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KNOWING A THOUSAND FORMULAS IS NOT THE SAME AS KNOWING MATHEMATICS

How Students at Different Levels of Mathematical Studies View, Understand and Study Mathematics

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Abstract

This research is about how students at different levels of mathematical studies view, understand and study mathematics. The research is based on two theories on mathematical understanding, James Hiebert and Patricia Lefevre's (1986) theory on mathematical knowledge as divided into procedural knowledge and conceptual knowledge and Anna Sfards' (1991) theory on mathematical understanding as the duality of structural understanding and operational understanding. Rooted in these theories, the mathematical understanding, views of mathematics, and how mathematics is studied by students at the gymnasium, student teachers and students studying advanced mathematics at university in Sweden, are researched. It is analyzed whether views, understandings and ways to study differ between different levels and whether students' views and understandings impact students' ways of studying. The method used is mixed methods, involving both qualitative and quantitative methods. The students answered a questionnaire and their answers were summarized and analyzed with the help of coding and qualitative content analysis. The results point out that the most common way to view mathematics differs quite a lot between students at the gymnasium and students at the university and so do the ways of studying and the ways of understanding mathematics. The results show a correlation between the way students at different levels view mathematics, how they understand it and how they study it, with the students at the gymnasium having more of a procedural approach to mathematics than students at university.

Keywords: Conceptual knowledge, procedural knowledge, structural understanding, operational understanding, views, ways to study, levels of study

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1 Introduction

What is mathematics? This is a question that I, as a mathematics student and teacher, have asked myself many times. It is a question that will be answered differently depending on who you ask. Searching for *mathematics* at Wikipedia it says: “Mathematics (from Greek: μάθημα, máthēma, 'knowledge, study, learning') includes the study of such topics as quantity (number theory), structure (algebra), space (geometry), and change (analysis). It has no generally accepted definition.” (“Mathematics”, 2021, june 18). This illustrates how different people can view the subject in different ways and brings us to the next set of questions: does the way people view mathematics affect how they understand the subject? What does it mean to understand mathematics? What does it mean to have mathematical knowledge? Does solving mathematical exercises correctly mean that one has understood the mathematical content? Does a person who knows a thousand formulas by heart know mathematics?

These are questions without simple answers as it is hard to pinpoint what it means to actually understand maths, or what it means to have mathematical knowledge. Whatever the answers are, it is true that mathematics is a subject that is given a lot of time and place in the Swedish school system and a subject that we use in many ways in today's society. It is a subject that is being taught from early ages and yet a subject that some students have a hard time getting a grip of. Why is it that there is such a divided understanding of mathematics?

One possible explanation for the divided understanding of mathematics is that math is abstract, it consists of abstract objects that one can’t see with the eye, but still need to work with. Another explanation is that mathematics is multifaceted. One facet of maths is the computations and operations that one has to master to solve exercises. Another facet is becoming friends with the notions and concepts of maths and understanding the relations between mathematical objects and how different parts of mathematics are connected. Yet another facet of maths is the formality of the language, of the axioms, definitions, theorems and proofs. All of these aspects are important in maths and an understanding of all of them are needed to get a full view of mathematics.

A common problem for students in learning mathematics seems however to be to connect the aspects of mathematics. For example, Schoenfeld (1989) takes up the conflict between students learning the formal mathematics by rote so that they make great scores at tests, but when they are given a problem that builds on the formal mathematics they do not know how to use it. He writes in one of his papers that students, when asked to reconstruct a proof they had learnt by heart and then asked to apply the theory of the proof to a construction problem, “first produced correct proofs of the deductive results and then conjectured a solution to the construction problem that flatly violated the results they had just proven.” (Schoenfeld, 1989, p.340). Studies such as this show that procedures in mathematics sometimes are learned by rote by students, that students don’t always understand why to use the procedure they choose and that students sometimes give unreasonable answers to exercises without realizing it themselves. To put it in Skemp’s (1976) words, students sometimes learn *rules without reasons* or they learn *how* to solve mathematical tasks and not *how and why*.

How then should one study to be able to connect the different aspects of mathematics? Do students of mathematics aim to connect the different aspects of mathematics when studying it?

Does the way students view mathematics affect how they study it and how they understand the subject? Do the views of mathematics and how students study mathematics differ between different levels of mathematical studies? These are questions that will be processed in this thesis.

1.1 Aim and research questions

The aim of this study is to research how students at different levels of mathematical studies view mathematics, what kind of understanding they are searching for when learning mathematics and how they study to reach this kind of understanding.

The main questions this thesis aim to answer are:

1. What views of mathematics do students at different levels of mathematical studies express? How do their views differ from each other?

2. What kind of mathematical knowledge do students seek: do they just want to know *how* to solve mathematical tasks or do they want to understand *how and why*?

3. How do students study mathematics? Does it differ depending on what level they are studying at, what kind of mathematical knowledge they seek and what views they have of mathematics?

2 Theoretical background

To answer the research questions we will have to start with looking into what is already explored in this area of mathematics. In this chapter the theoretical background for this research will be presented. Firstly the learning theory constructivism that this research is based on will be explained. Secondly the theory of James Hiebert and Patricia Lefevre (1986) on procedural and conceptual knowledge from the book *Conceptual and procedural knowledge: the case of mathematics* is presented. Thirdly the paper *On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin* (1991) by Anna Sfard where she introduces the notions operational and structural knowledge as well as how mathematical knowledge is developed through three steps - interiorisation, condensation and reification - is summarized. Lastly, research on how mathematics is taught at different levels of schooling and how students experience the transition from studying mathematics at the gymnasium to university is presented.

2.1 Constructivism

Constructivism is a learning theory that has Jean Piagets' theory of cognitive development as a foundation and has also been influenced by the cognitive theory of Lev Vygotsky (Fosnot & Perry, 1996). Piaget wanted to explain how children develop and therefore he studied a child from a biological point of view (Phillips & Soltis, 2015). He explained that a child in development is busy building cognitive structures - different cognitive schemas for each action. This building is something the child does itself and so the child constructs its own knowledge. Piaget introduces three words from biology to explain the process; assimilation, accommodation and equilibrium. When someone is in a learning situation this person uses the cognitive structures that have been established so far. If the new situation corresponds to the learner's cognitive structure one assimilates the new information into the already existing schemata. However, if the new situation doesn't resonate with the learner's cognitive structure something new has been introduced that disturbs the cognitive equilibrium the learner earlier had reached and so the learner has to reconstruct the cognitive structure to again reach the cognitive equilibrium. Piaget called this reconstruction of the cognitive structure accommodation. Thus, the child constructs knowledge and puts it into different structures when interacting with the world. Although there is criticism of Piagets' work today, the main ideas are solid and they have influenced how we view learning and knowledge.

Vygotsky also believed that learning is constructive, in the way that it is the learner that has to construct and develop knowledge (Fosnot & Perry, 1996). However, he also focuses on social interaction and the culture of learning and the potential a child has to learn when being led by persons of greater knowledge. Vygotsky talked about a child's *zone of proximal development* to describe the gap between the potential a child has to learn by itself and the potential it has to learn under the guidance of a person with greater knowledge. The assistance a teacher or an adult or another knowledgeable person can give to the child in the learning process, that supports the zone of proximal development, Vygotsky called *scaffolding* (Kalina & Powell, 2009) and this is an important part of the child's learning process.

Piaget's thoughts have been applied to learning in general, and out of it the learning theory we nowadays call constructivism has emerged (Phillips & Soltis, 2015). Vygotsky's theories are also considered a part of constructivism, although this part of constructivism is often labeled social constructivism. The main focus of constructivism is that it is the learner that constructs the knowledge when experiencing the world and social constructivism emphasizes that an important part of this learning process are the interactions with others (Kalina & Powell, 2009). Thus, constructivism states that knowledge is not something you can give to another person or force upon someone, it has to be the person itself who attains it. However, people around us are an important part of our learning since we can make sense of things through discussions with other people or by getting new perspectives on the things we already know. Fosnot and Perry (1996) explains constructivism in the following way:

> Rather than behaviors or skills as the goal of instruction, cognitive development and deep understanding are the foci; rather than stages being the result of maturation, they are understood as constructions of active learner reorganization. Rather than viewing learning as a linear process, it is understood to be complex and fundamentally nonlinear in nature (p. 3).

This research is based on the constructivist thinking that learners *construct* new understanding and knowledge and integrate the new knowledge with what they already know. From this perspective there is interest in studying how students interpret mathematics, and how they study it, and this is also what the main aim of this research is. In the coming sections two theories about how one understands math will be presented and both these theories are to be understood in the light of constructivism.

2.2 Conceptual and procedural knowledge

James Hiebert and Patricia Lefevre (1986) present the notions procedural and conceptual knowledge in mathematics and the important relationship between the two kinds of knowledge in the book *Conceptual and procedural knowledge: the case of mathematics*. Many philosophers, mathematicians and teachers have through the years been dividing mathematical knowledge into different categories, probably since "the relationship between the ability of performing a task and to understand both the task and why the action is appropriate" (Murray & Rodney Sharp, 1986, p.xi) is complicated. Different people have labeled their categories differently, such as principles and skills (Gelman & Gallistel, 1978), meaningful and mechanical knowledge (Baroody & Ginsburg, 1986), instrumental and relational understanding (Skemp, 1976) but Hiebert and Lefevre (1986) choose to divide mathematical knowledge into the categories conceptual and procedural knowledge. They do themselves point out that this isn't a universal division that everyone agrees upon, but that the existence of different breakdowns "are a sign of a healthy, vital discussion about very complex issues" (p.9).

Hiebert and Lefevre (1986) points out that the distinction between conceptual and procedural knowledge is not disjoint, some knowledge fits into both categories and some in neither one, however, they also argue that the distinction can help us make sense of how students learn mathematics and to easier make out what causes their failures and successes. Moreover they point out that what they are focusing on is the relationship between the two types of mathematical knowledge because in the relationship lies insights into how mathematical competence develops. Thus, the study of procedural and conceptual knowledge is important, but even more so the study of the relationship between the two. Hiebert and Wearne (1986) even write that "mathematical competence is characterized by connections between conceptual and procedural knowledge. More specifically, we argue that mathematical incompetence often is due to an absence of connections between conceptual and procedural knowledge." (p.199). Thus, to research students' mathematical competence it can help to study the connection between their conceptual and procedural knowledge.

Conceptual knowledge

Conceptual knowledge is defined by Hiebert and Lefevre (1986) as knowledge that is rich in connections. It is characterized by the existence of relations between individual facts and propositions that together make up a network such that all pieces of information are connected to some other. Thus one part of conceptual knowledge cannot be an isolated piece of knowledge.

A person develops conceptual knowledge through constructing relations between different pieces of knowledge, either between two pieces already existing in memory or between something that is already known and something that is learned at the moment. Conceptual knowledge can also develop when already existing networks that up until now have been seen as separate knowledge gets connected and new relationships and connections are being crystallized. The development of conceptual knowledge involves what Piaget called assimilation of new material to already existing knowledge structures. This process has also been described by others but with labels such as *meaningful* learning (Brownell, 1935) or simply *understanding* (Davis, 1984; Skemp, 1976).

Procedural knowledge

Procedural knowledge deals with the procedures in mathematics and Hiebert and Lefevre (1986) divide this kind of knowledge into two parts. One part consists of the formal language and symbols that are used in mathematics. At more advanced levels, when studying theorems and proofs, the syntactic way of writing proofs is also included in the formal language part of procedural knowledge. Hiebert and Lefevre point out that the knowledge of symbols and syntax doesn't imply that the person has understood the mathematics they are using.

The second part of procedural knowledge consists of the procedures and rules that are being used to solve mathematical problems, meaning the step-by-step instructions that dictate how to solve a mathematical task. These recipe-like instructions that is the nature of procedures separate this kind of knowledge from other sorts. All procedures are however not of the same kind, some treat written symbols, others operate on diagrams or other objects but it holds for different kinds of procedures that they are structured in their core.

Hiebert and Wearne (1986) summarizes the definitions of conceptual and procedural knowledge in the following way:

> conceptual knowledge is knowledge that is rich in relationships but not rich in techniques for completing tasks. Procedural knowledge is rich in rules and strategies for completing tasks but not rich in relationships. Metaphorically speaking, conceptual knowledge is the semantics of mathematics and procedural knowledge is the syntax. (p.201).

Meaningful and rote learning

As mentioned earlier, Hiebert and Lefevres' (1986) definition of conceptual knowledge - that it is rich in connections and that it reflects a relationship between different pieces of information - is similar to what others have called *understanding* or *meaningful learning* (Brownell, 1935; Davis, 1984; Skemp, 1976). Thus the phrase *meaningful learning* has often been used in the same sense as Hiebert and Lefevre (1986) have defined conceptual knowledge which implies that conceptual knowledge has to be learned meaningfully. On the other hand procedures *can* be learned without meaning, just by rote. Procedural knowledge can also be learnt with meaning, but this is not the only way to learn procedural knowledge, as it is with conceptual knowledge.

Hiebert and Lefevre (1986) propose that procedures learnt with meaning are procedures that are connected to conceptual knowledge while procedures learnt by rote produce knowledge that does not have relations to other parts of mathematics and is closely tied to the context it is learnt in. Thus, procedures *can* be learned by rote learning, and a student may use a procedure without understanding why it yields the correct answer to a mathematical task. This relates to what Skemp (1976) describes as only knowing *how*, and not knowing *how and why*. Procedures learnt by rote may however later on be connected to conceptual knowledge.

Relation between conceptual and procedural knowledge

Hiebert and Lefevre (1986) stresses that it is the relationship between conceptual and procedural knowledge that holds the key to understanding mathematics. They write that the conceptual and procedural knowledge are essential for each other, and that it is the relationship between the two that makes for mathematical understanding: "When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing." (p.9). Thus the relationship between the two is important, and this goes in both directions: procedural knowledge needs conceptual knowledge to be meaningful and conceptual knowledge needs procedural knowledge to be able to make something with the knowledge, for example to compute something.

Both the symbolic part of procedural knowledge and the procedural part of it is benefitted by the relationship with conceptual knowledge. For example, it is possible to grow knowledge of symbols without understanding what they mean but to develop symbols with meaning they must be connected to the conceptual knowledge they represent. Moreover, in mathematics one uses tons of different procedures, which all of them are too many to memorize on their own. It is going to be easier to remember the different procedures if they are connected to the conceptual knowledge of the underlying mathematical concept. It will be easier to understand how and why a procedure works and therefore also easier to remember the procedure. It will also be easier to understand different problem situations in different ways if the procedural knowledge is connected to conceptual knowledge and thus it will be easier to choose an effective procedure to solve the problem. Moreover one can more easily change between different procedures and even reduce the number of procedures required whilst solving a problem if the procedures are related to conceptual knowledge. Thus a relationship between conceptual and procedural knowledge benefits the procedural knowledge.

On the other hand, conceptual knowledge also benefits from procedural knowledge. For example, the formal language system of mathematics is there for a reason and it is a powerful tool for dealing with abstract notions. Symbols can represent complex ideas or concepts, which helps us when thinking about these complex objects. Hiebert and Lefevre (1986) even claim that mathematical symbols can help develop conceptual knowledge; for example, how we notate place-value can help to grasp how numbers are related. Moreover, procedural knowledge is important for solving problems more efficiently. The first time one meets a mathematical problem where there is no routine procedure one knows about, it takes time and energy to solve it. However, when solving similar problems one can start transforming the conceptual knowledge to procedural knowledge and it will be easier for the problem solver to complete the task. Thus procedures can help reduce the mental effort it takes to solve a mathematical task, which will help when solving even more complex tasks. Just as conceptual knowledge can lead to procedural knowledge it works the other way around as well.

Problems with constructing the relationship

To construct a relationship between the two types of mathematical knowledge is not easily done, in fact students sometimes have a hard time connecting the two types of knowledge. Therefore the relationship of the two is worth examining, since being competent in mathematics involves both kinds of knowledge and being able to relate them. Hiebert and Lefevre (1986, pp 17-18) write about three factors for why students do not always construct a relationship between the two kinds of knowledge, namely

1. *Deficits in the Knowledge Base*
   One cannot construct a relationship between knowledge if the knowledge doesn't exist. For example, Carpenter (1986) claims that children become proficient in addition problems if they have a rich conceptual knowledge base that they can tie the procedures they learn to. If they don't, the meaning of the procedures cannot be developed.
2. *Difficulties of Encoding Relationships*
   Children have a tendency to fail to notice or to see relationships, even relationships that might be obvious for adults. Thus, even though the relationship between mathematical knowledge might be taught with appropriate methods students might anyway have a hard time picking it up.
3. *Tendency to Compartmentalize Knowledge*
   When acquiring new knowledge it is bound to the context, which can obstruct the possibility to see similarities between the new knowledge and knowledge already acquired. If this new knowledge continues to be separated from other similar knowledge when it develops it becomes *compartmentalized*. It might even be the case that the different knowledge compartments speak against each other, that they don't add up, without the person noticing it, because they use the different knowledge in different situations.

One effect of students' problems to establish relations between procedural knowledge and conceptual knowledge is for example that they might be able to answer problems correctly, but they do not understand the meaning of their answer. Moreover, they might be able to solve routine exercises, but when meeting real world problems that don't resemble something they have done earlier they might not be able to solve it, although they know all they need to know to do it. Moreover, students might use a procedure they have learnt by rote and then if something goes wrong they might not see what they've done wrong as they have forgotten a step in a step-by-step procedure and suddenly don't know how to continue. Thus, once again, the relationship between conceptual and procedural knowledge is important for mathematical understanding and mathematical proficiency.

2.3 Operational and structural understanding

In the paper *On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin* (1991) Anna Sfard describes two different ways of understanding mathematics - operationally and structurally. She also describes how mathematical understanding is developed through three hierarchical steps:

1. Interiorization
2. Condensation
3. Reification.

These ways of understanding mathematics as well as how mathematical understanding is developed is explained in this section.

Sfard (1991) writes that one reason for the widespread difficulty to understand mathematics might be that its notions are abstract. For example, although a function can be seen as a graph it is not something that one can actually touch, see or experience with the senses. The notion of a function is something that each person has to think about in their minds, which makes it hard to grasp the true meaning of it. However, for a trained mind - like a mathematician that daily works with these abstract notions - these objects become as real as any other object in this world. According to Sfard the abstract notions of mathematics can either be conceived in a structural or an operational way, thus she divides mathematical understanding, similarly to Hiebert and Lefevre, into two categories - structural and operational understanding. The structural understanding means understanding mathematical notions as objects - like something that *is*, a structure - and the operational way means understanding them as a process - like something that's *being done*, an operation. To reconnect to the example of a function, it can either be seen as a set of ordered pairs, an object with specific qualities - a structural approach - or it can be seen as a computational process, that for each input-value in the mathematical expression, you will get an output-value - an operational approach.

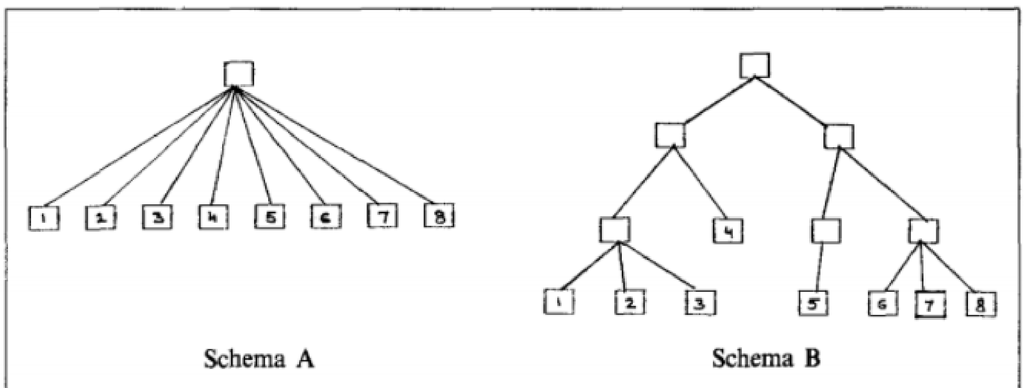
Operational and structural understanding are both needed

Sfard (1991) points out that the categorization of mathematical understanding into operational and structural understanding is not a dichotomy - two things that are entirely different - but a duality where both views are needed and complement each other. Sfard even quotes Paul Halmos (1985) who compared trying to define which way of approaching mathematics is most important as arguing about which of your feet are most important when taking a walk. Halmos goes on writing that "every mathematician must be both an effective calculator and an abstract thinker, and the relative importance of the two kinds of activities depends on the task at hand." (p.14). Sfard explains a similar view on structural and operational understanding of mathematics, and as can be seen by the name of the paper she sees these ways of understanding mathematics as *different sides of the same coin*. Thus, both these ways to conceive mathematics are essential for the understanding of the subject.

The operational way of thinking is in some ways similar to Hiebert and Lefevres (1986) category *procedural knowledge* and the structural understanding is more alike the *conceptual knowledge* - but the categorization is not entirely the same. However, Sfards' (1991) claim that the categorization is a duality corresponds to Hiebert and Lefevres' argument that it is the relationship between conceptual and procedural understanding that is important for the development of mathematical understanding.

Just like Hiebert and Lefevre (1986) argue that procedural knowledge can exist apart from conceptual knowledge Sfard (1991) explains that one could think about mathematics only from an operational point of view and that the notion of a *mathematical object* and the structural way of thinking might be seen as superfluous. Theoretically, one could continue from basic processes to higher-level processes without introducing any mathematical objects. This is even the way some parts of mathematics have been practiced through history, but mathematics of today is filled with mathematical objects and for a modern mathematician it is hard to imagine mathematics without them.

The operational information is not always easily processed and structural knowledge can help simplify the mental processes. One way to better understand how structural knowledge simplifies mental processes in mathematics is with the help of cognitive schemata. New information is stored in the memory in different kinds of cognitive schematas. These can be in their extremities either sequential, shallow and wide, see Schema A in figure 1, or deep, well-organized and narrow, see Schema B in figure 1. With a deep, well-organized and narrow schemata cognitive processes such as retrieval and storing information will be faster and easier than with a sequential, shallow and wide structure. Moreover, a well-organized schemata helps us get around the problem of our limited working memory.



Any information can be stored in many different schemata. For example, the two schemata pictured in this figure contain the same information (represented by (1, 2, 3, 4, 5, 6, 7, 8)). Schema A is sequential, shallow and wide. As a result of reification it can be reorganized into a deeper and narrower structure, such as Schema B. With the new organization, all cognitive processes (retrieval and storing) become much faster.

Figure 1. How Sfard (1991) describes the two different structures of schemata; p.27.

Sfard (1991) writes that strictly operational information is stored in sequential, wide schemata and that "the purely operational ideas must be processed in a piecemeal, cumbersome manner, which may lead to a great cognitive strain and to a disturbing feeling of only local - thus insufficient - understanding.” (p. 26). The structural way of looking at mathematics, however, leads to the construction of more well-organized, deeper schemata that simplifies the process. Thus, with the structural way of thinking, mental processes become simpler and it makes the operational knowledge easier to both remember and to use, as the structural way of thinking gets us around the problem of our limited working memory.

This is similar to what Hiebert and Lefevre (1986) writes as one of the advantages with connecting conceptual knowledge and procedural knowledge - it will be easier to remember different procedures if one understands the concepts they are tied to. Moreover, they write that one of the obstacles with connecting conceptual and procedural knowledge students face is that they might compartmentalize knowledge - they are not able to connect it to other mathematical ideas. This is similar to having a schemata that is sequential and this problem will be reduced when developing a well-organized, deep schemata. Thus, the structural understanding helps the operational understanding.

Sfard (1991) goes on writing that

> almost any mathematical activity may be seen as an intricate interplay between the operational and the structural versions of the same mathematical ideas: when a complex problem is being tackled, the solver would repeatedly switch from one approach to the other in order to use his knowledge as proficiently as possible. (p.28).

Thus, an interactivity between one’s structural and operational understanding is needed to competently use one's mathematical knowledge, where the structural understanding leads to reflections and insights while the operational understanding leads to actions and solutions.

Interiorization, condensation and reification

Understanding mathematics in a structural way is not necessarily a foregone conclusion. According to Sfard (1991) we develop structural understanding for mathematics through a three-step procedure - interiorization, condensation and reification. To give a better understanding of this procedure Sfard compares it with the development of different number sets in mathematics through the years. The concept of number - going from the natural numbers to the complex numbers - has developed through our need to count something that wasn’t possible with the already existing number sets. From this need the existing number set needed to expand. To give an example, let’s look at the expansion of the positive real numbers to all reals by introducing negative numbers.

The negative numbers were originally not regarded as numbers, they were merely a result of some procedure, and the need for an object that denotes that procedure did not exist at the time. However, as this procedure became used by more and more people and when the need to do higher-level procedures on this procedure grew, the need for an object evolved. Through this need and by the need to use procedures on these new numbers our schemata expanded and the object *negative number* developed. With it the reals expanded to include the negative numbers and so the number sets continued to grow. Now, when getting square root of -1 it first seemed absurd to objectify this notion, but later on it became established as a mathematical object and nowadays it is denoted *i* - the imaginary unit.

According to Sfard (1991) this developing process - going from an operational way of looking at something to developing an object and a structural understanding of the object - can be applied to how we as individuals develop structural understanding for mathematics. This is done through a three-step procedure: interiorization, condensation and reification.

Interiorization is the phase where the formation of a new concept begins. In this phase the learner becomes acquainted with a process that will result in a new concept and to do this, procedures are carried out on familiar objects. Gradually the learner becomes efficient in the performance of these procedures.

After the interiorization comes *condensation*. In this state processes and operations are turned into autonomous entities, so that the learner can start to “squeeze lengthy sequences of operations into more manageable units” (Sfard, 1991, p. 19). Now the learner becomes more capable of thinking of a process as a whole and doesn’t have to go through the details - a new concept is born.

The last state is the *reification*. This describes the development of the ability to see this new entity as an object-like whole. When an object is reified one can start with working out processes on this new object, and thus the interiorization and the three-step process can start all over again.

While both the interiorization- and the condensation-phases are states where the learner can stay for a long time the reification is defined as an ontological shift - something that happens suddenly and helps the learner to see something familiar in a totally new light. The reification is what gives the feeling of understanding a new object. Sfard (1991) writes:

> At a less technical, more philosophical level, we can say that in mathematics, transition from processes to abstract objects enhances our sense of understanding mathematics. After all, reification increases problem-solving and learning abilities, so the more structural our approach, the deeper our confidence in what we are doing. (p.29)

Sfard (1991) also points out that although this three-step-process starts with procedures and leads to structural conception it is often the case in mathematics that new concepts are introduced by structural definitions, without connecting these to underlying processes. She points out that for well-trained minds of mathematicians this way of approaching mathematics might work out fine, but it is more common for students to approach the new definition in an operational way and go through the three-step-procedure before being able to get structural understanding for the new concept.

The imperfect spiral of the three-step procedure

The three-step procedure - interiorization, condensation and reification - is like a spiral that starts over again when it is finished, as can be seen in figure 2. Thus, when an object has been reified the student is back at the interiorization-step but now the student has reached a new level of abstraction and has reified a new object on which interiorization can start again. However, Sfard (1991) points out that this is an imperfect spiral since to start with the interiorization of an object the former object has to be reified but at the same time there is usually no need to reify an object if we don’t have use of the object in some specific way. For example, one doesn't need to look at a function as an object until one wishes to manipulate it as a whole, and as this is true in general it makes sense to objectify a process only if one wishes to make some higher-level process on it. Thus it is a spiral where the last and the first step need each other in order to be carried out.

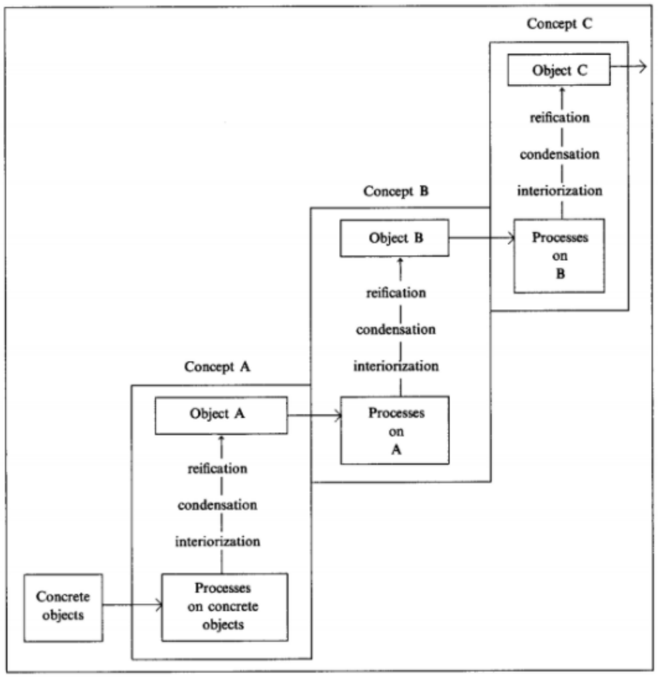


Figure 2. How new concepts are formed (Sfard, 1991:22).

Here again we meet the question of what should be practiced first, is it the drilling of new procedures or is it the concept development that has to come first? Sfards' (1991) answer to this is that

> one ability cannot be fully developed without the other: on one hand, a person must be quite skillful at performing algorithms in order to attain a good idea of the "objects" involved in these algorithms; on the other hand, to gain full technical mastery, one must already have these objects, since without them the processes would seem meaningless and thus difficult to perform and to remember. (p.32)

Thus, once again, the conclusion is that the different aspects of mathematics need to walk hand in hand to fully develop; sometimes the mechanical drilling of a procedure comes before the understanding for the concepts evolve, and sometimes the understanding of an object helps mastering the procedure.

The reification of an object might come when you least expect it, like a sudden flash. Before this happens however, there might be a time where the students doubt the meaningfulness and feel that they have lost understanding of the subject. Sfard (1991) goes on explaining that this flashing feeling of insight might show up while resting after an intense period of work with an object, and that this delay of the feeling of understanding might be what brings many to think that mathematics cannot be understood. If you are not willing to struggle for reaching the reification, you might end up feeling that the study of the subject is meaningless. However, the ones finding delight in the subject seem to be the ones that fights through the feeling of doubt and have the motivation and patience to work through the uneasy periods, to come out on the other side with a bigger picture, a higher level of abstraction of an object and a feeling of understanding.

2.4 Mathematics at the gymnasium and mathematics at the university

This research is focused on students' views on mathematics, what kind of mathematical knowledge they seek and how their views and aims affect their way of studying. Worth mentioning is that the research is taking place in Sweden, and thus, it is done in a Swedish context. Certainly there are many things that affect students' views and experiences of mathematics, one such thing being how mathematics is taught at different levels of education and how students experience the transition from studying mathematics at the gymnasium to the university in Sweden. Research about this is presented in this section.

How mathematics is being taught

Stadler (2009) writes that there is a difference in the view of the nature of mathematics in the gymnasium and at university and she calls the two different views *school mathematics*[1] and *academic mathematics*[2]. School mathematics is common at the gymnasium and is characterized by the teaching consisting of short lectures where the teacher introduces a new concept and shows some examples. The rest of the lesson students get to work in their books, solving exercises, where the exercises are similar to the ones the teacher showed. The exercises should be solved with a certain method that the teacher has introduced, and the exercises have a correct answer that the students aim to reach. It is common for these exercises to be separated into smaller, easier exercises that lead the student along the way to the correct answer. This is done with good intentions, to help the students solve exercises and to understand little by little. However, it might be hard for the students to get a greater view of the solution and to get a deeper understanding for the treated content, since they are only given small, easy parts at a time.

When entering the university students meet academic mathematics instead of school mathematics (Stadler, 2009) and now the focus is on the students getting familiar with mathematical notions and how to apply these. Academic mathematics is characterized by focusing on proofs, a wish to explore mathematical results and using specialized, technical expressions. Usually the students need to learn many new, abstract concepts in a short amount of time and the concepts are built on each other such that to get an understanding for a new concept, the former ones have to be understood. At the university students don't get led through exercises in the same way as at the gymnasium, now they need to get the bigger picture and to themselves be able to identify different connections and ways to solve complex problems.

Transition from mathematics at the gymnasium to the university

The difference in views on mathematics leads to students entering university expecting to be guided while the university expects the students to work autonomously (Liebendörfer & Hochmuth, 2015) which might lead to students having a hard time transitioning. Stadler (2009) writes that students that transition from studying mathematics at the gymnasium to studying mathematics at university often experience that there is a significant difference in the way of studying and that the transition is complex. Thunberg and Filipsson (2005) identify a gap in the

[1] Swedish original: *Skolmatematik*
[2] Swedish original: *Akademisk matematik*

content[3] between what universities expect their new students to know and what content is treated during math courses at the gymnasium. The gap includes both content that the university expects their students to know that isn't a part of the teaching at the gymnasium at all and content that is usually treated during the gymnasium, but is treated differently and not as thoroughly as is expected and required at the university. Thunberg and Filipsson point out that there is not only a difference in what content students learn at the gymnasium and are expected to know when entering university, but also a difference in the culture between the two institutions. There is a breach in what is considered mathematical workmanship at the gymnasium and the university, for example, at the gymnasium students usually get to have a sheet of formulas and identities during exams while students at the university are expected to know these things by heart. de Guzmán et.al. (1998) also agrees with this, and they argue that when entering university there is an expectation of increased depth both in procedural efficiency and conceptual understanding where the mathematical thinking needs to be shifted from elementary to advanced. Suddenly it is not enough for the students to reproduce methods and procedures but they need to investigate, analyze and produce themselves, and to do this deeper thought processes are needed. Moreover, when introduced to new concepts at the university it is not necessarily possible for the students to any longer connect the new concepts to real-world experiences or to understand them with the help of intuition but they need to work with formal definitions and through logical deduction come to conclusions.

Stadler et.al. (2013) points out in their paper on differences in the approach to learning maths between beginning and experienced university students that "beginners rely heavily on the teacher, while experienced students re-orient themselves from the teacher to other kinds of mathematical resources" (p.2436). Thus, when entering university students have to find new ways to study to get a grip of the mathematics they are faced with.

[3] Swedish original: *stoffgap*.

3 Method

As mentioned previously, this research is based on the constructivist learning theory, and from a constructivist point of view it is relevant to research how students view mathematics, how they understand mathematics and what consequences it brings to their way of study, as it is assumed that the students construct their knowledge. To investigate what views students have of mathematics, what kind of mathematical knowledge they seek and how they study mathematics a digital self-administered questionnaire (Fink, 2003) was made and sent/handed out to three groups of students. In this chapter the method used in this research is explained. More precisely, in this chapter it is explained what mixed methods are, how the questionnaire was built, how the answers were analyzed, what groups of students the questionnaire was handed out to, why the research was done in this way and how the validity of the research and research ethics have been considered.

3.1 Mixed methods

The method chosen to use in this research is a combination of qualitative and quantitative methods which goes by the name mixed methods and is considered the third major research approach (the first two being qualitative and quantitative methods) (Johnson et al., 2007). Johnson et.al. (2007) defines mixed methods as a method that "combines elements of qualitative and quantitative research approaches (e.g., use of qualitative and quantitative viewpoints, data collection, analysis, inference techniques) for the broad purposes of breadth and depth of understanding and corroboration." (p.123). This is also why this method has been chosen in this research; to get a breadth and a depth of understanding in order to better reflect reality. Greene (2007) writes that "a mixed methods way of thinking is thus generative and open, seeking richer, deeper, better understanding of important facets of our infinitely complex social world" (p.20) and since learning and understanding are complex matters it appeared natural to research these things from multiple different angles and approaches.

When trying to understand the students' ways of looking at mathematics and the kind of understanding they are searching for one will have to settle for collecting the students' own expression of their view of mathematics as well as trying to analyze these answers and categorise them. Sfard (1991) writes that

> It seems that we have no choice but to describe each phase in the formation of abstract objects in terms of such *external* characteristics as student's *behaviours*, *attitudes*, and *skills*. The resulting specification will hopefully be clear enough to serve as a tool for diagnosing, maybe even measuring, student's ability to think structurally about a concept at hand. (p.18).

This was kept in mind while designing the questionnaire. How should the questions be formulated to get a clear enough view of how student's view mathematics, whether they look at mathematical objects structurally or operationally or both and to what extent this affects how they study mathematics? Therefore, the questionnaire included a mix of open and closed questions (Fink, 2003). In the open questions the students were asked to give their view with their own words and answer the questions freely. The closed questions were multiple-choice questions either where the answers consisted of different views of a mathematical object or where students got to choose to what extent they agreed with a certain claim.

3.2 The survey

The questionnaire was designed to help answer the research questions of this thesis and it consisted of four parts[4]. The first part consisted of five open questions where the students themselves could express their thoughts on different views on mathematics and how they usually study mathematics. The second part consisted of four multiple-choice questions and one open question about different mathematical objects. This second part of the questionnaire was intended to investigate how students thought of some mathematical objects, such as a circle, a function and a fraction. The third part of the survey consisted of 15 claims about mathematics and how to study mathematics where the students were asked to grade how much they agreed with a claim on the scale: *I fully disagree*, *I do not totally agree, I agree to some extent* and *I agree completely*. The fourth and last part consisted of questions about the students, such as where the student was studying and what math courses the student was taking at the time.

Reasons for the questionnaire's design

The mix of open and closed questions on the questionnaire was used to get as much information from the students as possible. The open questions were chosen to be able to collect unanticipated answers from the students and to get an understanding of how the students think about and study mathematics, and not just letting them choose from preselected answers (Fink, 2003). However, since it can be difficult to interpret answers to open questions and to compare these answers to each other, the survey also consisted of closed questions to complete the open questions. Thus, the closed questions were designed to help interpret the answers to the open questions, to help understand the views of the students more precisely, and also to catch the opinions and thoughts of the students that have a harder time expressing themselves and their thoughts in writing (Fink, 2003).

Students answering the survey

This research is taking place in Sweden, and the three groups of students that participated in answering the questionnaire are partly from a Swedish gymnasium, where students study their 10-12th year (16-19 years old), and partly from a Swedish university. The students at the gymnasium are studying a natural science program and are studying their first year at the gymnasium, reading their second mathematics course - mathematics 2c. The two groups of students at the university are attending the same university and are either student teachers or students studying mainly mathematics. Several criteria led to the choice of these three groups. Firstly, the groups were chosen because it was possible to reach students in all three of these groups. Secondly the students in the three groups have experienced mathematics in different ways as they study at different levels, and one part of the research is about how the view of mathematics differs between different levels of study. Lastly the groups were chosen because all three groups consist of students that have enough interest in mathematics to choose a program where they study mathematics a lot and thus they presumably have some interest in taking time to answer a questionnaire about their views on mathematics.

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[4] To view the entire questionnaire, see Appendix. Appendix A contains the Swedish original while Appendix B contains the English translation.

The questionnaire was first handed out as a pilot study to three students that study courses similar to mathematics 2c in the gymnasium, two newly examined math teachers and two students that have been studying advanced mathematics courses at university. After the pilot study was done a few questions on the questionnaire were deleted and some rewritten and then the survey went out a second time. This time it was sent/handed out to students at the gymnasium reading mathematics 2c, students at the university that are studying to become math teachers and students at the university that study mainly mathematics.

At the gymnasium the questionnaire was handed out during a mathematics lesson to a class where 32 students were present. It was explained that the questionnaire was optional to fill in, and 27 students answered the questionnaire. The survey was furthermore posted at the common virtual learning environment for about 120 student teachers (it is not known how many of these students saw the post) and 16 students, spread between the first year of studying to become math teachers to the last year of studying, answered. Lastly, the questionnaire was shared in a facebook group for math students containing about 360 members - where the members are university students at different levels that mainly studies mathematics (it is not know how many of these students saw the post) - with the wish that students that either at the moment were studying an advanced level mathematics course, or that had been studying such a course in the last two years should answer the survey. It is not known how many in this group fit the requested description, but there were 16 students that answered, with a background ranging from one year of studying mathematics at university level to six or more years of mathematics studies. The majority of these students were either studying mathematics courses at an advanced level at the time for the questionnaire or had been doing it earlier and will therefore henceforth be called advanced mathematics students.

3.3 Analysis of the answers to the questionnaire

The analysis of the answers to the questionnaire has been done with the help of qualitative methods and quantitative methods. The qualitative methods used are called coding and qualitative content analysis (Jacobsson & Skansholm, 2019). This means that while reading the answers to the open questions on the questionnaire, repeated themes have been identified - coding - and when this is done, the themes are made into categories that represent different answers - qualitative content analysis. The quantitative methods on the other hand consist of numerically compiling how many students have given a certain answer and making a table or a chart to present this. These methods will be presented in more detail here.

Qualitative methods - categorizing the answers to the open questions

When reading the answers to the open questions on the questionnaire it was possible to identify some common themes among the answers. These themes were identified and separated into different categories that reflected different kinds of answers. For example, there were two open questions on the survey intended to help get a picture of students' views on mathematics, namely;

- *What is the most fun part of mathematics?*
- *When thinking of the mathematics you have encountered so far, what would you say mathematics is all about?*

When looking at the answers to these questions the different answers were sorted into some main categories that emerged. For example, many students answered that mathematics is about solving problems and many answered that mathematics is about patterns and relationships, thus one category was named *Mathematics is about solving problems* and the other *Mathematics is about patterns and relationships*.

The two categories just mentioned were easy to take out and label, as the answers from the students included the same words that named the categories. However, some of the categories were more difficult to take out and more difficult to label. To give an example of a more difficult category to make, we can look at the sixth of the categories of students' views of mathematics: *Mathematics is about abstracting to get a better understanding and about deep truths*. The name of this category was changed multiple times during the course of the analysis, as it was hard to put a label that reflected the answers that were sorted into this category. The core in the answers was that mathematics is about abstraction, complexity and about truths, but the answers differed some. In the end the label was chosen close to how two of the students put it themselves, one writing that mathematics is about finding *the right level of abstraction to understand* and the other wrote that it is about the *real deep truths*.

Thus, the qualitative content analysis was not always straightforward and some categorizations were not obvious to make, while other categories "made themselves". To be transparent about the categorization process each category is presented together with quotes from the students' answers that are counted into that category, to show what a typical answer in that category looks like. An answer could contain several different aspects, and show a multifaceted view on mathematics, and if that was the case the answer was placed into all of the categories that it expressed. No answer to the first open questions were left out - every answer to the open questions was sorted into a category. However, the answers that were unique and didn't fit into any of the main categories were placed in a catchall-category called *other answers*.

This kind of categorization was done with all the answers to the open questions on the survey, to better get a picture of the students' answers. When the categories were set the frequency of the answers ending up in different categories was evaluated and denoted, to be able to compare how many students at different levels gave an answer that fit in a certain category. All the categories are presented in chapter 4: Results. The student answers that are quoted as well as the questions on the questionnaire that are cited have been translated from Swedish, as that was the language of the questionnaire.

Quantitative methods - summarizing the answers to the closed questions

The answers to the closed questions were compiled and made into charts. For example, the third part of the questionnaire consisted of claims where the students were asked to express to what extent they agreed to a certain claim by choosing one of the answers: *I fully disagree, I do not totally agree, I agree to some extent* and *I agree completely.* The answers to each claim were summarized and made into circle charts for each group of students that showed how many students had agreed to the claim to the different extents. Thus one circle chart was made to show how the students at the gymnasium answered the first claim, one chart to show how student teachers answered the first claim and one chart to show how the advanced math students answered the first claim. These kinds of charts were made for every closed question on the questionnaire to better get an overview of how the students answered these questions. Most of the charts are presented in chapter 4: Results, however, as there were many closed questions on the questionnaire only the relevant charts are presented.

3.4 Research ethics and validity

During the work of this research choices have consequently been made to strengthen the reliability and the validity of the research. The ethics of the research have also been carefully considered. How this has been considered will be presented in this section.

Reliability

Reliability says something about how similar the results of the research would be if the same research method was used another time, in the same way but in different circumstances (Priest & Robert, 2006). Thus in this research it is important to analyze if the survey is done in a reliable way. The work to strengthen the reliability of this research has been in the design of the questionnaire. However, since this study has a qualitative aspect it is not possible to talk about frequency of answers among students in general, and since the answering students are not chosen in a way that the conclusions drawn could be generalized to students in general, it has been important to reflect over how to put the questions on the questionnaire in such a way so that if it were to be given in a different kind of context it would be answered similarly. To affect this a lot of thought has been given into how the questions have been written to assure that they were formulated in a clear way so that the students knew what was asked for. Therefore a lot of time was spent designing the questions and when that was done a pilot study was made, to see how students interpreted the questions. When reading the answers to the pilot study some of the questions that were unclear were rewritten or removed and the questions that the students interpreted in the way they were intended were kept. Thus, even though the answering frequency cannot be generalized the questions have been designed so that the answers can still be considered representative.

Another thing that can affect the reliability is if the survey is done in a rush (Jacobsson & Skansholm, 2019). Therefore, the questionnaire was given to the student teachers and the advanced mathematics students to answer when they had time and to the students at the gymnasium it was handed out in the beginning of a lesson, one lesson after they had a test, so that none of the groups had to answer the questionnaire in a rush, but the students had the time they needed to fill it in.

Validity

Validity is about how well a research measures what it is intended to measure (Priest & Robert, 2006). The validity of a research is a hard thing to guarantee, but one can strive to get it as valid as possible. This has naturally been the aim of this research as well. One part of validity is the reliability, and how this has been taken into account has already been explained, but moreover the validity has been considered in the following ways.

Firstly, the questions on the survey were carefully chosen to help answer the research questions. To assure this it was important that the students interpreted the questions and what was asked for in the same way (or very similarly) to what the questions intended to research. When the pilot study was handed out it became clear that some questions were hard for the students to understand - either because of the formulation of the question, or because it was hard to interpret what was asked about - and these questions were later on reformulated or deleted. The questions that brought answers that were aligned with what the questions intended to research were kept. Thus, the questions were designed to research what they should, and not any other things.

Secondly, one problem with using a questionnaire is how to formulate questions that will give answers that not only shows how good students are at expressing themselves in words but that reflects their actual thoughts. To tackle this challenge the method chosen for the research was mixed methods, having both a qualitative and a quantitative approach, which resulted in having a mix of open and closed questions on the questionnaire to better catch the true thoughts of the students. This helped the students who had a harder time expressing themselves in text, since they could express their thoughts through the answers to the closed questions instead, and while analyzing the answers it was possible to compare how well the students' answers to the open questions were consistent with the answers to the closed ones.

Research ethics

Like with all research where people are included one has to consider the ethics of the research (Jacobsson & Skansholm, 2019). In this study the ethics especially needed to be taken into consideration when students were given the questionnaire.

First and foremost, when given the questionnaire all students were informed of what kind of research they were going to contribute to and that their answers might be used in this thesis. After being informed they got a choice if they wanted to contribute or not. Thus, no one has been forced to answer the questionnaire, but all the students that answered it did it voluntarily.

Secondly, the questionnaire was anonymous, which the students also were informed of. This was important so that the students felt that they could answer the questionnaire honestly, and so that their answers won't affect them in any way, no matter who reads this thesis.

Lastly the collected data has only been used for this research, and not in any other way, and will not be used in any other way in the future either.

4 Results

The results from the survey are presented in this chapter. This is done with the research questions as a basis, where each section treats one research question. Answers from students that help answer the research question of that section are presented in that section. The questions on the questionnaire were designed to answer specific research questions, thus there is no crossover of the answers to the questionnaire, and it is obvious how to divide the answers between the different research questions.

The results presented in this chapter have been coded, categorized and put together into tables and charts. Some comments are written about the results in this chapter, but the results are analyzed and discussed further in chapter 5: Discussion. Throughout the tables and charts the students from the gymnasium will be called “SGY”, students from the university that are studying to become math teachers “STE” and students from the university that are studying maths on an advanced level “SMA”. In the tables there are sometimes quotes from student answers presented and at the end of every quote it will be denoted which student gave this answer. For example, a quote ending with “SGY5” means that this quote is taken from the answer of the fifth student in the gymnasium that answered the survey.

4.1 Question 1 - What views of mathematics do students express?

Two of the open questions on the questionnaire were designed to help answer the first research question; what views of mathematics do students express? The third part of the survey was made up of claims about mathematics and some of these claims were also designed to show to what extent students agreed with some possible views of mathematics. The answers to these questions and claims are presented in this section.

Different views of mathematics

The first and the fifth questions on the survey were stated as follows: *What is the most fun part of mathematics according to you?* and *When thinking of the mathematics you have encountered so far, what would you (shortly) say mathematics is all about?* These two questions were designed to gain insight into what views students have of mathematics. The latter of the two questions gave the students the opportunity to formulate their view on mathematics while the former helped give an indication of how the students view mathematics, since if one thinks something is fun with a subject, this often mirrors how one views the subject.

The answers to the two questions have been sorted into 7 different categories which are presented here. For each category some student answers, or part of student answers, are given to show what typical answers in that category look like. Many students answered in such a way that their answers fit in several of the categories and if that is the case the answers are counted into multiple categories. Thus, a student answer may appear in more than one category if it reflects multiple views on mathematics.

Category 1: Mathematics is about numbers, formulas, rules, methods and counting

The first category consists of answers that somehow say that mathematics is about numbers, formulas, rules, methods and counting. More than half of all the answers from students in the gymnasium are placed in this category, one fourth of the answers from the student teachers and none of the answers from the advanced mathematics students.

Table 1

Mathematics is about numbers, formulas, rules, methods and counting

1: Mathematics is about numbers, formulas, rules, methods and counting.
SGY: 14/27 = 51,8%   STE: 4/16 = 25,0%   SMA: 0/16 = 0,0%   Totally: 18/59 = 30,5%
To calculate something. To find an unknown value I find is the majority of mathematics... SGY8 *...It's about numbers and different ways to use them. SGY10* *Put two and two together. There are so many rules in mathematics that one has to learn before one can count. SGY11* *...generally it is about counting and finding different ways to solve an exercise. SGY12* *Mathematics is about methods. Lots of different methods to solve the same things. There are many ways to solve exercises but at the same time there is only right or wrong... SGY20*

Category 2: Mathematics gives us tools to model the world and explain the world

The second category consists of answers about how mathematics can help in modelling and explaining the world, and answers in this group reflect that mathematics is used for describing things and is applicable in the real world. This is a common view among the students in the gymnasium and the student teachers - over 40% of respondents in these groups have answers in this category - and one fourth of the advanced mathematics students agrees with this view.

Table 2

Mathematics gives us tools to model the world and explain the world around us.

2: Mathematics gives us tools to model the world and explain the world around us.
SGY: 11/27 = 40,7%   STE: 7/16 = 43,8% SMA: 4/16 = 25,0%   Totally: 22/59 = 37,3%
Explanations and aids to more or less everything that is part of our world. Money, figures, weight etc. SGY16 *Mathematics is about ways to represent or simulate real happenings/objects. SGY24* *...I do however understand that for others mathematics is primarily useful and is about explaining how the world works and about making the right decisions, either if it is about maximizing profit or minimizing accidents or anticipating some aspect of the future as exact as possible… STE14* *A way of describing things. SMA16*

Category 3: Mathematics is about solving problems

The third category consists of answers that explain that mathematics is about solving problems. This is the category where answers from students from all three different levels of study are agreeing the most - more or less 40% of the students in all three levels have in some way included in their answer that mathematics is about solving problems - even 50% of the student teachers state this. This is also the category that holds the biggest percentage of the total number of answers - 42,3% of all the answers fit into this category.

Table 3
Mathematics is about solving problems

3: Mathematics is about solving problems.
SGY: 11/27 = 40,7%   STE: 8/16 = 50,0% SMA: 6/16 = 37,5%   Totally: 25/59 = 42,3%
...Problem solving in everyday-life situations is quite common as well. SGY8 *Mathematics is about problem solving, to effectively find solutions to solve problems. SGY13* *...solve problems... STE3* *To try to solve abstract but clearly formulated problems (that might be possible to apply to reality (but that is not the interesting part)). SMA1*

It is worth mentioning that even though some of the answers consist solely of mathematics being about solving problems, a lot of the students have given an answer that maths is about something else but then they include that it is *also* about solving problems. Thus, a lot of the students hold the view that problem solving is an important part of mathematics, regardless if they for example think of mathematics as numbers, formulas etc, as a language or as patterns and relationships. Thus, that problem solving is a part of mathematics is a shared view among the students.

Category 4: Mathematics is about patterns and relationships

The fourth category consists of answers that explain that mathematics is about patterns, connections and relationships. This is the most common view of mathematics among the student teachers - more than 50% of them agree with this view. A fourth of the advanced mathematics students agree with this view as well, while not as many of the students at the gymnasium answer this.

Table 4
Mathematics is about patterns and relationships

4: Mathematics is about patterns and relationships.
SGY: 5/27 =18,5%   STE: 9/16 = 56,3%   SMA: 4/16 = 25,0%   Totally: 18/59 = 30,5%
Mathematics is about counting with different numbers and sort of finding relations between different things and to try to simplify these relations/resemblances…SGY19 *Much logic and how things are connected to each other. SGY21* *To see contexts and to connect. STE4* *Mathematics is about patterns and to describe relationships. STE6* *Mathematics is (often) about studying complicated objects by means of its local/global symmetries and patterns. SMA8*

Category 5: Mathematics is a logical system or a language

The fifth category holds answers that suggest that mathematics is a logical system or a language. The group with the highest percentage with this kind of answer are the advanced math students, where 50% gave answers coherent with this view. One fourth of the students at the gymnasium and the student teachers hold this view as well.

Table 5
Mathematics is a logical system or a language

5: Mathematics is a logical system or a language.
SGY: 7/27 = 25,9%   STE: 4/16 = 25,0%   SMA: 8/16 = 50,0%   Totally: 19/59 = 32,2%
Mathematics is about logic, it just works. It just makes sense! SGY17 *Logic, understanding - if we want to be extreme one could say that mathematics is a language that expresses the world in different ways. STE10* *Mathematics is a logical system. SMA4* *To understand what a reasoning is and what cornerstones one uses when carrying out a reasoning. SMA7* *I would say mathematics is like a language, but instead of communicating ideas, feelings etc one communicates patterns with mathematics. SMA10* *Mathematics is a language to me. It allows us to, but only in this way, very precise and unequivocally describe everything. It makes it very difficult since us people otherwise do things half-assed and equivocally. SMA12*

Category 6: Mathematics is about abstracting to better understand and about deep truths

The sixth category is a bit difficult to give a suitable title, since the answers in this category are similar to each other but still differ some. The core of the answers in this category is that they explain how maths is about abstracting to get a better understanding of something, and how maths is about the deep truths, or a tool to help finding the truth. More than 40% of the advanced mathematics students' answers fall into this category, whilst none of the answers from the students in the gymnasium is in this category, and only one answer from the student teachers.

Table 6

Mathematics is about abstracting to get a better understanding and about deep truths

6: Mathematics is about abstracting to get a better understanding and about deep truths.
SGY: 0/27 = 0%   STE: 1/16 = 6,3%   SMA: 7/16 = 43,8%   Totally: 8/59 = 13,6%
...and about proving things with absolute certainty… STE14 *I think it's hard to capture in a sentence, but if I should try maybe the following "Given an object, find the right level of abstraction to understand the object better" With object in a wide sense:* - *If I need to count the number of things, for example horses, the only quality I need in the horse is the fact that it is a horse. Now I can add, subtract, exactly like with the natural numbers.* - *If I want to understand how things change I want to do it in terms of differential calculus.* - *If I want to rationate about what is the most believable thing to happen in the future based on a set of events that have already occured, I introduce probabilistic measures and further equipment…. SMA2* *Hard to tell, sometimes one gets a feeling that it is about the real deep truths… SMA3* *The level of abstraction on easy 'problems'. SMA6 (to answer "what is the most fun part in mathematics")* *The most fun part is the feeling of a deep order… SMA14 (to answer "what is the most fun part in mathematics")*

Category 7: Other answers

The last category consists of all the answers that don't fit into any of the other categories, and that are too unique to form an additional category. Some of the answers from all groups of students are in this category, but what is remarkable is that as much as 43,8% of the answers from the advanced maths students include some part that puts them in this category. This is why the answers from the advanced mathematics students were quite characteristic and not always easy to categorise.

Table 7

Other answers

7: Other answers (that didn't fit into any of the categories 1-6 and are unique)
SGY: 6/27 = 22,2%   STE: 6/16 = 37,5%   SMA: 7/16 = 43,8%   Totally: 19/59 = 32,2%
Learn how to think and use one's brain. SGY2 *That in mathematics you get to use your creativity and expand your intuitions. SMA8 (to answer "what is the most fun part in mathematics")* *...I also see mathematics as an art. SMA12* *Systematic optimization. SMA13*

Summary

The results show that there is a divergence of views of mathematics from the students at the different levels of studying mathematics. Although some answers overlap - for example students in all three groups agree that mathematics is about solving problems - there is a difference between what category the majority of the answers lie in at each level of studying. At the gymnasium a lot of the students claim that mathematics is about counting, numbers, formulas etc, whilst the student teachers focuses on patterns and relationships, and the advanced mathematics students views mathematics mostly as a logical system or a language as well as they think that mathematics is about abstracting to understand and about deep truths.

Students' answers to claims about views of mathematics

The third part of the survey was made up of claims where the students should answer to what extent they agreed with different claims. Some of these claims were designed to show to what extent students agreed with views on mathematics and these answers are presented here.

Table 8

Claims about what views students have of mathematics

Claim	SGY	STE	SMA
Mathematics is all about getting the right answer on exercises.			
Mathematics is thought provoking and useful.			
I fully disagree	I do not totally agree	I agree to some extent	I agree completely

A majority of the students at the gymnasium agreed that *mathematics is all about getting the right answer on exercises*. 50% of the student teachers agreed with this as well, while the advanced mathematics students were disagreeing with this statement to a bigger extent.

Everyone of the advanced math students agrees however with the claim that *mathematics is thought provoking and useful*. The majority of the students in the gymnasium as well as the student teachers agree with this statement as well, although not as many agree completely and there are also some that do not totally agree with the claim in these groups of students.

4.2 Question 2 - What kind of mathematical knowledge do students seek?

Some of the open questions on the survey were designed to help answer the second research question; what kind of mathematical knowledge do students seek? Furthermore, the second part of the questionnaire contained claims about how students view some mathematical objects, where the answers say something about what kind of mathematical knowledge students have. Lastly, the third part of the survey that consisted of claims included some claims that were designed to help unravel what kind of knowledge students seek. The answers to the open questions, to the questions about views on mathematical objects and to the relevant claims are presented in this section.

Is memorization important or are there other things that are more important?

The first open question relevant to answering research question two was whether memorization is important in mathematics or if there are other things that are more important, and if that is the case, the students were asked to explain it, and give an example. Answers to this question have been divided into three different groups of answers, where the answers in each group share the same core. The groups are the following:

1. Yes, memorization is important.
2. It is more important to understand how to use mathematics or how the mathematics is connected.
3. Other answers.

Some of the students gave answers that fit into more than one of the groups, for example;

> *Yes and no. Yes, because results can sometimes be hard to absorb from the beginning, and then you have to try to create some kind of comprehension of what they say and just remember them, hopefully to be able to come back to them later and understand them. No, because I mean that curiosity and perseverance are the important things in mathematics; not memorization. If you have the curiosity to understand something and the perseverance to actually do the needed work to understand it, then you will automatically remember the things as well. SMA7*

These kinds of answers have been counted into more than one group, thus a student answer might be located in more than one group.

Group 1 - Yes, memorization is important

The first group consists of the student answers that agreed with the claim that memorization is important in mathematics. More than half of the students at the gymnasium and of the student teachers agreed that memorization is important, while less than half of the advanced math students agreed with this.

Table 9
Memorization is important

Yes, memorization is important.
SGY: 16/27 = 59,3% STE: 9/16 = 56,3% SMA: 7/16 = 43,8% Totally: 32/59 = 54,2%
I think it is good to memorize as much as possible as it will be easier on tests for example if one knows some of the formulas by rote so that one doesn't have to waste time from the test to look at the sheet of formulas. SGY2 *I think it is important to remember formulas such as the pq-formula, the distance-formula etc. SGY12* *Absolutely, it is important to memorize things in mathematics, especially things that can easily be applied to reality such as different types of problem solving. SGY26* *To get at good flow when counting it is easier if one knows the formulas and easy computations (for example the multiplication table) so that one doesn't have to stop all the time. So yes - it is important with memorization. STE2* *Yes, it is extremely important to memorize. Mathematics is a language. A language without glossaries is impossible to understand. STE3* *Yes, I think it is important to memorize things in mathematics. No, I don't think there are any other things that are more important. But I also believe that memorization is the foundation to all knowledge. Nothing is more important than memorization when one learns things in general - that is my opinion (but I also know that people usually think I am a bit rigid when I say this). SMA1*

The answers in this group reflect that memorization is an important part of mathematics, but the reason for why it is important varies, as can be seen in table 9. The need to "not waste time" influences some of the answers where students explain that it is important to memorize things in mathematics to become more effective. Another argument is that it is important to memorize because maths is like a language and to learn a language one has to practice glossaries. Moreover there are some that argue that memorization helps the understanding, and that one has to memorize things to be able to understand them.

Group 2 - It is more important to understand how to use mathematics or how the mathematics is connected

The second group consists of answers that say something about how it is more important to understand how to use mathematics and understand how the mathematics is connected than to memorize things in mathematics. Over 50% of the student teachers expressed this opinion, as well as over 40% of the advanced mathematics students but less than 40% of the students at the gymnasium.

Table 10

It is more important to understand how to use mathematics or how the mathematics is connected

It is more important to understand how to use mathematics or how the mathematics is connected.
SGY: 10/27 = 37,0% STE: 9/16 = 56,3% SMA: 7/16 = 43,8% Totally: 26/59 = 44,1%
I don't think one has to memorize things, the things that are more factual or harder to remember like for example the area to different figures or similarly are written on the sheet of formulas. It is more important that one understands what one is working with, to memorize an exercise won't help at a test but to have understood the exercise can help as then you can have use of it in similar exercises. Thus, it is more important to understand than to memorize, if one has understood how something works one knows how to solve exercises that are diffuse but if one had just memorized something one won't see the likeness in the exercises and be able to solve them. SGY19 *I don't think memorization is as important as understanding the subject. Often there is room in an exercise to prove what you otherwise could have memorized. SGY24* *To understand mathematics for real. For example, to count with the area of a circle, to do that it is enough to memorize r*r*pi, but to understand why one does this makes it both easier to remember the formula and the mathematics becomes reasonable and it doesn't feel as much as it is just made up. STE5* *I think it is much more important to get a deep understanding. An example is the unit circle and the "known angles" you were encouraged to learn by heart during the gymnasium. To learn these by rote learning never worked for me, I had to look at the sheet of formulas every time, which actually isn't that much trouble. When I started to use the unit circle however, and understood that every point on the circle can be expressed by two coordinates whose squares has the sum one, and at the same time is sine and cosine for an angle I learnt better how it was connected. After that I could with greater security retrieve sine and cosine for the known angles without looking at the sheet of formulas and also easier and more confidently write them in different ways.* *All things one gets encouraged to learn by rote comes from somewhere. It is not taken out of thin air, but to be "forced" to learn them by heart instead of getting to know where they come from or how to get there by yourself gives an impression that it is taken out of thin air. Then I can understand that some things are used so often that it can be good to learn to recognize it so one doesn't have to deduce it every time, and it may be good to get that pointed out and then it can be a good tool for some to learn it by heart. It just shouldn't be disconnected or be used instead of understanding. STE14* *I think it is much more important to understand than to memorize. What I mean by that is that a student that knows 1000 formulas is not a student I would say knows mathematics, since that student probably couldn't derive these formulas with mathematical deduction. STE16* *I think it is more important to understand. Then one doesn't have to memorize. SMA3*

The answers in this group state that it is more important to understand mathematics and how to use mathematics than to memorize things, and some of the students also argue that if one understands one remembers easier. Some of the answers from the students at the gymnasium also state that it is not important to memorize formulas etc. since they are written on the sheet of formulas they get to have during exams.

Group 3 - Other answers

Some of the answers the students gave didn't fit into either group 1 or group 2 and these answers are gathered here in group 3. As many as 37,5% of the advanced mathematics students' answers, 31,3% of the student teachers' answers and less than a fourth of the answers from the students at the gymnasium are in this group.

Table 11

Other answers

Other answers.			
SGY: 6/27 = 22,2%	STE: 5/16 = 31,3%	SMA: 6/16 = 37,5%	Totally: 17/59 = 28,8%
Maths is like a sport, either you get better at it by practicing or you are a natural talent. SGY7 *I don't like memorization. I believe it puts an unnecessary dimension to mathematics. I appreciate things that you learn that then sticks. SGY8* *The principals; the logic in the thinking. What you actually benefit from in today's society without having to put the subject in extreme situations. SGY15* *It is not important to memorize things. It is better to have a feeling for mathematics, then one usually knows what is possible and what is not, and one can look up the right formula or method if one has forgotten it. STE6* *I think it's more important to be able to meet mathematical problems and to "read" and interpret mathematics. STE7* *No, I rather see memorization as a tool to save time. The most important thing I believe is creativity in problem solving; to not get stuck in thought patterns but to be able to look at a problem from different angles. SMA10*			

Most of the answers in this group are about how memorization is not the most important thing, but something else - that is not the same as in group 2 - is more important. This was a common answer among the advanced mathematics students, who thought that things like general problem solving and to not get stuck in thought patterns but be creative are important parts of mathematics.

Summary

All three groups think memorization is important, but the students at the gymnasium are the group where the biggest percentage of the students agree with the statement. However, what one can see in the quotes from the students in table 9, 10 and 11 is that many more of the answers from the students at the gymnasium are solely that memorization is important, while the student teachers as well as the advanced math students more often give a dual answer that says that memorization is important but there are other things that also are important or even are more important. One can see this as well by studying the percentages in table 9, 10 and 11, since there are as many students among the student teachers and the advanced maths students that answer that memorization is important as there are that answer that understanding the mathematics is more important and there are still students giving *other answers.* Thus very few university students' answers are in only one of the groups while this is more common among the students at the gymnasium.

To understand

A lot of the students answered the survey-question *what is the most fun part of mathematics according to you?* with variants of *understanding something.* This was a very common answer to the question, but since this is not necessarily an answer to what mathematics *is*, these answers are presented here instead of getting their own category about views on mathematics.

Over 50% of the students in the gymnasium answered in some way that understanding mathematics is the most fun part of it, and about 40 % of the student teachers and the advanced mathematics students stated this as well.

Table 12
To understand

To understand
SGY: 14/27 = 51,9%   STE: 7/16 = 43,7%   SMA: 6/16 = 37,5%   Totally: 27/59 = 45,8%
The most fun part is when you feel that you understand and solve a difficult problem SGY4 *When you don't understand something and then you understand or when you work with a difficult exercise that you don't really know how to solve but then you just start and get the correct answer. Thus when you get the right answer and you understand something that you earlier thought was hard. SGY20* *When one finally understands something in mathematics. It can be anything, but the feeling when one starts to understand what one is counting and feels that it starts to be simple. To find out how to solve an exercise, and then be correct. That is the most fun part of mathematics. SGY22* *When one sits with an exercise that is challenging and all puzzle pieces fall into place so one can solve it. STE1* *When you have been thinking for a long time and realise that what you have in front of you isn't just gibberish. SMA2* *...to understand how things are related. SMA14*

What students mean by understanding something is not always clear, but one can see some patterns at the different levels of studying in table 12. At the gymnasium the students often answer that the most fun part is when they understand something and *solve different exercises, get the right answer* or *be correct*. Thus it seems like the feeling of understanding for the students at the gymnasium is tightly connected with getting the right answer to a task.

Some of the answers from the student teachers and the advanced math students are similar with those of the students at the gymnasium, but it is even more common in these groups that the understanding is described like a deeper understanding of a whole, for example like when *puzzle pieces fall into place*, when you suddenly see that *what you have in front of you isn't just gibberish* or you *understand how things are related*. There are students at the gymnasium that explain understanding in this manner as well, but there are considerably fewer of these answers at the gymnasium than the answers that ties understanding with solving exercises and getting the correct answers. Thus what it means to understand something seems to differ a bit between the students at the gymnasium and the university students.

Students' views on some mathematical objects

In the second part of the survey some questions were asked about students' views on some mathematical notions. These questions were given with alternatives, where one alternative was reflecting Sfards' (1991) division of a structural way of looking at the notion, while the other option reflected an operational way of looking at the same notion. For example, one question was;

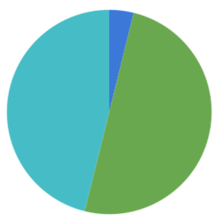
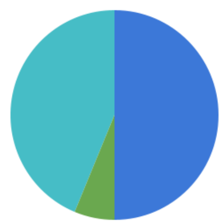
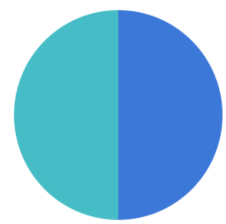
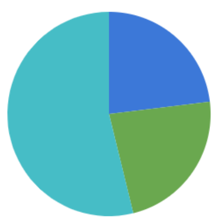
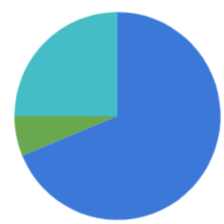
> *Which of the following descriptions of a circle is most coherent with your view of what it is? If you have another view, please write it down instead.*
> - *All points that have the same distance to a given point.*
> - *The geometric form one gets when drawing a round ring.*
> - *Both the descriptions above are coherent with my view of a circle.*

To this question, the first description; *All points that have the same distance to a given point* is regarded as the structural way of looking at a circle, while the second description; *The geometric form one gets when drawing a round ring* is regarded as the operational way of looking at the same object.

In table 13 the answers to the different questions about mathematical objects are presented, showing how big part of the students at the gymnasium, student teachers and advanced math students answered with a structural description - either the one given or if they wrote their own description that reflected a structural view -, an operational description and how many that answered that both the descriptions were coherent with their view of the object. The other mathematical notions that were asked about were functions and fractions, and the questions were put in similar ways as the question above, but with different alternatives except for the last alternative that included all descriptions. For the function the question was how students interpret $y = 3x^2$ with the alternatives being *An instruction that I for each value on x should take the square of it and multiply the answer with 3 to get the value of y* - the operational alternative - and *An expression that tells how x and y relate to one another* - the structural alternative. The last question was how students interpret ¾ with the alternatives being *It means that I should divide 3 by 4* - the operational alternative -, *It is a rational number* - the structural alternative - and this question had a third alternative, *It is 0,75,* as this is another common way to interpret ¾. However, the last alternative cannot really be described as either structural or operational, and thus, answers that consist solely of this answer have been called *numerical* in table 13 below.

Two examples of explanations the students wrote themselves instead of choosing one of the existing alternatives are, firstly of a circle, secondly of a fraction:
A continuous set where the distance from every element to the middle point is the same. STE6.
The second and the third alternative. I don't feel that I have to do anything when I see ¾. SMA2.

Table 13
Students' views on some mathematical objects

SGY	STE	SMA
Circle		
		
Function		
		
Fraction		
		
Structural	Operational	Both Numerical

Note: The question regarding fractions had an alternative that ¾ was the same as 0,75. Whether this choice reflects a structural or an operational view of ¾ is not obvious, and therefore the answers that consisted solely of that description have been called Numerical.

Worth noticing in table 13 is that with all the three mathematical objects the students at the gymnasium show more of a solely operational view than in both the other groups, and that among the students that study advanced mathematics almost no one shows a strictly operational view of any objects, but to a greater extent they chose the structural view of the mathematical object. Among the student teachers there are some that chose an operational view, several that chose a structural view, and a lot of students that state that they view the objects in both ways.

Students' answers to claims about what kind of knowledge they seek

Some of the claims on the third part of the survey were designed to show what kind of knowledge students seek when studying mathematics, and these claims and their answers are presented here.

Table 14

Claims about what kind of knowledge students seek, part 1

Claim	SGY	STE	SMA
Mathematics is mostly about rules and facts that I have to memorize.			
To solve a problem I have to learn the right procedure, otherwise it won't work.			
I fully disagree	I do not totally agree	I agree to some extent	I agree completely

The majority of students in all three groups of students do not agree with the claim that *mathematics is mostly about rules and facts that I have to memorize.* However, the group with the greatest agreement to this statement is the students at the gymnasium, and this agrees with how the students answered the question if memorization is important or if there are other things that are more important. There are also more university students that completely disagree with the statement, than at the gymnasium. However, among the advanced math students the view differs the most - here we have over a third of the students that fully disagrees, but one student agrees completely with the statement.

A majority of the students at the gymnasium agree with the claim that *to solve a problem I have to learn the right procedure, otherwise it won't work*. More than half of the advanced math students agree with this as well although there are also some of these students that fully disagree with the statement. Among the student teachers there are a majority that don't agree with the statement.

Table 15

Claims about what kind of knowledge students seek, part 2

Claim	SGY	STE	SMA
I think it is both important to know how to solve an exercise and why I should solve it in that way.			
For me it is enough to know how I should solve an exercise, I don’t put much thought into why I should do it that way.			
I fully disagree	I do not totally agree	I agree to some extent	I agree completely

A majority of the students in all three groups agree with the claim: *I think it is both important to know how to solve an exercise and why I should solve it in that way,* although there are also some students at the gymnasium that fully disagree with the statement.

All of the students studying advanced maths disagree with the statement that *for me it is enough to know how I should solve an exercise, I don't put much thought into why I should do it that way*, and a majority of the students at the other levels disagree with the statement as well. However, at the gymnasium and among the student teachers there are some students that agree with the claim and at the gymnasium there are even some that agree completely.

Table 16

Claims about what kind of knowledge students seek, part 3

Claim	SGY	STE	SMA
Mathematical problems that you face in real life can be solved by using logic, you don’t need the rules you learn in school.			
I often feel that I can answer the exercises in the mathematics book without understanding what I have done.			
I fully disagree	I do not totally agree	I agree to some extent	I agree completely

A majority of the students at the gymnasium agree with the claim that you don't need the rules you learn at mathematics classes in school when mathematical problems appear in real life, while a majority of the student teachers and advanced mathematics students disagree with the statement.

The student teachers stand out in their answers to the claim: *I often feel that I can answer the exercises in the mathematics book without understanding what I have done.* In both the other groups more than half of the students disagree with this statement, and among the advanced math students there are very few that agree with it, but among the student teachers around 75% agree with the statement.

4.3 Question 3 - How do students study mathematics?

On the survey there were questions and claims designed to help answer the first part of research question 3; how do students study mathematics? The answers to those survey questions and claims are presented in this section.

Different ways to study mathematics

The third and fourth open questions on the survey were designed to help find out how students in general study mathematics. The answers to these questions have been sorted into 7 different categories that reflect the different ways of studying mathematics that the students expressed. These categories are presented here. For each category some student answers, or part of answers, are also presented to show what typical answers in that category look like. An answer can be in many of the categories at the same time. For example, the following answer is a mix of almost all of the categories, and thus it is placed in all of the categories it is reflecting:

> *It's a mix of reading, listening and counting, the last part preferably in a group, but sometimes alone. If I listen to a lecture I listen actively and take notes of everything written on the board and if I happen to think of something interesting or something that is not totally clear. Usually there is time during the lecture to figure out how things connect, regardless if it is clear by the explanation or not. If I read I let it take time and I often shift focus, so I can assure myself of understanding every detail, but also how the details are connected and what it all means. When I have listened or read I do some exercises to test my knowledge, preferably with others to be able to explain and answer questions which helps me consolidate my own knowledge and motivates me to spend more time with mathematics. STE14*

Category 1: Doing exercises

The first category consists of answers that state that students study maths by doing exercises. A majority of the students at all three levels have answered that they study math in this way, and it is the most common answer in all three groups, except for the advanced math students, where as many answer that they study by reflecting over the mathematics and trying to understand it, category 6.

Table 17
Doing exercises

1: Doing exercises.
SGY: 21/27 = 77,8%   STE: 13/16 = 81,3%   SMA: 11/16 = 68,8%   Totally: 45/59 = 76,3%
Practice on as many exercises as possible. SGY1 *Do many exercises. SGY3* *Counting, hands down. SGY7* *...In third place I think it is important to work with mathematics. Just like with biking, you get better by doing, not by reading. STE16* *Primarily doing exercises until I meet obstacles in my understanding... SMA8*

Although it is a common answer from students in all three groups that they study by doing exercises, what the answer looks like varies a bit in the different groups, as can be seen in table 17. For example, many of the answers from students in the gymnasium are solely about doing exercises, and multiple times it's about doing *as many exercises as possible*. In the answers from the student teachers and the advanced math students, however, doing exercises are more often just one part of a longer, multifaceted answer. It's an important thing to do, but in general it is not the only way of studying.

Category 2: Studying solved examples

Some of the students stated that it is helpful to study solved examples to better get a grip on mathematics and thus, the second category consists of these answers. This is not a very common answer among any group, but it's more common among the student teachers than the other groups, where 25% states that they usually study mathematics by studying solved examples.

Table 18
Studying solved examples

2: Studying solved examples.
SGY: 2/27 = 7,4%   STE: 4/16 = 25,0%   SMA: 2/16 = 12,5%   Totally: 8/59 = 13,6%
I don't know, just by going through the example tasks and then using the same calculation in the exercises. When I have done that enough times it usually sticks. SGY11 *I usually look at the suggested solutions and use it to reflect over what steps one needs to get there. STE9*

Category 3: Listen during lectures and watch Youtube videos

The third category is made up of answers that stated something about studying by listening during lectures or watching some lecture by oneself, for example on Youtube. There are students in all three groups that claim that they are studying in this way and among the student teachers more than half of the students state that this is a way of studying mathematics for them.

Table 19
Listen during lectures and watch Youtube videos

3: Listen during lectures and watch Youtube videos.
SGY: 10/27 = 37,0%   STE: 10/16 = 62,5%   SMA: 7/16 = 43,8%   Totally: 27/59 = 45,8%
I usually focus a lot during the lectures and take notes because I have a hard time understanding without someone showing me a task and how to solve it step by step. In case I don't understand the lecture I watch Youtube videos to really understand. SGY14 *Watch videos on Youtube, read in books, listen to the lecture, count a lot. I usually need to do many of these to get a deeper understanding. STE4* *...Secondly I think Youtube videos or rehearsed lectures are very good. Many math teachers don't have an established routine or the sense to talk to people that is needed to have an informative lecture, therefore I usually complement it with alternative resources. STE16* *Watch a video or a lecture... SMA13*

Category 4: Talk about mathematics with other people

In category number four are answers that deal with talking about mathematics with other people, and this could be either by discussing mathematics, by asking for help or by explaining mathematics to others. Both the students in the gymnasium and the student teachers have answered this quite frequently, even though also in this category the student teachers are the most frequent to answer this way.

Table 20
Talk about mathematics with other people, explaining to others or asking questions

4: Talk about mathematics with other people, explaining to others or asking questions.
SGY: 11/27 = 40,7%   STE: 9/16 = 56,3%   SMA: 1/16 = 6,3%   Totally: 21/59 = 35,6%
Try to listen to the lesson and actually ask questions when I don't understand something... SGY5 *...I asked my teacher for help and we went through it out loud together. Then when I read the questions out loud and were able to directly get an answer to the things I didn't understand, it clicked. SGY11* *I like discussing, or trying to explain it to someone else. If I feel that I know it more or less, after a lecture or when I have counted a little, then I learn best by explaining it to someone else directly after. SGY21* *... Explain to others and discuss to hear how others think. STE11* *Primarily I think one learns mathematics best by talking with others and working together with them. Then one can teach each other and fill in each other's gaps, and one has someone to discuss ideas with. STE16*

The answers from the students at the gymnasium and the student teachers in this category differ a bit. The majority of the answers from students at the gymnasium is about asking the teacher for help, and getting the content explained another time. The student teachers' answers, however, seldom are about asking the teacher for help but are mostly focusing on *discussing* mathematics with one another and together get a better understanding of the subject.

Category 5: Read in the book, study notes and study the theory

The fifth category contains answers that are about studying the theory, mostly by reading some book or by studying notes. Half of all the students studying at an advanced level are studying in this way, while it is a less common answer in the other two groups.

Table 21
Read in the book, study notes and study the theory

5: Read in the book, study notes and study the theory.
SGY: 7/27 = 25,9%   STE: 5/16 = 31,3%   SMA: 8/16 = 50,0%   Totally: 20/59 = 33,9%
I study the laws and rules that show clearly what, how and why you should solve an exercise in that way. SGY19 *...then I work with the material from the lecture to make it easier to remember… SGY24* *I usually read student literature, putting time and focus on the things in the text that feels unclear and finally try to solve problems like the ones I have encountered in the litterature to confirm what I have been learning. SMA5* *Usually I count by myself and read theory… SMA14*

Category 6: Reflecting over the mathematics and trying to understand it

The sixth category is made up of answers that somehow deal with reflecting over the mathematics and trying to understand it, for example by asking oneself questions about the mathematics, by trying to visualize it or by trying to construct own examples when meeting new things in mathematics. This, together with the first category - doing exercises -, is the most common answer among the advanced mathematics students. It is not an answer that is as common in the other two groups.

Table 22
Reflecting over the mathematics and trying to understand it

6: Reflecting over the mathematics and trying to understand it.
SGY: 6/27 = 22,2% STE: 6/16 = 37,5% SMA: 11/16 = 68,8% Totally: 23/59 = 39,0%
I try to understand what I am counting, for example I ask myself questions about why one counts in that way. SGY2 *...I try to understand what the mathematical concept really means. SGY17* *I think what is in common is that I ask questions, what is it that I want this thing to be? What quality am I trying to capture with this object? What is the origin of this object? SMA2* *I usually try to think deeply about the foundations and try to really understand them. If I do I feel like more complex reasoning is easier to embrace. This takes time, but it is worth it in the end. SMA7* *I work with litterature and lectures, trying to go through proofs step by step, focusing my thoughts on the things I don't get at all. I believe it is important to get used to the feeling of not understanding things and to let it motivate you. SMA14*

Category 7: Other answers

The seventh and last category consists of unique parts of answers that didn't fit into any of the other categories, and that are too special to form an additional category. The student teachers are the group with the highest frequency of answers in this category, 37,5%, while fewer of the answers from the students at the gymnasium and the advanced mathematics students are in this category.

Table 23
Other answers

7: Other answers (that didn't fit into any of the categories 1-6 and are unique)
SGY: 4/27 = 14,8% STE: 6/16 = 37,5% SMA: 3/16 = 18,8% Totally: 13/59 = 22,0%
… I seldom ask for help but struggle with an exercise for literally hours and try to get to the answer. Then I have been experiencing the whole process and in this way I have really learnt it to the next time a similar exercise comes up. SGY16 *… My answer is actually that I need to go through the same thing in many different ways to deepen my knowledge about mathematics. STE12* *… Exercises until I am really stuck, then I sleep on it and then it usually has come off. SMA3*

Summary

Students at the different levels of mathematical studies have different ways to study mathematics, although the majority in all three groups state that they do exercises to learn mathematics. At the gymnasium this is the most common category and then it is quite well-spread between the other categories, with many stating that they learn by talking to other people about mathematics, and especially by asking the teacher for help. The student teachers have a high frequency in all of the categories, and in the first, third and fourth category more than half of the students' answers fit. This shows that many of the student teachers use multiple different ways to study. Lastly, among the advanced mathematics students there are as many that state that they learn mathematics by reflecting over it and by trying to understand it as there are that state that they learn by doing exercises. Moreover, reading the theory and listening to lectures are two other common ways of study for the advanced math students.

Students' answers to claims about how to study mathematics

In the third part of the survey there were some claims about ways to study mathematics and the answers to two of these claims are presented in table 24. The two claims presented here are about in what order you learn things in mathematics rather than about specific ways of studying mathematics.

Table 24

Claims about how to study mathematics

Claim	SGY	STE	SMA
First I learn how to solve a problem, after a while I understand why I should do it that way.			
I need to first understand why I should do it a certain way, later on I can learn how to do it.			
I fully disagree	I do not totally agree	I agree to some extent	I agree completely

A majority of both the students at the gymnasium and the student teachers agree with the claim: *First I learn how to solve a problem, after a while I understand why I should do it that way.* Among the advanced mathematics students there are as many that agree with the claim as there are that disagree with it.

Among the advanced mathematics students a majority agree with the claim: *I need to first understand why I should do it a certain way, later on I can learn how to do it*. Half of the student teachers agree with this claim as well, while a majority of the students at the gymnasium disagree with it.

4.4 Summary of the result

To get a clearer view of what the results say about how students at different levels view, understand and study mathematics this chapter ends with a summary of the results for each level of mathematical studies.

Students at the gymnasium

The most common way to view mathematics among the students at the gymnasium is that mathematics is about numbers, formulas, rules, methods and counting. Moreover, a majority of the students in this group agrees with the claim that *mathematics is all about getting the right answer on exercises*.

A majority of the students in this group hold the opinion that memorization is important and not as many think that other things are more important in mathematics. Many of the students that talk about *understanding* mention it together with getting the right answer to a mathematical task or solving exercises. Moreover, this group gives operational answers to how they view the mathematical objects in question to a bigger extent than students at the university. A majority of the students at this level agree with the claim that *mathematical problems that you face in real life can be solved by using logic, you don't need the rules you learn in school,* and although the majority of the students disagree with the claim: *For me it is enough to know how I should solve an exercise, I don't put much thought into why I should do it that way,* a larger proportion of students agree with this claim than among the students at the university.

Lastly, the students at the gymnasium usually study maths by doing exercises - and often by doing *a lot* of exercises -, by listening to lectures and by asking for help and a majority of the students in this group agree with the claim that they first need to learn how to solve a problem and after a while the understanding of why one should do it in that way comes.

Student teachers

The most common way to view mathematics among the student teachers is that mathematics is about patterns and relationships. To see mathematics as a tool to model our world and to regard mathematics as being about problem solving are also common views.

There are the same number of student teachers that agree that memorization is important in mathematics as there are student teachers that think that it is more important to understand how to use mathematics and how different parts of mathematics are connected. The answers about understanding mathematics are sometimes described as puzzle pieces that fit together. In this group of students we find both structural and operational views on the mathematical objects, with a bigger group answering in a structural way than an operational way, but it is also common to answer that both the descriptions of the object are coherent with their view. The majority of the student teachers agree with the claim: *I often feel that I can answer the exercises in the mathematics book without understanding what I have done.*

Lastly, the student teachers' answers to how they study mathematics are often a combination of multiple ways, where the most common ways are to study by doing exercises, listen during lectures and talk about mathematics with other people, often discussing it with others. All of these activities are often combined either with each other or with other ways of studying in the answers of the student teachers. A majority of the student teachers agree with the claim: *First I learn how to solve a problem, after a while I understand why I should do it that way.*

Advanced mathematics students

The most common way to view mathematics among the advanced mathematics students is that mathematics is a logical system or a language, but the view that mathematics is about abstracting to understand and about the deep truths is common as well. All of the students in this group agree that *mathematics is thought provoking and useful* and many disagree with the claim: *Mathematics is all about getting the right answer on exercises.*

The answers to the question whether memorization is important in mathematics were well spread between the answers that it is important, it is more important to understand mathematics, and that there are other things that are more important, so the answers to this question reflects a duality. The answers about understanding mathematics are often described as understanding relationships. The answers in this group about how they view mathematical objects are to a majority structural, or a combination of structural and operational. There is almost no answer that is purely operational. Furthermore, all of the advanced mathematics students disagree with the claim: *For me it is enough to know how I should solve an exercise, I don't put much thought into why I should do it that way,* and almost all of the students even disagree strongly with it. A majority of the students in this group also disagree with the claim: *I often feel that I can answer the exercises in the mathematics book without understanding what I have done*.

Lastly, in this group of students there are as many that answer that they study mathematics by doing exercises as there are that answer that they study mathematics by reflecting over the mathematics and trying to understand it. Listening to lectures and studying theory are two other common ways to study among the advanced mathematics students. Furthermore, a majority of the students in this group agree with the claim: *I need to first understand why I should do it a certain way, later on I can learn how to do it.*

5 Discussion

In this chapter the results of this study are discussed. To start with, the result will be related to the earlier presented theories. Thereafter the answers to the research questions, the chosen method and lastly future use of this result for me and other teachers will be discussed.

5.1 How the result and the presented theories relate to one another

To remind the reader of the definition of conceptual and procedural knowledge of mathematics I repeat how Hiebert and Wearne (1986) summarizes procedural and conceptual knowledge: "conceptual knowledge is knowledge that is rich in relationships but not rich in techniques for completing tasks. Procedural knowledge is rich in rules and strategies for completing tasks but not rich in relationships." (p.201). Furthermore, Sfard (1991) divides mathematical understanding into structural and operational, where the structural understanding means looking at mathematical notions as structures, or objects, while the operational understanding means understanding mathematical notions as processes, or operations. However, both Sfard and Hiebert and Lefevre (1989) emphasize the importance of the interaction between these ways of looking at mathematics, and stress that both kinds of knowledge are essential and needed to master the subject.

Students views of mathematics in the light of the presented theories

The views that the students hold of mathematics can be compared to the already existing theories. Some of the categories containing the students' answers to how they view mathematics are easily recognized as either procedural or conceptual knowledge, or as having a structural or an operational approach to mathematics, while others are harder to divide. According to Hiebert and Lefevre (1986) mathematical knowledge is not necessarily easy to separate into the two categories *procedural* and *conceptual* and this can be seen here. For example, the two categories *mathematics gives us tools to model the world and explain the world around us* as well as *mathematics is about solving problems* are not easy to categorize by comparing with the already existing theories. To view mathematics as a tool to model the world or explain the world could be reflecting both procedural and conceptual knowledge of mathematics, and the same goes for the view that mathematics is about problem solving. It depends on what the students mean when they say "problem" and how they work with mathematics to model the world. Thus, it can be different in each case.

However, the first category, that mathematics is about *numbers, formulas, rules, methods and counting,* are just the things that Hiebert and Lefevre (1986) express are regarded as procedural knowledge in mathematics. Thus, viewing mathematics in this way reflects a procedural view of mathematics.

Looking instead at the categories *mathematics is mostly about patterns and relationships* and *mathematics is a logical system or a language,* these categories reflect a conceptual view on mathematics. That mathematics is about patterns and relationships is close to Hiebert and Lefevres' (1986) definition of conceptual knowledge and so is the answer that it is a logical system or language, since a logical system is full of relationships and connections. Thus, these two categories reflect conceptual knowledge of mathematics. Moreover, the view that mathematics is about *abstracting to better understand and about deep truths* is relatable to Sfard's (1991) way of explaining reification of mathematical objects, where she expresses a need to abstract a process to get a structural understanding of a mathematical notion. One could therefore say that this category is leaning towards a structural understanding for the mathematical objects, and when students write that this is what mathematics *is about* it indicates that a structural understanding is regarded as important for them in mathematics.

Students ways of studying mathematics in the light of the presented theories

The students' ways of studying mathematics can also be compared with the already existing theories. To start with, let's look at the first category - studying by doing exercises. At first glance this category seems closely connected to procedural knowledge. However, one can do exercises in different ways and two different ways that have been expressed in the answers to the questionnaire are firstly doing *a lot of* exercises, or *only* study by doing exercises, and secondly solving exercises while reflecting over the mathematics, or doing exercises *and* studying in other ways.

There is a critical difference between these kinds of answers. Doing a lot of exercises as the only way of studying could mean that one learns the exercises by heart, or that one learns how to solve an exercise in a certain context but given a slightly different situation one does not know how to solve it, or one cannot connect the treated notions to other things one has learnt in mathematics. This is one of the things Hiebert and Lefevre (1986) write about as an obstacle to connect procedural and conceptual knowledge, the tendency to compartmentalize knowledge. Thus there is a possibility that one does not develop the relationships and connections between the mathematical content the exercise is about and other mathematical content, and thus only get a procedural understanding of the content. However, Sfard (1991) explains the process to develop structural understanding as beginning with an interiorization step, where procedures on familiar objects become efficient for the learner. Thus, this way of studying - doing a lot of exercises - might just as well be the start of the three-phase-process that develops structural understanding. However, like Hiebert and Lefevre say - just by working with procedures mathematics is not necessarily learnt with meaning. It can be, but not necessarily. To sum it up, the approach to studying by doing *a lot of* exercises is a procedural approach and even though this can be the starting point of developing structural understanding, and it might lead to a connection with conceptual knowledge along the way, this is not a given.

However, the answers that are about solving exercises as a part of many ways of studying or doing it while reflecting over the mathematics and trying to understand it, shows more clearly a strive for developing structural knowledge, or a strive to connect procedural knowledge to conceptual knowledge, and not being satisfied with learning procedures by rote.

It is not obvious how to compare the other categories of ways to study to the presented theories, but some main features are worth noticing. The second category was about *studying solved examples*. This category is a bit similar to the first one - to solve exercises - in the way that this might be done in a solely procedural way, or it might be done by trying to connect the procedural knowledge to conceptual knowledge. However, just at a first glance of the category it reflects more of a procedural approach than a conceptual one.

Category 4 was to *talk about mathematics with other people*. This category consists of answers that involve people in different ways, including both answers that expressed asking the teacher questions and answers that expressed discussing mathematics with other people. These answers could be considered either conceptual or procedural or a mix of both depending on the nature of the questions and how the discussions go.

The categories *studying theory* and *listening to lectures* can be considered two ways to conceptually approach mathematics. Certainly a lecture at the gymnasium looks different than a lecture at the university, but in both cases, and when studying theory, students are presented with new mathematical notions and it is often explained how these new notions relate to things they already know about. Thus it is a conceptual approach to learning mathematics. However, one of the obstacles to connect procedural and conceptual knowledge that Hiebert and Lefevre (1986) talk about is that the learner might have difficulties picking up on the relationships even though they are presented in appropriate ways. Thus, whether the students are able to construct the relationships and the knowledge for themselves is not certain, but at least these ways of studying indicate that students approach mathematics in a more conceptual way - trying to work out relationships - than procedural.

Lastly, the category *reflecting over the mathematics and trying to understand it*, points more to conceptual knowledge as the student tries to form relationships between the new material and other, already existing information. It could also point to a connection between procedural and conceptual knowledge, if it is procedures the students are studying and trying to understand. It could also reflect the reification process and the want to get structural knowledge, trying to understand the mathematical notions that one comes across.

5.2 Answers to question 1 - What views of mathematics do students express?

Having presented the results and discussed how the results and the existing theories relate to one another the answers to the research questions will be summarized and discussed in the following sections. The first research question is about what views students have of mathematics and if this view varies depending on what levels of mathematical studies the student attends. The views on mathematics expressed by the students in this survey are the seven categories presented in the results, namely:

Category 1: Mathematics is about numbers, formulas, rules, methods and counting
Category 2: Mathematics gives us tools to model the world and explain the world around us
Category 3: Mathematics is about solving problems
Category 4: Mathematics is about patterns and relationships
Category 5: Mathematics is a logical system or a language
Category 6: Mathematics is about abstracting to get a better understanding and about deep truths
Category 7: Other answers

Looking at how frequently the students at different levels answered, one can see a difference between how students at different levels of mathematical studies view mathematics. Of course, there are students at each level that answer differently than the majority of students at their level, but the results show that the view on mathematics from the students in this research tend to differ depending on their level of study.

More than half of the students at the gymnasium consider mathematics to be about numbers, formulas, rules, methods and counting and as explained earlier, this category reflects procedural knowledge of mathematics. Moreover, a majority of students at the gymnasium agree with the claim that *mathematics is all about getting the right answer on exercises*, which also shows a procedural view of mathematics. The other two most frequent answers among the students at the gymnasium are that mathematics is about solving problems and that mathematics gives us tools to model the world around us, which can reflect both procedural knowledge and conceptual knowledge, depending on the situation. However, the categories 4,5 and 6 that reflect conceptual knowledge are not common answers among the students at the gymnasium. Thus, we can conclude that many of the students at the gymnasium view mathematics procedurally.

The student teachers' most frequent view is that mathematics is about patterns and relationships, thus a conceptual view on mathematics. However, also category 2 and 3 are common answers among the student teachers, which can reflect either procedural or conceptual knowledge depending on the answer. Thus both procedural and conceptual knowledge is reflected in the student teachers' answers. Exactly half of the student teachers also agree with the claim: *Mathematics is all about getting the right answer on exercises* and thus it shows that some agree with this procedural view on mathematics and some don't. However, no student teacher fully disagrees with the statement.

The advanced mathematics students' most common answer is that mathematics is a logical system or a language and almost as many think that mathematics is about abstracting to get a better understanding and about deep truths. Thus, the students studying advanced mathematics have a conceptual approach to mathematics in general and regard structural understanding as an important part of mathematics. Moreover, all of the advanced mathematics students agree with the claim: *Mathematics is thought provoking and useful*, none of these students express that mathematics is about numbers, formulas, rules methods and counting, and a majority disagrees with the statement: *Mathematics is all about getting the right answer on exercises* (quite many students even *fully* disagrees with the statement) which shows that the advanced mathematics students in general do not hold a solely procedural view of mathematics.

Conclusion

The view on mathematics differs between the different levels of study. The students at the gymnasium hold a procedural view in general, or views that can be seen either as procedural or conceptual, depending on the situation. The student teachers hold a conceptual view but also views that can either be seen as procedural or conceptual Thus, their view is more conceptual than the students at the gymnasium, but still procedural to some extent. The advanced mathematics students more frequently answer in ways that are reflecting a conceptual view or a strive to reach structural understanding of mathematics, and are in general not viewing mathematics procedurally to the same extent. Thus, how students view mathematics differ between the levels of studies.

5.3 Answers to question 2 - What kind of mathematical knowledge do students seek?

To answer the second research question, what kind of mathematical knowledge do students seek, and if they are satisfied with just knowing *how* to solve mathematical tasks or if they want to know *how and why*, we will have to analyze both how they answer the question whether they think memorization is important, how they look at some mathematical objects, what they express that understanding is and to what extent they agree with different claims on the questionnaire.

Is memorization important in mathematics?

The group that most commonly agreed that memorization is an important part of mathematics were the students at the gymnasium (59,3%). However, the student teachers also agreed to a large extent (56,3%) and even though the advanced mathematics student agreed to a lesser extent (43,8%) it was still a quite common answer. Thus, students at all three levels believe memorization is important, but there is a difference in the answers to this question in the way that it is common among the students at the gymnasium to only answer that memorization is important while the student teachers and the advanced mathematics students *also* to a large extent express that it is more important to understand how to use mathematics and how the mathematics is connected, or give answers of other things they think are important in mathematics. Thus, the student teachers and the advanced mathematics students show a duality in their opinions whether memorization is important and express for example that memorization can be important, but it is also important to understand how to use mathematics or to be curious and persevering.

For example, one of the advanced mathematics students wrote:

> *Yes and no. Yes, because results can sometimes be hard to absorb from the beginning, and then you have to try to create some kind of comprehension of what they say and just remember them, hopefully to be able to come back to them later and understand them. No, because I mean that curiosity and perseverance are the important things in mathematics; not memorization. If you have the curiosity to understand something and the perseverance to actually do the needed work to understand it, then you will automatically remember the things as well. (SMA7).*

This duality of views reflects a desire to not only memorize for the sake of memorization, but to use memorization as a tool to understand mathematics. Like the student above put it, sometimes mathematical results can be *hard to absorb from the beginning, and then you have to try to create some kind of comprehension of what they say and just remember them, hopefully to be able to come back to them later and understand them.* This is similar to the way Sfard (1991) presents how mathematical objects are formed for us, starting with an interiorization phase where the learner gets familiar with a process which eventually will lead to the formation of a new concept. As the student above writes, memorizing something unfamiliar can be a part of becoming acquainted with a new mathematical notion and thus be a part of the interiorization phase and the starting point of the three-step-process to form mathematical concepts.

To answer that memorization is important but also point out that it is even more important to understand mathematics and that memorizing can be a part of that process is a quite common answer for both the student teachers and the advanced mathematics students. Thus, it seems like advanced mathematics students and student teachers strive to understand mathematical concepts and evolve their operational understanding into an also structural understanding.

The students at the gymnasium, however, more often solely answer that memorization is important. For example, one of the students at the gymnasium wrote: *Absolutely, it is important to memorize things in mathematics, especially things that can easily be applied to reality such as different types of problem solving. (SGY26)*. Another student writes that *I think it is important to remember formulas such as the pq-formula, the distance-formula etc. (SGY12).* For these students it seems like it is important to memorize things in mathematics as that is the key to use mathematics in different contexts. This is similar to the second part of procedural knowledge that Hiebert and Lefevre (1986) presents - the recipe-like, step-by-step instructions that spell out how to solve a task; one has to memorize the correct formulas to be able to use them, or the way to solve a problem to be able to do it. This also comports with the fact that a majority of the students at the gymnasium agree with the claim: *To solve a problem I have to learn the right procedure, otherwise it won't work.* Thus, these answers point to a procedural approach to mathematics.

It should however be pointed out that not all of the students at the gymnasium who answer that memorization is important give a solely procedural answer to the question. Some answers reflect the duality of the student teachers' and advanced mathematics students' answers but this is a less frequent way to answer among the students at the gymnasium compared to the other two groups of students. Thus, it seems like the students at the gymnasium have a greater tendency to learn things by rote or to memorize things as a way of being able to use them again, than students at the university. This indicates that students at the gymnasium to a bigger extent aim for procedural knowledge of mathematics, while the student teachers and the advanced mathematics students to a bigger extent tries to connect the procedural and conceptual knowledge. The quote from the advanced mathematics student above also indicates that the advanced mathematics students aim to reach a structural understanding of mathematics.

To see if this conclusion matches up with the students' answers to the other questions, the discussion will continue by discussing the student answers to how they look at some specific mathematical objects, how they describe understanding in mathematics as well as to what extent they agreed to different claims about mathematics and what is important in mathematics.

To understand

Sfard (1991) writes that the ones finding delight in mathematics are the ones that struggle with mathematics, fight through the periods of doubt and come out with a feeling of understanding. This statement is similar to the fact that a lot of the students in all three groups answered the question about what is the most fun part of mathematics with variants of *understanding something.* These answers reflect that the feeling of understanding is an important part in mathematical studies. However, reading the answers from the students, the characteristics of the understanding differ quite a bit between students at the gymnasium and at the university. Analyzing these differences will give us an understanding of how the students understand mathematics and what kind of mathematical knowledge they are seeking.

Students at the gymnasium write about understanding as solving exercises and answering correctly. For example, one student writes:

> *When one finally understands something in mathematics. It can be anything, but the feeling when one starts to understand what one is counting and feels that it starts to be simple. To find out how to solve an exercise, and then be correct. That is the most fun part of mathematics. SGY22.*

The view of understanding being about solving exercises correctly comports with the fact that many of the students at the gymnasium agree with the claim: *Mathematics is all about getting the right answer on exercises.* This view reflects a procedural view on mathematical knowledge, where understanding means having strategies for completing tasks and getting correct answers.

However, even though some answers from the student teachers are similar to the ones from the students at the gymnasium, students at university more commonly describe understanding as pieces of a puzzle that fall into place, or seeing how things are related. This reflects more of a conceptual understanding of mathematics, or of a strive to relate procedural and conceptual knowledge of mathematics, where the student searches to see relationships, connect the mathematical information to already existing knowledge and get a bigger picture of the things they study. In terms of Sfard (1991), the way students at university describe understanding is similar to deepening their cognitive schemata.

Thus there is a difference in students' views on understanding. Students at the gymnasium express more of a procedural understanding, and even though some of the student teachers do this as well, students at the university in general talk about a conceptual understanding, or strive to develop a relation between conceptual and procedural understanding. This is in line with how they answered whether memorization is important in mathematics.

The mathematical objects

The students answered the questions about how they view specific mathematical objects differently in the different groups. Among the students at the gymnasium it was quite common to answer with an operational understanding for the mathematical objects that were asked about, in particular they answered the operational option to a much larger extent than the students at the university. However, there were also quite many of the students at the gymnasium who answered that both statements agree with their view of the mathematical object and only a few chose the solely structural option. That the result looks like this indicates that not all of the students have reified the mathematical objects in question yet. This could be because they haven't yet had the need to reify them. For example, looking at the students at the gymnasiums' view of a circle, 50% of them answer the operational alternative. This might be because they haven't had any need to use any higher-level procedures on a circle, and therefore it is enough for them to view it as *the geometric form one gets when drawing a round ring* and they have no need for the structural view *all points that have the same distance to a given point* yet.

When looking at the answers of the students at the university they more seldomly answer with an operational view, but we see a difference between the answers of the student teachers and the advanced mathematics students as well. Among the student teachers there are answers that are operational and structural, more commonly structural than operational. However, many of the student teachers answer that they view the object in both the suggested ways, thus as a combination of an operational and a structural approach. This might imply that they flexibly switch between the ways of looking at the objects, depending on what situation they are in and depending on which way to look at the object benefits the situation the most.

Looking at the advanced mathematics students' answers there is a majority choosing the structural view, except for the circle, where as many choose the structural view as the combination of both views. However, almost no one of the advanced mathematics students chose the solely operational view of any of the objects. Why the students answered this way is of course debatable, but it might imply that the object in question is reified to the extent that the operational way of looking at it no longer reflects the core of the object to them. To strengthen this claim, we can look at what one of the students wrote to answer how he/she views the fraction ¾. He/she wrote that he/she agreed to the structural view and that *I don't feel that I have to do anything when I see ¾,* probably as a reaction to the first alternative that was *It means that I should divide three by four*. Thus, among the advanced mathematics students the structural view of these mathematical objects were the most common answer and this might imply that the objects in question have been reified for the students.

The things we can conclude by looking at the views of some mathematical objects of the students is that their views correspond to the conclusion that has been drawn from the answers to whether memorization is important and from what they express that understanding is. We concluded that the students at the gymnasium aim for procedural knowledge to a bigger extent than the other groups, and here we see that they view the mathematical objects in question in an operational way to a greater extent than the other groups. This implies that their focus is on procedures and operations to a greater extent than the students at the university. Moreover, the student teachers are spread in their views but quite many view objects as both operational and structural. The advanced mathematics students look at the mathematical objects structurally, which we have seen earlier are reflected in what knowledge they are searching for. Thus, the aim students' answers reflect and how they claim they view some mathematical objects match up. However, left to discuss are still some of the students' answers to the claim on the questionnaire to see if they are consistent with what we have concluded so far.

The claims

Not all of the claims on the questionnaire will be discussed in this section, but four that are relevant to the question about what kind of mathematical knowledge students seek will be discussed here.

Let's begin with looking at whether the students agree with the two claims: *I think it is both important to know how to solve an exercise and why I should solve it in that way* and *For me it is enough to know how I should solve an exercise, I don't put much thought into why I should do it that way.* A majority of the students in all three of the groups agree with the first of the claims and disagree with the second one. This shows that a majority of the students in all three groups are not satisfied with only learning procedures without meaning and only by rote. Thus, to put it in Skemps (1976) words they are not satisfied with only knowing *how,* but they think it is important to know *how and why.* Hiebert and Lefevre (1986) writes that procedures learnt with meaning are procedures that are connected to conceptual knowledge. Thus, the majority of the students in all three groups seem to be of the opinion that it is important to connect procedural knowledge to conceptual knowledge.

However, it differs between the groups how many students think that the *how* is the important part and the *why* is not as significant, or in other words, who are of the opinion that procedural knowledge is dominant. Almost a fourth of the students at the gymnasium reflect this view, a bit fewer of the student teachers and almost none of the students at an advanced level. Moreover, almost all of the students who study at an advanced level fully disagree with the second claim, that it is enough to know how, but not put much thought into why. The students in the other groups are not disagreeing as strongly. Thus, the importance of understanding *why* you use a certain method to solve a problem seems to increase the further in the education one gets.

Thus, in the answers to these claims we see that the students of the gymnasium, to a larger extent than the other two groups, reflect that procedural knowledge of mathematics is enough, even though we also see that a majority thinks it is important to connect procedural and conceptual knowledge. Moreover, the advanced mathematics students are more strongly reflecting that it is important to connect procedural and conceptual knowledge than the student teachers and the students at the gymnasium.

Another interesting claim to discuss is the claim: *Mathematical problems that you face in real life can be solved by using logic, you don't need the rules you learn in school.* More than half of the students at the gymnasium agree with this claim, more than a fourth of the student teachers and exactly a fourth of the advanced mathematics students. However, among the advanced mathematics students there is no one that completely agrees - in contrast to the other two groups - and there are more students that fully disagree in this group than in the others. An interesting aspect of many of the students at the gymnasium and quite many student teachers agreeing with this claim is that more than 40% of the answers from students at the gymnasium and student teachers to the question about what mathematics is about is in the second category: *Mathematics gives us tools to model the world and explain the world around us.* Thus, it seems like there are many of the students at the gymnasium and the student teachers that believe that mathematics is important for using in and explaining the world around us but what we learn in school cannot necessarily help us with this. This shows, in my eyes, a compartmentalization of mathematical knowledge, where the things you learn in school are only possible to use in school, and not if you encounter a mathematical problem in real life. This compartmentalization of knowledge is one of the three things that Hiebert and Lefevre (1986) point out as an obstacle to being able to construct a relationship between procedural and conceptual knowledge. Thus, the students at the gymnasium answer to a larger extent in a way that points to not relating the two types of mathematical knowledge, while the advanced mathematics students want to relate them and the opinions of the student teachers are somewhere in the middle of the two other groups.

The last claim that will be discussed here is the claim: *I often feel that I can answer the exercises in the mathematics book without understanding what I have done.* This is a claim that fewer than half of the students at the gymnasium agree with, almost none of the advanced mathematics students agree with but as many as three fourths of the student teachers agree with it to at least some extent. At a first glance it therefore might seem like the students at the gymnasium and the advanced mathematics students work with their exercises more thoroughly than the student teachers, who often feel that they haven't understood what they have done. However, the way students in the different groups have answered this claim indicates different things depending on the three groups and how the students in the group have explained what *understanding* means to them. Since the way students view understanding influences what makes them feel that they understand this is a necessary thing to take into account.

Many of the students at the gymnasium express that understanding is about solving exercises and answering correctly. Thus, these students will feel that they have understood mathematics if they answer a question in the book and it is correct, even though there might be concepts they haven't really grasped fully yet. They might as well solve the exercise with a procedure they have learned by heart and not connected to mathematical concepts and feel that they have understood the mathematics in question. That many of the students at the gymnasium disagree with the claim in question therefore only indicates that they are satisfied with their understanding when they see that they have gotten the right answer to the exercise they are working with. The students at the university on the other hand explained understanding as relating things or like puzzle pieces falling into place. Thus, to feel that they have understood what they have done they need to have understood the relationships between the different mathematical notions they work with and understood every step of the way. That a majority of the student teachers agree with the claim therefore indicates that they are not satisfied only with answering a question correctly, but they want to have understood what they have done. That they often feel that they can answer exercises without understanding what they have done reflects that they possess procedural knowledge for the mathematics in question but haven't yet connected it to conceptual knowledge. However, it is likely that student teachers are eager to understand every step of the way in completing exercises, since they later on are going to teach mathematics to others, and therefore need to have understood it themselves. Lastly, that a majority of the advanced mathematics students disagree with the claim reflects that they when answering exercises have understood relationships and they have been connecting the mathematics along the way.

Conclusion

What kind of mathematical knowledge students seek seems to differ depending on what level of mathematical studies the students are at. The students at the gymnasium seek procedural knowledge to a bigger extent than the students at the university. These two groups instead aim to connect conceptual and procedural knowledge, with the student teachers still having more focus on procedural knowledge and operational understanding than the advanced mathematics students. The answers from the advanced mathematics students reflect a strive to construct structural understanding for mathematics and a strive to connect conceptual and procedural knowledge. Thus, it differs some between the groups of students. What should be pointed out, however, is that these conclusions do not necessarily include every student in each group but holds for the groups in general.

5.4 Answers to question 3 - How do students study mathematics?

The last of the research questions is how students study mathematics and if it differs depending on what views they have of mathematics, what kind of mathematical knowledge they are seeking and at what level they are studying mathematics. The first part of this question has already been answered in the results, where the students' answers to the questionnaire resulted in seven categories consisting of ways that students study mathematics, namely:

Category 1: Doing exercises

Category 2: Studying solved examples

Category 3: Listen during lectures and watch Youtube videos

Category 4: Talk about mathematics with other people, explaining to others or asking questions

Category 5: Read in the book, study notes and study the theory

Category 6: Reflecting over the mathematics and trying to understand it

Category 7: Other answers

Thus, the thing left to discuss here is if the way of study depends on what views students have of mathematics, what kind of mathematical knowledge they seek and at what level they study.

That the way of studying mathematics differs depending on what level students study becomes clear from the result. A majority of the students in all three groups claim that they study by doing exercises (SGY 77,8%, STE 81,3%, SMA 68,8%). Although this is similar in all three groups, *how* they express that they do exercises differs between the groups. A majority of the student teachers and the advanced mathematics students express that they do exercises *and* study in other ways or reflect on the mathematics and try to understand it while doing exercises, while many of the students at the gymnasium express that their way of studying mathematics is by doing *a lot of* exercises, or to *only* do exercises. Thus, in general this answer from the students at the gymnasium are given in a more procedural manner than the student teachers and the advanced mathematics students who instead gave answers that reflected a strive to connect procedural knowledge with conceptual knowledge.

Apart from doing exercises students at the gymnasium also state that they study by talking to other people about mathematics, often by asking the teacher for help when they are stuck. Also, more than 50% of the answers of the student teachers include that they study by talking to other people about mathematics, but instead of the focus being on asking the teacher questions many of the student teachers explain that they like discussing mathematics with others. This difference in talking with other people about mathematics reflects what Stadler et.al. (2013) writes about the focus shifting from relying on the teacher in the gymnasium to other kinds of resources during university.

Additionally to answering that they study by doing exercises, the student teachers' answers are quite well spread between the other categories, which implies that the student teachers usually study in multiple different ways. One of them expresses that *...My answer is actually that I need to go through the same thing in many different ways to deepen my knowledge about mathematics. STE12.* Thus, the student teachers are in general not satisfied with only encountering mathematics in one way or getting one view of it, instead they want to get a big picture and try to connect procedural knowledge with conceptual knowledge.

For the advanced mathematics students it is common to answer, in addition to doing exercises, that they study by *reflecting over the mathematics and trying to understand it*, as well as it is to read theory and listen during lectures. All of these categories can be considered to be a conceptual approach to mathematics or an approach where the students seek to connect procedural and conceptual knowledge.

Sfard (1991) writes about how the aspects of mathematics walk hand in hand and about how sometimes the drilling of a procedure has to come before evolving understanding of the underlying concepts while at other points the understanding of an object helps in mastering the procedure. However, both the majority of the students at the gymnasium as well as the student teachers agree with the claim: *First I learn how to solve a problem, after a while I understand why I should do it that way.* Thus, it seems like both these groups of students have a need for first learning procedures while the understanding of *why* to use a certain procedure comes later. Half of the advanced mathematics students agree with this claim, so among them what comes first seems to differ more. However, a majority of the students at the gymnasium disagree with the claim: *I need to first understand why I should do it a certain way, later on I can learn how to do it*, while among the advanced mathematics students a majority of the students instead agree with this claim. Thus, it seems like the students at the gymnasium study mathematics by first approaching it operationally while the advanced mathematics students more often approach mathematics structurally and the student teachers do a little bit of both.

Conclusion

Students at different levels study mathematics in different ways. Students at the gymnasium usually have more of a procedural approach to studying mathematics and rely on the teacher to a bigger extent than the students at the university. The advanced mathematics students, on the other hand, study in ways that can be considered conceptual or strive to connect procedural and conceptual knowledge and approach mathematics structurally. The student teachers also strive to connect procedural knowledge to conceptual knowledge in their ways to study, but the approach seems to shift between being operational and structural.

5.5 Correlation between views, understanding and ways of study

Now, it is clear that the majority of the students at different levels of mathematical studies have different views on mathematics, different ways of understanding mathematics and different ways of studying it. Although there are similarities between the levels, and not all students at a certain level agree with a specific view, seek the same kind of mathematical understanding or study mathematics in a specific way, it is possible to see some more common views, understandings and ways to study in the different groups. This leaves us with two questions remaining, whether the way of study differs depending on what kind of mathematical knowledge students seek and whether it differs depending on what views they have of mathematics. These questions are harder to answer, and although it would be interesting to analyze every student's answers individually to see if there are any correlations between the views, goals and ways to study, this will not be done in this thesis. Instead, I have looked at the groups of students and tried to see if one can see a correlation between the answers. When doing this it is quite clear that there is a correlation between students' views on mathematics, what kind of mathematical knowledge they seek and their ways of studying mathematics.

The students at the gymnasium view mathematics as being about numbers, formulas and methods, they agree to a big extent that memorization is important in mathematics, explain understanding as getting the right answers to exercises, look at mathematical objects with an operational approach to a bigger extent than the other groups and they study mainly by doing exercises, often *a lot of* them. Thus, in answers to all of the questions the students at the gymnasium reflect a procedural or operational approach to mathematics to a bigger extent than the students in the other two groups. They view mathematics as though it is about processes, they strive to gain procedural knowledge, they view mathematical objects operationally and they study by solving exercises; a procedural approach to learning mathematics. Thus, one can see a correlation in the answers to what their views are, what mathematical knowledge they seek and their way of studying and it seems like the students at the gymnasium have more of a procedural view of mathematics and an operational approach in general than the students at the other levels.

The answers from the advanced mathematics students differ from the students at the gymnasium and this group answers that they view mathematics as a logical system or language and secondly that mathematics is about abstracting to understand and about deep truths. They state that although memorization is important there are other things that are more important, like understanding mathematics. When asked about their view of mathematical objects they either answer with a mix of an operational and structural understanding or with a solely structural understanding and although a majority study by doing exercises just as many express that they study by reflecting over the mathematics and are trying to understand it. Thus, the advanced mathematics students answer that they view mathematics in ways that are reflecting a conceptual view or structural understanding of mathematics, they aim to connect conceptual and procedural knowledge and strive to construct structural understanding. They view mathematical objects structurally and they study in ways that help them connect the conceptual and procedural knowledge. Thus, there seems to be a correlation between the way of viewing mathematics, understanding mathematics and the way of studying for these students as well - they reflect a wish to connect conceptual and procedural knowledge and strive to reach structural understanding for mathematical objects.

The answers from the student teachers are often “in between” the answers from the students at the gymnasium and the advanced mathematics students. Their main way to view mathematics is that it is about patterns and relationships and more than half of these students agree that memorization is important but just as many agree that understanding mathematics is more important. When asked about the mathematical objects they give both procedural and structural answers, but many also answer in both ways. When asked how they study a majority answers that they do exercises but *also* that they study in other ways, the most common being listening to lectures and discussing with other students. Thus, the student teachers are quite similar to the advanced mathematics students, striving to connect conceptual and procedural knowledge. However, there are more hints of procedural knowledge both in their views on mathematics, what kind of mathematical knowledge they seek and the way they are studying mathematics. Their answers reflect both procedural and conceptual knowledge of mathematics, and their approach to mathematical objects is mixed between structural and operational. Thus, there is a correlation between the view of mathematics, the sought kind of knowledge and the way to study also for the student teachers.

In summary, there is a correlation between the ways students at different levels view mathematics, what kind of mathematical knowledge they seek and how they study mathematics.

This conclusion leads us to another question - is it only a correlation one can see or is there also a causality - where the way of viewing mathematics and the understanding students seek have an impact on the way they study?

It would be reasonable to think that how you view mathematics and what kind of mathematical knowledge you seek impacts how you study. For example, students that view mathematics as being about numbers, formulas, etc. and believe that mathematical understanding lies in getting correct answers and master procedures will probably study by doing exercises and get fluent in procedures, since, for them, this is what mathematics is all about. Moreover, a student that views mathematics as being about patterns and relationships, or about abstracting to better understand and who believes that mathematical understanding lies in relating concepts and seeing a bigger picture, will probably study by reflecting over the mathematics while working with it, and by attacking it from multiple different angles to get a full picture of it. Thus, it is reasonable to believe that students' views and ways of understanding mathematics impacts their way of studying.

However, it could just as well be the other way around - that the way students study mathematics impacts their way of viewing mathematics and the kind of mathematical knowledge they seek. If they are taught to study by drilling procedures and solving step-by-step exercises they will probably get the view that mathematics is about numbers and procedures and that mathematics is about getting the right answers to exercises. And if a student is taught to study by asking questions about the content and trying to make out relationships between different mathematical notions it will most likely also follow that they view mathematics as being about relationships and that mathematical understanding is to see relationships, connect mathematical notions and to connect procedures to underlying concepts.

Thus, the view of, understanding of and way of studying mathematics seems to not only correlate but also in a deeper way relate to each other, but what impacts what is not possible to say from this study. It might also be that it is a cycle and that all of it impacts each other. That the views, understandings and ways to study have nothing to do with one another is however unlikely looking at the results of this study. To explore this relationship and causality further would be an interesting way to continue this research.

Reasons for the differences between the different levels

We have now seen that students at different levels view, understand and study mathematics in different ways. There are several different possible explanations for why the views, understandings and ways to study mathematics differ between the levels of mathematical studies, and four explanations will be discussed in this section.

The first thing the difference between different levels might reflect is that the way of teaching and the mathematical culture is different at different levels of study and that this impacts students’ views of and ways to study mathematics. For example, Stadler (2009) writes about school mathematics at the gymnasium in Sweden and that this kind of mathematics is focused on exercises. She writes that exercises at the gymnasium are often constructed in a way that leads the student slowly forward step-by-step, so that they finally arrive at the right answer. This way of viewing mathematics and working with exercises in such a way probably impacts students' views on mathematics, and it is natural that the most common view among the students at the gymnasium is that mathematics is about numbers and methods and that it is about getting the right answers to exercises. Moreover, when there is such a great focus on exercises at the gymnasium, it is natural to study by doing a lot of exercises for students at this level. This way of viewing mathematics and studying mathematics does not necessarily encourage the connection of procedures with conceptual knowledge.

At the university other things are expected from the students than are expected at the gymnasium (de Guzmán et.al., 1998). It is no longer enough to reproduce procedures and learn by rote at the university, but the students need to analyze and prove things themselves, which also calls for a need to connect conceptual and procedural knowledge, which is expressed by the students at the university in this study. Furthermore, Stadler (2009) writes that the academic mathematics that is common at university is focused on proofs, exploring mathematical results, and learning new, abstract mathematical notions. This view of mathematics is also reflected in the answers from the advanced mathematics students, who commonly answered that mathematics is about abstracting to understand. Working with abstract notions increases the need for personal reflection, which is how the advanced mathematics students claim that they study, together with doing exercises. Moreover, at the university, students are introduced to new abstract notions all the time, by structural definitions, and therefore students at the university are more trained in working with structural definitions than students at the gymnasium and therefore they probably also approach and look at mathematics structurally more effortlessly than students at the gymnasium.

To sum up, the approach to mathematics in teaching at different levels seems to be more procedural at the gymnasium than at the university for the students in this study, and according to the result of this study this is in agreement with students' views and ways to study. Thus, the results seem to reflect that the way of teaching impacts students' views of, understanding of and ways to study mathematics. However, this is only one of the things the results might point to.

The second thing the differences might reflect is that the students at the different levels have different purposes with their studies. For example, although the Swedish gymnasium is not compulsory education in Sweden it is very unusual to not attend it, and even though you choose a program when entering the gymnasium that program will still include many different subjects. Thus, although the students at the gymnasium in this research attend a natural science program, that is known to include a lot of mathematics, it is likely that there are people in the class that don’t have specific interest in mathematics, but are studying mathematics only because it is a part of the program. Thus, the purpose of studying mathematics for the students at the gymnasium might differ in the class, and the fact that the students at the gymnasium show a procedural view of mathematics might be because there are students that just want to pass the course and aren’t interested in understanding how the different mathematical notions are connected.

In contrast to the students at the gymnasium, the student teachers and the advanced mathematics students have chosen to continue specifically their mathematics studies. Thus, their interest in mathematics is great enough to continue studying the subject at university. However, also in these groups the purpose with their studies supposedly differ. The student teachers are studying to become teachers of mathematics, and as a teacher it is important to know as much of the subject that you are going to teach as possible. Thus, both the procedures, the concepts and how they are related are important for a teacher to know. Looking at the student teachers' most common view on mathematics in this study, it is that mathematics is about patterns and relationships, which is not surprising, as they need to understand the relations between different objects and to see the patterns to be able to explain it to their future students. If they are going to be able to teach the subject they themselves have to know how it is connected, and they have to see the patterns and the relationships.

The advanced mathematics students have chosen to study mathematics at an advanced level and have therefore gone further into the world of mathematics than both the other two groups of students. They study mainly mathematics and know that what they learn in a certain course is likely to come back later on in their studies. Thus, they have to work to deepen their schemata to be able to store and retrieve the mathematics they learn and thus they have to understand mathematics structurally to not always work operationally and process the mathematical ideas "in a piecemeal, cumbersome manner, which may lead to a great cognitive strain and to a disturbing feeling of only local - thus insufficient - understanding." (Sfard, 1991, p. 26).

Thus, the purpose students have with their studies seems to differ between the different levels of mathematical studies and it might be one of the reasons for why the views, understanding and ways of studying mathematics also differ between the different groups of students in this study.

The third thing the differences might reflect is that the students are at different levels in their mathematical understanding and that this understanding is something that grows while going further into mathematical studies. Sfard (1991) describes the progression of development of number sets as something that happens over time, starting with learning procedures to lastly reach reification of the procedures into structural objects. She also writes about this process happening with specific mathematical objects going from only an operational view of the objects to also having a structural view. Although Sfard talks about this process happening with specific mathematical objects the results of this study might point to this also being a process going on with the view of mathematics. That students at the gymnasium view mathematics more procedurally than students at the university is possible to interpret as though the view of mathematics goes through Sfards three-step-process to some extent, starting with students viewing mathematics as a subject that is about procedures and where mathematical objects are seen operationally. When they have enough experience of procedures and have gotten a lot of experience in reificating mathematical objects they will start viewing mathematics as being about structures and relationships as well. The more students encounter mathematics and work with it they start to evolve structural understanding for the objects and start to see that mathematics can be studied by understanding structures and relationships instead of only seeing mathematics as procedures. As students at the university have worked with mathematics for a longer time than the students at the gymnasium they have come further in this process and therefore look at mathematics structurally to a bigger extent than the ones at the gymnasium.

The last thing that we will discuss here that the differences might reflect is that the individuals that understand mathematics in a certain way at the gymnasium are the ones who choose to continue with mathematical studies at university. Sfard (1991) writes that it is easy to give up on mathematics before the reification comes, as this might be a period of meaninglessness and when the feeling of not understanding might show up. Moreover, Hiebert and Lefevre (1986) write that it is when you connect conceptual and procedural knowledge that you can master mathematics. Thus, it is possible that the students that already show more of a strive to connect conceptual and procedural knowledge and seek to not only have operational understanding but to also evolve structural understanding for mathematics at the gymnasium are the ones that have enough of an interest to continue with their studies at the university. In discussing this it is also important to remind the reader that all of the students answering the questionnaire chose to do it voluntarily and the asked groups of student teachers and advanced mathematics students were quite big. It might be that it was the students that find mathematics interesting, have a conceptual view of and wish to connect procedural and conceptual knowledge in these groups that were the ones that took the time to answer the survey. Thus, the conclusions in this thesis cannot be generalized to hold for all students at the gymnasium, all student teachers and all advanced mathematics students, but they hold for the students taking the time to answer the questionnaire in this research and they show a tendency of how it is at different levels.

If the chosen method had been a longitudinal study instead, it could probably help in understanding how well the alternatives above account for the different approaches to mathematics at different levels. However, this is not for me to say anything final about, but it is reasonable to think that the results point to a mix of all of them. The culture at the gymnasium compared to the culture at the university probably affects students' views on mathematics and how to study it in some way, the purpose with the students' mathematical studies probably affect their views to some extent, the understanding for mathematics as a subject is probably transforming for students going from the gymnasium to university to some extent and there are probably at least some students that already reflect a both conceptual and procedural understanding of mathematics at the gymnasium that later continues studying mathematics at the university. Thus, it is reasonable to believe that all four explanations for the differences of views, understandings and ways to study mathematics between different levels of study hold to some extent.

Are the differences necessary?

It has already been mentioned that this study is not done in a way that the conclusions can be generalized to hold for more students than the ones in the research, but still the study describes something important about differences between students at different levels that gives clues to how students at different levels in Sweden in general view, understand and study mathematics. Something worth discussing is whether these differences, in the ways of viewing mathematics and in the ways of studying mathematics at the gymnasium and the university, need to exist - if it is in the nature of things that students at the gymnasium view mathematics and study mathematics in another way than students at the university - or if it is something we could change by changing the teaching. This will be discussed here.

To begin with, this study is done in Sweden, so when describing differences in teaching at the gymnasium and the university this is described in a Swedish context. Thus, one way to answer the question whether the differences need to look like they do in this research would be to study how teaching at similar levels looks in other countries, and whether students' views, understandings and ways to study are the same although they have experienced other kinds of teaching. However, this is not something that has been done in this research, and therefore not something I will discuss further, but it would be an interesting way to proceed with the research.

Instead, I want to once again go back to Sfard’s (1991) three-step-procedure and the natural way to start with an operational view on mathematical objects to later be able to develop structural understanding as well. This procedure tells us that it is wise to start with becoming familiar with and accustomed to procedures before trying to understand objects structurally. Thus, it seems understandable that the development of the knowledge from the gymnasium to the university goes from focusing on operational understanding and procedural knowledge to trying to connect the procedures with underlying concepts and develop the operational understanding to a structural understanding to a greater extent as well.

However, Sfard (1991) writes about this three-step-process as something that happens for every mathematical object we meet and that we are always somewhere in the three-step-process. Moreover, developing structural understanding in addition to operational understanding develops our schemata and helps us to easily retrieve and store information. Thus, this is a process students at the gymnasium are also going through and a process that, when done, benefits them. Therefore I believe it would be helpful for students at the gymnasium to be exposed to contexts where they need to use the mathematics they learn in new settings and where they get to explore mathematics themselves so that the need to reify their knowledge comes up. However, it is natural to not introduce concepts of mathematics that are considered to be at a higher level than what should be introduced at the gymnasium already at the gymnasium and therefore, some objects might have to wait with the reification until reaching university, since there are no real need to reach the structural understanding of that object yet. However, this is also true at university - you don’t introduce new concepts until you do - but even so the results of this research indicates that students there find a greater need to develop structural understanding and to connect procedural and conceptual understanding. Maybe it is possible to reach this need and strive as well at the gymnasium with the right guidance from the teacher. This will be discussed further in the section about didactical consequences, section 5.7.

5.6 Method discussion

In this section the choices that have been made during the course of the research will be discussed. More specifically, the chosen research questions, the theories the research have been based on, the method used in the research and the way of analyzing the results will be discussed.

The research questions

The research questions were formulated from the desire to research students’ views on mathematics and how this affects their way of studying. They have worked as a guide throughout the research, helping in the formulation of survey questions, in the analysis of the answers to the questionnaire and in keeping track of what is important for the research and what is not. The questions were formulated in a concrete enough way to help with all of this, they pinpoint the aim of this paper and they have been interesting to work with. The questions have been answered, except for the last part of the third question about if the ways of study differ depending on views and ways to understand mathematics. It has been established that there is a correlation between these things, but the causality could just as well be the other way around - that the way of studying impacts views and understandings. Thus, although the last part of the third research question could not be fully answered in this research, the research questions have been appropriate and have been a guiding star through the whole process. However, during the course of the research a lot of new interesting questions have risen as well, that would also be interesting to research, such as *does this difference between views, understanding and ways to study at different levels have to exist* and *how does the teaching at the gymnasium and the university impact students' views on mathematics*. These questions would be interesting to investigate further in other research.

Relevance of the chosen theories

Even though constructivism hasn't been explicitly mentioned many times in the course of this thesis the learning theory has been underlying at all times. Taking a constructivist perspective has resulted in the research questions being formulated in the way they were with the focus on students' thoughts and views, and this holds for the questionnaire and the analysis as well, having the thoughts and opinions of the students in focus and seeing how their thoughts and views affect their learning. Moreover, both Hiebert and Lefevres' (1986) and Sfards' (1991) theories are relevant in the light of constructivism, where both theories talk about each student constructing mathematical knowledge and Sfard explaining the three-step-process for the development of structural understanding. Thus, the thesis is closely connected to a constructivist perspective, and with the chosen theories and the way of doing the research this seems to be a proper learning theory to have as a basis for this study.

Both Sfard's (1991) theory and Hiebert and Lefevres' (1986) theory have been important to better understand students' views of, understanding of and ways to study mathematics. Hiebert and Lefevres' division of mathematical knowledge into procedural and conceptual has been essential in the comparison of the answers students have given and has helped in the work to see if the views, understanding and ways to study correlate. It is a theory that is easy to understand, straightforward to apply in the analysis and even though it was written at the end of the 20th century we have seen in this research that it is relevant today.

Sfard's (1991) theory has also been useful in the work of finding correlations in the result, but it has also helped deepen the discussion. As well as helping find correlations between student answers Sfard's theory has been helpful in trying to understand these correlations and explain the differences. Moreover, Sfard's theory gives insights into how we develop mathematical understanding and why understanding mathematics can be difficult in a way that together with the results in this thesis have given me as a mathematics teacher thoughts on how to develop the teaching of mathematics. Some of these thoughts are presented in section 5.7: Didactical consequences.

Thus, the chosen theories have been relevant for this research, they have helped the analysis of the results and broadened the discussion.

Method and analysis

Using a mixed methods approach has been useful for the research and has helped in answering the research questions to a great extent. As mentioned in chapter 3: Method, the mixed methods approach was chosen to get an as accurate view of reality as possible by getting as much information from the students as possible. This has been a good way of working, and combining qualitative and quantitative data has deepened the perspectives. Having open and closed questions on the questionnaire has been essential for the validity of the research. While analyzing the result it has been substantial to have both the freely written answers from the students as well as their answers to the claims. These answers have completed each other and strengthened and confirmed conclusions that have been drawn.

As the study involves qualitative methods it is given a lot of attention to how students' answers are put. However, the study also involves quantitative methods and therefore the answer frequency in the different groups should also be discussed. Among the students at the gymnasium 27 out of 32 students answered the survey (about 84%). Among the student teachers there were only 16 out of about 120 students that answered the survey (about 13%), and the questionnaire was sent out to a group of about 360 students who mainly studies mathematics (although it is not known how many of these students fit the requested description of studying advanced mathematics) with 16 answering students (about 4%). In both of the groups at the university it is not known how many students actually read the posts, but there are many students that might have been reached by the questionnaire in both these groups, but not so many that have answered. Thus, the answer frequency at the gymnasium is high while it is low at the university. Since the answer frequency is low at the university, there might be people that are not represented in this study, and therefore the percentages of how many students in a group answer a certain way cannot be generalised, which this study never claims. However, having more time or being more than one person, it would be rewarding to collect more answers, either by doing a longitudinal study instead, or to ask a greater random sample of students to answer the survey instead of the sample that was chosen now. In this way the result could have been generalised to not only hold for the groups of students in this research.

However, even though the percentages of how many students at a certain level view mathematics in one way or another cannot be generalized to hold for all students at that level, there are more than one or two people that participate in every group, and thus we can see some tendencies and patterns in the groups anyway. Moreover, the conclusions drawn are drawn out of great percentual differences - sometimes it differs more than 50 percentage points between the answers in the groups - and not out of small variances. Thus, there is no reason to believe that other students in other settings would answer in a totally different way, as the questions asked are what brings out the answers. Anyway, I have worked to describe and analyze my methods in a transparent way, so that the reader itself can decide whether the conclusions are possible to generalise to another setting.

At the beginning the plan was to follow up the questionnaire with interviews with some of the students that answered, to strengthen the reliability by getting a clearer view of the students' thoughts in case the answers were flawed and inadequate to have as a basis for conclusions. Doing interviews would have given the students the opportunity to expand upon their answers and there might be things that show up during an interview that students don't have the time or energy to write in a questionnaire. However, when reading the answers to the questionnaire and starting the categorization of the answers I realized that the data was rich, the answers were detailed and they were enough to be able to see patterns in and draw conclusions from. Thus, the interviews would take more time and energy than what they would yield and therefore I chose not to conduct any interviews, which I believe was a good choice for this research. However, I still hold the opinion that interviewing students would be a good complement to sending out a questionnaire in other research.

As mentioned in chapter 3: Method, a few of the charts showing how the claims were answered have not been presented in this thesis as there were a lot of claims in the questionnaire and not all of them were relevant to this thesis. In the same way two questions on the second part of the questionnaire, about students' views on some mathematical notions, and their answers haven't been presented in this thesis. The first of these questions gave the students a chance to, in their own words, explain what a function is, and the second asked which one of two different kinds of exercises the students found most interesting. The first of these questions would have been interesting to analyze further and to use in another research, but in this research it was similar to the presented question about students' views of a function and would not add anything different to the conclusions of this research. Thus, the answers haven't been presented. The second question and its answers were difficult to use in any way, and therefore they were also left out.

The coding and quantitative content analysis that led to the categorization of students' answers was a good way to analyze the large quantity of collected data. It helped locate the essential information in a student's answer, and it helped to see structures in the collected data. Moreover, the categories that emerged were possible to compare to the chosen theories, and the comparison showed both the use of dividing mathematical knowledge into procedural and conceptual knowledge, since this helped to see correlations, as well as the difficulty of pinpointing what kind of knowledge an answer sometimes reflects, as for some categories the reflected knowledge depends on the situation.

In this thesis the correlations between views, understandings and ways to study mathematics are analyzed groupwise and not for each student individually due to lack of time. However, from the start the plan was to look at the answers of every individual student to see if one could see correlations in their views, understanding and ways to study, but since the analysis of the answers took a lot of time this had to be omitted. However, to look at every individual student's answer and see if there are correlations in views, understanding and ways to study would be an interesting way to continue the research.

5.7 Didactical consequences

Working with this thesis has awakened a lot of thoughts in me about mathematics and how to introduce new mathematical objects to students. Thoughts have arisen on how to take what the results show in this research and apply it to my future work as a math teacher at the gymnasium, but it has also been an interesting introspection to how I myself view mathematics and how my own experience of mathematics has been at the gymnasium compared to my experiences at the university. Some of these reflections will be presented in this section. Due to lack of time and the possibility that it is not of interest to the reader I will however not write about my own views of mathematics and my own experiences at different levels of studying mathematics in further detail.

As expressed earlier, this study does not show whether it is the views and understanding that affect students' ways of study, or if it is the other way around, but we have seen that students express views and understanding that are in agreement with the teaching at the gymnasium and the university. Thus, it seems suitable to start with the reflection that it is important for me as a teacher at the gymnasium - and for teachers at any level - to know what views of mathematics we hold as this is something that will be reflected in our teaching and that our students might pick up on. If I as a teacher contemplate on my views on mathematics and think over whether I value some part of mathematics more than another I will be aware of what my teaching is likely to reflect, and can focus on bringing to light the parts that I miss in my own views and understandings as well. After all, the curriculum at the Swedish gymnasium includes both learning procedures and concepts (Skolverket, 2021) and we have seen the necessity of connecting the two types of knowledge and thus, being aware if one as a teacher believes that the procedures are more important than concepts or the other way around, can make it easier to not be one-sided in one's teaching, but to help students get effective in their procedures as well as understanding the underlying concepts and connect the two types of knowledge.

Moreover, this study shows that students have different views of mathematics, and as a teacher I will meet students with different views of and different ways of understanding mathematics. Being aware of different ways to view and understand mathematics can help me as a teacher, since if I can see and meet the students where they are, instead of having my own views and understandings as a starting position in the teaching, it will be easier for the students to understand what I try to teach. Furthermore, to see mathematical knowledge as a combination of procedural and conceptual knowledge and to see that one can understand mathematical notions both structurally and operationally will help enable me to find out if students understand a mathematical object in one way, and then create situations that will help them to see it the other way as well. Even though they themselves have to construct the knowledge, I can scaffold them to find their zone of proximal development where they can develop the lacking understanding.

The didactical consequences mentioned above are consequences that can be applied to any teacher at any level. However, as has been seen in this research there is a difference in how students at the gymnasium, student teachers and advanced mathematics students view, understand and study mathematics, with the students at the gymnasium having more of a procedural approach than students at the university in general. Now, looking at what has been written about the transition from studying mathematics at the gymnasium to the university it is clear that this is a transition that students struggle with. This isn’t very surprising when seeing that the views, understandings and ways to study differ depending on the level of mathematical studies. Thus, looking at the transition in the light of this research we can draw the conclusion that one thing that might help ease the transition would probably be to reduce the gap between views and understandings of mathematics at different levels of studying. Since we have concluded that it is the connection of procedural and conceptual knowledge that leads to mathematical success, it is worth contemplating how to help students at the gymnasium to find this connection to a bigger extent, as this is a connection students at the university seem to seek.

Thus, that students at the gymnasium searched for procedural knowledge to a bigger extent than the students at the university and did not seek to connect the procedural knowledge to the conceptual knowledge to the same extent is something I would like to address as a teacher at the gymnasium. While students at the gymnasium in this study expressed that they study by doing lots of exercises, many of the advanced mathematics students expressed that they study mathematics by reflecting over it and some wrote that they did this by asking themselves questions about the mathematics they were studying. This seems to be an effective way to connect procedural and conceptual knowledge and therefore I want to help the students I teach to start asking these kinds of questions. This can be done by beginning with me asking them questions that makes them reflect over the mathematics they work with, that helps them connect procedures and methods to mathematical concepts, and helps them not compartmentalize knowledge. Hopefully this will eventually lead to the students themselves asking questions that lead to a connection of procedural and conceptual knowledge. Questions that might be a help in this is for example, *Is that a procedure that will always work? Why or why not? How are the things we talk about today connected to what we have discussed earlier? Can you give an example of this?* and *Can you explain why this works the way it does?*

Moreover, two claims that students at the gymnasium agreed with to a great extent is that *mathematics is about getting the right answers on exercises* and *mathematical problems that you face in real life can be solved by using logic, you don't need the rules you learn in school.* These two claims show that students at the gymnasium think the important thing with solving exercises is getting the right answers and that the mathematics they learn at school is not connected to their experiences with the world around them. This, together with the fact that Stadler (2009) writes that students at the gymnasium are used to step-by-step exercises leading them to the correct result makes me think that it is important to involve more explorative exercises that are inspired by real-world problems at the gymnasium, without an obvious procedure to use to solve it and not necessarily with one correct answer. Doing this will give the students an opportunity to struggle with the mathematics in a way where it isn't enough to only use procedures learnt by rote, but it will be necessary for the procedures to get connected to the underlying concepts and the mathematics to be learnt meaningfully. However, since this is a way of working that students aren't used to, the feelings of doubt that Sfard (1991) explains can manifest during the period right before reification might show up. Sfard also explains that the ones that fight through this feeling of doubt are the ones that find delight in the subject. Thus, to involve exploratory tasks in the teaching might bring doubts of understanding to the students, and to not having the students give up when the feelings of doubt shows up, there has to be a culture where the doubts are okay. However, how to create such a culture and help students in the doubtful periods is not something this thesis explores, but it is something that would be interesting to investigate further as this seems to be an important key to whether students enjoy mathematics or not.

The things mentioned above - to help students at the gymnasium to struggle with mathematics and not only get led step-by-step through exercises, to help students learn to ask themselves questions that lead to the connection of procedural and conceptual knowledge and to help students understand that it is okay to doubt their understanding of the subject, as the feeling of understanding can come when least expected - would likely help students at the gymnasium connect procedural and conceptual knowledge to a bigger extent. This would, in turn, lead to the gap between views and understanding at gymnasium and university being reduced. A reduced gap would most likely contribute to an ease in the transition from mathematical studies at the gymnasium to the university. Surely, there are other things that could help in this transition as well, not to mention that there are things that the university could do to help their students with the transition, but this will not be discussed further here. To research how universities can act to help students in the transition from studying mathematics at the gymnasium to studying it at university would however be another interesting way to continue this research.

5.8 Final words

To sum up, the result in this research shows that views of, understanding of and ways to study mathematics correlate for students at a certain level of mathematical studies, but they differ between different levels. The students at the gymnasium approach mathematics more procedurally than the students at the university, and the advanced mathematics students reflect a strive to reach structural understanding to a greater extent than the other two groups. These results answer the formulated research questions, but also give rise to new questions that would be interesting to continue to research, such as if there has to be these differences between different levels of study and how teaching affects students' views, understandings and ways to study mathematics.

Lastly, I want to highlight two things that I bring with me into my future work as a teacher, and my future mathematical journey in general. Firstly, it is that views, understanding and ways to study correlate, which implies that if I want to impact either one of them in any way it is a good idea to work with all of them, as one thing will probably affect the other. Secondly, it is that procedural knowledge and conceptual knowledge of mathematics are *essential* for each other and that it is in the connection of the two that mathematical success lies. Thus, knowing how things relate but not being able to do anything with it does not necessarily mean knowing mathematics and neither does having knowledge of a thousand formulas without understanding how they are related to other mathematical content. Mathematics and being a mathematics teacher is more complex than that.

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Appendix A

The original Swedish version of the questionnaire

Part 1

Masterarbete - matematik och lärande

Hej!
Jag håller på att skriva ett masterarbete som handlar om gymnasieelevers och universitetsstuderandes uppfattning av matematik, hur man studerar matematik och vad man önskar lära sig under en matematikkurs. Det hade hjälpt mig oerhört om du ville svara på frågorna nedan. Enkäten består av 4 delar och tar ca 10 minuter att fylla i. Du är helt anonym så försök att svara så sanningsenligt du kan. Svaren kan komma att användas i min masteruppsats. Tack för din hjälp!

Vad är det roligaste med matematik enligt dig?

Ditt svar

Tycker du att det är viktigt att memorera saker inom matematik? Finns det andra saker du tycker är viktigare? Ge gärna något exempel för att förtydliga ditt svar.

Ditt svar

Om du tänker på en gång då du verkligen kände att du förstod något inom matematiken, vad tänker du på då? Hur gjorde du för att uppnå den förståelsen?

Ditt svar

Hur brukar du göra för att lära dig matematik?

Ditt svar

Utifrån den matematik du känner till och har kommit i kontakt med - vad skulle du (kortfattat) säga att matematik handlar om?

Ditt svar

Part 2

Matematiska begrepp

Frågorna här handlar om hur du uppfattar olika matematiska begrepp.

Beskriv kort din uppfattning av vad en funktion är.

Ditt svar

Vilken av följande beskrivningar av en cirkel stämmer bäst överens med din bild av vad det är? Har du en annan uppfattning får du gärna skriva ner den där det står övrigt.

- ○ Alla punkter som har samma avstånd till en given punkt.
- ○ Den geometriska form jag får när jag ritar en rund ring.
- ○ Båda ovanstående beskrivningar passar in på min bild av en cirkel.
- ○ Övrigt: ____________

Vilken av följande beskrivningar passar bäst in på hur du tolkar $y=3x^2$? Har du en annan uppfattning får du gärna skriva ner den där det står övrigt.

- ◯ En instruktion för att jag för varje värde på x ska ta det i kvadrat och multiplicera med 3 för att få y:s värde.
- ◯ Ett uttryck för hur x och y förhåller sig till varandra.
- ◯ Båda ovanstående beskrivningar passar in på min tolkning av $y=3x^2$.
- ◯ Övrigt: ____________________

Vilken eller vilka av följande beskrivningar passar bäst in på hur du tolkar 3/4? Har du en annan uppfattning får du gärna skriva ner den där det står övrigt.

- ☐ Det innebär att jag ska dela 3 med 4.
- ☐ Det är ett rationellt tal.
- ☐ Det är 0,75.
- ☐ Alla ovanstående beskrivningar passar in på min tolkning av 3/4.
- ☐ Övrigt: ____________________

Vilken av följande uppgifter tycker du är intressantast att arbeta med?

- ◯ Beräkna 23% av 569.
- ◯ Du har 50 000 kr som du vill sätta in på ett sparkonto, och har två alternativ. Det första alternativet är ett konto där du årligen får 2% ränta. Det andra är ett konto där du månadsvis får 0,05% ränta. Vilket konto bör du välja om du tänker låta pengarna stå orörda i fem år och vill tjäna så mycket som möjligt?

Part 3

Påståenden

Nedan följer ett antal påståenden och jag vill att du kryssar i hur starkt du håller med om påståendet.

I vilken grad håller du med om följande påståenden?

	Jag håller inte med överhuvudtaget	Jag håller inte helt med	Instämmer till viss del	Instämmer fullständigt
Matematiken handlar om att komma fram till rätt svar på uppgifter	○	○	○	○
Matematiken handlar mest om regler och fakta som jag behöver memorera	○	○	○	○
Matematiken är tankeväckande och användbar	○	○	○	○
För att lösa ett problem behöver jag lära mig rätt procedur, annars går det inte	○	○	○	○
Jag lär mig av att härma den arbetsgång läraren visar	○	○	○	○
Jag tycker om att undersöka själv och att inte få allt presenterat för mig	○	○	○	○
Jag tycker att det både är viktigt att veta hur jag ska lösa en uppgift och varför jag ska lösa den på det sättet	○	○	○	○

För mig räcker det att veta hur jag ska lösa en uppgift, jag lägger ingen vikt vid att förstå varför jag ska göra på det sättet	○	○	○	○
Verklighetsbaserade matematiska problem kan man lösa med ren logik, utan att behöva använda de regler man lär sig i skolan	○	○	○	○
I matematik är saker antingen rätt eller fel	○	○	○	○
Jag lär mig först hur jag ska göra för att lösa ett problem och efter ett tag brukar jag förstå varför jag ska göra på det viset	○	○	○	○
Jag behöver först förstå varför jag ska göra på ett visst sätt, sen kan jag lära mig att göra det	○	○	○	○
Mycket av det jag lärde mig i matematiken på gymnasiet upplevde jag att jag förstod ordentligt först på universitetet	○	○	○	○
Jag känner ofta att jag kan svara på uppgifterna i matematikboken utan att förstå vad jag har gjort	○	○	○	○
Jag funderar alltid på om svaret jag angett är rimligt	○	○	○	○

Part 4

Frågor om dig

Studerar du på gymnasiet eller universitetet

- ◯ Gymnasiet
- ◯ Universitetet
- ◯ Övrigt: ______

Vilket år studerar du?

Ditt svar

Vad för matematikkurs studerar du just nu?

Ditt svar

Eftersom det i en enkät inte går att utveckla sina svar på samma sätt som vid en intervju hade jag velat intervjua några av er som svarat på enkäten efteråt. Hade det varit okej om jag kontaktar dig för en intervju? I så fall är du inte längre anonym, men det är bara jag som kommer att veta vem det är som har svarat. Vad heter du i så fall och vad är din email/ditt mobilnummer?

Ditt svar

Appendix B

The English translation of the questionnaire

Part 1

Master thesis - mathematics and learning

Hello!
I am in the process of writing a master thesis about students at the gymnasiums' and university students' views of mathematics, how to study mathematics and what one wishes to learn during a mathematics course. It would help me enormously if you wanted to answer the questions below. The questionnaire contains 4 parts and takes about 10 minutes to fill in. You are completely anonymous so try to answer as truthfully as possible. The answers might be used in my master thesis. Thank you for your help!

What is the most fun part of mathematics according to you?

Do you think it is important to memorize things in mathematics? Are there other things that are more important? Please give an example to clarify your answer.

If you think of a time when you really felt that you understood something in mathematics, what do you think of? What did you do to reach that understanding?

What do you usually do to learn mathematics?

When thinking of the mathematics you have encountered so far, what would you (shortly) say mathematics is all about?

Part 2

Mathematical notions

The questions here are about how you view some mathematical notions.

Shortly describe your idea of what a function is.

Which of the following descriptions of a circle is most coherent with your view of what it is? If you have another view, please write it down instead.

- All points that have the same distance to a given point.
- The geometric form one gets when drawing a round ring.
- Both the descriptions above are coherent with my view of a circle.
- Other:...

Which of the following descriptions is most coherent with your interpretation of $y = 3x^2$? If you have another view, please write it down instead.

- An instruction that I for each value on x should take the square of it and multiply the answer with 3 to get the value of y
- An expression that tells how x and y relate to one another
- Both the descriptions above are coherent with my interpretation of $y = 3x^2$.
- Other:...

Which of the following descriptions is most coherent with your interpretation of 3/4? If you have another view, please write it down instead.
- It means that I should divide 3 by 4
- It is a rational number
- It is 0,75.
- All the descriptions above are coherent with my interpretation of 3/4.
- Other:...

Which of the following exercises do you think is the most interesting to work with?
- Compute 23% of 569.
- You have 50000 kr that you want to put into a savings account and have two alternatives. The first alternative is an account where you get a yearly interest rate of 2%. The other is an account where you get an interest rate of 0,05% every month. Which account should you choose if you're planning on leaving the money untouched for five years and want to earn as much as possible?

Part 3

Claims

Below follows a number of claims where I want you to choose how strongly you are agreeing with the claim.

To what extent do you agree with the following claims?
(The levels of agreement being *I fully disagree, I do not totally agree, I agree to some extent, I agree completely*)
- ➢ Mathematics is all about getting the right answer on exercises.
- ➢ Mathematics is mostly about rules and facts that I have to memorize.
- ➢ Mathematics is thought provoking and useful.
- ➢ To solve a problem I have to learn the right procedure, otherwise it won't work.
- ➢ I learn by imitating the way of working that the teacher shows.
- ➢ I like to research myself and to not get everything presented to me.
- ➢ I think it is both important to know how to solve an exercise and why I should solve it in that way.
- ➢ For me it is enough to know how I should solve an exercise, I don't put much thought into why I should do it that way.
- ➢ Mathematical problems that you face in real life can be solved by using logic, you don't need the rules you learn in school.
- ➢ In mathematics things are either right or wrong.
- ➢ First I learn how to solve a problem, after a while I understand why I should do it that way.
- ➢ I need to first understand why I should do it a certain way, later on I can learn how to do

it.

- ➢ A lot of what I learned in earlier math courses I feel that I understood for real when I started studying at this level.
- ➢ I often feel that I can answer the exercises in the mathematics book without understanding what I have done
- ➢ I always reflect on whether the answer I give is reasonable.

Part 4

Questions about you

Are you studying at the gymnasium or at university?

- The gymnasium
- The university
- Other:....................................

What year are you studying?

What mathematics courses are you attending right now?

Since it isn't possible to expand upon one's answers in a questionnaire in the same way as at an interview I would like to interview some of you who answered the questionnaire afterwards. Would it be okay with you if I contact you for an interview? In that case you are no longer anonymous, but it is only me who will know who has answered. If that is the case, what is your name and what is your email/mobile number?