SKETCHES OF NONCOMMUTATIVE TOPOLOGY

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Contents

Date: November 4, 2022.

1. C^*-algebras

Given a compact Hausdorff space X, the set of continuous complex-valued functions $C(X)$ on X forms an algebra under pointwise multiplication. The algebra is unital as the constant function 1 is the multiplicative identity. Taking the pointwise complex conjugate of a function defines a $*$-operation that makes $C(X)$ into a $*$-algebra. We also define the supremum norm $\|f\| \in [0, \infty)$ of $f \in C(X)$ by

$$\|f\| = \max_{x \in X} |f(x)|.$$

Since a continuous real function on a compact set attain its maximum value, it is immediate that $\|f\|$ exists. Any Cauchy sequence of continuous functions in $C(X)$ converges to a continuous function, so the normed space $C(X)$ is complete. For $f, g \in C(X)$ we have

- $\|fg\| \leqslant \|f\|\|g\|$, so $C(X)$ is a Banach algebra.
- $\|f^*\| = \|f\|$, so $C(X)$ is a Banach $*$-algebra.
- $\|f^* f\| = \|f\|^2$, so $C(X)$ is a C^*-algebra.

Definition 1.1. A C^*-algebra is a Banach $*$-algebra over the field of complex numbers with the following properties:

- $\|xy\| \leqslant \|x\|\|y\|$;
- $\|x^*\| = \|x\|$;
- $\|xx^*\| = \|x\|^2$.

$C(X)$ is an example of a commutative unital C^*-algebra. The Gelfand theorem (see [1]) states that every commutative unital C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X.

Given a set G, a relation on G is a set R consisting of pairs (p, η) where p is a $*$-polynomial on G and η is non-negative real number. A representation of (G, R) on a Hilbert space $\mathcal{H}$ is function ρ from G to the algebra of bounded operators on $\mathcal{H}$ such that $\|p \circ \rho(R)\| \leqslant \eta$ for all (p, η) in R. The pair is called admissible if a representation exists and the direct sum of representations is also a representation. Then

$$\|a\|_{univ} = \sup\{\|\rho(a)\| : \rho \text{ is a representation of } (G, R)\}$$

is finite and defines a seminorm satisfying the C^*-norm condition on the free algebra on G.

Definition 1.2. The completion of the quotient of the free algebra by the ideal $\{a : \|a\|_{univ} = 0\}$ in norm $\|\cdot\|_{univ}$ is called the universal C^*-algebra of (G, R).

Example 1.3. Let $\mathcal{H}$ be a separable Hilbert space. The algebra $\mathbb{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ is a norm closed subalgebra of $\mathbb{B}(\mathcal{H})$. It is also closed under involution; hence it is a C^*-algebra.

Example 1.4 ([9])**.** Let $n \geqslant 2$ and $\mathcal{H}$ be a separable Hilbert space. Consider the C^*-algebra generated by n isometries $S_1, \ldots, S_n$ acting on $\mathcal{H}$ satisfying

$$\sum_{i=1}^{n} S_i S_i^* = 1.$$

The concrete C*-algebra generated by $S_1, \ldots, S_n$ is isomorphic to the universal C*-algebra $\mathcal{O}_n$ generated $G = \{1, s_1, \ldots, s_n\}$ and $R = \{1 = 1^* = 1^2,\ 1s_1 = s_1 1 = s_1, \ldots, 1s_n = s_n 1 = s_n,\ s_1^* s_1 = 1, \ldots, s_n^* s_n = 1, \sum_{i=1}^{n} s_i s_i^* = 1\}$, where $\eta = 0$ for each polynomial described in R. $\mathcal{O}_n$ is called the Cuntz algebra.

Example 1.5. Given a skew-symmetric matrix $n \times n$ matrix Θ the noncommutative torus $C(\mathbb{T}^n_\Theta)$ is defined as the universal C^*-algebra generated by n unitaries $u_1, \ldots, u_n$ subject to the relations

$$u_i u_j = e^{-2\pi i \Theta_{ij}} u_j u_i.$$

Properties of this C^*-algebra will be discussed in Section 7.

2. $C_0(X)$-structures

Let X be a locally compact Hausdorff space and let $C_0(X)$ be the C^*-algebra of continuous functions on X that vanish at infinity.

Definition 2.1 ([30])**.** A $C_0(X)$-structure on a C^*-algebra A is a monomorphism

$$\Phi : C_0(X) \to Z\mathcal{M}(A)$$

such that the ideal $\Phi(C(X)) \cdot A$ is dense in A.

For $x \in X$ consider closed two-sided ideal

$$I_x = \{\Phi(f) \cdot a,\ a \in A,\ f \in C_0(X) \text{ such that } f(x) = 0\}.$$

The fiber $A(x)$ of A over x is defined as

$$A(x) = A/I_x,$$

and the canonical quotient map $ev_x : A \to A(x)$ is called the evaluation map at x.

Definition 2.2. An action α of a locally compact group G on a $C_0(X)$-C^*-algebra A is said to be fibrewise if

$$\alpha_g(\Phi(f)a) = \Phi(f)(\alpha_g(a)),\ g \in G,\ a \in A,\ f \in C_0(X).$$

If an action α is fibrewise then it induces an action α^x of G on $A(x)$ for every $x \in X$ making the fiber restriction equivariant.

3. Actions of groups and crossed products

Let G be a locally compact group with a choice of (left) invariant Haar measure λ, A be a C^*-algebra and $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on A. A $*$-homomorphism $\varphi : A \to B$ between C^*-algebras with G-actions α and β is called equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for every $a \in A$, $g \in G$.

Definition 3.1. A covariant representation of (G, A, α) on a Hilbert space $\mathcal{H}$ is a pair (v, π) consisting of a unitary representation $v : G \to U(\mathcal{H})$ and a representation $\pi : A \to \mathbb{B}(\mathcal{H})$, satisfying the covariance condition

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$.

Let $C_c(G, A, \alpha)$ be the linear space of compactly supported continuous A-valued functions on G. Given $f, g \in C_c(G, A, \alpha)$ we define multiplication as the following twisted convolution product:

$$(f_1 \cdot_\alpha f_2)(g) = \int_G f_1(h)\alpha_h(f_2(h^{-1}g))d\lambda(h).$$

On a locally compact group G there exists unique scalar function Δ such that for every Borel subset $S \subset G$ it holds that $\lambda(g^{-1}S) = \Delta(g)\lambda(S)$. We define the $*$-operation on $C_c(G, A, \alpha)$ by

$$f^*(g) = \Delta(g)^{-1}\alpha_g(f(g^{-1})^*).$$

Definition 3.2. The integrated form of a covariant representation (v, π) is the representation $v \times \pi : C_c(G, A, \alpha) \to \mathbb{B}(\mathcal{H})$ given by

$$(v \times \pi)(f)\xi = \int_G \pi(f(g))v(g)\xi d\lambda(g).$$

Definition 3.3. The crossed product $A \rtimes_\alpha G$ is the completion of the $*$-algebra $C_c(G, A, \alpha)$ by the norm

$$\|f\|_u := \sup\{\|(v \times \pi)(f)\| : (v, \pi) \text{ is a covariant representation}\}.$$

If $A = \mathbb{C}$ then we call $\mathbb{C} \rtimes G$ the group C^*-algebra $C^*(G)$.

4. KK-theory

KK-theory (see [1]) is a bivariant functor that jointly generalizes operator K-theory and K-homology: for two C^*-algebras A,B, the KK-group $KK(A, B)$ is a homotopy equivalence class of (A, B)-Hilbert C^*-bimodules equipped with an additional Fredholm module structure. The KK-group $KK(A, B)$ behave as K-homology of A in the first argument and as operator K-theory of B in the second. For further references see [1].

Definition 4.1. For a C^*-algebra B, a Hilbert C^*-module over B is a complex vector space E equpped with an action of B from the right and a sesquilinear map $\langle\cdot,\cdot\rangle : E \times E \to B$ such that

(1) $\langle x, y\rangle^* = \langle y, x\rangle$;
(2) $\langle x, x\rangle \geqslant 0$;
(3) $\langle x, x\rangle = 0$ precisely if $x = 0$;
(4) $\langle x, y \cdot b\rangle = \langle x, y\rangle \cdot b$;
(5) E is complete with respect to the norm $\|x\|_E = \|\langle x, x\rangle\|_B^{\frac{1}{2}}$.

Definition 4.2. For a C^*-algebra B and E a Hilbert C^*-module over B, a $\mathbb{C}$-linear operator $F : E \to E$ is called adjointable if there is an adjoint operator $F^* : E \to E$ with respect to the A-valued inner product in the sense that

$$\langle F(a), b\rangle = \langle a, F^*(b)\rangle, \text{ for all } a, b \in E.$$

Adjointable operators are automatically bounded and B-linear. We denote the set of adjointable operators on a Hilbert B-module E by $\mathbb{B}_B(E)$.

Definition 4.3. For a Hilbert C^*-module E, an adjointable operator $T : E \to E$ is of finite rank if it is of the form

$$T : v \mapsto \sum_{i=1}^{n} w_i \langle v_i, v \rangle$$

for $v_i \in E$ and $w_i \in E$. T is called B-compact operator if it is in the norm-closure of finite-rank operators.

We denote the set of B-compact operators on a Hilbert C^*-module E over B by $\mathbb{K}_B(E)$.

Definition 4.4. For C^*-algebras A, B an (A, B)-Hilbert C^*-bimodule is a Hilbert C^*-module over B equipped with an action of A from the left such that all $a \in A$ are adjointable for the B-valued inner product

$$\langle a^* \cdot x, y \rangle = \langle x, a \cdot y \rangle.$$

Definition 4.5. For C^*-algebras A, B a Kasparov (A, B)-bimodule is a $\mathbb{Z}_2$-graded (A, B)-Hilbert bimodule E equipped with an adjointable oddly-graded bounded operator $F \in \mathbb{B}_B(E)$ such that for all $a \in A$

(1) $(F^2 - 1)\pi(a) \in \mathbb{K}_B(E)$.
(2) $[F, \pi(a)] \in \mathbb{K}_B(E)$.
(3) $(F - F^*)\pi(a) \in \mathbb{K}_B(E)$.

Definition 4.6. A homotopy between two Kasparov (A, B)-bimodules is a $(A, C([0,1], B))$-bimodule that interpolates between the two.

A homotopy of Kasparov bimodules is an equivalence relation.

Definition 4.7. We write $KK(A, B)$ for the set of equivalence classes of Kasparov (A, B)-bimodules under homotopy.

$KK(A, B)$ is an abelian group under direct sum of bimodules and operators. There is a composition operation $KK(A, B) \times KK(B, C) \to KK(A, C)$ called the Kasparov product. We will denote by $\circ : KK(A, B) \times KK(B, C) \to KK(A, C)$ the Kasparov product, and by $\boxtimes : KK(A, B) \times KK(C, D) \to KK(A \otimes C, B \otimes D)$ the exterior tensor product. We say that two C^*-algebras A and B are KK-isomorphic if there exists an invertible element in $KK(A, B)$ with respect to the Kasparov product. Given a homomorphism $\phi : A \to B$, put $[\phi] \in KK(A, B)$ to be the induced KK-morphism. For more details see [1, 15].

5. Rieffel deformation

Given a C^*-algebra A with an action α of $\mathbb{R}^n$ and a skew-symmetric matrix Θ one can consider the Rieffel deformation A^Θ of A, see [31]. Let $C_b(\mathbb{R}^n, A)$ be the C^*-algebra of bounded continuous A-valued functions on $\mathbb{R}^n$ equipped with the supremum norm. There is an action of $\mathbb{R}^n$ on $C_b(\mathbb{R}^n, A)$ by translations. Denote $B^A(\mathbb{R}^n)$ to be the set of smooth elements for the translation action of $\mathbb{R}^n$ on $C_b(\mathbb{R}^n, A)$. For $f, g \in B^A(\mathbb{R}^n)$ we define product by the following oscillatory integral

$$(f \times_\Theta g)(t) := \int_{\mathbb{R}^n \times \mathbb{R}^n} f(t + \Theta(z)) g(t + s) e^{2\pi i z \cdot s} dz ds,$$

where the integral is understood in the sense of Proposition 1.6 of [31]. Denote by $B^A_\Theta(\mathbb{R}^n)$ the algebra equipped with multiplication $\times_\Theta$.

Let $S^A(\mathbb{R}^n)$ denote the space of A-valued Schwartz functions. There is an A-valued inner product on $S^A(\mathbb{R}^n)$ given by

$$\langle f|g\rangle_A := \int_{\mathbb{R}^n} f(s)^* g(s) ds.$$

We denote by X the completion of $S^A(\mathbb{R}^n)$ in the norm $\|f\|_A := \|\langle f|f\rangle\|_A^{\frac{1}{2}}$. Then X is a right Hilbert A-module. Let $f \in B^A_\Theta(\mathbb{R}^n)$ act on $S^A(\mathbb{R}^n)$ as

$$\pi^\Theta(f)g = f \times_\Theta g.$$

Then $\pi^\Theta(f)$ is an adjointable bounded operator on X.

Equip $B^A_\Theta(\mathbb{R}^n)$ with the pre-C^*-norm

$$\|f\|_\Theta := \|\pi^\Theta(f)\|,$$

where $\|\cdot\|$ is the operator norm on $\mathbb{B}_A(X)$. Denote by $\mathcal{B}^A_\Theta(\mathbb{R}^n)$ the C^*-completion of $B^A_\Theta(\mathbb{R}^n)$ with respect to this norm.

Definition 5.1. Let α be an action of $\mathbb{R}^n$ on A and let $A^\infty \subset A$ be the subalgebra of smooth elements for the action α. For $a \in A$ denote f_a to be the function in $\mathcal{B}^A_\Theta(\mathbb{R}^n)$ defined by $f_a(t) = \alpha_{-t}(a)$. For $a \in A^\infty$ we have $f_a \in B^A_\Theta(\mathbb{R}^n)$. The Rieffel deformation of A with respect to (α, Θ) is the C^*-algebra A^Θ obtained by completing A^∞ in the norm

$$\|a\|_\Theta := \|\pi^\Theta(f_a)\|.$$

Thus, A^Θ is a C^*-algebra with multiplication given by

$$a \cdot_\Theta b := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{\Theta(x)}(a)\alpha_y(b)e^{2\pi i\langle x,y\rangle}dxdy, \ a, b \in A^\infty. \tag{1}$$

Example 5.2. Let $\alpha : \mathbb{R}^n \to \mathrm{Aut}(C(\mathbb{T}^n))$ be the action defined by

$$\alpha_{t_1,\dots,t_n}(u_i) = e^{2\pi i t_i} u_i.$$

Then for the C^*-algebra defined in the Example 3 it holds that

$$C(\mathbb{T}^n_\Theta) \simeq C(\mathbb{T}^n)^{\frac{\Theta}{2}}.$$

6. Classification of Kirchberg algebras

For more details on the definitions below see [2]

Definition 6.1. A C^*-algebra A is nuclear if there exists a sequence of finite-dimensional C^*-algebras $F_1, F_2, \dots$ and completely positive maps $\varphi_n : A \to F_n$, $\psi_n : F_n \to A$ such that

$$\lim_n \|\psi_n \circ \varphi_n(a) - a\| = 0.$$

Definition 6.2. A C^*-algebra A is simple if it contains no nontrivial closed two-sided ideals.

Definition 6.3. A simple unital C^*-algebra A of dimension at least 2 is purely infinite if for every non-zero element $a \in A$ there are elements $x, y \in A$ such that $xay = 1$.

Definition 6.4. A C^*-algebra is called Kirchberg algebra if it is separable, nuclear, unital, purely infinite and simple.

Theorem 6.5 (Kirchberg-Philips, [27]). *Let A and B be Kirchberg C^*-algebras, and suppose that there exists an invertible element $\eta \in KK(A,B)$ such that $[\iota_A]\circ\eta = [\iota_B]$, where $\iota_A : \mathbb{C} \to A$ is $\lambda \mapsto \lambda 1_A$, and similarly for ι_B. Then A and B are isomorphic as C^*-algebras.*

7. Noncommutative tori

By now no one has given a satisfactory definition for a noncommutative manifold, however a number of naturally arising examples are known. One of the most well-studied examples of a noncommutative manifold is noncommutative tori. Given a skew-symmetric $n \times n$ matrix Θ the noncommutative torus $C(\mathbb{T}^n_\Theta)$ is defined as a universal C^*-algebras generated by n unitaries $u_1, \dots, u_n$ subject to the relations

$$u_i u_j = e^{-2\pi i \Theta_{ij}} u_j u_i.$$

Various different aspects of $C(\mathbb{T}^n_\Theta)$ has been studied and it appeared to be useful not just as a toy-object used to study effects which appear in non-commutative geometry, but also outside of the subject of noncommutative geometry, for example in mathematical physics: quantum diffusion [14], quantum Hall effect [7], String theory [22], Yang-Mills theory [8], in number theory: Generalized theta functions as holomorphic elements of projective modules [32], parallel between the theory of elliptic curves with complex multiplication and the theory of noncommutative tori with real multiplication [24].

We call Θ irrational if whenever $p\Theta\mathbb{Z}^n \subset \mathbb{Z}$ for some $p \in \mathbb{Z}^n$ it follows that $p = 0$. If Θ is irrational then $C(\mathbb{T}^n_\Theta)$ is simple.

One has an action α of $\mathbb{T}^n$ on $C(\mathbb{T}^n_\Theta)$ given by

$$\alpha_z(u_i) = z_i u_i.$$

The action is ergodic, i.e. $C(\mathbb{T}^n_\Theta)^\alpha \simeq \mathbb{C}$. The conditional expectation with respect to this action is the trace τ defined from the following property

$$1 \cdot \tau(a) = \int_{\mathbb{T}^n} \alpha_z(a) dz, \;\; a \in C(\mathbb{T}^n_\Theta).$$

One can show that in case of that $C(\mathbb{T}^n_\Theta)$ is simple it is the unique trace up to scalar multiple.

The problem of classifying $C(\mathbb{T}^n_\Theta)$ up to C^*-isomorphism has been solved in the case when Θ is irrational, see [28]. In particular, for $n = 2$, $C(\mathbb{T}^2_{\theta_1}) \simeq C(\mathbb{T}^2_{\theta_2})$ iff $\theta_1 = \pm\theta_2 \mod \mathbb{Z}$. When Θ satisfies certain rationality condition a classification is given in [5] for general n.

8. Wick algebras

Consider a set of numbers $\{T^{kl}_{ij},\; i,j,k,l = \overline{1,d}\} \subset \mathbb{C}$ satisfying the condition $T^{kl}_{ij} = \overline{T}^{lk}_{ji}$. Let $\mathcal{H} = \mathbb{C}^d$ and e_1, ..., e_d be the standard orthonormal basis of $\mathcal{H}$. Construct

$$T\colon \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}, \quad Te_k \otimes e_l = \sum_{i,j=1}^{d} T^{lj}_{ik} e_i \otimes e_j.$$

The Wick algebra $W(T)$, see [16], is the $*$-algebra generated by elements a_j, a_j^*, $j = 1, \dots, d$ subject to the relations

$$a_i^* a_j = \delta_{ij}\mathbf{1} + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l a_k^*.$$

It was studied in [16] how the properties of $W(T)$ depend on a self-adjoint operator T. Notice that the subalgebra of $W(T)$ generated by $\{a_j\}_{j=1}^d$ is free and can be identified with the full tensor algebra $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$ via

$$a_{i_1} \dots a_{i_k} \mapsto e_{i_1} \otimes \dots \otimes e_{i_k} \in \mathcal{H}^{\otimes k}.$$

Definition 8.1. The Fock representation $\pi_{F,T}$ of $W(T)$ is a representation on Hilbert space $\mathcal{H}$ such that the following conditions hold: (i) $\pi_{F,T}$ is an irreducible $*$-representation; (ii) there exists a unit vector $\Omega \in \mathcal{H}$ (vacuum vector), such that

$$\pi_{F,T}(a_j^*)\Omega = 0, \ j = \overline{1,d}.$$

The Fock representation, if it exists, is unique up to a unitary equivalence. In general, the problem of existence of $\pi_{F,T}$ is non-trivial and is one of the central problems in representation theory of Wick algebras. Some sufficient conditions are collected in the following theorem, see [4, 17, 16].

Theorem 8.2. *The Fock representation $\pi_{F,T}$ of $W(T)$ exists if one of the conditions below is satisfied*

- *The operator of coefficients $T \geqslant 0$;*
- $||T|| < \sqrt{2} - 1$*;*
- *T is braided, i.e. $(\mathbf{1}\otimes T)(T\otimes \mathbf{1})(\mathbf{1}\otimes T) = (T\otimes \mathbf{1})(\mathbf{1}\otimes T)(\mathbf{1}\otimes T)$ on $\mathcal{H}^{\otimes 3}$ and $||T|| \leqslant 1$. Moreover, if $||T|| < 1$ then $\pi_{F,T}$ is a faithful representation of $W(T)$ and $||\pi_{F,T}(a_j)|| < (1 - ||T||)^{-\frac{1}{2}}$. If $||T|| = 1$, one can not guarantee boundedness of $\pi_{F,T}$ and in this case* $\ker \pi_{F,T}$ *is a $*$-ideal $\mathcal{I}_2$ generated as a $*$-ideal by* $\ker(\mathbf{1} + T)$*. Hence $\pi_{F,T}$ is a faithful representation of $W(T)/\mathcal{I}_2$.*

An important question in the theory of Wick algebras is the question of stability of isomorphism classes of $\mathcal{W}(T) = C^*(W(T))$ for the case $||T|| < 1$. The following problem was posed in [18].

Conjecture 8.3. *Let $T : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ be a self-adjoint braided operator and $||T|| < 1$. Then $\mathcal{W}(T) \simeq \mathcal{W}(0)$.*

In particular, the authors of [18] have shown that the conjecture holds for the case $||T|| < \sqrt{2} - 1$, for more results on the subject see [10], [19].

Consider the case $T = 0$ in a few more details. If $d = \dim \mathcal{H} = 1$, then $W(0)$ is generated by a single isometry s, $s^*s = 1$. In this case the universal C^*-algebra $\mathcal{E}$ of $W(0)$ exists and is isomorphic to the C^*-algebra generated by the unilateral shift S in $l_2(\mathbb{Z}_+)$. Notice also that $\pi_{F,0}(s) = S$, so the Fock representation of the C^*-algebra $\mathcal{E}$ is faithful. The ideal $\mathcal{I}$ in $\mathcal{E}$, generated by $\mathbf{1} - ss^*$ is isomorphic to the algebra of compact operators and $\mathcal{E}/\mathcal{I} \simeq C(S^1)$, see [6]. When $d \geqslant 2$, $W(0)$ is generated by s_j, s_j^*, such that

$$s_i^* s_j = \delta_{ij}\mathbf{1}, \quad i, j = \overline{1,d}.$$

The Fock representation $\pi_{F,d}$ acts on $\mathcal{F} := \mathcal{F}_d$ as follows

$$\pi_{F,d}(s_j)\Omega = e_j, \quad \pi_{F,d}(s_j)e_{i_1} \otimes \cdots \otimes e_{i_k} = e_j \otimes e_{i_1} \otimes \cdots \otimes e_{i_k},\ k \geqslant 1,$$
$$\pi_{F,d}(s_j^*)\Omega = 0, \quad \pi_{F,d}(s_j^*)e_{i_1} \otimes \cdots \otimes e_{i_k} = \delta_{ji_1} e_{i_2} \otimes \cdots \otimes e_{i_k},\ k \geqslant 1.$$

The universal C^*-algebra generated by $W(0)$ with $d \geqslant 2$ exists and coincides with the Cuntz-Toeplitz algebra $\mathcal{O}_d^{(0)}$. It is isomorphic to $C^*(\pi_{F,d}(W(0)))$, so the Fock representation of $\mathcal{O}_d^{(0)}$ is faithful, see [9]. Further, the ideal $\mathcal{I}$ generated by $1 - \sum_{j=1}^d s_j s_j^*$ is the unique largest ideal in $\mathcal{O}_d^{(0)}$. It is isomorphic to the algebra of compact operators on $\mathcal{F}_d$. The quotient $\mathcal{O}_d^{(0)}/\mathcal{I}$ is the Cuntz algebra $\mathcal{O}_d$.

9. Pseudodifferential operators

9.1. **Pseudodifferential operators.** Theory of Pseudodifferential operators starts from the following idea. Write the Fourier inversion formula:

$$f(x) = \int \widehat{f}(\xi) e^{ix\cdot\xi} d\xi,$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^n} \int f(x) e^{-ix\cdot\xi} dx.$$

After differentiation one obtains:

$$D^\alpha f(x) = \int \xi^\alpha \widehat{f}(\xi) e^{ix\cdot\xi} d\xi,$$

where $D^\alpha = D_1^{\alpha_1} \cdot D_n^{\alpha_n}$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Hence, if

$$p(x, D) = \sum_{|\alpha| \leqslant k} a_\alpha(x) D^\alpha$$

is a differential operator, we have

$$p(x, D) f(x) = \int p(x, \xi) \widehat{f}(\xi) e^{ix\cdot\xi} d\xi,$$

where

$$p(x, \xi) = \sum_{|\alpha| \leqslant k} a_\alpha(x) \xi^\alpha.$$

One uses the Fourier integral representation to define pseudodifferential operators, taking the function $p(x, \xi)$ to belong to one of a number of different classes of symbols.

Definition 9.1. Assume $\rho, \delta \in [0, 1]$, $m \in \mathbb{R}$, define $S^m_{\rho,\delta}$ to consist of C^∞-functions $p(x, \xi)$ satisfying

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leqslant C_{\alpha\beta}(1 + |\xi|^2)^{\frac{1}{2}(m - \rho|\alpha| + \delta|\beta|)},$$

for all α, β.

Usually the case of interest is $\rho = 1, \delta = 0$.

Definition 9.2. Suppose there are smooth $p_{m-j}(x,\xi)$, homogeneous in ξ of degree $m-j$ for $|\xi| \geqslant 1$, that is, $p_{m-j}(x,r\xi) = r^{m-j}p_{m-j}(x,\xi)$ for r, $|\xi| \geqslant 1$ such that for all N

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S_{1,0}^{m-N-1}.$$

Then we say that $p(x,\xi) \in S_{cl}^{m}$ is classical symbol of order m.

9.2. **Schwartz kernels.** To an operator $p(x,D) \in S_{\rho,\delta}^{m}$ corresponds a Schwartz kernel $K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, satisfying

$$\langle u{\cdot}v, K\rangle = \int\int u(x)p(x,\xi)\hat{v}(\xi)e^{ix\cdot\xi}d\xi dx = \frac{1}{(2\pi)^n}\int\int\int u(x)p(x,\xi)e^{i(x-y)\cdot\xi}v(y)dyd\xi dx.$$

Proposition 9.3. *Given symbol $p(x,\xi)$ the integral kernel which corresponds to it is*

$$K = \frac{1}{(2\pi)^n}\int p(x,\xi)e^{i(x-y)\cdot\xi}d\xi.$$

Important theorem for the theory of Pseudodifferential operators:

Theorem 9.4. *If $\rho > 0$, then K is C^∞ off the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$.*

Theorem 9.5. *If $\rho > 0$ and $\delta < 1$ then $p(x,D)$ has the pseudolocal property:*

$$\textit{sing supp}\, p(x,D)(u) \subset \textit{sing supp}\, u,\ \ u \in \mathcal{E}'(\mathbb{R}^n).$$

Definition 9.6. Define $\mathcal{H}_\mu^{\#}(\mathbb{R}^n)$ to be the space of distributions on $\mathbb{R}^n$, homogeneous of degree μ, which are smooth on $\mathbb{R}^n \backslash 0$.

Theorem 9.7. *Assume $L \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is a smooth function of x with values in $S_0'(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Let $j = 1,2,3,\ldots$. Then $K(x,y) = L(x,x-y)$ defines an operator in S_{cl}^{-j} if and only if*

$$L(x,z) \sim \sum_{l\geqslant 0}(q_l(x,z) + p_l(x,z)\log|z|),$$

where each $D_x^\beta q_l(x,\cdot)$ is a bounded continuous function of x with values in $\mathcal{H}_{j+l-n}^{\#}$, and $p_l(x,z)$ is a polynomial homogeneous of degree $j+l-n$ in z, with coefficients that are bounded, together with all their x-derivatives.

9.3. **Adjoints and products.** Given $p(x,\xi) \in S_{\rho,\delta}^{m}$, we obtain that the adjoint is given by

$$p(x,D)^*v = \frac{1}{(2\pi)^n}\int p(y,\xi)^* e^{i(x-y)\cdot\xi}v(y)dyd\xi.$$

However presence of $p(y,\xi)^*$ makes the expression be in a nonstandard form. Instead there is an asymptotic expansion.

Proposition 9.8. *If $p(x,D) \in S_{\rho,\delta}^{m}$ then*

$$p(x,\xi)^* \sim \sum_{\alpha\geqslant 0}\frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha D_x^\alpha p(x,\xi)^*.$$

There is also an asymptotic expansion formula for the symbol of product of two pseudodifferential operators.

Proposition 9.9. *Given* $p_1(x,D) \in S^{m_1}_{\rho_1,\delta_1}$ *and* $p_2(x,D) \in S^{m_2}_{\rho_2,\delta_2}$*, suppose*

$$0 \leqslant \delta_2 < \rho \leqslant 1, \ \ \rho = \min(\rho_1, \rho_2).$$

Then

$$p_1(x,D)p_2(x,D) = q(x,D) \in S^{m_1+m_2}_{\rho,\delta},$$

with $\delta = \max(\delta_1, \delta_2)$ *and*

$$q(x,\xi) \sim \sum_{\alpha \geqslant 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi p_1(x,\xi) D^\alpha_x p_2(x,\xi).$$

9.4. **Elliptic operators.** We say that $p(x,D) \in S^m_{\rho,\delta}$ is elliptic if for some $r < \infty$

$$|p(x,\xi)^{-1}| \leqslant C(1+|\xi|^2)^{-\frac{m}{2}}, \ \ |\xi| \geqslant r.$$

Thus if $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ is equal to 0 for $|\xi| \leqslant r$, 1 for $|\xi| \geqslant 2r$, it follows easily from the chain rule that

$$\psi(\xi)p(x,\xi)^{-1} = q_0(x,\xi) \in S^{-m}_{\rho,\delta}.$$

Applying asyptotic expansion formula for products we obtain

$$q_0(x,D)p(x,D) = I + r_0(x,D),$$

$$p(x,D)q_0(x,D) = I + \widetilde{r}_0(x,D),$$

with

$$r_0(x,\xi), \widetilde{r}_0(x,\xi) \in S^{-\rho+\delta}_{\rho,\delta}.$$

Using the formal expansion

$$I - r_0(x,D) + r_0(x,D)^2 - \ldots \sim I + s(x,D) \in S^0_{\rho,\delta}$$

and setting $q(x,D) = (I + s(x,D))q_0(x,D) \in S^{-m}_{\rho,\delta}$, we have

$$q(x,D)p(x,D) = I + r(x,D), \ \ r(x,\xi) \in S^{-\infty}.$$

Similarly, we obtain $\widetilde{q}(x,D) \in S^{-m}_{\rho,\delta}$ satisfying

$$p(x,D)\widetilde{q}(x,D) = I + \widetilde{r}(x,D), \ \ \widetilde{r}(x,\xi) \in S^{-\infty}.$$

But evaluating

$$(q(x,D)p(x,D))\widetilde{q}(x,D) = q(x,D)(p(x,D)\widetilde{q}(x,D))$$

yields $q(x,D) = \widetilde{q}(x,D) \mod S^{-\infty}$, so in fact

$$q(x,D)p(x,D) = I \mod S^{-\infty},$$

$$p(x,D)q(x,D) = I \mod S^{-\infty}.$$

We say that $q(x,D)$ is a two-sided parametrix for $p(x,D)$.

The parametrix can establish the local regularity of a solution to

$$p(x,D)u = f.$$

Suppose u, f are tempered distributions and $p(x,D) \in S^m_{\rho,\delta}$ is elliptic. Constructing $q(x,D) \in S^{-m}_{\rho,\delta}$ we have

$$u = q(x,D)f - r(x,D)u.$$

Thus $u = q(x,D)f \mod C^\infty$.

Proposition 9.10. *If* $p(x,D) \in S^m_{\rho,\delta}$ *is elliptic then for any* u *being tempered distribution,*

$$\textit{sing supp } p(x,D)u = \textit{sing supp } u.$$

9.5. **Pseudodifferential operators on manifolds.**

Definition 9.11. If X is a smooth manifold and $C_c^\infty(X) \subset C^\infty(X)$ is the space of C^∞ functions of compact support, then, for any $m \in \mathbb{R}$, $\Psi^m(X)$ is the space of linear operators

$$A : C_c^\infty(X) \to C^\infty(X)$$

with the following properties. First, if $\phi, \psi \in C^\infty(X)$ have disjoint supports then exists $K \in C^\infty(X \times X, \Omega_R)$ such that for all $u \in C_c^\infty(X)$,

$$\phi A \psi u = \int_X K(x,y)u(y),$$

and secondly if $F : W \to \mathbb{R}^n$ is a coordinate system in X and $\psi \in C_c^\infty(X)$ has support in W then there exists B in $\Psi_\infty^m(\mathbb{R}^n)$ with support in $F(W) \times F(W)$ such that $\psi A \psi u$ restricted to W is $F^*(B((F^{-1})^*(\psi u)))$ for all $u \in C_c^\infty(X)$.

9.6. **Symbol of an operator.** Let D be a differential operator on manifold M of order k. For $(x, \xi) \in T_x^* M$ take $g \in C^\infty(M, \mathbb{R})$ with $g(x) = 0$ and $dg(x) = \xi$. Then

$$\sigma_D(x,\xi) = D(\frac{1}{k!}g^k)(x).$$

10. Functional analytic properties of a pseudodifferential operator

Many functional analytic properties of a pseudodifferential operators can be extracted from its symbol.

10.1. **Boundedness.**

Theorem 10.1. *Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy the following condition: there is a constant $C > 0$ such that*

$$\|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leqslant C,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ with $\alpha_j = 0$ or 1, $\beta_j = 0$ or 1.

Then $a(x, D)$ is continuous from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with its norm bounded by $C_n\|a\|$ where C_n is a constant depending only on n and $\|a\|$ is the smallest C such that the inequality above holds.

10.2. **Compactness.**

Theorem 10.2. *Let $a(x, D) \in S^0_{\rho,\delta}$ such that the kernel of $a(x, D)$ has compact support and $\sup_x |a(x, \xi)| \to 0$ as $\xi \to \infty$. Then $a(x, D)$ extends to a compact operator on L^2.*

Theorem 10.3. *Classical pseudodifferential operator on a compact manifold is compact if and only if it has negative order.*

10.3. **Selfadjointness, normalness and unitarity.**

Theorem 10.4. *Let $a(x, D) \in S^0_{1,0}$. Then it is self-adjoint operator on $L^2(\mathbb{R}^n)$ iff for all $\xi, \eta \in \mathbb{R}^n$*

$$\int_{\mathbb{R}^n} e^{2\pi i\langle \xi - \eta, y\rangle}(a(y,\xi) - \overline{a(y,\eta)})dy = 0.$$

Theorem 10.5. *Let $a(x, D) \in S^0_{1,0}$. Then it is normal operator on $L^2(\mathbb{R}^n)$ iff for all $\xi, \eta \in \mathbb{R}^n$*

$$\int_{\mathbb{R}^n} e^{2\pi i\langle \xi-\eta, y\rangle}(a(y,\xi)\overline{a(y,\eta)} - a^*(y,\xi)\overline{a^*(y,\eta)})dy = 0.$$

Theorem 10.6. *Let $a(x, D) \in S^0_{1,0}$. Then it is unitary operator on $L^2(\mathbb{R}^n)$ iff for all $\xi, \eta \in \mathbb{R}^n$*

$$\int_{\mathbb{R}^n} e^{2\pi i\langle \xi-\eta, y\rangle}(a(y,\xi)\overline{a(y,\eta)} - a^*(y,\xi)\overline{a^*(y,\eta)})dy = \delta_{\xi,\eta}.$$

Theorem 10.7. *Let $a(x, D) \in S^0_{1,0}$. Then it is unitary operator on $L^2(\mathbb{R}^n)$ iff for all $\xi, \eta \in \mathbb{R}^n$*

$$\{a(\cdot,\xi),\ \xi \in \mathbb{R}^n\} \text{ and } \{a^*(\cdot,\xi),\ \xi \in \mathbb{R}^n\}$$

are orthonormal bases for $L^2(\mathbb{R}^n)$.

10.4. **Essential normality.**

Theorem 10.8. *Let $a(x, D) \in S^0_{1,0}(M)$ for a compact manifold M. Then $a(x, D)$ is automatically essentially normal! To understand this you can read article of Shahla Molahajloo called "A Characterization of Compact Pseudo-Differential Operators on S^1", where in Proposition 2.2 he gives an argument for $M = S^1$, but the argument works for arbitrary compact manifold.*

10.5. **Spectrum.**

Theorem 10.9. *Suppose A is a self-adjoint elliptic pseudodifferential operator on a smooth manifold M. Then there exists a complete orthonormal system of $C^\infty(M)$ functions which are eigenfunctions of A.*

Theorem 10.10. *Suppose $A \in S^m_{\rho,\delta}$ is elliptic with $m > 0,\ 1-\rho \leqslant \delta < \rho$. Then for the spectrum $\sigma(A)$ there are two possibilities:*

(1) $\sigma(A) = \mathbb{C}$ (which, in particular, is the case when $\operatorname{ind}(A) \neq 0$).
(2) $\sigma(A)$ is discrete.

10.6. **Fredholmness.**

Theorem 10.11. *Let M be a closed manifold and $A \in S^m_{\rho,\delta}(M)$ is elliptic, $1-\rho \leqslant \delta < \rho$. For any $s \in \mathbb{R}$ construct the operator $A_s \in \mathbb{B}(H^s(M), H^{s-m}(M))$ - the extension of A be continuity to Sobolev spaces. Then*

(1) A_s is Fredholm.
(2) $\ker A_s \subset C^\infty(M)$, therefore $\ker A_s$ does not depend on s.
(3) index(A_s) does not depend on s.
(4) if $D \in S^{m'}_{\rho,\delta}$, where $m' < m$ then index$(A + D) =$ index(A).

11. Pseudodifferential operators and BDF theory

11.1. **BDF theory.** The story of Brown–Douglas–Fillmore theory begins with the Weyl–von Neumann theorem, which, in one of its formulations, says that a bounded self-adjoint operator $T = T^*$ on an infinite-dimensional separable Hilbert space $\mathcal{H}$ is determined up to compact perturbations, modulo unitary equivalence, by its essential spectrum. (The essential spectrum is the spectrum $\sigma(\pi(T))$ of the image $\pi(T)$ of T in the Calkin algebra $\mathbb{Q}(\mathcal{H}) = \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$; it is also the spectrum of the restriction of T to the orthogonal complement of the eigenspaces of T for the eigenvalues of finite multiplicity. In other words, unitary equivalence modulo the compacts $\mathbb{K}(\mathcal{H})$ washes out all information about the spectral measure of T, and only the essential spectrum remains. This result was extended to normal operators by I.D. Berg and W. Sikonia, working independently. However, the theorem is not true for operators that are only essentially normal, in other words, for operators T such that $T^*T - TT^* \in \mathbb{K}(\mathcal{H})$. Indeed, the "unilateral shift" S satisfies $S^*S = 1$ and $SS^* = 1 - P$, where P is a rank-one projection, yet S cannot be a compact perturbation of a normal operator since its Fredholm index is non-zero. L.G. Brown, R.G. Douglas and P.A. Fillmore (known to operator theorists as "BDF") showed that this is the only obstruction: an operator T in $\mathbb{B}(\mathcal{H})$ is a compact perturbation of a normal operator if and only if T is essentially normal and $\operatorname{ind}(T - \lambda) = 0$ for every $\lambda \notin \sigma(\pi(T))$.

11.2. **Classification of classical pseudodifferential operators.** Let M be a closed Riemannian manifold.

Theorem 11.1. *The following diagram commutes, has exact rows and the vertical maps are injections:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{-1}_{cl}(M) & \longrightarrow & S^0_{cl}(M) & \xrightarrow{\sigma_0} & C^\infty(S^*M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{K}(L^2(M)) & \longrightarrow & \mathbb{B}(L^2(M)) & \longrightarrow & \mathbb{B}(L^2(M))/\mathbb{K}(L^2(M)) & \longrightarrow & 0 \end{array}$$

Every 0-order classical pseudodifferential operator is essentially normal. Thus by BDF theory, up to unitary equivalence modulo compact they are classified by

(1) The essential spectrum.
(2) The set $(\operatorname{ind}(T - \lambda))_{\lambda \notin \sigma_{ess}(T)}$.

The following proposition follow from the commutative diagram above.

Proposition 11.2. *Let $A \in S^0_{cl}(M)$. Then*

$$\sigma_{ess}(A) = \operatorname{im} \sigma_0(A).$$

Proposition 11.3. *Let M be closed connected manifold. Let $A, B \in S^0_{cl}(M)$. Then there exists a unitary operator U on $L^2(M)$ and $K \in \mathbb{K}(L^2(M))$ such that $UAU^* = B + K$ iff*

(1) $\operatorname{im} \sigma^0_A = \operatorname{im} \sigma^0_B$.
(2) For some $\lambda \notin \operatorname{im} \sigma^0_A$ holds $\operatorname{ind}(A - \lambda) = \operatorname{ind}(B - \lambda)$.

Proposition 11.4. *Let M be a closed connected manifold. Then up to scaling there is unique elliptic classical pseudodifferential operator of order 0 with discrete spectrum up to unitary equivalence modulo compacts.*

Proof. So:

(1) Pseudodifferential operators of order 0 are essentially normal.
(2) Essential spectrum is a subset of spectrum.
(3) If M is connected, then the essential spectrum is connected. Thus essential spectrum is a point in $\mathbb{C}$.
(4) By BDF theory, up to unitary equivalence modulo compacts the operator is determined by index (which is 0) and the essential spectrum (which is one point), so up to scaling the operator is unique.

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Corollary 11.5. *Let M be a closed connected manifold. Then scalar multiples of identity are the only elliptic pseudodifferential operators of order 0 with discrete spectrum modulo compact operators.*

Corollary 11.6. *Let M be a closed connected manifold. If elliptic pseudodifferential operator of order 0 has index 0 then it is scalar multiple of identity modulo compact operator.*

Corollary 11.7. *If M is a closed connected odd-dimensional manifold then every elliptic $A \in \Psi^0(M)$ has form*

$$A = \lambda I + K,\ K \in \Psi^{-1}(M).$$

Corollary 11.8. *If M is a compact connected manifold and $P \in \Psi^0(M)$ is a projection then either P or $1 - P$ is compact.*

Corollary 11.9. *On a compact connected manifold there is unique pseudodifferential projection up to unitary equivalence modulo compacts.*

Corollary 11.10. *On a compact connected manifold there are $2\aleph_0$ projections modulo unitary equivalence:*

$$P_0, 1 - P_0, P_1, 1 - P_1, \dots,$$

in every pair one projection is compact and the other is elliptic.

And so on! There are numerous ways to play with BDF theory and pseudodifferential operators.

11.3. **Conjectural approximation of abstract essentially normal operators by pseudodifferential operators.** Since mapping $[D] \mapsto \operatorname{ind} D$ is surjective, one can approximate every essentially normal operator N with connected essential spectrum with a sequence of pseudodifferential operators $A_1, A_2, \dots$, for which we have topological index formula. Thus we get topological index formula for abstract operators in functional analysis:

$$\operatorname{ind}(N) = \lim_{n\to\infty} \int_M \tau \circ \mathsf{ch}[\sigma_{A_n}] \wedge \mathsf{Td}(T^*M).$$

There are several problems of course:

(1) How to realize arbitrary operator as operator on $L^2(M)$?
(2) How to find sequence of pseudo-differential approximations?
(3) What should we do with non-connected essential spectrum?

12. Dirac operators

12.1. **Clifford algebra.** Let V be a finite-dimensional, real vector space, g a quadratic form on V. We allow g to be definite or indefinite if nondegenerate; we even allow g to be degenerate.

Definition 12.1. The Clifford algebra $Cl(V, g)$ is the quotient algebra of the tensor algebra

$$\bigotimes V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

by the ideal $I \subset \bigotimes V$ generated by

$$\{v \otimes w + w \otimes v - 2\langle v, w\rangle \cdot 1 : v, w \in V\},$$

where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form on V arising from g.

Thus, in $Cl(V, g)$, V occurs naturally as a linear subspace, and there is the anti-commutation relations

$$vw + wv = 2\langle v, w\rangle \cdot 1, \ v, w, \in V.$$

Definition 12.2. Vector space E is a Clifford module if there exists $\nu : V \to \mathrm{End}(E)$ a linear map from V into the space of endomorphisms of a vector space E such that

$$\nu(v)^2 = \langle v, v\rangle \cdot I, \ v \in V.$$

In this case ν extends uniquely to an algebra homomorphism

$$\nu : Cl(V, g) \to \mathrm{End}(E), \ \nu(1) = I.$$

Definition 12.3. Clifford module E with a Hermitean metric is a Hermitean Clifford module if $\nu(v) = \nu(v)^*$.

Definition 12.4. Hermitean Clifford module E with a $\mathbb{Z}/2\mathbb{Z}$-grading $E = E_0 \oplus E_1$ is a graded Hermitean Clifford module if $\nu(v)(E_i) \subset E_{i+1}$.

12.2. **Operators of Dirac type.** Let M be a Riemannian manifold, $E_j \to M$ vector bundles with Hermitean metrics.

Definition 12.5. A first-order, elliptic differential operator

$$D : C^\infty(M, E_0) \to C^\infty(M, E_1)$$

is said to be of Dirac type provided D^*D has scalar principal symbol, i.e.

$$\sigma_{D^*D}(x, \xi) = g(x, \xi)I : E_{0,x} \to E_{0,x},$$

where $g(x, \xi)$ is a positive quadratic form on $T^*_x M$.

If $E_0 = E_1$ and $D = D^*$, we say D is a symmetric Dirac-type operator. Given a general operator D of Dirac type, if we set $E = E_0 \oplus E_1$ and define $\widetilde{D}$ on $C^\infty(M, E)$ as

$$\widetilde{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix},$$

then D is a symmetric Dirac-type operator.

Let $\nu(x, \xi)$ denote the principal symbol of a symmetric Dirac-type operator. With $x \in M$ fixed, set $\nu(\xi) = \nu(x, \xi)$. Thus ν is a linear map from $T^*_x M$ into $\mathrm{End}(E_x)$ satisfying

$$\nu(\xi) = \nu(\xi)^*, \ \nu(\xi)^2 = \langle \xi, \xi\rangle I.$$

Example 12.6. If M is a Riemannian manifold, the exterior derivative operator

$$d : \Lambda^j M \to \Lambda^{j+1} M$$

has a formal adjoint

$$\delta = d^* : \Lambda^{j+1} M \to \Lambda^j M,$$

so $d + \delta$ acts on $\Lambda^* M$. One can show that

$$(d + \delta)^*(d + \delta) = -\Delta,$$

where Δ is the Hodge Laplacian, so $d + \delta$ is a symmetric Dirac-type operator.

This operator is used to deduce Gauss-Bonnet formula from the Atiyah-Singer index formula.

Example 12.7. Suppose $\dim M = 2k$ is even. In terms of the Hodge star operator, δ is defined on $\Lambda^j M$ as follows:

$$\delta = d^* = (-1)^{j(n-j)+j} * d* = *d * .$$

On complexification $\Lambda^*_{\mathbb{C}} M$ define

$$\alpha : \Lambda^j_{\mathbb{C}} M \to \Lambda^{n-j}_{\mathbb{C}} M,\ \ \alpha = i^{j(j-1)+k} * .$$

Then $\alpha^2 = 1$ and $\alpha(d + \delta) = -(d + \delta)\alpha$, so we can write eigenspaces of α:

$$\Lambda^*_{\mathbb{C}} = \Lambda^+ M \oplus \Lambda^- M$$

and

$$D^{\pm}_H = d + \delta : C^\infty(M, \Lambda^{\pm}) \to C^\infty(M, \Lambda^{\mp})$$

are operators of Dirac type.

This operators are used to deduce Hirzebruch signature formula from the Atiyah-Singer index formula.

Example 12.8. Let M be Riemannian manifold, $T^*_x M$ has an induced inner product, giving rise to bundle $Cl(M) \to M$ of Clifford algebas. We suppose that $E \to M$ is a Hermitian vector bundle such that each fiber is a Hermitian $Cl_x(M)$-module. Let $E \to M$ have a connection ∇, so

$$\nabla : C^\infty(M, E) \to C^\infty(M, T^* \otimes E).$$

If E_x is a $Cl_x(M)$-module, the inclusion $T^*_x \to Cl_x$ gives rise to a linear map

$$m : C^\infty(M, T^* \otimes E) \to C^\infty(M, E),$$

called Clifford multiplication. Let

$$D = i \cdot m \circ \nabla : C^\infty(M, E) \to C^\infty(M, E).$$

For $v \in E_x$,

$$\sigma_D(x, \xi)v = m(\xi \otimes v) = \xi \circ v,$$

so $\sigma_D(x, \xi)$ is $|\xi|_x$ times an isometry on E_x. Hence D is of Dirac type.

13. Many incarnations of the Atiyah-Singer index theorem

13.1. **Atiyah-Singer.**

Theorem 13.1. *Let D be an elliptic differential operator on a closed manifold M. Then*

$$\mathsf{ind}(D) = \int_M \tau \circ \mathsf{ch}([\sigma_D]) \wedge \mathsf{Td}(T^* M),$$

where

- $\mathsf{Td}(T^* M) \in H^*(M)$ *is the Todd class.*
- $[\sigma_D] \in K_*(T^* M)$ *is a K-theory class of the elliptic complex given by multiplication with σ_D on $T^* M$.*
- $\mathsf{ch}([\sigma_D]) \in H^*(T^* M)$ *is the image of the Chern character.*
- $\tau : H^*(T^* M) \to H^*(M)$ *is the Thom isomorphism.*

13.2. **Toeplitz index theorem.** Let M be an odd dimensional closed Spin^c manifold with Dirac operator D acting on sections of the spinor bundle S. If E is a smooth $\mathbb{C}$ vector bundle on M, D^E denotes D twisted by E. The closure $\overline{D}^E$ of D^E is an unbounded self-adjoint operator on the Hilbert space $L^2(M, S \otimes E)$ of L^2-sections of $S \otimes E$. $\overline{D}^E$ has discrete spectrum with finite dimensional eigenspaces. Denote by $L^2_+(M, S \otimes E)$ the Hilbert space direct sum of the eigenspaces of $\overline{D}^E$ for eigenvalues $\lambda \geqslant 0$. P^E_+ denotes the orthogonal projection

$$P^E_+ : L^2(M, S \otimes E) \to L^2_+(M, S \otimes E).$$

Suppose that α is an automorphism of E, and $I_S \otimes \alpha$ the resulting automorphism of $S \otimes E$. M_α is the bounded invertible operator on $L^2(S \otimes E)$ obtained from $I_S \otimes \alpha$. The Toeplitz operator T_α is the composition of $M_\alpha : L^2_+ \to L^2$ with $P^E_+ : L^2 \to L^2_+$,

$$T_\alpha = P^E_+ \circ M_\alpha : L^2_+(M, S \otimes E) \to L^2_+(M, S \otimes E).$$

The Toeplitz operator T_α is a Fredholm operator.

Theorem 13.2. *Let M be an odd dimensional compact Spin^c manifold without boundary. If E is a smooth $\mathbb{C}$ vector bundle on M, and α is an automorphsim of E, then*

$$\operatorname{ind}(T_\alpha) = (\mathsf{ch}(E, \alpha) \cap \mathsf{Td}(M))[M]).$$

13.3. **Gauss-Bonnet.** If you plug $d + d^*$ into the Atiyah-Singer index formula, you get Gauss-Bonnet.

Theorem 13.3. *Let M be a Riemannian even-dimensional compact orientable manifold, Ω - Riemannian curvature. Then the Euler characteristics $\chi(M)$ can be computed as*

$$\chi(M) = \frac{1}{(2\pi)^n} \int_M \mathsf{Pf}(\Omega).$$

Corollary 13.4. *In dimension $2n = 4$, we get*

$$\chi(M) = \frac{1}{8\pi^2} \int_M |Riem|^2 - 4|Ric|^2 + R^2,$$

where Riem is Riemannian curvature, Ric is the Ricci curvature and R is the scalar curvature.

13.4. **Riemann-Roch.** Take X to be a complex manifold with a holomorphic vector bundle V. We let the vector bundles E and F be the sums of the bundles of differential forms with coefficients in V of type $(0, i)$ with i even or odd, and we let the differential operator D be the sum

$$\overline{\partial} + \overline{\partial}^*.$$

restricted to E. Then the analytical index of D is the holomorphic Euler characteristic of V:

$$\operatorname{index}(D) = \sum_p (-1)^p \dim H^p(X, V).$$

The topological index of D is given by

$$\operatorname{index}(D) = \operatorname{ch}(V) \operatorname{Td}(X)[X],$$

the product of the Chern character of V and the Todd class of X evaluated on the fundamental class of X. By equating the topological and analytical indices we get the Hirzebruch–Riemann–Roch theorem.

13.5. **Hirzebruch index theorem.** The Hirzebruch signature theorem states that the signature of a compact oriented manifold X of dimension $4k$ is given by the L genus of the manifold. This follows from the Atiyah–Singer index theorem applied to the following signature operator.

The bundles E and F are given by the $+1$ and 1 eigenspaces of the operator on the bundle of differential forms of X, that acts on k-forms as$i^{k(k-1)}$

times the Hodge $*$ operator. The operator S is the Hodge Laplacian

$$D \equiv \Delta := (\mathbf{d} + \mathbf{d}^*)^2$$

restricted to E, where d is the Cartan exterior derivative and d^* is its adjoint.

The analytic index of D is the signature of the manifold X, and its topological index is the L genus of X, so these are equal.

13.6. **Gromov-Lawson-Rosenberg conjecture.** A result of Lichnerowicz states that there are spin manifolds which do not admit positive scalar curvature metrics. Indeed, by the Lichnerowicz formula, the existence of such a metric implies that the index of the Dirac operator vanishes. This, combined with the Atiyah-Singer index theorem implies that $\widehat{A}$ genus, which is a linear combination of the Pontrjagin classes of the manifold, vanishes. The $\widehat{A}$ obstruction was generalized by Hitchin to an obstruction $\alpha(M) \in KO_n$, where α denotes the Atiyah-Bott-Shapiro homomorphism. This agrees with $\widehat{A}$ in dimensions $0 \mod 4$, but is in fact a strict generalization, and indeed Hitchin constructed exotic spheres admitting no metric of positive scalar curvature in dimensions $n \equiv 1, 2 \mod 8$. Letting π denote any fundamental group, the homomorphism α gives rise to a transformation of cohomology theories

$$\alpha : \Omega_n^{spin}(B\pi) \to KO(B\pi)$$

and Gromov and Lawson conjectured that $\alpha(M) = 0$ was also a sufficient condition for M to admit a metric of positive scalar curvature. Rosenberg later generalized this further, showing that if a spin manifold M with fundamental group π admitted a metric of positive scalar curvature, then $\mathrm{ind}([M, u]) = 0$, where u is the classifying map of the universal cover of M, and ind maps to the real $K-$theory of the reduced C^* algebra of π:

$$\mathrm{ind} = A \circ \alpha : \Omega_n^{spin}(B\pi) \to KO(C^*(\pi)_{red}).$$

This can be thought of as an equivariant generalized index, and the map A is the assembly map of Baum-Connes. Modifying the Gromov-Lawson conjecture, Rosenberg conjectured that the converse was true also; namely that a compact spin manifold M with $\pi_1(M) = \pi$ and $n \geqslant 5$ admits a positive scalar curvature metric if and only if $\mathrm{ind}[u : MB\pi] = 0 \in KO_n(C^*(\pi)_{red})$. The conjecture has been proven in the simply connected case, if π has periodic cohomology, and if π is a free group, free abelian group, or the fundamental group of an orientable surface. It is also known to be false in general, for example if $\pi = \mathbb{Z}_4 \times \mathbb{Z}_3$, and for a large class of torsion free groups.

13.7. **Stolz conjecture.** The Stolz conjecture asserts that if X is a closed manifold with String structure which furthermore admits a Riemannian metric with positive Ricci curvature, then its Witten genus vanishes.

One part of the reasoning that motivates the conjecture is the idea that string geometry should be a "delooping" of spin geometry, and that the Witten genus is roughly like the index of a Dirac operator on loop space. Now for spin geometry the Lichnerowicz formula implies that for positive scalar curvature there are no harmonic spinors on a Riemannian manifold X, and hence that the index of the Dirac operator vanishes. One might then expect that there is a sensible concept of scalar curvature of smooth loop space obtained by integrating the Ricci curvature on X along loops (transgression). Therefore, in this reasoning, a positive Ricci curvature of X would imply a positive scalar curvature of the smooth loop space, thus a vanishing of the index of the "Dirac operator on smooth loop space", hence a vanishing of the Witten genus.

14. Summary of Paper I

In Paper I, "Faithfulness of the Fock representation of C*-algebra generated by q_{ij}-commuting isometries" we consider universal C^*-algebra $Isom_{q_{ij}}$ generated by n isometries $a_1, \ldots, a_n$ such that

$$a_i^* a_j = q_{ij} a_j a_i^*.$$

This algebra has a distinguished representation called the Fock representation. It is a unique up to unitary equivalence representation π_F on the Fock space $\mathbb{C}\{\Omega\} \oplus \mathbb{C}^n \oplus \mathbb{C}^{n^2} \oplus \ldots$ such that $a_i^*(\Omega) = 0$. This representation exists for every Wick algebra and it is a question if π_F is faithful. It is known however that it is faithful $*$-algebraically.

In order to prove faithfulness of π_F for $Isom_{q_{ij}}$ we examine the fixed point subalgebra under the action of $\mathbb{T}^n$. It appears to be an AF-algebra with Bratelli diagram which looks like multidimensional Pascal tetrahedron. Using integration trick we reduce problem of faithfulness of π_F on $Isom_{q_{ij}}$ to the problem of faithfulness of π_F restricted to $Isom_{q_{ij}}^{\mathbb{T}^n}$. Since AF-algebras are limits of finite-dimensional algebras and faithfulness is proven on algebraic level, we show faithfulness of π_F on C^*-algebraic level.

It is expected that $Isom_{q_{ij}}$ is in fact isomorphic to the Cuntz-Toeplitz algebra $\mathbb{K}\mathcal{O}_n$, which has unique maximal ideal $\mathbb{K}$. For $\mathbb{K}\mathcal{O}_n$ generator is very simple, but for $Isom_{q_{ij}}$ it is not so obvious. We construct generated for ideal $\mathbb{K}$ explicitly.

15. Summary of Paper II

In Paper II, "On q-tensor product of Cuntz algebras" we consider the C^*-algebra $\mathcal{E}_{n,m}^q$ generated by the Wick algebra for T described as follows. Let $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^m$, $|q| \leqslant 1$ and

$$\begin{aligned} &Tu_1 \otimes u_2 = 0, \quad Tv_1 \otimes v_2 = 0, \quad u_1, u_2 \in \mathbb{C}^n,\ v_1, v_2 \in \mathbb{C}^m, \\ &Tu \otimes v = qv \otimes u, \quad Tv \otimes u = \overline{q} u \otimes v, \quad u \in \mathbb{C}^n,\ v \in \mathbb{C}^m. \end{aligned}$$

We consider cases $|q| < 1$ and $|q| = 1$ separately. In the case $|q| < 1$ we write explicit formula for the isomorphism

Theorem 15.1. *For any $q \in \mathbb{C}$, $|q| < 1$, one has an isomorphism $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0$.*

In the case $|q| = 1$ the C^*-algebra $\mathcal{E}_{n,m}^q$ decomposes into the following short exact sequence

$$0 \to \mathcal{M}_q \to \mathcal{E}_{n,m}^q \to (\mathcal{O}_n \otimes \mathcal{O}_m)^{\Theta_q} \to 0,$$

where for $q = e^{2\pi i\phi}$ we define $\Theta_q = \begin{pmatrix} 0 & \frac{\phi}{2} \\ -\frac{\phi}{2} & 0 \end{pmatrix}$ and $\mathcal{M}_q$ is the maximal ideal described below.

We prove that $(\mathcal{O}_n \otimes \mathcal{O}_m)^{\Theta_q}$ is a Kirchberg algebra KK-isomorphic to $\mathcal{O}_n \otimes \mathcal{O}_m$. Then we use Kirchberg-Philips classification theorem in order to conclude the next theorem

Theorem 15.2. *The C^*-algebras $(\mathcal{O}_n \otimes \mathcal{O}_m)^{\Theta_q}$ and $\mathcal{O}_n \otimes \mathcal{O}_m$ are isomorphic for any $|q| = 1$.*

We show that the ideal $\mathcal{M}_q$ of $\mathcal{E}^q_{n,m}$ further decomposes as

$$0 \to \mathbb{K} \to \mathcal{M}_q \to \mathcal{O}_m \otimes \mathbb{K} \oplus \mathcal{O}_n \otimes \mathbb{K} \to 0.$$

The extension happens to be essential which allows us to use the Voiculescu theorem to prove

Theorem 15.3. *For any $q \in \mathbb{C}$, $|q| = 1$, one has $\mathcal{M}_q \simeq \mathcal{M}_1$.*

16. Summary of Paper III

In Paper III, "Classification of irrational Θ-deformed CAR C^*-algebras" we consider the C^*-algebra CAR_Θ defined as the universal enveloping C^*-algebra of $*$-algebra generated by $a_1, \ldots, a_n$ subject to the relations

$$a_i^* a_i + a_i a_i^* = 1,$$

$$a_i^* a_j = e^{2\pi i \Theta_{ij}} a_j a_i^*,$$

$$a_i a_j = e^{-2\pi i \Theta_{ij}} a_j a_i.$$

CAR_Θ has an action α of $\mathbb{T}^n$ given by

$$\alpha_z(a_i) = z_i a_i.$$

We express it as a Rieffel deformation of the tensor product of n copies of 1-dimensional CAR with respect to the action α and the skew-symmetric matrix Θ.

Proposition 16.1.

$$\mathsf{CAR}_\Theta \simeq (\mathsf{CAR}_1^{\otimes n})^{\frac{\Theta}{2}}$$

Denote Cl_n to be the complex Clifford C^*-algebra on n generators. The C^*-algebra CAR_1 possess $C[0, \frac{1}{2}]$-structure and its fibers are the following:

$$(\mathsf{CAR}_1)(0) \simeq Cl_2,$$
$$(\mathsf{CAR}_1)(x) \simeq C(\mathbb{T}) \rtimes_\sigma \mathbb{Z}_2,\ 0 < x < \frac{1}{2},\ \sigma(f)(z) = f(-z),$$
$$(\mathsf{CAR}_1)(\frac{1}{2}) \simeq C(\mathbb{T}).$$

Action α of $\mathbb{T}$ on CAR_1 is fibrewise with respect to the $C[0, \frac{1}{2}]$-structure, which allows us to conclude the following:

Proposition 16.2. CAR_Θ *possess a* $C[0,\frac{1}{2}]^n$*-structure with fibers given by* $\mathsf{CAR}_\Theta(x) \simeq (\mathsf{CAR}_1^{\otimes n}(x))^{\frac{\Theta}{2}}$*. Given* $x = (x_1,\dots,x_n) \in [0,\frac{1}{2}]^n$*, let*

$$L_x = \{i \in \{1,\dots,n\} : x_i = 0\},$$
$$M_x = \{i \in \{1,\dots,n\} : 0 < x_i < \frac{1}{2}\},$$
$$R_x = \{i \in \{1,\dots,n\} : x_i = \frac{1}{2}\}.$$

Then

$$\mathsf{CAR}_\Theta(x) \simeq Cl_{2|L_x|} \otimes C(\mathbb{T}^{|M_x|+|R_x|}_{\Theta_{M_x \sqcup R_x}}) \rtimes \mathbb{Z}_2^{|M_x|}.$$

Any irreducible representation of a C^*-algebra A equipped with a $C_0(X)$-structure factors through an irreducible representation of a fiber $\mathcal{A}(x)$. This fact combines with Proposition 2 give us the next theorem

Theorem 16.3. *Any irreducible representation of* CAR_Θ *is unitary equivalent to a representation* τ_x*,* $x \in [0,\frac{1}{2}]^{M_x}$*, given on* $(\otimes_{k\in L_x}\mathbb{C}^2) \bigotimes (\otimes_{k\in M_x}\mathbb{C}^2) \bigotimes H$ *by*

$$\begin{aligned}
\tau_x(a_i) &= \prod_{k\in L_x}(e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) e_i \otimes 1, i \in L_x,\\
\tau_x(a_i) &= \prod_{k\in L_x}(e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) \otimes \Bigg(\Bigg(\prod_{k\in M_x, k<i}(e_k^* e_k + e^{2\pi i \Theta_{i,k}} e_k e_k^*) \otimes 1_H\Bigg)\\
&\times \Bigg(\sqrt{x_i}\prod_{k\in M_x, k\geqslant i}(e_k^* e_k + e^{4\pi i \Theta_{i,k}} e_k e_k^*) e_i \otimes v_i + \sqrt{1-x_i}\, e_i^* \otimes 1_H\Bigg)\Bigg), i \in M_x,\\
\tau_x(a_i) &= \prod_{k\in L_x}(e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) \otimes \prod_{k\in M_x}(e_k^* e_k + e^{2\pi i \Theta_{i,k}} e_k e_k^*) \otimes \frac{1}{\sqrt{2}} v_i\ , i \in R_x.
\end{aligned} \tag{2}$$

where $(v_i)_{i\in M_x \sqcup R_x}$ *defines an irreducible representation of* $C(\mathbb{T}_\Sigma^{M_x\sqcup R_x})$ *on* H*, where*

$$\Sigma_{i,j} = \begin{cases} 4\Theta_{i,j} & i,j \in M_x \\ 2\Theta_{i,j} & (i,j) \text{ or } (j,i) \in M_x \times R_x \\ \Theta_{i,j} & i,j \in R_x \end{cases}$$

Moreover, two such irreducible representations τ_x *and* τ_y *are unitary equivalent if and only if* $x = y$ *and the corresponding representations of* $C(\mathbb{T}_\Sigma^{M_x\sqcup R_x})$ *are unitarily equivalent.*

Finally, we give a partial classification result for CAR_Θ with irrational Θ

Theorem 16.4. *Let* Θ_1 *and* Θ_2 *be irrational.*

(1) If P *is a signed permutation matrix then* $\Theta_1 = P\Theta_2P^t$ *implies* $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$*.*
(2) If $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$ *then* $(\Theta_2)_{i,j} = \pm(\Theta_1)_{\sigma(i,j)} \mod \mathbb{Z}$ *for a bijection* σ *of the set* $\{(i,j) : i < j,\ i,j = 1,\dots,n\}$*.*

In particular, for $n = 2$ we get

Corollary 16.5. *If* θ_1*,* θ_2 *are irrational numbers then* $\mathsf{CAR}_{\theta_1} \simeq \mathsf{CAR}_{\theta_2}$ *iff* $\theta_1 = \pm\theta_2 \mod \mathbb{Z}$*.*

17. Summary of paper IV

In paper IV, "CCR and CAR algebras are connected via a path of Cuntz-Toeplitz algebras" we prove conjecture of Jorgensen, Schmidt and Werner about independence of C^*-isomorphism class of universal C^*-algebra $\mathfrak{E}_q$ generated by $a_1, \ldots, a_n$ subject to relations

$$a_i^* a_j = \delta_{ij} + q a_j a_i^*$$

from number q. It was conjectured that $\mathfrak{E}_q \simeq \mathfrak{E}_0$, but the prove was done only for $|q| < \sqrt{2} - 1$. We prove it for $|q| < 1$ using recent developments in classification of C^*-algebras, Kirchberg-Philips theory and works of Rosenberg from 70s.

18. Summary of paper V

Everybody knows Atiyah-Singer index formula, here it is:

$$\operatorname{ind}(D) = \int_M \tau \circ \mathsf{ch}[\sigma_D] \wedge \mathsf{Td}(T^*M).$$

It works only for elliptic pseudodifferential operators on compact manifolds. But there might be not so many of them! Eric van Erp gave (not for the first time in history probably!) an example of non-elliptic differential operator which is hypoelliptic, Fredholm and everything else you might want from a differential operator, you can observe it below:

$$D = X^2 + Y^2 + \gamma Z,$$

where $[X, Y] = Z$ and γ is a function. It appears that D is Fredholm if and only if image of γ does not contain odd integers. This differential operator lives naturally in a nonclassical pseudodifferential calculus called Heisenberg calculus. Heisenberg calculus can be introduced for contact manifolds, which locally look like Heisenberg Lie algebra. Contact manifolds have a distinguished vector subbundle $H \subset TM$ of codimension 1, quotient TM/H let's denote by Z. Eric van Erp and Paul Baum modified Atiyah-Singer index formula to work for Heisenberg elliptic operators on contact manifolds:

$$\operatorname{ind}(D) = \int_Z \tau \circ \mathsf{ch}([\sigma_D^H] \otimes \mathcal{H}) \wedge \mathsf{Td}(Z).$$

In paper V, "Index theory of hypoelliptic operators on Carnot manifolds" we modify this formula even further. Carnot manifolds are manifolds with filtration of TM by subbundles:

$$\{0\} = T^0M \subset T^1M \subset \ldots \subset T^kM = TM,$$

such that $[C^\infty(M, T^iM), C^\infty(M, T^jM)] \subset C^\infty(M, T^{i+j}M)$. Carnot manifolds locally resemble certain nilpotent Lie algebras. For such manifolds there is a corresponding pseudodifferential calculus and notion of ellipticity, called Rockland condition. For pseudodifferential operators with Rockland condition our formula now is

$$\operatorname{ind}(D) = \int_\Gamma \tau \circ \mathsf{ch}([\sigma_D^{\mathcal{F}}] \otimes \mathcal{H}) \wedge \mathsf{Td}(\Gamma),$$

where Γ is a fiber bundle over the manifold with fiber being a dense stratum of spectrum of the nilpotent Lie algebra associated to the Carnot manifold.

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