

THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Global residue currents and the Ext functors

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ABSTRACT

This thesis concerns developments in multivariable residue theory. In particular we consider global constructions of residue currents related to work by Andersson and Wulcan. In the first paper of this thesis, we consider global residue currents defined on projective space, and we show that these currents provide a tool for studying polynomial interpolation.

Polynomial interpolation is related to local cohomology, and by a result known as local duality, there is a close connection with certain Ext groups. The second paper of this thesis is devoted to further study of connections between residue currents and the Ext functors. The main result is that we construct a global residue current on a complex manifold, and using this we give an explicit formula for an isomorphism of two different representations of the global Ext groups on complex manifolds.

Keywords: Complex spaces, Residue currents, Polynomial interpolation, Ext groups

PREFACE

This thesis is based on the following papers:

- **Paper I.** Jimmy Johansson, *Polynomial interpolation and residue currents*, Complex Variables and Elliptic Equations (2021)
- **Paper II.** Jimmy Johansson and Richard Lärkäng, *An explicit isomorphism of different representations of the Ext functor using residue currents*, preprint
- **Paper III.** Jimmy Johansson, *A residue current associated with a twisting cochain: duality and comparison formula*, preprint

The following paper is not included in this thesis:

- Jimmy Johansson and Lyudmila Turowska, *The Shilov boundary for a q -analog of the holomorphic functions on the unit ball of 2×2 symmetric matrices*, Operators and Matrices **12** (2018), no. 1, 39–53

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1. INTRODUCTION

This thesis concerns developments of the theory of *residue currents* with applications to problems in algebraic geometry. Classically, algebraic geometry is the branch of mathematics concerned with the study of solutions to systems of polynomial equations. The set of all solutions to a system of polynomial equations is called an *algebraic set*. As an example we have that the unit circle is an algebraic set, since it is the set of points $(x, y) \in \mathbb{R}^2$ that satisfy the equation $x^2 + y^2 = 1$. In mathematical notation, we can express this set as

$$X_1 = \{(x, y) : x^2 + y^2 = 1\}.$$

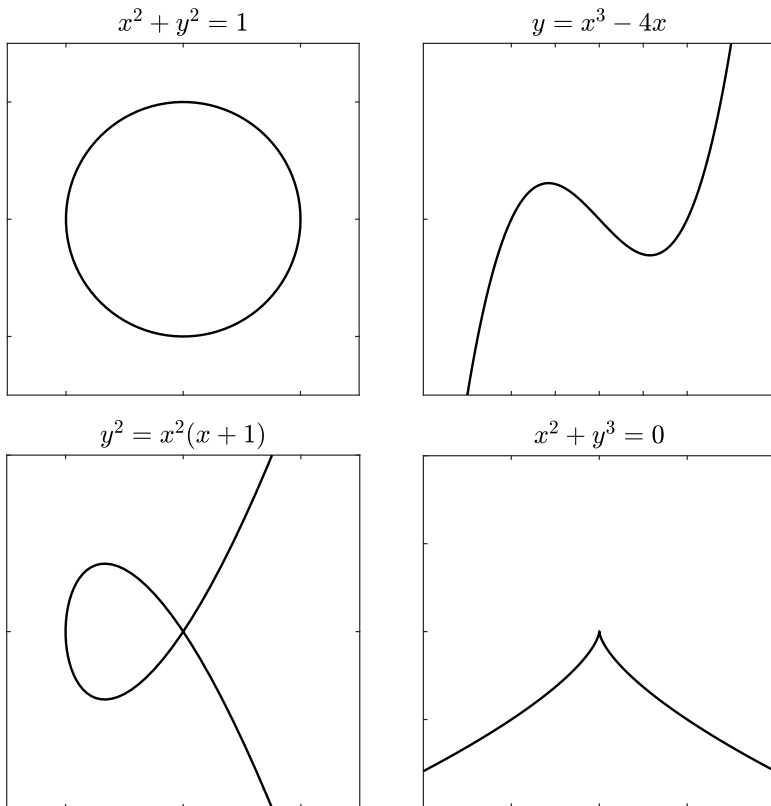


FIGURE 1. Examples of algebraic sets

Other examples of algebraic sets in the plane are

$$\begin{aligned} X_2 &= \{(x, y) : y = x^3 - 4x\}, \\ X_3 &= \{(x, y) : y^2 = x^2(x + 1)\}, \\ X_4 &= \{(x, y) : x^2 + y^3 = 0\}. \end{aligned}$$

These algebraic sets are shown in Figure 1.

Just by looking at the pictures, we can draw some conclusions about the geometries of these sets. We see that X_1 and X_2 are smooth. This roughly means that if we look close enough, the curve will look like a straight line. However, X_3 and X_4 are not smooth at the origin since no matter how close we look we will always see a cross and a cusp, respectively.

There are many ways that one can study geometric properties of algebraic sets. One way, as the name suggests, is to study them algebraically. For simplicity, let X be an algebraic subset of \mathbb{R}^2 . If f is a polynomial in the variables x and y , then f determines a function from X to \mathbb{R} . Let $A(X)$ denote the set of all such functions with the convention that two polynomials f and g are considered equal if they agree as functions from X to \mathbb{R} , i.e., $f|_X = g|_X$. Note that elements in $A(X)$ can be added and multiplied just like ordinary polynomials. Thus $A(X)$ is an instance of an algebraic object known as a ring, and it is referred to as the coordinate ring of X . The geometry of X can be studied by studying the ring $A(X)$ using tools from algebra.

Another way, which belongs to the field of mathematical analysis, is to use residue currents, which is the topic of this thesis. The theory of residue currents is a generalization of the classical notion of residues found in the field of complex analysis. A proper definition of residue currents, how they are related to geometry, and what kind of problems they can be used to solve are the main topics of this introductory part of this thesis. For now we will simply think of residue currents as powerful analytical tools that can be used to construct explicit solutions to many problems that concern algebraic sets. One such problem concerns interpolation with respect to sets that consist of finitely many points in \mathbb{R}^n or, more generally, \mathbb{C}^n . Such sets are also algebraic sets, and, although they might look simple, they are important examples of algebraic sets that give rise to many interesting problems.

Let us describe what we mean by interpolation. Let $X = \{p_1, p_2, \dots, p_r\}$ be a set consisting of r points in \mathbb{R}^n , and let g be any real-valued function defined on X . If G is a polynomial defined on \mathbb{R}^n , then we say that

G interpolates g if $G = g$ on X , i.e.,

$$G(p_1) = g(p_1),$$

$$G(p_2) = g(p_2),$$

...

$$G(p_r) = g(p_r).$$

Figure 2 shows an example of a function g defined on a set of 5 points in \mathbb{R} and an interpolant G of degree 3.

We are motivated by the following question: What are the conditions on a function $g \in A(X)$ for the existence of an interpolant of degree at most d ? It is not difficult to see that, in order for g to have an interpolant of degree at most d , the values of g at the points p_j must satisfy a system of linear equations. The main application of the results of Paper I is to show how one can explicitly compute this system of equations using residue currents.

Example 1.1. Let

$$X := \{-2, -1, 0, 1, 2\} \subseteq \mathbb{R},$$

and let $g : X \rightarrow \mathbb{R}$ be the function defined by

$$g(-2) = 0, \quad g(-1) = 3, \quad g(0) = 0, \quad g(1) = -3, \quad g(2) = 0.$$

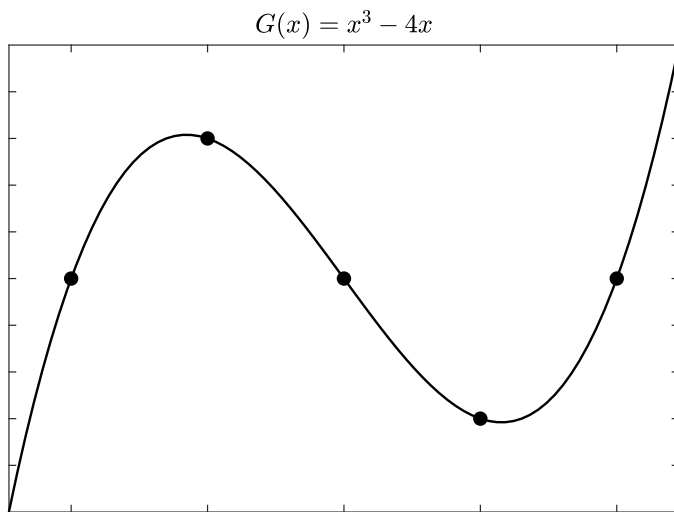


FIGURE 2. A polynomial of degree 3 that interpolates a function defined on 5 points in \mathbb{R} .

It turns out that there exists a polynomial of degree 3 that interpolates g , namely $G(x) = x^3 - 4x$, see Figure 2. Clearly, there is no polynomial of lower degree that interpolates g .

Let now $g : X \rightarrow \mathbb{R}$ be an arbitrary function, and suppose that we want to find a polynomial of degree at most 1 that interpolates G . Clearly, this is possible if and only if the points

$$(-2, g(-2)), (-1, g(-1)), \dots, (2, g(2)) \in \mathbb{R}^2$$

lie on a straight line. A calculation shows that this is the case if and only if the values of g satisfy the linear system of equations given by

$$\begin{aligned} g(0) &= g(-2) + 2(g(-1) - g(-2)) \\ g(1) &= g(-2) + 3(g(-1) - g(-2)) \\ g(2) &= g(-2) + 4(g(-1) - g(-2)). \end{aligned}$$

In this case it is easy to derive the linear system of equations that needs to be satisfied. In general, especially when we move to higher dimensions, this task becomes more difficult.

Another related problem is to determine the smallest integer d such that *any* function $g \in A(X)$ has an interpolant of degree at most d . This number is referred to as the *interpolation degree* of X . The solution to this problem will depend on the geometry of X . Let us consider an example of two sets in \mathbb{R}^2 .

Example 1.2. Consider the following sets in \mathbb{R}^2 consisting of three points each. Let

$$X_1 = \{(0, 0), (1, 0), (2, 0)\},$$

and

$$X_2 = \{(0, 0), (1, 0), (0, 1)\}.$$

A function $g \in A(X_1)$ has an interpolant of degree at most 1 if and only if its graph lies on a line. On the other hand, any $g \in A(X_1)$ has an interpolant of degree at most 2 since we can take

$$G(x, y) = g(0, 0) + (g(1, 0) - g(0, 0))x + \left(\frac{g(2, 0)}{2} - g(1, 0) + \frac{g(0, 0)}{2} \right) x(x-1).$$

Thus the interpolation degree of X_1 is 2.

For X_2 we have that any $g \in A(X_2)$ has an interpolant of degree at most 1, since we can take

$$G(x, y) = g(0, 0) + (g(1, 0) - g(0, 0))x + (g(0, 1) - g(0, 0))y.$$

Thus it follows that X_2 has interpolation degree 1.

For arbitrary sets, there is a known formula for the interpolation degree. In Paper I we will give a proof of this formula using residue currents.

So far we have only discussed interpolation with respect to sets consisting of finitely many points. If things are interpreted suitably, then one can also consider interpolation with respect to more general algebraic sets. This will be the point of view that we take in Paper I, but we will keep in mind that our primary motivation comes from polynomial interpolation with respect to points.

The interpolation problem can also be studied using tools from algebra. Let $X \subseteq \mathbb{R}^n$ be an algebraic set. Let \mathbb{P}_d denote the set of polynomials in n variables of degree at most d . Any polynomial in \mathbb{P}_d can be thought of as a function defined on X by restricting the polynomial to a function on X . In other words, there is a map $\pi_d : \mathbb{P}_d \rightarrow A(X)$ given by $\pi_d(f) = f|_X$. The questions posed above can then be stated in terms of this map as follows. When is $g \in A(X)$ in the image of π_d , and when is π_d surjective?

Closely related to $A(X)$ is the *homogeneous coordinate ring* S_X . With it one can associate its first *local cohomology module*

$$H_{\mathfrak{m}}^1(S_X) = \bigoplus_d H_{\mathfrak{m}}^1(S_X)_d.$$

This object has the property that for each nonnegative integer d there exists a surjective map $\delta_d : A(X) \rightarrow H_{\mathfrak{m}}^1(S_X)_d$ with the property that $g \in A(X)$ is in the image of π_d if and only if $\delta_d(g) = 0$ in $H_{\mathfrak{m}}^1(S_X)_d$. In algebraic terminology we say that the sequence of maps

$$\mathbb{P}_d \xrightarrow{\pi_d} A(X) \xrightarrow{\delta_d} H_{\mathfrak{m}}^1(S_X)_d \longrightarrow 0 \quad (1.1)$$

is *exact*. Thus the functions $g \in A(X)$ that have an interpolant of degree at most d are precisely those that are mapped to 0 in $H_{\mathfrak{m}}^1(S_X)_d$. Moreover, each element of $H_{\mathfrak{m}}^1(S_X)_d$ corresponds to a function in $A(X)$ which does not have an interpolant of degree at most d .

The first local cohomology module $H_{\mathfrak{m}}^1(S_X)$ is closely related to a certain *Ext functor* denoted $\text{Ext}^n(S_X, S)$. (Here S denotes the ring of homogeneous polynomials.) In Paper II, we investigate the connection between a certain class of Ext functors and residue currents.

1.1. An elementary introduction to residue currents. The theory of residue currents is one way to generalize the notion of residues in complex analysis in one variable to the multivariable case. In this section we will attempt to explain, as elementary as possible, how ordinary residues can be viewed as so-called currents. In Section 2 we will then continue with the multivariable case.

Recall that if f is a holomorphic function in a punctured disk with center $a \in \mathbb{C}$, then

$$\text{Res}(f; a) := \frac{1}{2\pi i} \int_{C(a, r)} f(\zeta) d\zeta,$$

where $C(a, r)$ is the circle with center a and sufficiently small radius r . Let $\Omega \subseteq \mathbb{C}$ be a simply connected open subset, and suppose that f is a holomorphic function defined on $\Omega \setminus \{a_1, \dots, a_r\}$. The residue theorem implies that

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i \sum_{k=1}^r \text{Res}(f; a_k),$$

where γ is any simple closed curve that encircles the a_k .

We will think of residues as objects that “act” on functions. Suppose that $0 \in \Omega$, and suppose that f does not vanish outside 0. Let $\mathcal{O}(\Omega)$ denote the space of holomorphic functions on Ω , and consider the linear functional $R_f : \mathcal{O}(\Omega) \rightarrow \mathbb{C}$ defined by

$$R_f(h) := \text{Res}\left(\frac{h}{f}; 0\right),$$

i.e., R_f is a function that as input takes a holomorphic function h and outputs a complex number denoted by $R_f(h)$. For another holomorphic function ϕ we can form the linear functional ϕR_f which is defined by

$$(\phi R_f)(h) := R_f(\phi h).$$

The functional R_f satisfies the following fundamental property called the *duality principle*.

Proposition 1.3. *Let $\phi \in \mathcal{O}(\Omega)$ be a holomorphic function. There exists a holomorphic function $g \in \mathcal{O}(\Omega)$ such that $\phi = fg$ if and only if $\phi R_f = 0$.*

When we write $\phi R_f = 0$ we mean that ϕR_f is the zero functional, i.e., $(\phi R_f)(h) = 0$ for all holomorphic functions h , or, equivalently, $R_f(\phi h) = 0$ for all holomorphic functions h .

Proof. Suppose that $\phi = fg$. Then

$$\begin{aligned} (\phi R_f)(h) &= R_f(\phi h) \\ &= \text{Res}\left(\frac{\phi h}{f}; 0\right) \\ &= \text{Res}(gh; 0) = 0. \end{aligned}$$

Since this holds for all h , we have that $\phi R_f = 0$.

Conversely, suppose that $\phi R_f = 0$. By Cauchy's integral formula we have that

$$\phi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta.$$

for any closed simple curve γ that contains z in its interior. By writing

$$\frac{1}{\zeta - z} = \frac{f(z) + f(\zeta) - f(z)}{f(\zeta)(\zeta - z)},$$

we get that

$$\phi(z) = \frac{f(z)}{2\pi i} \int_{\gamma} \frac{\phi(\zeta)}{f(\zeta)(\zeta - z)} d\zeta + \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(\zeta)(f(\zeta) - f(z))}{f(\zeta)(\zeta - z)} d\zeta. \quad (1.2)$$

For each z we have that $f(\zeta) - f(z) = (z - \zeta)u(\zeta)$ for some holomorphic function u . Since $\phi R_f = 0$, we thus get that the second term vanishes, and hence $\phi(z) = f(z)g(z)$, where g is the holomorphic function

$$g(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(\zeta)}{f(\zeta)(\zeta - z)} d\zeta. \quad \square$$

It turns out that it is beneficial to let R_f act not only on holomorphic functions but instead of all smooth functions ξ . To this end, we redefine R_f as

$$R_f(\xi) := \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{C(0, \epsilon)} \frac{\xi}{f} d\zeta,$$

where ξ is a smooth function. Note that this agrees with the original definition of R_f if ξ is holomorphic. Such a functional that acts on the space of smooth functions is called a residue current. In Section 2 we will continue with how one can generalize this idea to the multivariable case, and we will show that the duality principle holds in this case as well.

2. RESIDUE CURRENTS

There has been a lot of work in generalizing the theory of residues to the multivariable case. In this section we will give a brief introduction to the current aspect introduced by and further developed by Coleff, Dickenstein, Herrera, Lieberman, Passare, Sessa, Tsikh, Yger, et al, see [CH], [DS], [HL], [P], [PTY]. In particular we will focus on more recent developments by Andersson, Lärkäng, and Wulcan.

Throughout this thesis, let $\chi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a smooth function that satisfies $\chi(t) = 0$ in a neighborhood of zero and $\chi(t) = 1$ when $t \geq 1$.

Let f be a holomorphic function on \mathbb{C}^n . Using Hironaka's desingularization theorem [H2], it can be shown that the following limit exists and

defines a current referred to as the principal value current:

$$\left[\frac{1}{f} \right] := \lim_{\epsilon \rightarrow 0} \chi \left(\frac{|f|^2}{\epsilon} \right) \frac{1}{f}. \quad (2.1)$$

The limit should be interpreted in the sense that the action on a test form ξ is given by

$$\left\langle \left[\frac{1}{f} \right], \xi \right\rangle = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}^n} \chi \left(\frac{|f|^2}{\epsilon} \right) \frac{\xi}{f}. \quad (2.2)$$

If χ is taken to be the indicator function of $[1, \infty)$, then this is in fact precisely the definition of the principal value current given by Herrera and Lieberman in [HL].

From (2.2) it is clear that

$$f \left[\frac{1}{f} \right] = 1. \quad (2.3)$$

The $(0, 1)$ current

$$R^f := \bar{\partial} \left[\frac{1}{f} \right]$$

is called the *residue current* of f . Since $\frac{1}{f}$ is holomorphic outside the zero set of f , $V(f) := f^{-1}(0)$, R^f has support on $V(f)$. We have that its action on a test form ξ is given by

$$\begin{aligned} \left\langle \bar{\partial} \left[\frac{1}{f} \right], \xi \right\rangle &= - \left\langle \left[\frac{1}{f} \right], \bar{\partial} \xi \right\rangle = \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}^n} \chi \left(\frac{|f|^2}{\epsilon} \right) \frac{\bar{\partial} \xi}{f} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}^n} \bar{\partial} \chi \left(\frac{|f|^2}{\epsilon} \right) \frac{\xi}{f}, \end{aligned} \quad (2.4)$$

where the last equality follows by Stokes' theorem. Therefore

$$\bar{\partial} \left[\frac{1}{f} \right] = \lim_{\epsilon \rightarrow 0} \bar{\partial} \chi \left(\frac{|f|^2}{\epsilon} \right) \frac{1}{f}.$$

Theorem 2.1. *Let ϕ be a holomorphic function. There exists a holomorphic function ψ such that $\phi = \psi f$ if and only if $\phi R^f = 0$.*

Proof. Suppose that $\phi = \psi f$ for some holomorphic function ψ . Then by using (2.3),

$$\phi R^f = \psi \bar{\partial} \left(f \left[\frac{1}{f} \right] \right) = \psi \bar{\partial}(1) = 0.$$

Conversely, if $\phi R^f = 0$, then

$$\bar{\partial} \left(\phi \left[\frac{1}{f} \right] \right) = \phi R^f = 0.$$

By elliptic regularity of the $\bar{\partial}$ -operator there is a holomorphic function ψ that is equal to

$$\phi \left[\frac{1}{f} \right]$$

as currents. In view of (2.3), it follows that $\phi = \psi f$. \square

Let $f = (f_1, \dots, f_p) : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a holomorphic mapping. If f defines a complete intersection, i.e., $V(f) = f^{-1}(0)$ has codimension p , then Coleff and Herrera showed in [CH] how one can make sense of the expression

$$\mu^f := \bar{\partial} \left[\frac{1}{f_p} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{f_1} \right].$$

Since there is no general notion of products of distributions, let alone wedge products of currents, a priori, this is not well-defined. However, Coleff and Herrera showed that with their definition this expression behaves very much as if μ^f was the wedge product of the $(0,1)$ -currents $\bar{\partial} \left[\frac{1}{f_j} \right]$, in the sense that μ^f is a $\bar{\partial}$ -closed $(0,p)$ -current with support on $V(f)$ that is anticommuting in the f_j . We will not give Coleff and Herrera's definition here. Instead, we will use the more recent theory of *pseudomeromorphic* currents, see [AW2] and [AW3], to define μ^f and give a proof of these facts in Theorem 2.4. Moreover, following [A2], we will give a proof of the following theorem.

Theorem 2.2. *A holomorphic function ϕ belongs locally to the ideal generated by f if and only if $\phi \mu^f = 0$.*

This result is known as the duality principle, and it was independently proved by Dickenstein–Sessa in [DS] and Passare in [P].

Let

$$\left[\frac{1}{z_{j_1}^{m_1}} \right] \dots \left[\frac{1}{z_{j_k}^{m_k}} \right] \bar{\partial} \left[\frac{1}{z_{j_{k+1}}^{m_{k+1}}} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{z_{j_r}^{m_r}} \right] \quad (2.5)$$

denote the tensor product of the one-variable currents $\left[\frac{1}{z_k^{m_k}} \right]$ and $\bar{\partial} \left[\frac{1}{z_\ell^{m_\ell}} \right]$.

Let τ be a current of the form (2.5), and let α be a smooth form with compact support. We say that a current of the form $\tau \wedge \alpha$ is an *elementary pseudomeromorphic current*. We say that a current is *pseudomeromorphic* if it is given by a locally finite sum of push-forwards of elementary pseudomeromorphic currents under holomorphic mappings, see [AW3, Theorem 2.15]. We denote by $\mathcal{PM}(X)$ the sheaf of pseudomeromorphic currents on a complex manifold X .

Let T be a pseudomeromorphic current. Then $\bar{\partial}T$ is also pseudomeromorphic. Moreover, the class of pseudomeromorphic currents is closed

under multiplication by smooth forms. One important property of pseudomeromorphic currents is that they satisfy the following *dimension principle*.

Proposition 2.3. *Let T be a pseudomeromorphic $(*, q)$ -current on X with support on a subvariety Z . If $\text{codim } Z > q$, then $T = 0$.*

Given a holomorphic function h we define

$$\left[\frac{1}{h} \right] T = \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon) \frac{1}{h} T. \quad (2.6)$$

This is well-defined by [AW3, Section 2.2]. Moreover, we define

$$\bar{\partial} \left[\frac{1}{h} \right] \wedge T = \bar{\partial} \left(\left[\frac{1}{h} \right] T \right) - \left[\frac{1}{h} \right] \bar{\partial} T, \quad (2.7)$$

so that Leibniz's rule holds.

With these definitions, we define the Coleff–Herrera product recursively as

$$\begin{aligned} \mu^{f_1} &:= \bar{\partial} \left[\frac{1}{f_1} \right], \\ \mu^f &= \mu^{(f_p, \dots, f_1)} := \bar{\partial} \left[\frac{1}{f_p} \right] \wedge \mu^{(f_{p-1}, \dots, f_1)}. \end{aligned}$$

We wish to think of (2.6) and (2.7) as products and wedge products, respectively. However these product do not, in general, satisfy the properties that one would expect. For example, it is not in general true that

$$\left[\frac{1}{f} \right] \bar{\partial} \left[\frac{1}{g} \right] = \bar{\partial} \left[\frac{1}{g} \right] \cdot \left[\frac{1}{f} \right]$$

and

$$\bar{\partial} \left[\frac{1}{f} \right] \wedge \bar{\partial} \left[\frac{1}{g} \right] = -\bar{\partial} \left[\frac{1}{g} \right] \wedge \bar{\partial} \left[\frac{1}{f} \right].$$

For example, it follows from (2.6) that

$$\left[\frac{1}{z} \right] \bar{\partial} \left[\frac{1}{z} \right] = 0,$$

but in view of (2.7) we have that

$$\bar{\partial} \left[\frac{1}{z} \right] \left[\frac{1}{z} \right] = \bar{\partial} \left[\frac{1}{z^2} \right].$$

Moreover, by (2.6) and (2.7), in \mathbb{C}^2 we have that

$$\bar{\partial} \left[\frac{1}{zw} \right] \wedge \bar{\partial} \left[\frac{1}{z} \right] = 0$$

but

$$\bar{\partial} \left[\frac{1}{z} \right] \wedge \bar{\partial} \left[\frac{1}{zw} \right] = \bar{\partial} \left[\frac{1}{z^2} \right] \wedge \bar{\partial} \left[\frac{1}{w} \right] \neq 0.$$

To see this, note that

$$\left[\frac{1}{z} \right] \bar{\partial} \left[\frac{1}{zw} \right] = \lim_{\epsilon \rightarrow 0} \chi(|z|^2/\epsilon) \frac{1}{z^2} \bar{\partial} \left[\frac{1}{w} \right],$$

Thus outside $\{z = 0\}$ we have that

$$\left[\frac{1}{z} \right] \bar{\partial} \left[\frac{1}{zw} \right] = \frac{1}{z^2} \bar{\partial} \left[\frac{1}{w} \right],$$

and outside $\{w = 0\}$ we have that

$$\left[\frac{1}{z} \right] \wedge \bar{\partial} \left[\frac{1}{zw} \right] = 0.$$

Thus

$$T := \left[\frac{1}{z} \right] \wedge \bar{\partial} \left[\frac{1}{zw} \right] - \left[\frac{1}{z^2} \right] \wedge \bar{\partial} \left[\frac{1}{w} \right]$$

has support at the origin, and hence $T = 0$ by the dimension principle. Applying $\bar{\partial}$ to T gives the statement.

On the other hand, suppose that $f, g : \mathbb{C}^2 \rightarrow \mathbb{C}$ are holomorphic and $V(f, g)$ has codimension 2. Note that

$$T := \left[\frac{1}{f} \right] \bar{\partial} \left[\frac{1}{g} \right] - \bar{\partial} \left[\frac{1}{g} \right] \left[\frac{1}{f} \right]$$

has support on $V(f, g)$, and hence $T = 0$ by the dimension principle. Thus

$$\left[\frac{1}{f} \right] \bar{\partial} \left[\frac{1}{g} \right] = \bar{\partial} \left[\frac{1}{g} \right] \left[\frac{1}{f} \right].$$

Applying $\bar{\partial}$, we get that

$$\bar{\partial} \left[\frac{1}{f} \right] \wedge \bar{\partial} \left[\frac{1}{g} \right] = -\bar{\partial} \left[\frac{1}{g} \right] \wedge \bar{\partial} \left[\frac{1}{f} \right].$$

This idea is the key to prove that the Coleff–Herrera product is anticommutative in the f_j .

Theorem 2.4. *The Coleff–Herrera product μ^f is $\bar{\partial}$ -closed, has support on $V(f)$ and is anticommuting in the f_j . Moreover,*

$$f_p \left(\left[\frac{1}{f_p} \right] \mu^{(f_{p-1}, \dots, f_1)} \right) = \mu^{(f_{p-1}, \dots, f_1)}, \quad (2.8)$$

and

$$f_j \mu^f = 0. \quad (2.9)$$

Proof. We argue by induction over p . From the discussion in the beginning of this section it follows that the theorem holds if $p = 1$. Suppose that the theorem is proved for $p = k$ and consider

$$T := \left[\frac{1}{f_{k+1}} \right] \mu^{(f_k, \dots, f_1)} - \bar{\partial} \left[\frac{1}{f_k} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{f_1} \right] \left[\frac{1}{f_{k+1}} \right]. \quad (2.10)$$

Note that both terms have support on $V(f_1, \dots, f_k)$, and hence T has support there as well. Outside $V(f_{k+1})$ we have that

$$T = \left[\frac{1}{f_{k+1}} \right] \mu^{(f_k, \dots, f_1)} - \mu^{(f_k, \dots, f_1)} \left[\frac{1}{f_{k+1}} \right] = 0$$

since $\left[\frac{1}{f_{k+1}} \right]$ is smooth there. Thus T has support on $V(f_1, \dots, f_{k+1})$.

Since this set has codimension $k + 1$ and T has bidegree $(0, k)$ it follows from the dimension principle that $T = 0$. Applying $\bar{\partial}$ to (2.10) we get that

$$\bar{\partial} \left[\frac{1}{f_{k+1}} \right] \wedge \bar{\partial} \left[\frac{1}{f_k} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{f_1} \right] = (-1)^k \bar{\partial} \left[\frac{1}{f_k} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{f_1} \right] \wedge \bar{\partial} \left[\frac{1}{f_{k+1}} \right].$$

From this together with the induction hypothesis it follows that μ^f is anticommuting in the f_j .

Using the induction hypothesis it follows that

$$T := \mu^{(f_k, \dots, f_1)} - f_{k+1} \left[\frac{1}{f_{k+1}} \right] \mu^{(f_k, \dots, f_1)}$$

has support on $V(f_1, \dots, f_{k+1})$. As before, this set has codimension $k + 1$ and T has bidegree $(0, k)$, and hence it follows from the dimension principle that $T = 0$. Therefore (2.8) holds. Applying $\bar{\partial}$ to T and using the induction hypothesis one sees that (2.9) holds. \square

Proof of Theorem 2.2. If ϕ is in the ideal, then $\phi \mu^f = 0$ by Theorem 2.4.

For the converse statement, we begin by introducing some formalism. Let E be a trivial vector bundle of rank p , and let e_1, \dots, e_p be a global frame. We define a map

$$D : \bigwedge^r E \rightarrow \bigwedge^{r-1} E$$

by

$$De_{i_1} \wedge \dots \wedge e_{i_r} := \sum_{j=1}^r (-1)^{j+1} f_{i_j} e_{i_1} \wedge \dots \wedge \widehat{e}_{i_j} \wedge \dots \wedge e_{i_r}.$$

It can be shown that $D^2 = 0$. We extend this map to the sheaf of $(\bigwedge^\bullet E)$ -valued $(0, q)$ -currents, $\mathcal{C}^{0,q}(\bigwedge^\bullet E)$, as follows. If ω is a $(0, q)$ -current we define

$$D(\omega e_{i_1} \wedge \dots \wedge e_{i_r}) := (-1)^q \omega D e_{i_1} \wedge \dots \wedge e_{i_r}.$$

With this definition we have that $\bar{\partial}D = -D\bar{\partial}$. We combine D and $\bar{\partial}$ into an operator ∇ on $\mathcal{C}^{0,\bullet}(\bigwedge^\bullet E)$:

$$\nabla := D - \bar{\partial}.$$

We can now carry on with the proof. Define

$$U := \left[\frac{1}{f_1}\right] e_1 + \left[\frac{1}{f_2}\right] \mu^{f_1} e_1 \wedge e_2 + \cdots + \left[\frac{1}{f_p}\right] \mu^{(f_{p-1}, \dots, f_1)} e_1 \wedge \cdots \wedge e_p.$$

Then, in view of (2.8), we have that

$$\nabla U = 1 - \mu^f e_1 \wedge \cdots \wedge e_p.$$

Suppose that $\phi\mu^f = 0$. Since ϕ is holomorphic we have that

$$\nabla(\phi U) = \phi(\nabla U) = \phi - \phi\mu^f e_1 \wedge \cdots \wedge e_p = \phi.$$

Set $v := \phi U$, and let v_r denote the $\bigwedge^r E$ -valued component of v . We have that v is a current solution to the system of equations,

$$\begin{aligned} Dv_1 &= \phi, \\ Dv_{r+1} &= \bar{\partial}v_r, \quad r \geq 1. \end{aligned}$$

Locally, we can find a section w that satisfies the system of $\bar{\partial}$ -equations

$$\bar{\partial}w_r = v_r + Dw_{r+1}.$$

This is because the right-hand side is $\bar{\partial}$ -closed, i.e., we have

$$\bar{\partial}(v_r + Dw_{r+1}) = \bar{\partial}v_r - D\bar{\partial}w_{r+1} = 0.$$

Now $\psi = v_1 + Dw_2$ is a holomorphic section that satisfies $D\psi = \phi$, i.e.,

$$\psi_1 f_1 + \cdots + \psi_p f_p = \phi,$$

and hence we are done. \square

In the next section we will describe a construction due to Andersson and Wulcan, which associates a residue current R with an arbitrary ideal sheaf, that is in general not a complete intersection, that satisfies the duality principle. Recall that this means that a holomorphic section ϕ belongs to \mathcal{J} if and only if $R\phi = 0$. The basic idea is quite similar to the construction above. One starts with a locally free resolution of $\mathcal{O}_X/\mathcal{J}$, that plays the role of the complex $\bigwedge^\bullet E$ above, and constructs a current U such that $\nabla U = \text{id} - R$. Before this we will introduce some additional theory concerning pseudomeromorphic currents.

Let T be a pseudomeromorphic current, and let $V \subseteq X$ be a subvariety. In [AW2], the authors defined the restriction of T to $X \setminus V$ denoted by $\mathbf{1}_{X \setminus V} T$. This is a pseudomeromorphic current on X defined by

$$\mathbf{1}_{X \setminus V} T := \lim_{\epsilon \rightarrow 0} \chi(|F|^2/\epsilon) T,$$

where F is a section of a Hermitian holomorphic vector bundle with metric $|\cdot|$ such that $V = \{F = 0\}$. A pseudomeromorphic current T on X is said to have the *standard extension property* if $\mathbf{1}_{X \setminus V} T = T$ for any subvariety V of positive codimension.

Let s be a holomorphic section of a Hermitian holomorphic line bundle L . Just as in (2.1), the principal value current $\left[\frac{1}{s}\right]$ can be defined as

$$\left[\frac{1}{s}\right] := \lim_{\epsilon \rightarrow 0} \chi(|s|^2/\epsilon) \frac{1}{s}.$$

A current is said to be *semimeromorphic* if it is of the form $\omega \left[\frac{1}{s}\right]$, where ω is a smooth form with values in L . A current A is *almost semimeromorphic* on X , written $A \in ASM(X)$, if there is a modification $\pi : X' \rightarrow X$ such that A is the push-forward of a semimeromorphic current, i.e.,

$$A = \pi_* \left(\omega \left[\frac{1}{s}\right] \right).$$

The almost semimeromorphic currents on X form an algebra over smooth forms.

Note that an almost semimeromorphic current is in particular a global pseudomeromorphic current. It follows from the dimension principle and the fact that the restriction commutes with multiplication by smooth forms that almost semimeromorphic currents have the standard extension property. In particular, if a is a smooth form on $X \setminus V$, and a has an extension as an almost semimeromorphic current A on X , then the extension is given by

$$A = \lim_{\epsilon \rightarrow 0} \chi(|F|^2/\epsilon) a. \quad (2.11)$$

Let A be an almost semimeromorphic current on X . Let Z be a subvariety of positive codimension such that A is smooth outside of Z . By [AW3, Proposition 4.16], $\mathbf{1}_{X \setminus Z} \bar{\partial} A$ is almost semimeromorphic on X . The *residue* $R(A)$ of A is defined by

$$R(A) := \bar{\partial} A - \mathbf{1}_{X \setminus Z} \bar{\partial} A. \quad (2.12)$$

Note that

$$\text{supp } R(A) \subseteq Z. \quad (2.13)$$

Let $F \neq 0$ be a section of a holomorphic vector bundle such that $\{F = 0\} \supseteq Z$, and set

$$\chi_\epsilon := \chi(|F|^2/\epsilon).$$

Since A is almost semimeromorphic, and thus has the standard extension property, it follows by (2.11) that

$$R(A) = \lim_{\epsilon \rightarrow 0} (\bar{\partial}(\chi_\epsilon A) - \chi_\epsilon \bar{\partial}A) = \lim_{\epsilon \rightarrow 0} \bar{\partial}\chi_\epsilon \wedge A. \quad (2.14)$$

If ω is a smooth form, then

$$R(\omega \wedge A) = (-1)^{\deg \omega} \omega \wedge R(A). \quad (2.15)$$

3. RESIDUE CURRENTS OF GENERICALLY EXACT COMPLEXES

In this section we will describe the construction, due to Andersson and Wulcan, of a residue current associated with a generically exact complex of holomorphic vector bundles. Let

$$0 \longrightarrow F^{-N} \xrightarrow{a} \dots \xrightarrow{a} F^{-1} \xrightarrow{a} F^0 \longrightarrow 0 \quad (3.1)$$

be a generically exact complex of holomorphic Hermitian vector bundles over a complex manifold X of dimension n , i.e., (3.1) is pointwise exact outside a subvariety $Z \subseteq X$ of positive codimension. In [AW1], Andersson and Wulcan constructed a $\text{Hom}(\bigoplus_r F^r, \bigoplus_r F^r)$ -valued current R . If (3.1) is exact as a complex of \mathcal{O}_X -modules at each level $r < 0$, then the authors proved that R takes values in $\text{Hom}(F^0, \bigoplus_{r>0} F^{-r})$, and that if ϕ is a holomorphic section of F^0 , then $\phi \in \text{im } a$ if and only if $R\phi = 0$. If F^0 is the trivial bundle and (3.1) is a locally free resolution of $\mathcal{O}_X/\mathcal{J}$, where \mathcal{J} is a coherent ideal sheaf, then we have that (3.1) is generically exact and a section ϕ of \mathcal{O}_X is a section of \mathcal{J} if and only if $R\phi = 0$.

In this section we will recall this construction. We will in fact generalize this construction slightly, because, for the purposes of this thesis, it is vital to construct a residue current associated with a locally free resolution of an arbitrary sheaf \mathcal{F} . However, if (3.1) is a locally free resolution of an arbitrary \mathcal{O}_X -module \mathcal{F} , then (3.1) is in general only generically exact at each level $r < 0$. For this reason we will generalize Andersson and Wulcan's construction in order to construct a $\text{Hom}(\bigoplus_r F^r, \bigoplus_r F^r)$ -valued current R associated with a complex that is generically exact at each level $r < 0$. We will show that if the complex is exact as a complex of \mathcal{O}_X -modules at each level $r < 0$, then a holomorphic section ϕ of F^0 is in $\text{im } a$ if and only if $R\phi = 0$.

We will begin by establishing a sign convention that will be used throughout this thesis. We will build upon these conventions as we progress through this thesis, see Section 6.1 and Section 7.1.

3.1. Homological preliminaries I. Let $E := \bigoplus_r E^r$, $F := \bigoplus_r F^r$, and $G := \bigoplus_r G^r$ be graded holomorphic vector bundles such that only a finite number of the E^r , F^r , and G^r are nonzero. Note that the vector

bundle $\mathrm{Hom}(F, G)$ has a natural grading given by

$$\mathrm{Hom}^r(F, G) = \bigoplus_k \mathrm{Hom}(F^k, G^{k+r}).$$

For a $\mathrm{Hom}^r(F, G)$ -valued $(0, q)$ -current f , we define its degree as $\deg f := q + r$. Given a section f of $\mathcal{C}^{0,q}(\mathrm{Hom}^r(F, G))$ and g of $\mathcal{C}^{0,q'}(\mathrm{Hom}^{r'}(E, F))$, we define their product fg as a section of $\mathcal{C}^{0,q+q'}(\mathrm{Hom}^{r+r'}(E, G))$ via

$$fg := (-1)^{rq'} f \wedge g, \quad (3.2)$$

provided that the wedge product of f and g exists. We have that $\bar{\partial}$ acts as an operator of degree 1 on $\mathcal{C}^{0,\bullet}(\mathrm{Hom}^\bullet(F, G))$, and with the product defined as above it can be shown that

$$\bar{\partial}(fg) = (\bar{\partial}f)g + (-1)^{\deg f} f(\bar{\partial}g).$$

By identifying E with $\mathrm{Hom}(\mathcal{O}_X, E)$, where \mathcal{O}_X is viewed as a graded vector bundle concentrated in degree 0, we can also define multiplication of sections of $\mathcal{C}^{0,\bullet}(\mathrm{Hom}^\bullet(E, F))$ and $\mathcal{C}^{0,\bullet}(E)$, and the product will be a section of $\mathcal{C}^{0,\bullet}(F)$.

Let $a \in \mathrm{End}^1 F$ be an endomorphism of degree 1 such that $aa = 0$, i.e., a is a differential. In other words, the pair (F, a) can be interpreted as a complex of the form (3.1). Suppose $b \in \mathrm{End}^1 E$ is another differential. We define an operator D of degree 1 on $\mathcal{C}^{0,\bullet}(\mathrm{Hom}^\bullet(E, F))$ by

$$Df := af - (-1)^{\deg f} fb.$$

It can be shown that D is an antiderivation with respect to the product structure, i.e.,

$$D(fg) = (Df)g + (-1)^{\deg f} f(Dg),$$

and moreover we have that $D^2 = 0$. It can also be shown that D and $\bar{\partial}$ anticommute, i.e., $D\bar{\partial} = -\bar{\partial}D$.

If u is a section of $\mathcal{C}^{0,\bullet}(\mathrm{End} F)$, then in particular we have that

$$Du = au - (-1)^{\deg u} ua.$$

Moreover, if ϕ is a section of $\mathcal{C}^{0,\bullet}(F) = \mathcal{C}^{0,\bullet}(\mathrm{Hom}(\mathcal{O}_X, F))$, then since the only differential on the complex \mathcal{O}_X is the zero map, we have that

$$D\phi = a\phi,$$

so for a section of F , D is just the differential of the complex.

We combine D and $\bar{\partial}$ into an operator ∇ of degree 1 on $\mathcal{C}^{0,\bullet}(\mathrm{Hom}^\bullet(E, F))$ by

$$\nabla = D - \bar{\partial},$$

which, since D and $\bar{\partial}$ anticommute, satisfies $\nabla^2 = 0$. Moreover, it is clear that ∇ is an antiderivation as well, i.e.,

$$\nabla(fg) = (\nabla f)g + (-1)^{\deg f} f(\nabla g). \quad (3.3)$$

For a $\text{Hom}(E, F)$ -valued section f , we denote by f_k^ℓ the part of f that takes values in $\text{Hom}(E^{-\ell}, F^{-k})$, and we set $f_\bullet^\ell = \sum_k f_k^\ell$.

3.2. Residue currents and the duality principle. Our starting point is a complex of Hermitian holomorphic vector bundles (3.1) that is generically exact for each level $r < 0$. We define the graded holomorphic vector bundle $F := \bigoplus_r F^r$. Here we have that $F^r = 0$ for $r < -N$ and $r > 0$. We can think of the differential of the complex as a map $a \in \text{End}^1 F$.

Let $Z \subseteq X$ be the subvariety where a is not pointwise exact. On $X \setminus Z$, let σ be the minimal right-inverse of a , i.e., if $\xi \in \text{im } a$ then $\sigma\xi$ is the solution to $a\eta = \xi$ such that η has minimal pointwise norm and if ξ is orthogonal to $\text{im } a$, then $\sigma\xi = 0$. It can be shown that σ is smooth. Note that σ satisfies the properties $a\sigma a = a$, $\sigma|_{(\text{im } a)^\perp} = 0$, and $\text{im } \sigma \perp \ker a$. In other words, σ is the Moore–Penrose inverse of a . From the last two properties it follows that $\sigma\sigma = 0$.

Proposition-Definition 3.1. *Define*

$$u := \sigma + \sigma(\bar{\partial}\sigma) + \cdots + \sigma(\bar{\partial}\sigma)^n.$$

Let $h \neq 0$ be a section of a holomorphic vector bundle such that $Z \subseteq \{h = 0\}$. Then

$$U := \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon)u \quad (3.4)$$

is an almost semimeromorphic extension of u . Moreover,

$$R := \text{id} - \nabla U \quad (3.5)$$

is pseudomeromorphic and ∇ -closed, i.e., $\nabla R = 0$.

Proof. By similar arguments as in [AW3], it follows that the terms $\sigma(\bar{\partial}\sigma)^k$ have almost semimeromorphic extensions, and hence it follows that u has an almost semimeromorphic extension given by (3.4).

We have that U and id are pseudomeromorphic, and since $\mathcal{PM}(X)$ is closed under $\bar{\partial}$ and multiplication by smooth functions, R is pseudomeromorphic as well. Since $\nabla \text{id} = 0$ and $\nabla^2 = 0$, we have that $\nabla R = 0$. \square

We shall refer to R as the *residue current* associated with (3.1). It is related to the exactness of the complex in the following way.

Theorem 3.2. *Let ϕ be a holomorphic section of $F^{-\ell}$.*

- (i) *If $a\phi = 0$ and $R\phi = 0$, then ϕ belongs to $\text{im}(F^{-(\ell+1)} \rightarrow F^{-\ell})$, i.e., locally there is a holomorphic section ψ of $F^{-(\ell+1)}$ such that $a\psi = \phi$.*
- (ii) *Conversely, if ϕ belongs to $\text{im}(F^{-(\ell+1)} \rightarrow F^{-\ell})$ and $R_\bullet^{\ell+1} = 0$, then $R\phi = 0$.*

Proof. (i) By the assumption on ϕ we have that $\nabla\phi = 0$. By (3.3) and (3.5) we have that

$$\nabla(U\phi) = (\nabla U)\phi - U(\nabla\phi) = \phi - R\phi = \phi.$$

Thus $v = U\phi$ is a current solution to the system of equations,

$$\begin{aligned} av_{\ell+1} &= \phi, \\ av_{\ell+k+1} &= \bar{\partial}v_{\ell+k}, \quad k > 1. \end{aligned}$$

Locally, we can find a section w that satisfies the system of $\bar{\partial}$ -equations

$$\bar{\partial}w_{\ell+k} = v_{\ell+k} + aw_{\ell+k+1}.$$

Now $\psi = v_{\ell+1} + aw_{\ell+2}$ is a holomorphic section that satisfies $a\psi = \phi$.

(ii) Locally, we can write $\phi = a\psi = \phi$. Since $\nabla R = 0$, by (3.3), we have that $R\phi = R(\nabla\psi) = \nabla(R\psi) = \nabla(R_{\bullet}^{\ell+1}\psi) = 0$. \square

Proposition 3.3. *Let R be the current defined by (3.5). Then R can be decomposed as*

$$R = R(U) + R', \quad (3.6)$$

where R' takes values in $\text{Hom}(F^0, F)$ and is the almost semimeromorphic extension of $\text{id} - D\sigma - uD\bar{\partial}\sigma$. Moreover, if (3.1) is generically exact, then $R' = 0$.

The proof uses the following lemma that will be useful later on as well.

Lemma 3.4. *Let A and B be $\text{Hom}(E, F)$ -valued almost semimeromorphic currents that are smooth outside a variety Z . If $\nabla A = B$ outside Z , then*

$$R(A) = B - \nabla A. \quad (3.7)$$

Proof. We have that $\bar{\partial}A = DA - B$ outside Z , and hence

$$\mathbf{1}_{X \setminus Z} \bar{\partial}A = \mathbf{1}_{X \setminus Z} (DA - B) = DA - B,$$

where the last equality follows by the standard extension property. Thus

$$R(A) = \bar{\partial}A - (DA - B),$$

which gives (3.7). \square

Proof of Proposition 3.3. Since σ is a right-inverse and the complex is generically exact except at level 0 we have that, when we restrict to $\bigoplus_{r < 0} F^r$,

$$D\sigma = a\sigma + \sigma a = \text{id}.$$

Thus $D\sigma - \text{id}$ takes values in $\text{Hom}(F^0, F)$. By applying $\bar{\partial}$ we get that $D\bar{\partial}\sigma$ does as well. Thus R' takes values in $\text{Hom}(F^0, F)$ as well.

Note that

$$(\text{id} - \bar{\partial}\sigma)^{-1} = \text{id} + \bar{\partial}\sigma + \cdots + (\bar{\partial}\sigma)^n,$$

and hence $u = \sigma(\text{id} - \bar{\partial}\sigma)^{-1}$. Applying ∇ to both sides of

$$(\text{id} - \bar{\partial}\sigma)^{-1}(\text{id} - \bar{\partial}\sigma) = \text{id}$$

and rearranging gives

$$\begin{aligned} \nabla(\text{id} - \bar{\partial}\sigma)^{-1} &= (\text{id} - \bar{\partial}\sigma)^{-1} D\bar{\partial}\sigma(\text{id} - \bar{\partial}\sigma)^{-1} \\ &= (\text{id} - \bar{\partial}\sigma)^{-1} D\bar{\partial}\sigma, \end{aligned}$$

where the last equality follows since $D\bar{\partial}\sigma$ takes values in $\text{Hom}(F^0, F)$. Therefore we have that

$$\begin{aligned} \nabla u &= \nabla\sigma(\text{id} - \bar{\partial}\sigma)^{-1} - \sigma\nabla(\text{id} - \bar{\partial}\sigma)^{-1} \\ &= (D\sigma - \text{id} + \text{id} - \bar{\partial}\sigma)(\text{id} - \bar{\partial}\sigma)^{-1} - \sigma(\text{id} - \bar{\partial}\sigma)^{-1} D\bar{\partial}\sigma \\ &= (D\sigma - \text{id})(\text{id} - \bar{\partial}\sigma)^{-1} + \text{id} - uD\bar{\partial}\sigma \\ &= D\sigma - uD\bar{\partial}\sigma, \end{aligned}$$

where the last equality follows since $D\sigma - \text{id}$ takes values in $\text{Hom}(F^0, F)$. The decomposition (3.6) now follows by (3.5) and Lemma 3.4. \square

The following proposition gives conditions for which parts of $R(U)$ that vanish. Let Z^k be the subvariety where the differential $a : F^{-k} \rightarrow F^{-k+1}$ does not have optimal rank, i.e., less than the generic rank.

Proposition 3.5. *If $\text{codim } Z^{\ell+m} \geq m + 1$ for $m = 1, \dots, k - \ell$, then $R(U)_k^\ell = 0$.*

Proof. We need to prove that $R(\sigma(\bar{\partial}\sigma)^{k-\ell-1})_k^\ell = 0$. By applying $\bar{\partial}$ to the equality $\sigma\sigma = 0$ it follows that $\sigma(\bar{\partial}\sigma)^q = (\bar{\partial}\sigma)^q\sigma$ for all q , and we may thus equivalently prove that

$$R((\bar{\partial}\sigma)^{k-\ell-1}\sigma)_k^\ell = 0. \quad (3.8)$$

We claim that (3.8) holds for $k = \ell + 1$. Indeed, $\sigma_{\ell+1}^\ell$ is smooth outside $Z^{\ell+1}$, and hence $R(\sigma)_k^\ell$ has support on this set. Since $Z^{\ell+1}$ has codimension at least 2 and $R(\sigma)_k^\ell$ has bidegree $(0, 1)$, we have that $R(\sigma)_k^\ell = 0$ by the dimension principle.

For $k > \ell + 1$ we will prove (3.8) by induction over k . To this end, assume that $R((\bar{\partial}\sigma)^{k-\ell-2}\sigma)_{k-1}^\ell = 0$. Outside Z^k we have that σ_k^{k-1} is smooth, and thus

$$R((\bar{\partial}\sigma)^{k-\ell-1}\sigma)_k^\ell = (\bar{\partial}\sigma)R((\bar{\partial}\sigma)^{k-\ell-2}\sigma)_{k-1}^\ell = 0$$

by (2.15) and the induction hypothesis. Thus $R((\bar{\partial}\sigma)^{k-\ell-1}\sigma)_k^\ell$ has support on Z^k , which has codimension at least $k - \ell + 1$. Since it has bidegree $(0, k - \ell)$, it must vanish by the dimension principle. \square

Theorem 3.6. *Assume that (3.1) is exact as a complex of \mathcal{O}_X -modules at each level $r < 0$, and let R be the associated residue current. Then $R_\bullet^\ell = 0$ for all $\ell > 0$, i.e., R takes values in $\text{Hom}(F^0, F)$. Moreover,*

$$Ra = (Ra)_\bullet^1 = 0.$$

Proof. By [E1, Theorem 20.9], for $j \geq 1$,

$$\text{codim } Z^j \geq j. \quad (3.9)$$

Thus if $\ell \geq 1$, then $Z^{\ell+m} \geq \ell + m \geq m + 1$, and hence $R(U)_k^\ell = 0$ by Proposition 3.5. This combined with the fact that R' takes values in $\text{Hom}(F^0, F)$ proves the first statement. The last statement follows from the fact that $\nabla R = 0$, since

$$0 = (\nabla R)_\bullet^1 = (aR)_\bullet^1 + (Ra)_\bullet^1 - (\bar{\partial}R)_\bullet^1 = (Ra)_\bullet^1. \quad \square$$

Complexes that are exact as \mathcal{O}_X -modules at each level $r < 0$ arise in practice from locally free resolutions of coherent \mathcal{O}_X -modules \mathcal{F} :

$$0 \longrightarrow F^{-N} \longrightarrow \cdots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow \mathcal{F} \longrightarrow 0. \quad (3.10)$$

Proposition 3.7. *Suppose that $\text{codim } \mathcal{F} > 0$. Then $\text{supp } R \subseteq \text{supp } \mathcal{F}$ and $R_k^0 = 0$ for $k < \text{codim } \mathcal{F}$.*

Proof. Since $\text{codim } \mathcal{F} > 0$, we have that $R = R(U)$ by Proposition 3.3. We have that $\text{supp } \mathcal{F} = Z^1$, and thus $\text{supp } R \subseteq \text{supp } \mathcal{F}$. Moreover, by [E1, Corollary 20.12], we have that

$$Z^{j+1} \subseteq Z^j$$

for all j , and hence the second statement follows by Proposition 3.5. \square

By combining Theorem 3.2 and Theorem 3.6 we get the duality principle for residue currents:

Theorem 3.8 (Duality principle). *Let R be the residue current associated with the complex (3.10). For a holomorphic section ϕ of F^0 , we have that ϕ belongs to $\text{im}(F^{-1} \rightarrow F^0)$ if and only if $R\phi = 0$.*

In particular, if $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ and F^0 is the trivial bundle, then we have that $\text{im}(F^{-1} \rightarrow F^0) = \mathcal{J}$, and hence a holomorphic function ϕ belongs to \mathcal{J} if and only if $\phi R = 0$.

3.3. The comparison formula. Let (E, b) and (F, a) be locally free resolutions of coherent \mathcal{O}_X -modules \mathcal{E} and \mathcal{F} , and let $f : \mathcal{E} \rightarrow \mathcal{F}$ be a morphism. It is well known that f can be lifted to a chain map $\varphi : (E, b) \rightarrow (F, a)$. The comparison formula due to Lärkäng, [L2], relates the associated residue currents R^E and R^F via φ . In our slightly more general situation, where we do not assume that the complexes are generically exact at level 0, the theorem can be stated as follows.

Theorem 3.9. *Let $\varphi : (E, b) \rightarrow (F, a)$ be a chain map of Hermitian complexes that are generically exact at each level $r < 0$. Let R^E and R^F be the associated residue currents. Set*

$$M' := (R^F)' \varphi U^E - U^F \varphi (R^E)',$$

and let M be defined as

$$M := M' + R(U^F \varphi U^E).$$

Then

$$R^F \varphi - \varphi R^E = \nabla M.$$

Note that if the complexes are generically exact, then $M' = 0$. This is the formulation of the theorem found in [L2].

Proof. Let Z be a variety that contains the sets where (E, b) and (F, a) are not pointwise exact. Outside Z we have that $\nabla U^E = \text{id} - (R^E)'$ and $\nabla U^F = \text{id} - (R^F)'$. Moreover, since φ is a chain map, it follows that $\nabla \varphi = D\varphi = a\varphi - b\varphi = 0$. Thus

$$\nabla(U^F \varphi U^E) = (\text{id} - (R^F)') \varphi U^E - U^F \varphi (\text{id} - (R^E)')$$

By (3.7) we get that

$$R(U^F \varphi U^E) = \varphi U^E - U^F \varphi - M' - \nabla(U^F \varphi U^E),$$

and hence

$$\begin{aligned} \nabla M &= \nabla(M' + R(U^F \varphi U^E)) = \varphi(\text{id}_E - R^E) - (\text{id}_F - R^F)\varphi \\ &= R^F \varphi - \varphi R^E. \end{aligned} \quad \square$$

We have the following result on which parts of the residue part of M that vanish. The proof is similar to Proposition 3.5 and can be found in [L2].

Proposition 3.10. *Let $Z^{E,k}$ and $Z^{F,k}$ be as in Proposition 3.5. If*

$$\text{codim } Z^{E,\ell+m} \geq m + 1 \text{ for } m = 1, \dots, k - \ell - 1 \text{ and}$$

$$\text{codim } Z^{F,\ell+m} \geq m \text{ for } m = 2, \dots, k - \ell,$$

then $R(U^F \varphi U^E)_k^\ell = 0$.

From this we get the following corollary, cf. [L2, Corollary 3.7].

Corollary 3.11. *If (E, b) and (F, a) are locally free resolutions of \mathcal{E} and \mathcal{F} , then*

$$M_\bullet^\ell = 0, \text{ for } \ell \geq 1,$$

and if \mathcal{E} and \mathcal{F} have codimension greater than or equal to k , then

$$M_k^0 = 0.$$

Moreover,

$$Mb = M_\bullet^0 b = 0.$$

4. PAPER I

Paper I concerns interpolation with respect to a closed complex subspace of \mathbb{P}^n . Let $i : X \rightarrow \mathbb{P}^n$ be a closed complex subspace of \mathbb{P}^n . Let Φ and φ be global holomorphic sections of $\mathcal{O}(d)$ and $\mathcal{O}_X(d) := i^*\mathcal{O}(d)$, respectively. (Recall that $\mathcal{O}(d)$ denotes the d th tensor power of the dual of the tautological line bundle over \mathbb{P}^n .) We say that Φ interpolates φ if $i^*\Phi = \varphi$.

We have that X is defined by a saturated homogeneous ideal in the graded ring $S := \mathbb{C}[z_0, \dots, z_n]$. From a minimal graded free resolution of the homogeneous coordinate ring of X , $S_X := S/I$,

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow S \longrightarrow S_X \longrightarrow 0, \quad (4.1)$$

where $F_k = \bigoplus_{\ell} S(-\ell)^{\beta_{k,\ell}}$, we obtain a locally free resolution of \mathcal{O}_X of the form

$$0 \longrightarrow \bigoplus \mathcal{O}(-\ell)^{\beta_{n,\ell}} \longrightarrow \dots \longrightarrow \bigoplus \mathcal{O}(-\ell)^{\beta_{1,\ell}} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0. \quad (4.2)$$

Here we use notation and terminology concerning graded S -modules from [E2].

Let R be the residue current associated with this complex, and let R_n be the component of bidegree $(0, n)$. We will prove the following necessary and sufficient condition for the existence of an interpolant.

Theorem 4.1. *A global holomorphic section φ of \mathcal{O}_X has an interpolant if and only if $R_n\varphi$ is $\bar{\partial}$ -exact, i.e., there exists a current η such that $R_n\varphi = \bar{\partial}\eta$.*

Let $R_{n,\ell}$ denote the $\mathcal{O}(-\ell)^{\beta_{n,\ell}}$ -valued component of R_n , and let ω be a nonvanishing holomorphic $\mathcal{O}(n+1)$ -valued n -form. Using Serre duality, see, e.g., [D, Theorem 7.3], we will prove the following equivalent condition for the existence of an interpolant.

Corollary 4.2. *A global holomorphic section φ of $\mathcal{O}_X(d)$ has an interpolant if and only if for each ℓ it holds that*

$$\int_{\mathbb{P}^n} R_{n,\ell}\varphi \wedge h\omega = 0 \quad (4.3)$$

for all global holomorphic sections h of $\mathcal{O}(\ell - d - n - 1)$.

We define the *interpolation degree* of X as

$$\inf\{d : \text{all global holomorphic sections of } \mathcal{O}_X(d) \text{ has an interpolant}\}.$$

Using Corollary 4.2 we get an analytic proof of the following upper bound on the interpolation degree.

Corollary 4.3. *The interpolation degree of X is less than or equal to*

$$\sup\{k : \beta_{n,k} \neq 0\} - n.$$

(We use the convention that the supremum of the empty set is $-\infty$.)

4.1. Local cohomology and Ext. As mentioned in the introduction, interpolation is connected with local cohomology. This is in turn related with a certain Ext functor by a result known as *local duality*. (Local duality is closely related to Serre duality, see [S] and [M2].)

Consider the complex of graded S -modules

$$0 \longrightarrow S_X \xrightarrow{d^0} \bigoplus_{j=0}^n S_X[z_j^{-1}] \xrightarrow{d^1} \bigoplus_{i < j} S_X[z_i^{-1} z_j^{-1}],$$

where d^0 is the obvious map, and d^1 is given by

$$d^1(\varphi)_{ij} = \varphi_j - \varphi_i.$$

The quotient $H_{\mathfrak{m}}^1(S_X) := \ker d^1 / \operatorname{im} d^0$ is called the first local cohomology module of S_X . Note that d^0 is injective since I is assumed to be saturated. Moreover, from the definition of d^1 , it is easy to see that

$$\ker d^1 \cong \bigoplus_d H^0(\mathbb{P}^n, \mathcal{O}_X(d)).$$

Thus we have the following short exact sequence of graded S -modules

$$0 \longrightarrow S_X \longrightarrow \bigoplus_d H^0(\mathbb{P}^n, \mathcal{O}_X(d)) \longrightarrow H_{\mathfrak{m}}^1(S_X) \longrightarrow 0.$$

From this we obtain we obtain the exact sequence (1.1).

By the local duality theorem, see [E2, A1.9], we have that $H_{\mathfrak{m}}^1(S_X)$ can be identified with the graded dual of $\operatorname{Ext}^n(S_X, S(-n-1))$. We will elaborate on the theory of the Ext functors in the next section, but for now we will simply state the fact that $\operatorname{Ext}^n(S_X, S(-n-1))$ is given by

$$\operatorname{Hom}(F_n, S(-n-1)) \cong \bigoplus_{\ell} S(\ell - n - 1)^{\beta_{k,\ell}}$$

modulo the image of the map

$$\operatorname{Hom}(F_{n-1}, S(-n-1)) \rightarrow \operatorname{Hom}(F_n, S(-n-1))$$

given by $h \mapsto hf_n$. Thus the pairing

$$([\varphi], [h]) \mapsto \int_{\mathbb{P}^n} R_{n,\ell} \varphi \wedge h \omega$$

can be seen as an explicit realization of the duality between $H_{\mathfrak{m}}^1(S_X)$ and $\operatorname{Ext}^n(S_X, S(-n-1))$. Note that $[\varphi] = 0$ means that φ has an interpolant, and hence $R_{n,\ell} \varphi$ is $\bar{\partial}$ -exact so that the integral becomes zero. If $[h] = 0$, then $h = gf_n$ for some g , and hence the integrand is $\bar{\partial}$ -exact since $\nabla R = 0$. Thus the integral becomes zero in this case as well. This shows that the pairing is well-defined.

Note that, by Corollary 4.2, we get an analytic proof of the fact that the pairing is nondegenerate with respect to the first slot.

5. RESIDUE CURRENTS AND THE EXT FUNCTORS

In Section 4.1 we saw that residue currents provided us with a pairing that realized a duality between the first local cohomology group and the n th Ext group. These kinds of results of representing Ext functors in terms of residue currents was first explored by Andersson in [A1]. We will now continue by further exploring the connection between residue currents and the Ext functors. Our goal is to give a representation of the global Ext groups in terms of residue currents, and this will be the topic of Paper II.

Roughly speaking, the Ext functors are right derived functors of certain Hom functors. We will therefore begin with a short survey of the topic of derived functors. The Ext functors are then defined in Section 5.2. In Section 5.3 we will discuss a general connection between the Ext functors and currents, and in Section 5.4, with [A1] as our starting point, we will begin to explore the connection between residue currents and the Ext functors.

5.1. Derived functors. Fix an abelian category \mathfrak{A} . In this thesis we will mostly be concerned with the abelian category of abelian groups and the abelian category of \mathcal{O}_X -modules. Let F be a covariant left-exact functor from \mathfrak{A} to an abelian category. Recall that this means that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence, then applying F yields the exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C).$$

The right derived functors $R^j F$ of F measures how far F is from being exact in the sense that one can continue the above exact sequence to the following long exact sequence

$$\begin{aligned} 0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow \\ \rightarrow R^1 F(A) \rightarrow R^1 F(B) \rightarrow R^1 F(C) \rightarrow R^2 F(A) \rightarrow \dots \end{aligned} \quad (5.1)$$

In order to construct the right derived functors, we recall the notion of injective objects. An object I is said to be *injective* if every morphism $A \rightarrow I$ factors through each monomorphism $A \rightarrow B$. The category \mathfrak{A} is said to have enough injectives if, for every object in \mathfrak{A} , there exists a monomorphism into an injective object. If \mathfrak{A} has enough injectives it follows that every object A has an injective resolution:

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

We can now state the definition of right derived functors. Let \mathfrak{A} be an abelian category with enough injectives, and let F be a covariant left-exact functor from \mathfrak{A} to an abelian category. For each object A in \mathfrak{A} we choose an injective resolution I^\bullet of A . The right derived functors $R^j F(A)$ are defined as the cohomology of the complex,

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow \dots,$$

i.e., $R^j F(A) := H^j(F(I^\bullet))$.

The definition of derived functors is quite abstract, and it is in general not very useful for actual computations. However, in many situations one can replace injective resolutions with resolutions of more concrete objects. This fact can be formulated as follows.

Proposition 5.1. *Let*

$$0 \longrightarrow A \longrightarrow C^0 \xrightarrow{\varepsilon^0} C^1 \xrightarrow{\varepsilon^1} \dots$$

be an exact sequence where the C^k are such that

$$R^j F(C^k) = 0 \tag{5.2}$$

for all $j > 0$. Then

$$R^j F(A) \cong H^j(F(C^\bullet)). \tag{5.3}$$

An object that satisfies (5.2) is said to be F -acyclic. Note that, in particular, injective objects are acyclic. In Section 5.3 we will use the fact that the sheaves of currents are acyclic in order to compute the Ext functors.

Sketch of proof. The statement follows by applying F to the short exact sequences

$$\begin{aligned} 0 \longrightarrow A \longrightarrow C^0 \longrightarrow \ker \varepsilon^1 \longrightarrow 0, \\ 0 \longrightarrow \ker \varepsilon^1 \longrightarrow C^1 \longrightarrow \ker \varepsilon^2 \longrightarrow 0, \end{aligned}$$

etc, and using the long exact sequence for derived functors, (5.1). \square

5.2. Ext groups and sheaves. In this section we will recall the definition and basic properties of the Ext functors that we shall study in this thesis, namely the Ext groups and Ext sheaves.

Let \mathcal{F} and \mathcal{G} be \mathcal{O}_X -modules. Recall that $\text{Hom}(\mathcal{F}, \mathcal{G})$ denotes the group of morphisms $\mathcal{F} \rightarrow \mathcal{G}$, and the \mathcal{O}_X -module $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is defined by

$$\mathcal{U} \mapsto \text{Hom}(\mathcal{F}|_{\mathcal{U}}, \mathcal{G}|_{\mathcal{U}}).$$

It is easy to see that $\mathcal{H}om(\mathcal{F}, -)$ and $\text{Hom}(\mathcal{F}, -)$ are left exact covariant functors from the category of \mathcal{O}_X -modules to the categories of \mathcal{O}_X -modules and abelian groups, respectively. Since the category of \mathcal{O}_X -modules has enough injectives, we can define their right derived functors, which are denoted $\mathcal{E}xt^k(\mathcal{F}, -)$ and $\text{Ext}^k(\mathcal{F}, -)$, respectively. Let us

write this out explicitly. Let

$$0 \longrightarrow \mathcal{G} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

be an injective resolution of \mathcal{G} . Then

$$\mathcal{E}xt^k(\mathcal{F}, \mathcal{G}) = H^k(\mathcal{H}om(\mathcal{F}, I^\bullet))$$

and

$$\text{Ext}^k(\mathcal{F}, \mathcal{G}) = H^k(\text{Hom}(\mathcal{F}, I^\bullet)).$$

For the long exact sequences we have the following. Let

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of \mathcal{O}_X -modules. We get the long exact sequences

$$0 \longrightarrow \mathcal{H}om(\mathcal{E}, \mathcal{G}) \longrightarrow \mathcal{H}om(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om(\mathcal{H}, \mathcal{G}) \longrightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{G}) \longrightarrow \dots$$

and

$$0 \longrightarrow \text{Hom}(\mathcal{E}, \mathcal{G}) \longrightarrow \text{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(\mathcal{H}, \mathcal{G}) \longrightarrow \text{Ext}^1(\mathcal{E}, \mathcal{G}) \longrightarrow \dots$$

Note that $\text{Hom}(\mathcal{F}, \mathcal{G})$ is the group of global sections of the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. In general there is no analogous relation between the Ext groups and Ext sheaves. However, the Ext groups and sheaves can be related using the Grothendieck local-to-global spectral sequence, see, e.g., [W].

Theorem 5.2. *Let \mathcal{F} and \mathcal{G} be \mathcal{O}_X -modules. There is a spectral sequence (E_r) with*

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \text{Ext}^{p+q}(\mathcal{F}, \mathcal{G}).$$

If \mathcal{F} has a locally free resolution (F, a) , then one can also represent the Ext sheaves $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G})$ by applying $\mathcal{H}om(-, \mathcal{G})$ to F and taking cohomology.

Proposition 5.3. *Let \mathcal{F} and \mathcal{G} be \mathcal{O}_X -modules, and suppose that \mathcal{F} has a locally free resolution (F, a) . Consider the complex*

$$0 \longrightarrow \mathcal{H}om(F^0, \mathcal{G}) \longrightarrow \mathcal{H}om(F^{-1}, \mathcal{G}) \longrightarrow \dots$$

with differential given by ξa where ξ is a section of $\mathcal{H}om(F^{-k}, \mathcal{G})$. Then

$$\mathcal{E}xt^k(\mathcal{F}, \mathcal{G}) \cong H^k(\mathcal{H}om(F^{-\bullet}, \mathcal{G})). \quad (5.4)$$

Proof. Let

$$0 \longrightarrow \mathcal{G} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \quad (5.5)$$

be an injective resolution of \mathcal{G} . Consider the double complex

$$E^{p,q} := \mathcal{H}om(F^{-p}, I^q).$$

Since the I^q are injective we have that

$$H^p(E^{\bullet, q}) \cong \begin{cases} \mathcal{H}om(\mathcal{F}, I^q), & p = 0, \\ 0, & p \geq 1, \end{cases}$$

and

$$H^q(E^{p, \bullet}) \cong \begin{cases} \mathcal{H}om(F^{-p}, \mathcal{G}), & q = 0, \\ 0, & q \geq 1. \end{cases}$$

Thus both spectral sequences degenerate, and each inclusion into the total complex yields an isomorphism with the cohomology of the total complex. \square

If \mathcal{F} is an arbitrary coherent \mathcal{O}_X -module, then it is in general not possible to find a locally free resolution of \mathcal{F} , see, e.g., [V]. However, one can always find a cover $\mathfrak{U} = (\mathcal{U}_\alpha)$ of X and locally free resolutions $(F_\alpha^\bullet, a_\alpha)$ of $\mathcal{F}|_{\mathcal{U}_\alpha}$. We can then compute $\mathcal{E}xt^k(\mathcal{F}|_{\mathcal{U}_\alpha}, \mathcal{G}|_{\mathcal{U}_\alpha})$ over each \mathcal{U}_α using Proposition 5.3, and these sheaves then glue to $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G})$. Let us discuss this last point in more detail.

Dual to the notion of injective objects is the notion of projective objects. An object P is said to be *projective* if every morphism $P \rightarrow F$ factors through each epimorphism $E \rightarrow F$. It is well known that an $\mathcal{O}_{X,x}$ -module is projective if and only if it is free, see, e.g., [GH]. This together with an application of Cartan's Theorem B, see, e.g., [T], gives the following proposition.

Proposition 5.4. *Let X be a Stein manifold. Locally free \mathcal{O}_X -modules are projective in the category of coherent \mathcal{O}_X -modules.*

If \mathcal{F} is coherent, then, using the Syzygy Theorem, one can find a Stein cover $\mathfrak{U} = (\mathcal{U}_\alpha)$ of X and locally free resolutions $(F_\alpha^\bullet, a_\alpha)$ of $\mathcal{F}|_{\mathcal{U}_\alpha}$. Over an intersection $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, which is Stein, we have that $(F_\alpha^\bullet, a_\alpha)$ and $(F_\beta^\bullet, a_\beta)$ are locally free resolutions of $\mathcal{F}|_{\mathcal{U}_{\alpha\beta}}$. Using Proposition 5.4 and [E1, Proposition A3.13], we get that $(F_\beta^\bullet, a_\beta)$ and $(F_\alpha^\bullet, a_\alpha)$ are chain homotopy equivalent via a chain map

$$\varphi_{\alpha\beta} : F_\beta^\bullet \rightarrow F_\alpha^\bullet.$$

It is easy to see that $\varphi_{\alpha\beta}$ induces an isomorphism between the representations

$$H^k(\mathcal{H}om(F_\alpha^{-\bullet}, \mathcal{G})) \xrightarrow{\cong} H^k(\mathcal{H}om(F_\beta^{-\bullet}, \mathcal{G}))$$

of $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G})$ via

$$[\xi_\alpha] \mapsto [\xi_\alpha \varphi_{\alpha\beta}].$$

5.3. Ext functors and currents. When \mathcal{F} is a coherent \mathcal{O}_X -module and \mathcal{G} is a locally free \mathcal{O}_X -module, then the theory of currents gives more explicit ways to compute the Ext groups and Ext sheaves. Recall that a locally free \mathcal{O}_X -module \mathcal{G} is the sheaf of holomorphic sections of a holomorphic vector bundle G . By slight abuse of notation we shall simply write G both for the vector bundle and its locally free sheaf of holomorphic sections. We let $\mathcal{C}^{p,q}(G)$ denote the sheaf of G -valued currents of bidegree (p, q) .

Let \mathcal{F} be a coherent \mathcal{O}_X -module, and let G be a locally free \mathcal{O}_X -module. We will now show that

$$\mathcal{E}xt^k(\mathcal{F}, \mathcal{C}^{0,q}(G)) = 0,$$

and

$$\mathrm{Ext}^k(\mathcal{F}, \mathcal{C}^{0,q}(G)) = 0$$

for $k \geq 1$. Thus

$$0 \longrightarrow G \longrightarrow \mathcal{C}^{0,0}(G) \xrightarrow{\bar{\partial}} \mathcal{C}^{0,1}(G) \xrightarrow{\bar{\partial}} \dots$$

is an acyclic resolution of G for the functors $\mathcal{H}om(\mathcal{F}, -)$ and $\mathrm{Hom}(\mathcal{F}, -)$. Thus we have the representations

$$\mathcal{E}xt^k(\mathcal{F}, G) \cong H^k(\mathcal{H}om(\mathcal{F}, \mathcal{C}^{0,\bullet}(G))), \quad (5.6)$$

$$\mathrm{Ext}^k(\mathcal{F}, G) \cong H^k(\mathrm{Hom}(\mathcal{F}, \mathcal{C}^{0,\bullet}(G))). \quad (5.7)$$

By a result due to Malgrange, [M1, Theorem VII.2.4], one has that $\mathcal{C}^{0,q}(G)$ is stalkwise injective, i.e., each stalk $\mathcal{C}^{0,q}(G)_x$ is an injective $\mathcal{O}_{X,x}$ -module. Since \mathcal{F} is coherent,

$$\mathcal{E}xt^k(\mathcal{F}, \mathcal{C}^{0,q}(G))_x \cong \mathrm{Ext}^k(\mathcal{F}_x, \mathcal{C}^{0,q}(G)_x) = 0,$$

for $k \geq 1$, where the isomorphism follows by, e.g., [H1, Proposition III.6.8], and the last equality follows since $\mathcal{C}^{0,q}(G)_x$ is injective. Thus

$$\mathcal{E}xt^k(\mathcal{F}, \mathcal{C}^{0,q}(G)) = 0$$

for $k \geq 1$.

In addition, since $\mathcal{C}^{0,q}$ is fine, $\mathcal{H}om(\mathcal{F}, \mathcal{C}^{0,q}(G))$ is as well, so

$$H^k(X, \mathcal{H}om(\mathcal{F}, \mathcal{C}^{0,q}(G))) = 0,$$

for $k \geq 1$. This together with the vanishing of the Ext sheaves gives

$$H^r(X, \mathcal{E}xt^s(\mathcal{F}, \mathcal{C}^{0,q}(G))) = 0$$

for $r + s \geq 1$. By the local-to-global spectral sequence for Ext, we get that

$$\mathrm{Ext}^k(\mathcal{F}, \mathcal{C}^{0,q}(G)) = 0$$

for $k \geq 1$.

5.4. Ext sheaves and residue currents. In this section we relate two different representations of the Ext sheaves via residue currents. This construction is originally due to Andersson [A1], and it generalizes earlier work by Dickenstein–Sessa [DS].

If $\mathcal{G} = G$ is a holomorphic vector bundle, i.e., considered as a sheaf, it is a locally free \mathcal{O}_X -module, then we have the following isomorphism between the representations (5.4) and (5.6).

Theorem 5.5. *Let \mathcal{F} be a coherent \mathcal{O}_X -module, and let G be a locally free \mathcal{O}_X -module. Suppose that \mathcal{F} has a locally free resolution of the form (3.10), and let R be the associated residue current. Then there is an isomorphism*

$$H^k(\mathrm{Hom}^\bullet(F, G)) \cong H^k(\mathrm{Hom}(\mathcal{F}, \mathcal{C}^{0,\bullet}(G)))$$

that is given by

$$[\xi] \mapsto [\phi \mapsto \xi R\phi_0],$$

where ϕ_0 is a section of F^0 that represents ϕ in $\mathcal{F} \cong F^0/\mathrm{im} a$.

Proof. Note that the morphism $\phi \mapsto \xi R\phi_0$ is well-defined since if $\phi \in \mathrm{im} a$, then $R\phi = 0$.

Consider the double complex

$$E^{p,q} := \mathrm{Hom}(F^{-p}, \mathcal{C}^{0,q}(G)).$$

For similar reasons as in Proposition 5.3 we have that

$$H^p(E^{\bullet,q}) \cong \begin{cases} \mathrm{Hom}(\mathcal{F}, \mathcal{C}^{0,q}(G)), & p = 0, \\ 0, & p \geq 1, \end{cases}$$

and we have that

$$H^q(E^{p,\bullet}) \cong \begin{cases} \mathrm{Hom}(F^{-p}, G), & q = 0, \\ 0, & q \geq 1. \end{cases}$$

Thus both spectral sequences degenerate, and each inclusion into the total complex yields an isomorphism with the cohomology of the total complex. In view of (3.5), we have that

$$\nabla((-1)^k \xi U) = \xi - \xi R,$$

and hence we get the desired isomorphism. \square

As before, if \mathcal{F} is an arbitrary coherent \mathcal{O}_X -module, we can choose an open cover $\mathcal{U} := (\mathcal{U}_\alpha)$ of Stein open sets and locally free resolutions $(F_\alpha^\bullet, a_\alpha)$ of $\mathcal{F}|_{\mathcal{U}_\alpha}$.

Suppose that we are on an intersection of two such sets \mathcal{U}_α and \mathcal{U}_β . Let F_α^\bullet and F_β^\bullet be locally free resolutions of \mathcal{F} over these sets, and let R_α and R_β be the associated residue currents. Consider a section of

$\mathcal{E}xt^k(\mathcal{F}, G)$ represented by ξ_α and ξ_β in $\mathcal{H}om^\bullet(F_\alpha, G)$ and $\mathcal{H}om^\bullet(F_\beta, G)$, respectively. Thus the maps

$$\phi \mapsto \xi_\alpha R_\alpha \phi_\alpha \quad (5.8)$$

and

$$\phi \mapsto \xi_\beta R_\beta \phi_\beta, \quad (5.9)$$

must be $\bar{\partial}$ -cohomologous. Here ϕ_α and ϕ_β are representatives of ϕ in F_α^0 and F_β^0 , respectively. Using the comparison formula we can show this explicitly.

Since F_α^\bullet and F_β^\bullet are resolutions of $\mathcal{F}|_{\mathcal{U}_{\alpha\beta}}$, and $\mathcal{U}_{\alpha\beta}$ is Stein, they are homotopy equivalent via a chain map $\varphi_{\alpha\beta} : F_\beta^\bullet \rightarrow F_\alpha^\bullet$. By the comparison formula we have that

$$R_\alpha \varphi_{\alpha\beta} - \varphi_{\alpha\beta} R_\beta = \nabla M_{\alpha\beta},$$

and hence

$$\begin{aligned} \xi_\alpha R_\alpha \varphi_{\alpha\beta} - \xi_\alpha \varphi_{\alpha\beta} R_\beta &= \xi_\alpha \nabla M_{\alpha\beta} \\ &= \xi_\alpha a_\alpha M_{\alpha\beta} + \xi_\alpha M_{\alpha\beta} a_\beta - \bar{\partial}(\xi_\alpha M_{\alpha\beta}) \\ &= -\bar{\partial}(\xi_\alpha M_{\alpha\beta}), \end{aligned}$$

since $\xi_\alpha a_\alpha = 0$ by hypothesis and $M_{\alpha\beta} a_\beta = 0$ by Corollary 3.11. Since ϕ_α and ϕ_β represents ϕ , we have that $\phi_\alpha - \varphi_{\alpha\beta} \phi_\beta \in \text{im } a_\alpha$. Moreover, we have that $\xi_\alpha \varphi_{\alpha\beta} - \xi_\beta = \eta_\beta a_\beta$ for some η_β . Thus

$$\begin{aligned} \xi_\alpha R_\alpha \phi_\alpha &= \xi_\alpha R_\alpha \varphi_{\alpha\beta} \phi_\beta \\ &= \xi_\alpha \varphi_{\alpha\beta} R_\beta \phi_\beta - \bar{\partial}(\xi_\alpha M_{\alpha\beta} \phi_\beta) \\ &= \xi_\beta R_\beta \phi_\beta + \eta_\beta a_\beta R_\beta \phi_\beta - \bar{\partial}(\xi_\alpha M_{\alpha\beta} \phi_\beta) \\ &= \xi_\beta R_\beta \phi_\beta + \bar{\partial}(\eta_\beta R_\beta \phi_\beta - \xi_\alpha M_{\alpha\beta} \phi_\beta). \end{aligned}$$

This shows that the maps (5.8) and (5.9) are $\bar{\partial}$ -cohomologous with $\bar{\partial}$ -potential

$$\phi \mapsto \eta_\beta R_\beta \phi_\beta - \xi_\alpha M_{\alpha\beta} \phi_\beta.$$

Note that this is well-defined by the duality principle and Corollary 3.11.

In conclusion, we have that the square of isomorphisms

$$\begin{array}{ccc} H^k(\mathcal{H}om^\bullet(F_\alpha, G)) & \longrightarrow & H^k(\mathcal{H}om(\mathcal{F}, \mathcal{C}^{0,\bullet}(G))) \\ \downarrow & & \parallel \\ H^k(\mathcal{H}om^\bullet(F_\beta, G)) & \longrightarrow & H^k(\mathcal{H}om(\mathcal{F}, \mathcal{C}^{0,\bullet}(G))) \end{array}$$

given by

$$\begin{array}{ccc} [\xi_\alpha] & \longmapsto & [\phi \mapsto \xi_\alpha R_\alpha \varphi_{\alpha\beta} \phi_\beta] \\ \downarrow & & \parallel \\ [\xi_\alpha \varphi_{\alpha\beta}] & \longmapsto & [\phi \mapsto \xi_\alpha \varphi_{\alpha\beta} R_\beta \phi_\beta] \end{array}$$

commutes as expected.

6. TWISTING COCHAINS AND THE EXT GROUPS

We now turn to the Ext groups. Let X be a complex manifold, and let \mathcal{F} and \mathcal{G} be coherent \mathcal{O}_X -modules. In this section we will give a representation of the Ext groups $\text{Ext}^k(\mathcal{F}, \mathcal{G})$ that is analogous to the representations of the Ext sheaves $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G})$ given in Proposition 5.3, i.e., a representation that involves a locally free resolution of \mathcal{F} . This representation is due to Toledo and Tong, see [TT]. It is in general not possible to find global locally free resolutions of arbitrary coherent \mathcal{O}_X -modules, see, e.g., [V]. However, as discussed previously, it is possible to find a Stein cover of X and a locally free resolution of \mathcal{F} on each of the open sets. In [TT], Toledo and Tong introduced the notion of a *twisting cochain*, which is a device for keeping track of the locally free resolutions and how they relate to each other on the overlaps.

6.1. Homological preliminaries II. Let $\mathfrak{U} = (\mathcal{U}_\alpha)$ be a covering of X by Stein open sets. We will use the notation $\mathcal{U}_{\alpha_0 \dots \alpha_p} := \mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_p}$. For each α , let $E_\alpha = \bigoplus_r E_\alpha^r$, $F_\alpha = \bigoplus_r F_\alpha^r$, and $G_\alpha = \bigoplus_r G_\alpha^r$ be bounded graded holomorphic vector bundles over \mathcal{U}_α . We will use the letters E , F , and G to denote the families (E_α) , (F_α) , and (G_α) , respectively.

We will consider a sort of Čech cochains that take values in these vector bundles. Let

$$C^p(\mathfrak{U}, \text{Hom}^r(F, G)) := \prod_{(\alpha_0, \dots, \alpha_p)} \mathcal{H}om^r(F_{\alpha_p}, G_{\alpha_0})(\mathcal{U}_{\alpha_0 \dots \alpha_p}),$$

where $\mathcal{H}om^r(F_{\alpha_p}, G_{\alpha_0})$ denotes the sheaf of holomorphic sections of the vector bundle $\text{Hom}^r(F_{\alpha_p}, G_{\alpha_0})$. For an element $f \in C^p(\mathfrak{U}, \text{Hom}^r(F, G))$, we define its degree as $\deg f = p + r$, and we call p the Čech degree and r the Hom degree.

There is a bilinear pairing

$$C^p(\mathfrak{U}, \text{Hom}^r(F, G)) \times C^{p'}(\mathfrak{U}, \text{Hom}^{r'}(E, F)) \rightarrow C^{p+p'}(\mathfrak{U}, \text{Hom}^{r+r'}(E, G)), \quad (6.1)$$

which maps (f, g) to the product fg defined by

$$(fg)_{\alpha_0 \dots \alpha_{p+p'}} := (-1)^{rp'} f_{\alpha_0 \dots \alpha_p} g_{\alpha_{p+1} \dots \alpha_{p+p'}},$$

where the product on the right-hand side is composition of vector bundle maps.

We will consider the following coboundary operator

$$\delta : C^p(\mathfrak{U}, \text{Hom}^r(F, G)) \rightarrow C^{p+1}(\mathfrak{U}, \text{Hom}^r(F, G)),$$

which is defined by

$$(\delta f)_{\alpha_0 \dots \alpha_{p+1}} := \sum_{k=1}^p (-1)^k f_{\alpha_0 \dots \widehat{\alpha}_k \dots \alpha_{p+1}} |_{\mathcal{U}_{\alpha_0 \dots \alpha_{p+1}}}.$$

Note that δ is similar to the usual Čech coboundary, but in the sum, it is necessary to omit $f_{\alpha_1 \dots \alpha_{p+1}}$ and $f_{\alpha_0 \dots \alpha_p}$ since these are not sections of $\text{Hom}^r(F_{\alpha_{p+1}}, G_{\alpha_0})$. However, we still have that δ is a differential and an antiderivation with respect to the product (6.1), i.e., $\delta^2 = 0$, and

$$\delta(fg) = (\delta f)g + (-1)^{\deg f} f(\delta g).$$

6.2. Twisting cochains and twisted resolutions. We are now ready to define the notion of a twisting cochain.

Definition 6.1. A *twisting cochain* $a \in C^\bullet(\mathfrak{U}, \text{Hom}^\bullet(F, F))$ is an element

$$a = \sum_{k \geq 0} a^k,$$

where $a^k \in C^k(\mathfrak{U}, \text{Hom}^{1-k}(F, F))$, such that

$$\delta a + aa = 0, \tag{6.2}$$

and $a^1_{\alpha\alpha} = \text{id}_{F_\alpha}$ for all α . For simplicity we shall simply refer to the pair (F, a) as a twisting cochain.

To better understand the meaning of a twisting cochain, it is helpful to look at (6.2) component-wise. In particular, we have that a must satisfy

$$a^0_\alpha a^0_\alpha = 0 \tag{6.3}$$

$$a^0_\alpha a^1_{\alpha\beta} = a^1_{\alpha\beta} a^0_\beta \tag{6.4}$$

$$a^1_{\alpha\gamma} - a^1_{\alpha\beta} a^1_{\beta\gamma} = a^0_\alpha a^2_{\alpha\beta\gamma} + a^2_{\alpha\beta\gamma} a^0_\gamma. \tag{6.5}$$

The first equation says that (F_α, a^0_α) is a chain complex, the second says that $a^1_{\alpha\beta}$ defines a chain map $(F_\beta|_{\mathcal{U}_{\alpha\beta}}, a^0_\beta) \rightarrow (F_\alpha|_{\mathcal{U}_{\alpha\beta}}, a^0_\alpha)$, and the third says that, over $\mathcal{U}_{\alpha\beta\gamma}$, $a^1_{\alpha\gamma}$ and $a^1_{\alpha\beta} a^1_{\beta\gamma}$ are chain homotopic, with the homotopy given by $a^2_{\alpha\beta\gamma}$. In particular, from the condition $a^1_{\alpha\alpha} = \text{id}_{F_\alpha}$, it follows that $a^1_{\alpha\beta}$ and $a^1_{\beta\alpha}$ are chain homotopy inverses to each other. Thus for each α we have cohomology sheaves $\mathcal{H}^\bullet_{a^0_\alpha}(F_\alpha)$ over \mathcal{U}_α , and over each intersection $\mathcal{U}_{\alpha\beta}$ we have an isomorphism

$$H(a^1_{\alpha\beta}) : \mathcal{H}^\bullet_{a^0_\beta}(F_\beta)|_{\mathcal{U}_{\alpha\beta}} \rightarrow \mathcal{H}^\bullet_{a^0_\alpha}(F_\alpha)|_{\mathcal{U}_{\alpha\beta}}$$

such that over each $\mathcal{U}_{\alpha\beta\gamma}$, $H(a_{\alpha\beta}^1)H(a_{\beta\gamma}^1) = H(a_{\alpha\gamma}^1)$. We denote by \mathcal{H}_a^\bullet the sheaf that we obtain by gluing the sheaves $\mathcal{H}_{a_\alpha^0}^\bullet$ via these isomorphisms.

Suppose that $F_\alpha = F_\alpha^0$, i.e., each F_α is a complex concentrated in degree 0. Then we have that $a = a^1$, and (6.2) amounts to the single equation

$$a_{\alpha\gamma}^1 = a_{\alpha\beta}^1 a_{\beta\gamma}^1.$$

This together with $a_{\alpha\alpha}^1 = \text{id}_{F_\alpha^0}$ gives $F = (F_\alpha^0)$ the structure of a holomorphic vector bundle. Thus a twisting cochain can in fact be thought of as a generalization of a holomorphic vector bundle.

The twisting cochains that are of main interest to us arise in the following way. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Using the Syzygy Theorem, one can find a Stein cover $\mathfrak{U} = (\mathcal{U}_\alpha)$ such that for each α there exists a free resolution of $\mathcal{F}|_{\mathcal{U}_\alpha}$,

$$\dots \xrightarrow{a_\alpha^0} F_\alpha^{-1} \xrightarrow{a_\alpha^0} F_\alpha^0 \longrightarrow \mathcal{F}|_{\mathcal{U}_\alpha} \longrightarrow 0,$$

of length at most $\dim X$. Over each intersection $\mathcal{U}_{\alpha\beta}$ one can find a chain map

$$a_{\alpha\beta}^1 : (F_\beta|_{\mathcal{U}_{\alpha\beta}}, a_\beta^0) \rightarrow (F_\alpha|_{\mathcal{U}_{\alpha\beta}}, a_\alpha^0)$$

that extends the identity morphism on $\mathcal{F}|_{\mathcal{U}_{\alpha\beta}}$, and which can be chosen to be the identity if $\alpha = \beta$. Since $a_{\alpha\gamma}^1$ and $a_{\alpha\beta}^1 a_{\beta\gamma}^1$ are chain maps $(F_\gamma|_{\mathcal{U}_{\alpha\beta\gamma}}, a_\gamma^0) \rightarrow (F_\alpha|_{\mathcal{U}_{\alpha\beta\gamma}}, a_\alpha^0)$ that extends the identity morphism on $\mathcal{F}|_{\mathcal{U}_{\alpha\beta\gamma}}$, there exists a chain homotopy $a_{\alpha\beta\gamma}^2$ between these maps. As explained in [OTT1, Section 1.3], one can proceed inductively to construct a twisting cochain $a = \sum_k a^k$. Note that $\mathcal{H}_a^k = 0$ if $k > 0$ and $\mathcal{H}_a^0 \cong \mathcal{F}$. We shall refer to (F, a) as a *twisted resolution* of \mathcal{F} .

Consider two twisting cochains (F, a) and (G, b) . We define an operator D of degree 1 on $C^\bullet(\mathfrak{U}, \text{Hom}^\bullet(F, G))$,

$$Df := \delta f + bf - (-1)^{\deg f} fa.$$

We have that $D^2 = 0$, and

$$D(fg) = (Df)g + (-1)^{\deg f} f(Dg). \quad (6.6)$$

6.3. A representation of the Ext groups. Following the proof of Theorem 2.9 in [TT], we will now show how the Ext groups $\text{Ext}^k(\mathcal{F}, \mathcal{G})$ can be represented as the cohomology of a complex formed from a twisted resolution of \mathcal{F} .

Theorem 6.2. *Let (F, a) be a twisted resolution of a coherent \mathcal{O}_X -module \mathcal{F} . Then for any coherent \mathcal{O}_X -module \mathcal{G} , there is an isomorphism*

$$H^k \left(\bigoplus_{p+r=\bullet} C^p(\mathfrak{U}, \text{Hom}^r(F, \mathcal{G})) \right) \cong \text{Ext}^k(\mathcal{F}, \mathcal{G}).$$

Proof. Let

$$0 \longrightarrow \mathcal{G} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \quad (6.7)$$

be an injective resolution of \mathcal{G} . Consider the double complex

$$E^{k,q} := \bigoplus_{p+r=k} C^p(\mathfrak{U}, \text{Hom}^r(F, I^q))$$

where the differential in the k -direction is given by D and the differential in the q -direction is the differential of (6.7).

We claim that

$$H^q(E^{k,\bullet}) \cong \begin{cases} \bigoplus_{p+r=k} C^p(\mathfrak{U}, \text{Hom}^r(F, \mathcal{G})), & q = 0, \\ 0, & q \geq 1, \end{cases}$$

and, moreover, we claim that

$$H^k(E^{\bullet,q}) \cong \begin{cases} \text{Hom}(\mathcal{F}, I^q), & k = 0, \\ 0, & k \geq 1. \end{cases}$$

Under the assumption that these claims are true, we get that both spectral sequences degenerate, and hence it follows that

$$H^k \left(\bigoplus_{p+r=\bullet} C^p(\mathfrak{U}, \text{Hom}^r(F, \mathcal{G})) \right) \cong H^k(\text{Hom}(\mathcal{F}, I^\bullet)) = \text{Ext}^k(\mathcal{F}, \mathcal{G}),$$

where the isomorphism is induced by the inclusions into the total complex. From this the statement of the theorem follows.

Let us prove the two claims. For the first claim we have that

$$H^q(E^{k,\bullet}) = \bigoplus_{p+r=k} \prod_{(\alpha_0, \dots, \alpha_p)} H^q(\mathcal{H}om(F_{\alpha_p}^r, I^\bullet)(\mathcal{U}_{\alpha_0 \dots \alpha_p})).$$

Using the definition of Ext together with [H1, Proposition 6.7] and Cartan's Theorem B, we get that

$$\begin{aligned} H^q(\mathcal{H}om(F_{\alpha_p}^r, I^\bullet)(\mathcal{U}_{\alpha_0 \dots \alpha_p})) &= \text{Ext}^q(F_{\alpha_p}^r|_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}, \mathcal{G}|_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}) \\ &\cong \text{Ext}^q(\mathcal{O}_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}, (F_{\alpha_p}^r)^\vee \otimes \mathcal{G}|_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}) \\ &\cong H^q(\mathcal{U}_{\alpha_0 \dots \alpha_p}, (F_{\alpha_p}^r)^\vee \otimes \mathcal{G}|_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}) \\ &\cong \begin{cases} \text{Hom}(F_{\alpha_p}^r|_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}, \mathcal{G}|_{\mathcal{U}_{\alpha_0 \dots \alpha_p}}), & q = 0, \\ 0, & q \geq 1. \end{cases} \end{aligned}$$

Here $(F_{\alpha_p}^r)^\vee := \mathcal{H}om(F_{\alpha_p}^r, \mathcal{O}_{\mathcal{U}_{\alpha_p}})$ denotes the dual of $F_{\alpha_p}^r$. Thus

$$H^q(E^{k,\bullet}) \cong \begin{cases} \bigoplus_{p+r=k} C^p(\mathfrak{U}, \text{Hom}^r(F, \mathcal{G})), & q = 0, \\ 0, & q \geq 1. \end{cases}$$

For the second claim we define

$$A^{p,r} := C^p(\mathfrak{U}, \text{Hom}^r(F, I^q)),$$

and we let A be the complex $A := \bigoplus_{p+r=k} A^{p,r}$ with differential D . Consider the spectral sequence (A_s, d_s) associated with the filtration

$$F^p A = \bigoplus_{p' \geq p} A^{p', \bullet}.$$

We have that $A_1^{p,r}$ vanishes for $r > 0$. Moreover, a straightforward computation shows that

$$A_1^{p,0} \cong \prod_{(\alpha_0, \dots, \alpha_p)} \mathcal{H}om(\mathcal{F}, I^q)(\mathcal{U}_{\alpha_0 \dots \alpha_p}),$$

i.e., the group of ordinary Čech cochains, and the map $d_1 : A_1^{q,0} \rightarrow A_1^{q+1,0}$ is the ordinary Čech differential under this identification.

Thus

$$A_2^{p,0} \cong \check{H}^p(\mathfrak{U}, \mathcal{H}om(\mathcal{F}, I^q)),$$

which vanishes for $p > 0$, since $\mathcal{H}om(\mathcal{F}, I^q)$ is a flabby sheaf, and is equal to $\text{Hom}(\mathcal{F}, I^q)$ for $p = 0$. Thus the only nonvanishing term on the second page is $A_2^{0,0}$ and thus the second claim follows. \square

From the bigraded complex

$$E_0^{p,q} := C^p(\mathfrak{U}, \mathcal{H}om^q(F, \mathcal{G})),$$

we also recover the local-to-global spectral sequence for Ext . Consider the spectral sequence associated with the filtration with respect to p . We have shown that this spectral sequence converges to $\text{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$. Taking cohomology in the q -direction and then in the p -direction, we get that the first page of the spectral sequence is

$$E_1^{p,q} = C^p(\mathfrak{U}, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})),$$

and the second page is given by

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})).$$

7. PAPER II

Let X be a complex manifold. Let \mathcal{F} be a coherent \mathcal{O}_X -module, and let G be a locally free \mathcal{O}_X -module. Recall that we have the representations

$$\text{Ext}^k(\mathcal{F}, G) \cong H^k(\text{Hom}(\mathcal{F}, \mathcal{C}^{0,\bullet}(G))).$$

Let (F, a) be a twisted resolution of \mathcal{F} . By Theorem 6.2, we have that

$$\text{Ext}^k(\mathcal{F}, G) \cong H^k \left(\bigoplus_{p+r=\bullet} C^p(\mathfrak{U}, \text{Hom}^r(F, G)) \right).$$

The aim of Paper II is to provide an explicit isomorphism between these two representations of the Ext groups.

7.1. Homological preliminaries III. Let as in Section 6.1 $\mathfrak{U} = (\mathcal{U}_\alpha)$ be a covering of X by Stein open sets, and for each α let $E_\alpha = \bigoplus_r E_\alpha^r$, $F_\alpha = \bigoplus_r F_\alpha^r$, and $G_\alpha = \bigoplus_r G_\alpha^r$ be bounded graded holomorphic vector bundles over \mathcal{U}_α . We will now combine the material from Section 3.1 and Section 6.1, and hence we will consider current-valued analogues of $C^p(\mathfrak{U}, \text{Hom}^r(F, G))$.

To this end, we define

$$C^p(\mathfrak{U}, \mathcal{C}^{0,q}(\text{Hom}^r(F, G))) := \prod_{(\alpha_0, \dots, \alpha_p)} \mathcal{C}^{0,q}(\text{Hom}^r(F_{\alpha_p}, G_{\alpha_0}))(\mathcal{U}_{\alpha_0 \dots \alpha_p}). \quad (7.1)$$

For $f \in C^p(\mathfrak{U}, \mathcal{C}^{0,q}(\text{Hom}^r(F, G)))$, we define its degree as $\deg f = p + q + r$.

For elements $f \in C^p(\mathfrak{U}, \mathcal{C}^{0,q}(\text{Hom}^r(F, G)))$ and $g \in C^{p'}(\mathfrak{U}, \mathcal{C}^{0,q'}(\text{Hom}^{r'}(E, F)))$ we define their product $fg \in C^{p+p'}(\mathfrak{U}, \mathcal{C}^{0,q+q'}(\text{Hom}^{r+r'}(E, G)))$ by

$$(fg)_{\alpha_0 \dots \alpha_{p+p'}} := (-1)^{(q+r)p'} f_{\alpha_0 \dots \alpha_p} g_{\alpha_{p+1} \dots \alpha_{p+p'}},$$

where the product on the right-hand side is defined by (3.2) provided that it exists. We let the $\bar{\partial}$ -operator act as an operator of degree 1 on $C^\bullet(\mathfrak{U}, \mathcal{C}^{0,\bullet}(\text{Hom}^\bullet(F, G)))$ by

$$(\bar{\partial}f)_{\alpha_0 \dots \alpha_p} := (-1)^p \bar{\partial} f_{\alpha_0 \dots \alpha_p}. \quad (7.2)$$

With this definition we have that $\bar{\partial}(fg) = (\bar{\partial}f)g + (-1)^{\deg f} f(\bar{\partial}g)$, and as usual $\bar{\partial}^2 = 0$. Note that $C^p(\mathfrak{U}, \text{Hom}^r(F, G))$ can be identified with the subgroup of $\bar{\partial}$ -closed elements of $C^p(\mathfrak{U}, \mathcal{C}^{0,0}(\text{Hom}^r(F, G)))$.

Next we define

$$\delta : C^p(\mathfrak{U}, \mathcal{C}^{0,q}(\text{Hom}^r(F, G))) \rightarrow C^{p+1}(\mathfrak{U}, \mathcal{C}^{0,q}(\text{Hom}^r(F, G)))$$

precisely as in Section 6.1, i.e.,

$$(\delta f)_{\alpha_0 \dots \alpha_{p+1}} := \sum_{k=1}^p (-1)^k f_{\alpha_0 \dots \hat{\alpha}_k \dots \alpha_{p+1}} |_{\mathcal{U}_{\alpha_0 \dots \alpha_{p+1}}}.$$

Let (F, a) and (G, b) be twisting cochains. We define an operator D of degree 1 on $C^\bullet(\mathfrak{U}, \mathcal{C}^{0,\bullet}(\text{Hom}^\bullet(F, G)))$ in the same way as in Section 6.2, i.e.,

$$Df := \delta f + bf - (-1)^{\deg f} fa.$$

We combine D and $\bar{\partial}$ into an operator

$$\nabla = D - \bar{\partial}$$

of degree 1 on $C^\bullet(\mathfrak{U}, \mathcal{C}^{0,\bullet}(\text{Hom}^\bullet(F, G)))$. It can be shown that $\bar{\partial}D = -D\bar{\partial}$, and from this it follows that $\nabla^2 = 0$. Moreover, we have that

$$\nabla(fg) = (\nabla f)g + (-1)^{\deg f} f(\nabla g). \quad (7.3)$$

7.2. A residue current associated with a twisted resolution. We are now ready to describe the main idea on how one can construct an explicit isomorphism between the above described representations of the Ext groups.

Let (F, a) be a twisted resolution of a coherent \mathcal{O}_X -module \mathcal{F} . The main idea in Paper II is the construction of two elements U and R of $C^\bullet(\mathfrak{U}, C^{0,\bullet}(\text{Hom}^\bullet(F, F)))$ of degree -1 and 0 , respectively, that satisfy

$$\nabla U = \text{id} - R.$$

We refer to R as the residue current associated with the twisted resolution of \mathcal{F} .

In order to formulate the isomorphism we shall also need the following operator. Let (ρ_α) be a partition of unity subordinate to \mathfrak{U} . We define an operator

$$v : C^p(\mathfrak{U}, C^{0,q}(\text{Hom}^r(F, G))) \rightarrow C^{p-1}(\mathfrak{U}, C^{0,q}(\text{Hom}^r(F, G)))$$

by

$$(vf)_{\alpha_0 \dots \alpha_{p-1}} := \sum_{\alpha} \rho_{\alpha} f_{\alpha \alpha_0 \dots \alpha_{p-1}}$$

for $p \geq 1$ and $vf := 0$ otherwise. Here $\rho_{\alpha} f_{\alpha \alpha_0 \dots \alpha_{p-1}}$, which is defined on $\mathcal{U}_{\alpha \alpha_0 \dots \alpha_{p-1}}$, is extended by 0 to $\mathcal{U}_{\alpha_0 \dots \alpha_{p-1}}$.

The main result of Paper II can now be expressed as follows.

Theorem 7.1. *Let \mathcal{F} be a coherent \mathcal{O}_X -module, and let G be a locally free \mathcal{O}_X -module. Let (F, a) be a twisted resolution of \mathcal{F} , and let R be the associated residue current. There is an isomorphism between the above mentioned representations of $\text{Ext}^k(\mathcal{F}, G)$:*

$$H^k \left(\bigoplus_{p+r=\bullet} C^p(\mathfrak{U}, \text{Hom}^r(F, G)) \right) \xrightarrow{\cong} H^k(\text{Hom}(\mathcal{F}, C^{0,\bullet}(G))) \quad (7.4)$$

given by

$$[\xi] \mapsto \left[\phi \mapsto \sum_j (\bar{\partial}v)^j (\xi R)^j \phi^0 \right], \quad (7.5)$$

where ϕ^0 is a 0-cochain that represents ϕ . Moreover, $(\xi R)^j$ denotes the component of Čech degree j , and $(\bar{\partial}v)^j$ denotes the composition $\bar{\partial} \circ v$ repeated j times.

The theorem is proved using a spectral sequence argument where one constructs a ∇ -potential involving U .

8. PAPER III

In Paper II we constructed a residue current associated with a twisted resolution of a coherent \mathcal{O}_X -module. More generally, given a complex \mathcal{F}^\bullet of coherent \mathcal{O}_X -modules, one can also speak about a twisted resolution of \mathcal{F}^\bullet . A twisted resolution of \mathcal{F}^\bullet consists of a twisting cochain (F, a) such that for each α there is a quasi-isomorphism $F_\alpha^\bullet \rightarrow \mathcal{F}^\bullet|_{\mathcal{U}_\alpha}$. In Paper III we give a construction of a residue current associated with a twisted resolution, and, more generally, an arbitrary twisting cochain. The main results are a duality principle analogous to Theorem 3.2, and an analogous comparison formula.

Recall that using the Syzygy Theorem, one can prove that, given a coherent \mathcal{O}_X -module \mathcal{F} , there exists a twisted resolution. Analogously, we have that, given a complex \mathcal{F}^\bullet of coherent \mathcal{O}_X -modules, one can always find a twisted resolution: Using the method outlined in [E1, Exercise 3.53], one first finds a cover $\mathfrak{U} = (\mathcal{U}_\alpha)$ by Stein open sets and for each \mathcal{U}_α a complex $(F_\alpha^\bullet, a_\alpha^0)$ that is quasi-isomorphic with $\mathcal{F}^\bullet|_{\mathcal{U}_\alpha}$. By [OTT2, Proposition 1.2.3], a^0 can be extended to a twisting cochain (F, a) which is then a twisted resolution of \mathcal{F}^\bullet .

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