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# A Model for Central Counterparty Risk with Stochastic Default Intensities

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## Abstract

In this thesis we use a dynamic model to compute several margins required by a central counterparty, the central clearing house (CCP), to the participants, called clearing members (CM). These margins form the so called default waterfall. In this market only credit default swaps (CDS) are exchanged. CDS are financial instruments that work as an insurance against counterparties default. The CCP is the main infrastructure: it takes on counterparty credit risk between members and provides clearing and settlement services in the trading activity. The first layer of the waterfall is the variation margin (VM). It is defined as the value of a member's portfolio in the previous period (usually one day) and it is also needed for the next two layers computations. The second layer is the initial margin ($IM$), which represents an added collateral for market fluctuations and has a greater time horizon, equal to 5-10 days. The third layer is the default fund ($DF$). It represents the last margin of interest in this work and it is used in order to cover losses deriving from defaults of one or more CM. This is the most challenging margin because there is no consensus on the way it should be computed and distributed among the members. Our final goal is focused on finding the optimal $DF/IM$ ratio for every CM. For the study we consider 8 CM and 4 CDS contracts. First, we compute the values of the CDS contracts, that are subsequently pooled together composing the portfolios of the CM. The margins of the waterfall are computed using dynamic time consistent risk measures and, in order to take into account the riskiness of the CDS contracts and CM, we extend the model of [Bielecki et al., 2018] to the case of stochastic default intensities, in particular using Cox-Ingersoll-Ross (CIR) processes. The results on $DF/IM$ ratio for each CM suggest that our model is able to scale appropriately the number of members and to take into account their riskiness.

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# Glossary

| | |
|---|---|
| CCP | Central Clearing Counterparty |
| CDS | Credit Default Swap |
| CIR | Cox-Ingersoll-Ross |
| CM | Clearing Member |
| DF | Default Fund |
| E | CCP Equity |
| EP | CCP Exposure for Default Fund |
| ES | Expected Shortfall |
| IM | Initial Margin |
| MPOR | Margin Period of Risk |
| MtM | Mark-to-Market |
| OTC | Over-the-Counter |
| $u$DF | Unfunded Default Fund |
| $r$ | Credit Default Swap Spread |
| $Rec$ | Recovery Rate |
| V | Portfolio Value |
| VaR | Value at Risk |
| VM | Variation Margin |
| X | CCP Exposure for Initial Margin |
| $Z^*$ | Radon-Nikodym Derivative |
| $\alpha$ | Significance level for Initial Margin |
| $\beta$ | Significance level for Default Fund |
| $\lambda$ | Default Intensity |
| $\tau$ | Default Time CM |
| $\phi$ | Default Time CDS |

# 1 Introduction

The global financial crisis of 2007-08 is one of the main event of the recent history. It brought down the GDP of several countries around the globe and people had been suffering its effects several years after the official end.
On August 2007, the first signs of trouble were quite clear, with the housing prices that started to fall after a huge increase in the earlier period.
On September 2008, the Lehman Brothers, the American fourth-largest investment bank at that time, declared bankruptcy. This event represented the tip of the iceberg of a sick financial system and had a huge impact also in the public opinion, considering that Lehman Brothers' bankruptcy was and remains the largest in U.S. history.
Meantime the Dow Jones Industrial Average (DJIA) was falling and, particularly, on September $29^{th}$, the index fell 777.68 points in intraday trading, the largest drop at that time. The highest pre-recession peak was reached by the DJIA on October $9^{th}$ 2007 and, by March $5^{th}$ 2009, it had dropped more than 50%. The DJIA therefore had lost half of its total value in only 18 months.

The financial instruments that caused the crisis where traded in the so called uncleared over-the-counter (OTC) markets. OTC networks are markets in which participants trade securities without the intermediation of a broker or a central exchange and have less stringent regulation.
Moreover, OTC markets can be centrally cleared, namely a central authority brings together supply and demand of securities and settles the trading activity between members and these markets are then referred to as "cleared OTC markets", in order to distinguish them from the uncleared OTC case. Clearing is defined by the European Association of CCP Clearing Houses as "the process of guaranteeing financial market transactions between the execution of transaction and its settlement". The settlement is the end of a transaction and takes place when a security changes in ownership against a payment.
After the crisis, at Pittsburgh summit in 2009, G-20 leaders agreed that all standardised derivatives contracts should be cleared through a central authority, named central clearing counterparty (CCP), which is an infrastructure in the market that distributes the risk and is responsible for netting [1] all the operations among the participants.
Moreover, several further reforms such as Dodd–Frank Wall Street Reform and Consumer Protection Act., enacted in 2010, Basel III agreed in November 2010 and European Market Infrastructure Regulation (EMIR) in 2012 have thereafter contributed to extend the regulatory system, in particular in the direction of fixing risk management standards.

[1] The activity of reducing transfers of funds to a net amount.

The decision of imposing a CCP was due to the fact that the presence of a third party to regulate better the trading activity was necessary, considering the crisis was triggered by the financial system. The primary issue for regulators was to create a working and exhaustive outline to ensure stability in the overall financial system.

The main actors in a centrally cleared OTC market are therefore the participants in the trading activity, also called clearing members (CM), and the aforementioned CCP.
In a centrally cleared market, every contract between two counterparties (i.e. clearing members) is replaced by two contracts between CCP and each counterparty. Hence CCP represents the pivotal infrastructure in the trading activity and acts as the buyer to every seller and the seller to every buyer. In case of a member's default, CCP activates the default management procedure and uses the funds collected from all the clearing members in order to avoid other defaults and keep the market working.
The goal of this thesis is to study the so called *default waterfall* for a CCP, that is the sequence of funds that CCP uses to cover the loss resulting from the default of one or more clearing members and to guarantee the stability and the efficient functioning of the market. In particular, the objective here is to quantify these funds under the relations presented below. In this thesis we will work with and partly extend the CCP model in [Bielecki et al., 2018].
The variation margin (VM) is the first layer of the waterfall and is equal to the value of member's cleared portfolio of derivatives and it is daily transferred to the CCP.
When the VM is not sufficient to cover the losses at defaults, the initial margin (IM), posted by the defaulted CMs, is used. The goal of the IM is to cover potential future market fluctuations of the cleared derivatives over a horizon usually of 5/10 days.
If the IM is not enough to cover the losses at defaults, then the CCP draws from the default fund (DF) which is aimed to cover potential losses in case of default of one or more members. The DF is a loss mutualization tool where the survived clearing members share the loss burden of the defaulted members, if the defaulted members' VM, IM and DF do not cover their losses. In a bilateral contract, the protection buyer is exposed to the protection seller's risk. The CCP, as central authority, groups all the resources of the sellers and makes the losses from the default of an individual seller shared across all protection buyers if the VM and IM of the defaulted clearing member do not cover the loss caused by CM.
Moreover, other two layers (that are not analyzed in this thesis) form the default waterfall, namely the CCP equity (E), also known as "skin-in-the-game", which is the amount pledged by the CCP to cover losses, and the unfunded DF ($u$DF), that is additional amounts paid by surviving members.

The following scheme, taken from [Bielecki et al., 2018], summarises the default waterfall.

IM of the defaulted member(s) (IM)

$\Downarrow$

(pre-funded) DF contribution of the defaulted member(s) (DF)

$\Downarrow$

CCP equity, or skin-in-the-game (E)

$\Downarrow$

(pre-funded) DF of the surviving members (DF)

$\Downarrow$

Unfunded DF from the surviving members ($u$DF) and/or some additional capital from CCP

Looking at the waterfall summarized above, we see that, in case of default, first the IM and DF of the defaulting members are used and then, in case this amount is insufficient to cover the total loss, also the surviving members' DF are used.
At the writing moment, the DF represents the most problematic layer in the CCP literature, because there is not a common stand on how to compute it (see, for instance, [Bielecki et al., 2018]). Moreover, there are other layers in the default waterfall, but they will not be studied quantitatively in this thesis but will be mentioned later on for completeness' sake.
The computation of IM and DF are usually done by using the so called risk measures and in this thesis, just as in [Bielecki et al., 2018], we will focus on consistent dynamic risk measures. Risk measures are statistical measures that assign a value to a risk. Moreover, consistent dynamic risk measures are measures that evaluate risk at different times and present the time consistency property, that is the property of not having contradictory evaluations of risk at different time points. Every risk measure presents attached a given significance level, usually expressed in percentage, that indicates how sure we are about the value of risk obtained.
In order to compute IM and DF via risk measures, we need two significance levels, $\alpha$ and $\beta$, respectively for IM and DF.

Concerning the financial instruments, we wil in this thesis, just as [Bielecki et al., 2018], only consider *credit default swap* (CDS). A CDS is a credit derivative in which the seller pays a certain amount the buyer in case of default of a reference entity, against a periodic payment made by the buyer expressed as a percentage of a notional. These swaps gained recently great popularity because of their role in the financial crisis of 2007-2008 and, from then on, they have been remaining a trending topic in the financial literature.
In order to study the underlying CDS entity's default time, we will use the *reduced form*

*model*, that will be explained in Section 4 and such models will also be applied to the default times of the CCP clearing members.

In this thesis we work with and partly extend the model of [Bielecki et al., 2018] to a more realistic framework. [Bielecki et al., 2018] introduces a model where the default intensities of the CDS are constant and clearing members default intensities are piecewise constant and change according to the switching time of a Markov chain [2]. In this thesis, instead, we introduce stochastic default intensities to study both the default behaviour of clearing members and CDS contracts. In particular in our case the default intensities are modelled following a CIR process, which is a process belonging to the family of *affine models*, that allows for closed form representations of several important quantities, such as the default probabilities and conditional default probabilities.
The choice of CIR processes is motivated by the fact that the hypothesis of constant intensities is very simplistic, in particular for long time horizons. In order to model real life riskiness, piecewise constant default intensities are often used as an approximation. In this study, instead, default intensities change continuously and randomly in time.
Moreover, in order to take into account dependence among members' default times, we consider copulas, which are functions able to separate the marginal distributions to dependence structure. More precisely, we use a *one-factor Gaussian copula model*, that will be analyzed in the next sections.

In the section devoted to the numerical results, we conduct an extensive numerical analysis to confirm our theoretical findings and discuss the type of effects that can be caught by our model. We compute VM, IM and DF and address the problem of the allocation of DF among clearing members. As in [Bielecki et al., 2018], our final outcome of interest, that summarizes the whole waterfall, is the $DF/IM$ ratio and its behaviour depending on the risk and the portfolio value of the members. The $DF/IM$ ratios obtained change significantly between the members, that present different riskiness, and therefore show the ability of our model to take into account correctly the risk and to compute consistently the margins of the default waterfall.
Apart from the main setting with 8 clearing members, we also consider the case of 4 and 6 clearing members and test robustness of our model by performing some sensitivity analysis. The results attest that our model is able to scale appropriately the $DF/IM$ ratio according to the number of clearing members. The capability to scale the $DF/IM$ ratio in an appropriate way is particularly important because a widely used methodology for computing DF,

[2]It is a memoryless process characterized by transitions from one state to another according to specific probability rules.

the *Cover 1/Cover 2* principle, is not able to do so.
For 8 clearing members, we consider different derivatives portfolios. Our study shows how the ratio changes according to the composition of members' portfolios. Moreover, some sensitivity analysis on the significance levels $\alpha$ and $\beta$ of the dynamic consistent risk measure chosen is performed. Using three-dimensional bar charts, we see that the $DF/IM$ ratio increases significantly in case of $\alpha$ and $\beta$ equal to 0.999%, while it remains quite flat in all the other cases.

## Structure of the Thesis

The rest of the thesis is structured as follows. Section 2 contains the literature review, in which we present the previous studies made on the topic which we started from in order to carry out the thesis. Section 3 explores the concepts we presented in the introduction and, in particular, the notion of credit risk, CDS and its regulation, CCP and default waterfall are expounded. Section 4 deals with the model used in order to perform the study, presenting the theoretical context and its assumptions. First, the mathematics behind default intensities is presented, then we introduce some starting assumptions and finally there are all the formulas of the default waterfall. Section 5 shows the numerical analysis and it presents more in detail the formulas and tools used to perform the research and primarily the results of the thesis, with comments and arguments with the aim of understanding in detail the topic under study. Section 5 highlights our extension of the model in [Bielecki et al., 2018] in which CCP related calculations in a model not previously used in clearing contexts. More specific, we use a model with stochastic default intensities, results of each layer of the default waterfall are shown and some sensitivity analysis is performed.
Finally, Section 6 closes the thesis, mapping out some conclusions and trying to suggest potential future developments in this research field.

# 2 Literature Review

In this section we analyze the existing literature on the main topics of the thesis. We divide the section in two parts, the first analyzes the literature about CCP and the related topics, while the second focuses on risk measures and capital allocation [3].

## 2.1 Central Clearing, Counterparty Risk and Margins

There is a huge branch of literature in credit risk on CCPs, since their regulation has been changing a lot in the last few years.
[Cecchetti et al., 2009] is one of the key paper on the topic and was published immediately after the first changes arising out of the financial crisis of 2007-08. It is one of the first papers to detect the prospective role of CCP as stabilising factor in the financial system.
Among papers that date back to the first stage of the studies after the crisis, [Pirrong, 2009] analyzes the effect of central clearing counterparties on the overall system, exploring pros and cons and it concludes saying that the benefits of central clearing are sometimes lower than costs.
In [Cont and Kokholm, 2012] there is a comparison between bilateral and multilateral netting. In particular, over-the-counter trades are analyzed with respect to different asset classes, with heterogeneous characteristics.
Other important papers on the effect of central clearing are [Pirrong, 2011], which gives a general overview on the topic, [Amini et al., 2016], with a focus on systemic risk [4], and [Faruqui et al., 2018], which particularly explores the CCP-bank nexus.
[Duffie and Zhu, 2011] compares non-centrally cleared vs centrally cleared OTC trading and concludes that sometimes you need a lot more CM to make the centrally cleared OTC market better than without a CCP. The impact of central clearing counterparties is studied also in [Loon and Zhong, 2014], with a focus on the CDS market.
Margins required to the participants are studied, for instance, in [Nahai-Williamson et al., 2013], where the authors concentrate on the resources a CCP should have to face possible members' defaults and the amount of initial margins and default fund in a stylised setting. In particular, it is shown that, in case of low probability of clearing member $i$'s default and high volatility of the underlying asset, the initial margin is preferred, while, in the opposite situation, that is an high member's probability of default and low volatility of the asset, the default fund is a better resource.

[3] Capital allocation is considered in our case as the process of distributing financial resources among the clearing members participating in a centrally cleared market.

[4] It is the risk of a collapse of the system rather than of individual members.

[Cumming and Noss, 2013] focuses on the same basic problems of [Nahai-Williamson et al., 2013] and moreover proposes a "top-down" approach to assess the funds' sizes, in contrast with the common "bottom-up" stress testing for the clearing members in extreme but plausible conditions, as is given in [Russo et al., 2013]. A specific detection of the *Cover 1/Cover 2* principle is made in [Murphy and Nahai-Williamson, 2014], where the authors specifically study the CCP's buffers and how the losses caused by defaulting members are allocated in the overall system.
[Duffie et al., 2015] analyzes the impact of new clearing and margin regulations on collateral demand, using group of bilateral CDS. The relation between new informations available and potential default contagion between members is considered in [Biais et al., 2016], where the authors focus in particular on the fact that bad news, increasing expected liabilities, can potentially reduce risk-prevention incentives and therefore can cause instability, as mentioned earlier in Section 2.1.
The main reference to our work is [Bielecki et al., 2018]. In this paper the authors introduce a dynamic model to describe default waterfall for derivatives CCPs and develop a new risk sensitive methodology for the evaluation of the IM and the DF based on consistent risk measures.
Further recent researches on CCP can be found in e.g. [Heckinger et al., 2016], [McPartland et al., 2017] and [Giulia et al., 2018], amongst others.

## 2.2 Risk Measures and Capital Allocation

Risk measures and their applications is a classical topic in quantitative finance. In [Delbaen, 1998] the authors present market and non-market risks and introduce some risk measures used to assess their magnitude in a static model. Later, [Delbaen, 2000] extends the previous paper's field to a general probability space. Other important papers are [Artzner et al., 2002], whose attention is on multiperiod risk measures and [Bielecki et al., 2017], where the authors provide an overview on the time consistency property of dynamic risk and performance measures. The latter extends the previous literature to a dynamic setup and is aimed to adapting the measurements throughout time at the current information available.
Capital allocation is a major issue in risk management and is established by Basel accords [5] in finance and solvency agreements [6] in insurance. Other previous studies about Euler allocation are provided in [Koyluoglu and Stoker, 2002], [Urban et al., 2004] and the book

[5] Group of accords issued by the Basel Committee on Banking Supervision (BCBS) concerning primarily banking regulation. The last agreement, known as Basel III, was reached in November 2010 and its implementation has been extended repeatedly because of continuous new updates. The next implementation date is scheduled for January 2022.

[6] Directives in European Union law aimed to harmonise and modernise rules used by EU insurance companies. Known also as "Basel for insurers", after a series of delays, its last version, Omnibus II Directive, became effective on January 2016.

[McNeil, 2005]. More recently, [Embrechts et al., 2017] study the problem of risk sharing among participants by using a two parameter class of quantile [7] based risk measures.

[7] A quantile is a value that represents the division of a variable in a group of intervals.

# 3 The Credit Risk Market and the Central Clearing Counterparty

In this section, we introduce the elements which will be studied in the subsequent parts of this thesis. In particular, the main attention is on the central counterparty and the so called *default waterfall*.
The first two parts of the section are dedicated respectively to the credit risk framework, its definition and classifications and then to an overview of the *credit default swaps*, the financial instruments object of study. Then, a more detailed part about CCP is presented, and the last subsection deals with the default waterfall framework and specifies all the funds required to the clearing members.

## 3.1 Credit Risk

Credit risk is the risk that at least one counterpart involved in a financial transaction fail to honour its obligation, for instance due to default (see, e.g. [Hull, 2017]). Whether the contemporary measurement and management tools for credit risk turns out to be inappropriate is a pivotal concern for financial institutions and regulators, particularly after the financial crisis of 2007-08.
According to [Schönbucher, 2003], we may distinguish four components of credit risk, that are:

- **Arrival Risk**. The risk connected to whether or not the default happens in a given interval.
- **Timing Risk**. The risk regarding when default occurs, given that the default happens.
- **Recovery Risk**. The risk related to the amount to be recovered at default.
- **Default Dependency Risk**. The risk of observing a cascade of defaults due to financial relations among several obligors.

## 3.2 Credit Default Swaps

Nowadays, the most popular financial instrument used to cover for the risk of a counterparty default is represented by the *credit default swap* (CDS). This instrument can be seen as a form of insurance against counterparty's default. In particular, a CDS is an agreement between two counterparties, the protection buyer and the protection seller. Looking at Figure 1, the *protection buyer* $\boldsymbol{A}$ has the right to sell the bonds issued by a company $\boldsymbol{C}$,

also called the *reference entity*, to the *protection seller* $\boldsymbol{B}$ for their face value, or *notional principal* $N$, immediately after the default of the issuer, know as *credit event*, if the credit event happens before the maturity $T$ of the CDS contract. For this default protection $\boldsymbol{A}$ pays a quarterly fee, called CDS *spread* $R(T)$, expressed in basis points (bps) of the notional principal up to time $T$ or until the credit event, whichever comes first.
Figure 1 and Figure 2, taken from [Herbertsson, 2019], illustrate the functioning of a CDS.

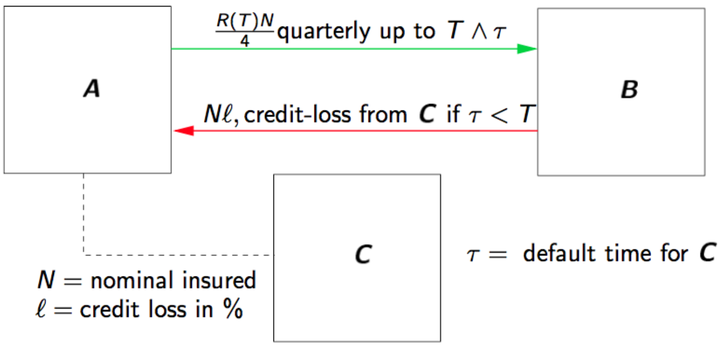



Figure 1: The structure of a CDS contract. Source: Herbertsson (2019)

The protection buyer $\boldsymbol{A}$ pays (quarterly) an amount equal to $\frac{R(T)N}{4}$ to $\boldsymbol{B}$, who pays the credit loss of $N\ell$, if default of the reference entity $\boldsymbol{C}$ occurs before time $T$.
Figure 2 illustrates the CDS cash flows due to the default of $\boldsymbol{C}$ at time $\tau$, where $\tau < T$. $\boldsymbol{B}$ pays to $\boldsymbol{A}$ the amount that the defaulted reference entity is not able to repay to the protection buyer.

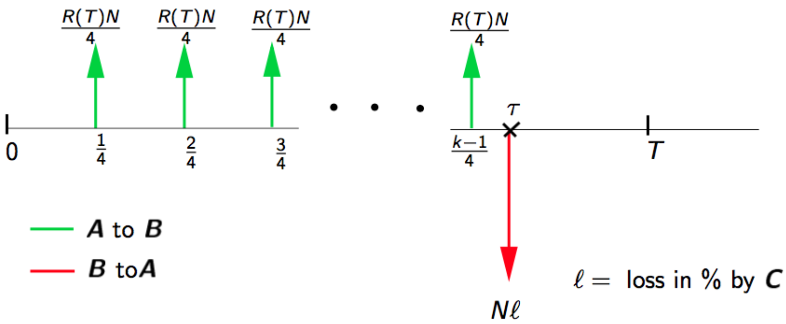



Figure 2: Description of the flow in case of a credit event. Source: Herbertsson (2019)

If the default of $\boldsymbol{C}$ does not occur before the CDS, the spread is paid until the end of the contract, at time $T$. Figure 1 and Figure 2 show the functioning of a CDS contract where the CDS spread is paid quarterly. In the pricing of a CDS contract, we can also assume the spread is paid continuously, as we will see in Section 4.3.1.

For any time point $t$ with $0 \leq t \leq T$ we define the value $S_t$ of a CDS contract as the difference between the present value of the protection leg ($PVprot_t$), that is the amount paid by the protection buyer $\boldsymbol{A}$, and the present value of the premium leg ($PVprem_t$), namely the amount paid by the protection seller $\boldsymbol{B}$ in case of reference entity's default. Hence, $S_t$ is given by

$$S_t = PVprot_t - PVprem_t.$$

Furthermore, the CDS spread $R(T)$ is determined so that $S_0 = 0$, that is $R(T)$ is set so that the expected present value of the cash flows between $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal at the start of the CDS contract.

With the great increase in their trading during the period 2001-2007, CDS became very popular credit derivatives among investors and naked positions, namely the possession of the CDS without the security to insure, were very common.

In the period concurrently with the financial crisis of 2007-2008, CDS were used mainly for speculating. The chaotic increase of these financial instruments, their poor regulation and inappropriate pricing models, which underestimated default probabilities and barely took into account contagion costs in case of defaults, made the CDS one of the main responsible of the economic collapse of 2007-08.

After the crisis, the market structure of the CDS changed rapidly due to the increasing importance of the central counterparties, as shown by Figure 3, taken from the BIS Quarterly Review of June 2018 ([Aldasoro and Ehlers, 2018]).

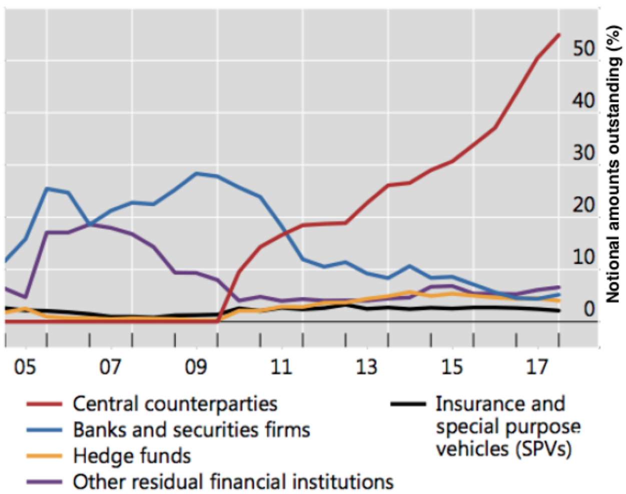



Figure 3: The outstanding gross-notional value of different OTC-derivatives traded via different intermediaries for 2005-2017. Source: BIS Quarterly Review (June 2018)

Figure 3 shows the trend of the market share of CCPs in comparison with the inter-dealer trades as a percentage of the notional amounts outstanding, in the period 2005-2017.

### 3.2.1 CDS Regulation

We dedicate a specific section to the regulation of CDS, because of the huge impact it had on the functioning of these swaps.
Since the credit default swaps are a type of OTC credit derivatives [8], they were initially poorly regulated. After the financial crisis of 2007-2008 a lot of changes have been made with the main goal to increase the stability and transparency of the CDS market. In March 2008, as argued by [White, 2013], the Working Group on Financial Markets outlined three aspects to work on in order to make the market more efficient after the financial crisis. These points are

- **Trade compression**. The goal is reducing the amount outstanding and use net compensation among participants.
- **Timely trade matching**. In order to make the market faster, trade processing needs to be reduced up to the trade date.
- **Central clearing**. The third goal is creating a centrally cleared market.

The Security and Exchange Commission (SEC) is the regulatory agency for securities trading in USA, while in Europe, since 2011, there is the European Securities and Markets Authority (ESMA).
In April 2009, one of the biggest change was made by CDS market participants that agreed on standardize the contracts. The reform, also known as the "Big Bang", was made by the International Swaps and Derivatives Association (ISDA), that gathers into one all the participants in OTC derivatives market. The Big Bang reform, as reported in [White, 2013], standardized mainly three aspects, that are the legal effective date, the accrued interests and the coupons. With the Big Bang, now the CDS contracts are traded with fixed spreads, namely spreads set *a priori* and not decided by the market, equal to 100 or 500 bp in the United States and 25, 100, 500 or 1000 bp in Europe. Therefore, an upfront payment, that is the amount required to make the contract fair (CDS contract's value equal to zero), is paid and therefore the contracts' offset is made easier.

## 3.3 The Central Clearing Counterparty

The central clearing counterparty, in short CCP, is a financial market infrastructure (FMI) that takes on counterparty credit risk between parties in a transaction and provides

[8] Type of derivatives that enable to handle credit risk.

clearing and settlement services for trades in foreign exchange, securities, options, and derivative contracts. As central institution, it is usually referred to be "the buyer to every seller and the seller to every buyer", as said in [Russo et al., 2013].
CCP also performs risk checking among members and it must guarantee the overall stability of the system, preventing and managing the potential defaults among its members.
Our main focus is on the clearing and settlement activity of the CCP. In contrast with bilateral netting, in which every party settles its own position singularly with the other members, the CCP becomes the counterparty to both the buyer and the seller, and defines what is required from each party to reduce counterparty credit risk and to guarantee the settlement of the operation, even in case of defaults.
The number of members that a CCP manages varies a lot ; for example, as reported in [Bielecki et al., 2018], we go from ICE Clear US, that handles 28 members, to OCC that has more than 100.
Figure 4 focuses on the mentioned central role of CCP, which represents a connection point for the OTC network. In particular it typically connects large financial institutions to each other, which in turn have connections with the rest of the financial system, namely with smaller firms and, finally, the end-users.

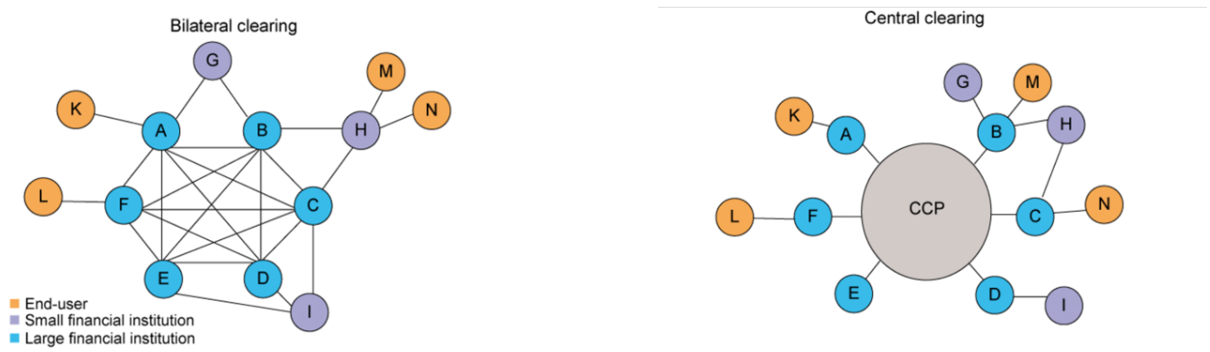



Figure 4: Bilateral v. Central Clearing. Source: Reserve Bank of Australia, Central Clearing of OTC Derivatives in Australia (June 2011)

The conditions for having a centrally cleared transaction are listed in [Gregory, 2020] and are summarized as follows:

- **Standardisation**. Products traded in the centrally cleared market must be standardised, in order to have an easier management of transactions.

- **Complexity**. Centrally cleared transactions only apply to vanilla financial products, which, citing [Gregory, 2020], are "easily and robustly valued". Complex products often cause problems in the margins calculations and make the overall system less efficient.

- **Liquidity**. Products must be liquid [9] in order to have exact pricing informations that can be used for the required margins and also because liquid products can be more efficiently replaced by the CCP in default scenarios.
- **Wrong-way Risk**. Centrally cleared products are not subject to wrong-way risk (WWR), that is defined by the International Swaps and Derivatives Association (ISDA) as the risk that arises whenever "exposure to a counterparty is adversely correlated with the credit quality of that counterparty".
- **Market Volume**. The market volume must always be high enough to let the central institution work without losses in the product development process.

CCP has another important task, that is to require to the clearing members a given amount of money, a collateral − whose relevance has increased since the financial crisis of 2007-08 − as a stability tool and an incentive to make the members behaving correctly. All the members' collaterals are pooled and controlled by the CCP, which in case of member defaults are used to absorb the loss among survived clearing members. Hence, central clearing allows mutualization of counterparty risk, while margins provide incentives to avoid counterparty risk. In particular, as stated in [Biais et al., 2016], margins are more worthwhile than CCP's equity capital, because they potentially eliminate moral hazard.
An important difference between central clearing counterparties and commercial banks is the following: while commercial banks are mainly in the business of taking risks, CCPs are in the business of sharing risks. As reported in [Faruqui et al., 2018], almost two thirds of over-the-counter interest rate derivative contracts are now cleared through central counterparties.

## 3.4 Default Waterfall

The *default waterfall*, also known as *loss waterfall*, is a pivotal concept in central counterparty risk management as it hopefully guarantees stability through capital requirements. The default waterfall represents the sequence of funds that must be used to cover the loss resulting from the default of one or more clearing members and it defines how counterparty risk is allocated and managed.
The waterfall requirements, claimed to each member in the market, are generally set according to the Basel Committee on Banking Supervision (BCBS) for Europe, while in US the regulatory work follows the indications of Federal Reserve System (FED) or the Securities and Exchange Commission (SEC).

[9] Liquid products are widely traded and therefore easily converted into cash.

In the default waterfall there are several funds, used for absorbing losses from defaulted clearing members, explained in [BCBS, 2014]. Here we summarize the mechanism.
The first two amounts which are used by the CCP in case of a member defaulting are the variation margin (VM) and the initial margin (IM).
The VM is the daily margin account with the CCP and is equal to the mark-to-market [10] (MtM) of the open positions [11]. The VM quantity is sometimes computed intraday, that is several times during a trading day. The VM is used to cover everyday market fluctuations of participants' portfolios.
The next layer of resources available for the CCP in the default waterfall is the IM, whose goal is to cover the risk exposure of the clearing members towards the central counterparty, with a time horizon usually of 5 or 10 days [12].

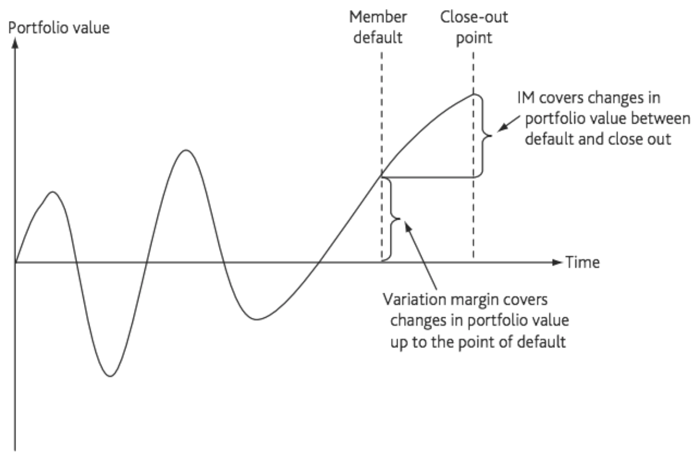



Figure 5: How VM and IM are used by CCP. Source: Nahai-Williamson *et al.* (2013)

Figure 5 shows the relation between VM and IM. We see that in case of default, first the loss is covered by the VM, while the IM, that enters in the waterfall in case the first margin VM is not sufficient, covers the loss resulting from the fluctuation of the portfolio value in the period between the default and the close-out point, namely the time when the portfolio is liquidated.
The variation margin is notably required by the CCP to cover the portfolio's net present value (NPV) of the clearing members derivatives portfolios, which means that the variation margin aims to cover the difference between the present value of the inflows and the outflows deriving from the specific portfolio of OTC-derivatives. In our specific case the only OTC-derivatives will be CDS. The IM is added in order to cover for the potential exposure of the CCP in the time period between the last VM is collected and the point at which the

[10] Method to measure security's value over time by considering the current market price.
[11] An open position is a trade that has been established, but not closed yet.
[12] This time horizon is called margin period of risk (MPOR)

CCP liquidates the defaulting member's position, i.e. the close-out point (see Figure 5).
Another contribution is required to form the pre-funded default fund (DF), usually collected every month. According to the *Cover 1/Cover 2* principle of the CPSS and IOSCO (see [Russo et al., 2013]), DF is defined as the necessary amount to cover for the default of the two members "that would potentially cause the largest aggregate credit exposure for the CCP in extreme but plausible market conditions".

Note that the pre-funded DF is usually defined in a qualitative way. As a matter of fact, see [Ghamami, 2015], while VM and IM depend on the value of the underlying asset, the pre-funded DF is often computed with an ad-hoc method based on a priori quantities.

Then, if VM and IM are not able to cover the total losses, there are other funds, the defaulter-pay layers, that enter in the waterfall.

The pre-funded default fund contribution is paid by all clearing members while the so called unfunded default fund ($uDF$) contribution is only paid by the clearing members that still are alive after all of the prefunded default funds have been wiped out.

Finally, there is the CCP's equity contribution E, often referred as "skin-in-the-game" such as in [Bielecki et al., 2018], as an incentive to use proper risk management, in which CCP pledges usually 20-25% of its total equity value.

Figure 6 below shows a scheme of the margins and the likelihood of reaching the different layers of the waterfall.

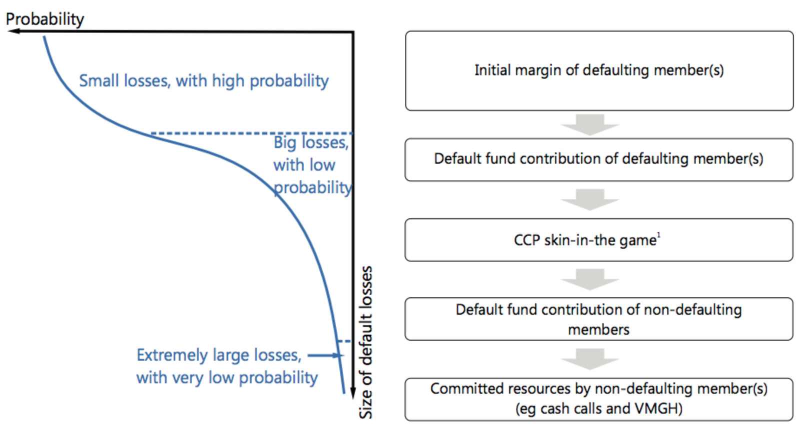



Figure 6: Default waterfall and probabilities. Source: Faruqui *et al.* (2018)

In [Nahai-Williamson et al., 2013], the authors state that the IM is less *collateral efficient* than the Default Fund, because of the way it is used by CCP. As a matter of fact, IM covers only for the losses of the member that posts it, whereas the DF are pooled into a common fund used to potentially absorb losses from all clearing members.

# 4 A Stochastic Model for Central Cleaning Counterparty

In this section we present the theory used for the default times in our CCP model, which are modelled using so called reduced form models via so called default intensities. First, we give a brief discussion of so called structural models and reduced form models.

## 4.1 Structured Models vs Reduced Form Models

In order to study entity's default time, the literature proposes several models, which can be grouped in *structural models* and *reduced form models*.
Concerning structural models, the central idea is that a default happens when the entity's assets value goes below a given threshold and therefore it is not able to pay back its debt. Thus, the model provides an intuitive, endogenous explanation of the default. In this type of models the asset value is typically represented by a diffusion process, which implies that the default times are predictable.
In reduced form models, often called intensity based models, the default time is modeled using an exogenous jump process [13] and in particular it is characterized by the use of so called default intensities $\lambda$, representing the instantaneous "rate" default. In reduced form models, the default time is a totally inaccessible random time, i.e. it is not predictable because of its jump nature (see, e.g. [McNeil, 2005]). In this thesis we follow the reduced form approach to describe the riskiness and default times of the CDS contracts and the clearing members. Then we address the topic of default intensities, analyzing the case of constant, deterministic and stochastic intensities. Then the dependence structure among the clearing members' default times is presented. Next we move to the starting assumptions of our model, analyzing the CDS pricing formula and finally we present the mathematics behind the default waterfall, dividing the margins of the waterfall in subsections.

## 4.2 Default Intensities

In this subsection we provide the construction of a default time according to the intensity approach.
The notation used is taken from [Herbertsson, 2019] and follows the theory presented in [Lando, 2004].
Let default intensity $\lambda_t$ be a non-negative stochastic process, i.e. $\lambda_t \geq 0$ for each $t \geq 0$, and

[13] A type of stochastic process in which the changes between discrete states, called jumps, happen at random arrival times.

let the market information available at time $t$ be represented by the $\sigma$-algebra [14] $\mathcal{F}_t$ [15]. We assume that

$$\mathbb{P}[\tau \in [t, t+\Delta t) | \mathcal{F}_t] \approx \lambda_t \Delta t, \text{ if } \tau > \text{t}, \tag{1}$$

This condition implies that the default intensities $\lambda_t$ models the rate of the conditional probability of default in the time period $[t, t+\Delta t)$, given that the default has not happened up to time $t$.
Let $(X_t)_{t\geq 0}$ be a d-dimensional stochastic process and assume that $\lambda_t$ is of the form $\lambda_t = \lambda(X_t)$ for some non-negative function $\lambda : \mathbb{R}^d \mapsto (0, \infty)$.
Let $E$ be an exponentially distributed random variable, with $\mathbb{E}[E] = 1$, independent on the process $(X_t)_{t\geq 0}$. The default time $\tau$ is defined as

$$\tau = \inf \left\{ t \geq 0 : \int_0^t \lambda(X_s)\, ds \geq E \right\}. \tag{2}$$

The definition of $\tau$ in Equation (2) is called the canonical construction of a default time and $\tau$ is thus the first time the integrated intensity process reaches the random level $E$. Given the construction of $\tau$ in Equation (2), one can prove that

$$P[\tau > t] = \mathbb{E}[\exp(-\int_0^\infty \lambda(X_s)\, ds)], \tag{3}$$

see e.g. [Lando, 2004], [McNeil, 2005] or [Herbertsson, 2019].
The intensity based approach for default times is often used in the credit literature and in the industry, as it has several advantages. In particular, the reduced form approach allows to calibrate the parameters from the market, while the firm value approach requires several firm's values, such as assets value or total debt, usually hard to know and to compute regularly.
In the following subsections, we will describe the intensity based approach under three different assumptions on the default intensities: constant, as done by [Bielecki et al., 2018] for the CDS contracts, deterministic and stochastic default intensities.

### 4.2.1 Constant Default Intensities

We briefly describe here the setting of [Bielecki et al., 2018], under constant default intensities $\lambda$. Under constant default intensities, the default time $\tau$ is given by

$$\tau = \inf \{ t \geq 0 : \lambda t \geq E \},$$

[14] A $\sigma$-algebra on a set X is a collection $\Sigma$ of subsets of $X$ that includes $X$ itself, is closed under complement, and is closed under countable unions.

[15] The filtration $\mathcal{F}_t$ represents the information available up to time $t$, generated by some stochastic process.

which allows for a simple closed formula for the default time is available, that is

$$\tau = \frac{E}{\lambda}.$$

For what concerns the default probability up to time $t$, the formula is

$$\mathbb{P}(\tau < t) = 1 - \exp\left(-\lambda\, t\right).$$

and we thus clearly see that in the case of constant default intensity $\lambda$ for $\tau$, then $\tau$ is exponentially distributed with density $f_\tau(t)$ given by $f_\tau(t) = \lambda \exp\left(-\lambda\, t\right)$ for $t \geq 0$.
In the next subsection, we change the basic assumption, developing a more realistic environment through a deterministic function $\lambda(t)$.

### 4.2.2 Deterministic Default Intensities

Before studying stochastic default intensities, we analyze an intermediate step, where $\lambda(t)$ is now a deterministic function of time $t$, where we write $\lambda(t)$ to distinguish it from the general stochastic process notation $\lambda_t$ used in the general setup in Equation (1) and Equation (2). Thus, Equation (2) can then be rephrased as

$$\tau = \inf\left\{ t \geq 0 : \int_0^t \lambda(s)\, ds \geq E \right\}$$

and, from Equation (3), the default probability up to time $t$ is then given by

$$\mathbb{P}(\tau < t) = 1 - \exp\left(-\int_0^t \lambda(s)\, ds\right). \tag{4}$$

### 4.2.3 Default Intensities by a CIR Process

We next present the case with stochastic default intensities and we will focus on so called CIR processes. Such CIR processes will in this thesis be used in order to simulate the proceeding of the default intensities for the clearing members. Let the default intensities $\lambda_t$ be driven by the so called Cox-Ingersoll-Ross (CIR) model, which means that $\lambda_t$ satisfies

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma_v \sqrt{\lambda_t} dW_t, \tag{5}$$

where $W_t$ in our case is a Brownian motion under the risk neutral measure $\mathbb{P}$ and $\kappa$, $\theta$ and $\sigma_v$ are all positive parameters.
The CIR process was first introduced in [Cox et al., 1985] and it is also known as a *one factor model*, because its behaviour is described through one source of market risk only.
We consider the CIR model because of its property of being non-negative. The preference for

the CIR model is straightforward since the intensities cannot be negative, because, referring to Equation (1), it would mean that the probability of default is negative.
The parameter $\kappa$ in Equation (5) represents the speed of the adjustment of the process towards the central location or long term value $\theta$ and $\sigma_v$ is the volatility.
Following [Broadie and Kaya, 2006], one can show that the process $\lambda_t$ has a transition law as follows

$$\lambda_t \stackrel{d}{=} \frac{\sigma_v^2(1-e^{-k(t-u)})}{4k} \chi_d^2 \left( \frac{4ke^{-k(t-u)}}{\sigma_v^2(1-e^{-k(t-u)})} \lambda_u \right), \tag{6}$$

where, given $\frac{4ke^{-k(t-u)}}{\sigma_v^2(1-e^{-k(t-u)})}\lambda_u = a$, $\chi_d^2(a)$ represents a noncentral chi-squared random variable, with $a$ as noncentrality parameter and $d = 4\theta k/\sigma_v^2$ degrees of freedom.
According to [Broadie and Kaya, 2006], given $\lambda_u$, $\lambda_t$ is distributed as $\frac{\sigma_v^2(1-e^{-k(t-u)})}{4k}$ times a noncentral chi-squared distribution with parameter $d$.
Moreover, the condition $2\kappa\theta > \sigma^2$ guarantees that $\lambda_t$ stays strictly positive, see [Cox et al., 1985].
Equation (6) is very useful when one needs to simulate $\lambda_t$, since we can avoid more problematic simulations of the intensities based on discretization of Equation (5), which for example can lead to negative values of the approximation to $\lambda_t$. Such negative values will never happen with Equation (6).
Let $\mathcal{F}_t$ be the total information available at time $t$, defined as

$$\mathcal{F}_t = \mathcal{G}_t^W \bigvee \mathcal{H}_t$$

where $\mathcal{G}_t^W = \sigma(W_s; s \le t)$ is the filtration generated by the Brownian motion $(W_s)_{s\ge 0}$ up to time $t$ and

$$\mathcal{H}_t = \sigma(\mathbb{1}_{\phi \le s}; s \le t)$$

is the filtration generated by the indicator function $\mathbb{1}_{\phi\le s}$ where we remind the reader that for any event $A$ then

$$\mathbb{1}_A = \begin{cases} 1, & \text{if } A \text{ happens;} \\ 0, & \text{if } A^c \text{ happens.} \end{cases}$$

Then, given the construction of $\tau$ in Equation (2), one can prove that for any $T \ge t$, the conditional survival probability $\mathbb{P}[\tau > T|\mathcal{F}_t]$ is given by

$$\mathbb{P}[\tau > T|\mathcal{F}_t] = \mathbb{1}_{\tau \ge t}\, \mathbb{E}\left[ \exp\left( -\int_t^T \lambda_s\, ds \right) | \mathcal{G}_t^W \right], \tag{7}$$

see, for example, [Lando, 2004], [McNeil, 2005], [Bielecki and Rutkowski, 2013], [Herbertsson, 2019].
Under a Cox-Ingersoll-Ross determination, one can prove that, see e.g. [McNeil, 2005],

$$\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s\,ds\right) \mid \mathcal{G}_t^W\right] = \exp(A(t,T) - B(t,T)\lambda_t), \tag{8}$$

where

$$\begin{aligned} A(t,T) &= \frac{2\kappa\theta}{\sigma_v^2}\ln\left(\frac{2\gamma e^{(\gamma+\kappa)(T-t)/2}}{(\gamma+\kappa)(e^{\gamma(T-t)}-1)+2\gamma}\right), \\ \gamma &= \sqrt{\kappa^2 + 2\sigma_v^2}\,, \\ B(t,T) &= \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa)(e^{\gamma(T-t)}-1)+2\gamma}\,. \end{aligned}$$

By taking the expected value of Equation (8) and using Equation (7) and the law of iterated expectations (see e.g [McNeil, 2005]), we immediately get

$$\mathbb{P}[\tau > T] = \exp(A(T) - B(T)\lambda_0)\,, \tag{9}$$

which provides the unconditional survival probability.
Note that the CIR is an affine process. An affine process is a stochastic process which has a characteristic function to the integrated process which looks like Equation (8). For a detailed analysis on affine processes, see [Duffie et al., 2000].
Starting from the CIR process just presented, we are able to construct the default times for the clearing members. In particular, in the next section we present the topic of modelling the members' default times so that the dependence is taken into account.

## 4.3 Dependent Default Times

In order to take into account default dependence among members, we use the so called *one-factor Gaussian copula model*. The notation and setup in this setting is taken from [Herbertsson, 2019] and similar outlines can also be found e.g. in [McNeil, 2005].
Let $F_i(t)$ be the default distribution of the $i$-th member so that $F_i(t) = \mathbb{P}[\tau_i \leq t]$.
Let $Y_i$ be a standard normal for each member $i$, namely with zero mean and unit variance, and let $Z$ be a standard normal independent of the variable $Y_i$. Then, the random variable $X_i$ is defined as

$$X_i = \sqrt{\rho}\,Z + \sqrt{1-\rho}\,Y_i, \tag{10}$$

where $\rho$ is a correlation coefficient and $\rho \in [0, 1]$. Next, the threshold $D_i(t)$ is introduced as

$$D_i(t) = N^{-1}(F_i(t)), \tag{11}$$

namely the threshold is the inverse of the default distribution for the member $i$. Finally we define the member's default time $\tau_i$ as

$$\tau_i = \inf \ \{ \ t > 0 : X_i \leq D_i(t)\} \tag{12}$$

so the default times $\tau_i$ are constructed as the first time $D_i(t)$ exceeds the random level $X_i$. It is important to say that this construction of default times is independent on the model considered. In our case, the distribution $F_i(t)$ will be given by the formulas presented in Section 4.2.3, about the CIR process. From Equation (12) we have that

$$\tau_i \leq t \Longleftrightarrow X_i \leq D_i(t)$$

and consequently

$$\mathbb{P}[\tau_i \leq t] = \mathbb{P}[X_i \leq D_i(t)].$$

In view of the construction of $D_i(t)$ in Equation (11), we have that

$$\begin{aligned}
\mathbb{P}[\tau_i \leq t] &= \mathbb{P}[X_i \leq D_i(t)] \\
&= N(D_i(t)) \\
&= N(N^{-1}(F_i(t))) \\
&= F_i(t)
\end{aligned}$$

and hence the construction of the default time $\tau_i$ is consistent with the marginal default distribution considered. Recalling the definition of $X_i$ in Equation (10) and the default time in Equation (12), we have that

$$\tau_i \leq t \Longleftrightarrow \sqrt{\rho}\, Z + \sqrt{1-\rho}\, Y_i \leq D_i(t)$$

which implies that the default time $\tau_i$ is driven by a common factor $Z$, that creates the default dependence among members, and a specific factor $Y_i$, unique for each member $i$. Concluding this section, we see that, conditional on the aforementioned common factor $Z$, the default times $\tau_i$ are independent, namely

$$\mathbb{P}[\tau_1 \leq t, \tau_2 \leq t, \ldots, \tau_m \leq t|Z] = \prod_{i=1}^{m} \mathbb{P}[\tau_i \leq t\,|Z].$$

The default times $\tau_i$ are thus conditionally independent given the variable $Z$.
Using the setting presented in this subsection, we are able to construct default times so that there is dependence between the clearing members.

## 4.4 CDS Pricing Formulas

In this section, we aim to compute the mark-to-market [16] (MtM) of the CDS contracts. We study the CDS pricing formulas under three different assumptions on the default intensities, namely constant, deterministic and stochastic intensities and in later sections CDS formulas will be used in our numerical studies.
As presented in the sections above, we model the time of default using an intensity-based approach and we first assume the default intensity is constant. Let $Rec$ be the recovery rate, namely the part of the notional is recovered in case of default, which is also assumed to be constant.
The CDS spread is is given by some constant $R(T)$ which we will denote by $R$, i.e. $R(T) = R$. The discount factor $\beta$ is assumed to be 1, therefore we have interest rates equal to zero, just as in the numerical examples in [Bielecki et al., 2018].
The default time of the reference entity underlying the CDS is represented by $\phi$. We define the default indicator process by $H_t = \mathbb{1}_{\phi \leq t}$, $t \geq 0$ and $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ represents its natural filtration. Recalling Section 3.2 about credit default swaps, the CDS value is the difference between the present value of the protection leg ($PVprot_t$) and the present value of the premium leg ($PVprem_t$). We will consider CDS with continuous premium payments, as done in [Bielecki et al., 2018], in contrast with the quarterly payments considered in Section 3.2. Let the loss given default, i.e. the amount of money lost when the borrower defaults be denoted by $LGD = 1 - Rec$. Then, the general formula for the CDS value with continuous premium payments is given by

$$PVprot_t = \mathbb{E}[LGD\,\mathbb{1}_{t<\phi\leq T} \mid \mathcal{F}_t] \tag{13}$$

$$PVprem_t = \mathbb{E}[(T \wedge \phi - t)\, R\, \mathbb{1}_{\phi>t} | \, \mathcal{F}_t] \tag{14}$$

$$S_t = PVprot_t - PVprem_t \tag{15}$$

[16] The mark-to-market of a derivative contract is the net present value of the cash flows under the risk neutral pricing measure discounted using current forward market interest rates.

where we remind the reader that $R(T) = R$ is the CDS spread. Hence, in view of (13), (14) and (15), we have that

$$S_t = \mathbb{E}[\, LGD \mathbb{1}_{t<\phi\leq T} - (T \wedge \phi - t)\, R\, \mathbb{1}_{\phi>t} |\, \mathcal{F}_t] \tag{16}$$

where, as in [Bielecki et al., 2018], we have used zero interest rates. We will consider CDS with continuous premium payments, as done in [Bielecki et al., 2018], in contrast with the quarterly payments considered in Section 3.2. Note that the assumption of continuous premium payments leads to an analytical expression for $PVprem_t$ that is more compact compared with the case of quarterly payments. In real life, quarterly payments are used and continuous payments are not used in trading and only exist in theory.
We here also remark that Equation (16) is a modified version of the corresponding CDS formula (C.1) in Appendix C in [Bielecki et al., 2018]. More specific, in Equation (C.1) in [Bielecki et al., 2018], the indicator function $\mathbb{1}_{\phi>t}$ is missing, which is wrong, since the premium payments are only done as long as the reference entity is alive at time $t$, that is as long as $\phi > t$. By construction of $R = R(T)$, it should hold that $S_0 = 0$, that is, in view of Equation (15), $R$ is set so that

$$PVprot_0 = PVprem_0. \tag{17}$$

Next note that in Equation (16), we have two sources of risks, that are the default process and the Brownian motion, with filtrations $\mathbb{H}$ and $\mathbb{G} = (\mathcal{G}_t^W,\, t \geq 0)$ (recall the notation in Equation (7) and the description of $\mathcal{F}_t$, $\mathcal{G}_t^W$ and $\mathcal{H}_t$). Still referring to Equation (16), in the CDS evaluation the CDS spread $R$ is assumed to be paid continuously, as done in [Bielecki et al., 2018].
For constant and deterministic $\lambda$, the only source of risk is represented by the indicator function $\mathbb{1}_{\phi>t}$. If the default intensities are constant or deterministic, filtration $\mathbb{F}$ coincides with the natural filtration of the default process $\mathbb{H}$, because there is no other source of risk. Therefore we have

$$S_t = \mathbb{E}[\, LGD \mathbb{1}_{t<\phi\leq T} - (T \wedge \phi - t)\, R(T)\, \mathbb{1}_{\phi>t} |\, \mathcal{H}_t] \tag{18}$$

and recall that $R(T)$ is set so that $S_0 = 0$ at time $t = 0$, that is

$$R(T) = \frac{LGD\, \mathbb{P}[\phi < T]}{\mathbb{E}[min(T, \phi)]}. \tag{19}$$

In the case with constant default intensities one can derive a closed formula for $S_t$, see in [Herbertsson, 2021a] and [Herbertsson, 2021b] so that

$$S_t = LGD\,\mathbb{1}_{\phi>t}\,[1-e^{-\lambda(T-t)}] \;-\; R(T)\,\mathbb{1}_{\phi>t}\left[\frac{(\lambda t+1)e^{-\lambda t}-e^{-\lambda T}}{\lambda e^{-\lambda t}} - t\right] \tag{20}$$

and

$$R(T) = LGD\,\lambda \tag{21}$$

where the relation (21) is the so called "credit triangle", connecting $\lambda$, $R(T)$ and $LGD$. For details about the derivation of (20) and (21) see [Herbertsson, 2021a] and [Herbertsson, 2021b]. Note that letting $t = 0$ in Equation (20) together with Equation (21) give that $S_0 = 0$, which is in line with Equation (17). We here also remark that [Bielecki et al., 2018] in the constant intensity case states a closed form expression for $S_t$, see in Equation (C.2) and (C.3) in Appendix C. However, we note that their formula for $S_0$ does not satisfy $S_0 = 0$, where we remind that $\mathbb{1}_{\phi>t}$ is missing in Equation (C.1) in [Bielecki et al., 2018]. In fact, in the constant intensity case $\lambda(t) = \lambda$, by using Equation (21), one can show that

$$PVprot_t = PVprem_t = \mathbb{1}_{\phi>t}LGD(1-e^{-\lambda(T-t)}) \;\text{ for }\; 0 \le t \le T \tag{22}$$

and this in Equation (20) will then yield that

$$S_t = 0 \;\text{ for }\; 0 \le t \le T \tag{23}$$

in the constant default intensity case. For details, see [Herbertsson, 2021a] and [Herbertsson, 2021b]. Equation (23) can also be obtained by inserting Equation (21) into Equation (20) with some trivial computations. In the case with general deterministic default intensities $\lambda(t)$ one can prove that $S_t$ is given by, see [Herbertsson, 2021a] and [Herbertsson, 2021b], that

$$S_t = LGD\,\mathbb{1}_{\phi>t}\,[1-e^{\int_t^T \lambda(s)ds}] - R(T)\mathbb{1}_{\phi>t}\left[\frac{\int_t^T s\lambda(s)e^{\int_0^s \lambda(u)du}ds + T(1-\int_0^T \lambda(s)e^{\int_0^s)\lambda(u)du}ds}{e^{\int_0^t \lambda(s)ds}} - t\right] \tag{24}$$

and

$$R(T) = \frac{LGD(1-e^{(-\int_0^T \lambda(s)ds)})}{\int_0^T s\lambda(s)e^{-\int_0^u \lambda(u)du}ds + T(1-\int_0^T \lambda(s)e^{-\int_0^s \lambda(u)du}ds)} \tag{25}$$

for more details, see [Herbertsson, 2021a] and [Herbertsson, 2021b].
Again, note that $S_t$ in Equation (20) satisfies $S_0 = 0$ which is in line with Equation (15). Equation (16) will be used from now on for the computations of the MtM value of CDS contracts.

## 4.5 The Default Waterfall

In the next section we aim to compute VM, IM, DF and its allocation. We now introduce some further notation. The IM and VM are computed at discrete times $\mathcal{T}^f = \{0, \delta_f, 2\delta_f, ..., T - \delta_f\}$, with $\delta_f$ equal to one business day, while DF is computed according to the discrete tenor $t \in T^F = \{T_0, T_1, T_2, ..., T - \Delta_f\}$ and where $T_j = j\Delta_f$, with $j = 0, 1, 2, ...$ and with $\Delta_f$ as a multiple of $\delta_f$, is made to have 30 business days length. The MPOR $\delta$ is the period during which, in case of member's default, its portfolio is either liquidated or auctioned. In our case we have $\delta = 10$ business days.

### 4.5.1 Variation Margin

The VM of each member $i$ at time $t_k$ is the mark-to-market (MtM) of the member's portfolio at time $t_{k-1}$, i.e.

$$VM^i_{t_k} = V^i_{t_{k-1}}, \tag{26}$$

for every $k$, where $V^i_{t_{k-1}}$ is the $i$-th member's portfolio value at time $t_{k-1}$.
In practice, only values above a specific threshold are transferred from the members to the CCP. For the VM layer, computations are very straightforward and do not require further reasoning. Numerical quantification of variation margins are explained in detail in Section 5.

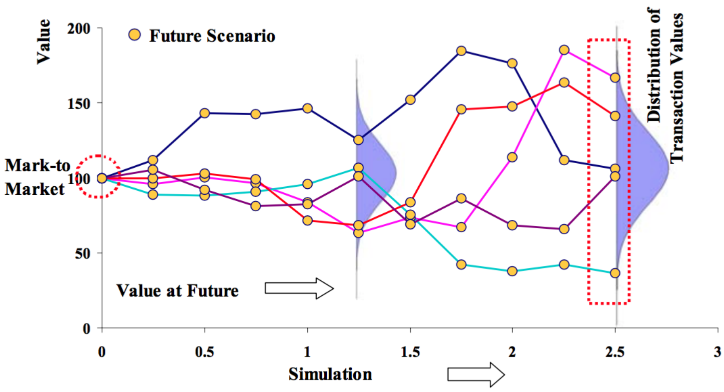



Figure 7: Simulations of MtM. Source: Lomibao and Zhu (2005)

Following [Corelli, 2019], the fundamental steps for computing the portfolio value of member $i$ (and consequently $VM^i_{t_k}$) are:

- **Scenario Generation**. Future market factors are simulated.
- **Instrument Valuation**. Based on the market values, instrument valuation is performed for each date time and each simulation.

- **Portfolio Aggregation**. All the instruments are pooled and the portfolio's value $V_i$ is computed.

### 4.5.2 Initial Margin

The first amount to be computed is the net exposure of the CCP towards each clearing member. The amount $X^i_{t_k}$ captures this quantity, subtracting to the portfolio value of clearing member $i$ the sum of possible dividends $D^i$ and the variation margin $VM^i$, that is

$$X^i_{t_k} = \beta^{-1}_{t_k}\beta_{t_{k+\delta}}V^i_{t_{k+\delta}} + \beta^{-1}_{t_k}\sum_{u=t_k}^{t_{k+\delta}}\beta_u D^i_u - VM^i_{t_k} \tag{27}$$

where all the factors $\beta$ represent discount factors that in our case will be equal to 1, considering that we hypothesize zero interest rates, as pointed out in Section 4.4.
Then, according to [Acciaio and Penner, 2011] and [Bielecki et al., 2017], a coherent risk measure [17] is applied to the outcome of Equation (27) and, finally, the initial margin is computed, that is

$$IM^i_{t_k} = \psi_{t_k}(-(X^i_{t_k})^+) \tag{28}$$

where $\psi_{t_k}$ is the coherent risk measure aforementioned. We consider a time horizon of 10 business days for the initial margins' computations.
Recalling the notation of [Bielecki et al., 2018], it is noteworthy focusing on the amount attached to the risk measure $\psi$, i.e. the quantity $-(X^i_{t_k})^+$. It means that, starting from the CCP's exposure, we consider only the losses in the distribution, namely the positive values of $X^i_{t_k}$, and then we change the sign, in order to have only negative values that represent the loss for the CCP. That means we are interested in the distribution's left tail of $-(X^i_{t_k})^+$.

**Expected Shortfall as Coherent Risk Measure** Two of the most widely used risk measures in practice, which are also suggested in Basel accords, are value at risk (VaR) and Expected Shortfall (ES). VaR does not satisfy sub-additivity in general and hence is not coherent. ES is obtained in a sense by averaging values at risk that exceed a certain confidence level and it is well known to be coherent and hence applies to our scope.

[17] A risk measure is said to be coherent if, according to [Artzner et al., 1999], the following four properties are satisfied:

- Monotonicity: for $L_1 \leq L_2$, we have $\rho(L_1) \leq \rho(L_2)$
- Sub-additivity: $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$
- Positive homogeneity: $\rho(aL) = a\rho(L)$
- Translation invariance: $\rho(L + a) = \rho(L) + a$

Our goal is to provide an empirical evaluation of VaR and ES from data. To this we will first need to estimate the empirical distribution of X and its $\alpha$-quantile, $q_\alpha$
The empirical distribution is given by

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{[X_k,\infty)}(x).$$

Hence, the VaR at level $\alpha$ is given by

$$q_\alpha(F_n) = \inf\{x \in \mathbb{R} : F_n(x) \geq \alpha\} = F_n^{\leftarrow}(\alpha),$$

where $F_n^{\leftarrow}(\alpha)$ indicates the pseudo-inverse [18] of the empirical distribution of $x$.
Sorting the sample $X_1, ..., X_n$ from the highest to the lowest value, namely $X_{1,n} \geq ... \geq X_{n,n}$, the empirical $\alpha$ quantile is given by

$$q_\alpha(F_n) = X_{[n(1-\alpha)]+1,n}$$

which in turn implies that the empirical ES is given by

$$\hat{ES}_\alpha(F) = \frac{\sum_{k=1}^{[n(1-\alpha)]+1} x_{k,n}}{[n(1-\alpha)]+1}.$$

### 4.5.3 Default Fund

In this section, we present the general framework for the total default fund's computations and next a specific part is dedicated to the allocation scheme.
The default fund is the amount formed by all members' contributions via a mutualization of the loss in order to prevent the default of one or more of the parties. The default fund layer is aimed to cover only extreme losses and it is used only when the initial margin is not sufficient. Figure 8 shows the dynamic elapsing the IM and the DF.

[18] Suppose $f : A \to B$ is a function with range R. A function $g : B \to A$ is a pseudo-inverse of $f$ if for all $b \in R$, $g(b)$ is a preimage of $b$.

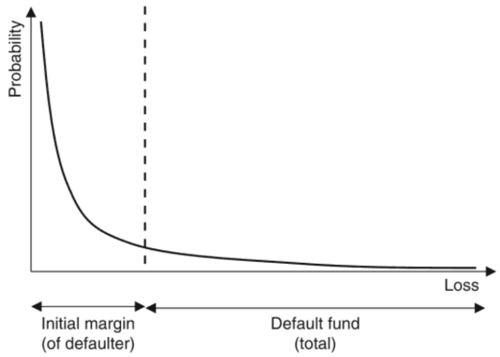



Figure 8: Relation between IM and DF. Source: Gregory (2020)

Looking at Figure 8, for small losses the IM is able to cover everything, while, in case of higher, rarer losses, the DF enters in the waterfall. According to [Bielecki et al., 2018] the exposure of the CCP towards the $i$-th member at time $T_k$, net of VM and IM is

$$EP^i_{T_k} = \beta^{-1}_{T_k} \sum_{t_m=T_k+\delta_f}^{T_k+1} (\beta_{t_m+\delta}[V^i_{t_m+\delta} - \hat{V}^i_{t_m+\delta}] + \sum_{u=t_m}^{t_m+\delta} \beta_u D^i_u \\ -\beta_{t_m}[VM^i_{t_m} + IM^i_{t_m}])^+ \mathbb{1}_{\tau_i=t_m} \tag{29}$$

where $V^i_{t_m+\delta} - \hat{V}^i_{t_m+\delta}$ in Equation (29) represents the actual portfolio value lost by the CCP, considering that $\hat{V}^i_{t_m+\delta} = Rec\, V^i_{t_m+\delta}$, where $Rec$ is the recovery rate. Furthermore, recall that from the definition of discrete tenor $t \in T^F$ in Section 4.4, we know that $EP$, the exposure just presented in Equation (29), is computed every $T_k$, namely 30 business days.
As stated in [Bielecki et al., 2018], in the current regulatory practice the exposure does not take into account the possible amount recovered during the margin period of risk, therefore we do not consider the recovery rate $Rec$ in our study and hence the actual $EP$ value is

$$EP^{i,reg}_{T_k} = \beta^{-1}_{T_k} \sum_{t_m=T_k+\delta_f}^{T_k+1} (\beta_{t_m+\delta} V^i_{t_m+\delta} - \beta_{t_m}(VM^i_{t_m} + IM^i_{t_m}))^+ \mathbb{1}_{\tau_i=t_m}. \tag{30}$$

For the sake of simplicity, from now on, we will always refer to $EP^{i,reg}_{T_k}$ as $EP^i_{T_k}$.
The values $VM^i_{t_m}$ and $IM^i_{t_m}$ are computed daily, and the indicator function $\mathbb{1}_{\tau_i=t_m}$ takes into account the possible default of the clearing member $i$, namely it is equal to 1 in case the default occurs and 0 otherwise. In particular, be reminded that member's default time $\tau_i$ is constructed according to Equation (12) and in general to the procedure explained in Section 4.3.
Following [Bielecki et al., 2018], all individual exposures $EP^i_{T_k}$ are added together in order to create the total exposure $EP_{T_k}$, whereby the total DF at time $T_k$ is calculated, namely

$$DF_{T_k} = \eta_{T_k}(-\sum_{i \in I} EP^i_{T_k}), \tag{31}$$

where $\eta_{T_k}$ represents a coherent, dynamic time-consistent monetary risk measure, as in the previous case with the initial margins.

### 4.5.4 Default Fund Allocation

[Bielecki et al., 2018] proposes an alternative criterion with respect to the classic practice, which involves the allocation of the total default fund proportionally to the initial margins. The problem is that, applying the risk measure $\eta_{T_k}$ to the single $CM_i$, we would not satisfy the *full allocation property*, which requires that the total amount of the default fund is allocated among the clearing members and it must run out in the allocation. Put in other words,

$$\sum_{i \in I} DF^i_{T_k} = DF_{T_k}.$$

Therefore we need a different allocation scheme than a simple application of a risk measure to all the $EP_i$s. The scheme proposed in [Bielecki et al., 2018] first assumes the following representation for the risk measure $\eta$, namely

$$\eta_t(X) = \text{ess sup}\, \mathbb{E}^Q[-X|\mathcal{F}_t], \tag{32}$$

where $\mathcal{F}_t$ is the available $\sigma$-algebra at time $t$, $Q \in \mathcal{Q}$, $\mathcal{Q}$ is a set of probability measures which are absolutely continuous with respect to $P$ [19] and ess sup is the essential supremum of $\mathbb{E}^Q[-X|\mathcal{F}_t]$. It is clear that if we consider $X$ instead of $-X$ in Equation (32), we will have an essential infimum. Essential infimum and essential supremum have basically the same meaning of infimum and supremum in mathematics, but adapted to measure theory. [Bielecki et al., 2018] next proposes that the allocation in the default fund is given by

$$\eta_{T_k}(-\sum_{i \in I} EP^i_{T_k}) = \mathbb{E}[Z^*_{T_k} \sum_{i \in I} EP^i_{T_k} | \mathcal{F}_{T_k}].$$

Here $Z^*$ is the Radon-Nikodym derivative of $Q^*$ with respect to $P$, where $Q^*$ is the measure that provides the essential infimum, and hence we have $Z^*_{T_k} = dQ^*_{T_k}/dP$ [20]. Therefore, according to the default fund allocation scheme, we have

[19] A probability measure $Q$ is said to be absolutely continuous with respect to $P$ if, for every event $A \in \mathcal{F}_t$ such that $P(A) = 0$, then $Q(A) = 0$.

[20] $Z^*$ is a strictly positive martingale that provides the density of the measure $Q^*$ with respect to $P$.

$$DF_{T_k} = \sum_{i \in I} \mathbb{E}[Z^*_{T_k} EP^i_{T_k} | \mathcal{F}_{T_k}]$$

and for clearing member $i$, we then get

$$DF^i_{T_k} = \mathbb{E}[Z^*_{T_k} EP^i_{T_k} | \mathcal{F}_{T_k}]. \tag{33}$$

The allocation rule presented above satisfies the full allocation property, that is

$$DF_{T_k} = \sum_{i \in I} DF^i_{T_k}.$$

**Radon-Nikodym Derivative for the Allocation** We dedicate a specific paragraph to the Radon-Nikodym derivative and how it is constructed in the model following the notation and setup of [Bielecki et al., 2018]. The significance level is indicated as $\beta$. The martingale $Z^*$ is given by

$$Z^*_{T_k} = \frac{1}{\beta}(\mathbb{1}_{X_{T_k} < q^{\pm}_{\beta}(X|\mathcal{F}_{T_k})} + \epsilon \mathbb{1}_{X_{T_k} = q^{\pm}_{\beta}(X|\mathcal{F}_{T_k})}), \tag{34}$$

where $q^{\pm}_{\beta}(X|\mathcal{F}_{T_k})$ is the conditional upper/lower $\beta$ quantile of $X$, that in our case is the loss, at time $T_k$, which is the sample of the $n$ simulations performed.
The quantity $\epsilon$ is equal to

$$\epsilon = \begin{cases} 0, & \text{if } \ P(X_{T_k} = q^{\pm}_{\beta}(X|\mathcal{F}_{T_k})) = 0 \\ \frac{\beta - P(X_{T_k} < q^{\pm}_{\beta}(X|\mathcal{F}_{T_k}))}{P(X_{T_k} = q^{\pm}_{\beta}(X|\mathcal{F}_{T_k}))}, & \text{otherwise} \end{cases}$$

Since $\epsilon$ is equal to 0 in the event $X_{T_k} = q^{\pm}_{\beta}(X|\mathcal{F}_{T_k})$, we can rewrite Equation (34) omitting the second indicator function $\mathbb{1}_{X_{T_k} = q^{\pm}_{\beta}(X|\mathcal{F}_{T_k})}$.
As pointed out in [Bielecki et al., 2018], with this choice for $Z^*$, the allocation scheme coincides with the Euler allocation scheme for $\eta_t(\,\cdot\,) = ES_{\beta}(\,\cdot\,|\mathcal{F}_t)$ [21].

[21] The Euler allocation scheme in case of ES has a closed form (see [Tasche, 2007]), that is:
$\mathrm{ES}_{\beta}(X_i|X) = -\beta^{-1}\,\mathbb{E}[X_i\,\mathbb{1}_{X \le q_{\beta}(X)}]$.

# 5 Numerical Analysis

This section is dedicated to the numerical implementation of our model. In order to compute the margins of the default waterfall, first we present the computations of the default intensities for the CDS contracts and the CM. The CDS pricing and the exposure of the CCP towards the members are then shown. Next, the computations of the default waterfall margins are presented, with a focus on the $DF/IM$ ratio. Finally, some sensitivity analysis is performed to check the robustness of the model. The numerical analysis here is done using the software MATLAB® .
In order to compute the amounts required to the clearing members we followed [Bielecki et al., 2018].

## 5.1 Default Intensities and Default Times

First of all, we start with the default intensities of the CDS contracts and the clearing members. We present the numerical results deriving from the simulation of the Cox–Ingersoll–Ross (CIR) model for the default intensity process presented in Section 4.1.3. See, for instance, [Cox et al., 1985] and [Broadie and Kaya, 2006] for the theoretical part.

### 5.1.1 CDS Contracts

This section analyzes a modification from the proceeding presented in [Bielecki et al., 2018], where, for the CDS contracts, they use constant default intensities, while we use CIR intensities.
In order to have a better reasoning on the impact of the hypothesis of the stochastic default intensities, we consider default intensities being constant, deterministic and then stochastic. For what concerns the deterministic default intensities, we consider their proceeding being described first by a simple increasing functions and then by periodic functions.
As said before, we consider four different CDS contracts as done in [Bielecki et al., 2018]. In all the four settings, we consider default intensities $\lambda(0)$ at time 0 for the reference entities, as given in Table 1

| $CDS_1$ | $CDS_2$ | $CDS_3$ | $CDS_4$ |
|---|---|---|---|
| 0.02 | 0.03 | 0.045 | 0.075 |

Table 1: $\lambda(0)$ for every CDS contract.

In the constant default intensities hypothesis the values is obviously the same for the whole time period. The reasoning behind the values in Table 1 is that we pick the different $\lambda(0)$ in order to have, within one year, realistic default probabilities for the four CDS contracts and we get the following results, rounded to the fourth decimals

$$\begin{aligned}
\mathbb{P}[\tau_1 < T] &= 0.0198, \\
\mathbb{P}[\tau_2 < T] &= 0.0296, \\
\mathbb{P}[\tau_3 < T] &= 0.0440, \\
\mathbb{P}[\tau_4 < T] &= 0.0723,
\end{aligned} \tag{35}$$

with $T = 1$ year, where we used Equation (4) on p.25. Then, we hold these as starting values for the other three cases.
The proceedings for the first three cases, namely constant, increasing and periodic default intensities are now presented. The three types of default intensities that will be used in the CDS contracts have the following three functions

$$\lambda(t) = \lambda, \tag{36}$$

$$\lambda(t) = \lambda + a\,t, \tag{37}$$

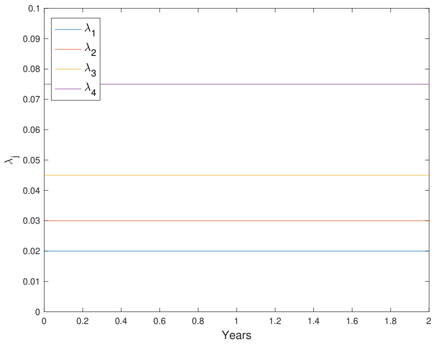
$$\lambda(t) = \lambda + b \sin(c\,t) \tag{38}$$

where the parameters $a$, $b$, $c$ have been calibrated to match the default probabilities in (33). The constant case in (36) is easily obtained via Equation (21), that is $\lambda = \frac{R(T)}{LGD}$. Then, we show in Table 1 the starting values for $\lambda$ used in all the three cases, constant, increasing and periodic default intensities. Then, for what concerns increasing and periodic default intensities, we calibrated our model in order to have, after one year, the default probabilities shown in (33). In Table 2 we present, besides the parameters $\lambda(0)$ already shown, the parameters $a$, $b$ and $c$ used.

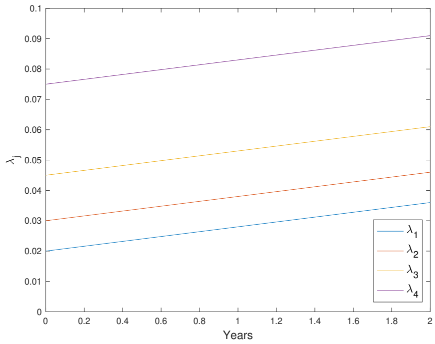
| | $\lambda(0)$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $CDS_1$ | 0.02 | 0.008 | 1 | 0.01 |
| $CDS_2$ | 0.03 | 0.008 | 1 | 0.01 |
| $CDS_3$ | 0.45 | 0.008 | 1 | 0.01 |
| $CDS_4$ | 0.75 | 0.008 | 1 | 0.01 |

Table 2: Parameters for the functions (34), (35) and (36) so that (33) is satisfied

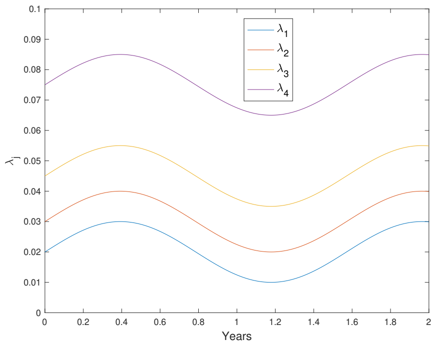
Then, the graphs for the constant,increasing intensities and periodic default intensities are shown graphically.




(a) Constant default intensities.




(b) Increasing default intensities.




(c) Periodic default intensities.

Figure 9: Default intensities trend depending on the assumption about the default intensities.

Looking at (c) in Figure 9, we see that the cycle is concluded around at $T = 1.5$. We will further study the different assumptions, when we will compute the CDS spreads and the MtM values. Then, in the next section the simulations of CIR processes for the CM are made.

### 5.1.2 Clearing Members

We assume that the clearing members have default intensities that follow CIR processes as discussed in Section 4.2.3. We will calibrate the CIR parameters for each CM via default probabilities obtained from rating agencies. This implies that we set the parameters such that, depending on the rating of clearing member $i$, its default probability within a year is coherent with the rating. In the calibration, we use Equation (9) and from this we got the parameters for the CIR process. Figure 10 displays displays the default probabilities, taken from a report by Scope, one of the major European rating agencies, which we use in our calibration of the CIR parameters.

| Scope | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| AAA | 0.003% | 0.007% | 0.015% | 0.029% | 0.049% |
| AA+ | 0.003% | 0.010% | 0.025% | 0.047% | 0.078% |
| AA | 0.007% | 0.025% | 0.056% | 0.101% | 0.159% |
| AA- | 0.014% | 0.032% | 0.073% | 0.135% | 0.216% |
| A+ | 0.024% | 0.056% | 0.128% | 0.227% | 0.351% |
| A | 0.041% | 0.107% | 0.211% | 0.345% | 0.505% |
| A- | 0.071% | 0.164% | 0.321% | 0.520% | 0.754% |
| BBB+ | 0.122% | 0.341% | 0.612% | 0.924% | 1.270% |
| BBB | 0.211% | 0.574% | 0.998% | 1.467% | 1.975% |
| BBB- | 0.364% | 1.066% | 1.846% | 2.667% | 3.516% |
| BB+ | 1.142% | 2.284% | 3.426% | 4.559% | 5.682% |
| BB | 1.778% | 3.557% | 5.335% | 7.053% | 8.709% |
| BB- | 2.541% | 5.082% | 7.624% | 10.027% | 12.294% |
| B+ | 4.604% | 9.208% | 12.559% | 15.381% | 17.903% |
| B | 5.941% | 11.882% | 16.063% | 19.491% | 22.490% |
| B- | 9.232% | 17.941% | 23.403% | 27.554% | 30.997% |
| CCC | 24.731% | 39.066% | 47.904% | 54.528% | 59.961% |
| CC | 47.076% | 61.650% | 71.569% | 79.703% | 86.929% |
| C | 83.919% | 100.000% | 100.000% | 100.000% | 100.000% |

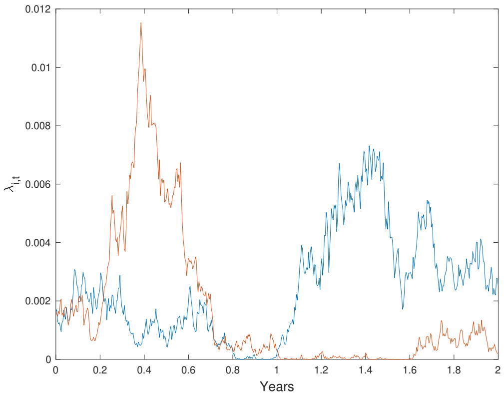
Figure 10: Default cumulative probabilities of a firm for 5 years depending on the rating. Source: Scope (2019)

For our computations, we pick ratings BBB, B, B- and CCC for our clearing members and the calibration then implies the following CIR parameters $\kappa$, $\theta$, $\sigma$ and $\lambda_0$ displayed in Table 3.

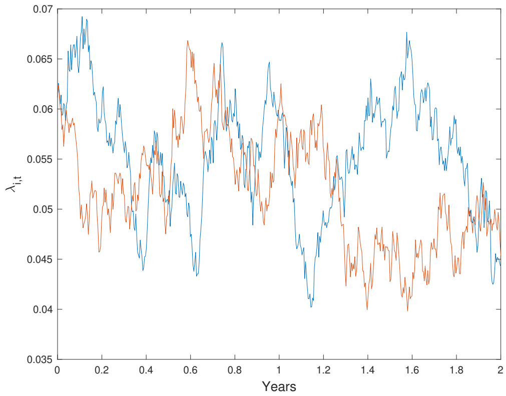
| | $\kappa$ | $\theta$ | $\sigma$ | $\lambda_0$ |
|---|---|---|---|---|
| BBB | 0.0097 | 0.0995 | 0.1003 | 0.0016 |
| B | 0.0100 | 0.0998 | 0.1001 | 0.0612 |
| B- | 0.0100 | 0.1000 | 0.1000 | 0.0970 |
| CCC | 0.0085 | 0.1009 | 0.0993 | 0.2854 |

Table 3: Parameters for the CIR processes depending on the rating.

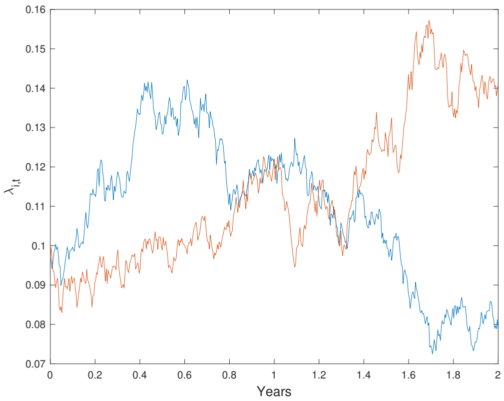
Clearing members have the following ratings: $CM_1$ and $CM_5$ have a BBB rating, $CM_2$ and $CM_6$ a B, $CM_3$ and $CM_7$ a B- and finally $CM_4$ and $CM_8$ have a CCC rating.
Figure 11 displays one (of our $10^5$ simulated) trajectories of the CIR default intensities $\lambda_{i,t}$ for our 8 clearing members ($i$=1,2,...,8) with parameters given as in Table 3.



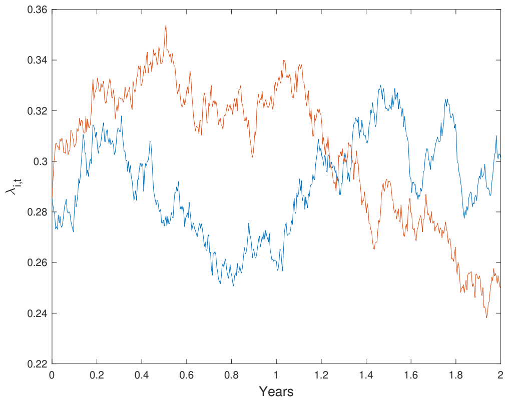

(a) $CM_1$ and $CM_5$, rating BBB.




(b) $CM_2$ and $CM_6$, rating B.




(c) $CM_3$ and $CM_7$, rating B-.




(d) $CM_4$ and $CM_8$, rating CCC.

Figure 11: CIR process for the clearing members' default intensities.

The four panels in Figure 11 represent the four different classes of rating in which the clearing members are organized. The biggest difference looking at the four graphs is in the starting values $\lambda_{i,0}$ of the processes, coherent with the different CMs' riskiness.
Recalling Section 4.2 in the CIR setting, we can obtain the default distribution $F_i(t)$, required to have $D_i(t)$ in Equation (10) and, finally, the dependent default times $\tau_i$ are computed according to Equation (11).

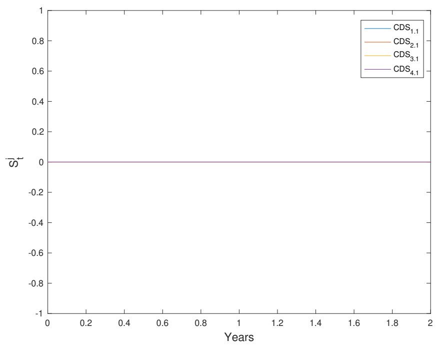
## 5.2 Mark-to-Market Value

Before studying the MtM values, we present the CDS spreads obtained in our different types of deterministic default intensities discussed in Section 5.1.1.
The different assumptions on the default intensities behaviours have an impact on the corresponding CDS spreads computed in all the settings.
Hence, we show the CDS spreads for the three deterministic default intensities cases specified in Equation (36), (37) and (38).

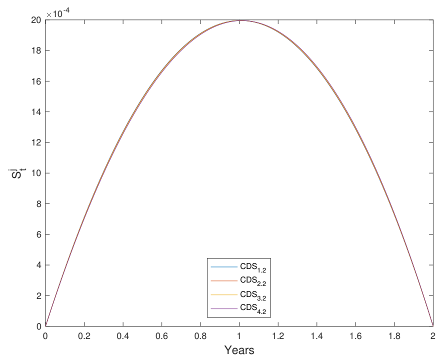
| $\lambda$ | $CDS_1$ | $CDS_2$ | $CDS_3$ | $CDS_4$ |
|---|---|---|---|---|
| Constant | 0.0100 | 0.0150 | 0.0225 | 0.0375 |
| Increasing | 0.0140 | 0.0189 | 0.0264 | 0.0414 |
| Periodic | 0.0107 | 0.0157 | 0.0232 | 0.0382 |

Table 4: CDS spreads depending on the assumption on the default intensities behaviour.

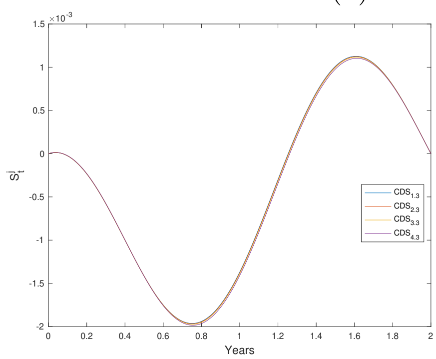
Table 4 shows that the highest CDS spreads are in the case of increasing default intensities. This is due to the higher overall riskiness of the contracts, that have the same starting values as in the other cases, but in this case default intensites are intended to increase constantly. The different cases for the default intensities have also an impact on the MtM values $S_t$. In Section 4.4 we displayed explicit or semi-explicit formulas for $S_t$ with the different equations for two cases, namely constant and deterministic default intensities.
Next, we present the graphs for $S_t$ in three different models for $\lambda(t)$ given of the upfront payments. We plot the graphs below in order to have an insight on the different proceeding of the MtM values depending on the assumptions made on the default intensities behaviour.




(a) Constant default intensities.




(b) Increasing default intensities.




(c) Periodic default intensities.

Figure 12: The CDS MtM values $S_t$ for a 2-year CDS with $\phi > 2$ for the different default intensities.

All the three graphs in Figure 12 show $S_t = 0$ for $t = 0$ and $t = 2$, namely at the beginning and at the end of the protection period for the CDS which has a maturity of $T = 2$ years. Also recall that from Equation (23) $S_t = 0$ for 0, which explain the flat lines shown in Figure 12 (a).

## 5.3 CCP Exposure

In this section the matrix concerning the positions of the CCP towards the clearing members is presented. Let $H_{ij}$ be the matrix

$$H_{8\times 4} = \begin{pmatrix} 6 & -1 & -3 & -2 \\ 2 & -1 & 2 & -3 \\ 1 & 3 & -1 & -3 \\ 5 & 1 & -1 & -5 \\ -5 & 1 & 1 & 3 \\ -3 & 1 & -1 & 3 \\ -7 & -1 & 5 & 3 \\ 1 & -3 & -2 & 4 \end{pmatrix}, \tag{39}$$

where $i$ represents the $i$-th CM, while $j$ is the $j$-th contract.
The matrix $H$ in Equation (39) shows the position of each CM for every contract, with the perspective of the clearing house. The positive values in the matrix represent the CCP is buying the contract (long position), while the negative values mean that the CCP is selling (short position). In general, from now on, all the clash flows are seen from the CCP's perspective and therefore a positive value corresponds to an inflow for it, while a negative value means an expense. Since the CCP runs a matched order book it must hold that

$$\sum_{i\in I} H_{ij} = 0. \tag{40}$$

In the following section, each subsection focuses on a specific part of the default waterfall. In addition, there is a part for the default fund allocation and its computations and besides, there is a specific section concerning the $DF/IM$ ratio.

## 5.4 Default Waterfall

In the following subsections, we adopt other assumptions of [Bielecki et al., 2018]. We assume 252 business days in a year, the fundamental unit of time is $\delta_t$, equal to one business day and the margin period of risk (MPOR) , represented by $\delta$, is equal to 10 business days. For what concerns the DF's time window $\Delta_t$, it is 30 business days. The recovery rate is $Rec = 0.5$ and the default intensities follow a CIR process.

It is important to note that the margins depend also on the position, long or short, assumed by the clearing member toward the CCP.

### 5.4.1 Variation Margin

We begin with the computations of the variation margin (VM). The portfolio value at time $t_k$ for each of clearing member $i$ is given by

$$V^i_{t_k} = H_{ij} \times S^j_{t_k}, \quad i = 1, ..., 8 \tag{41}$$

where $S^j_{t_k}$ denotes the MtM value of the CDS contract for entity $j$ and $S^j_{t_k}$ is given by the formulas in (20) and (22) for different parametrizations of $\lambda(t)$ given in (34), (35) and (36). Figure 13 represents the portfolio values at time $t$ for each of the eight clearing members given that at time $t$ all the reference entities are still alive. The default intensities used from now on for the CDS contracts are periodic, following Equation (38).

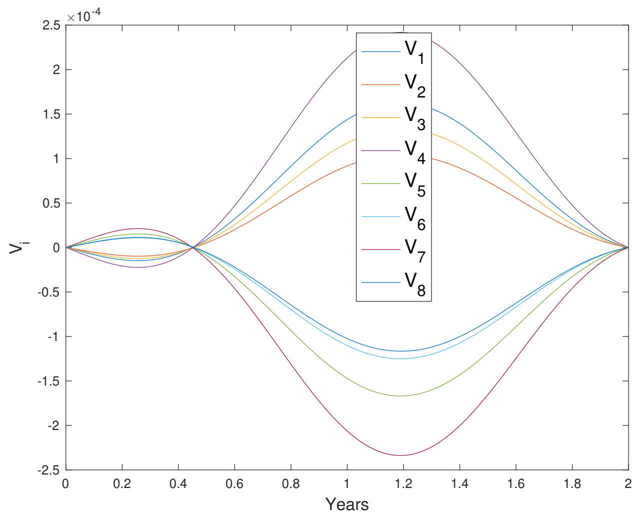



Figure 13: Portfolio value for every CM obtained via Equation (38) with same parameters as in Table 2.

It is worth saying that $VM^i_{t_k}$ cannot be negative, because this would mean the clearing member has to receive an amount by the CCP, that is impossible. Therefore, variation margins' values are the positive part of the portfolio values, that is

$$VM^i_{t_k} = max(V^i_{t_{k-1}}, 0) \tag{42}$$

and Figure 14 displaying $VM^i_{t_k}$ is obtained from Figure 13 via Equation (38).

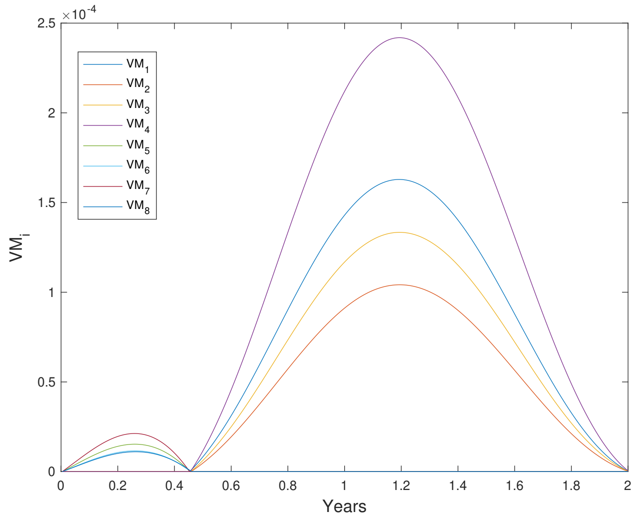



Figure 14: Variation margin for every CM obtained via Equation (38) with same parameters as in Figure 13.

Figure 14 shows the proceeding of the individual VM for all the clearing members and, as we said before, we consider only positive values for this margin, otherwise this would mean the CCP has to pay something to the members. This fact explains the graph with only values greater or equal to zero.

### 5.4.2 Initial Margin

In this subsection we will follow the notation and setup as in [Bielecki et al., 2018]. Be reminded that all the CDS contracts have maturity of 2 years. In order to compute the initial margin ($IM$), we start from the mark-to-market's empirical distribution of the $j$-th CDS contract. Recall that, as in our CDS computations in Section 4.4, we assume, without loss of generality, that interest rate is equal to zero. Let $\check{S}_t^j$ denote the difference between the MtM value of a CDS contract on reference entity $j$ at time $t_{k+\delta}$ and the MtM value at time $t_{k+1}$, namely the CCP exposure for the entity $j$ at time $t_k$, as stated in [Bielecki et al., 2018]. Then, the CCP exposure for one such CDS contract on reference entity $j$ towards the clearing members is given by

$$\check{S}_{t_k}^j = S_{t_{k+\delta}}^j + \sum_{u=t_k}^{t_{k+\delta}} d_u^j - S_{t_{k-1}}^j, \tag{43}$$

which refers to the $j$-th contract. Note that Equation (43) is essentially the same as (27), without the discount factors $\beta$, because of the zero interest rates hypothesis. The variable $d_u^j$ is the dividend of the CDS contract on reference entity $j$ at time $t_k$. If the next coupon payment is at $T_D > t_k$ and the previous payment occurred at $t_D \leq t_k$, we have that the factor $\sum_{u=t_k}^{t_{k+\delta}} d_u^j$ is equal to

$$\sum_{u=t_k}^{t_{k+\delta}} d_u^j = \begin{cases} -R(T)(T_D - t_D)\mathbb{1}_{t_k<T_D\le t_k+\delta}, & \phi^j > t_k+\delta \\ -R(T)(\phi^j - t_D), & t_k < \phi^j \le t_k+\delta \end{cases}$$

We assume that at time $t_k$ every CDS contract is alive and the clearing members are alive too. Hence we can simply use $\check{S}^j_{t_k}$ in the formulas so that Equation (43) can be rewritten as

$$\check{S}^j_{t_k} = \begin{cases} S^j_{t_{k+\delta}} - R(T)(T_D - t_D)\mathbb{1}_{t_k<T_D\le t_k+\delta} - S^j_{t_{k-1}}, & \phi^j > t_k+\delta \\ Rec - R(T)(\phi^j - t_D) - S^j_{t_{k-1}} \approx Rec - R(T)(t_k - t_D) - S^j_{t_{k-1}}, & t_k < \phi^j \le t_k+\delta \end{cases} \tag{44}$$

If a default of reference entity happens during the period $(t_k, t_{k+\delta}]$, considering that our unit of time is small, we can approximate the default time $\phi^j$ with the valuation date $t_k$, which leads to the second line in (41).
Hence, we end up with $10^5$ trajectories for the CDS in case of no default prior to $t_{k+\delta}$ (first line) and in case of default over the time interval $(t_k, t_{k+\delta}]$ (second line), for every $t_k$.
Combining each value with its corresponding probability yields

$$\begin{aligned} &P(\check{S}^j_{t_k} = \tilde{S}^j_{t_{k+\delta}} - R(T)(T_D - t_D)\mathbb{1}_{t_k<T_D\le t_k+\delta} - \tilde{S}^j_{t_{k-1}} |\ \phi^j > t_k) = p_j, \\ &P(\check{S}^j_{t_k} = Rec - R(T)(\phi^j - t_D) - \tilde{S}^j_{t_{k-1}} \approx Rec - R(T)(t_k - t_D) - \tilde{S}^j_{t_{k-1}} |\ \phi^j > t_k) = 1 - p_j \end{aligned}$$

see also Equation (4.2) - (4.4) in [Bielecki et al., 2018].
Finally, we take the expected value for every $t_k$ and we get $\check{S}^j_{t_k}$ for each time point. Considering what we said so far, we still have a group of trajectories for any time point $t_k$ for the quantity $\check{S}^j_{t_k}$. Then, the exposure $X^i_{t_k}$ is computed using the matrix $H_{8\times 4}$ presented at (37). The formula is

$$X^i_{t_k} = \sum_{j=1}^{4} H_{ij}\check{S}^j_{t_k}. \tag{45}$$

Finally, the ES of $X^i_{t_k}$ is computed and for the initial margins we use a 99% significance level $\alpha$, that is

$$IM^i = ES_\alpha(X^i_{t_k})$$

and the results are shown in Figure 15 below. Be reminded that the results shown are obtained using periodic default intensities for the reference entities underlying the CDS contracts according to (36) and then using CIR processes for the clearing members.

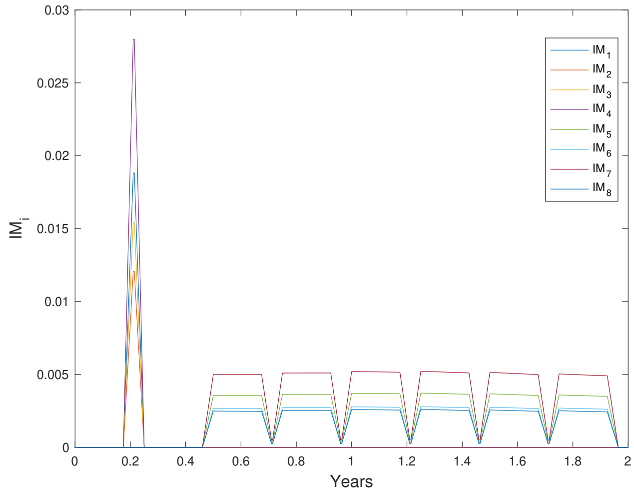



Figure 15: Initial margin for every CM.

Figure 15 shows the initial margins for all the clearing members. We see an initial peak and then a periodic trend that ends at $T = 2$ years.

### 5.4.3 Default Fund

Next we compute the default fund (DF) for the clearing members with CIR parameters as in Table 3. We start with the CCP exposure towards the clearing members net of VM and IM, as presented in Equation (30). The computations of the exposure produce 17 values, one for every month for the 2 years of the CDS contracts. Then all the individual exposure are added together (and the sign is changed).

Having total exposure available, the total default fund is computed, using the ES as risk measure, with a significance level $\beta$ equal to 99%.

Figure 16 shows the total default fund depending on the correlation coefficient $\rho$ we used in Section 4.3 for the one-factor Gaussian copula. Be reminded that the CDS contracts have maturity of 2 years.

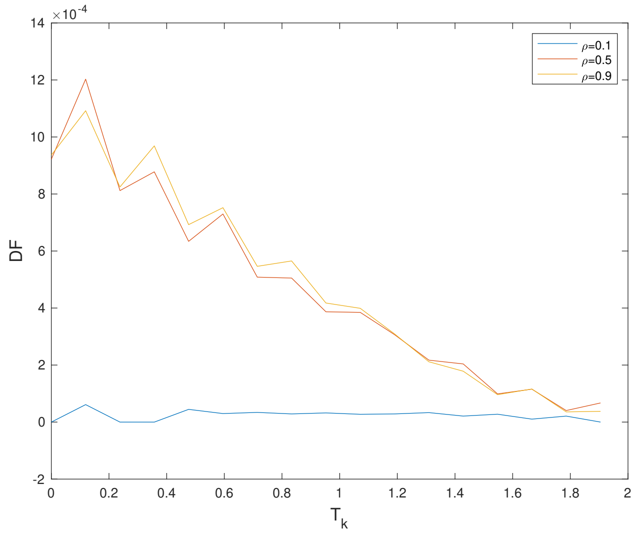



Figure 16: Total default fund depending on the default correlation $\rho$.

Looking at Figure 16, we see a huge increase in the total DF when the default correlation coefficient $\rho$ switches from 0.1 to 0.5. This results shows the importance of taking into account the relation between the potential defaults of all the clearing members. We do not see a great difference between $\rho = 0.5$ and $\rho = 0.9$. For the DF allocation, we consider the case of $\rho = 0.5$. In order to find $Z^*_{T_k}$, we have to compute first $X = -\sum_{i \in I} EP^i_{T_k}$, following the procedure shown in Section 4.5.2. Now, Equation (33) is used to find the individual default funds and we show the results in Figure 17.

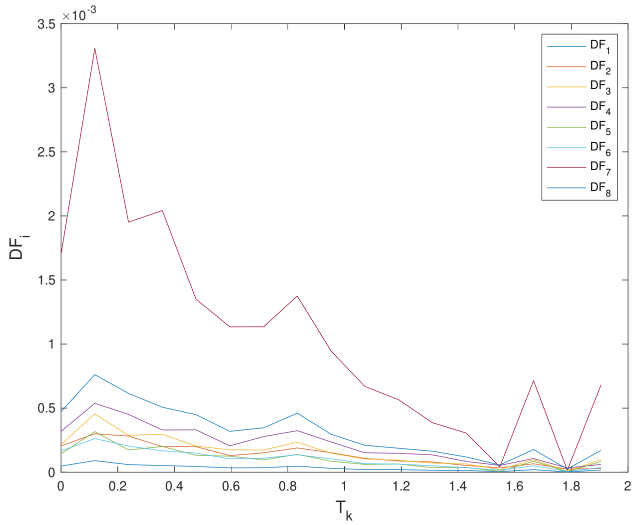



Figure 17: Default fund for every CM.

Individual default funds are aggregated together in order to check that the full allocation property is satisfied.

### 5.4.4 Default Fund over Initial Margin

The last subsection of the section is dedicated to the $DF/IM$ ratio to further investigate the relation between these two layers of the default waterfall. The results for the $DF/IM$ ratios are shown for $\alpha = 0.99$ and $\beta = 0.99$. We want to detect how this ratio changes according to the clearing members rating of belonging (recall from Section 5.1.2 that members' ratings are different).
Table 5 gives the average of the ratios, one for each $CM_i$.

| $CM_1$ | $CM_2$ | $CM_3$ | $CM_4$ |
|---|---|---|---|
| 0.0001 | 0.0015 | 0.0029 | 0.0092 |

Table 5: $DF/IM$ ratio for the first 4 CM for p-values $\alpha$ and $\beta$ equal to 0.99 and for correlation coefficient $\rho = 0.5$.

The results at Table 5 show a $DF/IM$ ratio equal to 0.0001 for $CM_1$, $CM_2$ shows a ratio of 0.0015, $CM_3$ presents a $DF/IM$ of 0.0029 and $CM_4$ of 0.0092. Hence, we see how this ratio changes according to the 4 ratings considered. The main problem in the default waterfall is due to the DF computations, considering that there is no accordance among practitioners on the model that should be used for this layer of the waterfall.
Due to the assumption on the classification in ratings for the clearing members, we see the results are consistent with the individual riskiness. As a matter of fact, from $CM_1$ to $CM_4$ we see an increasing $DF/IM$ ratio.
Therefore, our model is able to differentiate from the single members' position, being the default funds' weight with respect to the initial margin gradually increasing as the rating decreases.

# 6 Conclusions

This thesis deals with the problem of computing the margins that compose the default waterfall in a market with a CCP and CMs who can only trade CDS. Starting from [Bielecki et al., 2018], we extend the model setting in two directions by considering a more realistic framework where default intensities of CDS and CMs are stochastic and represented via CIR processes. To assess the individual riskiness, the reference paper [Bielecki et al., 2018] uses constant default intensities for CDS contracts, while it uses a Markov chain for the clearing members default intensities. Being the constant default intensities an unrealistic hypothesis, we apply the CIR process in both cases, in order to study deeper its impact on the riskiness evaluation. Moreover, we use the reduced model also to assess the members' riskiness, in order to have a more uniform model.
First of all, for all the 8 clearing members, we construct balanced portfolios, namely portfolios that for each member have an overall exposure equal to 0 offsetting long and short positions. We focus on the first 4 CM, which have the 4 ratings into account (BBB, B, B-, CCC). We construct their portfolios such that they have positive and similar values. In this way, we are able to obtain $DF/IM$ ratios whose differences are mainly due to the different riskiness of the CM.
Our model represents an alternative to the aforementioned *Cover 1/Cover 2* principle of the CPSS and IOSCO, that, despite its convenience due to the simplicity of its definition, suffers from some drawbacks that we have been described in the thesis.
It is worth noticing that some aspects of this work should be studied deeper. The first one is related to the underlying probability space. In our case we directly started working under the risk neutral measure. However how this measure is specified is not addressed. The market, in fact, is clearly incomplete, and hence infinitely many risk measures exist. This issue is left unsolved in [Bielecki et al., 2018] as well. Moreover, we focused on IM and DF, but CCP equity (E) and the unfunded DF ($u$DF) should be studied deeper and included in the whole discussion, in order to have a complete analysis on the default waterfall.

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