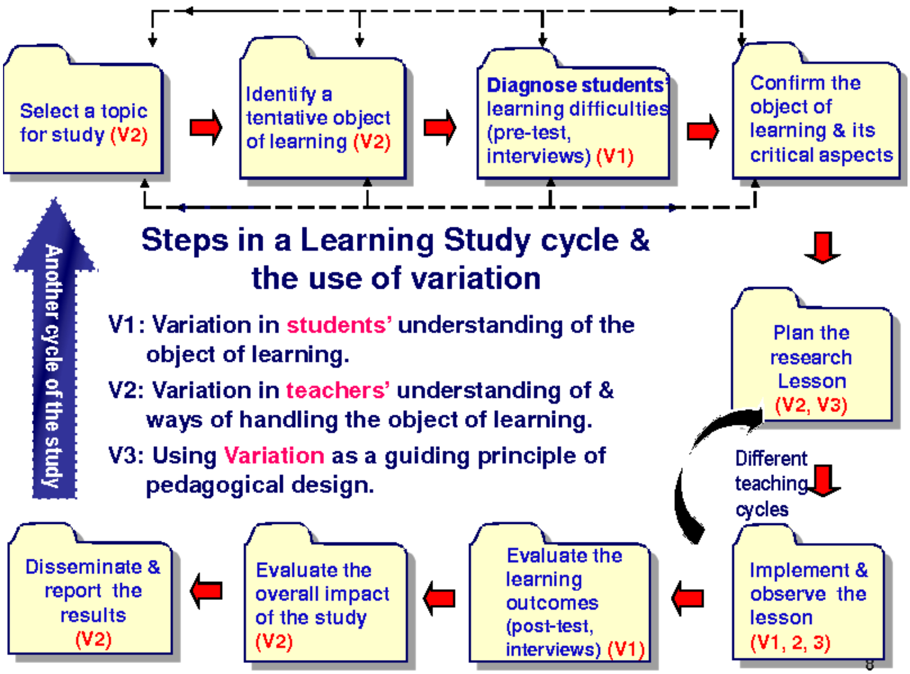

Figure 2. Steps in a learning study and the use of variation (Cheng & Lo, 2013).

The process of analysing and revising the teaching is then repeated with each new class according to the cyclical process. Usually, at least three cycles with one or two lessons are conducted in a learning study. After that, a written document is produced by the teacher team and the researcher, describing the aim, process, and results of the learning study. The results can be used, tested and developed by other teachers and researchers in other contexts (Lo et al., 2005; Morris & Hiebert, 2011; Pang & Marton, 2003).

Having an explicit learning theory, such as variation theory, adds value to the learning study process, as the theory serves as a source of guiding principles when engaging in pedagogical design, lesson analysis and evaluation. The theory makes it possible to explain the relationship between what has taken place in the classroom and what the student learns, and to identify ways to improve student learning further, in a learning study setting (Cheng & Lo, 2013; Pang & Lo, 2012).

Chapter 3 Review of Literature

The chapter begins with a description of the mathematical topic being taught, linear and non-linear relationships in proportionality and geometry, and a review of research on teaching and learning of the topic. This is followed by a review of research on whole-class and small-group discussions in mathematics teaching and learning.

3.1 The mathematics topic being taught – linear and non-linear relationships with respect to proportionality and the "illusion of linearity"

Proportionality is seen as a cornerstone in a wide variety of areas. In the field of mathematics, it occurs in many contexts and can thus be seen as a central mathematical principle (Lamon, 2007). A common way of presenting proportionality in mathematics is by pointing to two main categories of proportional problems (Miyakawa & Winslöw, 2009; Lamon, 2007). One category consists of a finite number of pairs of values with a constant ratio which can be expressed as $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots = \frac{a_n}{b_n}$. The second category involves proportionality between two variables and defines a relationship, called a linear relation, $y = k \cdot x$, where k is the proportionality constant and is non-zero. Lamon (2007) also pointed out an exponential proportionality, a non-linear relationship such as $y = k \cdot x^2$or, $y = k \cdot x^3$where k is the proportionality constant and is non-zero.

In this thesis, the mathematical topic, enlarging and reducing two-dimensional geometric figures and handling the scale factor correctly, is in focus, and this involves linear and non-linear relationships. A brief overview is given of the difference between proportional and non-proportional relationships in relation to linear and non-linear relationships (see Table 1).

	Linear relationships	**Non-linear relationships**
Proportional relationships	$y = k \cdot x$ Relationships whose graph is a straight line through the origin. For the study, the linear relationship in question is length scale.	
Exponential proportional relationships		$y = k \cdot x^2$ $y = k \cdot x^3$ Relationships whose graph is a curve through origin. For the study, the relationship in question is the square relation.
Non-proportional relationships	$y = k \cdot x + m$ Relationships that do not cut the origin.	$y = k \cdot x^2 + m$ $y = k \cdot x^3 + m$ Relationships that do not cut the origin.

Table 1. Overview of proportional and non-proportional relationships related to the object of learning for the study, enlarging and reducing two-dimensional geometric figures (marked with grey).

3.1.1 A proportional image and the scale factor

The concept of scale in mathematics has to do with proportional imaging, and is an example of a proportional relationship. Scale, which is taught in connection with maps and construction, is synonymous with the length scale, and shows the linear relationship between the lengths of two objects (Bentley, 2008). If scaling refers to an enlargement of an object in reality, it is written by specifying the number of times the magnification is done. It is often used in problems of similarity in geometry, and means that the image is such that the distance between two arbitrary points is multiplied by a constant. Bentley (2008) expresses the definition of scale as: “If the scaling is seen as an enlargement of an object, it could be described as the number of times the object is enlarged”. (Bentley, 2008, p. 42, my translation.). However, Bentley’s definition can be problematic when it comes to two- and three-dimensional geometric figures, i.e. how “object” can be interpreted in the expression “the number of times the object is enlarged”. The definition can be understood as being unclear, as 'object' can be interpreted as referring to all geometric objects in one, two or three dimensions, or only geometric objects in one dimension, i.e. distances. Scales usually relate to maps and distances but can also be used when enlarging and reducing figures with two or three dimensions. If a scale on a map or drawing is 1: 1000, it means that 1 mm on the map corresponds to 1000 mm in reality, i.e. reality has been reduced 1000 times. This way of reducing reality applies to all parts of the map and is therefore proportional. For example, in a drawing where the image is an enlargement by, for example, the scale factor 2:1, this means that all lengths have been made twice as long, but not the area of the drawing. The same relationship applies to the entire drawing, and the scale thus expresses a proportional relationship. If two triangles are similar and one of them is obtained by enlarging the other by the scale factor 2:1, it means that the ratio of corresponding sides in the triangles is the same (see Figure 3). The lengths of the distances in the larger triangle are twice as long as the corresponding distances in the smaller triangle.

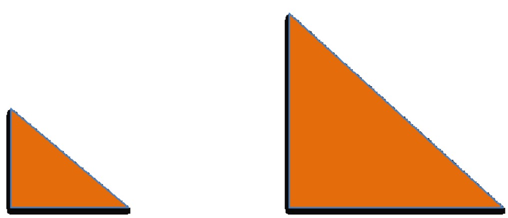

Figure 3. Similar triangles.

If the relationships between distances within the figure are maintained in the case of an enlargement or a reduction of the figure, this describes a proportionality. A proportionality can also be described in terms of the relationships between the corresponding distances in the two figures, i.e. that if one side is twice as long as another in the original figure, the same will be true of the corresponding sides in the enlarged or reduced figure. This ratio also indicates the length scale (Bentley & Bentley, 2011). The ratio within a figure or between figures are two ways of seeing at enlarged and reduced figures. Figure 4 illustrates how these two ways can be described in relation to the focus of this study.

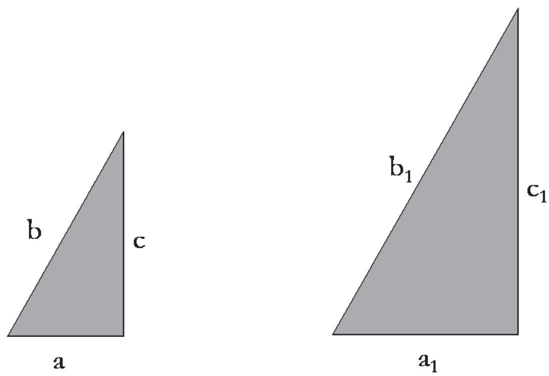

Figure 4. Triangles which together with the formulas below illustrate within and between proportionalities.

A proportionality within the figure, a within ratio, which in this thesis is related to proportional imaging, or similarity, and means that all conditions within the figures are maintained and can be described as follows:

$$\frac{a}{b} = \frac{a_1}{b_1}$$

$$\frac{a}{c} = \frac{a_1}{c_1}$$

The proportionality between figures, a between ratio, can be seen as a linear relationship, i.e. the ratio is the same between the figures, which in this thesis can be related to the handling of the scale concept, where k is the proportionality constant and constitutes the scale factor (Lamon, 2007; Vergnaud, 1988).

$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$$

and

$$a_1 = k \cdot a$$
$$b_1 = k \cdot b$$
$$c_1 = k \cdot c$$

A relationship between two quantities in the same frame or figure is a within ratio, e.g. the length-to-width ratio of a rectangle. For all similar rectangles, the corresponding "within ratio" is the same. A "between ratio" is a ratio of two corresponding quantities in different figures. The relationship between the length of a side in one rectangle and the corresponding length of a side in another similar rectangle (enlarged or reduced) is seen as a between ratio and relates to the scale factor. The relationship within each rectangle's sides relates to the concept of similarity, when enlarging or reducing rectangles.

3.2 Teaching and learning linear and non-linear relationships in geometry

There is a comprehensive body of research relating to students' difficulties in understanding and separating linear and non-linear relationships in mathematics. The research includes students in different age groups and refers to different topics in mathematics such as arithmetic word problems and geometry (e.g. De Bock et al., 1998; De Bock et al., 2002; Van Dooren et al., 2004; Hilton et al., 2013; Ayan & Bostan, 2018). A well-known problem in geometry teaching is that if a geometric figure is enlarged x times, students express that the figure's area and/or volume become x times larger too.

Ayan and Bostan (2018) investigated students' solution strategies for non-linear proportional problems regarding length, area, and volume of enlarged figures. Ayan and Bostan detected that a significant majority of students' answers were influenced by the obstacle of linearity, which was related to the idea that if the sides in a geometric figure were enlarged x times, the area and volume were enlarged x times too. The students implied that they needed to multiply the area and volume by 2 when all sides were enlarged by scale factor 2. One of the problems that students, aged 10-16, were working with in Hilton et al. (2013) was about a butterfly, and what happened with the area when the length and width were doubled. Fewer than 10% of the students, regardless of age, were able to answer the question correctly. There was also no significant difference in results between the younger and older students. Between 60-78% of the participating students responded that the area also becomes twice as large, as the lengths are twice as long. The results in both Ayan and Bostan (2018) and Hilton et al. (2013) indicated that a majority of the students could not separate the linear relationships from the non-linear relationships between two uniform figures. In Svanteson

Wester (2014), a similar result was noticed in a pre-test. Only a few students could separate change in length and change in area when scaling a two-dimensional figure, regardless the shape of figure.

Research shows that students have a strong tendency to use linear relationships uncritically, i.e. they tend to overgeneralise what they have experiencing as "true" in linear relationships, and apply this to non-linear relationships as well. This incorrect use of linearity in non-linear situations is often referred to as the "illusion of linearity". There is an extensive body of research on the phenomenon of the illusion of linearity in geometry and this research indicates that the illusion of linearity is widespread among students of many different ages and in several areas of mathematics. Just as the concept of linearity has many forms, its incorrect use also has many forms (e.g. De Bock et al., 1998; De Bock et al., 2002; Fernández et al., 2009; Gagatsis et al., 2009; Paic-Antunovic & Vlahovic-Stetic, 2011; Ayan & Bostan, 2018). Fernández et al. (2009) argue that since students often have only limited access to the strategies needed to enlarge and reduce figures, and to interpret scaled drawings, there are many possibilities for misunderstandings and the incorrect use of linearity. Van Dooren et al. (2004) point out that the acquisition of knowledge is hindered by the fact that students' previous knowledge may not be compatible with what is going to be learnt and that may contribute to misunderstandings. Some researchers argue that students' lack of knowledge in geometry may be one of the reasons why students have difficulty distinguishing linear relationships from non-linear relationships (De Bock et al., 2002; Gagatsis et al., 2006; Gagatsis et al., 2009). In order to acquire the ability to reason about proportional relationships in this area of mathematics, enlarging and reducing of two-dimensional geometric figures, these researchers argue, students need to have good geometry skills (e.g. the ability to discern different parts and properties of geometric figures, or to recognise the shift from measurement of one-dimensional units to the measurement of area). In geometry, constructing similar figures provides a visual representation of proportions and proportional thinking, and supports the understanding of similarity. A study by Gagatsis et al. (2006) shows that children do not always initially recognise parts and properties of a geometric figure they know the name of. Younger children may understand a figure as a whole and not as a sum of its parts and identify figures from their overall appearance. This is in line with Svanteson Wester (2014), where results indicated that the students had difficulties putting into words why an enlarged figure is similar or not. The students in the study articulated that the picture was enlarged but could not describe the mathematics behind this. One student answered the question about why an enlarged picture was similar by saying "It is the same. It is the same shape" (Svanteson Wester, 2014, p. 110). Gagatsis et al. (2006) studied younger students' strategies for designing enlarged two-dimensional geometric shapes in a growing series and found some useful implications for teaching geometry. The results of the study indicated that students' ability to identify geometric figures is not directly related to their ability to construct and transform geometric figures. Based on these results, Gagatsis et al. (2006) suggest that teaching aimed at developing the students' overall geometric knowledge needs to support both of these abilities, by providing not only activities where recognition of geometric figures is in focus, but also activities where the construction and transformation of figures is in the foreground, i.e. students should both recognise the figures and be able to change them, e.g. to do a proportional enlargement or a reduction. This points to the importance of letting the students do drawings or constructions of geometric figures when enlarging and reducing two-dimensional figures.

The following example from Van Dooren et al., (2004) shows it can be challenging for students to separate length and area in a two-dimensional geometric figure, as well as to discern and separate linear and non-linear relationships within and between geometric figures. One student raises her hand at the end of the lesson and asks the following question: "I really do understand now why the area of a square increases 9 times if the sides are tripled in length, since the enlargement of the area goes in two dimensions. But suddenly I start to wonder why this does **not** hold for the perimeter. The perimeter also increases in two directions, doesn't it?" (Van Dooren et al., 2004, p. 496). The fact that students struggle with the meaning of lengths, and with separating change in length from change in area when enlarging and reducing two-dimensional geometric figures was also found in Svanteson Wester (2014). As in the example of Van Dooren et al., (2004) the students in Svanteson Wester's study express their ways of experiencing the mathematical topic, which in Svanteson Wester proved to be crucial to increasing students' learning, as by hearing about the students' experiencing, the teachers got the opportunity to refine their strategies for how to handle the content and thereby develop the teaching. Students appeared to have difficulties in separating the change in length and change in area, but also in discerning the relationship between them. It seems to be important to address the epistemological obstacle of linearity, i.e. what is difficult, and also the illusion it generates, which Modestou et al., (2008) suggest can only be dealt with by means of a well-thought-out pedagogical teaching situation. Modestou et al., (2008) conducted a study in which they created situations where the spontaneous and uncritical use of the linear model could be questioned by the students. The situations, according to Modestou et al., (2008), were created in such a way that contestation of linearity arises spontaneously as a necessary tool in the solution of the problem. Also, Modestou et al., (2004) suggests that a more systematic didactic intervention regarding the teaching about non-linear relationships in geometric objects should be investigated and further developed.

In Modestou et al.'s (2004) study, 12- to 13-year-old Cypriot students worked with pictures of two-dimensional geometric figures. The students' results were slightly better after the intervention, but the problem remained because the students found it difficult to discern the common non-linear property of two- and three-dimensional figures, and therefore they handled the situation differently from the mathematically correct way. This is in line with Bentley's study from 2008. Bentley showed that it is challenging for teachers to get students to see the mathematics behind the concept being taught. In order for students to be able to develop their understanding of concepts effectively, Bentley emphasises, teachers need to know how students in the class experiencing the topic taught, and then use these experiencing during teaching in order to develop teaching and increase student learning. The students get the opportunity to interpret and evaluate both their own and others' thoughts, solutions and answers, which, according to Bentley, develops the students' own thinking and their understanding of the mathematics. Several studies have shown how classrooms discussions (in whole class, in small groups and in pairs), when students explain and justify solutions, contribute to students making connections between mathematical features which supports their learning in mathematics (Hiebert et al., 1997; Cengiz et al., 2011; Francisco, 2013).

3.2.1 Proportional reasoning

To be able to reason about proportional relationships is fundamental in the understanding of mathematical concepts and can thus be said to constitute a framework for further studies in mathematics. Comparisons between relationships is an important category of tasks for students in order to develop their understanding of proportionality (Lamon, 2007). Proportionality is to a large extent a part of everyday life, such as reading scaled maps, duplicating recipes, figuring out the "best deal", and sharing equally (Dole, 2010). Authentic contexts which are familiar to students in everyday life are important for the development of the ability to reason about proportional relationships. It is estimated that more than half of all adults have not had the opportunity to develop their understanding of proportionality, which means that it is unlikely that we acquire the ability to reason about proportional relationships by just getting older without being taught (Lamon, 2007). Lamon (2007) points to some features of reasoning about proportional relationships. She suggests that a person who has the ability to reason about proportionality understands relationships where two quantities change at the same time and can see how the change in one corresponds to the change in the other. Both in Dole et al. (2008) and in Lobato et al. (2011), the researchers were interested in the teachers' understanding of the concept of proportional relationships and how this was embodied in, and related to, the teachers' classroom practice. In Dole et al. (2008) the results show that the teachers were helped by a structured programme to train themselves to reflect on the students' answers. Lobato et al. (2011) show that teachers had a lack of deep understanding of proportionality and that they use mainly procedural calculation, for example, cross-multiplication. It is reasonable to imagine that teachers themselves need to become aware of their strategies and what they take for granted regarding students' understanding of the concept of proportionality when planning their teaching. Lo (2012) suggests that what is not so easily discerned by teachers is also usually what becomes the biggest barrier to student learning. In addition, it seems difficult for teachers to identify the aspects that will be challenging for students if the teachers do not themselves have trouble discerning these aspects, i.e. teachers find it difficult to see what could be an obstacle to students' learning. If teachers are unaware that they are ignoring these aspects, Lo (2012) writes, "it will result in a knowledge gap in the lesson that they may not notice" (p. 28).

The illusion of linearity is a recurring phenomenon that appears to be universal and resistant to various forms of support aimed at overcoming the phenomenon (e.g. De Bock et al., 2002). In this thesis, variation theory together with the learning study approach (Marton & Tsui, 2004; Marton, 2015) provide tools to examine and analyse teaching with small-group and whole-class discussions, and students' previous knowledge and experiencing of the topic taught, to enhance students' learning.

3.3 Teaching and students' learning in mathematics lessons when small-group discussions are used

A central issue in mathematic teaching is the role that discussions should play and can play in the mathematics classroom (e.g. Sfard, 2001). A mathematical discussion is a purposeful talk on a mathematical content with input from at least one of the students who assists in moving the discussion forward (Pirie & Schwarzenberger, 1988). The discussion is not about who has the right idea, but how seeds in each student's idea can contribute to the group's or class's understanding of the topic. Carpenter et al. (2003) maintain that students cannot learn mathematics with understanding and without participating in a discussion.

How discussions in whole class, in small groups or in pairs can contribute to students' learning and understanding in mathematics is an ongoing debate (e.g. Ball, 1993; Francisco, 2013; Lampert 1990; McCrone, 2005; Stein et al., 2008; Weber et al., 2008). Research studies have described the benefits of whole-class and small-group discussions in mathematics and results indicate that there is strong scientific support for the benefits of students' learning in mathematics during classroom discussions compared to traditional teaching methods, such as transmission of knowledge from teacher to students, or individual work, particularly at secondary school level (e.g. Carpenter et al., 2003; Francisco, 2013; Hoyles, 1985; Imm & Stylianon, 2012). However, studies indicate that whole-class and small-group discussions seem to have two functions related to students' learning: helping the students learn to develop social skills or mathematical communication skills through working together or serving as a means through which they can obtain new content knowledge (Forslund Frykedal & Hammar Chiriac, 2012).

The research in this thesis focused on student learning of a specific content in mathematics in a whole-class setting by the means of small-group discussions and on how the content is handled in those discussions and during the whole-class teaching. Studies investigating students' opportunities to learn mathematics in collaborative activities, such as whole-class, small-group and pair discussions, have primarily focused on generic skills, for example students' development of solutions, strategies and explanations during problem solving, and students' ability to express and validate solutions and mathematical ideas (e.g. Ryve et al., 2013; Webb, 2009; Yackel et al., 1991) or how to get students engaged in the discussions (e.g. Ayalon & Even, 2016; Cengiz et al., 2011; Henning et al., 2012; Hunter, 2014; Sjöblom, 2022). Other studies show that whole-class and small-group discussions can extend students' mathematical thinking about mathematical ideas (e.g. Ball, 1993; Cenzig et al., 2011; Francisco, 2013; Lampert, 1990; McCrone, 2005; Stein et al., 2008; Webb, 2009). When students discuss and elaborate on mathematical concepts, an opportunity to notice relationships between mathematical concepts and procedures may occur which can contribute to a deeper understanding of the mathematics (e.g. Stigler & Hiebert, 1999). Developing a deeper understanding is dependent on particular pedagogical approaches, purposefully focused on clarification of concepts and producing consensus in the classroom community, such as in whole-class or small-group discussions.

Gaining insights about different ways of experiencing or thinking about mathematical ideas, and reflecting on, elaborating on and clarifying those, is seen as fundamental for moving learning forward. Students who reflect on what they do with the mathematics when they solve problems, who jointly explore the content in small-group discussions or who give help to others in the classroom are in a better position to notice useful connections between mathematical ideas (Hiebert et al. 1997), compared to those who take part only in non-exploratory discussions or give/receive help without exploring the content, for example help that just involves directions on what steps to take (Webb, 1989, 1991). Students who discuss errors and other students' ways of seeing, and explore why a mathematical idea does or does not make sense to them, also seem to develop new mathematical insights (Bray, 2011).

Whole-class and small-group discussions, as a teaching strategy, help students to develop a deeper understanding of mathematical key concepts, and to develop the ability to be more explicit in expressing their mathematical thinking (e.g. Francisco, 2013). Through whole-class and small-group discussions, students are given an opportunity to articulate their own mathematical ideas and listen to those of others, and to compare their own ideas to those of others, thereby becoming aware of similarities and differences which may enhance their learning (e.g. Francisco, 2013; Hoyles, 1985; Mc Crone, 2005). However, having students discuss mathematical ideas in small groups does not guarantee that meaningful learning will occur. Students may discuss and compare different solutions without really discussing the mathematics involved (Ball, 1991; Kazemi & Stipek, 2001; McCrone, 2005). Hiebert & Wearne (1993) suggest that when analysing the value of a whole-class discussion or a small-group discussion in relation to student learning in mathematics, one must consider the depth and quality of mathematical ideas being discussed. To what extent do students elaborate on the mathematics? Sharing ideas does not necessarily generate a deeper understanding in mathematics.

However, few studies investigate student learning of a specific content in mathematics, nor the relation between students' learning opportunities and what was intended to be learnt during the lesson with small-group discussions. In this thesis, small-group discussions are related to the role of discussions as a means for the students to gain new content knowledge. The focus is on student learning and student learning opportunities with regard to a specific mathematical content, offered during lessons when both small-group and whole-class discussions are used. According to Cengiz et al. (2011), there appears to be a limited amount of research that examines the lesson as a whole, i.e. students' learning opportunities in whole-class teaching when small-group discussions are used during the lesson. Studies have primarily examined student learning in whole-class discussions and small-group discussions separately (Francisco, 2013; Weber et al., 2008). Though, in Yackel et al. (1991) lessons with small-group discussions with subsequent whole-class discussions are examined, but the focus is on the construction of class-rooms norms for cooperation and student learning opportunities during mathematical activities. Cengiz et al. (2011) and Francisco (2013) emphasise that there is a need to gain insight into teaching and the effect on students' learning about a specific content when both whole-class and small-group discussions are used in a lesson. In this thesis, how students, through whole-class or small-group discussions, develop their general abilities to solve mathematical problems, and to reason and communicate mathematically, are not examined. The focus is instead on the students' learning of a specific content in mathematics when small-group discussions is used during whole-class teaching.

3.3.1 Eliciting students' experiencing of the content as a part of classrooms discussions

Empirical studies since the 1970s have stressed the importance of letting students put their mathematical ideas into words, in order to increase the opportunity to develop their mathematical thinking (e.g. Francisco, 2013; Mercer, 2008). Another purpose of letting students reveal and discuss their ideas about the mathematical content is that when the teacher is listening to the students, teachers can gain insight into students' experiencing and misunderstandings about the topic taught (Ball, 1993; Even, 2005; Mason & Davis, 2013; Shaughnessy, 1977). Mason (2000) points out that in order for teachers to be able to ask relevant questions during teaching to enhance student learning, it is advantageous if the teachers have knowledge of students' experiencing about the topic taught. Staples (2007) emphasises that revealing student misunderstandings is not simply about errors being accepted, in the sense that everybody makes mistakes, but rather that examining errors or ideas that are not fully mathematically correct can contribute to fundamental learning opportunities for the whole class, by conjecturing, justifying and reconciling these errors or ideas by means of classroom discussions. Identifying the key conceptual points and students' common ways of seeing the topic taught, and then designing the teaching to focus on these points and other ways of seeing presents students with substantial challenges by exposing those other ways of seeing and exploring and resolving them through whole-class and small-group discussions. This exploration of students' initial experiencing can lead to productive results related to students' opportunities to learn (Ball, 1993; Bell, 1993; Staples, 2007; Bray, 2011).

In Bell (1993), results show the ineffectiveness of individual work in relation to student learning and the greater effectiveness with regard to student learning of the teaching strategy of using students' experiencing of the topic taught. In this teaching strategy, students' correct answers, incorrect answers and different ways of seeing the topic taught were used, and conflicts between them were investigated through small-group discussions with subsequent whole-class discussions. Incorrect answers about the topic taught in mathematics are, according to Schwarzenberger (1984), as important and significant as correct answers. In some cases, they are more significant. An incorrect answer yields more information about what the student is thinking about the topic than does a correct answer. Maunula (2018) investigated students' experiencing about the topic being taught, in terms of students' contribution in a whole-class setting during teaching in mathematics. Maunula (2018) showed that teachers' attention to students' experiencing of the topic taught, and how these experiencing was used in whole-class settings, were crucial in regard to the quality of learning opportunities constituted during the lesson. In Svanteson Wester (2014), this teaching strategy, i.e. where students' experiencing of the object of learning came to the fore and were explored during the lesson, seemed to contribute to increasing students' opportunities to learn what was intended. Several other studies also demonstrate that teachers' understanding of students' experiencing of what is going to be learnt can inform their instructional decision-making (e.g. Ball, 1993; Black et al., 2003; Even & Gottlib, 2011; Mason & Davis, 2013). It has been shown that it is mainly students with difficulties in school mathematics who have benefited when this teaching strategy (where students put their mathematical thinking into words and teachers use it in their teaching) is used. But there are also difficulties with this teaching strategy. In a study by McClain (2002), students' initial experiencing was used as a starting point for discussions in the classroom. The results show that the teacher did not take into account the extent of students' different ways of reasoning about the mathematical ideas. The teacher found it difficult to distance herself from her own way of reasoning and consider students' different ways of experiencing the mathematical ideas or their current understanding of the topic taught.

3.3.2 Teachers' role in classroom discussions during mathematics teaching

In several studies, teachers' role and actions have proved to be important in supporting students in classroom discussions (e.g. Ayalon & Even, 2016; Cengiz et al., 2011; McCrone, 2005; Shaughnessy et al., 2021; Webb, 2009, 2019). Cengiz et al., (2011) showed that it is of importance that teachers catch students' thinking (eliciting actions), extend students' thinking (extending actions) and support students' thinking (supporting actions) in order to expand students' understanding of mathematical ideas. The teacher can do this by evaluating a statement, comparing statements, and suggesting other methods of thinking and solving the problem. To support students' thinking, the teacher can steer the discussion toward reflecting on what is being discussed, remind the students of what they already know and relate it to new information, and keep the focus on what is being discussed. Elbers & Streefland (2000) studied mathematics lessons where the teacher's way of capturing and visualising students' mathematical ideas in whole class was in focus. Elbers & Streefland (2000) suggest that the teachers, by asking the students questions, help the students to focus and create opportunities to make the students' knowledge or mathematical ideas visible. The teacher explains which of the students' ideas need to be discussed in whole class and/or in smaller groups. This strategy creates a common context for the presentation of knowledge for the whole class. According to Hunter (2014), teachers' actions will gradually change student participation in discussions about mathematical ideas in the classroom. The teacher guides the students in conducting jointly exploratory discussions in small groups or with whole class and it seems to be, as Elbers (2003) has emphasised, crucial for the teachers to enact an atmosphere of mutual trust in which students discuss together without fear of failure, that students' own ideas are given space, and that these ideas are evaluated by the group.

However, according to Ball (1993), there is a dilemma inherent when using whole-class and small-group discussions to advance individual learning, while at the same time keeping an eye on where the class needs to be heading, i.e. what mathematics is going be learnt during the lesson. It seems to be a big challenge for teachers to highlight important mathematical ideas and relationships between the ideas when using small-group discussions (Ball, 1993; McCrone, 2005) and enact whole-class discussions based on students' experiencing of the topic taught (Boaler & Humphrey, 2005; Lampert et al., 2010; Stein et al., 2008). The teacher's knowledge or lack of knowledge about the students' current ways of seeing the topic taught (the students' prior knowledge) affects the development of the discussions in the classroom (Löwing, 2004).

The teacher's competence seems to be crucial in many ways for conducting classroom discussions about a mathematical idea. Findings in Bray (2011) showed that teachers' ability to respond to students' ways of seeing (correct and incorrect) the topic taught during whole-class discussions in ways that promote understanding of mathematical concepts was highly related to teachers' knowledge of relevant mathematical concepts, knowledge of students' ways of seeing the topic taught, knowledge of teaching strategies to support development of understanding of particular mathematical concepts, and ability to use mathematical knowledge to interpret students' work in the moment during whole-class discussions. According to Bray (2011), teachers particularly need support with envisioning how students' errors can be productively used as springboards for inquiring in the content of whole-class and small-group discussions. Another dimension of teacher competence is how it influences the students. Students seem to copy the teacher's way of asking questions of students and way of helping students. When the teacher is unable to develop answers given by the students, the students' ability to develop their mathematical thinking is thus limited.

Chapter 4 Licentiate study

The research in this thesis is based on previous findings in a licentiate study by Svanteson Wester (2014), in which a learning study in mathematics was analysed. The findings from the learning study, which was conducted in Grade 8, revealed critical aspects of students' learning about the object of learning, enlarging and reducing of two-dimensional geometric figures, and in what way differences in how the object of learning was handled during the lessons affected student leaning. For critical aspects to be explored and explicitly discerned, findings showed that it was important that the students' ways of seeing the object of learning came to the fore during the lesson. Students' ways of handling the object of learning in a small-group discussion gave the teachers insights about how the students experiencing the object of learning. Those insights turned out to be crucial for the refining of the lesson design, and for the whole class's learning about the object of learning. This chapter provides a description of the results from this learning study.

The rationale for exploring student learning about this specific content was the teacher team's own insights from previous teaching about scaling of two-dimensional geometric figures. The teacher team presupposed that if students were to be able to enlarge and reduce two-dimensional geometric figures, the students needed to discern both change in length and change in area, i.e. the students would have to separate length from area when handling scaling of two-dimensional geometric figures. This insight emanates from the teachers' prior teaching about the topic. The teachers noticed that students seemed to focus on the area when they worked with two-dimensional figures. An example is a sequence during one of the teacher's ordinary teaching about enlarging and reducing two-dimensional figures in Grade 8. By the means of a group task, students in the class were supposed to discuss an enlargement of a paperclip and decide the scale factor of the enlargement. The students got a paperclip and an enlargement of the paperclip on a piece of paper. The students did not know the length of the paperclip, nor did they have access to a ruler. One group of students answered that the paperclip had been enlarged three times in relation to the length and that the scale factor was 3:1. Another group of students answered that the paperclip had been enlarged nine times and that the scale factor was 9:1. The students in this group clearly described how they are experiencing this enlargement. The students illustrated that the paperclip fitted in exactly nine times on the enlarged paper clip and that the scale factor should be 9:1 During this sequence, the students opened up a dimension of variation regarding the scale factor, and the need arose to separate change in length and change in area when scaling two-dimensional geometric figures.

Based on the phenomenon "the illusion of linearity" (see Chapter 3), identified both by the participating teachers in previous teaching, and in previous research (e.g. De Bock et al., 1998; De Bock et al., 2002; Ayan & Bostan, 2018), the aim in Svanteson Wester (2014) was to investigate the relationship between teaching and students' learning regarding enlarging and reducing of two-dimensional geometric figures with a focus on: what the students needed to discern, how the content was handled during the lessons, the development of the lesson design, and how the teaching affected student learning. Learning study, guided by variation theory (Marton & Tsui, 2004), was used as a research approach, and, through its iterative process, provided an opportunity to generate empirical material that could provide a deeper understanding of the relationship between teaching and learning. The focus in the analysis was on the different patterns of variation that were enacted during the lessons and linking these patterns to the students' learning and the way in which the learning differed based on these different patterns of variation. The study was intended to answer the following research questions:

- What does the student need to discern in order to see both the linear (change in length) and non-linear (change in area) relationships when scaling two-dimensional geometric figures and, on the basis of this, handle the scale factor correctly?
- How does the handling of the content affect student learning?
- How does students' understanding change based on differences in how the content is handled during the teaching?

4.1 The learning study about scaling two-dimensional geometric figures

The study was conducted over a period of three months in March-May 2013 and consisted of three cycles with one class involved in each cycle. Three teachers and 45 students in total participated in the study. The three teachers who participated in the study had participated in learning studies before and usually use variation theory in their daily work. All students participated in a pre-test, two research lessons, a post-test and a delayed post-test. Each cycle included two research lessons, i.e. a total of six lessons. Each lesson was approximately 60 minutes long and was videotaped in its entirety and transcribed verbatim. As well as the pre- and post-tests, and delayed post-test (see Appendix 1), each cycle included meetings where the researcher, together with the teacher team, analysed how the content was handled during the lessons and examined this in relation to the students' learning, captured in the post-tests. A screening was also conducted in Grade 6 and in Grade 9. The screening was the basis for giving the researcher and the teacher team insights into the different ways in which students could experiencing the actual content. The results of the screening also formed the basis for the design of the pre-, post- and delayed post-tests. The three tests were identical throughout the study and took about 20-30 minutes for the students to complete. The tests contained 9 questions (see Appendix 1). The pre-test and post-test were performed a few days before and after the research lessons in each cycle. The delayed post-test was conducted approximately five weeks after the second research lesson in each cycle.

All lessons were video recorded using a video camera. The camera was placed in the middle of the classroom with a focus on the teacher and the whiteboard. This was because the study focused how the content was handled during the lessons and it was important to capture both the teacher's and the students' voices, and what was written on the white board. When the students worked in small groups with their group tasks, the camera was moved between these groups in order to be able to capture parts of the students' discussions when they solved tasks together.

4.1.1 The object of learning and the critical aspects

The researcher and the teacher team formulated the object of learning for the study as follows: *The ability to enlarge and reduce two-dimensional geometric figures and handle the scale factor correctly*. The basis for identifying potential critical aspects, in relation to the object of learning, consisted of the teachers' experiencing, the results from screening interviews and the difficulties that previous research has shown that students aged 12–16 have when enlarging and reducing multi-dimensional geometric figures. The results of the screening interviews showed that the majority of the students experiencing that if the lengths are doubled, the area will also be doubled when enlarging a two-dimensional geometric figure, regardless the shape of the figure, i.e. for these students, the illusion of linearity was in operation. This knowledge led the teachers to emphasise that students needed to be given the opportunity to separate the linear relationship (the change in length) and the non-linear relationship (the change in area), and also that the students needed to discern similarity between figures, i.e. the students need to discern lengths in a geometric figure and how these change when the figure is enlarged or reduced. In summary, the teacher team and the researcher formulated the following potential critical aspects for the study:

- Discern similarity
- Discern lengths in geometric figures
- Discern the change in length when scaling
- Discern the change in area when scaling

4.1.2 The lesson design

Based on the reasoning that one cannot learn more details without knowing what they are details of (Marton, 2015), the teacher team concluded that students should be given the opportunity to systematically break down the linear relationship and thus become aware of the simultaneous multi-dimensional effect when lengths change, which means that lengths, different features of length, scale factor and geometric figures should be varied in the teaching. The first two tasks were the same in all three cycles and focused the rectangle. In the third cycle, the circle was in focus in two different tasks. The lesson design and the activities involved (see Table 2) were jointly created by the researcher and the teacher team. The teacher team initially assumed that through the activities, the change in length, i.e. the linear relationship, would first be explored, and then the non-linear relationship, i.e. change in area, would be explored during the lesson.

Cycle 1	Cycle 2	Cycle 3
The photograph task	The photograph task	The photograph task
The plus-sign task	The plus-sign task	The plus-sign task
	The square task	The square task
		The circle task
The sheet of paper task (A4), scale factor 1:2 and 1:4	The sheet of paper task (A4), scale factor 1:2 and 1:4	The sheet of paper task (A4), scale factor 1:2 and 1:4
		The rectangle task
	The diagonal task	
The sheet of paper task (A4), scale factor 1:3	The sheet of paper task (A4), scale factor 1:3	The sheet of paper task (A4), scale factor 1:3
The pizza task	The pizza task	The pizza task
The doll's house task	The doll's house task	

Table 2. Planned activities for each cycle in the study – an overview

The teacher team also assumed that it was relevant to direct the involvement in this first task towards the students' qualitatively different ways of seeing the object of learning instead of focusing on a correct answer.

4.2 The analysis of the learning study

The unit of analysis in the study was lesson sequences. In the analysis of these sequences, the focus was on how the object of learning was handled and identifying qualitative differences in what was made possible to discern in relation to critical aspects and dimensions of variation. The focus was on pointing out crucial moments for students' learning, i.e. enacted patterns of variation that seemed to contribute to opportunities for the students to discern critical aspects. Besides the analysis of lesson sequences, pre-, post- and delayed post-tests were analysed to further explore the relationship between teaching and student learning outcomes. The teacher team and the researcher analysed the teaching and the students' tests before and between the cycles in order to further refine the planned patterns of variation and create new activities that enabled greater opportunities for the students to discern the critical aspects. A brief description of the results of the analysis of the lessons in the three cycles is provided in the next section. The aim of the description is to show similarities and differences between the three cycles related to the students' opportunities to learn what was intended to be learnt.

4.3 Students' learning during the learning study

The three classes were equal in terms of the pre-understanding of the object of learning, as shown in the pre-tests. In the pre-tests, a majority of the students answered that when the lengths of a figure become twice as long or four times longer, the area will be twice as big or four times bigger, i.e. the teachers noticed an illusion of linearity. The analysis also indicated that 75–80% of the students focused on the area when they reduced or enlarged two-dimensional figures using a given scale factor. The students used the given scale factor as an indication of how many times the area was supposed be enlarged or reduced. Also, most of these students did not have a focus on similarity between figures, and constructed figures on the pre-test that were a completely different shape from the original figures.

From the first to the third cycle, positive differences in students' learning could be identified. The students' learning increased between pre- and post-tests, and this was maintained in the delayed post-test. The clearest and most positive change was reached after Cycle 3. The results of pre-, post- and delayed post-tests, with a maximum of 9 points, showed that the students' learning outcome increased stepwise from the first to the third cycle. The increase of the mean value in Cycle 1 was from 2.4 to 4.6 (delayed post-test 4.7), in Cycle 2 from 2.6 to 6.5 (delayed post-test 6.2) and in Cycle 3 from 1.8 to 7.1 (delayed post-test 6.8). The results indicated that the class who participated in Cycle 3, to a greater extent than the classes in Cycles 1 and 2, developed the ability to separate linear and non-linear relationships when enlarging and reducing two-dimensional geometric objects, and thus to handle the concept of scale correctly. Overall, the test results showed that if the students were only able to discern each critical aspect separately, but missed the discernment of these aspects simultaneously, it resulted in students not understanding the object of learning in the intended way. Specifically, the results showed that it was the relationship between the two critical aspects, change in length and change in area, that was important to highlight and investigate, when scaling two-dimensional geometric figures. If the students were to increase their understanding of this specific object of learning, it seemed to be a question of whether the teacher made such a fusion possible for the students to discern. Comparisons between how the object of learning was handled and student learning follow in the next section.

4.3.1 The relationship between teaching and student learning – comparisons of the three cycles

From the analysis of the students' test results in relation to how the object of learning was handled during the lessons, it could be seen that the three cycles differed. Different patterns of variation were enacted and the discernment of critical aspects was made more or less explicit, which contributed to different spaces of learning being enacted in the different lessons.

Cycle 1

Based on the test results, the class in Cycle 1 showed that they had not discerned to any great extent the critical aspects of the object of learning. The enacted patterns of variation gave students limited opportunities to learn the object of learning the intended way. Analysis of the enacted patterns of variation showed that neither change in length nor change in area was in the foreground during the lessons in such a way that it allowed students to discern these aspects of the object of learning. The results in the post-test indicated that a majority of the students still experiencing the illusion of linearity, i.e. the idea that if the lengths became four times as long in a figure, the area would also be four times bigger. When the enacted patterns of variation were analysed, it was also shown that the students were not given the opportunity to experiencing the critical aspects simultaneously. The teacher kept the change in area in the background and the focus was on length and change in length in relation to a given scale factor. The teacher team and the researcher noticed and discussed that it was challenging to keep the change in area in the background because the aspects change in length and change in area were functions of each other. When the change in length was explored during the lesson, the students also noticed the change in area. This probably made it difficult for the students to separate these two aspects, but also to discern them simultaneously.

Cycle 2

The students' results in Cycle 2 indicated that the enacted patterns of variation gave the students a greater opportunity to discern lengths and change in length in relation to a given scale factor. In the first tasks in the post-test, a majority of the students showed that they discerned lengths and change in length in relation to the given scale factor. In some other tasks on the post-test (3, 4 and 5), where different geometric figures were to be enlarged, the students' performance was lower in the task where the circle was handled. When analysing the patterns of variation that were enacted during the lesson, the analysis showed that the pattern of variation was not structured as had been planned when the circle was handled (the pizza task). Several aspects varied simultaneously, which was actually planned, as the activity was seen as a fusion, but the presentation of students' different solutions and of which aspects were to be explored was not systematically made to the extent that would probably have been needed. The analysis also showed that the change in area was in the background in the first part of the lesson, and that it was mostly lengths and change in length that were in the foreground. Although the change in area was slightly more in focus in the second part of the lesson, it was not explicitly explored during the lesson. The fact that the students had problems focusing on both of these aspects simultaneously was identified when analysing the activity about the sheet of paper task (see Table 2), where the students discussed in small groups a reduction of a sheet of paper by scale factor 1:3. During this activity, when the students discussed in small groups, the teacher team and the researcher noticed that the students' different experiencing of the relationship between change in length and change in area needed to come to the fore and be explored in order for the students to experiencing the object of learning as intended. The results in the post-test showed that the students did not succeed so well on questions that required a discernment of both these aspects simultaneously. This insight was important when the lesson design was refined before the lessons in Cycle 3.

Cycle 3

In Cycle 3, the students performed significantly better than the students in Cycle 1 and 2, and specifically in the two test tasks (8 and 9) where both change in length and change in area were involved. A possible reason for this could be that more patterns of variation were enacted during Cycle 3 where these two aspects were kept in the foreground simultaneously, but also that patterns of variation were opened up in a more systematic way.

All three cycles started with the same activity, the photograph task. The teacher team assumed that it was relevant to direct the involvement in this first activity towards the qualitatively different ways of experiencing the object of learning and compare them, instead of focusing on a correct answer during the activity. By doing this, the teacher had an opportunity to contrast students' different ways of seeing the object of learning and from that identify critical aspects or offer opportunities for the students to discern critical aspects concerning the object of learning. In Cycle 2, there was a strong focus on the critical aspects, lengths and change in length, in relation to an enlargement or to a reduction of a rectangle, and to a given scale factor. The students' results on the post-test indicated that these patterns of variation were effective, and the teacher team decided to maintain this clear focus on this aspect in Cycle 3. What still seemed difficult for the students was the simultaneous discernment of the critical aspects, change in length and change in area. The teacher team planned for additional activities in Cycle 3, which would offer the students in a more systematic way the opportunity to separate change in length from change in area. A "common thread" was formulated to offer students the opportunity to first discern the change in length in relation to the scale factor and then problematise the change in area by enacting a generalisation where the geometric figure rectangle was involved. The relationship between change in length and change in area was also supposed to be taken into account.

So, in Cycle 3, the critical aspects, lengths and change in length in relation to similarity were focused on first, then the scale factor was picked up. In the next step, the change in area was explicitly highlighted, but with a simultaneous focus on the change in length. The relationship between these two aspects was also addressed in most of the activities during the two lessons in the cycle.

During one of the activities in the later part of the lesson in Cycle 3, enlargements of rectangles using scale factor 2:1 and scale factor 4:1 were handled. All the critical aspects of the object of learning came to the fore in a systematic way. The relationship between the critical aspects, change in length and change in area, was also problematised. The teacher drew the original rectangle and a rectangle enlarged by scale factor 2:1 on the whiteboard. Excerpts from the whole-class discussion from this activity will illustrate the need for a simultaneous focus on the critical aspects change in length and change in area.

> Student (Sofia): Well, the area will always be doubled?
>
> Teacher: Is it always double? You mean that two plus two is four?
>
> Student (Sofia): Yes. Or?
>
> Teacher: We can have this as a working hypothesis for a while.

Later, a contrast was made by means of an enlargement of a rectangle by scale factor 4:1. The question of whether the change in area can be twice as big as the change in length was raised again.

> Student (Johan): That is four time as long and that is also four time as long [points to the drawing on the whiteboard]
>
> Teacher: Well, the lengths, on each side, have become four times longer. What do you think has happened with the perimeter then? [points to the drawing on the whiteboard]
>
> [Several students are talking at the same time and the teacher repeats what the students say.]
>
> Teacher: Four times as big? Four times as long?
>
> Teacher: What has happened with the area then? Length and perimeter, you see is the same. They will be four times as big. [The teacher writes this on the whiteboard]
>
> Teacher: But what happens with the area?
>
> Student (Allan): Eight

When a contrast was enacted, as in the activity above, it was made possible for the students to discern lengths and change in length as well as change in area in relation to a correct enlargement and a given scale factor. Students' opportunities to discern the relationship between change in length and change in area increased. The patterns of variation systematically enacted by this activity were not a part of the lesson design in Cycles 1 and 2.

The fact that the students' experiencing of the object of learning were highlighted and became a part of how the object of learning was handled, contributed to powerful dimensions of variation being opened up, which seemed to be crucial for students' learning.

4.5 Conclusion of the learning study

Based on the results of the study, it can be concluded that it is important that the teacher has relevant knowledge about the object of learning, and that the teacher notices the students' experiencing of the object of learning and takes it into account when teaching. Teachers' insights about how students experiencing the object of learning during the lessons may change the planned lesson design. Kullberg et al. (2014) emphasise that students' and teachers' ways of asking questions and ways of responding to questions in the classroom can determine how the object of learning is handled during a lesson, i.e. the enacted object of learning. This study showed that through the iterativity of a learning study, the teacher team and the researcher could explore how the students' experiencing of the object of learning could be used to develop the teaching.

The communication between the teacher and students, and between students, contributed to spontaneous patterns of variation being enacted and critical aspects being explored in different ways. The interaction during the lessons could be seen as a dynamic content-related interaction (Marton, 2015) and took place between teachers, students and the object of learning, but also between students and the object of learning. The teachers in this learning study encouraged the students to communicate about the object of learning, both in whole-class and in small-group discussions. The students' different ways of experiencing the object of learning could thereby be problematised in such a way that the critical aspects were possible to discern. Questions asked at crucial stages of a lesson can focus students' attention on critical aspects of the object of learning, create a context that will help students to make sense of the object of learning and open up a space for exploration of an answer (Marton & Tsui, 2004). Other studies also show that students' participation is important when dimensions of variation are opened up (Maunula, 2018; Kullberg, 2010; Al-Murani & Watson, 2009). The studies show that variation generated by the students in the public domain contribute to the potential development of the enacted object of learning by making the variation available for all students. When different dimensions of variation are opened up in the teaching situation, a space of variation is created, a space of learning (Runesson, 2005), to which the teacher's and students' awareness is directed. Through a learning study's iterative approach, it was possible to try out, change and refine the systematics within and between the activities. The teachers were given the opportunity to reflect on the content and the information they used, which according to Kullberg et al. (2014) can result in activities that are better adapted. This may also have contributed to the students in Cycle 3 having a greater opportunity to develop the ability to enlarge and reduce two-dimensional geometric figures and deal with the length scale than the students in the other two cycles. A small-group discussion in Cycle 2 gave the teacher team and the researcher valuable insights into students' experiencing of the object of learning and thereby possible ways to handle the object of learning in Cycle 3. The use of small-group discussions in the study also contributed to new questions arising about the relationship between teaching and student learning of a specific content when small-group discussions are used in lessons.

The results from the licentiate thesis (Svanteson Wester, 2014) form the basis for the new research questions posed in this thesis. The research questions in the licentiate thesis were directed towards what the students needed to discern to see the object of learning in the intended way and in what way the object of learning had to be handled during the lessons in order for the students to discern what was necessary. The learning study in Svanteson Wester resulted in a lesson design where the critical aspects were explored jointly in a systematic way in whole-class discussions. Small-group discussions were just one element of the lesson design, used as a means for students to talk about the content. First, during the lesson in this lesson design, the students' way of seeing the object of learning was in the foreground and explored during the lesson. Then, the critical aspect change in length in relation to a proportional image was explored. After that, the change in length in relation to a scale factor was explored. Lastly, change in length and change in area were explored, simultaneously. In this last step, there was also a variation in the shape of two-dimensional geometric figures. However, one of the small-group discussions during the lesson in Cycle 2 contributed to students' way of seeing the object of learning coming to the fore in a more explicit way. The teachers identified students' way of seeing and made use of this in their teaching, and dimensions of variation that were crucial for students' learning of the object of learning in the intended way were opened up. This thesis is based on the lesson design from Svanteson Wester (2014), with the aim of further exploring the relationship between teaching and student learning, to contribute to a deeper understanding when small-group discussions are a part of whole-class teaching.

Chapter 5 The Design of the study

This chapter describes the research design with the purpose of giving an overview of the research process of the study. Ethical considerations, and the validity and reliability of the study are also discussed. The three articles are associated with a specific content in mathematics when small-group and whole-class discussions are used as a teaching arrangement in a whole-class setting. To achieve this purpose, a variation theory perspective on learning has been adopted (Marton, 2015). The methods used for production and analysis of data in this thesis have been chosen in accordance with this theoretical point of departure and are framed by learning study as a research approach (e.g. Pang & Lo, 2012). Pre- and post-tests were carried out, and lessons were planned and evaluated iteratively. The study was conducted in collaboration with a teacher team. The empirical data for analysing the relationship between teaching and student learning comes from video recorded teaching episodes in whole-class and small-group discussions, and students' pre- and post-tests.

5.1 A learning study with four cycles

The study consists of a learning study with four cycles. The first three cycles were part of a licentiate thesis (Svanteson Wester, 2014, see Chapter 4) and were carried out in Grade 8 during spring 2013 (see Table 3), with the aim of investigating the relationship between teaching and student learning about the object of learning, enlarging and reducing of two-dimensional geometric figures and handle the scale factor correctly (Svanteson Wester, 2014). Results from Svanteson Wester (2014) sparked further interest in questions concerning the relationship between teaching and student learning. This contributed to a re-analysis of data from the three cycles (pre- and posttests only and not the delayed posttest) being conducted in spring 2017. The analysis directed the focus towards the relationship between students' experiencing of the object of learning and how teachers' insights about the same emerged during the learning study process (Article 1). Findings from the re-analysis showed that the students' way of handling the object of learning during a small-group discussion in Cycle 2 proved to be decisive for how the object of learning was handled further in whole-class. The results of the re-analysis, over the three cycles, raised new questions about students' opportunities to learn from small-group and whole-class discussions in whole-class mathematics lessons. This contributed to additional data being collected. New empirical data can be seen as a fourth cycle in the learning study, but also as separate empirical data. The fourth cycle was carried out in Grade 8 in spring 2017, focusing on the same object of learning as in the first three cycles and using the critical aspects identified during those cycles. The new empirical data was generated to answer questions about what was made possible to learn in whole-class discussions and small-group discussions in whole-class teaching in mathematics, related to the object of learning, enlarging and reducing of two-dimensional geometric figures (Article 2). Finally, the relationship between teaching and student learning, when whole-class discussions and small-group discussions were used during the lessons, was examined through the four cycles, with a focus on how the object of learning was handled (Article 3) (see Table 3).

	Implemented	Teacher	Articles	
Cycle 1 ClassA N=17	March 2013	Teacher 1	Article 1	Article 3
Cycle 2 Class B N=21	April 2013	Teacher 2		
Cycle 3 Class C N=16	May 2013	Teacher 1		
Cycle 4 Class D N=21	May 2017	Teacher 1	Article 2	
Cycle 4 Class E N=17	March 2017	Teacher 2		

Table 3. Overview of the design of the study.

As mentioned before, learning study (Pang & Lo, 2012) has been used as a research approach in this study because it offers special conditions for enacting and making the object of learning visible in teaching. Learning study is iterative and is used as a method to create good conditions for developing teaching (Kullberg et al., 2019). Learning study makes it possible to explore students' learning opportunities in relation to how an object of learning is handled. The method is also used to generate qualitative research data for further analysis of the relationship between enacted objects of learning and students' learning, but also for further analysis of students' learning opportunities when small-group and whole-class discussions are used in mathematics teaching in a whole-class setting. Pre- and post-tests are used as indicators of differences in students' learning in relation to what was possible to learn during teaching.

5.1.1 Participants

The empirical data builds on ten mathematics lessons, conducted in Grade 8 in compulsory school in Sweden. The students at the school were from different socio-economic backgrounds, and approximately 30% of the students had a mother tongue other than Swedish. In total, three teachers and five classes (92 students) were involved in the study. The participating teachers had between 11 and 22 years of professional experiencing of working as teachers. The teachers were asked to participate based on the fact that the teachers had all participated in learning studies before and all three also use variation theory in their daily work, which most likely contributed positively in the design of lessons and teaching, but also in the analysis of the relationship between the enacted objects of learning and the teaching, and the evaluating of student learning outcomes. The teachers in the study were the same as in the licentiate study (Svanteson Wester, 2014), but one of them did not teach any of the research lessons in the study. As mentioned before, the teachers were colleagues of the researcher and they had previously participated in learning studies. This study could be seen as a continued collaboration between the teachers and the researcher. The choice of teachers for the study has also affected which classes participated in the study, as the teachers in the study were teaching the classes they normally teach. All students who participated in the study, and their parents, gave written consent to participate in the study (see Appendix 2) and the names of the students and teachers have been changed in the transcripts and the text.

5.1.2 The empirical data

The empirical data has been generated from ten video record lessons in mathematics. The length of the lessons was between 40-65 minutes. The object of learning for all the lessons was enlarging and reducing two-dimensional geometric figures, and the critical aspects were discerning a) similarity between figures, b) lengths in a geometric figure, c) change in length when scaling two-dimensional geometric figures and d) change in area when scaling two-dimensional geometric figures. The use of video recorded lessons made it the possibility to investigate the lesson as a whole, since both whole-class discussions and small-group discussions from the same lessons were video recorded. The benefits of video recorded lessons are numerous (Powell et al. 2003), in that they go beyond the human capacity to capture parts of an event in a lesson. In the study, one video camera was placed in the middle of the classroom focusing on the teacher and the whiteboard. The aim of this was to capture in detail the whole-class discussion and how the object of learning was handled in the classroom. It was important to document what was written on the white board, but also to include the teacher's and students' voices. During the first three cycles, there was one video camera in the classroom, and when the students worked in small-groups on a task during the lesson, the video camera was moved between these small-groups, meaning that the small-group discussions were only partially video recorded.

The aim with small-group discussions in the first three cycles was mostly to provide an opportunity for the students to talk about the object of learning, and the small-group discussions were not systematically integrated into the whole-class discussions. The whole-class discussions during the first three cycles have, however, been documented in their entirety through the video recorded lessons. In Cycle 4, two classes participated, and all small-group discussions in both classes were video recorded. Thus, there was one video camera on every small-group's table in the two classes. In Class D, there were six small groups and in Class E there were five small groups in the classroom. Each small-group participated in three group tasks, so in total 33 small-group discussions were video recorded in Cycle 4. All video recorded data has been transcribed verbatim. In Cycle 4 (see Table 4), the possibility of capturing the complexity in the relationship between teaching and student learning, in relation to how the mathematical content was handled in the classroom, increased, since how the object of learning was handled in both whole-class discussions and all small-group discussions in the two classes could be captured by video cameras. Since two classes participated in Cycle 4, this produces greater amount of data about student learning opportunities in lessons with small-group discussions. The two classes that participated in Cycle 4, called Class D and E (see Table 4), are referred to as Class A and Class C in Article 2.

The empirical data consists of transcripts from lessons of whole-class discussions and small-group discussions, and the students' written answers in the pre- and post-tests (see Table 4). Students' answers on pre- and post-test were analysed to answer the research questions in Articles 2 and 3, in order to get insights about students' learning when whole-class and small-group discussions were used in the classroom. To answer the research questions in Article 1, pre- and post-test were not needed.

	Carried out	Teacher	Pre- and post-tests	Video-recorded whole-class discussions	Video-recorded small-group discussions	Video-recorded in total
Cycle 1 Class A N=17	March 2013	Teacher 1	n=34	115 minutes	5 minutes	120 minutes
Cycle 2 Class B N=21	April 2013	Teacher 2	n=42	130 minutes	5 minutes	135 minutes
Cycle 3 Class C N=16	May 2013	Teacher 1	n=32	130 minutes	6 minutes	136 minutes
Cycle 4 Class D N=21	March 2017	Teacher 1	n=42	120 minutes	18x10-15 minutes	360 minutes
Cycle 4 Class E N=17	March 2017	Teacher 2	n=34	120 minutes	15x10-15 minutes	320 minutes

Table 4. An overview of empirical data from the study.

5.1.2 Pre- and post-test tasks

The pre- and post-tests (see Appendix 1) took about 30 minutes to complete and were conducted the day after the research lessons. There was no teaching about the specific content between the research lessons and the post-test. The test included six tasks, but task number 2 included four sub-tasks and the total number of points on the test was 9. The pre- and post-tests were the same for all five classes in the four cycles. The teacher team and the researcher decided just before Cycle 1 not to use rulers during the teaching, and this entailed that the tasks number 1, 6 and 7 were taken away from the pre- and post-tests. The first four tasks on the test (nos. 2, 3, 4 and 5) (see Appendix 1), required students to be able to keep the change in area in the background as the focus was the change in length in relation to a scale factor in different two-dimensional geometric figures. In the last two tasks (nos. 8 and 9) (see Appendix 1), change in area in relation to a given change in length or to a given scale factor was requested. These two tasks required students to be able to handle change in length and change in area simultaneously when scaling two-dimensional geometric figures. The first tasks were based on given pictures of different geometric figures (a square, a triangle, and a circle), which were supposed to be enlarged or reduced. Other tasks were written problems related to real-life situations, for example, in one task the students were supposed to solve a problem related to enlarging and reducing a horse paddock. The same ability was thus tested several times in different tasks using different contexts. Tasks 8 and 9 required careful reading, which could make the task more difficult for some students.

5.1.3 The lesson designs

It was assumed by the teacher team that students' experiencing of both linear and non-linear relationship were of decisive importance during the lesson. As mentioned before, previous research has shown that a majority of 12-16 years old students have difficulties in separating linear relationships from non-linear relationships when enlarging or reducing two-dimensional geometric figures (see Chapter 3). Simultaneous discernment of these two relationships was needed. The teacher team's idea regarding student learning opportunities related to linear and non-linear relationships was that students should experiencing a movement from an undifferentiated whole, through differentiation and integration, towards a differentiated and integrated whole. This means, on the one hand, that aspects have to be distinguished (separated), and on the other hand, that they have to be brought together (fusion), i.e. moving from an undifferentiated whole to a differentiated whole via a differentiating of critical aspects. The teacher team's idea in relation to the design of the lessons was that the students initially, in the first tasks during the lesson, should explore the undifferentiated whole of the object of learning, thereafter an opportunity to discern critical aspects separately. In the later part of the lesson, the students were to be offered more complex tasks which were supposed to offer a simultaneous discernment of the critical aspects and a movement towards a differentiated whole. The rationale for this was that, according to Marton (2015), one cannot learn more aspects without knowing what they are aspects of. Based on that statement, the teacher team and the researcher concluded that the students should be given the opportunity to systematically experiencing a differentiation of the object of learning, which means, for example, that a figure's lengths and area, and different features of lengths, should be discernible. Thus, students need to discern what a linear relationship between enlarged and reduced two-dimensional figures is, and how it is related to the scale factor, but also how this relates to situations where there is a non-linear relationship. Several tasks were designed by the teacher team and the researcher and used in the lessons in order to make critical aspects noticeable for the students. For example, the photograph task and the plus-sign task were used to discuss what a similar image is and is not, and to make it possible for the students to discern change in length when enlarging by scale factor 2:1. The lesson design in Cycle 4 was to a large extent the same as in Cycle 3. However, one more two-dimensional geometric figure was added in Cycle 4, an irregular heart shape, which was not discussed in the previous cycles (see Table 5). All four cycles started with the same activities, the photograph- task and the plus-sign task. The teacher team and the researcher assumed that it was relevant to direct the involvement in this first part of the lesson (the photograph task and the plus-sign task) towards the students' qualitatively different ways of experiencing the intended object of learning, and not to ask questions looking for quick and correct answers. Instead, students' different ways of seeing the object of learning should come to the fore and jointly be explored.

Cycle 1	**Cycle 2**	**Cycle 3**	**Cycle 4**
The photograph task	The photograph task	The photograph task	The photograph-task
The plus-sign task	The plus-sign task	The plus-sign task	The plus-sign task
	The square task	The square task	The square task
		The circle task	
The sheet of paper task (A4), scale factor 1:2 and 1:4	The sheet of paper task (A4), scale factor 1:2 and 1:4	The sheet of paper task (A4), scale factor 1:2 and 1:4	The sheet of paper task (A4), scale factor 1:2 and 1:4
		The rectangle task	
	The diagonal task		
A sheet of paper task (A4), scale factor 1:3	A sheet of paper task (A4), scale factor 1:3	A sheet of paper- task (A4), scale factor 1:3	A sheet of paper task (A4), scale factor 1:3
The pizza task	The pizza task	The pizza task	The pizza task
The doll house task	The doll house task		
			The heart task

Table 5. The lesson design through the learning study.

In the first three cycles, small-group discussions were used during the lesson, but they were not systematically integrated into the whole-class discussions. In Cycle 4, small-group discussions played a central role in the lesson design (see Table 6).

Cycle 4	
The photograph task	Whole-class discussion – video recorded
The plus-sign task	Whole-class discussion – video recorded
The square task	Whole-class discussion – video recorded
The sheet of paper task	Small-group discussions – video recorded
The sheet of paper task	Whole-class discussion – video recorded
The pizza task	Small-group discussions – video recorded
The pizza task	Whole-class discussion – video recorded
The heart task	Small-group discussions – video recorded
The heart task	Whole-class discussion – video recorded

Table 6. The lesson design in Cycle 4. Integrated small-group discussions in whole-class discussions.

The small-group discussions were systemically embedded in the whole-class lesson with the purpose, based on results in Svanteson Wester (2014), of extending the students' opportunities to investigate the object of learning and discern critical aspects.

5.2 The analysis

The analysis has been conducted on a micro-level. Concepts from variation theory (Häggström, 2008; Marton, 2015, Marton & Tsui, 2004) were used as analytical tools in the process of analysis. The detailed analysis of the data made it possible to describe differences in how the object of learning was handled. The unit of analysis was sequences from the lessons, in which the object of learning was discussed. The transcripts from the lessons were analysed in order to search for sequences in which the critical aspects of the object of learning were explored. The analysis of the sequences focused on qualitative differences in what was made possible to discern with respect to identified critical aspects. This micro-level analysis enabled the identification and description of subtle differences in the enactment of the same object of learning. The meaning was constituted between the researcher and the data material, which requires repeated reading of the entire transcribed material and students' written answers in the tests. The concept of critical aspects opened up as dimension of variation was used as an analytical tool for examining and comparing student learning opportunities offered in whole-class discussions and in small-group discussions. Different amounts of data have been used in relation to the different appended articles.

Article 1 relies upon a re-analysis of data carried out five years after the learning-study process of the first three cycles (six lessons and 54 students). The focus of the re-analysis was identifying students' different ways of experiencing the relationship between the two critical aspects, change in length and change in area, explored in whole-class and small-group discussions during the three first cycles in the learning study (six lessons). Another focus was how the teachers' knowledge about students' experiencing was reflected in the teaching. In order to examine students' experiencing of the relationship between those two aspects, transcripts from lessons were analysed in order to search for sequences in which change in length and change in area were in the foreground at the same time during the teaching. One example of such a sequence is when the teacher started to discuss what was twice as big in a figure (enlarged with the scale factor 2:1), and one of the students said "I thought, wouldn't the area be twice as big as well? If the length is going to be twice as big and the width twice as big." The analyses of students' and teachers' discussions about the object of learning during the lessons made it possible to categorise students' different ways of experiencing the relationship between the aspects change in length and change in area and also to notice how the teaching was developed in regard to students' experiencing of the object of learning. The analysis was focused on patterns of variation (e.g. contrast) enacted during the lessons.

Article 2 relies on analysis from Cycle 4 in the learning study. The analysis involved data generated from video recorded whole-class and small-group discussions and students' written pre- and post-tests in two classes (four lessons and 38 students). How the object of learning was handled during whole-class and small-group discussions was analysed, and different enacted objects of learning were identified. When analysing the discussions, critical aspects and dimensions of variation opened up were in focus. A description of qualitative differences between the objects of learning enacted in whole-class discussions and small-group discussions for the lessons was enabled. One example of a qualitative difference, related to the pizza task and the small-group discussions, was characterized by the fact that the students in one of the small-group discussions did not have the shape of a circle in the foreground when they tried to reduce the pizza by the scale factor 1:2. Instead they replaced the pizza with a square. This was identified in the analysis as an enacted object of learning with a lower quality (EOL 1), i.e. the critical aspects were not explicitly in the foreground and explored during the discussion. In another small group, the students discussed the pizza task in such a way that all the critical aspects were in the foreground, and an enacted object of learning with higher quality was identified (EOL 5). In this small group, the students illustrated the pizza as having the shape of a circle, which they were supposed to do, and they handled the scaling of the pizza with a focus on the diameter and the radius as being half as long, as well as handling the change in area as being four times smaller. In the article, the differences in results between students' pre- and post-tests were analysed and give insights about what the students had learnt during the lessons, i.e. the lived object of learning was also identified. The lived object of learning was compared to the different objects of learning enacted during the lesson as well as the lesson's intended object of learning.

Article 3 builds on the analysis carried out of the learning-study process and involved all four cycles (ten lessons). The data consists of transcribed video recorded lessons and written responses on pre- and post-tests by 92 students who participated in the learning study. In all four cycles, the relationship between how the object of learning was handled during the lessons in the public part, i.e. in the whole-class discussions and student learning outcomes in pre- and post-tests were analysed. The lessons were analysed in order to find similarities and differences regarding the teaching and tasks used, and how the object of learning was handled. The analysis was focused on critical aspects being explored and opened up as dimensions of variation by the teacher or by the students in the whole-class discussions during the lessons. The students' solutions and answers on the post-test were analysed in order to identify what the student had learnt (lived object of learning). A comparison was made between pre- and post-tests, in all four cycles, to identify differences related to student learning outcomes and how the object of learning was handled during whole-class discussions.

Since teaching and learning are described in commensurable terms (see Chapter 2), what the teacher wants the students to learn (the intended object of learning), what is taught (the enacted object of learning) and what is learned (the lived object of learning) can be related to each other (Runesson, 2005). In this thesis, the concepts critical aspects and dimensions of variation opened up by means of patterns of variation are used as an analytical tool, and this makes it possible to examine and compare the learning opportunities offered (the enacted objects of learning) during the lessons with the intended object of learning and the lived object of learning. Students' learning opportunities are described in relation to whether and which critical aspects were explored and whether dimensions of variation that corresponded to each aspect were opened up or not during the lessons. Differences in what students learned in one whole-class discussion or in one small-group discussion could compared with another whole-class discussion or small-group discussion as a result of differences in how the critical aspects were explored. The lived object of learning, based on students' answers on the post-tests, was compared to the enacted object of learning in order to determine which possibilities for learning were taken advantage of by the students.

The analysis of data has been continuously presented and discussed in research contexts at seminars or conferences with other researchers. By providing comprehensive descriptions of how the specific object of learning in mathematics was handled during the lessons, together with direct quotes from transcripts, an ambition has been to make the analyse as transparent as possible, so that the reader can assess the credibility. The study does not claim to provide a complete picture of all variation that occurred during the lessons but has been limited to describing the variation in the form of enacted patterns of variation that have been found to be most crucial for students' learning in terms of enlargement and reduction of two-dimensional geometric figures in relation to the concept of scale.

5.6 Ethical considerations

The study was conducted in accordance with the Swedish Research Council's ethical principles for research (Vetenskapsrådet, 2017). It is about contributing new valuable knowledge and at the same time taking responsibility for the participants in the research study. In that respect, the approach to teachers and students, as well as to the handling of data, is central. This means that the requirements for information, consent, and confidentiality (Vetenskapsrådet, 2017) have been considered in the research process. All data collected during the study has only been used for research purposes. The video recordings of lessons will be registered as personal records and stored encoded on a customized data storage server at the university of Gothenburg following the department's data management plan. Data in the form of tests will be stored locked in fireproof cabinets and scanned to a server intended for data storage.

The principal of the school gave his consent to the study. Students in the classes that were included in data collection, and their parents, were informed about the aim and practical implementation of the study in writing, that observations and video recorded lessons would be made, and that students' notes and pre- and post-test results would be saved. A letter of consent (see Appendix 2) with information about the research study was sent to the students and their parents. In the letter, there was information about the research study, that confidentiality was promised and that the research data would only be used for specific research purposes. The parents had the option to approve or not approve their child's participation and were asked to sign. For students who were 15 years of age or older their own signature was sufficient. The teachers who participated in the study gave their verbal consent. All teachers were positive towards participating and they saw their participation in the study as an opportunity to increase their own professional development. The researcher's relationship with the participating teachers in a study is an important ethical issue (Kvale, 1997). In this study, the relationship between teachers and the researcher can be described as both friends and colleagues and the research situation can be seen as mutual, in that both parties learn from the other. An open and critical reflection on the role as researcher in the intervention is important. Together, the researcher and the teachers contribute something novel to research and practice.

All data collected during the study has only been used for research purposes. One ethical dilemma relates to confidentiality. The number of participating teachers and students was quite small in the study, and the analyses were done on a highly detailed level. Even though no photos have been used and the teachers' and the students' names have been changed, it could be the case that a participating teacher or students might recognize themselves when reading this thesis or the articles. It was difficult to entirely overcome this anonymity problem. Well, the person whose face is part of the photo in the photograph task during the lesson was asked and has given his approval for the picture of him to be included in this study. However, changing the names to pseudonyms reduces the risk of this. The research lessons were not the basis for the teachers' assessment of the students' knowledge of mathematics. The students were informed that the research would not be used to assess their knowledge and that the participation would not affect the teachers' assessment of the students' knowledge. It is also important to emphasize that the focus of this thesis is on how the object of learning was handled. Even though sections of teaching from the different classes were compared, the teachers and students themselves were not the research interest, as the main focus was on the teaching and how the object of learning was handled in small-group and whole-class discussions. No sensitive personal data e.g., race, ethnic origin, health, or religion was collected.

5.7 Validity and reliability

High-quality in a study requires validity and reliability. Validity in qualitative analyses concerns how the research questions correspond to the method used, as well as the question of the depth of the analyses and the range of the data. It is the researcher's role to declare and, where possible, eliminate threats to the validity

(Cohen et al., 2011). Internal validity refers to the degree of confidence and whether the results really represent the reality of what has been studied. It is necessary to ask whether one has succeeded in capturing what the study aims to describe. In this study, the relationship between teaching and student learning about an object of learning is examined. Student learning is examined through pre- and post-tests. Cohen et al. (2011) believe that it is problematic to assess what students have learnt through their answers on a test, as their motivation and interest about the task can affect their performance. Cohen et al. (2011) also point out that it is difficult to know whether students respond to what they really have learnt, because it is known that students tend to respond in the way they think is expected of them. In this thesis, pre- and post-tests were designed in a way similar to tests that the students were used to. There was no reason to believe that students' answers would be affected to any great extent. Possibly the fact that the test was part of a research study would make the students strive to appear as good as possible. On the other hand, since the result on the tests is not included in the assessment of the students' regular schoolwork, it could also be that the students did not make an effort precisely because of this. Students' opportunities to learn were focused during the whole lesson. The whole lessons were therefore video recorded, both whole-class discussions and small-group discussions, which strengthens the validity of the study. Using sequences from lessons as the smallest unit of analysis has allowed a description of differences in how the object of learning was handled in both whole-class and small-group discussions. For instance, it was possible to identify the enacted object of learning in all small-group discussions and whole-class discussions during the lessons and also give a detailed description of this.

The knowledge generated through the data collected in this learning study are results from an in-depth analysis of what is made possible to learn, based on how the content is structured and handled in the teaching. How closely the collected empirical data, the analysis of the data and the conclusions drawn are linked is of importance, and that a clear description of the study is carried out (Kvale, 1997). Reliability in a study is about accuracy, which relates to ensuring that the researcher is as thorough as possible in the collection and analysis of data (Stenhouse, 1981). During the whole research process in this study, it was important to provide a transparent description of how the data was sampled and how the analysis was carried out. The data consists of video recorded lessons and pre- and post-tests, which could contribute to analyses and an in-depth description of the relationship between teaching and student learning about the object of learning. Repeated analyses both of lessons (video recorded whole-class discussions and small-group discussions) and of students' learning outcomes (pre- and post-tests) were seen as a necessity to be able to provide deeper analysis and greater accuracy in the claims. The analysis of the lessons as a whole, presented together with excerpts of illustrative teaching moments or situations, also contributed to increased credibility/trustworthiness. Although the researcher has worked independently, different elements of data, method issues and analyses have been continuously discussed with supervisors and colleagues to validate interpretations. In order to increase reliability and give a good insight into the empirical material and analysis, many excerpts from empirical data are included in the results sections of the articles. In this way, the study can be said to be more trustworthy. Describing the approach in a study in a clear way can be a way to facilitate the replicability of a study.

Chapter 6 Results

In this chapter, a summary of each of the thesis' articles is provided with an emphasis on the results of each article.

6.1 Article 1

Title: Understanding the relationship between length and area when changing the size of a two-dimensional geometric figure.
Authors: Svanteson Wester, J., & Kullberg, A. (2020).
Published in: Nordic Studies in Mathematics Education, 25(1), 89-109.

The data analysed in this paper was generated from three cycles in a learning study about geometry and proportionality (Svanteson Wester, 2014). The analysis of the lessons from the three cycles was conducted with the aim of contributing to a deeper understanding of students' ways of seeing the relationship between the critical aspects, change in length and change in area, when enlarging or reducing two-dimensional geometric figures, and how this object of learning could be taught in order to overcome "the illusion of linearity". The research questions were: *How are students' experiences of the relationship between change in length and change in area, when enlarging or reducing two-dimensional geometric figures, shown in lessons?* and *How is the knowledge that teachers gain about the students' experiences reflected in their teaching?* The results showed that the teachers gained insights about students' experiencing of the object of learning. The interplay between the teachers' teaching and the students' difficulties in regard to the object of learning was investigated. The analysis showed that the teachers refined their teaching using the knowledge gained about students' experiencing of the object of learning. Initially, the teacher team thought that it would be beneficial for student learning if the students first had the opportunity to discern the change in length in relation to a given scale factor, and then had the opportunity to discern that the area will not change in the same way as the lengths. The analysis of the data showed that the students, on the other hand, wanted to discuss how the change in length and change in area were connected, and not only that they were different. For example, in Cycle 1, the teacher started to discuss in whole class what was twice as long in the enlarged figure on the white board, when scaling with the scale factor 2:1. The students showed by their comments that they wanted to discuss the change in area at the same time as the change in length. However, the teacher did not take the students' comments about the change in area into account, and thus the relationship between the change in length and change in area was not jointly explored in the whole class. In the first cycle, the students did not get the opportunity to see how the two aspects change in length and change in area were connected. In Cycle 2, some of the students had, in a small-group discussion, an opportunity to explore the relationship between change in length and change in area. When the students worked on a group task about reducing a sheet of paper by scale factor 1:3, two different answers were brought up in whole-class by the students in one of the small-groups (see Figure 5). Those two different answers offered an opportunity for the students to explore the relationship between change in length and change in area.

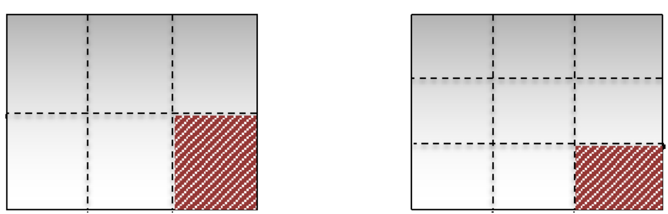

Figure 5. Two different answers to the group task about reducing a sheet of paper by scale factor 1:3 (Cycle 2).

When one student argued that the new area should be one-sixth of the original, another student instead argued that it was one-ninth. The rationale for the first student's answer may be related to the solution of a task from the beginning of the lesson, about reducing a sheet of paper by scale factor 1:2 (in this case the area was doubled due to the particular scale factor 1:2). The teachers found that the students, when reducing a sheet of paper by scale factor 1:2 and identifying the change in area as a fourth, understood the "four" in "one fourth of the original area" in different ways: as "two plus two", "two multiplied by two", or "two to the power of two", all of which are four. This made some students believe that the area was also reduced by half when encountering other scale factors (e.g. 1:3, 1:4). The contrast between a set of carefully designed tasks involving the following scale factors, first 1:2 and 2:1, then 1:4 and 4:1, followed by 1:3 and 3:1, proved to be successful, as it allowed the teachers to recognise that students had different ways of experiencing the number 2 in the scale factor 1:2 when identifying the change in area. The teachers' analysis from the video recorded small-group discussions in Cycle 2 played a significant role for the teachers, allowing them to notice how the students were experiencing the object of learning, and how to refine the teaching for the third cycle. The teachers came to the conclusion that they needed to handle the two critical aspects, change in length and change in area, simultaneously in Cycle 3, to make the change in length and change in area, and the relationship between them, come to the fore during the whole-class discussion.

6.2 Article 2

Title: Students' possibilities to learn from group discussions integrated in whole-class teaching in mathematics
Author: Svanteson Wester, J. (2021)
Published in: Scandinavian Journal of Educational Research, 65(6), 1020-1036

The data analysed in this paper was generated from lessons with two Grade 8 classes and students' pre-and post-tests. In total, 33 small-group discussions ([2]6 groups in Class A, 5 groups in Class C) and twelve whole-class discussions were analysed. The research questions were: *What objects of learning, related to enlarging and reducing two-dimensional geometric figures, are enacted in the small-group discussions and the whole-class discussions?* and *How do the enacted objects of learning in the small-group discussions and the whole-class discussions correspond to student learning outcomes?* The study aimed to better understand how teaching with small-group and whole-class discussions contributes to student learning, by focusing on what was made possible to learn (the enacted object of learning) in small-group and whole-class discussions in the same lesson, and how this was related to what the students were supposed to learn (the intended object of learning), and what the students learned (the lived object of learning). The results showed qualitative differences between the small groups concerning what object of learning was enacted during the discussions, and qualitative differences between small-group and whole-class discussions were also identified. During the small-group discussions, only six of eleven small groups enacted an object of learning that corresponded to the intended object of learning.

[2] In cycle four, the two classes have different designations depending on which article they are a part of. The classes are called A and C (article 2) and D and E (article 3), respectively.

Only in seven of the 33 small-group discussions was an object of learning enacted that corresponded to the intended object of learning (see Table 7).

Task	The photo IOL = EOL 4	The plus sign IOL = EOL 4	The square IOL = EOL 4	The sheet of paper (rectangle) IOL = EOL 4		The pizza (circle) IOL = EOL 5		The heart (irregular shape) IOL = EOL 6	
	Whole-class	Whole-class	Whole-class	Group	Whole-class	Group	Whole-class	Group	Whole-class
Class A	EOL 1	EOL 4	EOL 4		EOL 4		EOL 5		EOL 6
Gr. 1A				EOL 3		EOL 4		EOL 4	
Gr. 2A				EOL 4		EOL 4		EOL 4	
Gr. 3A				EOL 2		EOL 4		EOL 6	
Gr. 4A				EOL 2		EOL 2		EOL 1	
Gr. 5A				EOL 2		EOL 1		EOL 1	
Gr. 6A				EOL 3		EOL 1		EOL 3	
Class C	EOL 1	EOL 2	EOL 4		EOL 4		EOL 5		EOL 6
Gr. 7C				EOL 3		EOL 5		EOL 4	
Gr. 8C				EOL 4		EOL 4		EOL 5	
Gr. 9C				EOL 4		EOL 4		EOL 2	
Gr. 10C				EOL 4		EOL 1		EOL 6	
Gr. 11C				EOL 1		EOL 1		EOL 4	

Table 7. The enacted objects of learning (EOLs) identified in whole-class and small-group discussions. Dark grey indicates enacted objects of learning (EOLs) in the small-group discussions that correspond to the intended object of learning (IOL) for the task.

However, when students' learning outcomes (the lived objects of learning) from the post-tests were compared to the enacted objects of learning from the small-group discussions, it was shown that the students did have rich opportunities to learn what was intended. On the post-test, a majority of the students showed that they had discerned the critical aspects of the intended object of learning, i.e. the students' lived object of learning corresponded to the intended object of learning to a larger extent than the enacted objects of learning. Despite different enacted objects of learning during the small-group discussions, the students' learning outcomes were similar on the post-tests. The analysis showed that in the whole-class discussions that followed each episode of small-group discussions, the enacted objects of learning corresponded to the intended object of learning to a larger extent. The teachers' awareness of critical aspects of the object of learning meant that the teachers were prepared for how to respond to students' experiencing of the object of learning which took place in the small-group discussions, as presented by the students in the whole-class. The small-group discussions were used by the teacher, as a teaching arrangement, as a means to create a space in which the students had an opportunity to explore the object of learning. The whole-class discussions seemed to be a means to further explore the object of learning in a more public domain and thus widen the space of learning further for each student. With well-planned systematic teaching related to small-group discussions that are integrated into whole-class discussions and a focus on the object of learning during the lessons, this lesson design offered the students more opportunities to explore critical aspects, which contributed to an increase in dimensions of variation opened up and in opportunities to discern the critical aspects.

6.3 Article 3

Title: Using small-group discussions in a learning study about geometric scaling.

Author: Svanteson Wester, J.
Published in: Submitted

The learning study analysed in this article consisted of four cycles (ten lessons) with the aim of exploring the relationship between teaching and students' learning outcomes regarding enlarging and reducing two-dimensional geometric figures, when small-group discussions were used during the lessons. The research question was: *In what ways can the integration of small-group discussions into whole-class teaching enhance student learning?* Throughout all four cycles, small-group discussions were used during the lessons, but the researcher's analysis focused on how the object of learning was handled in the whole-class part of the lessons in relation to students' learning outcomes. The results of the pre- and post-tests (maximum 9 points) showed that students' learning outcomes increased from the first to the fourth cycle, with a significant increase between Cycles 3 and 4. One class each was involved in the three first cycles (Classes A, B and C), and in the fourth cycle there were two classes involved ([3]Classes D and E). The mean value increased, in Cycle 1 from 2.41 to 4.65, Cycle 2 from 1.62 to 5.14, Cycle 3 from 1.06 to 5.63, and Cycle 4 from 1.90 to 7.24 (in Class D) and 1.94 to 7.88 (in Class E). The results of the

[3] In cycle four, the two classes have different designations depending on which article they are a part of. The classes are called A and C (article 2) and D and E (article 3), respectively.

post-test in the fourth cycle also showed that the two classes in Cycle 4, Classes D and E, had similar learning gains.

The analyses indicated that the critical aspects related to the object of learning, enlarging and reducing two-dimensional geometric figures (Svanteson Wester, 2014) were in the foreground during the lessons in all four cycles but were explored in different ways. The results, related to the mean value on the pre- and post-test, showed a significant increase of students' learning outcomes between Cycle 3 and Cycle 4. In the analysis, differences in how the critical aspects were explored in the first three cycles on the one hand, and the fourth cycle on the other hand, were noticed. In addition, empirical evidence showed that the significant increase occurred in both classes in Cycle 4. One explicit difference was found between the lesson designs in the first three cycles and in the fourth cycle. In the lesson design in Cycle 4, a small-group activity was planned in which the task involved was about scaling of a sheet of paper by the scale factors 1:2, 1:4 and 1:3. The task was designed in such a way that the students in the small-groups were supposed to explore and discuss a given variation in other students' solutions, taken from the first three cycles and related to scaling of two-dimensional geometric figures, both mathematically correct and incorrect ones. The teacher in Cycle 4 followed up each of those small-group discussions with a whole-class discussion in which all small-groups were invited to participate to discuss the object of learning and explore the critical aspects further. How the object of learning was handled in this part of the lesson, when the students explored other students' ways of seeing the object of learning due to the design of the task, enabled a richer opportunity to open up dimensions of variation and the students seemed to be prepared for a whole-class discussion where the possibility of the critical aspects being explicitly discerned increased.

In comparison, in the first three cycles, the small groups were only exposed to their own group's ways of seeing the object of learning. The task did not involve presenting students with a variation in other students' ways of seeing the object of learning. Moreover, the presentation in the first three cycles was mostly a dialogue between the small group and the teacher, and did not involve the whole-class in an explicit whole-class discussion. Students in Cycle 4 were also offered a greater variation in regard to the geometric shape worked with. The last group task in Cycle 4 contained an irregular heart to be enlarged. This task was not included in the first three cycles. In sum, there were richer possibilities to explore the object of learning in Cycle 4 and the students were offered a greater and more systematic variation in ways of seeing the object of learning.

Thus, the differences related to the students' learning outcomes between Cycle 3 and Cycle 4 were not whether critical aspects were discernible or not, but rather how the critical aspects of the object of learning were explored during the lessons. The lesson design in Cycle 4, which consisted of planned small-group discussions with subsequent whole-class discussions, in which the same task was the basis for the discussion, most likely contributed to objects of learning being enacted in a more powerful way, with regard to critical aspects and dimensions of variation opened up. Another significant difference between Cycle 3 and Cycle 4 was that small-group discussions in Cycle 4 were used to systematically explore the variation in students' ways of seeing the object of learning, rather than only as a means for students to talk about the object of learning or to jointly solve the group task. The enacted whole-class discussion, supported by the teacher, contributed to the possibility of explicitly discerning critical aspects for all students. The variation used in the lesson design in Cycle 4 may illustrate the essential role of variation in teaching and learning. The study shows that with a systematic variation explored in small groups, the student learning increased (Marton & Tsui, 2004; Watson & Mason, 2006).

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Chapter 7 Discussion

The overall aim of this thesis is to contribute to a deeper understanding of the relationship between teaching and student learning in mathematics when small-group discussions are a part of whole-class teaching. The learning study model together with principles from variation theory were used to shed light on the relationship between teaching and learning and the learning opportunities constituted in lessons (Marton & Tsui, 2004; Marton, 2015). One point of departure is that all parts of the lesson need to be analysed in order to know what learning opportunities emerge during the lesson. To meet the aim, the following research questions were posed:

- What learning opportunities are constituted when small-group discussions are used in whole-class teaching?
- What is the significance of small-group discussions for student learning about scaling two-dimensional geometric figures?

The overall finding is that the use of small-group discussions as a planned and integrated part of whole-class teaching can contribute to widening the space of learning and thereby increase students' opportunities to learn what was intended. Lessons with pre-planned tasks for small-group discussion integrated into whole-class discussions seem to provide more powerful learning opportunities in relation to the object of learning compared to lessons with a less systematic use of small-group discussions (Article 3). In the study it was found that the small-group discussions on their own did not provide sufficient opportunities for students to learn what was intended. Instead, the interplay between small-group discussions (where different ways of experiencing the object of learning were made possible to explore) and a subsequent whole-class discussion (in which different ways to experience the object of learning were brought to the fore in relation to one another) seemed to contribute to students' learning (Article 2). The results also show how teachers benefit from listening to small-group discussions and students' reports of such discussions since it can be a means to develop their teaching practice (Articles 1 & 3). This was shown when the teachers changed their teaching in response to what they noticed about what could be critical for student learning about the intended object of learning.

7.1 Learning opportunities constituted

The concept learning opportunity is considered to be powerful for describing the relationship between teaching and learning (Hiebert & Grouws, 2007; Yackel et al., 1991). In this study, variation theory was used to analyse students' learning opportunities in terms of the enacted object of learning in whole-class and small-group discussions. A focus on how the content is handled in different parts of the lesson is rare in previous studies on learning opportunities in mathematics during classroom discussions. These studies investigate, for example, learning opportunities as they arise when students are engaged in collaborative activities (e.g. Yackel et al. 1991). Yackel et al. (1991) conclude that it is the verbalisation of students' thinking and justification of their solutions and clarifications in the interaction that takes place in discussions that provide the learning opportunities. Thus, the learning opportunities arise when the students attempt to reach a consensus as they work together. The results in this thesis are more detailed and show the learning opportunities in the small-group discussions and whole-class discussions. By differentiating between how the object of learning was enacted in the small-group discussions and the whole-class discussions, it was possible to identify differences in how the object of learning was enacted in all small-group discussions, as well as how the enacted object of learning in the whole-class discussions differed from the enacted object of learning in the small groups.

The results suggest that it was not the small-group discussion that provided the learning opportunities for the students. Instead, the interplay between small-group and whole-class discussions seemed to make a difference in the learning opportunities. The evidence for this claim is that the whole-class discussions which followed the small-group discussions in this study were necessary for enacting the object of learning more in line with the one intended. The analysis of the lessons in Cycle 4 showed that the interaction between small-group discussions and whole-class discussions was central.

The variation in how the object of learning was enacted in the small groups indicated that it was not enough that the students only made distinctions related to the intended object of learning by themselves with no support from the teacher (Marton, 2015). In most of the small-group discussions in this study, the critical aspects were not explicitly explored, and the enacted object of learning did not correspond to the intended one. Ball (1993) and Ball et al. (2008) argue that teachers need to have insights into students' thinking about the topic taught since those insights can provide a foundation for classroom discussions focusing on the topic taught. The results in this study showed that the teachers' insights about students' different ways of experiencing the object of learning and the teachers' knowledge about the critical aspects were necessary for enacting small-group and whole-class discussions in whole-class teaching in a powerful way (Ball, 1993; Ball et al., 2008). Cengiz et al. (2011) showed similarly that teachers' actions, such as highlighting students' thoughts and posing relevant questions to the small groups, were necessary to create conditions for extending students' mathematical thinking.

Marton (2015) stresses the importance of teachers paying attention to students' ways of seeing the object of learning during teaching. In this study, the teachers gained information about the students' lived object of learning from the small-group discussions when the small groups presented to the whole class. From this, the teacher could decide which variation was appropriate to offer students during the subsequent whole-class discussions so that critical aspects not yet discerned by all students would be possible to discern. This "exchange of variation" (Al Murani & Watson, 2009) enabled co-construction of the enacted object of learning in the whole-class discussion that corresponded to a larger extent to the intended object of learning compared to what was possible in the small-group discussions. The students' way of seeing the object of learning, in this study, came to the fore in small-group discussions and was further elaborated on in the subsequent whole-class discussions, directed by the teacher's awareness of critical aspects that needed to be discerned. This most likely increased the learning opportunities for all the students. The study indicates that although the small-group discussions did not always generate the intended learning, the discussions are identified as important for the learning opportunities constituted in the lesson as a whole. The small-group discussions became a significant part of the whole-class teaching and were a driving force in the object of learning being handled so that the critical aspects were exposed more explicitly during the fourth cycle compared to the first three cycles.

The results of this study are significant for understanding students' learning opportunities during lessons comprising small-group discussions and discussions in the whole class. In previous studies of mathematics teaching using variation theory as a theoretical framework, only the whole-class parts of lessons have been analysed (e.g. Ekdahl, 2019; Häggström, 2008; Maunula, 2018; Runesson, 1999).

In this way, the study provides a more comprehensive and complex picture of students' learning opportunities compared to previous studies using the same framework.

7.2 Small-group discussions in teaching for student learning about scaling two-dimensional geometric figures

In the following section, the results of the study concerning teaching and student learning are discussed from the point of view of: i) the use of small-group discussions, ii) the development of the group tasks, and iii) the specific content taught.

7.2.1 The use of small-group discussions

How small-group discussions were used during the lessons was developed by the teachers during the learning study process. The study shows a shift in the purpose of using small-group discussions. In the first three cycles, the small-group discussions were used primarily as an opportunity for students to express themselves about the content and as a way for students to solve problems together and compare their different solutions (pointed out as beneficial by, e.g. Francisco, 2013; McCrone, 2005; Ryve et al., 2013; Webb, 2009). In Cycle 4, the small group discussions were instead used as an opportunity for the students to systematically explore aspects of the content (Ball, 1993; Yackel et al., 1991). The small-group discussions in Cycle 4 were used to encourage in-depth mathematical discussions, i.e. aspects of the mathematical content were in focus (Yackel et al., 1991). This entailed changing the function of small-group discussions during the mathematics lessons. In the first three cycles, the function of small-group discussions was more about solving group tasks together and using students' own words to talk about the mathematics, whereas in the fourth cycle, the small-group discussions became a central and integrated part of the teaching with the purpose of contributing to the exploration of the object of learning. The reason for this change was based on insights from an extended analysis of the lessons in the second cycle.

During the learning study process, the teachers gained insights about students' way of experiencing the content, e.g. the relationship between critical aspects, when listening carefully to one of the small groups during the lesson (Article 1). The teachers made the relationship between the two critical aspects come to the fore in the lessons in the third cycle. However, it was not until the fourth cycle that the teachers purposefully changed the function of the group task to align with this insight, which in turn, had a positive significant effect on the students' learning opportunities. This points to the importance of empirically testing different ways of bringing critical aspects to the fore. Through demonstrating that shifting the function of small-group discussions in the overall teaching setup has the potential to be powerful in this respect, the study broadens the designs available for teachers and researchers to consider when developing teaching within learning studies, as well as in everyday teaching practice.

7.2.2 The refinement of group-tasks

The group tasks used in the lessons were designed to elicit critical aspects of the object of learning. The findings in the study show that the design of group tasks was important when developing the use of small-group discussions in whole-class teaching. In Cycle 4, the group tasks were refined so that there was a progression regarding the complexity of geometric figures used. Further, students' different ways of experiencing the object of learning came to the fore during the first group task and were explored to a greater extent (than in the first three cycles). The teachers' knowledge about students' different ways of experiencing the object of learning, gained from the three first cycles in the learning study, was of the utmost importance when refining the first group task in Cycle 4. In this task, students' different ways of experiencing the object of learning were shown with the aim that the students should explore these different solutions and different ways of experiencing the object of learning. The refinement of the group tasks implied that the students were asked to discuss correct and incorrect solutions when scaling two-dimensional geometric figures (Article 3). The lessons with a built-in progression of pre-planned small-group tasks in which students' different ways of experiencing the object of learning were in the foreground made it possible to discuss and explore the object of learning in such a way that the relationship between the critical aspects change in length and change in area came to the fore. The sequence of the various group tasks was enacted so the relationship between the critical aspects, change in length and change in area (in figures with increased complexity) would be systematically noticed. Different ways of seeing the relationship were in focus, which seemed to contribute to the relationship being examined explicitly in the small groups.

The importance of giving students the chance to discuss flawed solutions and errors to promote student learning has been suggested in previous research (e.g. Staples, 2007; Bray, 2011). In this study, the results indicated that the lesson design in Cycle 4, which involved group tasks with mathematically correct and incorrect ideas being discussed, contributed to the whole-class discussions becoming more powerful regarding the critical aspects being explicitly explored. This is in line with Staples' (2007) findings, which showed that conjecturing, adjusting, and reconciling ideas that are not fully mathematically correct by employing classroom discussions can contribute to learning for the whole class.

7.2.3 Student learning of mathematics

In this study, student learning of mathematics is investigated during a learning study with four cycles (Cheng & Lo, 2013), where each cycle involved two lessons that included classroom discussions. Developing rich and valuable mathematical discussions for students is a complex enterprise that involves multiple aspects, e.g. choice of tasks, the nature of the classroom environment, and communicative competencies. Thereby the focus in research is often on investigating students' development of communication and collaboration skills during longer periods of time (e.g. McCrone, 2005; Ball, 1993). In this study, the students only participated in two lessons designed by the teacher team. The results show that student learning about scaling two-dimensional geometric figures increased significantly after only two lessons. In this way, the study differs from previous research where student learning is related to students' participation in classroom discussions during a more extended period (e.g. McCrone, 2005; Ball, 1993). Only a few studies give insights into teaching that support student learning of scaling two-dimensional geometric figures. However, there is a vast body of research describing students' difficulties when scaling two-dimensional geometric figures and the "illusion of linearity" (e.g. Ayan & Bostan, 2018; De Bock et al., 1998; Hilton et al., 2013; Modestou et al., 2004). Findings from this study indicate that it was beneficial for student learning to discuss the relationship between the two critical aspects, change in length and change in area when scaling two-dimensional geometric figures. Tasks about comparing relationships are essential for students to develop their understanding of, for instance, proportionality (Lamon, 2007). Findings in this study suggest it is crucial to contrast the relationship between change in length and change in area with other relationships, e.g. additive and linear relationships. The results show that students' learning increased after each cycle in the learning study, but a significant increase was shown after the lessons in Cycle 4. In Cycle 4, classroom discussions were used systematically in which the relationship between the two aspects came to the fore and was explored both in small-group and whole-class discussions. It was also shown that students in Cycle 4 with both high and low achievements on the pre-test significantly increased their learning outcomes in the post-test compared to previous cycles. This was shown in pre-and post-test results for both classes in Cycle 4. Student learning outcomes on the post-tests indicated that the teaching involving small-group discussions integrated into whole-class discussions enhanced student learning about the object of learning.

7.3 Contributions of the study

This study primarily adds to previous research regarding: i) the function of small-group discussions in mathematics teaching, ii) the teaching of scaling two-dimensional geometric figures, and iii) the analysis using variation theory of lessons involving small-group and whole-class discussions. By using a different theoretical framework than previous studies (e.g. Forslund Frykedal & Hammar Chiriac, 2012; Francisco, 2013; Ryve et al., 2013; Sjöblom, 2022), it was possible to focus on the content taught and analyse what was made possible to learn in the small-group discussions and whole-class discussions, and the relationship between them. The teachers used the small-group discussions to make students' understanding of the object of learning visible, rather than what seems to be more common in previous research, where the objective has been to assist students in developing generic skills (e.g. communication skills) in mathematics through collaboration. By showing how teaching with a systematic use of small-group discussions integrated with whole-class discussions related to student learning, the findings add to previous research suggesting that both how the topic is taught, and the structure of the lesson have to be considered to facilitate and increase student learning in mathematics when classroom discussions are used (Kosko, 2012; Stein et al., 2008). The study shows that a planned and systematic interplay between small-group discussions and whole-class discussions can contribute to the content being explored more explicitly, increasing the opportunity for all students to learn what was intended.

The study also contributes knowledge about how the topic can be taught to increase student learning. Previous research (e.g. Ayan & Bostan, 2018; De Bock et al., 1998) has primarily focused on students' difficulties in separating proportional relationships from non-proportional relationships when scaling up or down two-dimensional geometric figures, and not primarily on teaching about those relationships. In this study, it was shown that the students seemed to focus on the relationship between the two aspects change in length and change in area when scaling two-dimensional geometric figures. When the small-group discussions about reducing a sheet of paper by scale factor 1:2 were analysed, different ways of seeing why the change in area became four times smaller were identified. The students saw the change in area as "two plus two", "two multiplied by two", or "two to the power of two", all of which are four. This became problematic when students encountered scale factor 1:3 in the next group task and they expressed that the area was also reduced by half, i.e. if the lengths were "a third", the area was "a sixth". During the learning study process, it was found that it was not enough to teach the two critical aspects, change in length and change in area, separately since the analysis of the lessons indicated that there was a need to investigate the relationship between them as well.

The study contributes to the theoretical framework used – variation theory – by proposing a way to analyse students' learning opportunities in all parts of a lesson, i.e. in small-group and whole-class discussions, and the relationship between them. Variation theory has, as mentioned before, primarily been used for analysing whole-class teaching parts of a lesson (e.g. Kullberg et al., 2017; Maunula, 2018; Ryberg, 2020) or small-group discussions as isolated from a whole-class setting (e.g. Berge & Ingerman, 2017). By using variation theory to analyse how the object of learning is handled, with a focus on dimensions of variation, it was possible to identify the enacted object of learning in the small groups and the whole-class discussions. When relating the enacted object of learning in the different groups and settings to the intended one, it was possible to identify more or less powerful learning opportunities in small-group and whole-class discussions.

7.4 Limitations and directions for future research

There are several limitations in this study. First, the study is based on only one learning study with four cycles, which involved three teachers, five classes, and 92 students. A larger number of lessons, teachers, and students would most likely have contributed to more reliable conclusions about teaching and student learning.

Second, this study showed that the students in the small-group discussions did not enact the object of learning in ways that to a greater extent corresponded to the intended object of learning than what was enacted in the whole-class teaching.

A new study may not show the same relation. Several studies are needed to get a more comprehensive picture of students' learning opportunities when small-group discussions are used. Moreover, only one specific topic was taught when studying small-group discussions in whole-class teaching. An analysis of learning studies using small-group discussions in the teaching of different topics in mathematics would have provided a more stable interpretation.

Third, the study makes claims about learning opportunities and students' learning of the specific topic. However, the evidence of student learning is limited to what can be analysed from pre-and post-tests. The pre-and post-tests have several weaknesses when it comes to validity and reliability. Designing tasks that capture specific knowledge (Carlgren et al., 2017) and interpreting students' answers is challenging. Interviews with students would most likely have contributed to an increased validity in interpreting student answers. The learning study process may have contributed to an increase in teachers' experience of teaching about the specific object of learning, which in turn may have contributed to the increase in student learning in the fourth cycle.

Finally, the data collection and analysis for this research were carried out by a single researcher. Although efforts were made to collect and analyse data consistently, a research project of greater scope would be strengthened by interpretations of multiple researchers to allow for data to be examined from a broader range of perspectives to increase credibility of findings. Various variables are uncontrolled in teaching and student learning in classroom environments. It can thus be difficult to fully establish whether it is how the object of learning is handled or some other variable that has influenced student learning and students' learning opportunities in this study. Criticism is expressed by Shavelson et al. (2003), who believe that many studies in classroom environments contain so many variables that it becomes difficult to determine what caused the result. However, what you analyse from the rich material that video-recorded lessons produce depends on choice. For instance, the students' verbal language, participation or collaborative skills during the small-group discussions were not analysed in this study. However, the specific research design and analysis with detailed descriptions strengthen the interpretation of the results as suggested by Larsson (2005).

There is a need for future research on student learning when small-group discussions are used in whole-class teaching. More studies are needed to get a more comprehensive picture of what the students learn in relation to a given content during lessons with small-group discussions, and what learning opportunities are constituted during different parts of the lesson. It would be valuable to know how this kind of detailed description about learning opportunities can be used or developed when other topics are taught and when small-group discussions are integrated into whole-class discussions. The study of other topics does not necessarily need to be carried out as a learning study. Another suggestion is that the lesson design in this study (small-group discussions integrated into whole-class discussions) with the specific topic taught or with another topic taught can advantageously be scaled up to involve more classes and teachers, in order to better understand aspects of students' learning when small-group discussions are used. Further, it would be beneficial to learn more about how to design tasks with a progression to be used during lessons with small-group discussions. Also, the tasks in the pre-and post-tests could be developed in order to reflect to a greater extent what the students learned from the lessons with small-group discussions. Students' post-tests could be complemented with interviews to provide a more nuanced picture of what students learn during lessons with classroom discussions.

7.5 Implications for teaching

The findings presented in this thesis have two main implications for teaching with small-group discussions. First, how small-group discussions are implemented in lessons are of importance for students' learning of the object of learning and learning opportunities constituted during the lesson. It was found that subsequent whole-class discussions played a major role for students' learning opportunities in lessons with small-group discussions. In the whole-class discussion, teachers are advised to bring to the fore what has been discussed in the small-group discussions, as this may provide the teacher with information about how students experiencing the object of learning, which can lead to the teacher being better prepared to meet students' difficulties. If the teachers do not know how the students are experiencing the object of learning during the lesson, they may not be able to give useful responses to students and thereby enact powerful whole-class discussions. Second, teachers' knowledge about critical aspects of the object of learning and insights about student ways of experiencing the object of learning were vital in this study. The critical aspects identified can be used by other teachers to create group tasks with a well-thought-out progression and to direct the subsequent whole-class discussion towards what is intended to be learnt. The findings showed that it was crucial for the teachers to obtain insights on how the students experienced the relationship between the two aspects change in length and change in area when enlarging and reducing two-dimensional geometric figures. It is suggested that insights about critical aspects of enlarging and reducing two-dimensional geometric figures and the use of students' ways of seeing the object of learning may increase the possibility of creating powerful teaching to overcome the illusion of linearity.

Summary in Swedish

Undervisning och lärande i matematik då smågruppsdiskussioner används – En learning study om skalning av geometriska figurer

Den här avhandlingen har tillkommit utifrån ett intresse av att bättre förstå relationen mellan undervisning och elevers lärande i matematik då smågruppsdiskussioner används i helklassundervisning. Avhandlingen har som avsikt att ge insikter om undervisning och elevers lärande av ett specifikt ämnesinnehåll samt de lärandemöjligheter som konstitueras då smågruppsdiskussioner används i helklassundervisning. Studien har också som avsikt att bidra med kunskap som är direkt användbar för matematiklärare i deras undervisning. Tidigare forskning visar att klassrumsdiskussioner (i par, i små grupper och helklass) ofta är en möjlighet för eleverna att få matematiska insikter och kunskaper (se t.ex. Ball, 1993; Cengiz et al., 2011; Hiebert & Waerne, 1993; Kosko, 2012). Kosko (2012) visar till exempel att elever som regelbundet deltar i klassrumsdiskussioner i matematik ökar sin förståelse för matematiska idéer i större utsträckning än elever som aldrig eller sällan deltar i klassrumsdiskussioner. Forskning visar också att de klassrumsdiskussioner som genomförs under matematiklektioner har olika syften. Till exempel kan smågruppsdiskussioner användas med syfte att ge eleverna en möjlighet att lösa matematiska problem tillsammans och därmed utveckla sitt sätt att resonera och kommunicera matematik då de under problemlösningen får förklara och kritiskt granska sina egna och andras resonemang (se t.ex. Francisco, 2013). Helklassdiskussioner kan ses som ett tillfälle för eleverna att engagera sig i en argumenterande aktivitet om ett matematiskt innehåll, vilket förmodas leda till ett lärande i matematik (t.ex. Ayalon & Even, 2016). Vad eleverna lär i relation till ett specifikt ämnesinnehåll när smågruppsdiskussioner används eller vilka lärandemöjligheter som konstitueras och på vilket sätt är mer sällan studerat. Det tycks också vara mer utbrett att undersöka elevernas lärande i smågruppdiskussioner då dessa inte ingår i en helklassundervisning utan som en isolerad händelse (t.ex. Weber et al., 2008). Flera forskare betonar att studier med fokus på hur och under vilka förhållanden diskussioner i små grupper och i helklass bidrar till elevernas lärande i matematik behövs. Det tycks även behövas studier med resultat som är direkt användbara för lärare i praktiken (t.ex. Francisco, 2013; McCrone, 2005, Weber et al., 2008). Det för studien valda ämnesinnehållet inom matematik, förstoring och förminskning av tvådimensionella geometriska figurer har varit föremål för ett flertal studier. Tidigare forskning har visat att en majoritet av 12–16-åriga elever har en tendens att utgå ifrån ett linjärt samband då de löser uppgifter av icke linjär karaktär. Fenomenet, som kallas 'illusionen av linjäritet', kommer av att elever då de ska förstora eller förminska flerdimensionella geometriska figurer intuitivt tenderar att utgå ifrån att om alla sidor görs dubbelt så långa blir även arean och volymen dubbelt så stor (e.g. De Bock et al. 1998; Van Dooren et al., 2004).

Syfte och frågeställning

Det övergripande syftet med avhandlingen är att bidra med djupare förståelse för relationen mellan undervisning och elevernas lärande då smågruppsdiskussioner används i helklassundervisning i matematik. I avhandlingen används det variationsteoretiska ramverket för att analysera relationen mellan undervisning och elevers lärande av det specifika ämnesinnehållet förstoring och förminskning av tvådimensionella geometriska figurer.

Följande frågeställningar avses att besvaras:

- Vilka lärandemöjligheter konstitueras när smågruppsdiskussioner används i helklassundervisning?
- Vilken betydelse har smågruppsdiskussioner för elevernas lärande avseende förstoring och förminskning av tvådimensionella geometriska figurer?

De tre artiklarna som ingår i avhandlingen har specifika syften och forskningsfrågor, som besvaras i respektive artikel.

Teori

Tidigare forskning visar att variationsteori är ett kraftfullt analytiskt verktyg för att analysera samband mellan undervisning och elevers lärande av ett lärandeobjekt i matematik i termer av hur lärandeobjektet behandlas under lektionen (t.ex. Häggström, 2008; Kullberg et al., 2017). Hur en elev erfar ett lärandeobjekt beror på vilka aspekter av lärandeobjektet som eleven urskiljer då lärandeobjektet behandlas. Enligt Marton (2015), kan lärande ses som en process där eleven ändrar sitt sätt att erfara lärandeobjektet genom att eleven urskiljer nya aspekter eller genom att urskilja en relation mellan aspekter eller en relation mellan aspekter och helheten. Vidare, de aspekter som är nödvändiga att urskilja för att erfara lärandeobjektet på ett visst sätt men som ännu inte har urskilts av eleven, benämns som kritiska aspekter (Marton, 2015). Vad som görs möjligt att läras i undervisningen kan förstås genom att undersöka vilka kritiska aspekter av lärandeobjektet som öppnas upp som dimensioner av variation (Marton & Tsui, 2004; Marton, 2015). En kritisk aspekt av ett lärandeobjekt som är iscensatt genom dimensioner av variation och kan användas för att jämföra de lärandemöjligheter som konstitueras för lärandeobjektet under lektionen. Tidigare har variationsteori endast använts för att analysera elevers lärande och lärandemöjligheter i den publika delen av helklassundervisning (se t.ex. Kullberg et al., 2017; Maunula, 2018; Pang & Marton, 2005). Elevers lärandemöjligheter i smågruppsdiskussioner har i ett fåtal studier analyserats isolerat, det vill säga separat från undervisning i helklassen. Till exempel Berge och Ingerman (2017) och Ingerman et al. (2009) undersökte elevernas möjligheter att utveckla konceptuell förståelse för kraft och friktion i newtonsk mekanik under diskussioner i små enskilda grupper och frånkopplat helklassundervisning. I den här studien görs en intervention i form av en learning study (t.ex. Lo et al., 2007; Pang & Marton, 2003) genomförd i samarbete med lärare. Analysen är inriktad på hur lärandeobjektet behandlas under hela lektionen, det vill säga i smågruppsdiskussionerna och i den publika delen av lektionerna (helklassdiskussioner) samt hur de förhåller sig till varandra och till lärandemöjligheterna som konstitueras. Dimensioner av variation och kritiska aspekter möjliga att urskilja används som analysenhet (i både smågrupps och helklassdiskussioner).

Licentiatuppsatsen

Utgångspunkten för avhandlingen är den tidigare genomförda learning study i tre cykler i Svanteson Wester's licentiatuppsats (Svanteson Wester, 2014). Även i den studien var relationen mellan undervisning och elevernas lärande avseende undervisningsinnehållet, förstoring och förminskning av tvådimensionella geometriska figurer i fokus. Undersökningen i studien riktades dock endast mot de publika delarna av lektionen vilket betyder att hur innehållet behandlades eller vilka lärandemöjligheter som skapades i de inslag av smågruppsdiskussioner som också fanns under lektionerna inte blev föremål för analys. Under lektionerna redovisade smågrupperna till helklassen hur de hade behandlat innehållet i de gruppuppgifter de löst tillsammans. Lärarna kunde på så vis få insikter i elevernas sätt att se på innehållet. I studien visade det sig att en av grupperna stötte på problem när de skulle förminska ett A4-papper i skala 1:3. Några elever i gruppen uttryckte att arean blev nio gånger så liten och andra elever i gruppen uttryckte att arean skulle bli sex gånger så liten. Eleverna som svarade att arean borde bli sex gånger så liten gjorde en jämförelse med uppgiften de genomfört tidigare under lektionen där ett A4-papper skulle förminskas i skala 1:2. Eleverna uttryckte att arean då borde bli fyra gånger så liten och att ”fyran” kom ifrån att 2+2 är fyra. Denna uppfattning tog de sedan med sig till uppgiften där skala 1:3 skulle behandlas och svarade att arean borde bli sex gånger så liten eftersom 3+3 är sex. Den här lektionssekvensen gav upphov till nya frågor om relationen mellan undervisning och elevernas möjligheter till lärande i matematik då helklass- och smågruppsdiskussioner används i helklassundervisning, vilka undersöktes vidare i avhandlingens studie.

Metod

De metoder som används för produktion och analys av data i avhandlingen har valts i enlighet med en variationsteoretisk utgångspunkt och ramats in av learning study som forskningsansats (t.ex. Marton & Runesson, 2015; Pang & Lo, 2012; Pang & Marton, 2003). Learning study är en systematisk undersökningsprocess som involverar planering, implementering, utvärdering och analys av klassrumsundervisning av ett specifikt ämnesinnehåll, ett s.k. lärandeobjekt, samt elevers lärande av detsamma (Cheng & Lo. 2013). Studien består av fyra cykler där de tre första cyklerna även ingick i licentiatuppsatsen (Svanteson Wester, 2014, se kapitel 4). Lektionerna i varje cykel planerades och utvärderades iterativt av lärargruppen där även författaren till den här avhandlingen ingick. Empiriska data för att analysera sambandet mellan undervisning och elevers lärande kommer från 10 videoinspelade helklasslektioner samt 33 videoinspelade smågruppsdiskussioner, och elevernas för- och eftertester. I resultat från Svanteson Wester (2014) identifierades fyra kritiska aspekter avseende lärandeobjektet a) urskilja likformighet mellan figurer, b) urskilja längder i en geometrisk figur, c) urskilja längdförändring vid förstoring och förminskning av tvådimensionella geometriska figurer och d) urskilja areaförändring vid förstoring och förminskning av tvådimensionella geometriska figurer. En ny analys genomfördes av de tre första cyklerna där fokus var relationen mellan elevernas sätt att se på lärandeobjekt och hur lärarnas insikter om detsamma (Artikel 1). Resultatet av en ny analys av de tre cyklerna väckte nya frågor om elevers lärandemöjligheter av det specifika ämnesinnehållet under helklass-lektioner där smågruppdiskussioner används. Detta bidrog till att ytterligare empiri samlades in. Den nya empirin kan ses som en fjärde cykel i learning studyn, men också som separat empiri. Den fjärde cykeln genomfördes i årskurs 8 våren 2017, med fokus på samma lärandeobjekt som under de tre första cyklerna och med de kritiska aspekter som identifierats under dessa cykler. Den nya empirin genererades för att besvara frågor om vad som gjordes möjligt att lära i smågruppsdiskussioner samt vad som gjordes möjligt att läras under lektioner som helhet där smågruppsdiskussioner var en central del i helklassundervisning. De lärandemöjligheter som iscensattes under lektionerna i cykel 4 analyserades (Artikel 2). Slutligen analyseras undervisning och elevernas lärande då smågruppsdiskussioner används under lektionerna genom alla fyra cykler (Artikel 3).

Analysen av innehållets behandling har genomförts på mikronivå. Begrepp från variationsteorin (Häggström, 2008; Marton, 2015, Marton & Tsui, 2004) användes som analysverktyg i analysprocessen. Den detaljerade analysen av underlaget gjorde det möjligt att beskriva skillnader i hur lärandeobjektet behandlades under lektionerna. Analysenheten var sekvenser från lektionerna där lärandeobjektet diskuterades. Transkriptioner från lektionerna analyserades för att söka efter sekvenser där de kritiska aspekterna av lärandeobjektet behandlades. I analysen fokuserades kvalitativa skillnader i vad som gjordes möjligt att urskilja med avseende på identifierade kritiska aspekter och dimensioner av variation som öppnades upp. Analysen möjliggjorde därmed identifiering och beskrivning av subtila skillnader i behandlingen av lärandeobjektet. Det var det möjligt att identifiera vilka lärandeobjekt som iscensattes under lektionens alla helklass- och smågruppsdiskussioner vilket i sin tur gjorde det möjligt att argumentera för mer eller mindre kraftfulla lärandemöjligheter.

Resultat

Det övergripande resultatet är att användningen av smågruppsdiskussioner som en planerad och integrerad del av helklassundervisningen kan bidra till att vidga läranderummet för hela klassen och därmed öka elevernas möjligheter att lära sig det som var tänkt. Lektioner med smågruppsdiskussioner integrerade i helklassklassdiskussioner verkar ge mer kraftfulla lärandemöjligheter i förhållande till lärandeobjektet jämfört med lektioner med en mindre systematisk användning av smågruppsdiskussioner. Sammantaget bidrar resultaten i de tre artiklarna till att fördjupa kunskapen om elevers lärande av ett specifikt ämnesinnehåll i matematik när smågruppsdiskussioner används i helklassundervisning. En utgångspunkt var att alla delar av lektionen, både smågruppsdiskussioner och helklassdiskussioner behöver analyseras för att få vetskap om vilka lärandemöjligheter som iscensätts under en lektion som helhet.

Resultaten visade hur lärare har nytta av att lyssna på de diskussioner som förs i smågrupperna samt till de redovisningar smågrupperna gör i helklass. Lärarna ändrade sin undervisning som svar på att de uppmärksammade hur eleverna behandlade lärandeobjektet i en av smågruppsdiskussionerna i cykel 2. I en av smågruppsdiskussionerna i cykel 2 framkom vad som tycktes vara avgörande för elevernas lärande avseende förstoring och förminskning av tvådimensionella geometriska figurer. Lärarna uppmärksammade att eleverna hade sambandet mellan de två kritiska aspekterna längdförändring och areaförändring i förgrunden då de förstorade och förminskade tvådimensionella geometriska figurer. Sambandet visade sig vara problematiskt, och det krävdes för att lärandeobjektet skulle förstås som det var tänkt, att längdförändring och areaförändring av figuren behandlades simultant under lektionen dels för att kunna skilja sambanden åt, och dels för att urskilja förhållandet dem emellan (Artikel 1).

Hur eleverna behandlade lärandeobjektet i en av smågruppsdiskussionerna under cykel 2 visade sig alltså vara avgörande för hur lektionsdesignen utvecklades vidare. I cykel 4 fick smågruppsdiskussionerna en central roll i helklassundervisningen då de utgjorde en integrerad del i helklassdiskussionerna. Emellertid visade resultat från cykel 4 att det iscensattes mindre kraftfulla lärandeobjekt under smågruppsdiskussionerna. Studien pekar på att smågruppsdiskussioner som en isolerad företeelse inte bidrar till kraftfulla lärandemöjligheter för eleverna. Endast ett fåtal av de 33 smågruppsdiskussionerna, som genomfördes i cykel 4, iscensatte lärandeobjekt som motsvarade det lärande som avsågs för lektionen. Däremot visade för- och efter-testresultat att elevernas lärande ökade markant under cykel 4. Analysen av helklassdiskussionerna i cykel 4 visade att de gemensamt iscensatta lärandeobjekten i helklassdiskussionerna i stor utsträckning motsvarade det som var tänkt att läras. Under helklassdiskussionerna öppnades dimensioner av variation upp och möjligheten för eleverna att urskilja de kritiska aspekterna ökade därmed. Resultatet indikerade att eleverna genom de välplanerade gruppuppgifterna i cykel 4 där de skulle undersöka en variation av möjliga sätt att se på lärandeobjektet gavs möjlighet att förbereda sig för ett fortsatt undersökande av lärandeobjektet i de efterkommande helklassdiskussionerna. Genom att elevernas olika sätt att se på lärandeobjektet kommer i förgrunden i smågruppsdiskussioner och sedan ligger som grund för hur innehållet behandlas vidare i helklassdiskussionerna indikerar resultatet i den här studien att smågruppsdiskussionerna kan ses som bränsle för helklassdiskussionerna avseende innehållets behandling. Detta tillsammans med lärarnas medvetenhet om de kritiska aspekterna tycktes resultera i att lärandemöjligheterna blev kraftfulla för alla elever i helklassen (Artikel 2).

Vid en jämförelse mellan de tre första cyklerna och den fjärde cykeln var det möjligt att se vilken betydelse smågruppsdiskussioner hade för elevernas lärandemöjligheter av det avsedda lärandeobjektet. Resultaten av de för- och eftertest som genomfördes i varje cykel indikerade att elevernas lärande av lärandeobjektet ökade successivt genom hela studien, men visade en markant ökning i de båda klasser som deltog i cykel 4. Skillnaden mellan de tre första cyklerna och den fjärde cykeln avseende hur de kritiska aspekterna utforskades och hur smågruppdiskussionerna användes i relation till helklassen framkom. I de tre första cyklerna fanns inslag av smågruppdiskussioner, men de var inte systematiskt integrerade i helklassdiskussionerna. I den fjärde cykeln förfinas användandet av smågruppsdiskussioner och utforskandet av kritiska aspekter. Elevernas sätt att se på lärandeobjektet, uttryckt under de tre första cyklerna, blev utgångspunkt för skapandet av gruppuppgifter till cykel 4. Gruppuppgifter skapades där eleverna tillsammans fick utforska olika sätt att se på lärandeobjektet. I den efterföljande helklassdiskussion undersöktes lärandeobjektet ytterligare, vilket tycktes öka möjligheten för kritiska aspekter att bli urskiljningsbara. I den här studien innebar det att aspekterna längdförändring och areaförändring och dess relation kom i förgrunden och undersöktes i större utsträckning än i de tre första cyklerna. Sammanfattningsvis visade lektionsdesignen i cykel 4, att en systematisk kombination av smågruppsdiskussioner och efterföljande helklassdiskussioner där elevernas sätt att se på innehållet kommer i förgrunden, ge eleverna kraftfulla möjligheter att utforska lärandeobjektet och kritiska aspekter att bli explicit urskiljningsbara. Därmed visade lektionsdesignen att utrymmet för lärande ökade för eleverna i klassrummet jämfört med lektioner där smågruppsdiskussioner inte användes systematiskt för att undersöka det ämnesinnehåll som skulle läras (Artikel 3).

Diskussion

Studien bidrar med kunskap om elevers lärandemöjligheter då klassrumsdiskussioner används i matematikundervisning. Forskning om elevers lärande då smågrupps- och helklassdiskussioner används har främst fokuserat på generiska förmågor, till exempel i) elevernas gemensamma utveckling av lösningsstrategier (t.ex. Cengiz et al., 2011; Francisco, 2013), ii) elevernas deltagande och engagemang i dialoger eller diskussioner (t.ex. Kosko, 2012; Sjöblom, 2022) eller iii) förmågan att formulera lösningar och matematiska idéer (t.ex. Ryve et al., 2013; Sjöblom, 2022; Webb, 2009). Elevers lärande i smågruppsdiskussioner har mestadels undersökts separat från helklassundervisning (t.ex. Forslund Frykedal & Hammar Chriac, 2011; Francisco, 2013). I studien visade det sig att de lärandemöjligheter som skapades genom smågruppdiskussioner inte var kraftfulla i relation till det som skulle läras under lektionen, vilket antydde att det inte var tillräckligt att eleverna endast gjorde distinktioner relaterade till det avsedda lärandeobjektet själva utan stöd från läraren. Ball (1993) och Ball et al. (2008) lyfter fram att klassrumsdiskussioner med ett fokus på ämnesinnehåll kräver att läraren har insikter i hur eleverna ser på det ämnesinnehåll som ska läras eftersom dessa insikter kan utgöra en grund för klassrumsdiskussioner med fokus på ämnesinnehåll. Resultatet i den här studien pekar mot att lärarnas insikter om elevernas sätt att se på lärandeobjektet var av betydelse för de lärandemöjligheter som skapades, men också att lärarna hade kunskap om kritiska aspekter i förhållande till lärandeobjektet. Lärarnas kunskap om kritiska aspekter tillsammans med insikter i hur eleverna såg på innehållet möjliggjorde att lektionerna i den här studien kunde genomföras på ett kraftfullt sätt avseende lärandemöjligheter av lärandeobjektet. Med kunskap om hur eleverna såg på ämnesinnehållet kunde lärarna, likt Al-Murani och Watson (2009), bestämma vilken variation som var lämplig att erbjuda eleverna under helklassdiskussionen så att kritiska aspekter skulle vara möjliga att urskilja.

Resultaten visar att elevernas lärande om förstoring och förminskning av tvådimensionella geometriska figurer ökade markant efter bara två lektioner. På så sätt skiljer sig studien från tidigare forskning där elevernas lärande är relaterat till elevers deltagande i klassrumsdiskussioner under en längre period (t.ex. McCrone, 2005; Ball, 1993). Endast ett fåtal studier ger insikter i undervisning som stödjer elevernas lärande av förstoring och förminskning av tvådimensionella geometriska figurer. Det finns dock en stor mängd forskning som beskriver elevernas svårigheter vid skalning av tvådimensionella geometriska figurer och ”illusionen av linjäritet” (t.ex. Ayan & Bostan, 2018; De Bock et al., 1998; Hilton et al., 2013; Modestou et al., 2004). Resultatet i den här studien indikerade att det var fördelaktigt för elevernas lärande att undersöka och jämföra relationer, vilket även Lamon (2007) menar är avgörande för att eleverna ska utveckla förståelse för till exempel proportionalitet. I den här studien betydde det att sambandet mellan de två kritiska aspekterna, ”förändring i längd” och ”förändring i area”, vid skalning av tvådimensionella geometriska figurer behövde lyftas fram i undervisningen för att eleverna skulle förstå lärandeobjektet som det var tänkt. Studiens resultat tyder på att både det ämnesinnehåll som ska läras och strukturen av lektionen måste fokuseras vid planering av undervisningen för elevernas lärande i matematik ska öka när klassrumsdiskussioner används (jfr, Kosko, 2012; Stein et al., 2008). Det systematiska samspelet mellan smågrupps- och helklassdiskussioner bidrog till att ge en struktur i hur lärare och elever tillsammans kunde undersöka lärandeobjektet. På så sätt bygger den utvecklade lektionsdesignen på elevernas sätt att se på lärandeobjektet både i smågruppsdiskussionerna och helklassdiskussionerna, det vill säga elevernas sätt att se det avsedda lärandeobjektet fanns i förgrunden, och utforskades under hela lektionen. Undersökningen och beskrivningen av lärandemöjligheter i den här studien adresserar Cai et als' (2020) uppmaning att utveckla forskningsdesigner som är särskilt känsliga för att identifiera och beskriva elevers lärandemöjligheter i relation till ett visst lärandemål för lektionen. Med ett tydligt fokus på det avsedda lärandeobjektet, kritiska aspekter som utforskas och dimensioner av variation som öppnas upp under lektionen, var det möjligt att identifiera olika lärandemöjligheter inom och mellan de olika smågruppdiskussionerna och helklassdiskussionerna. På en generell nivå bidrog forskningsdesignen, som inkluderar en learning study med variationsteori, till att möjliggöra att elevernas lärandemöjligheter av det specifika ämnesinnehållet kunde förbättras.

Studien bidrar till tidigare forskning genom att lyfta fram: i) funktionen av smågruppsdiskussioner i matematikundervisningen, ii) undervisningen om skalning av tvådimensionella geometriska figurer, och iii) variationsteori som analysverktyg för lektioner som involverar smågrupps och helklassdiskussioner. Lärarna använde smågruppsdiskussioner för att synliggöra elevernas förståelse av lärandeobjektet, snarare än att använda smågruppsdiskussioner som en hjälp för eleverna att utveckla generiska färdigheter (t.ex. kommunikationsförmåga) i matematik, vilket verkar vara vanligare i tidigare forskning. Genom att fokusera på elevernas lärandemöjligheter av ett specifikt ämnesinnehåll (t.ex skalning av två-dimensionella geometriska figurer) i helklassundervisning där smågruppsdiskussioner används kompletterar resultaten tidigare forskning och lyfter fram att både hur ämnet behandlas under lektionen och lektionens struktur måste beaktas för att öka elevernas lärande när klassrumsdiskussioner används (Kosko, 2012; Stein et al., 2008). Genom att använda ett annat teoretiskt ramverk än tidigare studier (t.ex. Forslund Frykedal & Hammar Chiriac, 2012; Francisco, 2013; Ryve et al., 2013; Sjöblom, 2022) var det möjligt att fokusera på det specifika ämnesinnehållet och analysera vad som var möjligt att lära i smågruppsdiskussioner och helklassdiskussioner, och relationen dem emellan.

Det finns flera begränsningar i den här studien. För det första är studien baserad på endast en learning study bestående av fyra cykler och som involverade tre lärare, fem klasser och 92 elever. Ett större antal lektioner, elever och lärare skulle med största sannolikhet ha bidragit till mer tillförlitliga slutsatser om undervisning och elevers lärande då smågruppsdiskussioner används. För det andra visade studien att eleverna inte i någon större utsträckning iscensatte lärandeobjekt i smågruppsdiskussionerna som motsvarade det avsedda lärandeobjektet, vilket det däremot gjordes i helklassdiskussionerna. En ny studie kanske inte visar samma samband. Det behövs flera studier för att få en mer heltäckande bild av elevernas lärandemöjligheter när smågruppsdiskussioner används. Dessutom baseras resultatet i den här studien endast på ett specifikt ämnesinnehåll. En analys av learning study där smågruppsdiskussioner används i undervisningen i olika ämnen inom matematik skulle ha givit en mer stabil tolkning.

Resultaten som presenteras i avhandlingen har två huvudsakliga implikationer för undervisning där smågruppsdiskussioner används. Det handlar dels om hur smågruppsdiskussioner kan implementeras i helklasslektioner och dels om betydelsen av lärares kunskap om kritiska aspekter och elevernas sätt att se på lärandeobjektet. Det visade sig vara av betydelse att smågruppdiskussionerna var systematiskt integrerade i helklassdiskussioner för att elevernas lärandemöjligheter skulle öka. Om lärare genom smågruppdiskussionerna får insikter om elevernas sätt att se på lärandeobjektet kan det leda till att lärare, tillsammans med kunskap om kritiska aspekter har större möjligheter att genomföra kraftfulla klassrumsdiskussioner avseende lärande av det specifika ämnesinnehållet. I den här studien handlade det om att det var avgörande att lärarna fick insikter om hur eleverna erfor förhållandet mellan de två aspekterna längdförändring och areaförändring vid förstoring och förminskning av tvådimensionella geometriska figurer för att skapa kraftfull undervisning för att övervinna illusionen av linjäritet.

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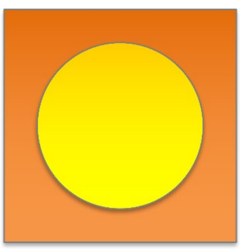
Appendix 1

Pre- and posttest

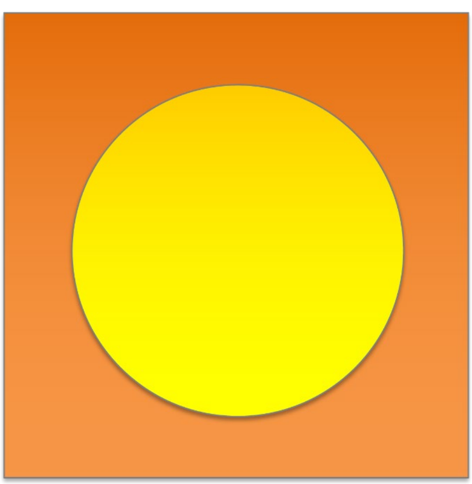
Nos 2

Look at the square figure on the left. Determine whether the following figures are an enlargement of the figure on the left by the scale factor 4:1. Explain your reasoning.



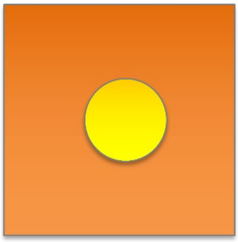
- ☐ Yes
- ☐ No
- ☐ I don't know

Explain your reasoning



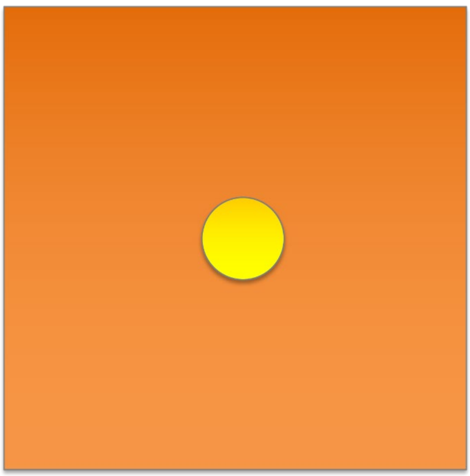
- ☐ Yes
- ☐ No
- ☐ I don't know

Explain your reasoning



- ☐ Yes
- ☐ No
- ☐ I don't know

Explain how you reason:

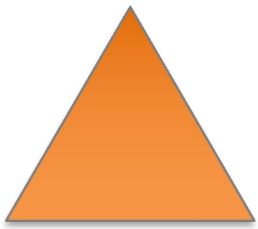


- ☐ Yes
- ☐ No
- ☐ I don't know

Explain how you reason:

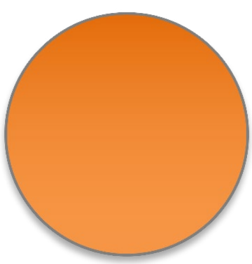
Nos 3

Make a uniform image of the triangle by the scale factor 4:1.



Nos 4

Make a uniform image of the circle by the scale factor 4:1.



Nos 5

Make a uniform image of the quadrate by the scale factor 2:1.

Nos 8

Martin is going to do a drawing of a 48 cm tall Molly Mouse on his daughter's wall. He has a drawing of a 12 cm tall Molly Mouse on a postcard and he is going to do an enlargement of that picture.
a) Which scale factor has he used to the enlargement?
b) How much larger is the area of the Molly Mouse on the wall compared to the one on the postcard?
Show how you solve the problem and explain your reasoning.

Nos 9

Stina has a fence around her quadratic garden plot. Stina's brother Olle has a fence around his quadratic plot as well. Olle's garden plot is bigger than Stina's garden plot and he needs a fence twice as long as Stina.
a) How much bigger is Olle's garden plot compared to Stina's?
b) Olle's garden plot is an enlargement of Stina's garden plot. Explain which is the scale factor of this enlargement?

Appendix 2

The consent of participation

Hej! Göteborg 161107

Mitt namn är Jenny Svanteson Wester och jag är anställd på skola x som matematik- och NO-lärare. Våren 2014 tog jag en licentiatexamen i pedagogiskt arbete med inriktning mot matematikdidaktik. Nu har jag fått möjlighet att fortsätta min utbildning mot en doktorsexamen vid Göteborgs universitet. Mitt intresse riktas fortfarande mot matematikundervisningens kärna - hur lärare behandlar det matematiska innehållet i undervisningen och vilka lärandemöjligheter det skapar för eleverna. Ni kanske känner till att svenska matematikundervisning debatteras mycket i media och att fokus ligger på att alltför många elever lär sig för lite matematik. För att kunna utveckla matematikundervisningen behöver vi som forskare få veta mer om den och det är i detta sammanhang som jag planerar att filma två av matematiklektionerna i ditt barns klass. Det som ska fångas på filmen är vad lärarna gör och säger i undervisningssituationen och hur barnens tankar bemöts och hanteras, samt hur eleverna behandlar det matematiska innehållet då de jobbar med gruppuppgifter. Jag kommer inte att ta med något material som kan uppfattas som kränkande eller som utlämnar enskilda barn. Jag kommer att analysera lektionerna för att på sikt kunna bidra med kunskap om vad i undervisningen som möjliggör lärande. Lektionerna har planerats med arbetslagets ordinarie lärare i matematik och ingår i den ordinarie undervisningen. Några av eleverna kommer, om de vill, bli intervjuade efter lektionerna. När studien är klar kommer den att presenteras som en del i min avhandling. Materialet kan även användas i utbildning av lärare, om ni godkänner det. Varken ditt barns namn eller skolans namn kommer nämnas i något sammanhang.

Jag hoppas att du och ditt barn ställer er positiva till att delta i detta forskningsprojekt. Har ni frågor eller undrar över något, går det bra att kontakta mig via telefon eller mail.

Vänligen,

Jenny Svanteson Wester
Doktorand vi Institutionen för didaktik och pedagogisk profession (IDPP), Göteborgs universitet. Forskarskolan CUL.

Mobil: xxxx-xxxxxx
Mail: jenny.svanteson@xxx.se

Handledare: Professor Åke Ingerman och Docent Angelika Kullberg

Barnets namn:

Barnets klass:

- Ja, mitt barn får delta i forskningsprojektet. Filmen får användas både i forskning och i utbildning av lärare.
- Ja, mitt barn får delta i forskningsprojektet, men filmen där mitt barn finns med får inte användas och visas i forskning och i utbildning av lärare.
- Nej, mitt barn deltar inte i forskningsprojektet.

Barnets underskrift:

Vårdnadshavares underskrift:

Translation from Swedish original:

Hi! Göteborg 161107

My name is Jenny Svanteson Wester and I am working at x school as a teacher in mathematics and science. In the spring of 2014, I took a licentiate degree in pedagogical work with a focus on teaching in mathematics. Now I have been given the opportunity to continue my education as a doctoral student at Gothenburg university. My interest is still directed towards the core of teaching mathematics - how teachers handle the mathematical content during teaching and what learning opportunities are constituted for the students. You might be aware that Swedish mathematics education is debated in the media and that the focus is that too many students learn too little mathematics. To be able to develop mathematics education, we need to gain more knowledge about it. This is the context in which I plan to videorecord two lessons in mathematics in your child's class. What should be captured on film is what the teachers do and say in the teaching situation and how the children's thoughts are responded to, as well as how the students handle the mathematical content when they work with group tasks. I will not include any material that may be perceived as offensive or that exposes individual children. I will analyze the lessons in order to contribute to the knowledge about learning opportunities. The lessons have been planned with the children's ordinary teachers in mathematics and are part of the ordinary lessons. Some of the students will, if they accept, be interviewed after the lessons. When the study is finish, it will be presented as part of my thesis. The material can also be used in teacher training if you approve. Neither your child's name nor the school's name will be mentioned in any context.

I hope that you and your child will approve to participate in this research project. If you have questions or anything you would to discuss, contact me by phone or mail.

Kind regards

Jenny Svanteson Wester
Doctoral student at the Department of Pedagogical, Curricular and Professional Studies, University of Gothenburg

Phone: xxxx xxxxxx
Mail: jenny.svanteson@xxx.se
Supervisors:
Professor Åke Ingerman
Docent Angelika Kullberg

The child's name:

The child's class: ´

- Yes, my child will participate in this research study. The recording may be used in both research and in teacher education.
- Yes, my child will participate in this research study. The recording may only be used in research context.
- No, my child will not participate in the research study

The child's signature:

The parent's signature:

The articles

Article 1
Svanteson Wester, J. & Kullberg, A. (2020). Understanding the relationship between length and area when changing the size of a two-dimensional geometric figure. *Nordic Studies in Mathematics Education*, *25*(1), 89–109.

Article 2
Svanteson Wester, J. (2021). Students' possibilities to learn from group discussions integrated in whole-class teaching in mathematics. *Scandinavian Journal of Educational Research, 6*(65), 1020-1036.

Article 3
Svanteson Wester, J. (Submitted). Using small-group discussions in a learning study about geometric scaling.

GOTHENBURG STUDIES IN EDUCATIONAL SCIENCES 471

This thesis studies how small-group discussions in a whole-class setting relate to student learning about scaling two-dimensional geometric figures. The learning study model together with principles from variation theory were used to shed light on the relationship between teaching and learning opportunities constituted in lessons with small-group discussions, and what students learned.

In the study it was found that small-group discussions on their own did not provide sufficient opportunities for students to learn what was intended. Instead, the interplay between small-group discussions where different ways of seeing the object of learning were made possible to explore and subsequent whole-class discussions in which different ways of seeing the object of learning were brought to the fore in relation to one another seemed to contribute to student learning. By showing how teaching with a systematic use of small-group discussions integrated with whole-class discussions related to student learning the findings add to previous research suggesting that both how the topic is handled during the lesson and the structure of the lessons have to be considered to facilitate and increase student learning when classroom discussions are used.

Jenny Svanteson Wester works as a teacher of mathematics and science in secondary school. Her main research interest concerns the relationship between teaching and student learning.

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