Sketches of Noncommutative Topology

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Abstract

This thesis thematically divided into two parts. In the first part we are mastering C^*-isomorphism problem by using various techniques applied to different examples of noncommutative algebraic varieties. In the second part we apply noncommutative homotopy theory to C^*-algebraic objects related to manifold theory, in such a way deriving results and formulas for such an object as differential operators.

In the first article we consider C^*-algebra $Isom_{q_{ij}}$ generated by n isometries $a_1, \ldots, a_n$ satisfying the relations $a_i^* a_j = q_{ij} a_j a_i^*$ with $\max |q_{ij}| < 1$. This C^*-algebra is shown to be nuclear. We prove that the Fock representation of $Isom_{q_{ij}}$ is faithful. Further we describe an ideal in $Isom_{q_{ij}}$ which is isomorphic to the algebra of compact operators.

In the second article we consider the C^*-algebra $\mathcal{E}_{n,m}^q$, which is a q-twist of two Cuntz-Toeplitz algebras. For the case $|q| < 1$, we give an explicit formula which untwists the q-deformation showing that the isomorphism class of $\mathcal{E}_{n,m}^q$ does not depend on q. For the case $|q| = 1$, we give an explicit description of all ideals in $\mathcal{E}_{n,m}^q$. In particular, we show that $\mathcal{E}_{n,m}^q$ contains a unique largest ideal $\mathcal{M}_q$. We identify $\mathcal{E}_{n,m}^q / \mathcal{M}_q$ with the Rieffel deformation of $\mathcal{O}_n \otimes \mathcal{O}_m$ and use a K-theoretical argument to show that the isomorphism class does not depend on q. The latter result holds true in a more general setting of multiparameter deformations.

In the third article we consider the universal enveloping C^*-algebra CAR_Θ of the $*$-algebra generated by $a_1, \ldots, a_n$ subject to the relations

$$a_i^* a_i + a_i a_i^* = 1,$$

$$a_i^* a_j = e^{2\pi i \Theta_{ij}} a_j a_i^*,$$
$$a_i a_j = e^{-2\pi i \Theta_{ij}} a_j a_i$$

for a skew-symmetric real nxn matrix Θ. We prove that CAR_Θ has a $C(K_n)$-structure, where $K_n = [0, \frac{1}{2}]^n$ is the hypercube and describe the fibers. We classify irreducible representations of CAR_Θ in terms of irreducible representations of a higher-dimensional noncommutative torus. We prove that for a given irrational skew-symmetric Θ_1 there are only finitely many Θ_2 such that $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$. Namely, $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$ implies $(\Theta_1)_{ij} = \pm(\Theta_2)_{\sigma(i,j)}$ for a bijection σ of the set $\{(i,j) : i <$

$j, i, j = 1, \ldots, n\}$. For $n = 2$ we give a full classification: $\mathsf{CAR}_{\theta_1} \simeq \mathsf{CAR}_{\theta_2}$ iff $\theta_1 = \pm\theta_2 \mod \mathbb{Z}$.

In the fourth article we consider the universal enveloping C^*-algebra of a $*$-algebra of q-canonical commutation relations (q-CCR) for $q \in \mathbb{R}$, $|q| < 1$, which is generated by $a_1, \ldots, a_n$ subject to the relations

$$a_i^* a_j = \delta_{ij} 1 + q a_j a_i^*.$$

It has a distinguished representation π_F called the Fock representation, which is believed to be faithful. In this article we denote the image of the universal enveloping C^*-algebra of q-CCR in the Fock representation by $\daleth_q$. The question whether C^*-isomorphism $\daleth_q \simeq \daleth_0$ holds has been considered in the literature and proved for $|q| < 0.44$. In this article we show that $\daleth_q \simeq \daleth_0$ for $|q| < 1$.

In the fifth article we study the index theory of hypoelliptic operators on Carnot manifolds – manifolds whose Lie algebra of vector fields is equipped with a filtration induced from sub-bundles of the tangent bundle. A Heisenberg pseudodifferential operator, elliptic in the calculus of van Erp-Yuncken, is hypoelliptic and Fredholm. Under some geometric conditions, we compute its Fredholm index by means of operator K-theory. These results extend the work of Baum-van Erp for co-oriented contact manifolds to a methodology for solving this index problem geometrically on Carnot manifolds. Under the assumption that the Carnot manifold is regular, i.e. has isomorphic osculating Lie algebras in all fibres, and admits a flat coadjoint orbit, the methodology derived from Baum-van Erp's work is developed in full detail. In this case, we develop K-theoretical dualities computing the Fredholm index by means of geometric K-homology a la Baum-Douglas. The duality involves a Hilbert space bundle of flat orbit representations. Explicit solutions to the index problem for Toeplitz operators and operators of the form "$\Delta_H + \gamma T$" are computed in geometric K-homology, extending results of Boutet de Monvel and Baum-van Erp, respectively, from co-oriented contact manifolds to regular polycontact manifolds. The existence and the precise form of the geometric duality constructed for the Heisenberg calculus relies on the representation theory in the flat coadjoint orbits of the osculating Lie groupoid. We address the technical issue of constructing a Hilbert space bundle of representations associated to the flat coadjoint orbits via Kirillov's orbit method. The construction intertwines the index theory of Heisenberg operators to characteristic classes constructed from the Carnot structure further clarifying the two opposite spin^c-structures appearing in Baum-van Erp's solution to the index problem on contact manifolds.

Keywords: Wick algebra, Rieffel's deformation, extension theory, K-theory.