Thesis for the Degree of Doctor of Philosophy

Differential forms and currents on non-reduced complex spaces with applications to divergent integrals and the $\bar{\partial}$-equation

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Abstract

This thesis consists of three papers in which we study differential forms and currents on complex spaces. An important tool for us is the theory of residue currents.

In Paper I we study divergent integrals over singular differential forms on a complex manifold. The differential form should have a pole along a complex hypersurface. To such a differential form we associate a residue form and a current with properties similar to residue currents. We connect the residue form and the current in a formula which can be thought of as a residue formula in this setting.

In Paper II we solve the $\bar{\partial}$-equation for (p, q)-forms on non-reduced complex spaces. It is not obvious what smooth differential forms and currents should be on a non-reduced space. We define these objects using residue calculus and show that we can (locally) solve the $\bar{\partial}$-equation.

In Paper III the setting is similar to that of Paper I but we now allow the differential form to be singular on a complex submanifold of higher codimension.

Preface

This thesis consists of the following papers.

- **Paper I.** Mattias Lennartsson, *"Residues and currents from singular forms on complex manifolds"*, preprint.
- **Paper II.** Mats Andersson, Richard Lärkäng, Mattias Lennartsson and Håkan Samuelsson Kalm, *"The $\bar{\partial}$-equation for (p, q)-forms on a non-reduced analytic space"*, preprint.
- **Paper III.** Mattias Lennartsson, *"Residues of singular differential forms on complex submanifolds"*, preprint.

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Mattias Lennartsson
Göteborg, November 2020

Contents

“The fundamental cause of the trouble
in the modern world today
is that the stupid are cocksure
while the intelligent are full of doubt”

Bertrand Russel

Introduction

This thesis is concerned with the field of complex analysis which is the study of derivatives and integrals of functions defined in the complex plane $\mathbf{C}$, or more generally $\mathbf{C}^n$ and even more generally on complex spaces. Important tools for us will be distributions and currents which yield a way of giving a meaning to derivatives of functions which are not differentiable.

1. Analysis in the complex plane

Let us begin by discussing some basic notions of complex analysis in one variable. The $\bar\partial$-operator applied to a function $f : \mathbf{C} \to \mathbf{C}$ is given by

$$\bar\partial f = \frac{\partial f}{\partial \bar z} \mathrm{d}\bar z, \tag{1}$$

where $\partial f/\partial\bar z = \big(\partial f/\partial x + \mathrm{i}\partial f/\partial y\big)/2$. If the function f is continuously differentiable then it is holomorphic if and only if $\bar\partial f = 0$, which is a compact way of writing the Cauchy–Riemann equations. The function f may of course be defined on some open subset of $\mathbf{C}$ but for simplicity we will often formulate results and formulas for holomorphic functions on all of $\mathbf{C}$ (or $\mathbf{C}^n$). We will do so throughout this introduction.

A function is said to be meromorphic if it is holomorphic everywhere except possibly at discrete points where it has poles. Given a meromorphic function f and a point $z_0 \in \mathbf{C}$ the residue

$\mathrm{Res}(f, z_0)$ is defined to be the z^{-1} coefficient in the Laurent series expansion of f around z_0. If f has a pole of order m at z_0, so that $(z - z_0)^m f(z)$ is holomorphic, then

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!}\Big(\frac{\partial^{m-1}}{\partial z^{m-1}}(z - z_0)^m f(z)\Big)\Big|_{z=z_0}. \tag{2}$$

The residue theorem says that for a simple, closed and positively oriented curve γ we have

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z\in \mathrm{Int}(\gamma)} \mathrm{Res}(f, z) \tag{3}$$

where $\mathrm{Int}(\gamma)$ denotes the bounded component of $\mathbf{C} \setminus \{\gamma\}$. We also need that f does not have any poles on the curve γ.
Let γ be a curve as above and f a holomorphic function in a neighbourhood of the closure of $\mathrm{Int}(\gamma)$. Cauchy's integral formula says that

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)\mathrm{d}w}{w - z} \quad \text{for } z \in \mathrm{Int}(\gamma). \tag{4}$$

This means that the Cauchy kernel reproduces holomorphic functions. More generally, if ϕ is smooth in a neighbourhood of $\overline{\Omega}$ then

$$\phi(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} \frac{\phi(w)\mathrm{d}w}{w - z} - \frac{1}{2\pi\mathrm{i}} \int_\Omega \frac{\bar\partial\phi \wedge \mathrm{d}w}{w - z} \quad \text{for } z \in \Omega. \tag{5}$$

A consequence of this formula is that the form $\frac{\mathrm{d}z}{2\pi i z}$ is a fundamental solution of the $\bar\partial$-operator, i.e. a distributional solution of the equation $\bar\partial u = [z = 0]$, where $[z = 0]$ denotes the Dirac distribution at the origin. This implies that if $v = \psi \mathrm{d}\bar z$ is a form of bidegree $(0, 1)$ with compact support then $u = \frac{\mathrm{d}z}{2\pi i z} * v$ is a solution of the equation $\bar\partial u = v$; here $*$ denotes convolution. Writing out what this means we have

$$v = \bar\partial\Big(\frac{1}{2\pi\mathrm{i}} \int_\Omega \frac{\mathrm{d}w \wedge v}{w - z}\Big). \tag{6}$$

2. Residue currents and residue classes

A natural question is whether or not one may define residues of meromorphic functions of several complex variables in such a way that we get a corresponding residue theorem as in (3).
Trying to define residues for meromorphic functions on $\mathbf{C}^n$, one main difference to the one variable case is that the polar set of a meromorphic function in several variables, if non-empty, cannot be compact. To get around this problem one may associate a residue current to a meromorphic function instead of merely a "residue number" as in the classical approach.
Recall that the space of distributions of $\mathbf{C}^n$ is the dual of the space of compactly supported smooth functions in $\mathbf{C}^n$. Similarly a current is an object acting on smooth differential forms with compact support. A current is said to have bidegree (p, q) if it acts on $(n-p, n-q)$-forms. Differential operators are defined on smooth forms and by duality they are also defined on currents, just as in distribution theory. For us the most important differential operator is the $\bar{\partial}$-operator. For a function ϕ in $\mathbf{C}^n$ it is defined as

$$\bar{\partial}\phi = \sum_{j=1}^{n} \frac{\partial \phi}{\partial \bar{z}_j} \mathrm{d}\bar{z}_j.$$

It can be extended to differential forms so that if ϕ and ψ are differential forms then

$$\bar{\partial}(\phi \wedge \psi) = (\bar{\partial}\phi) \wedge \psi + (-1)^k \phi \wedge \bar{\partial}\psi$$

where k is the degree of ϕ.
Let us illustrate the idea of using currents to define residues in the complex plane. In the complex plane we may study currents acting on $(0,1)$-forms $\psi \mathrm{d}\bar{z}$ where ψ is a smooth function of compact support. Let us see how one may define a current from the meromorphic form $\mathrm{d}z/z^m$ in $\mathbf{C}$. We define the *principal value*

current of $\mathrm{d}z/z^m$ by

$$\left\langle \left[\frac{\mathrm{d}z}{z^m}\right], \psi \mathrm{d}\bar{z} \right\rangle = \lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} \frac{\psi}{z^m} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{7}$$

By expanding ψ in its Taylor series one may show that the limit exists and defines a current. Let us compute the $\bar{\partial}$-image of this current.

$$\begin{aligned} \left\langle \bar{\partial}\left[\frac{\mathrm{d}z}{z^m}\right], \psi \right\rangle &= \left\langle \left[\frac{\mathrm{d}z}{z^m}\right], \frac{\partial \psi}{\partial \bar{z}} \mathrm{d}\bar{z} \right\rangle \\ &= \lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} \frac{\partial \psi}{\partial \bar{z}} \frac{1}{z^m} \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \lim_{\varepsilon \to 0} \int_{|z|=\varepsilon} \frac{\psi}{z^m} \mathrm{d}z, \end{aligned} \tag{8}$$

where the last equality follows from Stokes' theorem. If we Taylor expand ψ as $\psi = \sum_{k,\ell} C_{k,\ell} z^k \bar{z}^\ell + \mathcal{O}(|z|^N)$ then we may show, by for example changing to polar coordinates, that the only term which gives a contribution to the limit is the term $C_{m-1,0} z^{m-1}$. By Taylor's formula $C_{m-1,0} = \frac{\partial^{m-1} f}{\partial z^{m-1}}/(m-1)!$ where $\partial f/\partial z = (\partial f/\partial x - \mathrm{i}\partial f/\partial y)/2$. Hence we get that

$$\begin{aligned} \left\langle \bar{\partial}\left[\frac{\mathrm{d}z}{z^m}\right], \psi \right\rangle &= \frac{1}{(m-1)!} \lim_{\varepsilon \to 0} \int_{|z|=\varepsilon} \frac{\partial^{m-1}\psi}{\partial z^{m-1}}(0) \frac{\mathrm{d}z}{z} \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}\psi}{\partial z^{m-1}}(0) \lim_{\varepsilon \to 0} \int_{|z|=\varepsilon} \frac{\mathrm{d}z}{z} \\ &= \frac{2\pi \mathrm{i}}{(m-1)!} \frac{\partial^{m-1}\psi}{\partial z^{m-1}}(0), \end{aligned} \tag{9}$$

or equivalently that

$$\bar{\partial}\left[\frac{\mathrm{d}z}{z^m}\right] = \frac{2\pi \mathrm{i}(-1)^m}{(m-1)!} \frac{\partial^{m-1}}{\partial z^{m-1}} [z=0], \tag{10}$$

where this is now an equality of currents and $[z = 0]$ denotes the Dirac mass at the origin considered as a $(1,1)$-current.

Let us generalise this further. Suppose that f is a meromorphic function in $\mathbf{C}$ with polar set given by $P(f) = \{z_1, \ldots, z_k\}$. In analogy with (7) we define

$$\big\langle [f\mathrm{d}z], \psi \mathrm{d}\bar{z} \big\rangle = \lim_{\varepsilon \to 0} \sum_{j=1}^{k} \int_{|z - z_j| > \varepsilon} f\psi \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{11}$$

and a similar calculation to the one in (8) gives that

$$\big\langle \bar{\partial}[f\mathrm{d}z], \psi \big\rangle = \lim_{\varepsilon \to 0} \sum_{j=1}^{k} \int_{|z - z_j| = \varepsilon} f\psi \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{12}$$

From this we conclude that the support of the current $\bar{\partial}[f\mathrm{d}z]$ is contained in $P(f)$. Let us make a calculation to see how $\bar{\partial}[f\mathrm{d}z]$ behaves locally. In some neighbourhood of z_j we may write $f = g/(z-z_j)^m$, where g is a non-vanishing holomorphic function. In this neighbourhood of z_j we have, cf. (9),

$$\begin{aligned} \bar{\partial}[f\mathrm{d}z] &= \bar{\partial}\Big[\frac{g\mathrm{d}z}{(z - z_j)^m}\Big] \\ &= g\bar{\partial}\Big[\frac{\mathrm{d}z}{(z - z_j)^m}\Big] = g\frac{2\pi\mathrm{i}(-1)^m}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}[z = z_j]. \end{aligned} \tag{13}$$

The action of the right hand side of (13) on the function 1 is

$$\frac{2\pi\mathrm{i}}{(m-1)!}\frac{\partial^{m-1} g}{\partial z^{m-1}}(z_j),$$

which equals $\operatorname{Res}(f, z_j)$ in view of (2) since $g = (z - z_j)^m f$. If we let ψ in (12) be a smooth function with compact support which is identically equal to 1 in a neighbourhood Ω of the polar set $P(f)$ then we get that

$$\int_{\mathbf{C}} f \mathrm{d}z \wedge \bar{\partial}\psi = 2\pi\mathrm{i} \sum_{j=1}^{k} \mathrm{Res}(f, z_j).$$

Letting ψ tend to the characteristic function of Ω and writing $\gamma = \partial\Omega$ we recover (3). We thus have an alternative approach to residues and this approach may be generalised to several variables.

In [**HeLi**] Herrera and Lieberman constructed principal value currents and residue currents in quite a general setting. They proved the following theorem.

Theorem 0.0.1. *(Theorem 7.1 in [**HeLi**]) Given a holomorphic function g in $\mathbf{C}^n$, $g \not\equiv 0$, and a smooth form ψ with compact support the limit*

$$\lim_{\varepsilon \to 0} \int_{|g|>\varepsilon} \frac{\psi}{g} \tag{14}$$

exists and defines a current acting on ψ.

If $\{g = 0\}$ is a complex manifold then it is elementary to see that the limit in (14) exists, it essentially reduces to the one variable case. In general the set $\{g = 0\}$ is not a complex manifold, see Section 3 below. In general the theorem can be proved by first assuming that $\{g = 0\}$ has normal crossings, see Example 2 below, and then reducing to this case using Hironaka's theorem on resolution of singularities, see [**Hi**]. As far as we know there is no proof of the existence of the limit in (14) not using Hironaka's theorem. Herrera and Lieberman used Theorem 0.0.1 to define principal value currents on complex manifolds and even on reduced complex spaces. Principal value currents had been constructed in certain cases before by, for example, Dolbeault.

The limit (14) defines a current which we denote by $[1/g]$. One then defines the *residue current of* $1/g$ as $\bar{\partial}[1/g]$ and by Stokes' theorem we get that

$$\langle \bar{\partial}[1/g], \psi \rangle = \lim_{\varepsilon \to 0} \int_{|g|=\varepsilon} \frac{\psi}{g}. \tag{15}$$

In particular we see that $\bar{\partial}[1/g]$ has support contained in the zero set of g.

Another way of defining principal value currents and residue currents is through analytic continuations of divergent integrals. This idea originates with Bernstein–Gelfand in 1969 and Atiyah in 1970, see [**BeGe**] and [**At**]. The direct purpose of the construction is that given a holomorphic function f, or more generally a real analytic function, we want to find a concrete distribution u such that $fu = 1$.
In the context of residue currents this approach was developed by e.g. Barlet and Maire in [**BaMa**] and Berenstein, Gay and Yger in [**BGY**] and [**Yg**]. Residue currents have since then been extensively studied.
Let us be explicit on how to define currents through analytic continuation. Let X be a complex manifold, g a holomorphic function on X and ψ a smooth top degree form with compact support. For complex numbers λ with $\mathrm{Re}(\lambda) \gg 1$ the integral

$$\int_X \frac{|g|^{2\lambda}\psi}{g} \tag{16}$$

is convergent. Using Hironaka's theorem to reduce to the case when $\{g = 0\}$ has normal crossings one can show that as a function of λ it has an analytic continuation over the origin and then we define a principal value current of $1/g$, acting on ψ, by the value of this function at $\lambda = 0$. By Stokes' theorem the function

$$\lambda \mapsto \int_X \frac{\bar{\partial}|g|^{2\lambda}\psi}{g} \tag{17}$$

has an analytic continuation over the origin and we define a residue current of $1/g$, acting on ψ, by the value at $\lambda = 0$. It is not obvious that these currents coincide with the currents defined but Herrera and Lieberman, but in fact they do.

A different approach to residues was studied by Poincaré and Leray who defined *residue forms* and *residue classes*. Instead of merely focusing on meromorphic functions in $\mathbf{C}^n$ one may look at differential forms on a complex manifold X of dimension n which have a pole along a complex hypersurface. The primary residue form is the Poincaré residue which is defined for a meromorphic form α of bidegree $(n, 0)$ having a pole of order one along a smooth hypersurface D. If D is locally given by $g = 0$ then we may write

$$\alpha = \frac{\mathrm{d}g}{g} \wedge \widetilde{\alpha}, \tag{18}$$

where $\widetilde{\alpha}$ is holomorphic, and define the Poincaré residue of α as $\widetilde{\alpha}\big|_D$. The decomposition of α depends on the choice of function g but actually $\widetilde{\alpha}\big|_D$ is canonical.
Let us see how to relate the Poincaré residue to the residue currents defined above. To the form α we associate the principal value current $[\alpha] = \mathrm{d}g \wedge \widetilde{\alpha}[1/g]$; let us calculate $\bar{\partial}$ of this current. Since the hypersurface D is smooth the function g is a coordinate. Therefore we may let $g = z_1$ and bring the calculation back to the one-variable case to get

$$\bar{\partial}[\alpha] = \bar{\partial}\Big[\frac{\mathrm{d}g}{g}\Big] \wedge \widetilde{\alpha} = \bar{\partial}\Big[\frac{\mathrm{d}z_1}{z_1}\Big] \wedge \widetilde{\alpha} = 2\pi\mathrm{i}[g = 0] \wedge \widetilde{\alpha}. \tag{19}$$

From (19) we can actually see that the Poincaré residue does not depend on any choices: The left hand side does not depend on any choices and the current $[g = 0]$ is canonical and hence $\widetilde{\alpha}\big|_D$ does not depend on any choices. Furthermore, we see that the Poincaré residue shows up as a factor in a decomposition of the residue current.

Assume that X is compact and let Ω be a (tubular) neighbourhood of D. Further assume that ξ is a smooth, d-closed $(n-1)$-form in Ω. As in (19) we get $\bar{\partial}([\alpha] \wedge \xi) = 2\pi \mathrm{i}[g = 0] \wedge \widetilde{\alpha} \wedge \xi$ and applying this to a smooth function ψ which is 1 in a neighbourhood of D and has compact support contained in Ω gives that

$$\int_X \alpha \wedge \xi \wedge \bar{\partial}\psi = 2\pi \mathrm{i} \int_D \widetilde{\alpha} \wedge \xi. \tag{20}$$

Letting ψ tend to the characteristic function of Ω of D we get

$$\int_{\partial\Omega} \alpha \wedge \xi = 2\pi \mathrm{i} \int_D \widetilde{\alpha} \wedge \xi. \tag{21}$$

In the case that the dimension is one we can choose $\xi = 1$ and recover (3).

If α is a meromorphic form on $X \setminus D$ with a pole along D of higher order, then there is a cohomologous form β with a pole of order one along D. This makes it possible to define a residue for α but now as a de Rham cohomology class. If we interpret $\widetilde{\alpha}$ in (20) as this cohomology class then the formula in (21) holds for α.

So far we have discussed currents associated to differential forms which may locally be written α/g for some holomorphic form α and a holomorphic function $g \not\equiv 0$. In two of the papers of which this thesis consists we will mainly be interested in forms which may locally be written $\alpha/g\bar{h}$ where α is a smooth form and g and h are holomorphic functions which are not identically zero. These are forms which have real analytic singularities along complex hypersurfaces. We will be concerned with defining currents and residue classes from such forms using the method of analytic continuation of divergent integrals. Furthermore, we

connect these objects in a formula which is therefore, in some sense, a residue theorem in that setting.
To illustrate the main point let us look at the simplest case in one variable and we look at the form $\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{|z|^2}$. This form is obviously smooth away from the origin and therefore it defines a current α in $\mathbf{C}\setminus\{0\}$. We want to find an extension of α to $\mathbf{C}$. Such an extension cannot be unique since we may add derivatives of the Dirac distribution at the origin and we would still have a current extension of α. Let us apply the method of analytic continuation discussed above. For complex numbers λ with $\mathrm{Re}(\lambda)\gg 1$ and a smooth function ψ in $\mathbf{C}$ with compact support we get

$$\begin{aligned}\int_{\mathbf{C}}\frac{|z|^{2\lambda}\psi\mathrm{d}z\wedge\mathrm{d}\bar{z}}{|z|^2} &= \frac{1}{\lambda}\int_{\mathbf{C}}\frac{\partial}{\partial\bar{z}}\Big(\frac{|z|^{2\lambda}}{z}\Big)\psi\mathrm{d}z\wedge\mathrm{d}\bar{z}\\ &= -\frac{1}{\lambda}\int_{\mathbf{C}}\frac{|z|^{2\lambda}}{z}\frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z} \qquad (22)\end{aligned}$$

The integral

$$\int_{\mathbf{C}}\frac{|z|^{2\lambda}}{z}\frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z}$$

is holomorphic in λ in a neighbourhood of the origin. Let us calculate the first two terms in its Taylor expansion. The constant term is given by setting $\lambda=0$ and it is

$$\int_{\mathbf{C}}\frac{1}{z}\frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z}=2\pi\mathrm{i}\psi(0)$$

which follows from formula (5). The coefficient of λ is given by differentiating the integrand with respect to λ and then setting $\lambda=0$ which gives

$$\int_{\mathbf{C}}\frac{\log|z|^2}{z}\frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

Using this in the calculation (22) we get

$$\int_{\mathbf{C}} |z|^{2\lambda}\psi \frac{\mathrm{d}z\wedge \mathrm{d}\bar{z}}{|z|^2} = \frac{1}{\lambda}2\pi \mathrm{i}\psi(0) + \int_{\mathbf{C}} \frac{\log|z|^2}{z}\frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z\wedge \mathrm{d}\bar{z} + \mathscr{O}(\lambda) \tag{23}$$

From this we conclude: To the singular form $\frac{\mathrm{d}z\wedge \mathrm{d}\bar{z}}{|z|^2}$ we may associate two currents. The first one is the current $2\pi\mathrm{i}[z=0]$ and the second one is given by

$$\psi \mapsto \int_{\mathbf{C}} \frac{\log|z|^2}{z}\frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z\wedge \mathrm{d}\bar{z} \tag{24}$$

The second current is an extension of α to $\mathbf{C}$. This is seen by assuming that the test form ψ has support away from the origin and setting $\lambda = 0$ in (23).
However, as we have mentioned, the current extension is not canonical; we may add derivatives of the Dirac distribution at the origin. We think of the current $2\pi\mathrm{i}[z=0]$ as a residue current associated to the form $\frac{\mathrm{d}z\wedge \mathrm{d}\bar{z}}{|z|^2}$ and the current in (24) as the finite part of the form.
We will extend the above construction to forms with higher singularities and to several variables in Paper I and Paper III.

3. Complex spaces

Residue currents like $\bar{\partial}[1/g]$ have support on the zero set of a tuple of holomorphic functions. Such sets are called analytic sets. An analytic set X may be decomposed into X_{reg} and X_{sing} where the regular part, X_{reg}, consists of all points which have a neighbourhood in which X is a complex manifold. The singular part, X_{sing}, is defined as the complement of the regular part. For simplicity we here focus on analytic sets given by one holomorphic function.

***Example* 1.** Equip $\mathbf{C}^2$ with coordinates (z,w) and let $X = \{z^2 - w^3 = 0\}$. This space is usually called the cusp in $\mathbf{C}^2$. The only singular point of X is the origin. ✓

***Example* 2.** Equip $\mathbf{C}^n$ with coordinates $(z_1, \ldots, z_n)$ and let $X = \{z_1 \cdots z_k = 0\}$. The space X is a union of hyperplanes $\{z_j = 0\}$ and X_{sing} consists of all points where at least two such hyperplanes intersect. This is a typical example of an analytic set with so-called normal crossings. ✓

Properties of analytic sets are reflected by the holomorphic functions on these sets. However, in the case that X has singular points it is not obvious what a holomorphic function on X is. Suppose that $X \subset \mathbf{C}^n$ and let us define holomorphic functions on X. One natural notion of holomorphic functions on X is defined as follows: A function $h : X \to \mathbf{C}$ is holomorphic at $x \in X$ if there is an open neighbourhood U of x in $\mathbf{C}^n$ and a holomorphic function $\widetilde{h} : U \to \mathbf{C}$ such that $h = \widetilde{h}|_U$. This means that holomorphic functions on $X \cap U$ are given by holomorphic functions on U and we identify two such functions if they are equal on X. We denote the (sheaf of) holomorphic functions on $\mathbf{C}^n$ by $\mathscr{O}_{\mathbf{C}^n}$ and we let $\mathscr{I}_X \subset \mathscr{O}_{\mathbf{C}^n}$ be the holomorphic functions which vanish on X. We further denote the holomorphic functions on X by $\mathscr{O}_X$; by the above discussion we have that $\mathscr{O}_X = \mathscr{O}_{\mathbf{C}^n}/\mathscr{I}_X$. An analytic set together with this kind of holomorphic functions is a reduced complex space.

There are other notions of holomorphic functions on analytic sets. We illustrate this with an analytic set $X = \{g = 0\}$ where g is a holomorphic function in $\mathbf{C}^n$. We now define the holomorphic functions on X as $\mathscr{O}_X = \mathscr{O}_{\mathbf{C}^n}/(g)$. Here (g) denotes the ideal in $\mathscr{O}_{\mathbf{C}^n}$ generated by g. In the case that $\mathrm{d}g$ is non-zero on X_{reg} then this method produces the same set of holomorphic functions on X as the one above. If this is not the case then the ideal (g) is strictly smaller than $\mathscr{I}_X$. In this case the objects in $\mathscr{O}_X$ do not have a natural interpretation as functions on X. An analytic set equipped with such a notion of holomorphic functions is a non-reduced complex space. Let us illustrate the different notions of holomorphic functions with an example.

***Example* 3.** Let $X = \{0\} \subset \mathbf{C}$ and let $g = z$. A holomorphic function h in $\mathbf{C}$ is zero in $\mathscr{O}_{\mathbf{C}^n}/(g)$ precisely when $h(0) = 0$. Therefore $h + (g) \in \mathscr{O}_{\mathbf{C}^n}/(g)$ may be identified with the value of h on X.
Now instead let $g = z^2$. A holomorphic function h in $\mathbf{C}$ is zero in $\mathscr{O}_{\mathbf{C}^n}/(g)$ if and only if $h(0) = h'(0) = 0$. This can be seen by Taylor expanding h around the origin. In this case the set of equivalence classes in $\mathscr{O}_{\mathbf{C}^n}/(g)$ is not determined by the value of the functions on X. Therefore we cannot identify the quotient with functions on X. ✓

As mentioned, residue currents have support on analytic sets. Actually, the residue current $\bar\partial[1/g]$ contains all the information about the possibly non-reduced complex space $(X, \mathscr{O}_X)$, where $X = \{g = 0\}$ and $\mathscr{O}_X = \mathscr{O}_{\mathbf{C}^n}/(g)$, as shown in the following proposition.

Proposition 0.0.2. *Let h be a holomorphic function in $\mathbf{C}^n$. Then $h \in (g)$ (locally) if and only if $h\bar\partial[1/g] = 0$.*

Proof. Let us first notice that as currents we have $g[1/g] = 1$. This follows easily since the limit in (14) defines the current $[1/g]$. Assume that $h \in (g)$. Then $h = ag$ for some holomorphic function a and hence

$$h\bar\partial[1/g] = \bar\partial\big(h[1/g]\big) = \bar\partial\big(ag[1/g]\big) = \bar\partial a = 0.$$

Now instead assume that $h\bar\partial[1/g] = 0$. Then for the current

$u = h[1/g]$ we have $\bar\partial u = h\bar\partial[1/g] = 0$. Hence u is a holomorphic function and $gu = gh[1/g] = h$ which means that $h \in (g)$. $\square$

Proposition 0.0.2 gives an analytic description of the ideal generated by g which induces an analytic description of the quotient $\mathscr{O}_X = \mathscr{O}_{\mathbf{C}^n}/(g)$. In [**AnWu**] Andersson and Wulcan introduced currents similar to $\bar\partial[1/g]$ which generalise the so-called Coleff–Herrera currents, see e.g. [**CoHe**], and which describe general complex spaces. This opens up for the possibility of doing analysis on non-reduced spaces. Let us indicate how this may be done for the complex space $(X, \mathscr{O}_X)$ when $\mathscr{O}_X = \mathscr{O}_{\mathbf{C}^n}/(g)$. We want to define what smooth $(0,q)$-forms are on a non-reduced space X. Let us denote the (sheaf of) smooth (p,q)-forms on $\mathbf{C}^n$ by $\mathscr{E}^{p,q}_{\mathbf{C}^n}$ and the (sheaf of) currents in $\mathbf{C}^n$ of bidegree (p,q) by $\mathscr{C}^{p,q}_{\mathbf{C}^n}$. We define a map $\Psi : \mathscr{E}^{0,q}_{\mathbf{C}^n} \to \mathscr{C}^{0,q+1}_{\mathbf{C}^n}$ by

$$\phi \overset{\Psi}{\mapsto} \phi \wedge \bar\partial[1/g]$$

and then let $\mathscr{E}^{0,q}_X = \mathscr{E}^{0,q}_{\mathbf{C}^n}/\mathscr{K}er(\Psi)$. This is a natural definition in view of Proposition 0.0.2 above.
Since $\bar\partial(\phi \wedge \bar\partial[1/g]) = \bar\partial\phi \wedge \bar\partial[1/g]$ we also get a well-defined $\bar\partial$-operator $\bar\partial : \mathscr{E}^{0,q}_X \to \mathscr{E}^{0,q+1}_X$. In Paper II we define further analytic objects on non-reduced spaces and develop some theory for these objects.

The purpose of Paper I is to make sense of divergent integrals of the form $\int_X \alpha \wedge \overline{\beta}$ when α and β have poles along a hypersurface D on a complex manifold X. The problem is motivated by physics, see for example [**Wi**], but it is also a generalisation of the theory of residue currents in the sense that we do not just look at currents associated to $1/g$ but also at currents associated to $1/g\overline{f}$ where both f and g are holomorphic. If f and g vanish to a higher order along D then this may be thought of as currents associated with a non-reduced structure on D.

4. Koppelman formulas

Let us look at several variable analogues of the formula (5). In $\mathbf{C}^n$ one defines the *Bochner–Martinelli kernel* k_{BM} by

$$k_{BM}(z) = \frac{1}{(2\pi \mathrm{i})^n} \frac{\partial |z|^2 \wedge (\bar\partial \partial |z|^2)^{n-1}}{|z|^{2n}} = c_n \sum_{j=1}^{n} \frac{(-1)^{j+1} \bar z_j}{|z|^{2n}} \mathrm{d}z \wedge \widehat{\mathrm{d}\bar z_j}$$

where $c_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi \mathrm{i})^n}$, $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ and

$$\widehat{\mathrm{d}\bar z_j} = \mathrm{d}\bar z_1 \wedge \cdots \wedge \mathrm{d}\bar z_{j-1} \wedge \mathrm{d}\bar z_{j+1} \wedge \cdots \wedge \mathrm{d}\bar z_n.$$

The form k_{BM} is locally integrable and therefore it defines a current. Notice that if $n = 1$ then

$$k_{BM} = \frac{1}{2\pi \mathrm{i}} \frac{\partial |z|^2}{|z|^2} = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}z}{z}$$

which is the Cauchy kernel. The crucial property of the Bochner–Martinelli kernel is that

$$\bar\partial k_{BM} = [z = 0]. \tag{25}$$

where $[z = 0]$ is the Dirac distribution at the origin considered as a top degree current. Letting $\pi : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$ be the map $\pi(z, w) = z - w$ and $K_{BM} = \pi^* k_{BM}$ we get that $\bar\partial K_{BM} = [z = w]$. We denote by $K_{BM}^{p,q}$ the part of K_{BM} which is of bidegree (p, q) in z and hence of bidegree $(n - p, n - q - 1)$ in w.
With this setup one may prove Koppelman's formula, see e.g. [**Kop**]: For any $v \in \mathscr{E}^{p,q}(\overline{\Omega})$, $\Omega \subset \mathbf{C}^n$ a bounded domain,

$$v(z) = \int_{\partial\Omega} K_{BM}^{p,q}(z, w) \wedge v(w) + \bar\partial \int_{\Omega} K_{BM}^{p,q-1}(z, w) \wedge v(w) + \int_{\Omega} K_{BM}^{p,q}(z, w) \wedge \bar\partial v(w).$$

This is a several variable analogue of formula (5). We shall now discuss weighted integral formulas. The presentation will be a bit sketchy but the point is to illustrate the main ideas.

4.1 Weighted integral formulas. To incorporate weights into the integral formulas it is convenient to introduce the full Bochner–Martinelli form as

$$B = \sum_{k=1}^{n} \frac{1}{(2\pi\mathrm{i})^k} \frac{\partial|z-w|^2 \wedge (\bar\partial\partial|z-w|^2)^{k-1}}{|z-w|^{2k}}.$$

A weight adapted to a domain $\Omega \subset \mathbf{C}^n$ is a certain kind of smooth form γ which is given as a sum of terms $\gamma_{0,0}, \ldots, \gamma_{n,n}$ in $\Omega \times \Omega$. The subscript means bidegree and the term $\gamma_{0,0}$ should be 1 on the diagonal in $\Omega \times \Omega$. We will not go further into the details of the precise definition but the following example gives a hint of what weights can look like.

***Example* 4.** Let χ be a cut-off function in $\mathbf{C}$ which is 1 in a neighbourhood of the closure of a bounded open subset $\Omega \subset \mathbf{C}$. A weight for Ω is given by

$$\gamma = \chi(w) + \bar\partial\chi(w) \wedge \frac{\mathrm{d}(z-w)}{2\pi\mathrm{i}(w-z)}.$$ ✓

We define a weighted integral kernel by

$$K_\gamma = (\gamma \wedge B)_{n,n-1}$$

where $(\text{—})_{n,n-1}$ now means that we pick out the part of the form which has total bidegree $(n, n-1)$. In [**And**] the following weighted Koppelman formula is proved:

$$v(z) = \int_\Omega \gamma_{n,n} \wedge v(w) + \bar\partial \int_\Omega K_\gamma(z,w) \wedge v(w) + \int_\Omega K_\gamma(z,w) \wedge \bar\partial v(w). \tag{26}$$

In the case that Ω is a bounded pseudoconvex domain it is possible to choose a weight γ which is holomorphic in z and does not contain any $\mathrm{d}\bar z_j$, cf. Example 4. Assume that v is a $\bar\partial$-closed (p,q)-form in Ω. From the formula (26) we get the following:

(a) If $q = 0$ then we get a representation formula for holomorphic p-forms:

$$v = \int_\Omega \gamma_{n,n} \wedge v(w).$$

(b) If $q \geqslant 1$ then

$$v(z) = \bar\partial \int_\Omega K_\gamma(z,w) \wedge v(w).$$

The latter statement implies that if v is a smooth $\bar\partial$-closed (p,q)-form, with $q \geqslant 1$, then locally there is a smooth $(p,q-1)$-form ϕ such that $\bar\partial\phi = v$. Hence the same conclusion holds locally on any complex manifold. One alternative way of stating this is that for a complex manifold X the complex of sheaves

$$0 \to \Omega_X^p \xrightarrow{\bar\partial} \mathscr{E}_X^{p,1} \xrightarrow{\bar\partial} \mathscr{E}_X^{p,1} \xrightarrow{\bar\partial} \dots \tag{27}$$

is exact, which is the so-called Dolbeault–Grothendieck lemma, see [**Dol**]. Here Ω_X^p denotes the holomorphic p-forms and recall that $\mathscr{E}_X^{p,q}$ is the smooth (p,q)-forms on X. In particular, for $p = 0$ we get a resolution of the sheaf of holomorphic functions $\mathscr{O}_X$.

We may also use weights to get so called division-interpolation formulas in the following way: Take a holomorphic function g in $\mathbf{C}^n$ and let us for simplicity assume that $\{g = 0\}$ is smooth so that g is just a coordinate. For $\mathrm{Re}(\lambda) \gg 1$ we may then define a weight by

$$\gamma = \frac{\bar\partial |g(w)|^{2\lambda}}{g(w)} \wedge H + g(z)\frac{|g(w)|^{2\lambda}}{g(w)} H$$

where H is a so-called Hefer form which we will not define here, but it is actually a holomorphic form. Given the weight γ we let

$K_\gamma = (\gamma \wedge B)_{n,n-1}$ and get a Koppelman formula:

$$\begin{aligned} v(z) &= \int_\Omega \Big(\frac{\bar\partial |g(w)|^{2\lambda}}{g(w)} \wedge H\Big)_{n,n} \wedge v(w) \\ &+ \bar\partial \int_\Omega \Big(\frac{\bar\partial |g(w)|^{2\lambda}}{g(w)} \wedge H \wedge K\Big)_{n,n-1} \wedge v(w) \\ &+ \int_\Omega \Big(\frac{\bar\partial |g(w)|^{2\lambda}}{g(w)} \wedge H \wedge K\Big)_{n,n-1} \wedge \bar\partial v(w) + g(z)\psi_\lambda. \end{aligned}$$

where ψ_λ is smooth as long as $\mathrm{Re}(\lambda) \gg 1$. If we pull this equality back to $\{g = 0\}$ then the last term vanishes and one may show that the right hand side can be analytically continued over the origin. Setting $\lambda = 0$ gives a Koppelman formula on $\{g = 0\}$. To solve the $\bar\partial$-equation on $\{g = 0\} \cap \Omega$, where Ω is a pseudoconvex bounded domain, one can incorporate an additional weight adapted to Ω.
In Paper II we construct analogous Koppelman formulas for non-reduced complex spaces.

5. Summary of the papers

As we have mentioned the purpose of Paper I is to study integrals of the form $\int_X \alpha \wedge \overline{\beta}$ where X is a complex manifold of dimension n and α and β are meromorphic n-forms with poles along a hypersurface D with normal crossings. To make sense of the integral we pick a section $s : X \to L$, where L is the line bundle associated to D, and a metric $|\cdot|$ on L. For every smooth function ϕ with compact support and every complex number λ with $\mathrm{Re}(\lambda) \gg 1$ the integral

$$\int_X |s|^{2\lambda} \phi \alpha \wedge \overline{\beta}$$

is convergent and holomorphic in λ. Let $D_1, \dots, D_k$ be the irreducible components of D. Suppose for simplicity that each D_j is smooth and that both α and β have poles along each D_j.

Theorem 0.0.3. *Let κ be the maximal number of D_j:s that intersect. Then the function*

$$\lambda \mapsto \int_X |s|^{2\lambda} \phi \alpha \wedge \overline{\beta}$$

has a meromorphic continuation to $\mathbf{C}$ with poles contained in $\mathbf{Q}$. The Laurent expansion around $\lambda = 0$ is given by

$$\frac{1}{\lambda^\kappa} \mu_\kappa . \phi + \cdots + \frac{1}{\lambda} \mu_1 . \phi + \mu_0 . \phi + \mathscr{O}(|\lambda|) \tag{28}$$

where μ_j are currents with support on the set where j components of D intersect.

In Paper I we further show that μ_κ, the leading term, is independent of the choices of section s and metric $|\cdot|$. Therefore we call μ_κ the canonical current associated to $\alpha \wedge \overline{\beta}$. The other currents $\mu_{\kappa-1}, \dots, \mu_0$ however do depend on the choices we have made.
Let $Y = D_1 \cap \cdots \cap D_k$ and suppose that X is compact. In Paper I we further define an Aeppli cohomology class $\operatorname{Res}^Y_A(\alpha \wedge \overline{\beta})$ associated to $\alpha \wedge \overline{\beta}$ which is a class on Y such that

$$\langle \mu_\kappa, 1 \rangle = (-2\pi \mathrm{i})^\kappa \int_Y \operatorname{Res}^Y_A(\alpha \wedge \overline{\beta}).$$

In the case that $\kappa = 1$, i.e. when the hypersurface is smooth, and α and β have poles of order one then the Aeppli residue is given by $\operatorname{Res}(\alpha) \wedge \overline{\operatorname{Res}(\beta)}$ where Res denotes the Poincaré residue discussed in Section 2. In the case that α has a pole of higher order but β still has a pole of order one then the Aeppli residue may be described by the Felder–Kazhdan residue defined in [**FeKa**].

The main objective of Paper II is to solve the $\bar{\partial}$-equation on a possibly non-reduced space. Let Z be a analytic subset of a pseudoconvex domain $D \subset \mathbf{C}^N$. Suppose that $\mathscr{J}$ is an ideal in $\mathscr{I}_Z$ and let $\mathscr{O}_X = \mathscr{O}_D/\mathscr{J}$. We further suppose that $\mathscr{O}_X$ has pure dimension. This means that if $h \in \mathscr{O}_D$ is such that $h_x \in \mathscr{J}_x$ for generic $x \in Z$ then $h \in \mathscr{J}$.
A classical notion of holomorphic p-forms on the complex space X is the Kähler differentials given by

$$\Omega^p_{X,\text{Kähler}} := \frac{\Omega^p_D}{\mathscr{J}\Omega^p_D + \mathrm{d}\mathscr{J} \wedge \Omega^{p-1}_D} \tag{29}$$

It is possible that for $\phi \in \Omega^p_D$ we have that $\phi_x \in (\mathscr{J}\Omega^p_D + \mathrm{d}\mathscr{J} \wedge \Omega^{p-1}_D)_x$ for generic $x \in Z$ but still $\phi \notin (\mathscr{J}\Omega^p_D + \mathrm{d}\mathscr{J} \wedge \Omega^{p-1}_D)$. For technical reasons we want to exclude such forms; it is not a good property when doing analysis. To achieve this we enlarge the denominator in (29) to a sub-module $\mathscr{J}^p$ of Ω^p_D which coincides with $\mathscr{J}\Omega^p_D + \mathrm{d}\mathscr{J} \wedge \Omega^{p-1}_D$ for generic $x \in Z$ and with the property that if $\phi \in \mathscr{J}^p_x$ for generic x then $\phi \in \mathscr{J}^p$. We then define

$$\Omega^p_X = \Omega^p_D/\mathscr{J}^p$$

and call these forms the holomorphic p-forms on X.
By [**AnWu**] there is a residue current $\mathcal{R}$ such that for $\phi \in \Omega^p_D$ we have

$$\phi \in \mathscr{J}^p \text{ if and only if } \phi \wedge \mathcal{R} = 0,$$

cf. Proposition 0.0.2 above. Smooth (p,q)-forms on X are then defined as

$$\mathscr{E}^{p,q}_X = \frac{\mathscr{E}^{p,q}_D}{\{\phi \in \mathscr{E}^{p,q}_D : \phi \wedge \mathcal{R} = 0\}}.$$

This is strictly speaking not the definition given in the article but according to Proposition 3.9 in Paper II it is equivalent. In a similar way as in Section 3 above we get a well-defined $\bar{\partial}$-operator $\bar{\partial} : \mathscr{E}^{p,q}_X \to \mathscr{E}^{p,q+1}_X$. Hence we may study the $\bar{\partial}$-equation on our non-reduced space. However, even in the reduced case it is not always possible to solve the $\bar{\partial}$-equation smoothly when the complex space is not smooth. Our solution to this is to define an extension $\mathscr{A}_X^{p,q}$ of $\mathscr{E}_X^{p,q}$ in such a way that $\mathscr{A}_{X,x}^{p,q} = \mathscr{E}_{X,x}^{p,q}$ generically and so that $\mathscr{A}_X^{p,q}$ is an $\mathscr{E}_X^{0,q}$-module. In Paper II we prove the following theorem.

Theorem 0.0.4. *If $\phi \in \mathscr{A}_X^{p,q}$ is $\bar{\partial}$-closed then*

(a) if $q = 0$ then $\phi \in \Omega_X^p$,

(b) if $q \geqslant 1$ then there exists $\psi \in \mathscr{A}_X^{p,q}$ with $\bar{\partial}\psi = \phi$ locally on X.

We further prove that $\bar{\partial} : \mathscr{A}_X^{p,q} \to \mathscr{A}_X^{p,q+1}$. This together with theorem 0.0.4 implies that the complex of sheaves

$$0 \to \Omega_X^p \xrightarrow{\bar{\partial}} \mathscr{A}_X^{p,1} \xrightarrow{\bar{\partial}} \mathscr{A}_X^{p,2} \xrightarrow{\bar{\partial}} \cdots$$

is exact, cf. (27). Since $\mathscr{A}_X^{p,q}$ are $\mathscr{E}_X^{0,q}$-modules, in particular smooth partitions of unity are available, the abstract de Rham theorem implies that we get a representation of the sheaf cohomology:

$$H^q(X, \Omega_X^p) \simeq H^q(\mathscr{A}_X^{p,\bullet}(X)).$$

In Paper II we also find a similar description of the dual objects $H^q(X, \Omega_X^p)^*$ and this leads to an analytic version of Serre duality. This may be compared with the results in [**RSW**]. Our method is based on division–interpolation type formulas constructed using the residue current $\mathcal{R}$, similar to how we did in the end of Section 4. This gives integral operators on X and the sheaves $\mathscr{A}_X^{p,q}$ are given by iteratively applying these operators on smooth forms. This approach is basically a combination of the methods in [**AL**] and [**Sam**]. The idea to achieve a Dolbeault–Grothendieck lemma on complex spaces using the theory of residues currents originates in [**AS**].

In Paper III the setting is similar to the that of Paper I and we achieve results similar to the ones in [**FeKa2**]. We study divergent integrals on complex manifolds but now we begin with a differential form ω which is singular along a complex submanifold Y of possibly higher codimension. We let $(E, |\cdot|)$ be a hermitian vector bundle with a holomorphic section $s : X \to E$ which generates the ideal of holomorphic functions vanishing on Y. We assume that it is possible to choose such a metric and section so that $|s|^{2N}\omega$ is smooth on X for some integer $N \geqslant 0$. In Paper III we prove the following theorem.

Theorem 0.0.5. *The function*

$$\lambda \mapsto \int_X |s|^{2\lambda} \phi\omega$$

has a meromorphic continuation to some neighbourhood of the origin. The Laurent expansion around $\lambda = 0$ is given by

$$\frac{1}{\lambda}\langle \mu_1, \phi\rangle + \langle \mu_0(|s|^2), \phi\rangle + \mathscr{O}(|\lambda|) \tag{30}$$

where μ_1 and $\mu_0(|s|^2)$ are currents on X. The current μ_1 does not depend on the choice of section or metric and its support is contained in Y. If $\|\cdot\|$ is another metric on the vector bundle E then

$$\mu_0(|s|^2) = \mu_0(\|s\|^2) + \mu_1 \log \frac{|s|^2}{\|s\|^2}.$$

Let $\kappa = \mathrm{codim}(Y)$. If $N < \kappa$ then ω is locally integrable and in this case $\mu_1 = 0$ and $\mu_0 = \omega$, where ω is considered as a current. If $|s|^{2\kappa}\omega$ is smooth then there is a smooth form $\mathrm{res}(\omega)$ on Y such that

$$\langle \mu_1, \phi \rangle = \kappa(2\pi \mathrm{i})^\kappa \int_Y \mathrm{res}(\omega)\phi. \tag{31}$$

In the case that $|s|^{2N}\omega$ is smooth for $N > \kappa$ then there is a de Rham cohomology class $\mathrm{res}(\omega)$ on Y such that (31) holds.

Bibliography

[**AL**] M. Andersson and R. Lärkäng. "The $\bar{\partial}$-equation on a non-reduced analytic space". In: *Math. Ann.* 374.1-2 (2019).

[**And**] M. Andersson. "Integral representation with weights I." In: *Math. Ann.* 326.1-18 (2003).

[**AnWu**] M. Andersson and E. Wulcan. "Residue currents with prescribed annihilator ideals". In: *Ann. Sci. École Norm. Sup. (4)* 40.6 (2007), pp. 985–1007.

[**AS**] M. Andersson and H. Samuelsson. "A Dolbeault–Grothendieck lemma on complex spaces via Koppelman formulas." In: *Invent. Math.* 190.2 (2012).

[**At**] M. Atiyah. "Resolution of Singularities and Division of Distributions". In: *Communications on Pure and Applied Mathematics* 23 (1970).

[**BaMa**] D. Barlet and H.-M. Maire. "Developpement asymptotique des foncions obtenues par integration sur les fibres". In: *Journal of Functional Analysis* 83 (1982).

[**BeGe**] J. Bernstein and S.I. Gelfand. "Meromorphy of the function P^λ". In: *Functional Anal. Appl.* 3 (1969).

[**BGY**] C. Berenstein, R. Gay, and A. Yger. "Analytic Continuation of Currents and Division Problems". In: *Forum Math.* 1 (1989).

[**CoHe**] N. Coleff and M. Herrera. *Les courants residuels associes a une forme meromorphe.* Springer, 1978.

[**Dol**] P. Dolbeault. "Sur la cohomologie des variétés analytiques complexes". In: *C. R. Acad. Sc. Paris* 236 (1953).

[**Fe**] G. Felder and D. Kazhdan. "Divergent integrals, Residues of Dolbeault Forms and asymptotic Riemann Mappings". In: *International mathematics research notices* 08 (2016).

[**FeKa**] G. Felder and D. Kazhdan. "Divergent integrals, Residues of Dolbeault Forms and asymptotic Riemann Mappings". In: *International mathematics research notices* 08 (2016).

[**FeKa2**] G. Felder and D. Kazhdan. "Regularization of divergent integrals". In: *Selecta Mathematica* 24.1 (2018).

[**HeLi**] M. Herrera and D. Lieberman. "Residues and Principal Values on Complex Spaces". In: *Math. Ann.* 194 (1971).

[**Hi**] H. Hironaka. "Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II." In: *Annals of Mathematics.* 79 (1964).

[**Kop**] W. Koppelman. "The Cauchy integral for differential forms". In: *Bull. Amer. Math. Soc.* 73.4 (1967).

[**RSW**] J. Ruppenthal, H. Samuelsson Kalm, and E. Wulcan. "Explicit Serre duality on complex spaces." In: *Adv. Math.* 305 (2017).

[**Sam**] H. Samuelsson Kalm. *The $\bar{\partial}$-equation, duality, and holomorphic forms on a reduced complex space.* arXiv: 1506.07842.

[**Wi**] E. Witten. *Perturbative String Theory Revisited.* 2012. arXiv: 1209.5461.

[**Yg**] A. Yger. “Formules de Division et Prolongement Méromorphe”. In: *Séminaire d’Analyse P. Lelong–P. Dolbeault–H. Skoda (années 1985/1986)* (1987).

I

Paper I

Mattias Lennartsson, *"Residues and currents from singular forms on complex manifolds"*, preprint.

Residues and currents from singular forms on complex manifolds

MATTIAS LENNARTSSON

Abstract
Using methods from the theory of residue currents we provide asymptotic expansions of certain divergent integrals on complex manifolds. We express the coefficients in these expansions with the conjugate Dolbeault residue, introduced by Felder and Kazhdan in [**Fe**], and define a new residue which we call the Aeppli residue.

1. INTRODUCTION

Suppose X is a compact complex manifold of dimension d and $D \subset X$ is a smooth hypersurface. Motivated by perturbative string theory, in [**Fe**] Felder and Kazhdan discuss regularisations of divergent integrals of the form

$$\int_X \alpha \wedge \overline{\beta}$$

where α and β are $(d,0)$-forms which are smooth on $X \setminus D$, α has a pole along D and β has a pole of order one along D. In their paper they use cut-off functions, i.e. functions χ which are zero on D and otherwise positive, and prove the asymptotic expansion

$$\int_{\chi \geqslant \varepsilon} \alpha \wedge \overline{\beta} = \log \varepsilon \, I_0 + I_1(\chi) + \mathcal{O}(\varepsilon)$$

where $I_0 = \int_D \operatorname{Res} \alpha \wedge \overline{\operatorname{Res} \beta}$ does not depend on the cut-off function (here Res denotes the classical Leray residue which we discuss later). They also show that $I_1(\chi)$ depends linearly on χ and give an explicit expression for it in terms of the *conjugate Dolbeault residue*, $\operatorname{Res}_{\bar\partial}$, defined in the same paper. In a second paper, [**Fe2**], the same authors generalise the results to smooth manifolds and forms which have singularities on submanifolds determined by Morse–Bott functions. In particular they consider the case of a complex hypersurface with normal crossings. They also study analytic continuations of these divergent integrals.

In this paper we take the analytic continuation of divergent integrals as starting point. This means that we have a different method of regularising the divergent integrals and this will give us more explicit formulas. We allow D to be a hypersurface with normal crossings and α and β to be semi-meromorphic forms with poles along D of any order. If $s : X \to L$ is a holomorphic section of some line bundle such that $D = \{s = 0\}$ and $|\cdot|$ is a metric on L we define a function by

$$\lambda \mapsto \int_X |s|^{2\lambda} \alpha \wedge \bar{\beta}.$$

This function is a priori only defined for complex numbers λ with $\operatorname{Re} \lambda$ large enough but we will see that it has a meromorphic extension to $\mathbf{C}$ which is holomorphic when $\operatorname{Re} \lambda$ is large enough. We get a Laurent expansion at 0, cf. Theorem 2.3,

$$\int_X |s|^{2\lambda} \alpha \wedge \bar{\beta} = \lambda^{-\kappa} C_{-\kappa} + \cdots + \lambda^{-1} C_{-1} + C_0 + \mathcal{O}(\lambda) \tag{1}$$

where κ is defined in Section 2. Changing $\alpha\wedge\bar\beta$ to $\alpha\wedge\bar\beta\wedge\xi$, where ξ is a test function, we get currents $C_{-j}(\xi)$ of bidegree (d,d). We will focus on the leading coefficient $C_{-\kappa}$, which we call the *canonical current* associated to $\alpha\wedge\bar\beta$, and we denote it by $\{\alpha\wedge\bar\beta\}$. The motivation for this construction comes from the study of residue currents in complex geometry. Then one looks at so called semi-meromorphic forms α, i.e. locally $\alpha=\widetilde{\alpha}/f$ for some smooth form $\widetilde{\alpha}$ and some holomorphic function f such that $f\not\equiv 0$. Given such a form one can use this method to define the *principal value current* $[\alpha]$. We will recall more precisely how this is done in Section 2.

In the third section we discuss cohomological residues. Given a semi-meromorphic $(d,d-1)$-form α on X which is polar along a smooth hypersurface D the conjugate Dolbeault residue $\mathrm{Res}_\partial(\alpha)$ is a class in the conjugate Dolbeault cohomology group $H^{d-1,d-1}_\partial(D)$, see Definition 3.2 below. We then define a new residue, which we call the *Aeppli residue*, and denote it by Res_A. Given semi-meromorphic $(d,0)$-forms α and β which are polar along D the Aeppli residue $\mathrm{Res}_A(\alpha\wedge\bar\beta)$ is a class in the Aeppli cohomology group $H^{d-1,d-1}_A(D)$. We relate these residues to the currents defined from analytic continuations of divergent integrals. The following result relates principal value currents and the conjugate Dolbeault residue.

Theorem A. *For a semi-meromorphic form α which is polar along a smooth hypersurface D we have, for every test form ξ,*

$$\big\langle \bar\partial[\alpha],\xi\big\rangle=\big\langle[\bar\partial\alpha],\xi\big\rangle+2\pi\mathrm{i}\int_D \mathrm{Res}_\partial(\alpha\wedge\xi).$$

In the same spirit we can relate the canonical current to the Aeppli residue. We prove a more general result in Theorem 3.9 but a special case is the following.

Theorem B. *For semi-meromorphic forms α and β, polar along a smooth hypersurface D, we have for every test form ξ,*

$$\big\langle\{\alpha\wedge\bar\beta\},\xi\big\rangle=-2\pi\mathrm{i}\int_D \mathrm{Res}_A(\alpha\wedge\bar\beta\wedge\xi).$$

Theorem A and B concerns the leading coefficient in expansions such as (1). In Section 4 we use the previous results to describe the other coefficients, see Theorem 4.1 below. One of the main points of Theorem 4.1 is the following informally stated result.

Theorem C. *The coefficient C_{-r} in the asymptotic expansion (1) depends polynomially of degree $\kappa-r$ on the chosen metric.*

We finally note that asymptotic expansions similar to (1) have been studied before, see e.g. [**Bar**; **Bar2**], but to our understanding these results are not directly related to our residues.

2. Currents from singular forms

We recall some facts about semi-meromorphic forms and how to define principal value currents from them. In Section 2.2 we define currents from more general forms. Throughout X will be a complex manifold of dimension d.

2.1. Semi-meromorphic forms. We denote by $\mathcal{SM}(X)$ the semi-meromorphic forms, i.e. forms α which can be written locally as $\alpha=\widetilde{\alpha}/f$ where $\widetilde{\alpha}$ is a smooth form and f

a holomorphic function such that $f \not\equiv 0$. We write $P(\alpha)$ for the polar set of α, which consists of the points where α is not smooth. Given the local description above we get $P(\alpha) \subset \{f = 0\}$. For a hypersurface D we write $\mathcal{E}(*D)$ for the semi-meromorphic forms which have a polar set contained in D and $\mathcal{E}^{p,q}(*D)$ for the ones of bidegree (p, q). Since the pole of a semi-meromorphic form is determined locally by a holomorphic function, locally the order of the pole is well defined.

One way to define principal value currents from semi-meromorphic forms is the following cf. [**And**; **Ber**; **Sa**]: suppose $\alpha \in \mathcal{E}(*D)$ has a hypersurface D with normal crossings as polar set and $D = \{s = 0\}$ where $s : X \to L$ is a holomorphic section of some line bundle L. Let $|\cdot|$ be a metric on L and ξ a test form of complementary degree. The function

$$\lambda \mapsto \int_X |s|^{2\lambda} \alpha \wedge \xi$$

is a priori only defined when $\operatorname{Re}\lambda \gg 1$. One can show, however, that the function has an analytic continuation to $\operatorname{Re}\lambda > -\varepsilon$ for some $\varepsilon > 0$. Thus we may define the principal value current $[\alpha]$ by

$$\big\langle [\alpha], \xi \big\rangle = \Big(\int_X |s|^{2\lambda} \alpha \wedge \xi \Big)\Big|_{\lambda=0}.$$

The current does not depend on the choice of metric $|\cdot|$ or section s.

2.2. Quasi-meromorphic forms. We let $\mathcal{QM}(X)$ denote forms ω which can be written locally as $\omega = \widetilde{\omega}/f\bar{g}$ where $\widetilde{\omega}$ is a smooth form and f and g are holomorphic functions which are not identically zero. We call these forms *quasi-meromorphic* and they are smooth forms except that they can have real analytic singularities along (local) complex hypersurfaces.

For $\omega \in \mathcal{QM}(X)$ we define its polar set, denoted by $P(\omega)$, as the set of points where ω is not smooth. When ω has a polar set contained in a hypersurface D we write $\omega \in \mathcal{E}(*\bar{*}D)$, we call D the polar set even though ω may be smooth on parts of D. We will focus on forms in $\mathcal{E}(*\bar{*}D)$, for some D, since it is notationally more convenient. We write $\mathcal{E}^{p,q}(*\bar{*}D)$ for the forms in $\mathcal{E}(*\bar{*}D)$ which have bidegree (p, q).

The polar set of a quasi-meromorphic form has different parts between which we need to distinguish. We define the subset $P^{1,0}(\omega) \subset P(\omega)$ as follows. A point x in the polar set is *not* in $P^{1,0}(\omega)$ if around this point there is holomorphic function g, with $g \not\equiv 0$, such that $\bar{g}\omega$ is smooth. In the same spirit we define the set $P^{0,1}(\omega)$ to be the subset of polar points around which there is *not* a holomorphic function f, with $f \not\equiv 0$, such that $f\omega$ is smooth. We say that $P^{1,0}(\omega)$ is the set where ω has holomorphic singularities and $P^{0,1}(\omega)$ is the set where ω has anti-holomorphic singularities. We have that

$$P(\omega) = P^{1,0}(\omega) \cup P^{0,1}(\omega)$$

but $P^{1,0}(\omega) \cap P^{0,1}(\omega)$ need not be empty; it is the set where ω has both holomorphic and anti-holomorphic singularities. The order of the holomorphic (and anti-holomorphic) pole is locally well defined.

If $\omega \in \mathcal{E}(*\bar{*}D)$ then $P^{1,0}(\omega)$ and $P^{0,1}(\omega)$ are hypersurfaces contained in D and we temporarily set $H(\omega)$ to be the codimension one components of $P^{1,0}(\omega) \cap P^{0,1}(\omega)$. Since this is an analytic set there is a natural stratification, see Proposition II.5.6 in [**Dem**],

$$H(\omega)_d \subset H(\omega)_{d-1} \subset \cdots \subset H(\omega)_1 \subset H(\omega)_0 \tag{2}$$

where

(i) $H(\omega)_0 = X$,

(ii) $H(\omega)_1 = H(\omega)$,

(iii) if $k = 2, \dots, d$ then $H(\omega)_k$ is $\big(H(\omega)_{k-1}\big)_{\text{sing}}$ together with all the components of $H(\omega)_{k-1}$ with codimension greater than or equal to k.

Notice that $H(\omega)_k \setminus H(\omega)_{k+1}$ is a $(d-k)$-dimensional complex manifold which is possibly empty.

Definition 2.1. With the stratification as above we define the integer $\kappa(\omega)$ to be the largest number k such that $H(\omega)_k$ is non-empty. We further let $E(\omega) := H(\omega)_{\kappa(\omega)}$. ◇

The integer $\kappa(\omega)$ in some sense measures how bad the singularities of ω are. By definition $E(\omega)$ is a complex submanifold of dimension $d - \kappa(\omega)$.

***Example* 1.** To clarify these notions we give an example in $\mathbf{C}^3$ in the case of normal crossings. For

$$\omega = \frac{1}{z_1 \bar{z}_1 (z_1 - 1) z_2 \bar{z}_3}$$

we have

$$P^{1,0} = \{z_1 = 0\} \cup \{z_1 = 1\} \cup \{z_2 = 0\},$$
$$P^{0,1} = \{z_1 = 0\} \cup \{z_3 = 0\}.$$

Thus $P^{1,0} \cap P^{0,1} = \{z_1 = 0\} \cup \{z_1 = 1, z_3 = 0\}$ and hence $H(\omega) = \{z_1 = 0\}$. Since this is smooth we get that $\kappa(\omega) = 1$ and $E(\omega) = \{z_1 = 0\}$. ✓

For a semi-meromorphic form α we have $H(\alpha) = \varnothing$. Hence all components except $H(\alpha)_0 = X$ in the stratification are empty. Thus $\kappa(\alpha) = 0$ and $E(\alpha) = X$.

For a form $\omega \in \mathcal{E}(*\bar{*}D)$, where D has normal crossings, there is a more explicit description of $\kappa(\omega)$. Around any point $x \in X$ there are local coordinates $(z_1, \dots, z_d)$ with D given by $z_1 z_2 \cdots z_k = 0$. Then there are multi-indices J and K so that $z^J \bar{z}^K \omega$ is smooth. Choosing J and K minimal we define

$$\kappa_x(\omega) = \#\{j : J_j \neq 0 \text{ and } K_j \neq 0\}$$

and then

$$\kappa(\omega) = \max_{x \in X} \kappa_x(\omega).$$

Now suppose $s : X \to L$ is a holomorphic section such that $D = \{s = 0\}$ has normal crossings and that $\omega \in \mathcal{E}(*\bar{*}D)$. Around any point $x \in X$ there are coordinates $(z_1, \dots, z_d)$ so that $H(\omega)$ is given by $z_1 z_2 \cdots z_\ell = 0$. In a local holomorphic frame the section is given by $s = z^I \phi$ for some holomorphic ϕ which is non-vanishing on $H(\omega)$. We define

$$o_{\omega,x}(s) = \prod_{j=1}^{\ell} I_j. \tag{3}$$

and note that this does not depend on the choices of local coordinates or the frame.

Definition 2.2. For a holomorphic section $s : X \to L$ which defines a hypersurface D with normal crossings and $\omega \in \mathcal{E}(*\bar{*}D)$ we let

$$o_\omega(s) = \max_{x \in X} o_{\omega,x}(s).$$ ◇

Notice that in (3) we only multiply with the vanishing order for s on the local components on which ω has both holomorphic and anti-holomorphic poles. For ω semi-meromorphic $o_\omega(s) = 1$ for all sections s since then the product is empty.

We are now assuming that the polar set of ω is a hypersurface with normal crossings. For a test form ξ of complementary degree and $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) \gg 1$ we let

$$F_\xi(\lambda) = o_\omega(s) \int_X |s|^{2\lambda} \omega \wedge \xi. \tag{4}$$

The following theorem gives a first description of the function F_ξ.

Theorem 2.3. *Suppose $\omega \in \mathcal{QM}(X)$ has a hypersurface D with normal crossings as a polar set. The function F_ξ has the following properties*

(a) F_ξ has a meromorphic extension to $\mathbf{C}$,

(b) the possible poles of F_ξ are at $\mathbf{Q} \subset \mathbf{R}$,

(c) the order of the pole of F_ξ at the origin is $\leqslant \kappa(\omega)$.

To prove Theorem 2.3 we need the following lemma, the proof of which is a simple exercise.

Lemma 2.4. *For $\lambda \in \mathbf{C}$ and multi-indices I, J, K such that if $I_j = 0$ then $J_j = 0$ and $K_j = 0$ we have*

$$\frac{|z^I|^{2\lambda}}{z^J \bar{z}^K} = \frac{h(\lambda)}{\lambda^p} \frac{\partial^{J+K} |z^I|^{2\lambda}}{\partial z^J \partial \bar{z}^K}$$

where

$$h(\lambda) = \Big(\prod_{J_j \neq 0} I_j(\lambda I_j - 1) \cdots (\lambda I_j - J_j + 1) \Big)^{-1} \Big(\prod_{K_j \neq 0} I_j(\lambda I_j - 1) \cdots (\lambda I_j - K_j + 1) \Big)^{-1}$$

and $p = \#\{j : J_j \neq 0\} + \#\{j : K_j \neq 0\}$.

Notice that this means that $h(\lambda)$ has poles in

$$\lambda = \frac{1}{I_j}, \frac{2}{I_j}, \ldots, \frac{J_j - 1}{I_j} \quad \text{for } j \text{ with } J_j > 1$$

and

$$\lambda = \frac{1}{I_j}, \frac{2}{I_j}, \ldots, \frac{K_j - 1}{I_j} \quad \text{for } j \text{ with } K_j > 1.$$

Proof of Theorem 2.3. We may suppose that ξ has support in a coordinate chart and so we study the integral over, say, a polydisc $\Delta \subset \mathbf{C}^d$. Since D has normal crossings we may find coordinates so that the section s is a monomial, say $s = z^I = z_1^{I_1} \cdots z_d^{I_d}$ and we write the metric as $|\cdot| = |\cdot| \mathrm{e}^{-\phi}$ for some function ϕ. Furthermore, we write

$$\omega \wedge \xi = \frac{\psi}{z^J \bar{z}^K} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

where $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d$ and ψ is some smooth function with support in Δ. The integral in (4) may now be written

$$F_\xi(\lambda) = o_\omega(s) \int_\Delta \frac{|z^I|^{2\lambda}}{z^J \bar{z}^K} \mathrm{e}^{-2\lambda\phi} \psi \, \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{5}$$

We now prove (a). For integers $N \geqslant 0$ we can use Lemma 2.4 and Stokes' theorem to simplify the integral in (5) as

$$\begin{aligned}
F_\xi(\lambda) &= o_\omega(s) \int_\Delta \frac{|z^I|^{2\lambda+2N}}{z^{J+NI}\bar{z}^{K+NI}} \mathrm{e}^{-2\lambda\phi}\psi \,\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{o_\omega(s)h(\lambda)}{\lambda^{p_N}} \int_\Delta \frac{\partial^{J+K+2NI}|z^I|^{2\lambda+2N}}{\partial z^{J+NI}\partial\bar{z}^{K+NI}} \mathrm{e}^{-2\lambda\phi}\psi \,\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{(-1)^{|J+NI|+|K+NI|}o_\omega(s)h(\lambda)}{\lambda^{p_N}} \int_\Delta |z^I|^{2\lambda+2N} \frac{\partial^{J+K+2NI}}{\partial z^{J+NI}\partial\bar{z}^{K+NI}} \big(\mathrm{e}^{-2\lambda\phi}\psi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}.
\end{aligned}$$

The last integral in the above expression is holomorphic in $\operatorname{Re}\lambda > -N - \varepsilon$ for some $\varepsilon > 0$. Furthermore, the function h, which is given by Lemma 2.4 but here depends on N, is meromorphic in $\mathbf{C}$. Hence F_ξ has a meromorphic extension to $\mathbf{C}$, as N may be chosen arbitrarily large, and we have proven (a).

Now let us prove (b). The fact that the poles are located at rational numbers follows from the proof of (a) and Lemma 2.4 which describes the locations of the poles of h.

Finally we prove (c). Choosing $N = 0$ gives

$$F_\xi(\lambda) = \frac{(-1)^{|J|+|K|}o_\omega(s)h(\lambda)}{\lambda^p} \int_\Delta |z^I|^{2\lambda} \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\mathrm{e}^{-2\lambda\phi}\psi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{6}$$

Notice that Lemma 2.4 in particular gives that h does not have a pole at 0. We define a function g from the integral above by

$$g(\lambda) = \int_\Delta |z^I|^{2\lambda} \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\mathrm{e}^{-2\lambda\phi}\psi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Then g is holomorphic in $\operatorname{Re}\lambda > -\varepsilon$ for some ε. To show that F_ξ has a pole of order κ we need to show that g has a zero of order $p - \kappa$ at the origin. We have that

$$p - \kappa = \#\{j : J_j \neq 0 \text{ or } K_j \neq 0\} = \#\{j : I_j \neq 0\}.$$

Repeated use of the product rule for derivatives gives

$$g^{(k)}(0) = \sum_{\ell=0}^{k} \binom{k}{\ell} (-2)^{k-\ell} \int_\Delta \big(\log|z^I|^2\big)^\ell \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\psi\phi^{k-\ell}\big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{7}$$

and using the multinomial theorem we get

$$\begin{aligned}
&\int_\Delta \big(\log|z^I|^2\big)^\ell \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\psi\phi^{k-\ell}\big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \sum_M \binom{\ell}{M} \int_\Delta \prod_{j=1}^{d} \big(I_j \log|z_j|^2\big)^{M_j} \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\psi\phi^{k-\ell}\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}.
\end{aligned} \tag{8}$$

The sum is over multi-indices $M = (M_1, \ldots, M_d)$ such that $I_j = 0$ implies that $M_j = 0$, all $M_j \geqslant 0$ and $\sum_j M_j = \ell$. Thus we have to study integrals of the form

$$\int_\Delta \prod_{j=1}^{d} \big(I_j \log|z_j|^2\big)^{M_j} \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\psi\phi^{k-\ell}\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{9}$$

Suppose first that $I_1 \neq 0$ but $M_1 = 0$. Then the integral in (9) may be written

$$\int_{\Delta'} \prod_{j=2}^{d} \big(I_j \log|z_j|^2\big)^{M_j} \bigg(\int_{\Delta_1} \frac{\partial^{J+K}}{\partial z^J \partial\bar{z}^K} \big(\psi\phi^{k-\ell}\big) \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1\bigg) \mathrm{d}z' \wedge \mathrm{d}\bar{z}'$$

where $\Delta = \Delta_1 \times \Delta'$. But since $I_j \neq 0$ implies that $J_1 \neq 0$ or $K_1 \neq 0$ the inner integral vanishes using Stokes' theorem. Hence we get the following:

if $I_j \neq 0$ but $M_j = 0$ then the integral in (9) vanishes.

Now we suppose $k < p - \kappa$ and we want to show that $g^{(k)}(0) = 0$. From (7) and (8) we know that $g^{(k)}(0)$ is a sum of integrals as in (9). For each of these integrals there are an integer ℓ and a multi-index M such that

$$\sum M_j = \ell < p - \kappa = \#\{j : I_j \neq 0\}.$$

Hence, for each of the integrals, there is some j so that $I_j \neq 0$ but $M_j = 0$. Then, as explained above, all of the integrals are zero and thus $g^{(k)}(0) = 0$ for $k < p - \kappa$. Therefore g has a zero of order $p - \kappa$ at the origin which was what we wanted to prove. □

We use Theorem 2.3 (c) to make the following definition.

Definition 2.5. For $\omega \in \mathcal{E}(*\bar{*}D)$, where D has normal crossings, we define the *canonical current* $\{\omega\}$ *associated to* ω by

$$\langle\{\omega\}, \xi\rangle = \lambda^{\kappa(\omega)} F_\xi(\lambda)\Big|_{\lambda=0}. \qquad \diamond$$

A priori $\{\omega\}$ depends on choice of s and $|\cdot|$. Corollary 2.7, however, shows that this is not the case.

Remark. In the case that ω is semi-meromorphic $\{\omega\}$ is the principal value current of ω since then $\kappa(\omega) = 0$ and $o_\omega(s) = 1$.

2.3. Local calculations. We will make some calculations of canonical currents associated to quasi-meromorphic forms to hopefully clarify but also to show that they can behave a bit odd. Given a multi-index $J = (J_1, \dots, J_d)$ we write 1_J for the multi-index given by $(1_J)_j = 0$ if $J_j = 0$ and $(1_J)_j = 1$ if $J_j \neq 0$. We begin with a proposition.

Proposition 2.6. *For $\omega \in \mathcal{QM}(\mathbf{C}^d)$ and a test function ξ in $\mathbf{C}^d$ with support in Δ such that $\omega \wedge \xi = (\psi / z^J \bar{z}^K) \mathrm{d}z \wedge \mathrm{d}\bar{z}$ we have*

$$\langle\{\omega\}, \xi\rangle = \frac{(-1)^p}{(J - 1_J)!(K - 1_K)!} \int_\Delta \Big(\prod_{j: J_j + K_j \neq 0} \log |z_j|^2 \Big) \frac{\partial^{J+K} \psi}{\partial z^J \partial \bar{z}^K} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

where p is given by Lemma 2.4.

Proof. From the proof of Theorem 2.3 we know

$$\langle\{\omega\}, \xi\rangle = \lambda^{\kappa(\omega)} F_\xi(\lambda)\Big|_{\lambda=0} = \frac{o_\omega(s)(-1)^{|J|+|K|}}{(p - \kappa(\omega))!} h(0) g^{(p-\kappa(\omega))}(0)$$

and Lemma 2.4 gives

$$h(0) = \frac{(-1)^{|J|+|K|-p}}{(J - 1_J)!(K - 1_K)!} \Big(\prod_{j: J_j \neq 0} I_j \Big)^{-1} \Big(\prod_{j: K_j \neq 0} I_j \Big)^{-1}.$$

The equation (7) gives an expression for $g^{(p-\kappa(\omega))}(0)$ in terms of the integrals in (8). But just as in the proof of Theorem 2.3 these integrals vanish if $\ell < p-\kappa(\omega)$. For $\ell = p-\kappa(\omega)$ we must have all $M_j = 1$ for the integral not to vanish. Using this for $k = p-\kappa(\omega)$ we get

$$g^{(p-\kappa(\omega))}(0) = \Big(\prod_{j:I_j\neq 0} I_j\Big)(p-\kappa(\omega))! \int_\Delta \Big(\prod_{j:I_j\neq 0} \log|z_j|^2\Big)\frac{\partial^{J+K}\psi}{\partial z^J \partial \bar{z}^K} \mathrm{d}z\wedge \mathrm{d}\bar{z}.$$

This is the same integral as in the statement of the proposition. We only need to see what constant we get in front of it. This constant is

$$o_\omega(s)\frac{(-1)^p}{(J-1_J)!(K-1_K)!}\Big(\prod_{j:I_j\neq 0} I_j\Big)\Big(\prod_{j:J_j\neq 0} I_j\Big)^{-1}\Big(\prod_{j:K_j\neq 0} I_j\Big)^{-1}$$

but since $o_\omega(s) = \prod_{j:J_j\neq 0, K_j\neq 0} I_j$ this is precisely what is claimed. □

Corollary 2.7. *The canonical current* $\{\omega\}$ *does not depend on the choice of section* s *or metric* $|\cdot|$.

Proof. This follows immediately from Proposition 2.6 since the right hand side in that statement does not depend on the section s or the metric $|\cdot|$, as J and K do not. Hence (locally and thus also globally) this holds for $\{\omega\}$. □

Remark. We would not get the above corollary if we did not have the factor $o_\omega(s)$ in the definition of F_ξ.

When doing calculations we will get use of the following which is a consequence of Cauchy–Green's theorem: If ψ is a smooth function with compact support in $\Delta \subset \mathbf{C}$ then

$$\psi(0) = -\frac{1}{2\pi \mathrm{i}}\int_\Delta \log|z|^2 \frac{\partial^2\psi}{\partial z\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z}. \tag{10}$$

Corollary 2.8. *For* $\omega \in \mathcal{QM}(\mathbf{C}^d)$ *and a test function* ξ *in* $\mathbf{C}^d$ *with support in* Δ *we have*

(a) if $\omega\wedge\xi = (\psi/z_1^m\bar{z}_1^n)\mathrm{d}z\wedge\mathrm{d}\bar{z}$ *then*

$$\big\langle\{\omega\},\xi\big\rangle = -\frac{2\pi\mathrm{i}}{(m-1)!(n-1)!}\int_{\Delta\cap\{z_1=0\}}\frac{\partial^{m+n-2}\psi}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\mathrm{d}z'\wedge\mathrm{d}\bar{z}',$$

(b) if $\omega\wedge\xi = (\psi/z_1^{J_1}\dots z_k^{J_k}\bar{z}_1\dots\bar{z}_k)\mathrm{d}z\wedge\mathrm{d}\bar{z}$

$$\big\langle\{\omega\},\xi\big\rangle = \frac{(-2\pi\mathrm{i})^k}{(J-1_J)!}\int_{\Delta\cap\{z_1=\dots=z_k=0\}}\frac{\partial^{J-1_J}\psi}{\partial z^{J-1_J}}\mathrm{d}z''\wedge\mathrm{d}\bar{z}''$$

where $\mathrm{d}z'\wedge\mathrm{d}\bar{z}' = \mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2\wedge\dots\wedge\mathrm{d}z_d\wedge\mathrm{d}\bar{z}_d$ *and* $\mathrm{d}z''\wedge\mathrm{d}\bar{z}'' = \mathrm{d}z_{k+1}\wedge\mathrm{d}\bar{z}_{k+1}\wedge\dots\wedge\mathrm{d}z_d\wedge\mathrm{d}\bar{z}_d$.

Proof. This follows from Proposition 2.6 and (10). □

We now use Corollary 2.8 to make some explicit calculations.

***Example* 2.** Let $X = \mathbf{CP}^1$ with homogeneous coordinates $[z:w]$ and let 0 be the point where $z=0$ and ∞ the point where $w=0$. We let

$$\omega = \frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{z\bar{z}} = \frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{w\bar{w}} \quad \text{for } zw\neq 0,$$

which means that $\kappa(\omega) = 1$. In view of Corollary 2.8 (a), given a test function ξ, we get

$$\big\langle \{\omega\}, \xi \big\rangle = -2\pi \mathrm{i}\xi(0) - 2\pi \mathrm{i}\xi(\infty).$$

On the other hand, if $X = U$ for some open set $U \subset \mathbf{CP}^1$ which does not contain the origin or ∞ then $\kappa(\omega) = 0$ and therefore

$$\big\langle \{\omega\}, \xi \big\rangle = \int_U \frac{\xi(z)}{|z|^2} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \qquad \checkmark$$

Remark. The above example shows that for canonical currents we have the following property: in general $\chi\{\omega\} \neq \{\chi\omega\}$ for a smooth function χ. This means that when we define the canonical current associated to a form ω it is important to decide on what underlying space we consider it.

***Example* 3.** If we let $X = \mathbf{C}$ and apply Corollary 2.8 with $\omega = 1/(z^m \bar{z}^n)$ then we get that

$$z\Big\{\frac{1}{z^m \bar{z}^n}\Big\} = \Big\{\frac{1}{z^{m-1}\bar{z}^n}\Big\} \quad \text{and} \quad \bar{z}\Big\{\frac{1}{z^m \bar{z}^n}\Big\} = \Big\{\frac{1}{z^m \bar{z}^{n-1}}\Big\}$$

for $m, n \geqslant 2$. On the other hand

$$z^m\Big\{\frac{1}{z^m \bar{z}^n}\Big\} = 0 \quad \text{and} \quad \bar{z}^n\Big\{\frac{1}{z^m \bar{z}^n}\Big\} = 0$$

for $m, n \geqslant 1$. ✓

Theorem 2.3 (b) gives some insight about the poles of F_ξ but the following proposition gives more information.

Proposition 2.9. *The poles of the function F_ξ are located at rational numbers less than or equal to*

$$\max\Big\{ \min\Big\{\frac{J_j - 1}{I_j}, \frac{K_j - 1}{I_j}\Big\} : j = 1, \dots, d\Big\}.$$

Proof. First suppose $K_i = 0$ or $K_i = 1$ for all $i = 1, \dots, d$. We may assume that ξ has support in a local chart and so we can write down the integral locally as

$$\begin{aligned} F_\xi(\lambda) &= o_\omega(s) \int_\Delta \frac{|z^I|^{2\lambda}}{z^J \bar{z}^K} \mathrm{e}^{-2\lambda\phi} \psi \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \frac{h(\lambda)}{\lambda^p} \int_\Delta \frac{\partial^K |z^I|^{2\lambda}}{\partial \bar{z}^K} \frac{1}{z^J} \mathrm{e}^{-2\lambda\phi} \psi \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \frac{(-1)^{|K|} h(\lambda)}{\lambda^p} \int_\Delta \frac{|z^I|^{2\lambda}}{z^J} \frac{\partial^K \mathrm{e}^{-2\lambda\phi}\psi}{\partial \bar{z}^K} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \end{aligned}$$

We made a similar computation in the proof of Theorem 2.3, cf. Lemma 2.4, but now we only considered the anti-holomorphic derivatives. Since these are of at most order one the function h will not have any poles at all, see Lemma 2.4. But the integral in the last expression above is the principal value current of $1/z^J$ acting on $\frac{\partial^K \mathrm{e}^{-2\lambda\phi}\psi}{\partial \bar{z}^K} \mathrm{d}z \wedge \mathrm{d}\bar{z}$. This is known not to have any poles in the right half plane (and not at the origin). Hence F_ξ does not have any poles in $\mathrm{Re}(\lambda) > 0$.

Note that the above result would also hold as long as $J_i \leqslant 1$ or $K_i \leqslant 1$ for all i. Now suppose we are in the general case. Let $\mu = \lambda - M$ for some integer M. Then

$$\frac{|z^I|^{2\lambda}}{z^J \bar{z}^K} = \frac{|z^I|^{2\mu + 2M}}{z^J \bar{z}^K} = |z^I|^{2\mu} \frac{z^{MI} \bar{z}^{MI}}{z^J \bar{z}^K}$$

and choosing M so that $MI_i \geqslant J_i - 1$ or $MI_i \geqslant K_i - 1$ for each i we get from the above that F_ξ has no poles in $\mathrm{Re}(\mu) > 0$. That is, F_ξ has no poles in $\mathrm{Re}(\lambda) > M$. Choosing M so that this holds we get the proposition. □

One can note that by choosing higher powers I of the section s we can get the poles in the right half-plane arbitrarily close to the origin. Suppose $\omega = \alpha \wedge \bar{\beta}$ for semi-meromorphic forms α and β. Proposition 2.9 gives us a hint that the situation is a bit more well behaved when β only has poles of order one since then the proposition says that F_ξ does not have poles in the right half plane.

3. Cohomological residues

We will discuss the classical Leray residue, the conjugate Dolbeault residue and then define a residue for the Aeppli cohomology. Now X is assumed to be a compact complex manifold.

3.1. The conjugate Dolbeault residue. To define residues the classical setting is the following: suppose D is a smooth hypersurface and α a d-closed form in $X \setminus D$ with a holomorphic pole of order one along D. If $z_1 = 0$ is a local equation for D then α may locally be written as

$$\alpha = \frac{\mathrm{d}z_1}{z_1} \wedge \widetilde{\alpha} + \tau$$

for some forms $\widetilde{\alpha}$ and τ such that τ does not contain $\mathrm{d}z_1$. Certainly $\widetilde{\alpha}$ is smooth but it is well known that the closedness implies that τ is smooth. One defines the *Poincaré residue* by $\mathrm{Res}(\alpha) = \widetilde{\alpha}\big|_D$. It is easy to check that this gives a well defined closed form on D. If α is any closed form on $X \setminus D$ then there is a cohomologous form α' with a pole of order one along D, cf. [**Ch**, Thm. 6.3.3, p. 233]. The *Leray residue* is defined by

$$\mathrm{Res}(\alpha) = \big[\mathrm{Res}(\alpha')\big]_{\mathrm{dR}}$$

which gives a map

$$\mathrm{Res} : H^k(X \setminus D) \to H^{k-1}(D).$$

Since the groups $\mathcal{E}^{p,q}(*D)$ form a complex with the operator ∂ we get cohomology groups $H^{p,q}_\partial(*D)$. In [**Fe**] the *conjugate Dolbeault residue* was constructed as a map

$$\mathrm{Res}_\partial : H^{p,0}_\partial(*D) \to H^{p-1,0}_\partial(D).$$

We will give an alternative definition for forms in $H^{d,q}_\partial(*D)$ which is quite explicit. Given a (d,q)-form α in $\mathbf{C}^d$, with coordinates $z = (z_1, \dots, z_d)$, which has a holomorphic pole along $z_1 = 0$ we may write

$$\alpha = \frac{\mathrm{d}z_1 \wedge \widetilde{\alpha}_z}{z_1^m} \tag{11}$$

for some smooth form $\widetilde{\alpha}_z$ which does not contain $\mathrm{d}z_1$. To define a residue we need the following lemma. We do not give the proof since it is very similar to the proof of Lemma 3.4 below.

Lemma 3.1. *Let z and w be coordinates in $\mathbf{C}^d$ such that z_1/w_1 is a non-vanishing holomorphic function and let $D = \{z_1 = 0\}$. Suppose $\alpha \in \mathcal{E}^{d,q}(*D)$ has compact support and write*

$$\frac{\mathrm{d}z_1}{z_1^m} \wedge \widetilde{\alpha}_z(z) = \alpha = \frac{\mathrm{d}w_1}{w_1^m} \wedge \widetilde{\alpha}_w(w),$$

for some smooth forms $\widetilde{\alpha}_z(z)$ and $\widetilde{\alpha}_w(w)$ which does not contain $\mathrm{d}z_1$ or $\mathrm{d}w_1$.

(a) *If there is a form $\eta \in \mathcal{E}(*D)$ with compact support such that $\alpha = \partial\eta$ then there is a smooth form $\widehat{\eta}$ on D such that*

$$\left.\frac{\partial^{m-1}\widetilde{\alpha}_z}{\partial z_1^{m-1}}\right|_D = \partial\widehat{\eta},$$

with $\operatorname{supp}(\widehat{\eta}) \subset \operatorname{supp}(\alpha) \cap D$.

(b) *There is a smooth form β on D whose support is contained in* $\operatorname{supp}(\alpha) \cap D$ *such that*

$$\left.\frac{\partial^{m-1}\widetilde{\alpha}_z}{\partial z_1^{m-1}}\right|_D = \left.\frac{\partial^{m-1}\widetilde{\alpha}_w}{\partial w_1^{m-1}}\right|_D + \partial\beta.$$

Now suppose $\alpha \in \mathcal{E}^{d,q}(*D)$ and (ρ_j) is a partition of unity subordinate to a cover of X by charts with coordinates $\big(z_j = (z_{j,1}, z_{j,2} \dots, z_{j,d})\big)$ such that D is locally given by $z_{j,1} = 0$. We write

$$\alpha = \frac{\mathrm{d}z_{j,1} \wedge \widetilde{\alpha}_j(z)}{z_{j,1}^m} \quad \text{on } \operatorname{supp}(\rho_j),$$

and then define

$$R_{\rho,z}(\omega) = \sum_j \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}(\rho_j\widetilde{\alpha}_j)}{\partial z_{j,1}^{m-1}}\right|_D.$$

Using Lemma 3.1 one can prove that, for $\alpha \in \mathcal{E}^{d,q}(*D)$,

(a) $R_{\rho,z}(\alpha) = R_{\sigma,w}(\alpha) + \partial\beta$,

(b) $R_{\rho,z}(\partial\eta) = \partial\widehat{\eta}$.

The proof of (a) and (b) is very similar to the proof of Proposition 3.5 below. We can now make the following definition.

Definition 3.2. For a class $[\alpha] \in H_\partial^{d,q}(*D)$ we define its *conjugate Dolbeault residue* by

$$\operatorname{Res}_\partial(\alpha) = \big[R_{\rho,z}(\alpha)\big]_\partial. \qquad \diamond$$

The claims (a) and (b) above give that $\operatorname{Res}_\partial(\alpha)$ is well defined and independent of the choice of partition of unity and local coordinates. We now present a theorem which is not very related to the rest of the paper, but we think it is a nice application of the conjugate Dolbeault residue.

Theorem 3.3. *If $\alpha \in \mathcal{E}^{p,q}(*D)$, where D is a smooth hypersurface, and ξ a test form of bidegree $(d-p, d-q-1)$ then*

$$\big\langle \bar\partial[\alpha], \xi \big\rangle = \big\langle [\bar\partial\alpha], \xi \big\rangle + 2\pi\mathrm{i} \int_D \operatorname{Res}_\partial(\alpha \wedge \xi).$$

Proof. We may suppose ξ has support contained in a coordinate chart which is biholomorphic to the unit polydisc Δ and that D is there given by $z_1 = 0$. We may further suppose that $\alpha = \frac{a}{z_1^m}\mathrm{d}z_P \wedge \mathrm{d}\bar z_Q$ and $\xi = b\,\mathrm{d}z_R \wedge \mathrm{d}\bar z_S$ where $|P| = p$ and $|Q| = q$. Then we get

$$\begin{aligned}
\alpha \wedge \xi &= (-1)^{q(d-p)+s} \frac{ab}{z_1^m} \mathrm{d}z \wedge \mathrm{d}\bar z_Q \wedge \mathrm{d}\bar z_S, \\
\alpha \wedge \bar\partial\xi &= \sum_k (-1)^{(q+1)(d-p)+s+t} \frac{a}{z_1^m} \frac{\partial b}{\partial \bar z_k} \mathrm{d}z \wedge \mathrm{d}\bar z, \\
\bar\partial\alpha \wedge \xi &= \sum_k (-1)^{q(d-p)+d+q+s+t} \frac{\partial a}{\partial \bar z_k} \frac{b}{z_1^m} \mathrm{d}z \wedge \mathrm{d}\bar z,
\end{aligned}$$

where s and t are given by $\mathrm{d}z_P \wedge \mathrm{d}z_R = (-1)^s \mathrm{d}z$ and $\mathrm{d}\bar{z}_Q \wedge \mathrm{d}\bar{z}_k \wedge \mathrm{d}\bar{z}_S = (-1)^t \mathrm{d}\bar{z}$ (so t depends on k but we suppress this). For $k = 1$ we have

$$(-1)^t \mathrm{d}\bar{z} = \mathrm{d}\bar{z}_Q \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_S = (-1)^q \mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_Q \wedge \mathrm{d}\bar{z}_S$$

and hence

$$\mathrm{d}\bar{z}' := \mathrm{d}\bar{z}_2 \wedge \cdots \wedge \mathrm{d}\bar{z}_d = (-1)^{q+t} \mathrm{d}\bar{z}_Q \wedge \mathrm{d}\bar{z}_S.$$

This means that

$$\operatorname{Res}_\partial(\alpha \wedge \xi) = \frac{(-1)^{q(d-p)+q+s+t}}{(m-1)!} \frac{\partial^{m-1}(ab)}{\partial z_1^{m-1}} \mathrm{d}z' \wedge \mathrm{d}\bar{z}'.$$

We write $\Delta' = \Delta \cap \{z_1 = 0\} = \Delta \cap D$. Using Proposition 2.6 and the remark after Definition 2.5 we get

$$\begin{aligned}
\langle \bar{\partial}[\alpha], \xi \rangle &= (-1)^{p+q+1} \langle [\alpha], \bar{\partial}\xi \rangle \\
&= \sum_k \frac{(-1)^{q(d-p)+q+d+s+t}}{(m-1)!} \int_\Delta \log|z_1|^2 \frac{\partial^m}{\partial z_1^m} \Big(a \frac{\partial b}{\partial \bar{z}_k} \Big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{(-1)^{q(d-p)+q+d+s+t}}{(m-1)!} \int_\Delta \log|z_1|^2 \frac{\partial^{m+1}(ab)}{\partial z_1^m \partial \bar{z}_1} \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&\quad - \sum_k \frac{(-1)^{q(d-p)+q+d+s+t}}{(m-1)!} \int_\Delta \log|z_1|^2 \frac{\partial^m}{\partial z_1^m} \Big(\frac{\partial a}{\partial \bar{z}_k} b \Big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{2\pi\mathrm{i}(-1)^{q(d-p)+q+s+t}}{(m-1)!} \int_{\Delta'} \frac{\partial^{m-1}(ab)}{\partial z_1^{m-1}} \mathrm{d}z' \wedge \mathrm{d}\bar{z}' + \langle [\bar{\partial}\alpha], \xi \rangle \\
&= 2\pi\mathrm{i} \int_D \operatorname{Res}_\partial(\alpha \wedge \xi) + \langle [\bar{\partial}\alpha], \xi \rangle
\end{aligned}$$

□

3.2. A residue for the Aeppli cohomology. Recall that for a complex manifold X one defines the Bott–Chern cohomology groups by

$$H^{p,q}_{BC}(X) = \frac{\ker(\partial) \cap \ker(\bar{\partial})}{\operatorname{im}(\partial\bar{\partial})}$$

and the Aeppli cohomology groups by

$$H^{p,q}_{A}(X) = \frac{\ker(\partial\bar{\partial})}{\operatorname{im}(\partial) + \operatorname{im}(\bar{\partial})}.$$

Given a hermitian metric on X the induced Hodge star operator gives an isomorphism

$$* : H^{p,q}_{BC}(X) \to H^{n-p,n-q}_{A}(X)$$

so in this sense the Aeppli cohomology is dual to the Bott–Chern cohomology. We have the following natural maps

$$\begin{array}{ccccc}
 & & H^{p,q}_{BC}(X) & & \\
 & \swarrow & \downarrow & \searrow & \\
H^{p,q}_{\partial}(X) & & H^{p+q}_{dR}(X) & & H^{p,q}_{\bar{\partial}}(X) \\
 & \searrow & \downarrow & \swarrow & \\
 & & H^{p,q}_{A}(X) & &
\end{array}$$

and for a manifold on which the $\partial\bar{\partial}$-lemma holds all the outer maps are isomorphisms. In particular this is true for Kähler manifolds. For a more elaborate discussion on these facts we refer to [**Ang1**; **Ang2**; **Del**].

Restricting our attention to forms in $\mathcal{E}^{d,d}(*\bar{*}D)$ we consider the cohomology group $H_A^{d,d}(*\bar{*}D)$. To define a residue we need the following lemma.

Lemma 3.4. *Let z and w be coordinates in $\mathbf{C}^d$ such that z_1/w_1 is a non-vanishing holomorphic function and let $D = \{z_1 = 0\}$. Suppose $\omega \in \mathcal{E}^{d,d}(*\bar{*}D)$ has compact support and write*

$$\frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1}{z_1^m \bar{z}_1^n} \wedge \widetilde{\omega}_z(z) = \omega = \frac{\mathrm{d}w_1 \wedge \mathrm{d}\bar{w}_1}{w_1^{m'} \bar{w}_1^{n'}} \wedge \widetilde{\omega}_w(w),$$

for some smooth forms $\widetilde{\omega}_z(z)$ and $\widetilde{\omega}_w(w)$ which does not contain $\mathrm{d}z_1, \mathrm{d}\bar{z}_1$ or $\mathrm{d}w_1, \mathrm{d}\bar{w}_1$.

(a) If there are forms $\eta, \nu \in \mathcal{E}(*\bar{*}D)$ with compact support such that $\omega = \partial\eta + \bar{\partial}\nu$ then there are smooth forms $\widehat{\eta}$ and $\widehat{\nu}$ on D such that

$$\frac{\partial^{m+n-2}\widetilde{\omega}_z}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\bigg|_D = \partial\widehat{\eta} + \bar{\partial}\widehat{\nu},$$

with $\operatorname{supp}(\widehat{\eta}), \operatorname{supp}(\widehat{\nu}) \subset \operatorname{supp}(\omega) \cap D$.

(b) There are smooth forms $\widehat{\alpha}$ and $\widehat{\beta}$ on D whose support is contained in $\operatorname{supp}(\omega) \cap D$ such that

$$\frac{1}{(m-1)!(n-1)!}\frac{\partial^{m+n-2}\widetilde{\omega}_z}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\bigg|_D = \frac{1}{(m'-1)!(n'-1)!}\frac{\partial^{m+n-2}\widetilde{\omega}_w}{\partial w_1^{m'-1}\partial\bar{w}_1^{n'-1}}\bigg|_D + \partial\widehat{\alpha} + \bar{\partial}\widehat{\beta}.$$

Proof. We first prove (a) and suppose $\omega = \partial\eta$. If

$$\eta = \frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \eta_1 + \mathrm{d}\bar{z}_1 \wedge \eta_2}{z_1^{m-1}\bar{z}_1^n},$$

where η_1 and η_2 does not contain $\mathrm{d}z_1$ or $\mathrm{d}\bar{z}_1$, then

$$\omega = \partial\eta = \frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1}{z_1^m \bar{z}_1^n} \wedge \Big(-(m-1)\eta_2 + z_1\partial\eta_1 + z_1\frac{\partial\eta_2}{\partial z_1}\Big)$$

and therefore

$$\widetilde{\omega}_z = -(m-1)\eta_2 + z_1\partial\eta_1 + z_1\frac{\partial\eta_2}{\partial z_1}.$$

We get

$$\begin{aligned}
\frac{\partial^{m+n-2}\widetilde{\omega}_z}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\bigg|_D &= \frac{\partial^{m+n-2}}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\Big(-(m-1)\eta_2 + z_1\partial\eta_1 + z_1\frac{\partial\eta_2}{\partial z_1}\Big)\bigg|_D \\
&= (m-1)\Big(-\frac{\partial^{m+n-2}\eta_2}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}} + \frac{\partial^{m+n-3}\partial\eta_1}{\partial z_1^{m-2}\partial\bar{z}_1^{n-1}} + \frac{\partial^{m+n-2}\eta_2}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\Big)\bigg|_D \\
&= \partial\Big((m-1)\frac{\partial^{m+n-3}\eta_1}{\partial z_1^{m-2}\partial\bar{z}_1^{n-1}}\bigg|_D\Big).
\end{aligned}$$

The case $\omega = \bar{\partial}\nu$ is treated analogously. By linearity we get the case $\omega = \partial\eta + \bar{\partial}\nu$ and hence we have proven (a). Now we prove (b) and we first suppose $(m,n) = (m',n')$. The calculation

$$\begin{aligned}
\omega &= -\partial\Big(\frac{1}{m-1}\frac{\mathrm{d}\bar{z}_1 \wedge \widetilde{\omega}_z}{z_1^{m-1}\bar{z}_1^n}\Big) - \frac{1}{m-1}\frac{\mathrm{d}\bar{z}_1 \wedge \partial\widetilde{\omega}_z}{z_1^{m-1}\bar{z}_1^n} \\
&= -\partial\Big(\frac{1}{m-1}\frac{\mathrm{d}\bar{z}_1 \wedge \widetilde{\omega}_z}{z_1^{m-1}\bar{z}_1^n}\Big) + \frac{1}{m-1}\frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1}{z_1^{m-1}\bar{z}_1^n} \wedge \frac{\partial\widetilde{\omega}_z}{\partial z_1}
\end{aligned}$$

may be iterated and so we can write

$$\omega = \partial\alpha_1 + \bar\partial\beta_1 + \frac{1}{(m-1)!(n-1)!}\frac{\mathrm{d}z_1\wedge\mathrm{d}\bar z_1}{z_1\bar z_1}\wedge\frac{\partial^{m+n-2}\widetilde\omega_z}{\partial z_1^{m-1}\partial\bar z_1^{n-1}}.$$

Doing the same for the coordinate w we get that

$$\frac{\mathrm{d}z_1\wedge\mathrm{d}\bar z_1}{z_1\bar z_1}\wedge\frac{\partial^{m+n-2}\widetilde\omega_z}{\partial z_1^{m-1}\partial\bar z_1^{n-1}} - \frac{\mathrm{d}w_1\wedge\mathrm{d}\bar w_1}{w_1\bar w_1}\wedge\frac{\partial^{m+n-2}\widetilde\omega_w}{\partial w_1^{m-1}\partial\bar w_1^{n-1}} = \partial\alpha+\bar\partial\beta$$

for some α and β. Using (a) we get

$$\left.\frac{\partial^{m+n-2}\widetilde\omega_z}{\partial z_1^{m-1}\partial\bar z_1^{n-1}}\right|_D = \left.\frac{\partial^{m+n-2}\widetilde\omega_w}{\partial w_1^{m-1}\partial\bar w_1^{n-1}}\right|_D + \partial\widehat\alpha+\bar\partial\widehat\beta$$

which is what was to be proven. Now we treat the case that $(m,n)\neq(m',n')$ and for simplicity we suppose $m'\geqslant m$ and $n'\geqslant n$. We get

$$\begin{aligned}
&\frac{1}{(m'-1)!(n'-1)!}\left.\frac{\partial^{m'+n'-2}z_1^{m'-m}\bar z_1^{n'-n}\widetilde\omega_z}{\partial z_1^{m'-1}\partial\bar z_1^{n'-1}}\right|_D\\
&=\frac{1}{(m'-1)!(n'-1)!}\binom{m'-1}{m'-m}\binom{n'-1}{n'-n}\frac{\partial^{m'-m}z_1^{m'-m}}{\partial z_1^{m'-m}}\frac{\partial^{n'-n}\bar z_1^{n'-n}}{\partial\bar z_1^{n'-n}}\left.\frac{\partial^{m+n-2}\widetilde\omega_z}{\partial z_1^{m-1}\partial\bar z_1^{n-1}}\right|_D\\
&=\frac{1}{(m-1)!(n-1)!}\left.\frac{\partial^{m+n-2}\widetilde\omega_z}{\partial z_1^{m-1}\partial\bar z_1^{n-1}}\right|_D
\end{aligned}$$

since the restriction to D forces the correct amount of derivatives to land on $z_1^{m'-m}$ and $\bar z_1^{n'-n}$. This proves (b). ☐

For a form $\omega\in\mathcal{E}^{d,d}(*\bar*D)$ and a partition of unity (ρ_j) subordinate to a cover of X by charts with coordinates $\big(z_j=(z_{j,1},z_{j,2}\dots,z_{j,d})\big)$ such that D is locally given by $z_{j,1}=0$ and

$$\omega = \frac{\mathrm{d}z_{j,1}\wedge\mathrm{d}\bar z_{j,1}}{z_{j,1}^m\bar z_{j,1}^n}\wedge\widetilde\omega_j(z)\quad\text{on }\operatorname{supp}(\rho_j),$$

we let

$$\operatorname{Res}_{\rho,z}(\omega)=\sum_j\frac{1}{(m-1)!(n-1)!}\left.\frac{\partial^{m+n-2}(\rho_j\widetilde\omega_j)}{\partial z_{j,1}^{m-1}\partial\bar z_{j,1}^{n-1}}\right|_D.$$

Proposition 3.5. *For $\omega\in\mathcal{E}^{d,d}(*\bar*D)$ we have*

(a) $\operatorname{Res}_{\rho,z}(\omega)=\operatorname{Res}_{\sigma,w}(\omega)+\partial\alpha+\bar\partial\beta$,

(b) $\operatorname{Res}_{\rho,z}(\partial\eta+\bar\partial\nu)=\partial\alpha+\bar\partial\beta$.

Proof. We write

$$\operatorname{Res}^j_{\rho,z}(\omega)=\frac{1}{(m-1)!(n-1)!}\left.\frac{\partial^{m+n-2}(\rho_j\widetilde\omega_j)}{\partial z_{j,1}^{m-1}\partial\bar z_{j,1}^{n-1}}\right|_D$$

so that

$$\operatorname{Res}_{\rho,z}(\omega)=\sum_j\operatorname{Res}^j_{\rho,z}(\omega).$$

We have the following two identities:

(i) $\operatorname{Res}^j_{\rho,z}(\sigma_i\omega)=\operatorname{Res}^i_{\sigma,w}(\rho_j\omega)+\partial\alpha_{i,j}+\bar\partial\beta_{i,j}$,

(ii) $\operatorname{Res}^j_{\rho,z}(\omega) = \sum_i \operatorname{Res}^j_{\rho,z}(\sigma_i\omega)$.

The first is basically Lemma 3.5 (b) and (ii) is just an interchange of the differentiation and the sum. Using the claims we get

$$\begin{aligned}
\operatorname{Res}_{\rho,z}(\omega) &\overset{\text{def}}{=} \sum_j \operatorname{Res}^j_{\rho,z}(\omega) \\
&\overset{(ii)}{=} \sum_{j,i} \operatorname{Res}^j_{\rho,z}(\sigma_i\omega) \\
&\overset{(i)}{=} \sum_{i,j} \operatorname{Res}^i_{\sigma,w}(\rho_j\omega) + \partial\alpha_{i,j} + \bar\partial\beta_{i,j} \\
&\overset{(ii)}{=} \sum_i \operatorname{Res}^i_{\sigma,w}(\omega) + \sum_{i,j} \partial\alpha_{i,j} + \bar\partial\beta_{i,j} \\
&\overset{\text{def}}{=} \operatorname{Res}_{\sigma,w}(\omega) + \partial\Big(\sum_{i,j}\alpha_{i,j}\Big) + \bar\partial\Big(\sum_{i,j}\beta_{i,j}\Big)
\end{aligned}$$

since $\alpha_{i,j}$ and $\beta_{i,j}$ has support contained in $\operatorname{supp}(\rho_j\sigma_i)$. Thus we have proven (a). We further have

$$\begin{aligned}
\operatorname{Res}_{\rho,z}(\partial\eta + \bar\partial\nu) &= \operatorname{Res}_{\rho,z}\Big(\sum_i \partial(\sigma_i\eta) + \bar\partial(\sigma_i\nu)\Big) \\
&= \sum_i \operatorname{Res}_{\rho,z}\big(\partial(\sigma_i\eta) + \bar\partial(\sigma_i\nu)\big) \\
&= \sum_i \partial\alpha_i \\
&= \partial\Big(\sum_i \alpha_i\Big).
\end{aligned}$$

which proves (b). □

Using Proposition 3.5 we can give the following definition.

Definition 3.6. For $\omega \in H^{d,d}_A(*\bar{*}D)$ we define the *Aeppli residue* by

$$\operatorname{Res}_A(\omega) = [\operatorname{Res}_{\rho,z}(\omega)]_A \qquad \diamond$$

Remark. Our definition of the Aeppli residue is very similar to the definition of *the residue map* in [**Fe2**]. They define this in a different context and for forms with, what they call, tame singularities.

We thus have a map $\operatorname{Res}_A : H^{d,d}_A(*\bar{*}D) \to H^{d-1,d-1}_A(D)$.

Proposition 3.7. *(a) If $\omega \in H^{d,d}_A(*\bar{*}D)$ is semi-meromorphic then $\operatorname{Res}_A(\omega) = 0$.*

(b) If α and β are meromorphic $(d,0)$-forms with poles along a smooth hypersurface D and the pole of β is of order one then

$$\operatorname{Res}_A(\alpha \wedge \bar\beta) = (-1)^{d-1}\big[\operatorname{Res}_\partial \alpha \wedge \overline{\operatorname{Res}\beta}\big]_A$$

where the right hand side is a well defined class and $\operatorname{Res}\beta$ denotes the Poincaré residue.

Proof. We get (a) from Lemma 3.4 since we may choose $n \geqslant 1$. To prove (b) write locally $\alpha = (a/z_1^m)\mathrm{d}z$ and $\beta = (b/z_1)\mathrm{d}z$. Then $\alpha \wedge \bar{\beta} = (-1)^{d-1}\big(a\bar{b}/(z_1^m \bar{z}_1)\big)\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z' \wedge \mathrm{d}\bar{z}'$ and hence

$$\mathrm{Res}_A(\alpha \wedge \bar{\beta}) = (-1)^{d-1}\Big[\frac{\partial^{m-1}a}{\partial z_1^{m-1}}\bar{b}\,\mathrm{d}z' \wedge \mathrm{d}\bar{z}'\Big]_A$$

and $\mathrm{Res}_\partial(\alpha) = \big[\frac{\partial^{m-1}a}{\partial z_1^{m-1}}\mathrm{d}z'\big]_\partial$. The Poincaré residue $\mathrm{Res}\,\beta$ is meromorphic since β is. Letting $R = \frac{\partial^{m-1}a}{\partial z_1^{m-1}}\mathrm{d}z'$ we get that $(-1)^d R \wedge \overline{\mathrm{Res}\,\beta}$ is a representative of $\mathrm{Res}_A(\alpha \wedge \bar{\beta})$ and R is a representative of $\mathrm{Res}_\partial(\alpha)$. If we choose a different representative, say $R + \partial\gamma$, of $\mathrm{Res}_\partial(\alpha)$ we get

$$(R + \partial\gamma) \wedge \overline{\mathrm{Res}\,\beta} = R \wedge \overline{\mathrm{Res}\,\beta} + \partial(\gamma \wedge \overline{\mathrm{Res}\,\beta})$$

and therefore $\big[\mathrm{Res}_\partial\,\alpha \wedge \overline{\mathrm{Res}\,\beta}\big]_A$ is well defined. □

The next theorem relates the Aeppli residue to the canonical currents defined in Section 2.2. It gives an indication that canonical currents do not behave like principle value currents but rather as residue currents.

Theorem 3.8. *For $\omega \in \mathcal{E}(*\bar{*}D)$ with $\kappa(\omega) > 0$ and D a smooth hypersurface we have*

$$\big\langle \{\omega\}, \xi \big\rangle = -2\pi\mathrm{i}\int_D \mathrm{Res}_A(\omega \wedge \xi).$$

Proof. Choose a partition of unity (ρ_ι) subordinate to a cover consisting of charts which are mapped to the unit polydisc in which the hypersurface is given by $z_1 = 0$. Suppose the holomorphic pole has order m and the anti-holomorphic pole order n. Since $\kappa(\omega) > 0$ by assumption we have $m, n > 0$. Notice that $\kappa(\omega) > 0$ together with that D is smooth implies that $\kappa(\omega) = 1$. Write locally $\omega \wedge \xi = \psi/(z_1^m \bar{z}_1^n)\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Then, using Proposition 2.6, (10) and Definition 3.6 we get

$$\begin{aligned}\big\langle \{\omega\}, \xi \big\rangle &= \sum_\iota \frac{1}{(m-1)!(n-1)!}\int_\Delta \log|z_1|^2 \frac{\partial^{m+n}\rho_\iota\psi}{\partial z_1^m \partial \bar{z}_1^n}\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= -2\pi\mathrm{i}\sum_\iota \frac{1}{(m-1)!(n-1)!}\int_{\Delta\cap D} \frac{\partial^{m+n-2}\rho_\iota\psi}{\partial z_1^{m-1}\partial\bar{z}_1^{n-1}}\mathrm{d}z' \wedge \mathrm{d}\bar{z}' \\ &= -2\pi\mathrm{i}\int_D \mathrm{Res}_A(\omega \wedge \xi).\end{aligned}$$

□

We can define the Aeppli residue for (d,d)-forms which have poles along a hypersurface with normal crossings as follows. Suppose $D = D_1 \cup \cdots \cup D_k$ for smooth hypersurfaces $D_1, \ldots, D_k$ and that $\omega \in H_A^{d,d}(*\bar{*}D)$. Considering ω on $X \setminus D$ we may define its residue with respect to the hypersurface $D_1 \setminus \big(D_2 \cup \cdots \cup D_k\big)$ and we denote it $\mathrm{Res}_A^{D_1}(\omega)$. We should note here that, even though $X \setminus D$ is not compact, we can define the residue since the orders of the poles of ω are bounded, cf. the remark after Lemma 3.1.

The residue $\mathrm{Res}_A^{D_1}(\omega)$ is represented by a form which has poles along the hypersurfaces $D_1 \cap D_i$ and so in particular $\mathrm{Res}_A^{D_1}(\omega) \in H_A^{d-1,d-1}(*\bar{*}D_{\mathrm{sing}})$. We can make the same construction for every D_i and then let

$$\mathrm{Res}_A^D(\omega) = \mathrm{Res}_A^{D_1}(\omega) + \cdots + \mathrm{Res}_A^{D_k}(\omega).$$

By iterating this construction for the hypersurfaces $D_i \cap D_j$ in D and so on we may define the Aeppli residues for all normal crossings. In particular, writing $E = D_1 \cap \cdots \cap D_k$, we get a residue $\mathrm{Res}_A^E(\omega)$ which is now represented by a smooth form. We also set $\mathrm{Res}_A^X(\omega) = \omega$.

We get the following generalisation of Theorem 3.8.

Theorem 3.9. *For $\omega \in \mathcal{E}(**\bar{*}D)$ such that D has normal crossings we have*

$$\big\langle \{\omega\}, \xi \big\rangle_X = (-2\pi\mathrm{i})^{\kappa(\omega)} \Big\langle \big\{\mathrm{Res}_A^{E(\omega)}(\omega \wedge \xi)\big\}, 1\Big\rangle_{E(\omega)}.$$

Remark. In the above theorem we take the canonical current of a cohomology class which is *not* a well defined object. However, its action on 1 is.

Proof. Take a partition of unity with the same properties as in the proof of Theorem 3.8, but now the hypersurface will be given by $z^I = 0$. Suppose $E(\omega)$ is given by $z_1 = \cdots = z_\ell = 0$. Then we let $\mathrm{d}z' = \mathrm{d}z_{\ell+1} \wedge \cdots \wedge \mathrm{d}z_d$. Let R the multi-index which is 1 in the ℓ first positions and otherwise 0. If we write $p = 2\kappa(\omega) + p'$ then

$$p' = \#\{j : J_j = 0, K_j \neq 0\} + \#\{K_j \neq 0, J_j = 0\}.$$

Now, similar to the proof of Theorem 3.8, we get

$$\begin{aligned}
&\big\langle \{\omega\}, \xi \big\rangle \\
&= \sum_\iota \frac{(-1)^p}{(J-1_J)!(K-1_K)!} \int_\Delta \Big(\prod_{j: J_j + K_j \neq 0} \log|z_j|^2 \Big) \frac{\partial^{J+K} \rho_\iota \psi}{\partial z^J \partial \bar{z}^K} \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= (-2\pi\mathrm{i})^{\kappa(\omega)} \sum_\iota \frac{(-1)^{2\kappa(\omega)+p'}}{(J-1_J)!(K-1_K)!} \int_{\Delta \cap E(\omega)} \Big(\prod_{\substack{j: J_j = 0, K_j \neq 0 \\ \text{or } J_j \neq 0, K_j = 0}} \log|z_j|^2 \Big) \frac{\partial^{J+K-2R} \rho_\iota \psi}{\partial z^{J-R} \partial \bar{z}^{K-R}} \mathrm{d}z' \wedge \mathrm{d}\bar{z}' \\
&= (-2\pi\mathrm{i})^{\kappa(\omega)} \sum_\iota (-1)^{p'} \int_{\Delta \cap E(\omega)} \Big(\prod_{\substack{j: J_j = 0, K_j \neq 0 \\ \text{or } J_j \neq 0, K_j = 0}} \log|z_j|^2 \Big) \mathrm{Res}_A^{E(\omega)}(\omega \wedge \xi \rho_\iota) \mathrm{d}z' \wedge \mathrm{d}\bar{z}' \\
&= (-2\pi\mathrm{i})^{\kappa(\omega)} \big\langle \{\mathrm{Res}_A^{E(\omega)}(\omega \wedge \xi)\}, 1 \big\rangle_{E(\omega)}.
\end{aligned}$$

□

The right hand side of Theorem 3.9 is a bit messy but with one extra assumption we get a cleaner statement.

Corollary 3.10. *For $\omega \in \mathcal{E}(**\bar{*}D)$ such that D has normal crossings and $P^{1,0}(\omega) = P^{0,1}(\omega)$ we have*

$$\big\langle \{\omega\}, \xi \big\rangle = (-2\pi\mathrm{i})^{\kappa(\omega)} \int_{E(\omega)} \mathrm{Res}_A^{E(\omega)}(\omega \wedge \xi).$$

Proof. Under these assumptions $\mathrm{Res}_A^{E(\omega)}(\omega \wedge \xi)$ is smooth on $E(\omega)$ so the statement follows from Theorem 3.9. □

4. Analytic continuation of divergent integrals

We will use the results in the previous sections to describe asymptotic expansions coming from analytic continuations of divergent integrals. In this section we drop the point of view of currents of quasi-meromorphic forms. Instead we suppose we have two semi-meromorphic forms α and β, on a compact complex manifold X, which have poles along the same hypersurface D. As before we assume D to have normal crossings. We write

$$D_d \subset \cdots \subset D_1 \subset D_0$$

for the natural stratification of D, cf. (2) in Section 2. Recall that $D_0 = X$ and $D_1 = D$. Regularising the integral

$$\int_X \alpha \wedge \bar{\beta}$$

we use Theorem 2.3 to get the asymptotic expansion

$$\int_X |s|^{2\lambda}\alpha\wedge\bar{\beta} = \lambda^{-\kappa}C_{-\kappa} + \cdots + \lambda^{-1}C_{-1} + C_0 + \mathcal{O}\big(|\lambda|\big)$$

where $\kappa = \kappa(\alpha\wedge\bar{\beta})$. Interpreting Corollary 3.10 in this setting we get

$$C_{-\kappa} = \frac{(-2\pi \mathrm{i})^{\kappa}}{o(s)}\int_{D_\kappa}\mathrm{Res}_A\big(\alpha\wedge\bar{\beta}\big)$$

where $o(s) = o_{\alpha\wedge\bar{\beta}}(s)$. We will now make some calculations of the other coefficients and we will in particular see how they depend on the metric. The coefficients also depend on the choice of section but as long as we do not change the line bundle this can be seen as a change of metric. The result is the following theorem.

Theorem 4.1. *For the coefficients C_{-r} in the asymptotic expansion*

$$\int_X |s|^{2\lambda}\alpha\wedge\bar{\beta} = C_{-\kappa}\lambda^{-\kappa} + \cdots + C_{-1}\lambda^{-1} + C_0 + \mathcal{O}\big(|\lambda|\big)$$

we have

- *(a) C_{-r} depends polynomially of degree $\kappa - r$ on the metric. More precisely, if ϕ is the difference of two metrics then there are differential operators $Q_{r,j}$ with integrable coefficients such that*

$$C_{-r}(\phi) = \sum_{j=0}^{\kappa-r}\int_X Q_{r,j}(\phi^j).$$

- *(b) The term $\int_X Q_{r,\kappa-r}(\phi^{\kappa-r})$ may be written*

$$\frac{(-2\pi\mathrm{i})^{\kappa}(-2)^{\kappa-r}}{o(s)(\kappa-r)!}\int_{D_\kappa}\mathrm{Res}_A\big(\phi^{\kappa-r}\alpha\wedge\bar{\beta}\big),$$

- *(c) C_{-r} may be written as an integral over D_r, i.e. the codimension r components in the stratification of D.*

Proof. Similarly as in Section 2.2 we let

$$F(\lambda) = o(s)\int_X |s|^{2\lambda}\alpha\wedge\bar{\beta}$$

and from the proof of Theorem 2.3 we get

$$F(\lambda) = \frac{(-1)^{|J|+|K|}o(s)}{\lambda^p}h(\lambda)g(\lambda)$$

where

$$g(\lambda) = \sum_\iota\int_\Delta |z^I|^{2\lambda}\frac{\partial^{J+K}}{\partial z^J\partial\bar{z}^K}\big(\mathrm{e}^{-2\lambda\phi}\psi_\iota\big)\mathrm{d}z\wedge\mathrm{d}\bar{z},$$

ψ_ι is given by $\big(\psi_\iota/(z^J\bar{z}^K)\big)\mathrm{d}z\wedge\mathrm{d}\bar{z} = \rho_\iota\alpha\wedge\bar{\beta}$ and h and p is given by Lemma 2.4. We may choose J and K independent of ι. From now on we will suppress ι and ρ_ι. Since we have assumed that α and β have poles along the same hypersurface $p = 2\kappa$. From the proof of Theorem 2.3 we know that $g^{(k)}(0) = 0$ for $k = 0, \ldots, p-\kappa-1$. Taylor expanding hg we get, for $r = 0, 1\ldots, \kappa$,

$$C_{-r} = \frac{(-1)^{|J|+|K|}}{(p-r)!}\sum_{k=p-\kappa}^{p-r}\binom{p-r}{k}h^{(p-r-k)}(0)g^{(k)}(0).$$

Lemma 2.4 implies that the derivatives of h are combinatorial expressions involving J and K. From the proof of Theorem 2.3 we also get

$$g^{(k)}(0)=\sum_{\ell=\kappa}^{k}\binom{k}{\ell}(-2)^{k-\ell}\sum_{M}\binom{\ell}{M}\int_\Delta\prod_{j=1}^{d}\big(I_j\log|z_j|^2\big)^{M_j}\frac{\partial^{J+K}}{\partial z^J\partial\bar z^K}\big(\psi\phi^{k-\ell}\big)\mathrm{d}z\wedge\mathrm{d}\bar z$$

and hence we have proven the first part of (a), that $C_{-r}=\int_X\sum Q_{r,j}(\phi^j)$ for some differential operators $Q_{r,j}$. We further see that the highest power of ϕ is obtained when k is as large as possible and ℓ is as small as possible. Thus setting $k=p-r$, $\ell=\kappa$ and collecting the constants we get that the leading term is given by

$$\begin{aligned}&\frac{(-1)^{|J|+|K|}(-2)^{\kappa-r}}{(\kappa-r)!}h(0)\int_\Delta\prod_{j=1}^{d}\big(I_j\log|z_j|^2\big)^{M_j}\frac{\partial^{J+K}}{\partial z^J\partial\bar z^K}\big(\psi\phi^{\kappa-r}\big)\mathrm{d}z\wedge\mathrm{d}\bar z\\&=\frac{(-2\pi\mathrm{i})^{\kappa}(-2)^{\kappa-r}}{o(s)(\kappa-r)!}\int_{D_\kappa}\mathrm{Res}_A(\phi^{\kappa-r}\alpha\wedge\bar\beta)\end{aligned}$$

if we do a similar calculation as in the proof of Proposition 2.6. This proves the rest of (a) and (b).

To prove (c) we may suppose that $I_1,\dots,I_\kappa\neq0$ and $I_{\kappa+1},\dots,I_d=0$. We must show that we can reduce all the integrals in all the derivatives of g to an integral over D_r. Let us look at $g^{(k)}$ for $k=\kappa,\dots,p-r$. In the expression for the derivative we have a multi-index M such that $\sum_j M_j=\ell$, where $\ell\leqslant k$. We have seen that when $M_i=1$, so that we have $\log|z_i|^2$ in the integral, we may reduce it to an integral over $\Delta\cap\{z_i=0\}$.

First let $M_1=\dots=M_\kappa=1$. But then we need to add $\ell-\kappa$ to these indices, i.e. at most we need to add $p-r-\kappa=\kappa-r$. But if we add 1 to $\kappa-r$ different M_j there are still r number of M_j which are equal to one. And in these variables we may reduce the integrals r times, hence to codimension r. Adding more than one to some M_j only makes it better. □

Theorem 4.1 points out why we call the currents defined from quasi-meromorphic forms canonical; the currents come from the only coefficient in the asymptotic expansion which is independent of the metric. In the special case that D is a smooth hypersurface we get the following corollary.

Corollary 4.2. *If D is a smooth hypersurface then*

$$\int_X|s|^{2\lambda}\alpha\wedge\bar\beta=\lambda^{-1}C_{-1}+C_0+\mathcal{O}(|\lambda|)$$

with $C_{-1}=-\frac{2\pi\mathrm{i}}{o(s)}\int_D\mathrm{Res}_A(\alpha\wedge\bar\beta)$ *and*

$$C_0(\phi)=\frac{4\pi\mathrm{i}}{o(s)}\int_D\mathrm{Res}_A\big(\phi\alpha\wedge\bar\beta\big).$$

References

[**And**] M. Andersson. "Residue currents and ideals of holomorphic functions". In: *Bull. Sci. math.* 128 (2004).

[**Ang1**] D. Angella and A. Tomassini. "Inequalities à la Frölicher and cohomological decompositions". In: *J. Noncommut. Geom.* 9.2 (2015).

[**Ang2**] D. Angella and A. Tomassini. "On the $\partial\bar\partial$-lemma and Bott–Chern cohomology". In: *Invent. Math.* 192.1 (2013), pp. 71–81.

[**Bar**] D. Barlet. “Développement asymptotique des fonctions obtenues par intégration sur les fibres”. In: *Invent. math.* 68 (1982).

[**Bar2**] D. Barlet and H.-M. Maire. “Asymptotic Expansion of Complex Integrals via Mellin Transform”. In: *Journal of Functional Analysis* 83 (1989).

[**Ber**] C. Berenstein, R. Gay, and A. Yger. “Analytic continuation of currents and division problems”. In: *Forum Math.* 1 (1989), pp. 15–51.

[**Ch**] E. M. Chirka et al. *Introduction to Complex Analysis*. Springer, 1985.

[**Del**] P. Deligne et al. “Real homotopy theory of Kähler manifolds”. In: *Invent. Math.* 29.3 (1975), pp. 245–274.

[**Dem**] J.-P. Demailly. *Complex Analytic and Differential Geometry*. Available at http://www-fourier.ujf-grenoble.fr/∼demailly/manuscripts/agbook.pdf.

[**Fe**] G. Felder and D. Kazhdan. “Divergent integrals, Residues of Dolbeault Forms and asymptotic Riemann Mappings”. In: *International mathematics research notices* 08 (2016).

[**Fe2**] G. Felder and D. Kazhdan. “Regularization of divergent integrals”. In: *Selecta Mathematica* 24.1 (2018).

[**Sa**] H. Samuelsson. “Analytic continuation of residue currents”. In: *Ark. Mat.* 47 (2009).

Paper II

II

Mats Andersson, Richard Lärkäng, Mattias Lennartsson and Håkan Samuelsson Kalm, *"The $\bar{\partial}$-equation for (p, q)-forms on a non-reduced analytic space"*, preprint.

THE $\bar\partial$-EQUATION FOR (p,q)-FORMS ON A NON-REDUCED ANALYTIC SPACE

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Abstract. On any pure-dimensional, possibly non-reduced analytic space X we introduce sheaves $\mathscr{A}_X^{p,q}$ and show that the corresponding Dolbeault complex is exact, i.e., that the $\bar\partial$-equation is locally solvable in $\mathscr{A}_X$. The sheaves $\mathscr{A}_X^{p,q}$ are extensions of, and modules over the sheaves $\mathscr{E}_X^{p,q}$ of smooth (p,q)-forms, which are introduced as well.

We also introduce sheaves $\mathscr{B}_X^{n-p,n-q}$ of certain currents on X. These are dual to $\mathscr{A}_X^{p,q}$ in the sense of Serre duality. More precisely, we show that the compactly supported Dolbeault cohomology of $\mathscr{B}^{n-p,n-q}(X)$ in a natural way is the dual of the Dolbeault cohomology of $\mathscr{A}^{p,q}(X)$.

1. Introduction

It is natural to try to find concrete realizations of abstract objects like sheaf cohomology groups and their duals. On a smooth complex manifold X of dimension n the Ω_X^p-cohomology can be represented by Dolbeault cohomology. In fact, Dolbeault–Grothendieck's lemma states that the Dolbeault complex,

$$0 \to \Omega_X^p \to \mathscr{E}_X^{p,0} \xrightarrow{\bar\partial} \mathscr{E}_X^{p,1} \to \cdots, \tag{1.1}$$

is a fine resolution of Ω_X^p, and by standard arguments it follows that

$$H^{p,q}(X) := H^q(X, \Omega_X^p) \simeq H^q(\mathscr{E}^{p,\bullet}(X), \bar\partial). \tag{1.2}$$

If X is compact, then the duals of these groups are represented by $H^{n-p,n-q}(X)$ via the non-degenerate pairing

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}, \quad ([\phi],[\psi]) \to \int_X \phi \wedge \psi, \tag{1.3}$$

where ϕ and ψ are $\bar\partial$-closed (p,q) and $(n-p,n-q)$-forms, respectively. There are analogues of this so-called Serre duality even when X is not compact.

If X is a non-smooth reduced analytic space, then the complex (1.1) has a meaning but it is not exact in general except at $q=0$. Thus the direct analogue of (1.2) does not hold. However, there are fine sheaves $\mathscr{A}_X^{p,q}$ of (p,q)-currents, introduced in [7] for $p=0$ and in [24] for $p \geq 0$, that coincide with $\mathscr{E}_X^{p,q}$ on X_{reg}, such that

$$0 \to \Omega_X^p \to \mathscr{A}_X^{p,0} \xrightarrow{\bar\partial} \mathscr{A}_X^{p,1} \to \cdots \tag{1.4}$$

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are fine resolutions of[1] Ω^p_X. This leads to the representation

$$H^{p,q}(X) = H^q(X, \Omega^p_X) \simeq H^q(\mathscr{A}^{p,\bullet}(X), \bar\partial). \tag{1.5}$$

In the non-smooth case however the duality is more involved. Let ω^p_X be the sheaves of meromorphic $(p,0)$-forms which are $\bar\partial$-closed considered as currents on X. They were first introduced by Barlet in [13] in a slightly different way; see also [17]. In [23, 24] were introduced fine sheaves $\mathscr{B}^{p,q}_X$ of (p,q)-currents, that are smooth on X_{reg}, with the following properties: For each p we have a complex

$$0 \to \omega^p_X \to \mathscr{B}^{p,0}_X \xrightarrow{\bar\partial} \mathscr{B}^{p,1}_X \to \cdots \tag{1.6}$$

such that, given that X is compact, $H^{n-q}(\mathscr{B}^{n-p,\bullet}(X), \bar\partial)$ is the dual of $H^{p,q}(X)$, realized via the non-degenerate pairing

$$H^{p,q}(X) \times H^{n-q}(\mathscr{B}^{n-p,\bullet}(X), \bar\partial) \to \mathbb{C}, \quad ([\phi], [\psi]) \to \int_X \phi \wedge \psi, \tag{1.7}$$

where ϕ and ψ are $\bar\partial$-closed currents in $\mathscr{A}^{p,q}(X)$ and $\mathscr{B}^{n-p,n-q}(X)$, respectively. The complex (1.6) is exact at $q = 0$ but it is a resolution of ω^p_X if and only if Ω^p_X is Cohen–Macaulay.

The aim of this paper is to extend these results to the case when X is a non-reduced analytic space of pure dimension n. Already in [6] were defined a resolution of the structure sheaf $\mathscr{O}_X$, that is, (1.4) for $p = 0$, and as a consequence a representation (1.5) for $p = 0$. We thus have to extend this representation to $p \geq 0$ and find analogues of (1.6) and (1.7).

Let us describe various forms and currents on our non-reduced X. First recall that locally we have an embedding $i\colon X \to D \subset \mathbb{C}^N$ and a surjective sheaf mapping $i^*\colon \mathscr{O}^p_D \to \mathscr{O}^p_X$. This means more concretely that we have an ideal sheaf $\mathcal{J}_X \subset \mathscr{O}_D$ with zero set X_{red} such that i^* is the natural mapping $\mathscr{O}_D \to \mathscr{O}_D/\mathcal{J}_X \simeq \mathscr{O}_X$. There are similar surjective mappings $i^*\colon \Omega^p_D \to \Omega^p_X$ for $p \geq 1$. Moreover, we have the $\mathscr{O}_X$-sheaves $\mathscr{E}^{p,*}_X$ of smooth $(p,*)$-forms and natural surjective mappings $i^*\colon \mathscr{E}^{p,*}_D \to \mathscr{E}^{p,*}_X$. It turns out that i^* is a ring homomorphism as usual so that we natural products

$$\mathscr{E}^{p,q}_X \times \mathscr{E}^{p',q'}_X \to \mathscr{E}^{p+p',q+q'}_X, \quad (\phi, \psi) \mapsto \phi \wedge \psi. \tag{1.8}$$

We define the sheaf $\mathscr{C}^{p,q}_X$ of (p,q)-currents on X as the dual of the space of compactly supported sections of $\mathscr{E}^{n-p,n-q}_X$. Given the embedding $i\colon X \to D \subset \mathbb{C}^N$ we have natural injective mappings $i_*\colon \mathscr{C}^{p,q}_X \to \mathscr{C}^{N-n+p,N-n+q}_D$ so that the elements in $\mathscr{C}^{p,q}_X$ are identified with the ordinary $(N-n+p, N-n+q)$-currents in D that vanish on $\mathscr{K}\!er\, i^*$. In view of (1.8) we have natural products

$$\mathscr{E}^{p,q}_X \times \mathscr{C}^{p',q'}_X \to \mathscr{C}^{p+p',q+q'}_X, \quad (\phi, u) \mapsto \phi \wedge u. \tag{1.9}$$

We are mainly interested in subsheaves $\mathcal{W}^{p,q}_X$ of $\mathscr{C}^{p,q}_X$ where the elements have a certain regularity property; (1.9) holds also with $\mathscr{C}_X$ replaced by $\mathcal{W}_X$. The subsheaf of $\bar\partial$-closed members of $\mathcal{W}^{p,0}_X$ are denoted by ω^p_X; they are natural extensions to our non-reduced space X of the Barlet sheaves.

We are also interested in another class of non-smooth forms $\mathcal{V}^{p,q}_X$, which however are fundamentally different from $\mathscr{C}^{p,q}_X$. The sheaves $\mathcal{V}^{p,q}_X$ are extensions of $\mathscr{E}^{p,q}_X$ and contain for instance principal values of meromorphic forms. Generically on X elements in $\mathcal{V}^{p,q}_X$ are weak limits of elements in $\mathscr{E}^{p,q}_X$.

[1]In this paper Ω^p_X denotes the sheaf of Kähler differential p-forms modulo torion.

Remark 1.1. Notice that when X is reduced we have the inclusion $\Omega^p_X \subset \omega^p_X$ with equality if X is smooth. In the non-reduced case there is no such relation at all since the elements in ω_X, although holomorphic, are dual objects whereas elements in Ω^p_X have no natural interpretation as dual objects. However, if ϕ is in Ω^p_X and μ in $\omega^{p'}_X$, then $\phi \wedge \mu$ is in $\omega^{p+p'}_X$.

Here is our first main theorem.

Theorem A. *Let X be a non-reduced analytic space of pure dimension n. For each $p \geq 0$ there are fine*[2] *subsheaves $\mathscr{A}^{p,q}_X$ of $\mathcal{V}^{p,q}_X$ that coincide with $\mathscr{E}^{p,q}_X$ generically on X, such that* (1.4) *is a resolution of Ω^p_X.*

As an immediate corollary we get the representation (1.5) of sheaf cohomology.

For our second main theorem we must introduce an intrinsic notion of integration over X. If we have a current u on X of bidegree (n,n) with compact support, then there is a well-defined integral

$$\int_X u.$$

Given a local embedding as before and assuming that u has support in $D \cap X$ it is defined as the integral of $i_* u$ over D.

Theorem B. *Let X be a non-reduced analytic space of pure dimension n. Moreover assume that X is compact. There are fine subsheaves $\mathscr{B}^{p,q}$ of $\mathcal{W}^{p,q}_X$ such that*

(i) (1.6) *is a complex,*

(ii) (1.6) *is exact if and only if Ω^p_X is Cohen–Macaulay,*

(iii) the products $\phi \wedge \mu$ for ϕ in $\mathscr{A}^{p,*}_X$ and μ in $\mathscr{B}^{n-p,*}_X$ are well-defined in $\mathcal{W}^{n,*}_X$,

(iv) the pairing (1.7) *is well-defined and non-degenerate so that $H^{n-q}(\mathscr{B}^{n-p,\bullet}(X), \bar{\partial})$ is the dual of $H^{p,q}(X)$.*

There are variants of Theorem B even when X is not compact, see Section 7.

The construction of the new sheaves on X relies on the ideas in the previous papers [7, 24, 6]. The proofs of Theorems A and B relies on explicit Koppelman formulas for the $\bar{\partial}$-equation. The main novelty in this paper is the adaption of the ideas in [6] to the framework in [24]. We also believe that the non-reduced point of view sheds new light on Serre duality, even in the reduced case, cf. Remark 1.1. Finally, we think that the notions and result of this paper may serve as a basis for doing analysis on non-reduced spaces.

The paper is organized as follows. The main objects are introduced in Sections 3 and 4 and their basic properties are proved. In the rather technical Section 5 the integral operators used in the Koppelman formulas are defined and their basic mapping properties are shown. The sheaves $\mathscr{A}^{p,*}_X$ and $\mathscr{B}^{n-p,*}_X$ are introduced in Section 6 and Theorem A as well as Koppelman formulas are proved. In Section 7 we show Theorem B and in Section 8 some further examples are given.

[2] As in the reduced case, a sheaf is "fine" if it is closed under multiplication by smooth forms.

2. Preliminaries

Throughout this paper, unless otherwise said, $\mathcal{J}$ is a coherent pure n-dimensional ideal sheaf in a domain $D \subset \mathbb{C}^N$, Z is the zero set of $\mathcal{J}$, $i\colon X \hookrightarrow D$ is the (possibly) non-reduced analytic subspace with structure sheaf $\mathscr{O}_D/\mathcal{J}$, and $\kappa = N - n$.

Let $\iota\colon Z \to D$ be the inclusion. The sheaf of smooth (p,q)-forms on Z is $\mathscr{E}_Z^{p,q} := \mathscr{E}_D^{p,q}/\mathscr{K}er\,\iota^*$. It is well-known that this is an intrinsic notion, i.e., does not depend on the embedding $Z \to D$. The space of $(n-p, n-q)$-currents on Z is defined as the dual of $\mathscr{E}_Z^{p,q}$. More concretely, (p,q)-currents on Z can, via ι_*, be identified with $(\kappa+p, \kappa+q)$-currents μ in D such that $\mathcal{J}_Z\mu = d\mathcal{J}_Z\mu = \bar{\mathcal{J}}_Z\mu = d\bar{\mathcal{J}}_Z\mu = 0$. If $\pi\colon Z' \to Z$ is proper, μ a current on Z', and ψ is smooth on Z, then

$$\pi_*(\pi^*\psi \wedge \mu) = \psi \wedge \pi_*\mu. \tag{2.1}$$

In [9], see also [7], was introduced the sheaf $\mathcal{PM}_Z$ of *pseudomeromorphic currents*. A current τ in $U \subset \mathbb{C}^N$ is an elementary pseudomeromorphic current if $\tau = \varphi \wedge \tau'$, where φ is smooth with compact support in U and τ' is the tensor product of one-variable currents $1/z_k^{m_k}$ and $\bar\partial(1/z_\ell^{m_\ell})$. If Z is smooth, then, [10, Theorem 2.15], a current on Z is pseudomeromorphic if and only if it is a locally finite sum of currents of the form $f_*\tau$, where $f\colon U \to Z$ is holomorphic, $U \subset \mathbb{C}^N$, and τ is elementary. If Z has singularities the definition is slightly more involved. Pseudomeromorphic currents are closed under $\bar\partial$ and direct images of modifications, simple projections, and open inclusions.

Example 2.1. Recall that a current on Z is semi-meromorphic if it is of the form φ/s, where s is a generically non-vanishing section of some line bundle L and φ is a smooth form with values in L. If $|\cdot|$ is any Hermitian metric on L, then $\chi(|s|^2/\epsilon)\varphi/s \to \varphi/s$ as currents, where χ is a smooth approximation of the characteristic function of $[1,\infty) \subset \mathbb{R}$. Semi-meromorphic currents, and $\bar\partial$ of such, are sections of $\mathcal{PM}$.

We refer to [10] for properties of pseudomeromorphic currents. If $V = \{h = 0\}$ for some holomorphic tuple h in D and $\mu \in \mathcal{PM}(D)$, then

$$\mathbf{1}_{D\setminus V}\mu := \lim_{\epsilon\to 0}\chi(|h|^2/\epsilon)\mu. \tag{2.2}$$

The limit (2.2) exists, is in $\mathcal{PM}_D$, and is independent of such h and χ. Set

$$\mathbf{1}_V\mu := \mu - \mathbf{1}_{D\setminus V}\mu.$$

If $\pi\colon \widetilde{D} \to D$ is a modification or a simple projection and $\tau \in \mathcal{PM}(\widetilde{D})$ has compact support in the fiber direction, then

$$\mathbf{1}_V\pi_*\tau = \pi_*(\mathbf{1}_{\pi^{-1}V}\tau). \tag{2.3}$$

If $\mu \in \mathcal{PM}_D$ has support in Z then

$$\bar{\mathcal{J}}_Z\mu = d\bar{\mathcal{J}}_Z \wedge \mu = 0. \tag{2.4}$$

Dimension principle. *If $\mu \in \mathcal{PM}_Z$ has bidegree $(*,q)$ and support in a subvariety $V \subset Z$ such that $\operatorname{codim}_Z V > q$, then $\mu = 0$.*

A current $\mu \in \mathcal{PM}_D$ with support in Z has the *standard extension property* (SEP) with respect to Z if $\mathbf{1}_V\mu = 0$ for all germs of analytic sets V in D intersecting Z properly. The subsheaf of $\mathcal{PM}_D$ of $(N,*)$-currents with support in Z and the SEP with respect to Z is denoted $\mathcal{W}_D^{Z,*}$. The subsheaf of $\mathcal{PM}_Z$ of pseudomeromorphic currents on Z with the SEP with respect to Z is denoted $\mathcal{W}_Z$.

Remark 2.2. We will frequently consider $\mathscr{H}om$-sheaves, for instance like $\mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D)$. For future reference we notice that such sheaves in a natural way can be identified with sheaves of currents of bidegree $(N-p,*)$. For instance,

$$\mathcal{W}^{Z,(N-p,*)}_D \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D), \quad \mu \mapsto (\varphi \mapsto \varphi\wedge\mu),$$

where we temporarily let $\mathcal{W}^{Z,(N-p,*)}_D$ denote the sheaf of pseudomeromorphic $(N-p,*)$-currents in D with support on Z and the SEP with respect to Z. It is clear that if $\varphi\wedge\mu = 0$ for all $\varphi\in\Omega^p_D$ then $\mu = 0$. Hence, the mapping is injective. To see that it is surjective, let $\{dz_I\}$ be a basis of Ω^p_D and let $\{\partial/\partial z_I\}$ be the dual basis. If $u\in\mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D)$ then $u(dz_I)\in\mathcal{W}^{Z,*}_D$ and so, by [10, Theorem 3.7], there are $u_I\in\mathcal{W}^{Z,(0,*)}_D$ such that $u(dz_I) = dz\wedge u_I$, where $dz = dz_1\wedge\cdots\wedge dz_N$. Define $\mu_I\in\mathcal{W}^{Z,(N-p,*)}_D$ by $\mu_I = \pm(\partial/\partial z_I \lrcorner dz)\wedge u_I$, where $\pm$ is chosen so that $dz_I\wedge\mu_I = dz\wedge u_I$. Setting $\mu = \sum_I \mu_I$ it is straightforward to check that $\varphi\wedge\mu = u(\varphi)$ for all $\varphi\in\Omega^p_D$ since $\{dz_I\}$ is a basis of Ω^p_D.

In this paper we will use the $\mathscr{H}om$-notation but, keeping the identification in mind, we will for a $\mathscr{H}om$-element μ write $\varphi\wedge\mu$ (or possibly $\mu\wedge\varphi$) instead of $\mu(\varphi)$.

Suppose there are local coordinates (z,w) centered at some $z\in Z$ such that $Z=\{w=0\}$, i.e., Z is smooth in a neighborhood of z. Then, if $\mu\in\mathcal{W}^{Z,*}_D$, we have $\pi_*(w^\alpha\mu)\in\mathcal{W}^{n,*}_Z$, where $\pi(z,w)=z$ and $w^\alpha = w_1^{\alpha_1}w_2^{\alpha_2}\cdots$. Moreover, there is a unique representation

$$\mu = \frac{1}{(2\pi i)^\kappa}\sum_\alpha \pi_*(w^\alpha\mu)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}, \tag{2.5}$$

where the products are tensor products and $\bar\partial(dw/w^{\alpha+\mathbf{1}})$ is shorthand for $\bar\partial(dw_1/w_1^{\alpha_1+1})\wedge\bar\partial(dw_2/w_2^{\alpha_2+1})\wedge\cdots$. By [11, Proposition 3.12, Theorem 3.14] we have

Proposition 2.3. *If $u,\mu_1,\dots,\mu_\ell\in\mathcal{W}^{n,*}_Z$ and $u=0$ on the set where μ_j is smooth, then $u=0$.*

The sheaf $\mathscr{CH}^Z_D$ of *Coleff-Herrera currents* with support on Z was introduced by Björk, see [14]. An (N,κ)-current μ in D is in $\mathscr{CH}^Z_D$ if $\bar\partial\mu=0$, $\bar h\mu=0$ for any $h\in\mathcal{J}_Z$, and μ has the SEP with respect to Z. Alternatively, by [2], we have

$$\mathscr{CH}^Z_D = \{\mu\in\mathcal{W}^{Z,\kappa}_D;\ \bar\partial\mu=0\}. \tag{2.6}$$

Notice that $\mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D,\mathscr{CH}^Z_D)$ can be identified with Coleff–Herrera currents of bidegree $(N-p,\kappa)$ in view of Remark 2.2. Assume that there are local coordinates (z,w) such that $Z=\{w=0\}$ and set $\pi(z,w)=z$. Given $\mu\in\mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D,\mathscr{CH}^Z_D)$ there are unique $\mu_\alpha\in\Omega^{n-p}_Z$, $\mu_\alpha=0$ if $|\alpha|\gg 0$, such that

$$\mu = \sum_\alpha \mu_\alpha(z)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}. \tag{2.7}$$

The sheaf ω^{n-p}_Z was introduced by Barlet in [13] as the kernel of a certain map $j_*j^*\Omega^{n-p}_Z \to \mathscr{H}^1_{Z_{sing}}(\mathscr{E}xt^\kappa_{\mathscr{O}_D}(\mathscr{O}_Z,\Omega^{N-p}_D))$, where $j\colon Z_{reg}\to Z$ is the inclusion. It follows from [13] that sections of ω^{n-p}_Z are $\bar\partial$-closed meromorphic $(n-p)$-forms on Z, cf. [24, Section 4] and [17]. By [13, Lemma 4] we have

$$\iota_*\omega^{n-p}_Z = \{\mu\in\mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D,\mathscr{CH}^Z_D);\ \mathcal{J}_Z\mu = d\mathcal{J}_Z\wedge\mu = 0\}, \tag{2.8}$$

where $\iota\colon Z \to D$ is the inclusion, cf. [24, Section 4].

A current a on Z is *almost semi-meromorphic* if there are a modification $\pi\colon Z' \to Z$ and a semi-meromorphic current ν on Z' such that $a = \pi_*\nu$. In particular, a is generically smooth. Thus, if $\mu \in \mathcal{PM}_Z$, then $a \wedge \mu$ is generically well-defined. By [10, Theorem 4.8], there is a unique $T \in \mathcal{PM}_Z$ such that $T = a\wedge\mu$ where a is smooth and $\mathbf{1}_V T = 0$, where V is the Zariski closure of the singular support of a. Henceforth we let $a \wedge \mu$ denote the extension T. One can define $a \wedge \mu$ as

$$a \wedge \mu := \lim_{\epsilon\to 0} \chi(|h|^2/\epsilon) a \wedge \mu, \tag{2.9}$$

where h is a holomorphic tuple cutting out V. If $\mu \in \mathcal{W}_Z$, then $a \wedge \mu \in \mathcal{W}_Z$.

Let $E_j \to D$, $j = 0, \dots, N$, be complex vector bundles. Let $f_j\colon E_j \to E_{j-1}$ be holomorphic morphisms and suppose that we have a complex

$$0 \to E_N \xrightarrow{f_N} \cdots \xrightarrow{f_1} E_0 \to 0,$$

which is exact outside $Z \subset D$. Assume that the associated sheaf complex

$$0 \to \mathscr{O}(E_N) \xrightarrow{f_N} \cdots \xrightarrow{f_1} \mathscr{O}(E_0) \tag{2.10}$$

is exact and set $\mathscr{F} := \mathscr{O}(E_0)/\,\mathscr{Im}\ f_1$ so that (2.10) is a resolution of $\mathscr{F}$. Recall that $\mathscr{F}$ is Cohen–Macaulay if and only if there is a resolution (2.10) with $N = \kappa$. Let $Z_j^{\mathscr{F}} \subset D$ be the set where f_j does not have optimal rank. These sets are independent of the resolution and thus invariants of $\mathscr{F}$. These *singularity subvarieties* reflect the complexity of $\mathscr{F}$. It is well-known that

$$Z_N^{\mathscr{F}} \subset \cdots \subset Z_\kappa^{\mathscr{F}} = Z_{\kappa-1}^{\mathscr{F}} = \cdots = Z_1^{\mathscr{F}} = Z \tag{2.11}$$

and that $\mathrm{codim}_D Z_j^{\mathscr{F}} \geq j$, $j = \kappa, \kappa+1, \dots$. Moreover, [16, Corollary 20.14], $\mathscr{F}$ has pure codimension κ (i.e., no stalk of $\mathscr{F}$ has embedded primes or associated primes of codimension $> \kappa$) if and only if $\mathrm{codim}_D Z_j^{\mathscr{F}} \geq j+1$ for $j \geq \kappa+1$.

Assume that the E_j are equipped with Hermitian metrics and let $\sigma_j\colon E_{j-1} \to E_j$ be the Moore-Penrose inverse of f_j, i.e., the pointwise minimal inverse of f_j. The σ_j are smooth outside Z and are almost semi-meromorphic in D. Following [8], we define currents $U \in \mathcal{W}_D$ and $R \in \mathcal{PM}_D$ with support in Z and values in $\mathrm{End}\, E$, where $E = \oplus_j E_j$. Set $\sigma = \sigma_1 + \sigma_2 + \cdots$ and set $u = \sigma + \sigma\bar\partial\sigma + \sigma(\bar\partial\sigma)^2 + \cdots$ outside Z. Then $fu + uf - \bar\partial f = I_E$, where $f = \oplus_j f_j$. We extend u across Z as

$$U := \lim_{\epsilon\to 0} \chi(|F|^2/\epsilon) u,$$

where F is a (non-trivial) holomorphic tuple vanishing on Z. Alternatively, U can be defined in terms of the calculus of almost semi-meromorphic currents mentioned above. Since $fu + uf - \bar\partial f = I_E$,

$$R := I_E - (fU + Uf - \bar\partial f) = \lim_{\epsilon\to 0} \bar\partial\chi(|F|^2/\epsilon) \wedge u \tag{2.12}$$

has support in Z. It is proved in [8] that

$$R = R_\kappa + R_{\kappa+1} + \cdots,$$

where $R_j \in \mathcal{PM}_D^{0,j}$ has support in Z, takes values in $\mathrm{Hom}(E_0, E_j)$, and

$$fR = \bar\partial R. \tag{2.13}$$

Moreover, if $\varphi \in \mathscr{O}(E_0)$ then $R\varphi = 0$ if and only if $\varphi \in \mathscr{Im}\ f_1$. In particular, R induces an injective map from $\mathscr{F}$ to $(0,*)$-currents with values in E. We are interested in the case when $\mathscr{F}$ has pure codimension κ. It follows from [9] that, in this case, R has the SEP with respect to Z, i.e., $R \in \mathscr{H}\!om_{\mathscr{O}_D}(\Omega^N_D, \mathcal{W}^{Z,*}_D$ (cf. Remark 2.2), and that

$$R_\kappa \varphi = 0 \iff \varphi \in \mathscr{I}\!m\, f_1. \tag{2.14}$$

3. Holomorphic forms, smooth forms, and currents on X

3.1. **Holomorphic forms on X and associated residue currents.** Recall that the structure sheaf of holomorphic functions on X is defined as $\mathscr{O}_X = \mathscr{O}_D/\mathcal{J}$. In a similar way one defines the sheaves of Kähler differentials on X: Let Ω^p_D be the sheaf of holomorphic p-forms in D and set

$$\hat{\mathcal{J}}^p := \mathcal{J} \cdot \Omega^p_D + d\mathcal{J} \wedge \Omega^{p-1}_D, \quad \Omega^p_{X,\text{Kähler}} := \Omega^p_D/\hat{\mathcal{J}}^p.$$

Notice that $\hat{\mathcal{J}}^0 = \mathcal{J}$ so that $\Omega^0_{X,\text{Kähler}} = \mathscr{O}_X$. Notice also that $\Omega^p_{X,\text{Kähler}}$ is an $\mathscr{O}_X$-module. It is well-known that $\Omega^p_{X,\text{Kähler}}$ is intrinsic to X, i.e., that it does not depend on the embedding as a subspace of D. Since $\hat{\mathcal{J}}^0 = \mathcal{J}$ has pure dimension it follows that $\Omega^0_{X,\text{Kähler}} = \mathscr{O}_X$ is torsion-free. In general, $\Omega^p_{X,\text{Kähler}}$ has torsion.

The sheaf of *strongly holomorphic* p-forms on X is

$$\Omega^p_X := \Omega^p_{X,\text{Kähler}}/\text{torsion},$$

where torsion means $\mathscr{O}_X$-torsion. Notice that Ω^p_X is intrinsic and that it is the same considered as an $\mathscr{O}_X$-module or an $\mathscr{O}_D$-module.

Example 3.1. Let X be the subspace of $\mathbb{C}^2$ defined by $\mathcal{J} = \langle zw \rangle$. Then $\mathscr{O}_X = \mathscr{O}_z + \mathscr{O}_w$ and $d\mathcal{J} = \langle zdw + wdz \rangle$. For the 1-forms we have $\Omega^1_{X,\text{Kähler}} = \mathscr{O}_X\{dz, dw\}/\langle zdw + wdz \rangle$. When $w \neq 0$ then $\mathcal{J} = \langle z \rangle$ and therefore $zdw = wdz = 0$. By symmetry this also holds when $z \neq 0$. However, one easily checks that zdw and wdz are not zero as Kähler differentials and therefore they are torsion elements. If we mod these out the result is a torsion-free module which therefore is the strongly holomorphic 1-forms, i.e., $\Omega^1_X = \mathscr{O}_z\{dz\} + \mathscr{O}_w\{dw\}$.

An alternative definition of Ω^p_X is as follows. From a primary decomposition of $\hat{\mathcal{J}}^p$ one obtains coherent sheaves $\mathcal{J}^p$ and $\mathscr{S}^p$ such that $\hat{\mathcal{J}}^p = \mathcal{J}^p \cap \mathscr{S}^p$, $\mathcal{J}^p$ has pure dimension n, and $\mathscr{S}^p$ has dimension $< n$. Hence, $\Omega^p_D/\mathcal{J}^p$ has pure dimension and coincides with $\Omega^p_{X,\text{Kähler}}$ generically on Z. It follows that

$$\Omega^p_X = \Omega^p_D/\mathcal{J}^p.$$

If X is reduced and $j\colon X_{reg} \hookrightarrow D$ is the inclusion, then $\mathcal{J}^p = \{\varphi \in \Omega^p_D;\, j^*\varphi = 0\}$, see, e.g., [24].

Suppose that 0 is a smooth point of Z and choose local coordinates (z, w) for $\mathbb{C}^N$ such that $Z = \{w = 0\}$. Then we can identify $\mathscr{O}_Z$ with holomorphic functions of z. If $g(z)$ is holomorphic we let $\tilde{g}$ be the extension to ambient space given by $\tilde{g}(z, w) = g(z)$. In a neighborhood of 0 we can then define an $\mathscr{O}_Z$-module structure on Ω^p_X by setting $g\varphi := \tilde{g}\varphi$. Clearly this depends on the choice of local coordinates.

Proposition 3.2. *Assume that we have coordinates (z, w) so that $Z = \{w = 0\}$. Then, with the associated $\mathscr{O}_Z$-module structure, Ω^p_X is coherent. Moreover, if $\mathscr{O}_X$ is Cohen–Macaulay, then the following are equivalent*

(i) *Ω^p_X is Cohen–Macaulay as an $\mathscr{O}_X$-module,*
(ii) *Ω^p_X is a locally free $\mathscr{O}_X$-module,*
(iii) *Ω^p_X is Cohen–Macaulay as an $\mathscr{O}_Z$-module,*

(iv) Ω^p_X *is a locally free* $\mathscr{O}_Z$*-module.*

Proof. By the Nullstellensatz there is an $M \in \mathbb{N}$ such that $\mathcal{I} := \langle w^\alpha;\ |\alpha| = M\rangle \subset \mathcal{J}$. Set $\mathcal{I}^p := \mathcal{I}\Omega^p_D + d\mathcal{I} \wedge \Omega^{p-1}_D$ and let

$$A^p := \Omega^p_D/\mathcal{I}^p.$$

Clearly A^p is coherent both as an $\mathscr{O}_D$-module and an $\mathscr{O}_D/\mathcal{I}$-module, and these structures are the same. Moreover, the choice of coordinates makes A^p an $\mathscr{O}_Z$-module and one checks that it in fact is a free $\mathscr{O}_Z$-module. In particular, A^p is a coherent $\mathscr{O}_Z$-module. Since $\mathcal{I} \subset \mathcal{J}$ it follows that $\mathcal{I}^p \subset \hat{\mathcal{J}}^p \subset \mathcal{J}^p$ and so we have a natural surjective map of $\mathscr{O}_Z$-modules

$$A^p \to \Omega^p_X, \quad \varphi + \mathcal{I}^p \mapsto \varphi + \mathcal{J}^p.$$

The kernel $\mathscr{K}$ of this map is $\mathcal{J}^p/\mathcal{I}^p$. Since $\mathcal{J}^p$ is a coherent $\mathscr{O}_D$-module there are finitely many $\varphi_j \in \mathcal{J}^p$ generating $\mathcal{J}^p$ over $\mathscr{O}_D$. By Taylor expanding any $g(z,w) \in \mathscr{O}_D$ in the w-variables to order M we see that $\mathscr{K}$ is generated as an $\mathscr{O}_Z$-module by $w^\alpha\varphi_j + \mathcal{I}^p$ with $|\alpha| < M$. Since $\mathscr{K} \subset A^p$ and A^p is a coherent $\mathscr{O}_Z$-module it follows that $\mathscr{K}$ is coherent. Hence, $\Omega^p_X \simeq A^p/\mathscr{K}$ is a coherent $\mathscr{O}_Z$-module.

Claim 1: $\operatorname{depth}_{\mathscr{O}_X}\Omega^p_X = \operatorname{depth}_{\mathscr{O}_Z}\Omega^p_X$.

Claim 2: $n = \dim_{\mathscr{O}_X}\Omega^p_X = \dim_{\mathscr{O}_Z}\Omega^p_X$.

Recall that, for an R-module M, $\dim_R M := \dim_R(R/\operatorname{ann}_R M)$ and that M is Cohen–Macaulay if $\operatorname{depth}_R M = \dim_R M$.

We postpone the proofs of these claims and show that (i), (ii), (iii), and (iv) are equivalent if $\mathscr{O}_X$ is Cohen–Macaulay. Notice that it is a local (stalk-wise) statement; in what follows we suppress the point indicating stalk. Recall that if R is a Cohen–Macaulay ring and M is an R-module that has a finite free resolution over R, then the Auslander–Buchsbaum formula gives

$$\operatorname{depth}_R M + \operatorname{pd}_R M = \dim_R R,$$

where $\operatorname{pd}_R M$ is the length of a minimal free resolution of M over R, see [16, Theorem 19.9]. Thus, M is free over R if and only if $\operatorname{depth}_R M = \dim_R R$.

We now have that Ω^p_X is free over $\mathscr{O}_X$ if and only if $\operatorname{depth}_{\mathscr{O}_X}\Omega^p_X = \dim_{\mathscr{O}_X}\mathscr{O}_X$. But $\dim_{\mathscr{O}_X}\mathscr{O}_X = n = \dim_{\mathscr{O}_X}\Omega^p_X$, where we use the first equality of Claim 1, so (i) and (ii) are equivalent. In the same way, since $\mathscr{O}_Z$ is Cohen–Macaulay and n-dimensional, (iii) and (iv) are equivalent. Assume (i) so that $\operatorname{depth}_{\mathscr{O}_X}\Omega^p_X = \dim_{\mathscr{O}_X}\Omega^p_X$. Then by Claims 1 and 2 we get

$$\operatorname{depth}_{\mathscr{O}_Z}\Omega^p_X = \operatorname{depth}_{\mathscr{O}_X}\Omega^p_X = \dim_{\mathscr{O}_X}\Omega^p_X = \dim_{\mathscr{O}_Z}\Omega^p_X,$$

and so (iii) follows. In the same way, (iii) implies (i). It remains to prove Claims 1 and 2.

Proof of Claim 1: For notational convenience, set $R = \mathscr{O}_X$, $R' = \mathscr{O}_Z$, and $M = \Omega^p_X$; notice that R is a Noetherian local Cohen–Macaulay ring and that R' is a regular Notherian local ring. Since any function in $\mathcal{J}$ vanishes on Z we have an inclusion $R' \hookrightarrow R$ given by $g(z) \mapsto \tilde{g}(z,w) + \mathcal{J}$, where $\tilde{g}(z,w) = g(z)$; cf. the $\mathscr{O}_Z$-module structure on Ω^p_X. By "Miracle flatness", see, e.g., [16, Corollary 18.17] or [6, Proposition 3.1], R is a free R' module if and only if R is Cohen–Macaulay. Thus, R is a free R' module. By [16, Proposition 18.4] and the comment after Corollary 18.5, for a local ring $(A, \mathfrak{m})$ and an A-module N we have

$$\operatorname{depth}_A N = \min\{i;\ \operatorname{Ext}^i_A(A/\mathfrak{m}, N) \neq 0\}.$$

Notice that $R/\mathfrak{m} = \mathbb{C} = R'/\mathfrak{m}$. Claim 1 thus follows if we show that $\mathrm{Ext}^i_R(\mathbb{C}, M) = \mathrm{Ext}^i_{R'}(\mathbb{C}, M)$. To do this, let

$$0 \to M \to N^0 \to N^1 \to \cdots \tag{3.1}$$

be a resolution of M as an R-module by injective R-modules $N^\bullet$. Then $\mathrm{Ext}^i_R(\mathbb{C}, M) = H^i(\mathrm{Hom}_R(\mathbb{C}, N^\bullet))$.

The complex (3.1) is straightforwardly checked to be exact also considered as a complex of R'-modules. Moreover, by [19, p. 62], since R is a free R'-module, the $N^\bullet$ are injective R'-modules. Hence, $\mathrm{Ext}^i_{R'}(\mathbb{C}, M) = H^i(\mathrm{Hom}_{R'}(\mathbb{C}, N^\bullet))$. However,

$$\mathrm{Hom}_R(\mathbb{C}, N^\bullet) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, N^\bullet) = \mathrm{Hom}_{R'}(\mathbb{C}, N^\bullet)$$

and so $\mathrm{Ext}^i_R(\mathbb{C}, M) = \mathrm{Ext}^i_{R'}(\mathbb{C}, M)$.

Proof of Claim 2: We know from above that

$$\dim_{\mathscr{O}_X} \Omega^p_X = \dim_{\mathscr{O}_{\mathbb{C}^N}} \Omega^p_X = \dim_{\mathscr{O}_{\mathbb{C}^N}} \Omega^p_{X,\text{Kähler}} = n.$$

On the other hand, $\mathrm{ann}_{\mathscr{O}_Z} \Omega^p_X = \{0\}$ because if $g(z)\Omega^p_D \in \mathcal{J}^p$ then $g(z)|_Z = 0$. Hence,

$$\dim_{\mathscr{O}_Z} \Omega^p_X = \dim_{\mathscr{O}_Z}(\mathscr{O}_Z/\{0\}) = n.$$

□

Corollary 3.3. *Assume that there are coordinates (z, w) such that $Z = \{w = 0\}$, that $\mathscr{O}_X$ is Cohen–Macaulay, and that Ω^p_X is Cohen–Macaulay either as an $\mathscr{O}_X$-module or as an $\mathscr{O}_Z$-module. Then, locally there is an $M \in \mathbb{N}$ such that Ω^p_X is generated by*

$$\left\{ w^\alpha dz^\beta \wedge dw^\gamma + \mathcal{J}^p;\ |\alpha| < M, |\beta| + |\gamma| = p \right\} \tag{3.2}$$

over $\mathscr{O}_Z$ and a minimal set of generators is an $\mathscr{O}_Z$-basis.

See Example 8.1 below for a simple illustration of this Corollary.

Proof. Recall the module A^p from the proof of Proposition 3.2 and let $\varphi(z, w) \in \Omega^p_D$. Taylor expanding the coefficients of φ with respect to w to order M shows that A^p is generated as an $\mathscr{O}_Z$-module by (3.2) with $\mathcal{J}^p$ replaced by $\mathcal{I}^p$. Thus, Ω^p_X is generated by (3.2) over $\mathscr{O}_Z$. By a standard argument using Nakayama's lemma, a minimal generating set is a basis, cf., e.g., the proof of [21, Theorem 2.5]. □

Definition 3.4. We let $X_{p\text{-reg}}$ be the subset of Z_{reg} where $\mathscr{O}_X$ is Cohen–Macaulay and $\Omega^p_{X,\text{Kähler}}$ is Cohen–Macaulay.

Remark 3.5. The property of being Cohen–Macaulay is generic on Z so $X_{p\text{-reg}}$ is a dense open subset of Z_{reg}. Notice also that $\Omega^p_{X,\text{Kähler}}$ is torsion-free where it is Cohen–Macaulay. Hence,

$$\Omega^p_{X,\text{Kähler}} = \Omega^p_X \quad \text{on} \quad X_{p\text{-reg}}.$$

In view of Proposition 3.2, thus Ω^p_X and $\Omega^p_{X,\text{Kähler}}$ are locally free $\mathscr{O}_X$-modules and have locally a structure as a free $\mathscr{O}_Z$-module on $X_{p\text{-reg}}$.

Assume that (2.10) is a resolution of Ω^p_X and that $E_0 = T^*_{p,0}D$. If D is pseudoconvex, such resolutions exist since Ω^p_X is coherent, possibly after replacing D by a slightly smaller set. Notice that $\mathscr{O}(E_0) = \Omega^p_D$ and that $\mathscr{Im}\, f_1 = \mathcal{J}^p$. Let, for some choice of Hermitian metrics on E_j, $R = R_\kappa + R_{\kappa+1} + \cdots$ be the associated current. Recall from Section 2 that, since Ω^p_X has pure codimension, R is an injective homomorphism from Ω^p_D to E-valued pseudomeromorphic $(0,*)$-currents in D with support in Z and the SEP with respect to Z. Letting $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_N$ we define

$$\mathcal{R} := R \otimes d\zeta \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, E \otimes \mathcal{W}^{Z,*}_D). \tag{3.3}$$

Notice that in view of Remark 2.2, $\mathcal{R}$ can be identified with an $(N-p,*)$-current, cf. [24, Section 3]. We set $\mathcal{R}_\ell = R_\ell \otimes d\zeta$.

By construction, in view of (2.13), we have the following lemma.

Lemma 3.6. *The current $\mathcal{R} = \mathcal{R}_\kappa + \mathcal{R}_{\kappa+1} + \cdots$ has bidegree $(N-p,*)$, takes values in E, has the SEP with respect to Z, depends only on $d\zeta$ (and R), and*

$$f\mathcal{R} = \bar\partial \mathcal{R}.$$

If $\varphi \in \Omega^p_D$ then $\mathcal{R} \wedge \varphi = R \wedge \varphi \wedge d\zeta$.

3.2. **Smooth forms on X.** To begin with, in view of Remark 2.2 we notice that if $\mathcal{I} \subset \Omega^p_D$ is a submodule such that $\mathcal{J} \cdot \Omega^p_D \subset \mathcal{I}$, then we have the isomorphism

$$\{\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathscr{C}\mathscr{H}^Z_D);\, \mathcal{I} \wedge \mu = 0\} \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_D/\mathcal{I}, \mathscr{C}\mathscr{H}^Z_D), \tag{3.4}$$

$$\mu \mapsto (\varphi \mapsto \varphi \wedge \mu).$$

We remark that if $\mathscr{H}om_{\mathscr{O}_X}$ is replaced by $\mathscr{H}om_{\mathscr{O}_D}$ the assumption $\mathcal{J} \cdot \Omega^p_D \subset \mathcal{I}$ is superfluous.

Since $\mathcal{J}^p = \hat{\mathcal{J}}^p$ generically on Z and any $\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathscr{C}\mathscr{H}^Z_D)$ has the SEP with respect to Z, in view of (3.4) and Remark 2.2 we have

$$\begin{aligned} \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D) &= \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_{X,\text{Kähler}}, \mathscr{C}\mathscr{H}^Z_D) \\ &= \{\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathscr{C}\mathscr{H}^Z_D);\, \mathcal{J}\mu = d\mathcal{J} \wedge \mu = 0\}. \end{aligned} \tag{3.5}$$

Definition 3.7. We let

$$\mathscr{K}er_p\, i^* = \{\varphi \in \mathscr{E}^{p,*}_D;\, \varphi \wedge \mu = 0,\ \forall \mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D)\},$$

cf. Remark 2.2, and we define the sheaf of smooth $(p,*)$-forms on X by

$$\mathscr{E}^{p,*}_X := \mathscr{E}^{p,*}_D / \mathscr{K}er_p\, i^*.$$

Notice that if $\varphi \in \mathscr{K}er_p\, i^*$ then $\bar\partial\varphi \in \mathscr{K}er_p\, i^*$ and so $\bar\partial$ is well-defined on $\mathscr{E}^{p,*}_X$. We write i^* for the natural map $\mathscr{E}^{p,*}_D \to \mathscr{E}^{p,*}_X$. Notice that if X is reduced, then, in view of (2.8) and (3.5), a smooth $(p,*)$-form φ is in $\mathscr{K}er_p\, i^*$ if and only if $i^*\varphi = 0$. As in [6, Section 4] one shows that $\mathscr{E}^{p,*}_X$ is intrinsic, i.e., does not dependent on the embedding $i\colon X \to D$.

Proposition 3.8. *If $\varphi \in \mathscr{E}^{p,*}_D$ and $\varphi' \in \mathscr{E}^{p',*}_D$ then $i^*(\varphi \wedge \varphi')$ only depends on $i^*\varphi$ and $i^*\varphi'$. Setting $i^*\varphi \wedge i^*\varphi' := i^*(\varphi \wedge \varphi')$, $\mathscr{E}^{*,*}_X$ becomes a (bigraded) algebra. In particular, $\mathscr{E}^{p,*}_X$ is an $\mathscr{E}^{0,*}_X$-module.*

Proof. Assume that $\varphi \in \mathscr{K}er_p\, i^*$. We must show that $\varphi \wedge \varphi' \in \mathscr{K}er_{p+p'}\, i^*$. Suppose that $\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^{p+p'}_X, \mathscr{C}\mathscr{H}^Z_D)$. Then $\varphi' \wedge \mu$ is (a sum of terms) of the form $\xi \wedge \nu$, where $\xi \in \mathscr{E}^{0,*}_D$ and $\nu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D)$. Since, by definition of $\mathscr{K}er_p\, i^*$, $\varphi \wedge \nu = 0$ it follows that $\varphi \wedge \varphi' \wedge \mu = 0$, and hence $\varphi \wedge \varphi' \in \mathscr{K}er_{p+p'}\, i^*$. □

Proposition 3.9. *Let $\mathcal{R} = \mathcal{R}_\kappa + \mathcal{R}_{\kappa+1} + \cdots$ be the residue current associated with Ω^p_X defined in Section 3.1 and let $\varphi \in \mathscr{E}^{p,*}_D$. Then $\varphi \in \mathscr{K}er_p\, i^*$ if and only if $\mathcal{R}_\kappa \wedge \varphi = 0$.*

Proof. Recall the complex (2.10) that we used to define R and therefore also $\mathcal{R}$. Consider the dual complex:

$$\cdots \xleftarrow{f^*_{\kappa+1}} \mathscr{O}(E^*_\kappa) \xleftarrow{f^*_\kappa} \cdots \xleftarrow{f^*_1} \mathscr{O}(E^*_0) \leftarrow 0,$$

where f^*_j is the transpose of f_j. If $\xi \in \mathscr{O}(E^*_\kappa)$ and $f^*_{\kappa+1}\xi = 0$, then, in view of Lemma 3.6,

$$\bar\partial(\xi \cdot \mathcal{R}_\kappa) = \xi \cdot \bar\partial\mathcal{R}_\kappa = \xi \cdot f_{\kappa+1}\mathcal{R}_\kappa = f^*_{\kappa+1}\xi \cdot \mathcal{R}_\kappa = 0. \tag{3.6}$$

Hence, $\xi\cdot\mathcal{R}_\kappa$ is a $\bar\partial$-closed (scalar valued) pseudomeromorphic $(N-p,\kappa)$-current with support on Z. Moreover, since $\mathcal{J}^p\wedge\mathcal{R} = 0$, it follows that $\xi\cdot\mathcal{R}_\kappa \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D)$. If $\xi = f^*_\kappa\xi'$, since $\mathcal{R}_{\kappa-1} = 0$, a computation similar to (3.6) shows that $\xi\cdot\mathcal{R}_\kappa = 0$. Hence, we have a map

$$\mathscr{H}^\kappa(\mathscr{O}(E^*_\bullet), f^*_\bullet) \to \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D), \quad [\xi] \mapsto \xi\cdot\mathcal{R}_\kappa. \tag{3.7}$$

By [4, Theorem 1.5], this map is an isomorphism. If $\mathcal{R}_\kappa\wedge\varphi = 0$ thus $\varphi \in \mathscr{K}er_p\, i^*$.

Conversely, assume that $\varphi \in \mathscr{K}er_p\, i^*$. If Ω^p_X is Cohen–Macaulay and (2.10) is a resolution of minimal length, i.e., if $E_j = 0$ for $j > \kappa$, then $\bar\partial\mathcal{R}_\kappa = 0$ and so $\mathcal{R}_\kappa \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D)$. In this case, thus, $\mathcal{R}_\kappa\wedge\varphi = 0$. In general, Ω^p_X is Cohen–Macaulay generically on Z and the minimal resolution is a direct summand in any resolution. It follows, cf. the proof of [4, Theorem 1.2], that $\mathcal{R}_\kappa\wedge\varphi = 0$ generically on Z. By the SEP it then holds everywhere. □

Corollary 3.10. *There is a natural injective map $\Omega^p_X \hookrightarrow \mathscr{E}^{p,0}_X$.*

Proof. Since $\mathcal{J}^p \subset \mathscr{K}er_p\, i^*$, the inclusion $\Omega^p_D \subset \mathscr{E}^{p,0}_D$ induces a map $\Omega^p_X \to \mathscr{E}^{p,0}_X$. By Proposition 3.9, if $\varphi \in \mathscr{K}er_p\, i^*$, then $\mathcal{R}_\kappa\wedge\varphi = 0$ and so $\mathscr{K}er_p\, i^* \cap \Omega^p_D = \mathcal{J}^p$ in view of (2.14). It follows that $\Omega^p_X \to \mathscr{E}^{p,0}_X$ is injective. □

The following result is not necessary for this paper but is included here for future reference. We believe that it interesting in its own right since it shows that the de Rham operator $d = \partial + \bar\partial$ is well-defined on $\mathscr{E}_X$.

Proposition 3.11. *We have $\partial\colon \mathscr{E}^{p,q}_X \to \mathscr{E}^{p+1,q}_X$.*

Proof. We need to show that $\partial(\mathscr{K}er_p\, i^*) \subset \mathscr{K}er_{p+1}\, i^*$, i.e., that if $\varphi \in \mathscr{K}er_p\, i^*$ then $\partial\varphi\wedge\mu = 0$ for all $\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^{p+1}_X, \mathscr{C}\mathscr{H}^Z_D)$. Let $\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^{p+1}_X, \mathscr{C}\mathscr{H}^Z_D)$; cf. Remark 2.2 and (3.5). By [10, Theorem 3.7] we get $\partial\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,\kappa}_D)$ and we certainly have $\bar\partial\partial\mu = 0$. We also have $\mathcal{J}\partial\mu = \partial(\mathcal{J}\mu) \pm d\mathcal{J}\wedge\mu = 0$ and $d\mathcal{J}\wedge\partial\mu = \partial(d\mathcal{J}\wedge\mu) = 0$. Therefore $\partial\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D)$.

Let $\varphi \in \mathscr{K}er_p\, i^*$ and $\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^{p+1}_X, \mathscr{C}\mathscr{H}^Z_D)$. We have $\partial\varphi\wedge\mu = \partial(\varphi\wedge\mu)\pm\varphi\wedge\partial\mu$ and by the above the second term vanishes. Since $\varphi \in \mathscr{K}er_p\, i^*$, $\varphi\wedge\tilde\mu = 0$ for all $\tilde\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D)$ and therefore $\varphi\wedge\mu\wedge\alpha = 0$ for all $\alpha \in \Omega^1_D$. But then we must have $\varphi\wedge\mu = 0$ which shows that $\partial\varphi \in \mathscr{K}er_{p+1}\, i^*$. □

3.3. **Smooth forms on $X_{p\text{-reg}}$.** Here we give a more concrete description of $\mathscr{E}^{p,*}_X$ on $X_{p\text{-reg}}$. Choose local coordinates (z,w) centered at a point in $X_{p\text{-reg}}$ such that $Z = \{w = 0\}$. Recall that the local coordinates induce an $\mathscr{O}_Z$-module structure on Ω^p_X. On $X_{p\text{-reg}}$ we get a sequence of mappings

$$(\mathscr{O}_Z)^\nu \xrightarrow{\sim} \Omega^p_X \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_X}\big(\mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{C}\mathscr{H}^Z_D), \mathscr{C}\mathscr{H}^Z_D\big) \to (\mathscr{C}\mathscr{H}^Z_D)^m \to (\Omega^n_Z)^{m\tilde M} \tag{3.8}$$

defined as follows. On $X_{p\text{-reg}}$, Ω^p_X is a free $\mathscr{O}_Z$-module and the first mapping is the isomorphism given by an $\mathscr{O}_Z$-basis $\{b_k\} \subset \Omega^p_D$ of Ω^p_X.

The second mapping is defined on all of X and is the natural mapping into a double dual, $\varphi \mapsto (\mu \mapsto \varphi \wedge \mu)$, cf. Remark 2.2. It is injective since if $\varphi \in \Omega^p_D$ and $\varphi \wedge \mu = 0$ for all $\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{CH}^Z_D)$, then $\varphi \in \mathcal{J}^p$; cf. the proof of Corollary 3.10. It follows from a fundamental theorem of J.-E. Roos that the second mapping in fact is an isomorphism if and only if Ω^p_X is S_2, cf. [6, Theorem 7.3] and the discussion following it. On $X_{p\text{-reg}}$, Ω^p_X is Cohen–Macaulay, in particular S_2, and thus the second mapping is an isomorphism on $X_{p\text{-reg}}$.

The third mapping depends on a choice of generators μ_j, $j = 1, \dots, m$, of $\mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{CH}^Z_D)$. An element h of the double-$\mathscr{H}om$ then is mapped to the tuple $(h \wedge \mu_1, \dots, h \wedge \mu_m)$.

For the fourth mapping we choose $M > 0$ such that $w^\alpha \mu_j = 0$ for $j = 1, \dots, m$ and $w^\alpha \in \mathcal{J}$ if $|\alpha| \geq M$. Then a tuple $(\nu_j)_j \in (\mathscr{CH}^Z_D)^m$ is mapped to the tuple $(\pi_*(w^\alpha \nu_j))_{j,|\alpha|<M}$, where π is the projection $\pi(z, w) = z$. Since $w^\alpha \nu_j$ are $\bar\partial$-closed of bidegree (N, κ) in D, $\pi_*(w^\alpha \nu_j)$ are $\bar\partial$-closed of bidegree $(n, 0)$ on Z, i.e., holomorphic n-forms on Z.

We will see, Lemma 3.12, that the composition (3.8), denoted $\widetilde{T}$ from now on, is injective and $\mathscr{O}_Z$-linear and thus given by a matrix, also denoted $\widetilde{T}$, with Ω^n_Z-entries.

To analyze $\mathscr{E}^{p,*}_X$ on $X_{p\text{-reg}}$ we consider a variant of (3.8). First, as we did for Ω^p_X, cf. the paragraph before Proposition 3.2, we define a $\mathscr{E}^{0,*}_Z$-module structure on $\mathscr{E}^{p,*}_X$ by

$$\psi \wedge \varphi := \pi^* \psi \wedge \varphi, \quad \psi \in \mathscr{E}^{0,*}_Z,\ \varphi \in \mathscr{E}^{p,*}_X.$$

Corresponding to the mapping $\Omega^p_X \to (\mathscr{CH}^Z_D)^m$ of (3.8) we have the mapping

$$\mathscr{E}^{p,*}_X \to (\mathcal{W}^Z_D)^m, \quad \varphi \mapsto (\varphi \wedge \mu_1, \dots, \varphi \wedge \mu_m). \tag{3.9}$$

Notice that, by Definition 3.7, if $\varphi \in \mathscr{E}^{p,*}_X$ and $\varphi \wedge \mu_j = 0$ for all j, then $\varphi = 0$. Thus, (3.9) is injective. Corresponding to the mapping $(\mathscr{CH}^Z_D)^m \to (\Omega^n_Z)^{m\tilde{M}}$ of (3.8) we have

$$(\mathcal{W}^Z_D)^m \to (\mathcal{W}_Z)^{m\tilde{M}}, \quad (\tau_j)_j \mapsto \big(\pi_*(w^\alpha \tau_j)\big)_{j,|\alpha|<M}, \tag{3.10}$$

where $\widetilde{M}$ is the number of monomials w^α with $|\alpha| < M$. In view of [11, Proposition 4.1 and (4.3)] and [10, Theorem 3.5], (3.10) is injective. Composing (3.9) and (3.10) we get the injective map

$$T \colon \mathscr{E}^{p,*}_X \to (\mathcal{W}^{n,*}_Z)^{m\tilde{M}}, \quad T\varphi = \big(\pi_*(\varphi \wedge w^\alpha \mu_j)\big)_{|\alpha|<M, j=1,\dots,m}. \tag{3.11}$$

The restriction of T to Ω^p_X is (after the identification $\Omega^p_X \simeq (\mathscr{O}_Z)^\nu$) the mapping $\widetilde{T}$. For reference we notice that

$$\widetilde{T} \colon (\mathscr{O}_Z)^\nu \to (\Omega^n_Z)^{m\tilde{M}}, \quad (h_k)_{k=1,\dots,\nu} \mapsto \big(\pi_*(\sum_k h_k b_k \wedge w^\alpha \mu_j)\big)_{|\alpha|<M, j=1,\dots,m} \tag{3.12}$$

on $X_{p\text{-reg}}$.

Lemma 3.12. *The injective mappings T and $\widetilde{T}$ are $\mathscr{E}^{0,*}_Z$-linear and $\mathscr{O}_Z$-linear, respectively. Any $\varphi \in \mathscr{E}^{p,*}_X$ can be written*

$$\varphi = \sum_{k=1}^{\nu} \varphi_k \wedge b_k + \mathscr{K}er_p\, i^*, \quad \varphi_k \in \mathscr{E}^{0,*}_Z, \tag{3.13}$$

on $X_{p\text{-}reg}$ and T is given by matrix multiplication by $\widetilde{T}$, i.e., $T\varphi = \widetilde{T}(\varphi_1,\dots,\varphi_\nu)^t$.

Proof. Let $\psi \in \mathscr{E}_Z^{0,*}$. By definition of T and (2.1),

$$\begin{aligned} T(\psi\wedge\varphi) &= T(\pi^*\psi\wedge\varphi) = \big(\pi_*(\pi^*\psi\wedge\varphi\wedge w^\alpha\mu_j)\big)_{|\alpha|<M, j=1,\dots,m} \\ &= \psi\wedge\big(\pi_*(\varphi\wedge w^\alpha\mu_j)\big)_{|\alpha|<M,j=1,\dots,m}. \end{aligned}$$

Hence, T is $\mathscr{E}_Z^{0,*}$-linear. The same computation shows that $\widetilde{T}$ is $\mathscr{O}_Z$-linear and therefore given by a matrix with elements in Ω_Z^n. Explicitly, since any $\varphi\in\Omega_X^p$ can be written $\varphi = \sum_k \varphi_k b_k + \mathcal{J}^p$ for (unique) $\varphi_k\in\mathscr{O}_Z$,

$$\widetilde{T} = \begin{bmatrix} \pi_*(w^{\alpha_1}b_1\wedge\mu_1) & \dots & \pi_*(w^{\alpha_1}b_\nu\wedge\mu_1) \\ \vdots & \ddots & \vdots \\ \pi_*(w^{\alpha_{\tilde M}}b_1\wedge\mu_m) & \dots & \pi_*(w^{\alpha_{\tilde M}}b_\nu\wedge\mu_m) \end{bmatrix}. \tag{3.14}$$

Let $\varphi\in\mathscr{E}_X^{p,*}$ and let $\tilde\varphi\in\mathscr{E}_D^{p,*}$ be any representative. We can write $\tilde\varphi = \sum_i \tilde\varphi_i'\wedge\tilde\varphi_i''$, where $\tilde\varphi_i'\in\mathscr{E}_D^{0,*}$ and $\tilde\varphi_i''\in\Omega_D^p$. Moreover, we write $\tilde\varphi_i' = \phi_i+\psi_i$, where every term of ϕ_i contains a factor $d\bar w_j$ for some j and no term of ψ_i contains such a factor. Taylor expanding (the coefficients of) ψ_i with respect to w and $\bar w$ to the order M we get

$$\psi_i(z,w) = \sum_{|\alpha|<M}\frac{\partial^\alpha\psi_i}{\partial w^\alpha}(z,0)\frac{w^\alpha}{\alpha!} + \sum_{|\alpha|=M} w^\alpha\tilde\psi_{i,\alpha} + \mathcal{O}(\bar w),$$

where $\tilde\psi_{i,\alpha}\in\mathscr{E}_D^{0,*}$ and $\mathcal{O}(\bar w)$ is a sum of terms divisible by some $\bar w_j$. In view of (3.5) and (2.4), ϕ_i, $w^\alpha\tilde\psi_{i,\alpha}$, and $\mathcal{O}(\bar w)$ are in $\mathscr{K}er_p\, i^*$. Hence,

$$\tilde\varphi = \sum_{i,|\alpha|<M}\frac{\partial^\alpha\psi_i}{\partial w^\alpha}(z,0)\frac{w^\alpha}{\alpha!}\wedge\tilde\varphi_i'' + \mathscr{K}er_p\, i^*. \tag{3.15}$$

Since $w^\alpha\tilde\varphi_i''\in\Omega_D^p$ there are $\tilde\varphi_{\alpha,i,k}\in\mathscr{O}_Z$ such that $w^\alpha\tilde\varphi_i'' = \sum_k\tilde\varphi_{\alpha,i,k}(z)b_k + \mathcal{J}^p$, and so (3.13) follows from (3.15). By $\mathscr{E}_Z^{0,*}$-linearity,

$$T(\varphi_k\wedge b_k) = \varphi_k\wedge T|_{\Omega_X^p} b_k = \varphi_k\widetilde{T}(0,\dots,1_k,\dots,0)^t$$

and the last statement of the lemma follows. □

Notice that by this lemma, T is a map $\mathscr{E}_X^{p,*}\to(\mathscr{E}_Z^{n,*})^{m\widetilde{M}}$ on $X_{p\text{-reg}}$.

Proposition 3.13. *On $X_{p\text{-}reg}$, $\mathscr{E}_X^{p,*}$ is a free $\mathscr{E}_Z^{0,*}$-module, the representation* (3.13) *of an element $\varphi\in\mathscr{E}_X^{p,*}$ is unique, and*

$$\mathscr{E}_X^{p,*} = \mathscr{E}_D^{p,*}/\big(\mathcal{J}\mathscr{E}_D^{p,*} + d\mathcal{J}\wedge\mathscr{E}_D^{p-1,*} + \overline{\mathcal{J}_Z}\mathscr{E}_D^{p,*} + d\overline{\mathcal{J}_Z}\wedge\mathscr{E}_D^{p,*}\big),$$

where $\mathcal{J}_Z = \sqrt{\mathcal{J}}$.

Proof. Notice first that since $\widetilde{T}$ is injective and Ω_X^p is a free $\mathscr{O}_Z$-module on $X_{p\text{-reg}}$ it follows that, generically on $X_{p\text{-reg}}$, $\widetilde{T}$ is a pointwise injective matrix (times $dz_1\wedge\cdots\wedge dz_n$). Consider a representation (3.13) and assume that $\sum_k\varphi_k\wedge b_k\in\mathscr{K}er_p\, i^*$. Then $\widetilde{T}(\varphi_1,\dots,\varphi_\nu)^t=0$. Since $\widetilde{T}$ is generically pointwise injective on $X_{p\text{-reg}}$ it follows that $\varphi_j = 0$, $j=1,\dots,\nu$, on $X_{p\text{-reg}}$. Hence, the representation (3.13) is unique and $\mathscr{E}_X^{p,*}$ is a free $\mathscr{E}_X^{0,*}$-module.

It remains to see that

$$\mathscr{K}er_p\, i^* = \mathcal{J}\mathscr{E}_D^{p,*} + d\mathcal{J}\wedge\mathscr{E}_D^{p-1,*} + \overline{\mathcal{J}_Z}\mathscr{E}_D^{p,*} + d\overline{\mathcal{J}_Z}\wedge\mathscr{E}_D^{p,*} \tag{3.16}$$

on $X_{p\text{-reg}}$. Assume that φ is an element of the right-hand side and let $\mu \in \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{CH}^Z_D)$. In view of (2.4), the terms of φ contained in $\overline{\mathcal{J}_Z}\mathscr{E}^{p,*}_D + d\overline{\mathcal{J}_Z}\wedge\mathscr{E}^{p,*}_D$ annihilate μ so we may assume that $\varphi \in \mathcal{J}\mathscr{E}^{p,*}_D + d\mathcal{J}\wedge\mathscr{E}^{p-1,*}_D = \hat{\mathcal{J}}^p\wedge\mathscr{E}^{0,*}_D$. Write φ as (a sum of terms) $\varphi'\wedge\varphi''$, where $\varphi' \in \hat{\mathcal{J}}^p$ and $\varphi'' \in \mathscr{E}^{0,*}_D$. Since $\mathcal{J}^p \supset \hat{\mathcal{J}}^p$ we have $\varphi'\wedge\mu = 0$. Thus $\varphi\wedge\mu = 0$ and so $\varphi \in \mathscr{K}er_p\, i^*$.

Assume that $\varphi \in \mathscr{K}er_p\, i^*$ and write φ as (a sum of terms) $\varphi'\wedge\varphi''$, where $\varphi' \in \Omega^p_D$ and $\varphi'' \in \mathscr{E}^{0,*}_D$. As in the proof of Lemma 3.12, by Taylor expanding (the coefficients of) φ'' with respect to w to order M, we have

$$\varphi(z,w) = \varphi'\wedge\sum_{|\alpha|<M}\frac{\partial^\alpha\varphi''}{\partial w^\alpha}(z,0)\frac{w^\alpha}{\alpha!} + \varphi'\wedge\sum_{|\alpha|=M} w^\alpha\tilde{\varphi}'_\alpha + \mathcal{O}(\bar{w}, d\bar{w}), \tag{3.17}$$

where $\tilde{\varphi}'_\alpha \in \mathscr{E}^{0,*}_D$ and $\mathcal{O}(\bar{w}, d\bar{w})$ is a sum of smooth terms containing either some $\bar{w}_j$ or $d\bar{w}_j$. The second and the last term in the right-hand side of (3.17) belong to the right-hand side of (3.16). As in the proof of Lemma 3.12 again, this time by writing $w^\alpha\varphi' \in \Omega^p_D$ modulo $\mathcal{J}^p$ as a $\mathscr{O}_Z$-combination of the b_k on $X_{p\text{-reg}}$,

$$\varphi'\wedge\sum_{|\alpha|<M}\frac{\partial^\alpha\varphi''}{\partial w^\alpha}(z,0)\frac{w^\alpha}{\alpha!} = \sum_{k=1}^{\nu}\phi_k\wedge b_k + \mathcal{J}^p\wedge\mathscr{E}^{0,*}_Z, \tag{3.18}$$

where $\phi_k \in \mathscr{E}^{0,*}_Z$. Since $\hat{\mathcal{J}}^p = \mathcal{J}^p$ on $X_{p\text{-reg}}$ the last term on the right-hand side is contained in the right-hand side of (3.16). The sum S in the right-hand side of (3.18) is in $\mathscr{K}er_p\, i^*$ since, by the proof so far, φ and $\varphi - S$ are in $\mathscr{K}er_p\, i^*$. Thus, in view of Lemma 3.12, $\widetilde{T}(\phi_1,\dots,\phi_\nu)^t = TS = 0$. Since $\widetilde{T}$ is generically pointwise injective on $X_{p\text{-reg}}$, $\phi_j = 0$ on $X_{p\text{-reg}}$. Hence, the left-hand side of (3.18) belongs to the right-hand side of (3.16). Thus, all terms in the right-hand side of (3.17) do too, and so (3.16) follows. □

3.4. **Currents and structure forms on** X**.** The $(n-p, n-q)$-currents on X is the dual of the space of compactly supported sections of $\mathscr{E}^{p,q}_X$, cf. [18, Section 4.2]. The topology on $\mathscr{E}^{p,*}_X = \mathscr{E}^{p,*}_D / \mathscr{K}er_p\, i^*$ is the quotient topology. Notice that $\mathscr{K}er_p\, i^*$ is a closed subspace of $\mathscr{E}^{p,*}_D$ since it is defined as the annihilator of currents. It follows that the $(n-p, n-q)$-currents on X can be identified with the $(N-p, N-q)$-currents μ in D such that $\mu.\varphi = 0$ for all $\varphi \in \mathscr{K}er_p\, i^*$ with compact support. This holds if and only if $\varphi\wedge\mu = 0$ for all $\varphi \in \mathscr{K}er_p\, i^*$ since $\mathscr{K}er_p\, i^*$ is both a right and left $\mathscr{E}^{0,*}_D$-submodule of $\mathscr{E}^{p,*}_D$. If τ is an $(n-p, n-q)$-current on X we write $i_*\tau$ for the corresponding $(N-p, N-q)$-current in D. Notice that if $\varphi \in \mathscr{E}^{p,*}_D$, then $\varphi\wedge i_*\tau$ only depends on $i^*\varphi$ and we write

$$\varphi\wedge i_*\tau = i_*(i^*\varphi\wedge\tau).$$

Since $\bar\partial$ is well-defined on $\mathscr{E}^{p,*}_X$, $\bar\partial$ is defined on $(n-p, *)$-currents τ on X by $\bar\partial\tau.\varphi = \pm\tau.\bar\partial\mu$ and we have $\bar\partial i_*\tau = i_*\bar\partial\tau$.

If τ is an (n, n)-current on X with compact support we define

$$\int_X \tau := \tau.i^*1. \tag{3.19}$$

Notice that i^*1 is a well-defined element in $\mathscr{E}^{0,0}_X$ independent of the local embedding $i\colon X\to D$. Hence, (3.19) makes sense on any pure-dimensional X, not just embedded ones.

Let $\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D)$, cf. Remark 2.2, and assume that $\mathcal{J}^p \wedge \mu = 0$. Then $\hat{\mathcal{J}}^p \wedge \mu = 0$ and so, in view of (2.4) and (3.13), if $\varphi \in \mathscr{K}er_p\, i^*$ we have $\varphi \wedge \mu = 0$ on $X_{p\text{-reg}}$. Thus, by the SEP, $\varphi \wedge \mu = 0$. Hence, μ corresponds to an $(n-p,*)$-current on X.

Definition 3.14. The subsheaf $\mathcal{W}^{n-p,*}_X$ of the sheaf of currents on X is defined by

$$i_*\mathcal{W}^{n-p,*}_X = \{\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D);\ \mathcal{J}\mu = d\mathcal{J} \wedge \mu = 0\}. \tag{3.20}$$

Notice that, since $\hat{\mathcal{J}}^p = \mathcal{J}^p$ on $X_{p\text{-reg}}$ and currents in $\mathcal{W}^Z_D$ have the SEP with respect to Z, we have, cf. (3.5),

$$i_*\mathcal{W}^{n-p,*}_X = \{\mu \in \mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D);\ \mathcal{J}^p \wedge \mu = 0\}.$$

Recall that the current R associated with Ω^p_X has the SEP with respect to Z and $\mathcal{J}^p \wedge R = 0$. By (3.3), $\mathcal{R}$ has the same properties. Therefore, there is $\omega \in \mathcal{W}^{n-p,*}_X$ such that

$$\mathcal{R} = i_*\omega. \tag{3.21}$$

We say that ω is an $(n-p)$-*structure form* on X.

Definition 3.15. We let $\omega^{n-p}_X = \{\tau \in \mathcal{W}^{n-p,0}_X;\ \bar\partial\tau = 0\}$.

By Definition 3.14, in view of Remark 2.2, (2.6), and (3.5), we have

$$i_*\omega^{n-p}_X = \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \mathscr{CH}^Z_D), \tag{3.22}$$

cf. (2.8).

Proposition 3.16. *There is a tuple $\omega_0 = (\omega_{01}, \dots, \omega_{0r})$, where $\omega_{0i} \in \omega^{n-p}_X$, and a tuple $a_0 = (a_{01}, \dots, a_{0r})$ of E_κ-valued almost semi-meromorphic $(0,0)$-currents a_{0i} in D such that a_0 is smooth outside $Z^p_{\kappa+1} := Z^{\Omega^p_X}_{\kappa+1}$ and*

$$\mathcal{R}_\kappa = a_0 \cdot i_*\omega_0. \tag{3.23}$$

Moreover, for $j = 1, 2, \dots, n$, there are $Hom(E_{j-1}, E_j)$-valued almost semi-meromorphic $(0,j)$-currents a_j in D, smooth outside $Z^p_{\kappa+j} := Z^{\Omega^p_X}_{\kappa+j}$, such that

$$\mathcal{R}_{\kappa+j} = a_j \mathcal{R}_{\kappa+j-1}, \tag{3.24}$$

where the product is defined as in (2.9).

Proof. Since $\mathscr{K}er\, f^*_{\kappa+1} \subset \mathscr{O}(E^*_\kappa)$ is coherent, in particular finitly generated, there is a trivial vector bundle $F \to D$ and a morphism $g\colon \mathscr{O}(E_\kappa) \to \mathscr{O}(F)$ such that the image of the transpose $g^*\colon \mathscr{O}(F^*) \to \mathscr{O}(E^*_\kappa)$ equals $\mathscr{K}er\, f^*_{\kappa+1}$. Notice that $gf_{\kappa+1} = 0$ since $f^*_{\kappa+1}g^* = 0$. As in the proofs of [7, Proposition 3.3] and [25, Proposition 3.2], the pointwise minimal (with respect to some choice of metric) inverse, a_0, of g is smooth outside $Z^p_{\kappa+1}$, has an almost semi-meromorphic extension across $Z^p_{\kappa+1}$, and $R_\kappa = a_0 g R_\kappa$. Hence,

$$\mathcal{R}_\kappa = R_\kappa \otimes d\zeta = a_0 g \mathcal{R}_\kappa. \tag{3.25}$$

In view of Lemma 3.6 we have

$$\bar\partial g\mathcal{R}_\kappa = g f_{\kappa+1}\mathcal{R}_\kappa = 0,$$

$g\mathcal{R}_\kappa$ is an F-valued section of $\mathscr{H}om_{\mathscr{O}_D}(\Omega^p_D, \mathcal{W}^{Z,*}_D)$, and $\mathcal{J}^p \wedge g\mathcal{R}_\kappa = 0$. Thus, after a choice of frame of F, we can identify $g\mathcal{R}_\kappa$ with a tuple ω_0 of sections of ω^{n-p}_X, i.e., $g\mathcal{R}_\kappa = i_*\omega_0$. By the choice of frame of F, a_0 is a tuple of E_κ-valued almost semi-meromorphic currents. Hence, (3.23) follows from (3.25).

By [8, Theorem 4.4], in $D \setminus Z^p_{\kappa+j}$ there are smooth $(0,j)$-forms a_j such that $R_{\kappa+j} = a_j R_{\kappa+j-1}$. As in the proof of [7, Proposition 3.3] the a_j have almost semi-meromorphic extensions (also denoted a_j) across $Z^p_{\kappa+j}$ and $R_{\kappa+j} = a_j R_{\kappa+j-1}$ holds in D; here $a_j R_{\kappa+j-1}$ is defined as in (2.9), and we remark that for this last identity to hold in D it is necessary that Ω^p_X has pure dimension. Thus, (3.24) follows. □

4. The sheaf $\mathcal{V}^{p,*}_X$.

The sheaf $\mathcal{V}^{p,*}_X$ is an intrinsic sheaf on X that extends $\mathscr{E}^{p,*}_X$. In terms of our local embedding $i\colon X \to D$ the idea is as follows. Recall that $Z = X_{\text{red}}$ and that Ω^p_X locally on $X_{p\text{-reg}} \subset Z_{\text{reg}}$ is a free $\mathscr{O}_Z$-module, where the module structure depends on a choice of local coordinates. As in Section 3.3 we let $\{b_k\}_{k=1}^{\nu}$ be a local $\mathscr{O}_Z$-basis of Ω^p_X. By Lemma 3.12, each $\varphi \in \mathscr{E}^{p,*}_X$ has a representative $\sum_k \varphi_k \wedge b_k$ on $X_{p\text{-reg}}$, where $\varphi \in \mathscr{E}^{0,*}_Z$. One can define $\mathcal{V}^{p,*}_X$ on $X_{p\text{-reg}}$ as such sums with $\varphi_k \in \mathcal{W}^{0,*}_Z$ instead of $\mathscr{E}^{0,*}_X$ and require φ_k to transform under changes of coordinates and base $\{b_k\}$ as in the case of $\mathscr{E}^{p,*}_X$. However, we choose a more invariant approach. To motivate it we notice that each sum $\sum_k \varphi_k \wedge b_k$ with $\varphi_k \in \mathcal{W}^{0,*}_Z$ induces an $\mathscr{O}_X$-linear mapping $\omega^{n-p}_X \to \mathcal{W}^{n,*}_X$ as follows.

Let $\mu \in \omega^{n-p}_X$. Then $b_k \wedge i_*\mu$ is in $\mathscr{C}\mathscr{H}^Z_D$ and depends only on the class of b_k in Ω^p_X. Moreover, $\mathcal{J} b_k \wedge i_*\mu = 0$. If $\varphi_k \in \mathcal{W}^{0,*}_Z$ then, in view of (2.7), $\varphi_k \wedge b_k \wedge i_*\mu$ is well-defined in $\mathcal{W}^{Z,*}_D$ since $\varphi_k \wedge \bar\partial(dw/w^{\alpha+\mathbf{1}})$ exists as a tensor product. Moreover, $\mathcal{J}\varphi_k \wedge b_k \wedge i_*\mu = 0$ and so $\varphi_k \wedge b_k \wedge i_*\mu$ defines an element in $\mathcal{W}^{n,*}_X$. Hence, $\varphi_k \wedge b_k$ induces a mapping $\omega^{n-p}_X \to \mathcal{W}^{n,*}_X$.

With this in mind we make the following definition.

Definition 4.1. $\mathcal{V}^{p,*}_X := \mathscr{H}om_{\mathscr{O}_X}(\omega^{n-p}_X, \mathcal{W}^{n,*}_X)$.

Remark 4.2. The sheaf $\mathcal{V}^{0,*}_X$ was introduced in [6, Section 7] but was denoted $\mathcal{W}^{0,*}_X$ there. In this paper $\mathcal{W}^{0,*}_X$ naturally has another meaning, see Definition 3.14; cf. also Proposition 4.6 below.

If $\varphi \in \mathscr{E}^{p,*}_D$, then φ defines an element φ' in $\mathcal{V}^{p,*}_X$ by $\varphi'(\mu) = \tau$, where $i_*\tau = \varphi \wedge i_*\mu$. By Definition 3.7 and (3.22), $\varphi' = 0$ if and only if $\varphi \in \mathscr{K}er_p\, i^*$. Hence, we have a well-defined injection

$$\mathscr{E}^{p,*}_X \hookrightarrow \mathcal{V}^{p,*}_X.$$

In consistency with Remark 2.2, for $\varphi \in \mathcal{V}^{p,*}_X$ and $\mu \in \omega^{n-p}_X$ we write $\varphi \wedge \mu$ instead of $\varphi(\mu)$.

Definition 4.3. Let $\varphi, \psi \in \mathcal{V}^{p,*}_X$. We say that $\bar\partial\varphi = \psi$ if $\bar\partial(\varphi \wedge \mu) = \psi \wedge \mu$ for all $\mu \in \omega^{n-p}_X$.

Proposition 4.4. *Let $\varphi \in \mathcal{V}^{p,*}_X$. On $X_{p\text{-reg}}$ there are $\varphi_k \in \mathcal{W}^{0,*}_Z$ such that, for any $\mu \in \omega^{n-p}_X$,*

$$i_*\varphi \wedge \mu = \sum_{k=1}^{\nu} \varphi_k \wedge b_k \wedge i_*\mu = \sum_{k=1}^{\nu} \sum_{\alpha} \varphi_k \wedge \pi_*(w^\alpha b_k \wedge i_*\mu) \wedge \bar\partial \frac{dw}{w^{\alpha+\mathbf{1}}}. \tag{4.1}$$

Proof. Recall from (3.12) the matrix $\widetilde{T}$. We can choose a holomorphic matrix $\widetilde{A}$ such that

$$(\mathscr{O}_Z)^\nu \xrightarrow{\widetilde{T}} (\Omega_Z^n)^{m\widetilde{M}} \xrightarrow{\widetilde{A}} (\Omega_Z^n)^{M'} \tag{4.2}$$

is exact. Then also

$$(\mathcal{W}_Z^{0,*})^\nu \xrightarrow{\widetilde{T}} (\mathcal{W}_Z^{n,*})^{m\widetilde{M}} \xrightarrow{\widetilde{A}} (\mathcal{W}_Z^{n,*})^{M'} \tag{4.3}$$

is exact. To see this, notice first that (4.2) is generically pointwise exact. Take Hermitian metrics on the vector bundles underlying the free sheaves in (4.2) and let $\widetilde{B}$ and $\widetilde{S}$ be the Moore-Penrose inverses of $\widetilde{T}$ and $\widetilde{A}$, respectively. Then $\widetilde{B}$ and $\widetilde{S}$ are almost semi-meromorphic, cf. the definition of σ_j in connection to (2.10). Moreover, on the set where (4.2) is pointwise exact, $\widetilde{S}\widetilde{A} + \widetilde{T}\widetilde{B}$ is the identity on $(\Omega_Z^n)^{m\widetilde{M}}$. Thus, if $\mu \in (\mathcal{W}_Z^{n,*})^{m\widetilde{M}}$ and $\widetilde{A}\mu = 0$, we have $\mu = \widetilde{T}\widetilde{B}\mu$ since $\mathcal{W}$ is closed under multiplication by almost semi-meromorphic currents, cf. (2.9).

Let $\varphi \in \mathcal{V}_X^{p,*}$ and let μ_j, $j = 1,\dots,m$, be generators of ω_X^{n-p}. For notational convenience, we will identify $\varphi\wedge\mu_j$ and $i_*\varphi\wedge\mu_j$ as well as μ_j and the corresponding currents in $i_*\omega_X^{n-p}$. In view of (2.5),

$$\varphi\wedge\mu_j = \frac{1}{(2\pi i)^\kappa}\sum_\alpha \pi_*(w^\alpha\varphi\wedge\mu_j)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}. \tag{4.4}$$

We claim that the tuple $(\pi_*(w^\alpha\varphi\wedge\mu_j))_{\alpha,j} \in (\mathcal{W}_Z^{n,*})^{m\widetilde{M}}$ is in the image of $(\mathcal{W}_Z^{0,*})^\nu$ under $\widetilde{T}$. Given the claim, there are $\varphi_k \in \mathcal{W}_Z^{0,*}$ such that, cf. (3.14),

$$\pi_*(w^\alpha\varphi\wedge\mu_j) = (2\pi i)^\kappa\sum_k \varphi_k\wedge\pi_*(w^\alpha b_k\wedge\mu_j).$$

By (4.4), (4.1) follows with $\mu = \mu_j$. Since μ_j generate ω_X^{n-p}, (4.1) follows.

It remains to prove the claim. By exactness of (4.3) we need to show that

$$\widetilde{A}(\pi_*(w^\alpha\varphi\wedge\mu_j))_{\alpha,j} = 0. \tag{4.5}$$

In view of Proposition 2.3 it is enough to show (4.5) where $\pi_*(w^\alpha\varphi\wedge\mu_j)$ are smooth and (4.2) is pointwise exact. Fix such a point; for notational convenience, suppose it is 0.

Let (2.10) be a minimal resolution of Ω_X^p in a neighborhood of 0. Since Ω_X^p is Cohen–Macaulay on $X_{p\text{-reg}}$, $E_\ell = 0$ for $\ell > \kappa$, and the corresponding currents $R = R_\kappa$ and $\mathcal{R} = \mathcal{R}_\kappa$ are $\bar\partial$-closed. Since the mapping (3.7) is an isomorphism it follows that the components, μ_j, $j = 1,\dots,m$, of $\mathcal{R}$ (with respect to some frame of E_κ) generate $i_*\omega_X^{n-p}$. Let $(\mathscr{O}(E'_\bullet), f'_\bullet)$ be the Koszul complex of the regular sequence $z_1,\dots,z_n$ in D. Then $(\mathscr{O}(E'_\bullet), f'_\bullet)$ is a resolution of $\mathscr{O}_D/\langle z\rangle$, $\mathscr{O}(E'_0) = \mathscr{O}_D$, $\mathscr{O}(E'_n) = \mathscr{O}_D$, and the corresponding current is $R' = \bar\partial(1/z) := \bar\partial(1/z_1)\wedge\cdots\wedge\bar\partial(1/z_n)$.

Let $(\mathscr{O}(E''_\bullet), f''_\bullet)$ be the tensor product of the complexes $(\mathscr{O}(E_\bullet), f_\bullet)$ and $(\mathscr{O}(E'_\bullet), f'_\bullet)$, i.e., $E''_k = \oplus_{i+j=k}E_i\otimes E'_j$ and $f''_\bullet = f_\bullet\otimes\mathbf{1}_{E'} + \mathbf{1}_E\otimes f'_\bullet$. As the tensor product of minimal resolutions of properly intersecting Cohen–Macaulay modules, $(\mathscr{O}(E''_\bullet), f''_\bullet)$ is a resolution of $\mathscr{F} := \mathscr{O}(E''_0)/\mathscr{Im}\, f''_1$. Notice that $\mathscr{O}(E''_0) = \mathscr{O}(E_0)\otimes\mathscr{O}(E'_0) = \mathscr{O}(E_0) = \Omega_D^p$ and that $\mathcal{I} := \mathscr{Im}\, f''_1 = \mathscr{Im}\, f_1\cdot\mathscr{O}(E'_0) + \mathscr{Im}\, f'_1\cdot\mathscr{O}(E_0)$ so that

$$\mathscr{F} = \Omega_D^p/\mathcal{I} = \Omega_D^p/(\mathcal{J}^p + \langle z\rangle\Omega_D^p). \tag{4.6}$$

Clearly $\mathscr{F}$ is supported at 0 and since $(\mathscr{O}(E''_\bullet), f''_\bullet)$ has length $\kappa + n = N$, $\mathscr{F}$ is Cohen–Macaulay and $(\mathscr{O}(E''_\bullet), f''_\bullet)$ is a minimal resolution. Following [3, Section 4], the product $R\wedge R'$ makes sense and is the current R'' associated with $(\mathscr{O}(E''_\bullet), f''_\bullet)$. It follows that $\mu_j\wedge\bar\partial(1/z)$ generate $\mathscr{H}om_{\mathscr{O}_X}(\Omega^p_D/\mathcal{I}, \mathscr{CH}^Z_D)$. Since $\mathscr{F}$ is Cohen–Macaulay, for the same reason that the second map in (3.8) is an isomorphism on $X_{p\text{-reg}}$, the map

$$\mathscr{F}\to\mathscr{H}om_{\mathscr{O}_X}\big(\mathscr{H}om_{\mathscr{O}_X}(\Omega^p_D/\mathcal{I},\mathscr{CH}^Z_D), \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_D/(\mathcal{J}+\langle z\rangle),\mathscr{CH}^Z_D)\big), \tag{4.7}$$

$$\phi\mapsto(\mu\mapsto\phi\wedge\mu),$$

is an isomorphism. In view of (2.5), if $\phi\in\mathscr{F}$, then

$$\phi\wedge\mu_j\wedge\bar\partial(1/z)=\frac{1}{(2\pi i)^N}\sum_\alpha\pi'_*(w^\alpha\phi\wedge\mu_j\wedge\bar\partial\frac{1}{z})\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}\wedge\bar\partial\frac{dz}{z}, \tag{4.8}$$

where π' is the map $(z,w)\mapsto 0$. The tuple $(\pi'_*(w^\alpha\phi\wedge\mu_j\wedge\bar\partial(1/z)))_{\alpha,j}\in\mathbb{C}^{m\widetilde{M}}$ determines ϕ and we have the injective map

$$\mathscr{F}\to\mathbb{C}^{m\widetilde{M}}, \tag{4.9}$$

cf. (3.12). In view of (4.6), since b_k generate $\Omega^p_X=\Omega^p_D/\mathcal{J}^p$ over $\mathscr{O}_Z$, b_k also generate $\mathscr{F}$ over $\mathscr{O}_Z$. Hence, we have the surjective map $(\mathscr{O}_Z)^\nu\to\mathscr{F}$, $(h_k)_k\mapsto\sum_k h_kb_k$. Composing with (4.9), we get

$$\widetilde{\mathcal{T}}\colon(\mathscr{O}_Z)^\nu\to\mathbb{C}^{m\widetilde{M}},\quad \widetilde{\mathcal{T}}(h_k)_k=\big(\sum_k\pi'_*(w^\alpha h_kb_k\wedge\mu_j\wedge\bar\partial\frac{1}{z})\big)_{\alpha,j}.$$

Recall (again) the map $\widetilde{T}$ from (3.12) and (3.14) and write $\widetilde{T}=\widetilde{T}'dz$, where $\widetilde{T}'$ is a matrix of holomorphic functions. Let $\pi''\colon Z\to\{0\}$ and notice that $\pi'=\pi''\circ\pi$. We get

$$\begin{aligned}\pi'_*(w^\alpha h_kb_k\wedge\mu_j\wedge\bar\partial\frac{1}{z}) &= \pi''_*\big(\pi_*(w^\alpha b_k\wedge\mu_j)h_k\wedge\bar\partial\frac{1}{z}\big)=\pi''_*\big(\widetilde{T}'_{\alpha,j,k}h_kdz\wedge\bar\partial\frac{1}{z}\big)\\ &= \widetilde{T}'_{\alpha,j,k}(0)h_k(0).\end{aligned}$$

Hence, $\widetilde{\mathcal{T}}dz=\widetilde{T}(0)$. Since (4.2) is pointwise exact at 0 it follows that a tuple $(\lambda_{\alpha,j})\in\mathbb{C}^{m\widetilde{M}}$ is in the image of $\widetilde{\mathcal{T}}$ if and only if $\widetilde{A}(0)(\lambda_{\alpha,j})=0$. We remark that this implies that $\widetilde{T}$ is pointwise injective on $X_{p\text{-reg}}$.

Now, by (4.4), since $\pi_*(w^\alpha\varphi\wedge\mu_j)$ is smooth in a neighborhood of 0, we have

$$\varphi\wedge\mu_j=\sum_\alpha{\sum_{|L|=*}}'\phi_{j,\alpha,L}(z)\wedge d\bar z_L\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}, \tag{4.10}$$

where $\phi_{j,\alpha,L}(z)$ are smooth $(n,0)$-forms on Z and $d\bar z_L$ are a basis of $T^*_{0,*}Z$. Set

$$\phi_L(\mu_j\wedge\bar\partial(1/z)):=\sum_\alpha\phi_{j,\alpha,L}(z)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}\wedge\bar\partial\frac{1}{z}. \tag{4.11}$$

To see that ϕ_L is well-defined, recall that μ_j are the components of $\mathcal{R}$. Since (3.7) is an isomorphism it follows that the relations between the μ_j are generated by f^*_κ. In the same way, it follows that the relations between $\mu_j\wedge\bar\partial(1/z)$, which are the components of R'', are generated by $(f''_{\kappa+n})^*=f^*_\kappa\otimes\mathbf{1}_{(E'_n)^*}\oplus\mathbf{1}_{E^*_\kappa}\otimes(f'_n)^*$. Thus, if a_j are such that $\sum_j a''_j\mu_j\wedge\bar\partial(1/z)=0$, then we have that $a''_j=a_j+a'_j$, where $\sum_j a_j\mu_j=0$, and $a'_j\bar\partial(1/z)=0$. This implies that ϕ_L is well-defined.

Now, $\phi_L(\mu_j\wedge\bar\partial(1/z))$ is an (N,N)-current, in particular $\bar\partial$-closed, and it is annihilated by $\langle z\rangle$. Moreover, it is annihilated by $\mathcal{J}$ since $\varphi\wedge\mu_j$ is, and

$$\varphi\wedge\mu_j\wedge d\bar z_{L^c} = \pm d\bar z\wedge\sum_\alpha \phi_{j,\alpha,L}(z)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}},$$

where $L^c=\{1,\dots,n\}\setminus L$. Hence, $\phi_L(\mu_j\wedge\bar\partial(1/z))$ is in $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_D/(\mathcal{J}+\langle z\rangle),\mathscr{CH}_D^Z)$ and it follows that ϕ_L is in the right-hand side of (4.7). Since (4.7) is an isomorphism, ϕ_L is multiplication by an element, also denoted ϕ_L, in $\mathscr{F}$. In view of (4.8) and (4.11), the image under (4.9) of ϕ_L is the tuple

$$(2\pi i)^N\big(\phi_{j,\alpha,L}(0)\big)_{\alpha,j}.$$

It is in the image of $\widetilde{\mathcal{T}}$ and hence in the kernel of $\widetilde{A}(0)$. Thus,

$$0=\widetilde{A}(0)\big(\sum_{|L|=*}{}' \phi_{j,\alpha,L}(0)\wedge d\bar z_L\big)_{\alpha,j}.$$

However, in view of (4.4) and (4.10), $\sum'_{|L|=*}\phi_{j,\alpha,L}(0)\wedge d\bar z_L$ is the value of $\pi_*(w^\alpha\varphi\wedge\mu_j)$ at 0 and so (4.5) follows at 0. Hence, (4.5) follows at points where $\pi_*(w^\alpha\varphi\wedge\mu_j)$ are smooth and (4.2) is pointwise exact, concluding the proof of the claim. □

By Proposition 4.4, if $\varphi\in\mathcal{V}_X^{p,*}$ then, on $X_{p\text{-reg}}$, there are $\varphi_k\in\mathcal{W}_Z^{0,*}$ such that φ is given by multiplication by $\sum_k\varphi_k\wedge b_k$ in the way described in the second paragraph of this section. In this way we can identify $\mathcal{V}_X^{p,*}$ with such sums on $X_{p\text{-reg}}$.

The following lemma is proved in the same way as Lemma 7.7 and Corollary 7.8 are proved in [6].

Lemma 4.5. *Each $\varphi\in\mathcal{V}_X^{p,*}=\mathscr{H}om_{\mathscr{O}_X}(\omega_X^{n-p},\mathcal{W}_X^{n,*})$ has a unique extension to an element in $\mathscr{H}om_{\mathscr{E}_X^{0,*}}(\mathscr{E}_X^{0,*}\wedge\omega_X^{n-p},\mathcal{W}_X^{n,*})$. Moreover, if $\mu\in\mathcal{W}_X^{n-p,*}$ is such that $i_*\mu=\sum_\ell a_\ell\wedge i_*\mu_\ell$, where $\mu_\ell\in\omega_X^{n-p}$ and a_ℓ are almost semi-meromorphic in D and generically smooth on Z, then $\varphi\wedge\mu$ is well-defined in $\mathcal{W}_X^{n,*}$ by the formula*

$$i_*(\varphi\wedge\mu)=\sum_\ell(-1)^{deg\,a_\ell\cdot deg\,\varphi}a_\ell\wedge i_*(\varphi\wedge\mu_\ell),$$

where the product by a_ℓ is defined as in (2.9).

Notice that by this lemma $\mathcal{V}_X^{p,*}$ gets a natural $\mathscr{E}_X^{0,*}$-module structure, which is the same as the $\mathscr{E}_X^{0,*}$-module structure it inherits from $\mathcal{W}_X^{n,*}$.

4.1. **The sheaf $\mathcal{V}_X^{p,*}$ in case X is reduced.**

Proposition 4.6. *If $X=Z$ is reduced then $\mathcal{V}_X^{p,*}=\mathcal{W}_X^{p,*}$.*

Lemma 4.7. *If $\pi\colon\widetilde{Z}\to Z$ is a modification then $\pi_*\colon\mathcal{W}_{\widetilde{Z}}\to\mathcal{W}_Z$ is a bijection.*

Proof. Denote the exceptional set of the modification by E. If $\pi_*\tau=0$ then τ is zero on $\widetilde{Z}\setminus E$ and by the SEP τ is zero everywhere. Hence π_* is injective.

To show that the map is surjective pick $\nu\in\mathcal{W}_Z$. By [5, Proposition 1.2] there is a $\widetilde{\tau}\in\mathcal{PM}_{\widetilde{Z}}$ such that $\pi_*\widetilde{\tau}=\nu$. We have $\widetilde{\tau}\in\mathcal{W}_{\widetilde{Z}\setminus E}$ since π is a biholomorphism on $\widetilde{Z}\setminus E$. If we let $\tau:=\mathbf{1}_{\widetilde{Z}\setminus E}\widetilde{\tau}$ then $\tau\in\mathcal{W}_{\widetilde{Z}}$ since τ must have the SEP with respect to every subvariety. We also have $\pi_*\tau=\nu$ since both $\pi_*\tau$ and $\pi_*\widetilde{\tau}$ are in $\mathcal{W}_Z$ and they are equal generically and therefore equal everywhere. □

Lemma 4.8. *Given $\nu \in \mathcal{W}_Z^{n,q}$ and a generically non-zero $\mu \in \omega_Z^n$ there is a unique $\nu' \in \mathcal{W}_Z^{0,q}$ such that $\nu = \mu \wedge \nu'$.*

Proof. Let $\pi : \widetilde{Z} \to Z$ be a resolution of singularities. Then $\pi^*\mu$ is a generically non-zero meromorphic n-form on $\widetilde{Z}$. Moreover, by Lemma 4.7 there is a unique $\tau \in \mathcal{W}_{\widetilde{Z}}^{n,q}$ such that $\pi_*\tau = \nu$. In view of [10, Theorem 3.7], since $\widetilde{Z}$ is smooth, τ is a $K_{\widetilde{Z}}$-valued section of $\mathcal{W}_{\widetilde{Z}}^{0,q}$. Thus, $\tau' := \tau/\pi^*\mu$ is a section of $\mathcal{W}_{\widetilde{Z}}^{0,q}$, and $\tau = \pi^*\mu \wedge \tau'$, cf. (2.9). Then $\nu = \pi_*\tau = \pi_*(\pi^*\mu \wedge \tau') = \mu \wedge \pi_*\tau'$ and thus $\pi_*\tau'$ does the job.

If we have two currents satisfying the lemma then they are equal where μ is non-zero. By assumption this means that they are equal generically and then, by the SEP, they are equal everywhere. □

Remark 4.9. Any $h \in \mathcal{H}om_{\mathcal{O}_Z}(\omega_Z^{n-p}, \mathcal{W}_Z^{n,q})$ naturally extends to operate on forms $f\mu$, where f is a germ of a meromorphic function on Z, and $\mu \in \omega_Z^{n-p}$. The extension is unique and h becomes linear over the sheaf of meromorphic functions on Z. Notice that $f\mu$ is not necessarily in ω_Z^{n-p}.

Proof of Proposition 4.6. The currents in ω_Z^{n-p} are meromorphic and in particular almost semi-meromorphic. In view of (2.9) and the comment following it, $a\wedge\nu$ is well-defined and in $\mathcal{W}_Z$ for any almost semi-meromorphic current a on Z and any $\nu \in \mathcal{W}_Z$. Hence we can define a map $\Psi\colon \mathcal{W}_Z^{p,q} \to \mathcal{H}om_{\mathcal{O}_Z}(\omega_Z^{n-p}, \mathcal{W}_Z^{n,q})$ by $(\Psi\nu)(\mu) = \mu \wedge \nu$. If $\mu \wedge \nu = 0$ for all $\mu \in \omega_Z^{n-p}$ then $\nu = 0$ on Z_{reg}. But then, by the SEP, $\nu = 0$ on Z and hence Ψ is injective.

To show that Ψ is surjective take $h \in \mathcal{H}om_{\mathcal{O}_Z}(\omega_Z^{n-p}, \mathcal{W}_Z^{n,q})$. Suppose we have a local parametrization $Z \cap (\Delta_z \times \Delta_w) \to \Delta_z$ of Z, where Δ_z and Δ_w are polydiscs in $\mathbb{C}_z^n$ and $\mathbb{C}_w^\kappa$, respectively, so that $\{dz_I\}_{|I|=n-p}$ generically is a basis for ω_Z^{n-p}. This means that $\mu \in \omega_Z^{n-p}$ may be written $\mu = \sum_{|I|=n-p} f_I dz_I$ for some meromorphic functions f_I on Z. Therefore, by Remark 4.9, it suffices to find $\nu \in \mathcal{W}_Z^{p,q}$ so that $h(dz_I) = dz_I \wedge \nu$ for all I. By Lemma 4.8 there are unique $\nu_J \in \mathcal{W}_Z^{0,q}$ with $h(dz_J) = dz \wedge \nu_J$. We let $\nu = \sum_J dz_{J^c} \wedge \nu_J$, so that $\nu \in \mathcal{W}_Z^{p,q}$, and get $dz_I \wedge \nu = \sum_J dz_I \wedge dz_{J^c} \wedge \nu_J = dz \wedge \nu_I = h(dz_I)$. □

5. Integral operators on X

Given our local embedding $i\colon X \to D \subset \mathbb{C}^N$ as usual and a choice of local coordinates z in D we define integral operators and prove their basic mapping properties.

Let R and $\mathcal{R}$ be the currents associated with a resolution (2.10) of Ω_X^p such that $E_0 = T_{p,0}^* D$. The (full) Bochner-Martinelli form in $D_\zeta \times D_z$, where ζ and z are the same local coordinates in D, is

$$B = \sum_{j=1}^{N} \frac{1}{(2\pi i)^j} \frac{\partial|\zeta - z|^2 \wedge (\bar\partial\partial|\zeta - z|^2)^{j-1}}{|\zeta - z|^{2j}}$$

and we let B_j be the component of B of bidegree $(j, j-1)$. Let $H = H_0 + H_1 + \cdots$ be a holomorphic form in $D_\zeta \times D_z$ with values in $\mathrm{Hom}(E, T_{p,0}^*(D \times D))$, where H_j has bidegree $(j, 0)$ and values in $\mathrm{Hom}(E_j, T_{p,0}^*(D \times D))$, and let $g = g_0 + g_1 + \cdots$ be a smooth form in $D''_\zeta \times D'_z$, where g_j has bidegree (j, j) and $D', D'' \subset D$. The forms H and g will be specified in the next section.

If τ is a current in $D_\zeta \times D_z$ we let $(\tau)_N$ be the component of bidegree $(N,*)$ in ζ and $(0,*)$ in z. Let $\vartheta(\tau)$ be the current defined by

$$(\tau)_N = \vartheta(\tau)\wedge d\zeta.$$

Notice that, in view of (3.3),

$$(g\wedge HR)_N = \vartheta(g\wedge H)\mathcal{R};$$

here and for the rest of this section, $R = R(\zeta)$ and $\mathcal{R} = \mathcal{R}(\zeta)$. Similarly, outside the diagonal $\Delta \subset D_\zeta \times D_z$,

$$(B\wedge g\wedge HR)_N = \vartheta(B\wedge g\wedge H)\mathcal{R}.$$

Let $\varphi \in \mathcal{V}^{p,*}_X$ and let $\mu \in \mathcal{W}^{n-p,*}_X$. We give a meaning to

$$\vartheta(g\wedge H)\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z) \tag{5.1}$$

as follows. By Proposition 3.16, $\mathcal{R} = a\wedge i_*\omega_0$ where a is almost semi-meromorphic and generically smooth on Z. Therefore, by Lemma 4.5, $\mathcal{R}\wedge\varphi := a\wedge i_*(\varphi\wedge\omega_0)$ is well-defined and is in $\mathcal{W}^{Z,*}_D$. Since $\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z)$ exists as a tensor product and $\vartheta(g\wedge H)$ is smooth, (5.1) is defined. Notice that it is annihilated by both $\mathcal{J}(\zeta)$ and $\mathcal{J}(z)$, i.e., it is $\mathscr{O}_X$-linear both in φ and μ. Moreover, by [11, Corollary 4.7] it is in $\mathcal{PM}_{D''\times D'}$, has support in $Z\times Z$ and the SEP with respect to $Z\times Z$.

Let $\pi_i\colon D_\zeta\times D_z \to D$, $i=1,2$, be the natural projections on the first and second factor, respectively. If τ is a current in $D\times D$ such that π_i is proper on the support of τ, then $\pi_{i*}\tau$ is a current in D. Moreover, in view of (2.3), if $\tau\in\mathcal{PM}_{D\times D}$ has support in $Z\times Z$ and the SEP with respect to $Z\times Z$, then $\pi_{i*}\tau\in\mathcal{PM}_D$ has support in Z and the SEP with respect to Z.

Definition 5.1 (The operators P and $\check{P}$)**.** If g is smooth in $D\times D'$ and $\zeta\mapsto g(\zeta,z)$ has support in a fixed compact subset of D for all $z\in D'$, we define $P\colon \mathcal{V}^{p,*}(X)\to \mathcal{V}^{p,*}(X\cap D')$ by

$$i_*P\varphi\wedge\mu = \pi_{2*}\big(\vartheta(g\wedge H)\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z)\big),\quad \varphi\in\mathcal{V}^{p,*}(X),\ \mu\in\omega^{n-p}(X\cap D'). \tag{5.2}$$

If g is smooth in $D''\times D$ and $z\mapsto g(\zeta,z)$ has support in a fixed compact subset of D for all $\zeta\in D''$, we define $\check{P}\colon \mathcal{W}^{n-p,*}(X)\to\mathcal{W}^{n-p,*}(X\cap D'')$ by

$$i_*\check{P}\mu = \pi_{1*}\big(\vartheta(g\wedge H)\mathcal{R}\wedge i_*\mu(z)\big),\quad \mu\in\mathcal{W}^{n-p,*}(X). \tag{5.3}$$

If φ and μ have compact support in X, then $P\varphi$ and $\check{P}\mu$ are defined by (5.2) and (5.3), respectively, for any g.

Notice that $i_*P\varphi$ is a smooth $(p,*)$-form in D' since $\vartheta(g\wedge H)\mathcal{R}$ is smooth in z; if g is holomorphic in z, then $i_*P\varphi$ is holomorphic. Moreover, since $\mathcal{R}=\mathcal{R}(\zeta)$, it follows that $i_*\check{P}\mu = \psi\wedge\mathcal{R}$ for some smooth form ψ in D''.

To define the operators K and $\check{K}$ notice first that, in a similar way as for P and $\check{P}$, we can give a meaning to

$$\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z) \tag{5.4}$$

outside the diagonal $\Delta\subset D\times D$ since B is smooth there.

Lemma 5.2. *The current* (5.4) *has a unique extension to a current in* $\mathcal{PM}_{D\times D}$ *with support in* $Z\times Z$ *and the SEP with respect to* $Z\times Z$*. The extension is annihilated by both* $\mathcal{J}(\zeta)$ *and* $\mathcal{J}(z)$*, i.e., the extension depends* $\mathscr{O}_X$*-linearly on both* φ *and* μ*.*

Proof. The uniqueness is clear by the SEP since (5.4) a priori is defined in $D\times D\setminus\Delta$ and has support in $Z\times Z\setminus\Delta$.

Recall that $\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z)\in\mathcal{PM}_{D\times D}$ has support in $Z\times Z$ and the SEP with respect to $Z\times Z$. Since B is almost semi-meromorphic in $D\times D$, also $\vartheta(B\wedge g\wedge H)$ has these properties. Hence, cf. (2.9), $\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z)$ is in $\mathcal{PM}_{D\times D}$ with support in $Z\times Z$ and the SEP with respect to $Z\times Z$.

Clearly $\mathcal{J}(\zeta)$ and $\mathcal{J}(z)$ annihilate (5.4) outside Δ. Since the extension has the SEP with respect to $Z\times Z$ it is annihilated by $\mathcal{J}(\zeta)$ and $\mathcal{J}(z)$. □

We will use the notation (5.4) to denote the extension as well.

Definition 5.3 (The operators K and $\check{K}$)**.** If g is smooth in $D\times D'$ and $\zeta\mapsto g(\zeta,z)$ has support in a fixed compact subset of D for all $z\in D'$, we define $K\colon \mathcal{V}^{p,*}(X)\to\mathcal{V}^{p,*}(X\cap D')$ by

$$i_*K\varphi\wedge\mu=\pi_{2*}\big(\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi(\zeta)\wedge i_*\mu(z)\big),\quad \varphi\in\mathcal{V}^{p,*}(X),\ \mu\in\omega^{n-p}(X\cap D').$$

If g is smooth in $D''\times D$ and $z\mapsto g(\zeta,z)$ has support in a fixed compact subset of D for all $\zeta\in D''$, we define $\check{K}\colon\mathcal{W}^{n-p,*}(X)\to\mathcal{W}^{n-p,*}(X\cap D'')$ by

$$i_*\check{K}\mu=\pi_{1*}\big(\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge i_*\mu(z)\big),\quad \mu\in\mathcal{W}^{n-p,*}(X).$$

As with the operators P and $\check{P}$, if φ and μ have compact support in X, then $K\varphi$ and $\check{K}\mu$ are defined for any g.

Theorem 5.4. *(i) If $\varphi\in\mathcal{V}_X^{p,*}$ is in $\mathscr{E}_X^{p,*}$ in a neighborhood of a point $x\in X_{p\text{-}reg}$, then $K\varphi$ is in $\mathscr{E}_X^{p,*}$ in a neighborhood of x.*

(ii) If $\mu\in\mathcal{W}_X^{n-p,*}$ is such that, in a neighborhood of $x\in X_{p\text{-}reg}$, $i_*\mu=\sum_\ell\mu_\ell\wedge i_*\omega_\ell$, where $\mu_\ell\in\mathscr{E}_D^{0,*}$ and $\omega_\ell\in\omega_X^{n-p}$, then $i_*\check{K}\mu$ is of the same form in a neighborhood of x.

Recall that, by Proposition 4.4, in a neighborhood of $x\in X_{p\text{-reg}}$, any $\phi\in\mathcal{V}_X^{p,*}$ is represented by $\sum_k\phi_k\wedge b_k$ for some $\phi_k\in\mathcal{W}_Z^{0,*}$. That $\phi\in\mathcal{V}_X^{p,*}$ is smooth means, cf. Lemma 3.12, that $\phi_k\in\mathscr{E}_Z^{0,*}$. In view of this it is natural to call a $\mu\in\mathcal{W}_X^{n-p,*}$ with the property in (ii) smooth. Analogously to part (i), part (ii) of the theorem thus says that $\check{K}$ preserves the smooth elements of $\mathcal{W}_X^{n-p,*}$.

Proof. Notice that if $\varphi=\varphi(\zeta)\equiv 0$ in a neighborhood of x, then $K\varphi$ is smooth in a neighborhood of x since in that case $\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi$ is smooth for z in a neighborhood of x. To prove the first part of the theorem we may thus assume that φ has support in a small neighborhood of x. In this proof we let φ be also a fixed representative of φ in $\mathscr{E}_D^{p,*}$.

Let (z,w) and (ζ,τ) be two sets of the same local coordinates in D centered at x such that $Z=\{w=0\}=\{\tau=0\}$ in a neighborhood of x; these coordinates need not have any relation to our previous local coordinates which were used to define B. Suppose that φ has support where the coordinates (z,w) are defined. Let $\chi^\epsilon:=\chi(|\zeta-z|^2/\epsilon)$ and let, for any $\mu\in\mathcal{W}_X^{n-p,*}$,

$$T:=\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi(\zeta,\tau)\wedge i_*\mu(z,w). \tag{5.5}$$

Then, in view of (2.2),

$$\lim_{\epsilon\to 0}\chi^\epsilon T=\mathbf{1}_{D\times D\setminus\{\zeta=z\}}T.$$

By Lemma 5.2, T has the SEP with respect to $Z\times Z$ and so, since $\{\zeta=z\}\cap Z\times Z$ is a proper subset of $Z\times Z$, $\mathbf{1}_{\{\zeta=z\}}T=0$. Hence, $\mathbf{1}_{D\times D\setminus\{\zeta=z\}}T=T$ and thus $\chi^\epsilon T\to T$. Define $K^\epsilon\varphi$ by

$$K^\epsilon\varphi := \pi_{2*}\big(\chi^\epsilon\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi(\zeta,\tau)\big). \tag{5.6}$$

Then $K^\epsilon\varphi$ is smooth since $\chi^\epsilon\vartheta(B\wedge g\wedge H)\mathcal{R}\wedge\varphi(\zeta,\tau)$ is smooth in (z,w) and it follows that

$$K^\epsilon\varphi\wedge i_*\mu=\pi_{2*}(\chi^\epsilon T)\to\pi_{2*}T=i_*K\varphi\wedge\mu \tag{5.7}$$

as currents in D'.

By Lemma 3.12 there are $\phi_k^\epsilon\in\mathscr{E}_Z^{0,*}$ such that

$$K^\epsilon\varphi=\sum_k\phi_k^\epsilon\wedge b_k+\mathscr{K}\!er_p\, i^*$$

and by the proof of that lemma ϕ_k^ϵ are obtained by applying linear combinations of $\partial^{|\alpha|}/\partial w^\alpha$ to (the coefficients) of $K^\epsilon\varphi$ and evaluate at $w=0$. We claim that there are $\phi_k\in\mathscr{E}_Z^{0,*}$ such that $\phi_k^\epsilon\to\phi_k$ as currents on Z.

Given the claim we can conclude the proof of the first part of the theorem. Let $\mu\in\omega_X^{n-p}$. Then $b_k\wedge i_*\mu\in\mathscr{CH}_D^Z$ and so, in view of (2.7), there are $a_{k,\alpha}(z)\in\Omega_Z^n$ such that $b_k\wedge i_*\mu=\sum_\alpha a_{k,\alpha}(z)\wedge\bar\partial(dw/w^{\alpha+\mathbf{1}})$. Hence,

$$\begin{aligned}K^\epsilon\varphi\wedge i_*\mu &= \sum_k\phi_k^\epsilon\wedge b_k\wedge i_*\mu=\sum_{k,\alpha}\phi_k^\epsilon(z)\wedge a_{k,\alpha}(z)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}\\ &\to \sum_{k,\alpha}\phi_k(z)\wedge a_{k,\alpha}(z)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}=\sum_k\phi_k\wedge b_k\wedge i_*\mu\end{aligned}$$

as currents in D'. In view of (5.7), thus

$$i_*K\varphi\wedge\mu=\sum_k\phi_k\wedge b_k\wedge i_*\mu,$$

which means that $K\varphi\in\mathcal{V}_X^{p,*}$ is smooth.

To show the claim, notice that since $\mathcal{R}=\mathcal{R}_\kappa+\mathcal{R}_{\kappa+1}+\cdots$ we can replace H in (5.6) by $H_\kappa+H_{\kappa+1}+\cdots$. Hence, only B_j with $j\le N-\kappa=n$ contribute in (5.6). In view of Proposition 3.16, $\mathcal{R}=a\cdot i_*\omega_0$, where a is smooth on $X_{p\text{-reg}}$ and $\omega_0\in\omega_X^{n-p}$. Since $i_*\omega\in\mathscr{H}\!om_{\mathscr{O}_D}(\Omega_D^p,\mathscr{CH}_D^Z)$, in view of (2.7) it follows that $K^\epsilon\varphi$ is a sum of terms of the form

$$\pi_{2*}\big(\chi^\epsilon B_j\wedge\phi(\zeta,\tau,z,w)\wedge\bar\partial\frac{d\tau}{\tau^{\beta+\mathbf{1}}}\big), \tag{5.8}$$

where $j\le n$ and ϕ is smooth with support in a neighborhood of $(\zeta,\tau)=x$. It is proved in [6, Proposition 10.5], cf. in particular [6, Equation (10.5)], that after applying $\partial^{|\alpha|}/\partial w^\alpha$ to a term (5.8) and evaluating at $w=0$ the limit as $\epsilon\to 0$ is smooth in z. The claim thus follows.

The proof of part (ii) of the theorem is similar. First notice that if $\mu\equiv 0$ in a neighborhood of x, then $i_*\check K\mu$ equals $\mathcal{R}$ times a smooth form in a neighborhood of x. Since $\mathcal{R}=a\cdot i_*\omega_0$, where a is smooth on $X_{p\text{-reg}}$, the second part follows in this case. We can thus assume that μ has support in a small neighborhood of x.

Let T be given by (5.5) with $\varphi=1$. As above it follows that $\chi^\epsilon T\to T$. Set

$$u^\epsilon:=\pi_{1*}\big(\chi^\epsilon\vartheta(B\wedge g\wedge H)\wedge i_*\mu(z)\big).$$

Then u^ϵ is smooth and it follows that

$$u^\epsilon \wedge \mathcal{R} \to i_* \check{K}\mu \tag{5.9}$$

as currents in D''. As in the proof of Lemma 3.12 we have

$$u^\epsilon(\zeta,\tau) = \sum_{|\beta|<M} \frac{\partial^{|\beta|} u^\epsilon}{\partial \tau^\beta}(\zeta,0)\frac{\tau^\beta}{\beta!} + \mathcal{O}(|\tau|^M, \bar\tau, d\bar\tau),$$

where $\mathcal{O}(|\tau|^M, \bar\tau, d\bar\tau)$ is a sum of terms which are either $\mathcal{O}(|\tau|^M)$ or divisible by some $\bar\tau_j$ or $d\bar\tau_j$. Since $\mathcal{O}(|\tau|^M, \bar\tau, d\bar\tau)\wedge\mathcal{R} = 0$, if there are $u_\beta(\zeta) \in \mathscr{E}_X^{0,*}$ such that $\partial^{|\beta|}u^\epsilon(\zeta,0)/\partial\tau^\beta \to u_\beta(\zeta)$ as current on Z, it follows as above that

$$u^\epsilon \wedge \mathcal{R} \to \sum_{|\beta|<M} u_\beta(\zeta)\tau^\beta \wedge i_*\omega_0/\beta!$$

as currents in D''. Thus, by (5.9), $i_*\check{K}\mu$ has the desired form. To see that there are such u_β, notice that if $i_*\mu = \sum_\ell \mu_\ell \wedge i_*\omega_\ell$ then u^ϵ is a sum of terms

$$\pi_{1*}\big(\chi^\epsilon B_j \wedge \phi(\zeta,\tau,z,w)\wedge\bar\partial\frac{dw}{w^{\alpha+\mathbf{1}}}\big),$$

where $j \le n$ and ϕ is smooth, cf. (5.8) and the preceding argument. The existence of such u_β thus follows as above by [6]. □

The following lemma will be useful in the next section. The corresponding result, [6, Lemma 9.5], is formulated in terms of a λ-regularization of R whereas we here use an ϵ-regularization. However, in view of [20, Lemma 6], the proof of [6, Lemma 9.5] goes through in our case.

Lemma 5.5. *Let* $R^\epsilon := \bar\partial\chi(|F|^2/\epsilon)\wedge u$, *cf.* (2.12), *and let* $\mathcal{R}^\epsilon := R^\epsilon \otimes d\zeta$. *Then*

$$\lim_{\epsilon\to 0} \mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}^\epsilon = \mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R},$$

where the right-hand side is the product of the almost semi-meromorphic current $\vartheta(B\wedge g\wedge H)$ *by the tensor product* $\mathcal{R}(z)\wedge\mathcal{R}$, *cf. Lemma 5.2.*

6. Koppelman formulas and the sheaves $\mathscr{A}_X^{p,*}$ and $\mathscr{B}_X^{n-p,*}$

We assume now that $i\colon X \to D \subset \mathbb{C}^N$ is a local embedding into a pseudoconvex open set. Let z and ζ be two sets of the same local coordinates in D and let B be the corresponding Bochner–Martinelli form. We choose g and H in the definition of the integral operators of Section 5 to be a *weight*, in the sense of [1, Section 2], and a *Hefer morphism*, in the sense of [8, Section 5] and [1, Proposition 5.3], respectively.

Example 6.1 (Example 2 in [1])**.** Let $D' \Subset D$ and assume that $\overline{D}'$ is holomorphically convex. Let χ be a cutoff function in D such that $\chi = 1$ in a neighborhood of $\overline{D}'$. One can find a smooth $(1,0)$-form $s(\zeta,z) = \sum_j s_j(\zeta,z)d(\zeta_j - z_j)$, defined for ζ in a neighborhood of $\operatorname{supp}\bar\partial\chi$ and z in a neighborhood of $\overline{D}'$, such that $2\pi i\sum_j(\zeta_j - z_j)s_j(\zeta,z) = 1$ and $z \mapsto s(\zeta,z)$ is holomorphic. Then

$$g = \chi(\zeta) - \bar\partial\chi(\zeta)\wedge\sum_{k=1}^N s(\zeta,z)\wedge(\bar\partial s(\zeta,z))^{k-1}$$

is a weight with compact support in D_ζ, depends holomorphically on z in a neighborhood of $\overline{D}'$, and contains no $d\bar z_j$.

If D' is the unit ball we can take $s(\zeta,z)=\sum_j(\bar\zeta_j-\bar z_j)d(\zeta_j-z_j)/2\pi i(|\zeta|^2-z\cdot\bar\zeta)$

Let ω be an $(n-p)$-structure form on X and recall, see (3.21), that $i_*\omega=\mathcal{R}$ for some $\mathcal{R}$ associated to a resolution (2.10). Then by Proposition 3.16, $i_*\omega=a\cdot i_*\omega_0$ for some tuple ω_0 of elements in ω_X^{n-p} and a matrix of almost semi-meromorphic currents a which is smooth on $X_{p\text{-reg}}$. In view of Lemma 4.5 it follows that if $\varphi\in\mathcal{V}_X^{p,*}$, then $\varphi\wedge\omega$ is well-defined in $\mathcal{W}_X^{n,*}$.

Definition 6.2. If $\varphi\in\mathcal{V}_X^{p,*}$ we say that $\varphi\in\operatorname{Dom}\bar\partial_X$ if $\bar\partial(\varphi\wedge\omega)\in\mathcal{W}_X^{n,*}$ for any $(n-p)$-structure form ω on X.

Let us notice a few consequences. We can define $\bar\partial\colon\operatorname{Dom}\bar\partial_X\to\mathcal{V}_X^{p,*}$ as follows. Let $\mu\in\omega_X^{n-p}$. In view of (3.22), since the map (3.7) is an isomorphism, there is a current $\mathcal{R}$ and a holomorphic E^*-valued function ξ such that $i_*\mu=\xi\cdot\mathcal{R}$. Thus, by (3.21) there is an $(n-p)$-structure form ω such that $\mu=i^*\xi\cdot\omega$. If $\varphi\in\operatorname{Dom}\bar\partial_X$ it follows that $\bar\partial(\varphi\wedge\mu)\in\mathcal{W}_X^{n,*}$. Hence, for $\varphi\in\operatorname{Dom}\bar\partial_X$ we can define $\bar\partial\varphi\in\mathcal{V}_X^{p,*}$ by

$$\bar\partial\varphi\wedge\mu:=\bar\partial(\varphi\wedge\mu),\quad \mu\in\omega_X^{n-p}.$$

Since $\bar\partial\varphi\in\mathcal{V}_X^{p,*}$ if $\varphi\in\operatorname{Dom}\bar\partial_X$ it follows as in the paragraph preceding Definition 6.2 that $\bar\partial\varphi\wedge\omega$ is well-defined in $\mathcal{W}_X^{n,*}$ for any $(n-p)$-structure form ω. Moreover, if as above $i_*\omega=\mathcal{R}=a\cdot i_*\omega_0$, where $\mathcal{R}$ is associated to the resolution (2.10), then

$$\bar\partial\varphi\wedge\omega=-\nabla_f(\varphi\wedge\omega), \tag{6.1}$$

where $\nabla_f=f-\bar\partial$. In fact, by Lemma 3.6, $fa\cdot i_*\omega_0=\bar\partial(a\cdot i_*\omega_0)$ and so, since a is smooth on $X_{p\text{-reg}}$, in view of Lemma 4.5, we get

$$\begin{aligned}-\nabla_f(\varphi\wedge\omega)&=\bar\partial(\varphi\wedge i^*a\cdot\omega_0)-f(\varphi\wedge i^*a\cdot\omega_0)\\&=\pm i^*a\cdot\bar\partial(\varphi\wedge\omega_0)\pm\varphi\wedge\bar\partial(i^*a\cdot\omega_0)\mp\varphi\wedge fi^*a\cdot\omega_0\\&=\pm i^*a\cdot\bar\partial(\varphi\wedge\omega_0)=\bar\partial\varphi\wedge i^*a\cdot\omega_0=\bar\partial\varphi\wedge\omega\end{aligned}$$

on $X_{p\text{-reg}}$. Since both sides of (6.1) have the SEP, (6.1) holds everywhere.

We also notice that

$$\mathscr{E}_X^{p,*}\subset\operatorname{Dom}\bar\partial_X. \tag{6.2}$$

This follows since, as above, any $(n-p)$-structure form ω satisfies $\bar\partial\omega=f\omega$ for an appropriate f and hence, if $\varphi\in\mathscr{E}_X^{p,*}$, $\bar\partial(\varphi\wedge\omega)=\bar\partial\varphi\wedge\omega\pm\varphi\wedge f\omega\in\mathcal{W}_X^{n,*}$.

Proposition 6.3. *Let $D'\Subset D$ be a relatively compact open subset and set $X'=X\cap D'$. There are integral operators*

$$K\colon\mathscr{E}^{p,*+1}(X)\to\mathcal{V}^{p,*}(X')\cap\mathit{Dom}\,\bar\partial_X,\quad P\colon\mathscr{E}^{p,*}(X)\to\mathscr{E}^{p,*}(X')$$

such that for any $\varphi\in\mathscr{E}^{p,*+1}(X)$,

$$\varphi=\bar\partial K\varphi+K\bar\partial\varphi+P\varphi. \tag{6.3}$$

If $\varphi\in\mathscr{E}^{p,*+1}(X)$ has compact support in X one can choose K and P such that, additionally, $K\varphi$ and $P\varphi$ have compact support in X.

Proposition 6.4. *Let $D'\Subset D$ be a relatively compact open subset and set $X'=X\cap D'$. There are integral operators*

$$\check K\colon\mathcal{W}^{n-p,*+1}(X)\to\mathcal{W}^{n-p,*}(X'),\quad \check P\colon\mathcal{W}^{n-p,*}(X)\to\mathcal{W}^{n-p,*}(X')$$

such that if $i_*\mu=\sum_\ell\mu_\ell\wedge i_*\omega_\ell$ for some $\mu_\ell\in\mathscr{E}_D^{0,*}$ and $\omega_\ell\in\omega_X^{n-p}$, then

$$\mu=\bar\partial\check K\mu+\check K\bar\partial\mu+\check P\mu. \tag{6.4}$$

If μ in addition has compact support in X one can choose $\check{K}$ and $\check{P}$ such that $\check{K}\varphi$ and $\check{P}\varphi$ have compact support in X.

Proof of Propositions 6.3 and 6.4. Let $\mathcal{R}^\epsilon$ be as in Lemma 5.5. In the same way as in [24, Section 5], cf. also [7, Section 5] and [6, Eq. (9.16)], one obtains

$$\nabla_{f(z)}\big(\mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}^\epsilon\big)=\mathcal{R}\wedge[\Delta]-\mathcal{R}(z)\wedge\vartheta(g\wedge H)\mathcal{R}^\epsilon, \tag{6.5}$$

where $\nabla_{f(z)}=f(z)-\bar\partial$. Notice that, for $\epsilon>0$, all current products are well-defined as tensor products. Letting $\epsilon\to 0$ we get, by Lemma 5.5,

$$\nabla_{f(z)}\big(\mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}\big)=\mathcal{R}\wedge[\Delta]-\mathcal{R}(z)\wedge\vartheta(g\wedge H)\mathcal{R}. \tag{6.6}$$

To show the first statement of Proposition 6.3, choose g such that $\zeta\mapsto g(\zeta,z)$ has support in a fixed compact subset, containing D', of D for all $z\in D'$. Multiplying (6.6) by a $\tilde\varphi(\zeta)\in\mathscr{E}^{p,*+1}(D)$ such that $i^*\tilde\varphi=\varphi$ and applying π_{2*} we get

$$\nabla_f(\mathcal{R}\wedge i_*K\varphi)+\mathcal{R}\wedge i_*K(\bar\partial\varphi)=\mathcal{R}\wedge\tilde\varphi-\mathcal{R}\wedge i_*P\varphi,$$

i.e.,

$$\nabla_f(\omega\wedge K\varphi)+\omega\wedge K(\bar\partial\varphi)=\omega\wedge\varphi-\omega\wedge P\varphi. \tag{6.7}$$

In view of Definitions 5.1 and 5.3 all terms except $\nabla_f(\omega\wedge K\varphi)$ are in $\mathcal{W}_X^{n,*}$ and consequently $\nabla_f(\omega\wedge K\varphi)$ is too. Hence, since $f(\omega\wedge K\varphi)\in\mathcal{W}_X^{n,*}$ also $\bar\partial(\omega\wedge K\varphi)\in\mathcal{W}_X^{n,*}$, and so $K\varphi\in\operatorname{Dom}\bar\partial_X$. Thus, by (6.1), we can replace $\nabla_f(\omega\wedge K\varphi)$ in (6.7) by $-\omega\wedge\bar\partial K\varphi$. Multiplying the resulting equality by holomorphic E^*-valued ξ such that $f^*\xi=0$ we get, since the map (3.7) is an isomorphism,

$$\mu\wedge\varphi=\mu\wedge\bar\partial K\varphi+\mu\wedge K\bar\partial\varphi+\mu\wedge P\varphi,\quad\forall\mu\in\omega_X^{n-p}, \tag{6.8}$$

which is what (6.3) means.

If φ has compact support we can take a weight g such that $z\mapsto g(\zeta,z)$ has compact support. The preceding argument goes through unchanged and it is clear that $K\varphi$ and $P\varphi$ have compact support.

To show Proposition 6.4, let ξ_ℓ be holomorphic f^*-closed sections of E^* such that $i_*\omega_\ell=\xi_\ell\cdot\mathcal{R}$, so that $i_*\mu=\sum_\ell\mu_\ell\wedge\xi_\ell\cdot\mathcal{R}$. Since $\nabla_f\mathcal{R}=0$ and $\bar\partial\xi_\ell=0$ a simple computations gives

$$\begin{aligned}\mu_\ell\wedge\xi_\ell\cdot\nabla_{f(z)}\big(\mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}^\epsilon\big)\;&=\;\bar\partial\big(\mu_\ell\wedge\xi_\ell\cdot\mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}^\epsilon\big)\\&\quad+\bar\partial\mu_\ell\wedge\xi_\ell\cdot\mathcal{R}(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}^\epsilon.\end{aligned}$$

Hence, in view of Lemma 5.5, multiplying (6.5) by $\sum_\ell\mu_\ell\wedge\xi_\ell$ and letting $\epsilon\to 0$ we obtain

$$\bar\partial\big(i_*\mu(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}\big)+\bar\partial i_*\mu(z)\wedge\vartheta(B\wedge g\wedge H)\mathcal{R}=i_*\mu\wedge[\Delta]-i_*\mu\wedge\vartheta(g\wedge H)\mathcal{R}.$$

If g is chosen so that $z\mapsto g(\zeta,z)$ has support in a fixed compact for all $\zeta\in D'$, then (6.4) follows by applying π_{1*}. If μ has compact support we instead choose g such that $\zeta\mapsto g(\zeta,z)$ has compact support and apply π_{1*}. □

Definition 6.5. If $\mathcal{U}\subset X$ is open and $\varphi\in\mathcal{V}^{p,q}(\mathcal{U})$ we say that φ is a section of $\mathscr{A}_X^{p,q}$ over $\mathcal{U}$, $\varphi\in\mathscr{A}^{p,q}(\mathcal{U})$, if for every $x\in\mathcal{U}$ the germ φ_x can be written as a finite sum of terms

$$\xi_\nu\wedge K_\nu(\cdots\xi_2\wedge K_2(\xi_1\wedge K_1(\xi_0))\cdots), \tag{6.9}$$

where $\nu\geq 0$, $\xi_0\in\mathscr{E}_X^{p,*}$, $\xi_j\in\mathscr{E}_X^{0,*}$ for $j\geq 1$, K_j are integral operators as defined in Section 5, and ξ_j has compact support in the set where $z\mapsto K_j(\zeta,z)$ is defined.

Definition 6.6. If $\mathcal{U}\subset X$ is open and $\mu\in\mathcal{W}^{n-p,q}(\mathcal{U})$ we say that μ is a section of $\mathscr{B}_X^{n-p,q}$ over $\mathcal{U}$, $\mu\in\mathscr{B}^{n-p,q}(\mathcal{U})$, if for every $x\in\mathcal{U}$ the germ μ_x can be written as a finite sum of terms

$$\xi_\nu\wedge\check{K}_\nu(\cdots\xi_2\wedge\check{K}_2(\xi_1\wedge\check{K}_1(\xi_0\wedge\omega))\cdots), \tag{6.10}$$

where $\nu\geq 0$, ω is an $(n-p)$-structure form, $\xi_j\in\mathscr{E}_X^{0,*}$, $\check{K}_j$ are integral operators as defined in Section 5, ξ_j has compact support in the set where $\zeta\mapsto\check{K}_j(\zeta,z)$ is defined, and ξ_0 takes values in E^*.

Proposition 6.7. *The sheaf $\mathscr{A}_X^{p,*}$ has the following properties.*

- (a1) *It is a module over $\mathscr{E}_X^{0,*}$,*
- (a2) *if K is an integral operator as defined in Section 5 then $K\colon\mathscr{A}_X^{p,*+1}\to\mathscr{A}_X^{p,*}$,*
- (a3) *$\mathscr{A}_X^{p,*}\subset\text{Dom}\,\bar\partial_X$,*
- (a4) *$\bar\partial\colon\mathscr{A}_X^{p,*}\to\mathscr{A}_X^{p,*+1}$,*
- (a5) *$\mathscr{A}_X^{p,*}=\mathscr{E}_X^{p,*}$ on $X_{p\text{-}reg}$,*
- (a6) *(6.3) holds for $\varphi\in\mathscr{A}_X^{p,*}$.*

The sheaf $\mathscr{B}_X^{n-p,*}$ has the following properties.

- (b1) *It is a module over $\mathscr{E}_X^{0,*}$,*
- (b2) *if $\check{K}$ is an integral operator as defined in Section 5 then $\check{K}\colon\mathscr{B}_X^{n-p,*+1}\to\mathscr{B}_X^{n-p,*}$,*
- (b3) *$\bar\partial\colon\mathscr{B}_X^{n-p,*}\to\mathscr{B}_X^{n-p,*+1}$,*
- (b4) *if $\mu\in\mathscr{B}_X^{n-p,*}$ then on $X_{p\text{-}reg}$, $\mu=\sum_\ell\mu_\ell\wedge\omega_\ell$ for some $\mu_\ell\in\mathscr{E}_X^{0,*}$ and $\omega_\ell\in\omega_X^{n-p}$,*
- (b5) *(6.4) holds for $\mu\in\mathscr{B}_X^{n-p,*}$.*

To prove this proposition we need the following two lemmas. The first one is a variant of Propositions 6.3 and 6.4.

Lemma 6.8. *Let $\varphi\in\mathcal{V}^{p,*}(X)$. Assume that $\varphi,K\varphi\in\text{Dom}\,\bar\partial_X$ and that $\varphi\in\mathscr{E}_X^{p,*}$ on $X_{p\text{-}reg}$. Then (6.3) holds on X'. If in addition φ has compact support, then K and P can be chosen such that $K\varphi$ and $P\varphi$ have compact support.*

Let $\mu\in\mathcal{W}^{n-p,*}(X)$. Assume that $\bar\partial\mu,\bar\partial\check{K}\mu\in\mathcal{W}_X^{n-p,*}$ and that $i_*\mu=\sum_\ell\mu_\ell\wedge\omega_\ell$ on $X_{p\text{-}reg}$ for some $\mu_\ell\in\mathscr{E}_D^{0,*}$ and $\omega_\ell\in\omega_X^{n-p,*}$. Then (6.4) holds on X'. If μ in addition has compact support, then $\check{K}$ and $\check{P}$ can be chosen such that $\check{K}\mu$ and $\check{P}\mu$ have compact support.

Proof. Let h be a holomorphic tuple vanishing precisely on $X_{p\text{-sing}}$ and set $\chi^\epsilon=\chi(|h|^2/\epsilon)$. Then Proposition 6.3 applies to $\chi^\epsilon\varphi$ and hence

$$\chi^\epsilon\varphi=\bar\partial K(\chi^\epsilon\varphi)+K(\chi^\epsilon\bar\partial\varphi)+K(\bar\partial\chi^\epsilon\wedge\varphi)+P(\chi^\epsilon\varphi);$$

recall that this means that (6.8), with φ replaced by $\chi^\epsilon\varphi$, holds. Since $\varphi\in\mathcal{V}_X^{p,*}$ it follows that $\chi^\epsilon\varphi\to\varphi$, i.e., $\chi^\epsilon\varphi\wedge\mu\to\varphi\wedge\mu$ for all $\mu\in\omega_X^{n-p}$. By Lemma 5.2 the current (5.4) is in has the SEP with respect to $Z\times Z$ and therefore $K(\chi^\epsilon\varphi)\to K\varphi$. Similarly, $P(\chi^\epsilon\varphi)\to P\varphi$. Moreover, $\bar\partial\varphi\in\mathcal{V}_X^{p,*}$ since $\varphi\in\text{Dom}\,\bar\partial_X$ and so $K(\chi^\epsilon\bar\partial\varphi)\to K(\bar\partial\varphi)$. We claim that $\lim_{\epsilon\to0}K(\bar\partial\chi^\epsilon\wedge\varphi)=0$ on $X_{p\text{-reg}}$. Given the claim it follows that (6.3) holds on $X_{p\text{-reg}}$. Since $K\varphi\in\text{Dom}\,\bar\partial_X$ by assumption it follows by the SEP that (6.3) holds.

To show that claim we may assume that z is in a fixed compact subset of $X_{p\text{-reg}}$. Then $B \wedge \bar\partial\chi^\epsilon(\zeta)$ is smooth if ϵ is small enough. It thus suffices to show that

$$\mathcal{R}(\zeta) \wedge \bar\partial\chi^\epsilon(\zeta) \wedge \varphi(\zeta) \wedge i_*\mu(z) \to 0, \quad \forall \mu \in \omega_X^{n-p}.$$

Since this is a tensor product it suffices to see that $\omega \wedge \bar\partial\chi^\epsilon \wedge \varphi \to 0$. However, $\varphi \wedge \omega \in \mathcal{W}_X^{n,*}$ and, since $\varphi \in \operatorname{Dom}\bar\partial_X$, $\nabla_f(\varphi \wedge \omega) \in \mathcal{W}_X^{n,*}$. In view of (6.1) thus

$$\begin{aligned} \bar\partial\chi^\epsilon \wedge \varphi \wedge \omega &= \bar\partial(\chi^\epsilon\varphi) \wedge \omega - \chi^\epsilon\bar\partial\varphi \wedge \omega = -\nabla_f(\chi^\epsilon\varphi \wedge \omega) + \chi^\epsilon\nabla_f(\varphi \wedge \omega) \\ &\to -\nabla_f(\varphi \wedge \omega) + \nabla_f(\varphi \wedge \omega) = 0. \end{aligned}$$

The proof of the second part of the lemma is similar: By Proposition 6.4,

$$\chi^\epsilon\mu = \bar\partial\check{K}(\chi^\epsilon\mu) + \check{K}(\chi^\epsilon\bar\partial\mu) + \check{K}(\bar\partial\chi^\epsilon \wedge \mu) + \check{P}(\chi^\epsilon\mu).$$

Since $\mu \in \mathcal{W}_X^{n-p,*}$ we have $\chi^\epsilon\mu \to \mu$. By Lemma 5.2 the current (5.4) has the SEP with respect to $Z \times Z$ and therefore $\check{K}(\chi^\epsilon\mu) \to \check{K}\mu$. Similarly, $\check{P}(\chi^\epsilon\mu) \to \check{P}\mu$ and, since $\bar\partial\mu \in \mathcal{W}_X^{n-p,*}$, $\check{K}(\chi^\epsilon\bar\partial\mu) \to \check{K}\bar\partial\mu$. Hence, (6.4) holds modulo $\tau := \lim_{\epsilon\to 0}\check{K}(\bar\partial\chi^\epsilon \wedge \mu)$. Since $\bar\partial\check{K}\mu \in \mathcal{W}_X^{p,*}$ by assumption, all terms in (6.4) are in $\mathcal{W}_X^{n-p,*}$ and so (6.4) follows by the SEP if $\tau = 0$ on $X_{p\text{-reg}}$. For ζ in a fixed compact subset of $X_{p\text{-reg}}$, $B \wedge \bar\partial\chi^\epsilon(z)$ is smooth if ϵ is small enough. Thus, as above, to see that $\tau = 0$ on $X_{p\text{-reg}}$ it suffices to see that $\mathcal{R}(\zeta)\bar\partial\chi^\epsilon(z) \wedge \mu(z) \to 0$, which follows if $\bar\partial\chi^\epsilon \wedge \mu \to 0$. However, since $\bar\partial\mu \in \mathcal{W}_X^{n-p,*}$ by assumption, we have

$$\bar\partial\chi^\epsilon \wedge \mu = \bar\partial(\chi^\epsilon\mu) - \chi^\epsilon\bar\partial\mu \to \bar\partial\mu - \bar\partial\mu = 0.$$

□

The second lemma we need is (a version of) the crucial Lemma 6.2 in [7]. The proof of that lemma goes through in our case; cf. also the proof of [24, Lemma 5.3]. We remark that in these cited lemmas the statements and proofs are intrinsic on (Cartesian products of) X whereas we here formulate our version in (Cartesian products of) D. Let

$$k(\zeta, z) = \vartheta\big(B(\zeta, z) \wedge g(\zeta, z) \wedge H(\zeta, z)\big)\mathcal{R}(\zeta).$$

Let z^j be coordinates on the jth factor of D in $D \times \cdots \times D$. The current

$$T := \mathcal{R}(z^\nu) \wedge k(z^{\nu-1}, z^\nu) \wedge \cdots \wedge k(z^1, z^2) \tag{6.11}$$

is well-defined in $\mathcal{PM}_{D\times\cdots\times D}$, has support in $Z \times \cdots \times Z$ and the SEP with respect to $Z \times \cdots \times Z$ since it is the product of an almost semi-meromorphic current by the tensor product of the $\mathcal{R}$-factors, cf. Lemma 5.2. The different k-factors may correspond to different choices of B, g, H, and $\mathcal{R}$.

Lemma 6.9. *Let h be a holomorphic tuple which is generically non-vanishing on Z. Then*

$$\lim_{\epsilon\to 0}\bar\partial\chi(|h(z^j)|^2/\epsilon) \wedge T = 0.$$

Proof of Proposition 6.7. It is clear from the definition that $\mathscr{A}_X^{p,*}$ and $\mathscr{B}_X^{n-p,*}$ are modules over $\mathscr{E}_X^{0,*}$ and that $\mathscr{A}_X^{p,*}$ and $\mathscr{B}_X^{n-p,*}$ are closed under K-operators and $\check{K}$-operators, respectively. By Theorem 5.4 it follows that $\mathscr{A}_X^{p,*} = \mathscr{E}_X^{p,*}$ on $X_{p\text{-reg}}$ and that sections of $\mathscr{B}_X^{n-p,*}$ are of the claimed form on $X_{p\text{-reg}}$.

To show that $\mathscr{A}_X^{p,*} \subset \operatorname{Dom}\bar\partial_X$ assume that φ is given by (6.9), where the ξ_j are smooth, and let ω be a structure form. Then $i_*\omega = \mathcal{R}$ for some $\mathcal{R}$ and $i_*(\omega \wedge \varphi) = \pi_{\nu*}(T \wedge \xi)$, where T is given by (6.11), ξ is some smooth form in $D \times \cdots \times D$, and $\pi_\nu\colon D\times\cdots\times D\to D$ is the natural projection on the factor with coordinates z^ν. Let h be as in Lemma 6.9 and set $\chi^\epsilon=\chi(|h|^2/\epsilon)$. By Lemma 6.9 we have $\lim_{\epsilon\to 0}\bar\partial\chi^\epsilon\wedge\pi_{\nu *}T\wedge\xi=0$ and so, since $i_*(\omega\wedge\varphi)$ has the SEP with respect to Z, we get

$$
\begin{aligned}
(6.12)\quad \bar\partial i_*(\omega\wedge\varphi) &= \lim_{\epsilon\to 0}\bar\partial(\chi^\epsilon i_*(\omega\wedge\varphi))=\lim_{\epsilon\to 0}\bar\partial\chi^\epsilon\wedge\pi_{\nu *}T\wedge\xi+\lim_{\epsilon\to 0}\chi^\epsilon\bar\partial i_*(\omega\wedge\varphi)\\
&= \lim_{\epsilon\to 0}\chi^\epsilon\bar\partial i_*(\omega\wedge\varphi).
\end{aligned}
$$

Hence, $\bar\partial i_*(\omega\wedge\varphi)$ has the SEP with respect to Z and it follows that $\bar\partial(\omega\wedge\varphi)\in\mathcal{W}_X^{n,*}$.

In a similar way we show that if $\mu\in\mathscr{B}_X^{n-p,*}$, then $\bar\partial\mu\in\mathcal{W}_X^{n-p,*}$. Assume that μ is given by (6.10). Then $i_*\mu=\pi_{\nu *}T\wedge\xi$ for some smooth ξ. Replacing $\omega\wedge\varphi$ by μ in (6.12) it follows that $\bar\partial\mu\in\mathcal{W}_X^{n-p,*}$.

Since $\mathscr{A}_X^{p,*}$ is closed under K-operators, $\mathscr{A}_X^{p,*}\subset\operatorname{Dom}\bar\partial_X$, and $\mathscr{A}_X^{p,*}=\mathscr{E}_X^{p,*}$ on $X_{p\text{-reg}}$ the Koppelman formula (6.3) follows for sections of $\mathscr{A}_X^{p,*}$ by Lemma 6.8. Similarly, since $\mathscr{B}_X^{n-p,*}$ is closed under $\check K$-operators, $\bar\partial\mathscr{B}_X^{n-p,*}\subset\mathcal{W}_X^{n-p,*}$, and $\mu=\sum_\ell\mu_\ell\wedge\omega_\ell$ on $X_{p\text{-reg}}$ for any $\mu\in\mathscr{B}_X^{n-p,*}$ it follows from Lemma 6.8 that the Koppelman formula (6.4) holds for sections of $\mathscr{B}_X^{n-p,*}$.

It remains to see that $\mathscr{A}_X^{p,*}$ and $\mathscr{B}_X^{n-p,*}$ are closed under $\bar\partial$. Let $\varphi\in\mathscr{A}_X^{p,*}$ and assume that φ is given by (6.9). We show by induction over ν that $\bar\partial\varphi\in\mathscr{A}_X^{p,*}$. If $\nu=0$, then $\varphi=\xi_0\in\mathscr{E}_X^{p,*}$ and so $\bar\partial\varphi\in\mathscr{E}_X^{p,*}\subset\mathscr{A}_X^{p,*}$. If $\nu\geq 1$ we write $\varphi=\xi_\nu\wedge K\phi$, where ϕ is given by (6.9) with ν replaced by $\nu-1$. By the induction hypothesis, $\bar\partial\phi\in\mathscr{A}_X^{p,*}$. Hence, $K\bar\partial\phi\in\mathscr{A}_X^{p,*}$. Since the Koppelman formula (6.3), with φ replaced by ϕ, holds and since $P\phi\in\mathscr{E}_X^{p,*}$ it follows that $\bar\partial K\phi\in\mathscr{A}_X^{p,*}$. Hence, $\bar\partial\varphi=\bar\partial\xi_\nu\wedge K\phi+\xi_\nu\wedge\bar\partial K\phi\in\mathscr{A}_X^{p,*}$.

If $\mu\in\mathscr{B}_X^{n-p,*}$ is given by (6.10) we proceed in a similar way by induction over ν. If $\nu=0$ then $\mu=\xi_0\wedge\omega$. Then, by the computation showing (6.2), it follows that $\bar\partial\mu$ has the same form. If $\nu\geq 1$ we write $\mu=\xi_\nu\wedge\check K\mu'$ and the induction hypothesis gives $\bar\partial\mu'\in\mathscr{B}_X^{n-p,*}$. As before, since $\check P\mu'=\xi\wedge\omega$ for some smooth ξ, cf. Section 5, it follows from (6.4), with μ replaced by μ', that $\bar\partial\mu\in\mathscr{B}_X^{n-p,*}$. □

Proof of Theorem A. In view of Proposition 6.7 it only remains to show that (1.4) is a resolution of Ω_X^p. This is a local statement so we may assume that X is an analytic subspace of a pseudoconvex domain $D\subset\mathbb{C}^N$ and that the point in which we want to show that (1.4) is exact is 0. Let $\varphi\in\mathscr{A}^{p,q}(\mathcal{U}\cap X)$ be $\bar\partial$-closed, where $\mathcal{U}$ is a neighborhood of 0. Choose operators K and P corresponding to a choice of weight g such that $z\mapsto g(\zeta,z)$ is holomorphic in some neighborhood of 0 and $\zeta\mapsto g(\zeta,z)$ has support in a fixed compact subset of $\mathcal{U}$. Then $\vartheta(g\wedge H)\mathcal{R}(\zeta)$ has degree 0 in $d\bar z$. Since it has total bidegree (N,N) it must have full degree in $d\bar\zeta$. Hence, $P\varphi=0$ if $q\geq 1$. Since, by Proposition 6.7, the Koppelman formula (6.3) holds it follows that $\varphi=\bar\partial K\varphi$ if $q\geq 1$ and $\varphi=P\varphi$ if $q=0$. In the latter case, since $\vartheta(g\wedge H)$ is holomorphic in z, we get $\varphi\in\Omega_X^p$. □

Theorem 6.10. *The sheaf complex*

$$
(6.13)\qquad 0\to\omega_X^{n-p}\to\mathscr{B}_X^{n-p,0}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{B}_X^{n-p,n}\to 0
$$

is exact if and only if Ω_X^p is Cohen–Macaulay. In general $\mathscr{H}^q(\mathscr{B}_X^{n-p,\bullet},\bar\partial)\simeq\mathscr{E}xt_{\mathscr{O}_D}^{\kappa+q}(\Omega_X^p,\mathscr{O}_D)$.

Proof. Consider the free resolution (2.10) of Ω^p_X and let R and $\mathcal{R}$ be the associated currents. Let $(\mathscr{O}(E^*_\bullet), f^*_\bullet)$ be the dual complex of (2.10), cf. the proof of Proposition 3.9. It is well-known that

$$\mathscr{H}^{\kappa+q}(\mathscr{O}(E^*_\bullet), f^*_\bullet) \simeq \mathscr{E}xt^{\kappa+q}_{\mathscr{O}_D}(\Omega^p_X, \mathscr{O}_D). \tag{6.14}$$

Define the mapping

$$\varrho_\bullet \colon (\mathscr{O}(E^*_{\kappa+\bullet}), f^*_{\kappa+\bullet}) \to (\mathscr{B}^{n-p,\bullet}_X, \bar\partial), \quad i_*\varrho(\xi) = \xi \cdot \mathcal{R}_{\kappa+\bullet}.$$

Since $f\mathcal{R} = \bar\partial\mathcal{R}$ it follows that $\varrho_\bullet$ is a map of complexes and hence induces a map on cohomology. As in the proof of [24, Theorem 1.7], cf. also the proof of [23, Theorem 1.2], one shows that the map on cohomology is an isomorphism. Hence, the last statement of the theorem follows.

If Ω^p_X is Cohen–Macaulay we can choose the free resolution (2.10) of length κ. Thus, by (6.14), $\mathscr{E}xt^{\kappa+q}_{\mathscr{O}_D}(\Omega^p_X, \mathscr{O}_D) = 0$ for $q \geq 1$ and so $\mathscr{H}^q(\mathscr{B}^{n-p,\bullet}_X, \bar\partial) = 0$ for $q \geq 1$. Hence (6.13) is exact. Conversely, if (6.13) is exact then $\mathscr{E}xt^{\kappa+q}_{\mathscr{O}_D}(\Omega^p_X, \mathscr{O}_D) = 0$ for $q \geq 1$. Recall the singularity subvarieties $Z_k := Z^{\Omega^p_X}_k$ associated with Ω^p_X, cf. (2.11). From, e.g., the proof of [12, Theorem II.2.1] it follows that

$$Z_{\kappa+q} = \bigcup_{r \geq \kappa+q} \operatorname{supp} \mathscr{E}xt^r_{\mathscr{O}_D}(\Omega^p_X, \mathscr{O}_D).$$

Hence, $Z_{\kappa+q} = \emptyset$ for $q \geq 1$. It follows that $\operatorname{Im} f_{\kappa+1} \subset E_\kappa$ is a subbundle. Replacing E_κ by $E_\kappa/\operatorname{Im} f_{\kappa+1}$ and $E_{\kappa+q}$, $q \geq 1$, by 0 in (2.10) we obtain a free resolution of Ω^p_X of length κ. Thus, Ω^p_X is Cohen–Macaulay. □

7. Serre duality

In this section X is a pure n-dimensional analytic space. When considering local problems we tacitly assume that X is an analytic subset of some pseudoconvex domain $D \subset \mathbb{C}^N$.

Let $\varphi \in \mathscr{A}^{p,*}_X$ and $\mu \in \mathscr{B}^{n-p,*}_X$. By Proposition 6.7, on $X_{p\text{-reg}}$ φ is smooth and $\mu = \sum_\ell \mu_\ell \wedge \omega_\ell$, where μ_ℓ are smooth and $\omega_\ell \in \omega^{n-p}_X$. Hence, $\varphi \wedge \mu$ is well-defined in $\mathcal{W}^{n,*}_X$ on $X_{p\text{-reg}}$.

Theorem 7.1. *There is a unique map* $\wedge \colon \mathscr{A}^{p,*}_X \times \mathscr{B}^{n-p,*}_X \to \mathcal{W}^{n,*}_X$ *extending the wedge product on* $X_{p\text{-reg}}$. *If* $\varphi \in \mathscr{A}^{p,*}_X$ *and* $\mu \in \mathscr{B}^{n-p,*}_X$, *then* $\bar\partial(\varphi \wedge \mu) \in \mathcal{W}^{n,*}_X$ *and*

$$\bar\partial(\varphi \wedge \mu) = \bar\partial\varphi \wedge \mu + (-1)^{\deg\varphi}\varphi \wedge \bar\partial\mu. \tag{7.1}$$

Proof. This is a local statement. The uniqueness is clear by the SEP. Moreover, if $\varphi \wedge \mu \in \mathcal{W}^{n,*}_X$ and $\bar\partial(\varphi \wedge \mu) \in \mathcal{W}^{n,*}_X$ for all $\varphi \in \mathscr{A}^{p,*}_X$ and $\mu \in \mathscr{B}^{n-p,*}_X$, then (7.1) follows since it holds on $X_{p\text{-reg}}$ and both the left-hand side and the right-hand side have the SEP.

To show that $\varphi \wedge \mu \in \mathcal{W}^{n,*}_X$ and $\bar\partial(\varphi \wedge \mu) \in \mathcal{W}^{n,*}_X$ we represent φ and μ by (6.9) and (6.10), respectively. The case when $\nu = 0$ in (6.9) needs to be handled separately. In this case $\varphi \in \mathscr{E}^{p,*}_X$ and so clearly $\varphi \wedge \mu \in \mathcal{W}^{n,*}_X$. Moreover, since by Proposition 6.7, $\bar\partial\mu \in \mathscr{B}^{n-p,*}_X$ it follows that $\bar\partial(\varphi \wedge \mu) \in \mathcal{W}^{n,*}_X$.

Assume now that $\nu > 0$ in (6.9). Then, cf. (6.11),

$$i_*\varphi(\zeta) = \pi_*\big(k(w^\nu, \zeta) \wedge k(w^{\nu-1}, w^\nu) \wedge \cdots \wedge k(w^1, w^2) \wedge \xi\big) =: \pi_* T_\varphi,$$

where $\pi\colon D_\zeta\times D_{w^\nu}\times\cdots\times D_{w^1}\to D_\zeta$ is the natural projection, $\nu\geq 1$, and ξ is smooth. Hence, $\varphi=K\phi$ for an appropriate $\phi(w^\nu)\in\mathscr{A}_X^{p,*}$ and so, in view of Section 5, since $\mu\in\mathscr{B}_X^{n-p,*}\subset\mathcal{W}_X^{n-p,*}$ we have

$$i_*\mu\wedge\varphi=i_*\mu\wedge K\phi=\pi_*\big(\mu(\zeta)\wedge T_\varphi\big).$$

On $X_{p\text{-reg}}$, where φ is smooth, this coincides with the natural wedge product, cf. the proof of Theorem 5.4. In view of Definition 6.6 we may assume that

$$i_*\mu(\zeta)=\tilde{\pi}_*\big(\mathcal{R}(z^{\tilde{\nu}})\wedge k(z^{\tilde{\nu}-1},z^{\tilde{\nu}})\wedge\cdots\wedge k(\zeta,z^1)\wedge\tilde{\xi}\big)=:\tilde{\pi}_*T_\mu,$$

where $\tilde{\pi}\colon D_\zeta\times D_{z^{\tilde{\nu}}}\times\cdots\times D_{z^1}\to D_\zeta$ is the natural projection and $\tilde{\xi}=\tilde{\xi}(\zeta,z)$ is smooth. Hence,

$$i_*\mu\wedge\varphi=\pi_*\tilde{\pi}_*\big(T_\mu\wedge T_\varphi\big).$$

Since $T_\mu\wedge T_\varphi$ is of the form (6.11) (times $\xi\wedge\tilde{\xi}$) it follows that $\mu\wedge\varphi\in\mathcal{W}_X^{n,*}$. Moreover, by Lemma 6.9 and the computation (6.12), with ω and T replaced by μ and $T_\mu\wedge T_\varphi$, respectively, it follows that $\bar{\partial}(\mu\wedge\varphi)$ has the SEP. $\square$

Let $\mathscr{B}_c^{n-p,*}(X)$ be the vector space of sections of $\mathscr{B}_X^{n-p,*}$ with compact support. By Theorem 7.1 we have, cf. (3.19), a bilinear pairing

$$\mathscr{A}^{p,q}(X)\times\mathscr{B}_c^{n-p,n-q}(X)\to\mathbb{C}\quad(\varphi,\mu)\mapsto\int_X\varphi\wedge\mu, \tag{7.2}$$

which only depends on the class of μ in $H^{n-q}(\mathscr{B}_c^{n-p,\bullet}(X),\bar{\partial})$ and the class of φ in $H^q(\mathscr{A}^{p,q}(X),\bar{\partial})$. In particular, since $H^0(\mathscr{A}^{p,\bullet}(X),\bar{\partial})=\Omega^p(X)$ we have a pairing $\Omega_X^p(X)\times H^n(\mathscr{B}_c^{n-p,\bullet}(X),\bar{\partial})\to\mathbb{C}$.

Theorem 7.2. *Assume that X is an analytic subspace of a pseudoconvex domain $D\subset\mathbb{C}^N$. Then*

$$0\to\mathscr{B}_c^{n-p,0}(X)\xrightarrow{\bar{\partial}}\cdots\xrightarrow{\bar{\partial}}\mathscr{B}_c^{n-p,n}(X)\to 0$$

is exact except on level n. The pairing (7.2) *makes the cohomology group $H^n(\mathscr{B}_c^{n-p,\bullet}(X),\bar{\partial})$ the (topological) dual of $\Omega_X^p(X)$.*

Recall that the topology on $\Omega^p(X)=\Omega^p(D)/\mathcal{J}^p(D)$ is the quotient topology and that $\Omega^p(X)$ is a Fréchet space with this topology, see, e.g., [15, Ch. IX]. Notice that since convergence in $\Omega^p(D)$ is uniform convergence on compact subsets, a sequence $\varphi_\epsilon\in\Omega^p(X)$ converges to 0 if there are $\tilde{\varphi}_\epsilon\in\Omega^p(D)$ such that $\varphi_\epsilon=i^*\tilde{\varphi}_\epsilon$ and $\tilde{\varphi}_\epsilon\to 0$ uniformly on compacts. By the Cauchy estimates it follows that $\tilde{\varphi}_\epsilon\to 0$ in $\mathscr{E}^{p,0}(D)$.

Proof. Let $\mu\in\mathscr{B}_c^{n-p,n-q}(X)$ be $\bar{\partial}$-closed. Choose the weight g in the operators $\check{K}$ and $\check{P}$ of Section 5 such that $z\mapsto g(\zeta,z)$ is holomorphic in a neighborhood of the holomorphically compact closure of $\operatorname{supp}\mu$ and $\zeta\mapsto g(\zeta,z)$ has support in a fixed compact for all z in that neighborhood, cf. Example 6.1. Then $\vartheta(g\wedge H)\mathcal{R}(\zeta)$ has degree 0 in $d\bar{z}$ and so $\check{P}\mu=0$ if $q\geq 1$, cf. (5.3). Since by Proposition 6.7 the Koppelman formula (6.4) holds we conclude that $\mu=\bar{\partial}\check{K}\mu$ if $q\geq 1$. Since $\zeta\mapsto g(\zeta,z)$ has compact support also $\check{K}\mu$ has and the first statement of the theorem follows.

Suppose now that $\mu\in\mathscr{B}_c^{n-p,n}(X)$. Since convergence of a sequence in $\Omega^p(X)$ implies convergence in $\mathscr{E}^{p,0}(D)$ for a suitable sequence of representatives it follows that μ defines a continuous linear functional $\tilde{\mu}$ on $\Omega^p(X)$ via (7.2). This functional only depends on the cohomology class of μ and so we can, in view of (6.4), assume that $\mu=\check{P}\mu$. We have

$$\check{P}\mu=\pi_{1*}\big(\vartheta(g\wedge H)\mathcal{R}(\zeta)\wedge i_*\mu(z)\big)=\pm\pi_{1*}\big(\vartheta(g\wedge H)\wedge i_*\mu(z)\big)\mathcal{R}(\zeta),$$

where $\pi_1\colon D_\zeta \times D_z \to D_\zeta$. Since g and H are holomorphic for z in a neighborhood of the holomorphically compact closure of $\operatorname{supp}\mu$ it follows from the Oka-Weil theorem that if $\int_X \varphi\wedge\mu = 0$ for all $\varphi \in \Omega^p(X)$, then $\pi_{1*}\big(\vartheta(g\wedge H)\wedge i_*\mu(z)\big) = 0$. Hence, $\tilde\mu = 0$ implies $\mu = 0$, i.e., the map $\mu \mapsto \tilde\mu$ is injective.

To show surjectivity, let λ be a continuous linear functional on $\Omega^p(X)$. Then λ lifts to a functional, also denoted λ, on $\Omega^p(D)$. By the Hahn-Banach theorem this functional has to be carried by some compact $\mathbf{K} \subset D$, which we may assume is holomorphically convex, and there is an $(N-p,N)$-current ν in D of order 0 with compact support in a neighborhood V of $\mathbf{K}$ such that

$$\lambda(\varphi) = \int_D \varphi\wedge\nu, \quad \varphi \in \Omega^p(D).$$

Let P be an operator corresponding to a choice of weight g such that $z \mapsto g(\zeta,z)$ is holomorphic in V and $\zeta \mapsto g(\zeta,z)$ has support in a fixed compact subset of D for all $z \in V$. Then, if $\varphi \in \Omega^p(X)$, $P\varphi$ is an extension of φ to V. Let also $\phi \in \Omega^p(D)$ be an arbitrary representative of φ. Then

$$\begin{aligned}
\lambda(\varphi) &= \lambda(P\varphi) = \int_D P\varphi(z)\wedge\nu(z) = \int_D \pi_{2*}\big(\vartheta(g\wedge H)\mathcal{R}(\zeta)\wedge\phi(\zeta)\big)\wedge\nu(z)\\
&= \big(\vartheta(g\wedge H)\mathcal{R}(\zeta)\wedge\phi(\zeta)\wedge\nu(z)\big).1_{D\times D}\\
&= \pm\int_D \phi(\zeta)\wedge\pi_{1*}\big(\vartheta(g\wedge H)\wedge\nu(z)\big)\mathcal{R}(\zeta).
\end{aligned}$$

Since $\pi_{1*}\big(\vartheta(g\wedge H)\wedge\nu(z)\big)$ is smooth with compact support in D it follows that there is $\mu \in \mathscr{B}_c^{n-p,n}(X)$ such that

$$\pi_{1*}\big(\vartheta(g\wedge H)\wedge\nu(z)\big)\mathcal{R}(\zeta) = i_*\mu.$$

Hence,

$$\lambda(\varphi) = \int_D \phi\wedge i_*\mu = \int_X \varphi\wedge\mu.$$

□

Theorem 7.3. *Let X be a (paracompact) analytic space of pure dimension n. If $H^q(X,\Omega_X^p)$ and $H^{q+1}(X,\Omega_X^p)$ are Hausdorff, then the pairing*

$$H^q(\mathscr{A}^{p,\bullet}(X),\bar\partial)\times H^{n-q}(\mathscr{B}_c^{n-p,\bullet}(X),\bar\partial)\to\mathbb{C}, \quad ([\varphi],[\mu])\mapsto\int_X\varphi\wedge\mu \tag{7.3}$$

is non-degenerate so that $H^{n-q}(\mathscr{B}_c^{n-p,\bullet}(X),\bar\partial)$ is the dual of $H^q(\mathscr{A}^{p,\bullet}(X),\bar\partial)$.

Sketch of proof. Referring to, e.g., [23, Section 6.2] and [24, Section 7.3] for details we outline a proof showing that $H^{n-q}(\mathscr{B}_c^{n-p,\bullet}(X),\bar\partial)$ is the dual of $H^q(\mathscr{A}^{p,\bullet}(X),\bar\partial)$ via (7.3). The idea is to use Čech cohomology and homological algebra to reduce to the local duality of Theorem 7.2.

Let $\mathcal{U} = \{U_j\}_j$ be a locally finite covering of X such that U_j is an analytic subspace of some pseudoconvex domain D in some $\mathbb{C}^N$. Then $\mathcal{U}$ is a Leray covering for Ω_X^p. Let $(C^\bullet(\mathcal{U},\Omega_X^p),\delta)$ be the associated Čech cochain complex. Then

$$H^q(\mathscr{A}^{p,\bullet}(X),\bar\partial)\simeq H^q(C^\bullet(\mathcal{U},\Omega_X^p),\delta) \tag{7.4}$$

since both the left- and the right-hand sides are isomorphic to $H^q(X,\Omega_X^p)$. It is standard to make the isomorphism (7.4) explicit by solving local $\bar\partial$-equations.

The Fréchet topology on $\Omega^p(U_j)$ induces a natural Fréchet topology on $C^k(\mathcal{U},\Omega^p_X)$ and, consequently, on the cohomology of $(C^\bullet(\mathcal{U},\Omega^p_X),\delta)$. Recall that the standard topology on $H^q(X,\Omega^p_X)$ is defined as this topology. In view of, e.g., [22, Lemma 2] it follows that if $H^q(X,\Omega^p_X)$ and $H^{q+1}(X,\Omega^p_X)$ are Hausdorff, then

$$H^q(C^\bullet(\mathcal{U},\Omega^p_X),\delta)^*\simeq H^q(C^\bullet(\mathcal{U},\Omega^p_X)^*,\delta^*), \tag{7.5}$$

where $(C^\bullet(\mathcal{U},\Omega^p_X)^*,\delta^*)$ is the (topological) dual complex of $(C^\bullet(\mathcal{U},\Omega^p_X),\delta)$.

Let $C^{-k}(\mathcal{U},\mathscr{B}_c^{n-p,j})$ be the group of formal sums $\sum'_{|I|=k+1}\mu_I U_I^*$, where $\mu_I\in\mathscr{B}_c^{n-p,j}(\cap_{i\in I}U_i)$ and $U_I^*:=U_{i_0}^*\wedge\cdots\wedge U_{i_k}^*$ is a formal wedge product. It follows from Theorem 7.2 that

$$C^k(\mathcal{U},\Omega^p_X)^*\simeq H^n(C^{-k}(\mathcal{U},\mathscr{B}_c^{n-p,\bullet}),\bar\partial) \tag{7.6}$$

via the pairing induced by (7.2). The operator δ^* on $C^\bullet(\mathcal{U},\Omega^p_X)^*$ gives in a natural way an operator, also denoted δ^*, on $C^{-\bullet}(\mathcal{U},\mathscr{B}_c^{n-p,j})$. It turns out that this operator is formal interior multiplication by $\sum_\ell U_\ell$; μ_I is extended to $\cap_{i\in I\setminus i_\ell}U_i$ by 0. Thus we get the double complex

$$\big(C^{-\bullet}(\mathcal{U},\mathscr{B}_c^{n-p,\bullet}),\delta^*,\bar\partial\big). \tag{7.7}$$

In view of (7.6) we have

$$H^q(C^\bullet(\mathcal{U},\Omega^p_X)^*,\delta^*)\simeq H^q(H^n(C^{-\bullet}(\mathcal{U},\mathscr{B}_c^{n-p,\bullet}),\delta^*,\bar\partial)). \tag{7.8}$$

By Theorem 6.10, the $\bar\partial$-cohomology of (7.7) is trivial except on level n and, by, e.g., [23, Lemma 6.3], since the $\mathscr{B}_X$-sheaves are fine, the δ^*-cohomology of (7.7) is trivial except on level 0 where the cohomology is $\mathscr{B}_c(X)^{n-p,\bullet}$. By standard homological algebra it follows that

$$H^q(H^n(C^{-\bullet}(\mathcal{U},\mathscr{B}_c^{n-p,\bullet}),\delta^*),\bar\partial)\simeq H^{n-q}(\mathscr{B}_c(X)^{n-p},\bar\partial). \tag{7.9}$$

From (7.4), (7.5), (7.8), and (7.9) we see that $H^{n-q}(\mathscr{B}_c(X)^{n-p},\bar\partial)$ is the dual of $H^q(\mathscr{A}^{p,\bullet}(X),\bar\partial)$. To see that this duality is given by (7.3) one can make these isomorphisms explicit and use that (7.6) is induced by the pairing (7.2). □

Proof of Theorem B. Part (i) follows from Definition 6.6 and Proposition 6.7. Part (ii) follows from Theorem 6.10. Part (iii) follows from Theorem 7.1. Part (iv) follows from Theorem 7.3; indeed, if X is compact then we can replace $\mathscr{B}_c^{n-p,\bullet}(X)$ by $\mathscr{B}^{n-p,\bullet}(X)$ and, moreover, by the Cartan-Serre theorem, the cohomology of any coherent sheaf is finite-dimensional, in particular Hausdorff. □

8. Examples

We present two examples which illustrate our various notions of holomorphic forms and currents. The first one is rather straightforward whereas the second one is more elaborate.

Example 8.1. Let $D=\mathbb{C}^4$ with coordinates (z_1,z_2,w_1,w_2). For the ideal $\mathcal{J}=\langle w_1^2,w_2^2,w_1w_2\rangle$ we have $\sqrt{\mathcal{J}}=\langle w_1,w_2\rangle$ so that $Z=\{w=0\}$. In this case $d\mathcal{J}=$

$\langle w_1dw_1, w_2dw_2, w_2dw_1 + w_1dw_2\rangle$ and it is straightforward to check that

$$\begin{aligned} \mathscr{O}_X &= \mathscr{O}_Z\{1, w_1, w_2\}, \\ \Omega^1_X &= \Omega^1_{X,\text{Kähler}} = \Omega^1_Z\{1, w_1, w_2\} + \mathscr{O}_Z\{dw_1, dw_2, w_1dw_2 - w_2dw_1\}, \\ \Omega^2_X &= \Omega^2_{X,\text{Kähler}} \\ &= \Omega^2_Z\{1, w_1, w_2\} + \Omega^1_Z \wedge \{dw_1, dw_2, w_1dw_2 - w_2dw_1\} + \mathscr{O}_Z\{dw_1 \wedge dw_2\}. \end{aligned}$$

Since the underlying reduced space is smooth we may use Proposition 3.13 to describe the smooth forms on X. By this proposition we have, for example, that

$$\mathscr{E}^{2,*}_X = \mathscr{E}^{2,*}_D / \big(\mathcal{J}\mathscr{E}^{2,*}_D + d\mathcal{J} \wedge \mathscr{E}^{1,*}_D + \overline{\langle w_1, w_2\rangle}\mathscr{E}^{2,*}_D + d\overline{\langle w_1, w_2\rangle} \wedge \mathscr{E}^{2,*}_D\big).$$

We see that the denominator above, i.e., $\mathscr{K}\!er_2\, i^*$, contains all $\bar{w}_i$ and $d\bar{w}_i$ and what remains is

$$\mathscr{E}^{2,*}_X = \mathscr{E}^{2,*}_Z\{1, w_1, w_2\} + \mathscr{E}^{1,*}_Z \wedge \{dw_1, dw_2, w_1dw_2 - w_2dw_1\} + \mathscr{E}^{0,*}_Z \wedge \{dw_1 \wedge dw_2\},$$

very much in analogy with Ω^2_X above. Now let us look at currents on X; for simplicity we restrict to currents of bidegree $(2, *)$. If α is a $(2, *)$-current on X then, by definition, $i_*\alpha$ is annihilated by $\mathscr{K}\!er_0\, i^*$, which contains all $\bar{w}_i$ and $d\bar{w}_i$. It follows that

$$i_*\alpha = \sum_{k,\ell \geq 0} \alpha_{k,\ell}(z) dz_1 \wedge dz_2 \wedge dw_1 \wedge dw_2 \wedge \bar\partial \frac{1}{w_1^{k+1}} \wedge \bar\partial \frac{1}{w_2^{\ell+1}} \quad \text{with } \alpha_{k,\ell} \in \mathscr{C}^{0,*}_Z,$$

where $\mathscr{C}_Z$ is the sheaf of currents on Z. But we must also have that $i_*\alpha \wedge \mathcal{J} = 0 = i_*\alpha \wedge d\mathcal{J}$. The first equality implies that $k, \ell \leq 1$ and the second is automatically satisfied for degree reasons. We also see that $w_1w_2\bar\partial(dw_1/w_1^2) \wedge \bar\partial(dw_2/w_2^2) \neq 0$ and therefore

$$i_*\mathscr{C}^{2,*}_X = \mathscr{C}^{2,*}_Z \left\{ \bar\partial \frac{dw_1}{w_1} \wedge \bar\partial \frac{dw_2}{w_2},\ \bar\partial \frac{dw_1}{w_1^2} \wedge \bar\partial \frac{dw_2}{w_2},\ \bar\partial \frac{dw_1}{w_1} \wedge \bar\partial \frac{dw_2}{w_2^2} \right\}.$$

Writing $\mathcal{B}$ for the set of three basis elements above, we get $i_*\omega^2_X = \Omega^2_Z\mathcal{B}$.

Example 8.2. Let $D = \mathbb{C}^4$ with the same coordinates as above. Let $i\colon X \to D$ be defined by

$$\mathcal{J} = \langle w_1^2, w_2^2, w_1w_2, z_1w_2 - z_2w_1\rangle$$

and write $f = z_1w_2 - z_2w_1$ so that $d\mathcal{J} = \langle w_1dw_1, w_2dw_2, w_1dw_2 + w_2dw_1, df\rangle$. It is straightforward to see that

$$\mathscr{O}_X = \frac{\mathscr{O}_Z\{1, w_1, w_2\}}{\mathscr{O}_Z\{f\}}, \quad \Omega^1_{X,\text{Kähler}} = \frac{\Omega^1_Z\{1, w_1, w_2\} + \mathscr{O}_Z\{dw_1, dw_2, w_1dw_2 - w_2dw_1\}}{\Omega^1_Z\{f\} + \mathscr{O}_Z\{df\}}.$$

Here we write the quotient as a quotient of $\mathscr{O}_Z$-modules but to see how the multiplication in the rings work notice that $w_iw_j = 0$ and $w_idw_i = 0$.

We now describe the torsion elements of $\Omega^1_{X,\text{Kähler}}$. If $z_1 \neq 0$ then $w_2 = w_1z_2/z_1$ and since $w_1dw_1 = 0$ we get $w_2dw_1 = \big(z_2w_1/z_1\big)dw_1 = 0$. We have $w_1dw_2 + w_2dw_1 = 0$ everywhere and therefore we also get $w_1dw_2 = 0$ when $z_1 \neq 0$. By symmetry both w_1dw_2 and w_2dw_1 vanish when $z_2 \neq 0$. One may verify that neither w_1dw_2 nor w_2dw_1 is in $\Omega^1_Z\{f\} + \mathscr{O}_Z\{df\}$ and therefore they are torsion elements. These are actually the only torsion elements and hence

$$\Omega^1_X = \frac{\Omega^1_Z\{1, w_1, w_2\} + \mathscr{O}_Z\{dw_1, dw_2\}}{\Omega^1_Z\{f\} + \mathscr{O}_Z\{df\}}.$$

The way we check we found all torsion elements is to check that the module above is torsion-free. Similar calculations yield that

$$\Omega^2_X = \frac{\Omega^2_Z\{1,w_1,w_2\} + \Omega^1_Z\{dw_1,dw_2\}}{\Omega^2_Z\{f\} + \Omega^1_Z\{df\}}.$$

We now describe generators for ω^{2-p}_X. In [6, Example 6.9], it was shown that the generators for $i_*\omega^2_X$ are given by

$$\bar\partial\frac{1}{w_1}\wedge\bar\partial\frac{1}{w_2}\wedge dz\wedge dw \text{ and } \left(z_1\bar\partial\frac{1}{w_1^2}\wedge\bar\partial\frac{1}{w_2} + z_2\bar\partial\frac{1}{w_1}\wedge\bar\partial\frac{1}{w_2^2}\right)\wedge dz\wedge dw,$$

where $dz = dz_1\wedge dz_2$ and $dw = dw_1\wedge dw_2$. These correspond to intrinsic objects in ω^2_X, which can be considered as differential operators, and in view of the formula

$$\frac{1}{(2\pi i)^2}\bar\partial\frac{1}{w_1^{m_1+1}}\wedge\bar\partial\frac{1}{w_2^{m_2+1}}.\psi(w) = \frac{1}{m_1!m_2!}\frac{\partial^{m_1+m_2}}{\partial w_1^{m_1}\partial w_2^{m_2}}\psi\bigg|_{w=0},$$

they correspond (up to constants) to the form-valued differential operators

$$dz_1\wedge dz_2\mathrm{Id} \text{ and } dz_1\wedge dz_2\left(z_1\frac{\partial}{\partial w_1} + z_2\frac{\partial}{\partial w_2}\right)$$

followed by restriction to Z.

By a similar calculation as in [6, Example 6.9], one can obtain also generators for $i_*\omega^{2-p}_X$ for $p=1,2$. Indeed, if (E,f) is a locally free resolution of Ω^p_X, then ω^{2-p}_X is generated by all currents of the form ξR^E_2, where ξ is in $\mathscr{K}\!er\, f_3^*$ and R^E_2 is the part in $\mathrm{Hom}(E_0,E_2)$ of the residue current associated to (E,f). To calculate ξR^E_2, by the same argument as in [6, Example 6.9], if (F,g) is the direct sum of r_0 copies of the Koszul complex of (w_1^2,w_2^2), where $r_0 = \operatorname{rank} E_0 = \operatorname{rank}\Omega^p_{\mathbb{C}^4}$, and $a\colon (F,g)\to(E,f)$ is a morphism of complexes such that $a_0\colon F_0\to E_0$ is the identity, then

$$\xi R^E_2 h = \xi a_2(he)\wedge\bar\partial\frac{1}{w_2^2}\wedge\bar\partial\frac{1}{w_1^2}.$$

With the help of the software Macaulay2, we could calculate the morphism a_2 and generators for $\mathscr{K}\!er\, f_3^*$, and could thus calculate generators for $i_*\omega^{2-p}_X$. The sheaf $i_*\omega^1_X$ is generated by

$$dz_1\wedge dw_1\wedge dw_2\wedge\bar\partial\frac{1}{w_2}\wedge\bar\partial\frac{1}{w_1},\quad dz_2\wedge dw_1\wedge dw_2\wedge\bar\partial\frac{1}{w_2}\wedge\bar\partial\frac{1}{w_1},$$
$$((z_2w_1+z_1w_2)dz_2\wedge dw_1\wedge dw_2 + w_1w_2dz_1\wedge dz_2\wedge dw_2)\wedge\bar\partial\frac{1}{w_2^2}\wedge\bar\partial\frac{1}{w_1^2},$$
$$((z_2w_1+z_1w_2)dz_1\wedge dw_1\wedge dw_2 + w_1w_2dz_1\wedge dz_2\wedge dw_1)\wedge\bar\partial\frac{1}{w_2^2}\wedge\bar\partial\frac{1}{w_1^2},$$
$$(z_2dz_1\wedge dz_2\wedge dw_1 - z_1dz_1\wedge dz_2\wedge dw_2)\wedge\bar\partial\frac{1}{w_2}\wedge\bar\partial\frac{1}{w_1}.$$

These correspond (up to constants) to the differential operators

$$\begin{aligned}
&dz_1\mathrm{Id}, \quad dz_2\mathrm{Id},\\
&dz_2\left(z_2\frac{\partial}{\partial w_2}+z_1\frac{\partial}{\partial w_1}\right)+dz_1\wedge dz_2\wedge(dw_1\lrcorner),\\
&dz_1\left(z_2\frac{\partial}{\partial w_2}+z_1\frac{\partial}{\partial w_1}\right)+dz_1\wedge dz_2\wedge(dw_2\lrcorner),\\
&z_2dz_1\wedge dz_2\wedge(dw_2\lrcorner)-z_1dz_1\wedge dz_2\wedge(dw_1\lrcorner)
\end{aligned}$$

followed by restriction to Z. Finally, the sheaf $i_*\omega_X^0$ is generated by

$$\begin{aligned}
&dw_1\wedge dw_2\wedge\bar\partial\frac{1}{w_2}\wedge\bar\partial\frac{1}{w_1},\\
&((z_2w_1+z_1w_2)dw_1\wedge dw_2+w_1w_2dz_2\wedge dw_1-w_1w_2dz_1\wedge dw_2)\wedge\bar\partial\frac{1}{w_2^2}\wedge\bar\partial\frac{1}{w_1^2},\\
&(z_1dz_1\wedge dw_2-z_2dz_1\wedge dw_1)\wedge\bar\partial\frac{1}{w_2}\wedge\bar\partial\frac{1}{w_1},\\
&(z_1dz_2\wedge dw_2-z_2dz_2\wedge dw_1)\wedge\bar\partial\frac{1}{w_2}\wedge\bar\partial\frac{1}{w_1}.
\end{aligned}$$

These correspond (up to constants) to the differential operators

$$\begin{aligned}
&\mathrm{Id}, \quad z_2\frac{\partial}{\partial w_2}+z_1\frac{\partial}{\partial w_1}+dz_2\wedge(dw_2\lrcorner)+dz_1\wedge(dw_1\lrcorner),\\
&dz_1\wedge(z_1dw_1\lrcorner+z_2dw_2\lrcorner), \quad dz_2\wedge(z_1dw_1\lrcorner+z_2dw_2\lrcorner)
\end{aligned}$$

followed by restriction to Z.

We conclude this paper by putting the calculations of $\omega_X^\bullet$ into the context of Noetherian differential operators. Let as before $i\colon X\to D\subset\mathbb{C}^N$ be defined by $\mathcal{J}$ and $Z=Z(\mathcal{J})$. Recall that a holomorphic differential operator $\mathcal{L}\colon\mathscr{O}_D\to\mathscr{O}_Z$ is Noetherian for $\mathcal{J}$ if $\mathcal{L}\varphi=0$ for any $\varphi\in\mathcal{J}$. A set $\{\mathcal{L}_j\}_j$ is a complete set of Noetherian operators for $\mathcal{J}$ if $\varphi\in\mathcal{J}$ if and only if $\mathcal{L}_j\varphi=0$ for all j.

Assume that Z is smooth and that (z,w) are coordinates such that $Z=\{w=0\}$ and let $\pi\colon D\to Z$ be the projection $\pi(z,w)=z$. Given a set of generators $\mu=(\mu_1,\dots,\mu_m)$ of ω_X^{n-p} we construct a complete set of Noetherian type operators $\Omega_D^p\to\Omega_Z^n$ (acting as Lie derivatives) for $\mathcal{J}^p$ in a way similar to the construction of the mapping $\widetilde{T}$ in Section 3.3. Take $M>0$ large enough so that $w^\alpha\mu_j=0$ for all j if $|\alpha|\geq M$. We set

$$\mathcal{L}_{j,\alpha}\colon\Omega_D^p\to\Omega_Z^n,\quad \mathcal{L}_{j,\alpha}\varphi=\pi_*(w^\alpha\varphi\wedge i_*\mu_j);$$

that $\mathcal{L}_{j,\alpha}\varphi\in\Omega_Z^n$ follows as in Section 3.3. Moreover, $\varphi\wedge i_*\mu_j=0$ if and only if $\pi_*(w^\alpha\varphi\wedge i_*\mu_j)=0$ for all α. Since $\mathscr{K}er_p\, i^*$ is the annihilator of μ and $\mathscr{K}er_p\, i^*\cap\Omega_X^p=\mathcal{J}^p$, see the proof of Corollary 3.10, it follows that $\mathcal{L}_{j,\alpha}$, $j=1,\dots,m$, $|\alpha|<M$, is a complete set of Noetherian operators for $\mathcal{J}^p$.

References

[1] Andersson, Mats: Integral representation with weights. II. Division and interpolation. *Math. Z.* 254 (2006), no. 2, 315–332.

[2] Andersson, Mats: Uniqueness and factorization of Coleff-Herrera currents. *Ann. Fac. Sci. Toulouse Math. (6)* 18 (2009), no. 4, 651–661.

[3] Andersson, Mats: A residue criterion for strong holomorphicity. *Ark. Mat.* 48 (2010), no. 1, 1–15.
[4] Andersson, Mats: Coleff-Herrera currents, duality, and Noetherian operators. *Bull. Soc. Math. France* 139 (2011), no. 4, 535–554.
[5] Andersson, Mats: Pseudomeromorphic currents on subvarieties. *Complex Var. Elliptic Equ.* 61 (2016), no. 11, 1533–1540.
[6] Andersson, Mats; Lärkäng, Richard: The $\bar{\partial}$-equation on a non-reduced analytic space. *Math. Ann.* 374 (2019), no. 1-2, 553–599.
[7] Andersson, Mats; Samuelsson Håkan: A Dolbeault–Grothendieck lemma on complex spaces via Koppelman formulas. *Invent. Math.* 190 (2012), no. 2, 261–297.
[8] Andersson, Mats; Wulcan, Elizabeth: Residue currents with prescribed annihilator ideals. *Ann. Sci. École Norm. Sup. (4)* 40 (2007), no. 6, 985–1007.
[9] Andersson, Mats; Wulcan, Elizabeth: Decomposition of residue currents. *J. Reine Angew. Math.* 638 (2010), 103–118.
[10] Andersson, Mats; Wulcan, Elizabeth: Direct images of semi-meromorphic currents. *Ann. Inst. Fourier (Grenoble)* 68 (2018), no. 2, 875–900.
[11] Andersson, Mats; Wulcan, Elizabeth: Regularity of pseudomeromorphic currents. *Bull. Sci. Math.*, to appear. Available at arXiv:1703.01247 [math.CV].
[12] Bănică, Constantin; Stănăşilă, Octavian: Algebraic methods in the global theory of complex spaces. *Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney,* 1976. 296 pp.
[13] Barlet, Daniel: Le faisceau $\omega_X^\bullet$ sur un espace analytique X de dimension pure. *Fonctions de plusieurs variables complexes, III (Sém. Françcois Norguet, 1975–1977),* pp. 187–204, Lecture Notes in Math., 670, *Springer, Berlin,* 1978.
[14] Björk, Jan-Erik: Residues and $\mathscr{D}$-modules. *The legacy of Niels Henrik Abel,* 605–651, *Springer, Berlin,* 2004.
[15] Demailly, Jean-Pierre: Complex Analytic and Differential Geometry. Online book, available at https://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf
[16] Eisenbud, David: Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. *Springer-Verlag, New York,* 1995. xvi+785 pp.
[17] Henkin, Gennadi; Passare, Mikael: Abelian differentials on singular varieties and variations on a theorem of Lie-Griffiths. *Invent. Math.* 135 (1999), no. 2, 297–328.
[18] Herrera, M.; Lieberman, D.: Residues and principal values on complex spaces. *Math. Ann.* 194 (1971), 259–294.
[19] Lam, T. Y.: Lectures on modules and rings. Graduate Texts in Mathematics, 189. *Springer-Verlag, New-York,* 1999. xxiv+557 pp.
[20] Lärkäng, Richard; Samuelsson Kalm, Håkan: Various approaches to products of residue currents. *J. Funct. Anal.* 264 (2013), no. 1, 118–138.
[21] Matsumura, Hideyuki: Commutative ring theory. Second edition. Cambridge Studies in Advanced Mathematics, 8. *Cambridge University Press, Cambridge,* 1989. xiv+320 pp.
[22] Ramis, Jean-Pierre; Ruget, Gabriel: Complexe dualisant et théorèmes de dualité en géométrie analytique complexe. *Inst. Hautes Études Sci. Publ. Math.* No. 38 (1970), 77–91.
[23] Ruppenthal, Jean; Samuelsson Kalm, Håkan; Wulcan, Elizabeth: Explicit Serre duality on complex spaces. *Adv. Math.* 305 (2017), 1320–1355.
[24] Samuelsson Kalm, Håkan: The $\bar{\partial}$-equation, duality, and holomorphic forms on a reduced complex space. *J. Geom. Anal.*, to appear. Available at arXiv:1506.07842 [math.CV, math.AG].
[25] Sznajdman, Jacob: A Briançon-Skoda-type result for a non-reduced analytic space. *J. Reine Angew. Math.* 742 (2018), 1–16.

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Paper III

III

Mattias Lennartsson, *"Residues of singular differential forms on complex submanifolds"*, preprint.

Residues of singular differential forms on complex submanifolds

Mattias Lennartsson

Abstract

In this short note we investigate integration of differential forms which are singular along a complex submanifold.

1. Introduction

Let X be a complex manifold of dimension n, $Y \subseteq X$ a submanifold of codimension κ and ω a differential (n,n)-form which is smooth on $X \setminus Y$. Our aim is to make sense of divergent integrals on the form

$$\int_X \omega. \tag{1}$$

Such problems arise for instance in distribution theory and lead to notions such as the finite part and the principal value of an integral.

In this short note we will consider forms ω of the following form: We assume that there is a section $s : X \to E$ of a hermitian vector bundle $(E, |\cdot|) \to X$ such that $\{s = 0\} = Y$, $\mathrm{d}s$ has rank κ on Y and $|s|^{2N}\omega$ is smooth on all of X, for some integer $N \geqslant 0$. This means that s generates the ideal of all holomorphic functions which vanish on Y. Integrals of such forms show up naturally in modern physics, e.g. in perturbative string theory and renormalisation quantum field theory. The special case when Y is a hypersurface has been considered by [**Wi**], [**FK**] and by ourselves in [**ML**]. In [**FK2**] is considered the more general setup of real submanifolds. We will reach similar results and we use our special setup to get easier proofs and somewhat more explicit formulas.

Our approach is inspired by the theory of residue currents. Let ξ be a test function on X. We regularise the integral (1) by defining

$$F(\omega, \xi, |s|^2, \lambda) := \int_X |s|^{2\lambda}\omega\xi \tag{2}$$

for $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) \gg 1$. For such values of λ the integrand is smooth and compactly supported, the integral is well-defined and F is holomorphic in λ. The following theorem gives a further description of the function F.

Theorem 1.1. *The function F, defined in (2), has a meromorphic continuation to $\mathrm{Re}(\lambda) > -\varepsilon$, for some $\varepsilon > 0$, F has poles at $\lambda = 0, 1, \ldots, N-1$ and the pole at $\lambda = 0$ is of order at most one. The Laurent expansion of F at $\lambda = 0$ is*

$$F(\omega, \xi, |s|^2, \lambda) = \frac{1}{\lambda} C_1(\omega, |s|^2).\xi + C_0(\omega, |s|^2).\xi + \mathcal{O}(|\lambda|)$$

where the $C_j(\omega, |s|^2)$ are currents on X.

Suppose that $\|\cdot\|$ is another metric on E and define $F(\omega, \xi, \|s\|^2, \lambda)$ analogously as for $|\cdot|$. The function $F(\omega, \xi, \|s\|^2, \lambda)$ also has a pole at $\lambda = 0$ of order at most one and we denote the current coefficients in the Laurent expansion by $C_1(\omega, \|s\|^2)$ and $C_0(\omega, \|s\|^2)$. Then *(i)* $C_1(\omega, |s|^2) = C_1(\omega, \|s\|^2)$,

(ii) $C_0(\omega, |s|^2) = C_0(\omega, \|s\|^2) + C_1(\omega, |s|^2) \log \frac{|s|^2}{\|s\|^2}$.

Remark. (1) If $|s|^{2N}\omega$ is smooth for $N < \kappa$, then ω is integrable. In this case $C_1 = 0$ and $C_0.\xi = \int_X \omega\xi$.

(2) The function $\log \frac{|s|^2}{\|s\|^2}$ is not smooth but it is part of Theorem 1.1 that we may multiply $C_1(\omega, |s|^2)$ by it.

In the case that ω has compact support Theorem 1.1 gives us one way of making sense of the integral $\int_X \omega$, namely as $C_0(\omega, |s|^2).1$. However, this does depend on the choice of metric in accordance with Theorem 1.1 (ii).

By Theorem 1.1 (i) we may now write $C_1(\omega) := C_1(\omega, |s|^2)$ since this is independent of the choice of metric. We also see that the way the coefficient $C_0(\omega, |s|^2)$ depends on the metric is determined by $C_1(\omega)$. The next theorem describes $C_1(\omega)$.

Theorem 1.2. *Let ξ be a test function on X. If $|s|^{2\kappa}\omega$ is smooth in X then there is a smooth $(n-\kappa, n-\kappa)$-form* $\mathrm{res}(\omega)$ *on Y such that*

$$C_1(\omega).\xi = \kappa(2\pi\mathrm{i})^\kappa \int_Y \mathrm{res}(\omega)\xi.$$

If $|s|^{2N}\omega$ is smooth in X, for $N > \kappa$ then there is a de Rham cohomology class $\mathrm{Res}(\omega\xi) \in H^{n-\kappa,n-\kappa}(Y)$ *such that*

$$C_1(\omega).\xi = \kappa(2\pi\mathrm{i})^\kappa \int_Y \mathrm{Res}(\omega\xi).$$

Remark. In the first statement of Theorem 1.2 $\mathrm{res}(\omega)$ is the differential form $\widetilde{\omega}$, defined below in Lemma 2.1 (a), restricted to Y. In the second statement $\mathrm{Res}(\omega\xi) = [\mathrm{res}(|s|^{-2\kappa}Q(|s|^{2N}\omega\xi))]$ where Q is a differential operator on (n,n)-forms of order $2(N-\kappa)$ and the square brackets denote the de Rham cohomology class. The differential operator Q depends on the choice of local coordinates and a partition of unity but the de Rham class $\mathrm{Res}(\omega\xi)$ does not.

In the case that X is compact, given $|s|^2$ and ω as above it follows from the above results that the following

$$\int_{X,|s|^2} \omega := \lim_{\lambda\to 0} \left(\int_X |s|^{2\lambda}\omega - \frac{\kappa(2\pi\mathrm{i})^\kappa}{\lambda} \int_Y \mathrm{Res}(\omega) \right)$$

is well-defined. This is one way of making sense of the integral we started with. It depends on $|s|^2$; if $|s|^2$ is replaced with $\|s\|^2 = f|s|^2$, for some positive function f, then

$$\int_{X,\|s\|^2} \omega = \int_{X,|s|^2} \omega - C_1(\omega)\log(f).1.$$

2. Proofs

Proof of Theorem 1.1. We shall consider somewhat more general regularisations than (2). Let $\sigma : X \to \widehat{E}$ be a section of a hermitian bundle $(\widehat{E}, \|\cdot\|)$ such that $\{\sigma = 0\} = Y$ and $\mathrm{d}\sigma \neq 0$ on Y. This does not imply that $\|\sigma\|^{2N}\omega$ is smooth, but it is bounded. We then define $F(\omega, \xi, \|\sigma\|^2, \lambda)$ analogously to $F(\omega, \xi, |s|^2, \lambda)$ but with $|s|^2$ replaced by $\|\sigma\|^2$.

Let $\pi : \mathrm{Bl}_Y(X) \to X$ be the blow-up of X along Y. As $\|\sigma\|^{2\lambda}\omega\xi$ is integrable for $\mathrm{Re}(\lambda) \gg 1$ and π is biholomorphic outside a null set we have

$$F(\omega, \xi, \|\sigma\|^2, \lambda) = \int_{\mathrm{Bl}_Y(X)} \|\pi^*\sigma\|^{2\lambda}\pi^*(\omega\xi) \quad \text{for } \mathrm{Re}(\lambda) \gg 1.$$

Both $\pi^*\sigma$ and π^*s define the smooth hypersurface $D := \pi^{-1}(Y)$. We take a partition of unity $\{\rho_\iota\}$ on $\mathrm{Bl}_Y(X)$ subordinate to coordinate charts $\{U_\iota, z_\iota\}$ such that locally the hypersurface D is given by $z_{\iota,1} = 0$. Therefore locally $\|\pi^*\sigma\|^2 = |z_{\iota,1}|^2 \mathrm{e}^{-\phi_\iota}$ and $|\pi^*s|^2 = |z_{\iota,1}|^2 \mathrm{e}^{-\psi_\iota}$ for some locally defined smooth functions ϕ_ι and ψ_ι. Notice that $|\pi^*s|^{2N}\pi^*\omega$ is smooth and so it follows that $\|\pi^*\sigma\|^{2N}\pi^*\omega$ is also smooth. We now have

$$F(\omega, \xi, \|\sigma\|^2, \lambda) = \sum_\iota \int_{U_\iota} \frac{|z_{\iota,1}|^{2\lambda}}{|z_{\iota,1}|^{2N}} \mathrm{e}^{-\phi_\iota \lambda} |z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \quad \text{for } \mathrm{Re}(\lambda) \gg 1.$$

Dropping the index ι for the moment we have

$$\frac{|z_1|^{2\lambda}}{|z_1|^{2N}} = \frac{h(\lambda)}{\lambda} \frac{\partial^{2N-1}}{\partial z_1^{N-1} \partial \bar{z}_1^N} \left(\frac{|z_1|^{2\lambda}}{z_1} \right)$$

where $h(\lambda) = \big((\lambda-1)^2 \cdots (\lambda - N + 1)^2\big)^{-1}$. Notice that h is holomorphic when $\mathrm{Re}(\lambda) < 1$. Stokes' theorem now gives

$$F(\omega, \xi, \|\sigma\|^2, \lambda) = \frac{h(\lambda)}{\lambda} \sum_\iota \int_{U_\iota} \frac{|z_{\iota,1}|^{2\lambda}}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \big(\mathrm{e}^{-\phi_\iota \lambda} |z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \big) \quad \text{for } \mathrm{Re}(\lambda) \gg 1. \tag{3}$$

Denote the integral in (3) by $g_{\|\sigma\|^2}(\lambda)$ so that we have

$$F(\omega, \xi, \|\sigma\|^2, \lambda) = \frac{1}{\lambda} h(\lambda) g_{\|\sigma\|^2}(\lambda) \tag{4}$$

We see that $g_{\|\sigma\|^2}(\lambda)$ is defined and, by dominated convergence, holomorphic for $\mathrm{Re}(\lambda) > -1/2$. Hence $F(\omega, \xi, \|\sigma\|^2, \lambda)$ is meromorphic for $\mathrm{Re}(\lambda) > -1/2$, has a pole at $\lambda = 0$, and the order of the pole is at most one. Furthermore, the function $F(\omega, \xi, \|\sigma\|^2, \lambda)$ has poles at $\lambda = 1, 2, \ldots, N-1$ coming from the poles of h. In the view of (4) the Laurent expansion of $F(\omega, \xi, \|\sigma\|^2, \lambda)$ at $\lambda = 0$ is given by

$$F(\omega, \xi, \|\sigma\|^2, \lambda) = \frac{1}{\lambda} h(0) g_{\|\sigma\|^2}(0) + \Big(h'(0) g_{\|\sigma\|^2}(0) + h(0) g'_{\|\sigma\|^2}(0) \Big) + \mathcal{O}(|\lambda|). \tag{5}$$

We set

$$\begin{aligned} C_1(\omega, \|\sigma\|^2).\xi &:= h(0) g_{\|\sigma\|^2}(0), \\ C_0(\omega, \|\sigma\|^2).\xi &:= h'(0) g_{\|\sigma\|^2}(0) + h(0) g'_{\|\sigma\|^2}(0). \end{aligned}$$

and we get that

$$C_1(\omega, \|\sigma\|^2).\xi = h(0)g(0) = h(0) \sum_\iota \int_{U_\iota} \frac{1}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \Big(|z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \Big) \tag{6}$$

and the expression on the right hand side does not depend on the metric. This proves (i).

Now we look at the coefficient $C_0(\omega, \|\sigma\|^2)$. Differentiating $g_{\|\sigma\|^2}$ gives

$$\begin{aligned} g'_{\|\sigma\|^2}(\lambda) = &\sum_\iota \int_{U_\iota} \frac{\log |z_{\iota,1}|^2 |z_{\iota,1}|^{2\lambda}}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \big(\mathrm{e}^{-\phi_\iota \lambda} |z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \big) \\ &+ \sum_\iota \int_{U_\iota} \frac{|z_{\iota,1}|^{2\lambda}}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \big(-\phi_\iota \mathrm{e}^{-\phi_\iota \lambda} |z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \big) \end{aligned}$$

and letting $\lambda = 0$ we get

$$\begin{aligned} C_0(\omega, \|\sigma\|^2).\xi &= h(0) g'_{\|\sigma\|^2}(0) + h'(0) g_{\|\sigma\|^2}(0) \\ &= h(0) \sum_\iota \int_{U_\iota} \frac{\log |z_{\iota,1}|^2}{z_1} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \Big(|z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \Big) \qquad (7) \\ &- h(0) \sum_\iota \int_{U_\iota} \frac{1}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \Big(\phi_\iota |z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \Big) \qquad (8) \\ &+ h'(0) \sum_\iota \int_{U_\iota} \frac{1}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \Big(|z_{\iota,1}|^{2N} \pi^*(\omega\xi)\rho_\iota \Big). \qquad (9) \end{aligned}$$

Notice that of (7)-(9) only (8) depends on the metric. Doing the same calculations but using $|s|^2$ instead of $\|\sigma\|^2$ to regularise the integral we get similar coefficients in the Laurent expansion of $F(\omega, \xi, |s|^2, \lambda)$. We denote these coefficients by $C_1(\omega, |s|^2)$ and $C_0(\omega, |s|^2)$. We then get

$$\begin{aligned} C_0(\omega, |s|^2).\xi - C_0(\omega, \|\sigma\|^2).\xi &= h(0) \sum_\iota \int_{U_\iota} \frac{1}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \Big(|z_{\iota,1}|^{2N} (\phi_\iota - \psi_\iota) \pi^*(\omega\xi)\rho_\iota \Big) \\ &= h(0) \sum_\iota \int_{U_\iota} \frac{1}{z_{\iota,1}} \frac{\partial^{2N-1}}{\partial z_{\iota,1}^{N-1} \partial \bar{z}_{\iota,1}^N} \Big(|z_{\iota,1}|^{2N} \log \frac{|\pi^* s|^2}{\|\pi^*\sigma\|^2} \pi^*(\omega\xi)\rho_\iota \Big) \\ &= C_1(\omega, \|\sigma\|^2) \log \frac{|s|^2}{\|\sigma\|^2} \xi, \end{aligned}$$

where the last step follows in view of (6). In particular, choosing $\widehat{E} = E$ and $\sigma = s$ we get the statement in (ii). Finally, (6) also shows that $C_1(\omega, \|\sigma\|^2)$ is the push-forward of some current T on $\mathrm{Bl}_Y(X)$. The product $C_1(\omega, \|\sigma\|^2) \log \frac{|s|^2}{\|\sigma\|^2}$ should be interpreted as $C_1(\omega, \|\sigma\|^2) \log \frac{|s|^2}{\|\sigma\|^2} := \pi_*\big(T \log \frac{|\pi^* s|^2}{\|\pi^*\sigma\|^2}\big)$. Then the last step of the final calculation shows that this is well-defined. □

Remark. (1) The proof of Theorem 1.1 shows that given ω such that $|s|^{2N}\omega$ is smooth we may regularise F with any section σ such that $\{\sigma = 0\} = \{s = 0\} = Y$ and $\mathrm{d}\sigma \neq 0$ along Y.

(2) One may show that F has a meromorphic continuation to $\mathbf{C}$.

To prove Theorem 1.2 we need a lemma. Around a point in Y we pick local coordinates (z, w) of X such that $Y = \{w = 0\}$ and a local frame $\{e_j\}$ of E such that $s = \sum_j s_j e_j$. Let $\widetilde{H}$ be the hermitian matrix defined by $|s|^2 = (s_j)^t \widetilde{H} (\bar{s}_j)$. Since both w and s define the ideal of Y there is a holomorphic matrix A such that $(s_j) = Aw^t$. Letting $H = A^t \widetilde{H} \bar{A}$ we get

$$|s|^2 = w^t H \bar{w} \tag{10}$$

and H is a hermitian $(\kappa \times \kappa)$-matrix of rank κ.

Lemma 2.1. *(a) If $|s|^{2\kappa}\omega$ is smooth then there is a smooth $(n-\kappa, n-\kappa)$-form $\widetilde{\omega}$ in a neighbourhood of Y such that $\omega = \big(\frac{\bar\partial\partial|s|^2}{|s|^2}\big)^\kappa \wedge \widetilde{\omega}$.*

(b) There are real local coordinates $(t_1, \ldots, t_{2n})$ for X such that $Y = \{t_1 = \cdots = t_{2\kappa} = 0\}$ and $|s|^2 = t_1^2 + \cdots + t_{2\kappa}^2$.

(c) For the "local Laplacian" $\Delta_t = \frac{1}{4} \sum_{j=1}^{2\kappa} \frac{\partial^2}{\partial t_j^2}$ we have

$$\Delta_t^\ell |s|^{2(\lambda-k)} = d(\lambda) |s|^{2(\lambda-k-\ell)}$$

where $d(\lambda) = (\lambda-k)\cdots(\lambda-k-\ell+1)(\lambda-k-1+\kappa)\cdots(\lambda-k-\ell+\kappa)$. (Here Δ_t^ℓ means ℓ applications of Δ_t.)

Proof. (a) Let (z, w) be local coordinates in X such that $Y = \{w = 0\}$. By assumption we can write

$$\omega = \mathrm{d}w \wedge \mathrm{d}\bar{w} \wedge \frac{\omega'}{|s|^{2\kappa}} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

for some smooth function ω'. We also have $(\bar\partial\partial|s|^2)^\kappa = \pm\kappa! \det(H) \mathrm{d}w \wedge \mathrm{d}\bar{w}$ on Y; where H is the hermitian matrix defined in (10). Therefore $(\bar\partial\partial|s|^2)^\kappa = \big(\kappa! \det(H) + \mathcal{O}(|w|)\big) \mathrm{d}w \wedge \mathrm{d}\bar{w} + \mathcal{O}(\mathrm{d}z, \mathrm{d}\bar{z})$ where $\mathcal{O}(\mathrm{d}z, \mathrm{d}\bar{z})$ means any terms containing some $\mathrm{d}z_j$ or $\mathrm{d}\bar{z}_j$. Hence $(\bar\partial\partial|s|^2)^\kappa \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z} = \big(\kappa! \det(H) + \mathcal{O}(|w|)\big) \mathrm{d}w \wedge \mathrm{d}\bar{w} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}$ and therefore $(\bar\partial\partial|s|^2)^\kappa \big(\kappa! \det(H) + \mathcal{O}(|w|)\big)^{-1} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \mathrm{d}w \wedge \mathrm{d}\bar{w} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}$. Thus

$$\omega = (\bar\partial\partial|s|^2)^\kappa \big(\kappa! \det(H) + \mathcal{O}(|w|)\big)^{-1} \frac{\omega'}{|s|^{2\kappa}} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

and we may choose $\widetilde{\omega} = \big(\kappa!\det(H) + \mathcal{O}(|w|)\big)^{-1}\omega' \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}$ (or rather a sum of such expressions using a partition of unity).

(b) We write z and w in real local coordinates $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$. By noting that $w^t H \bar{w} = w^t \mathrm{Re}(H)\bar{w}$ and in view of (10) we get

$$|s|^2 = \begin{bmatrix} u^t & v^t \end{bmatrix} \begin{bmatrix} \mathrm{Re}(H) & \mathrm{Im}(H) \\ -\mathrm{Im}(H) & \mathrm{Re}(H) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Therefore the real Hessian of $|s|^2$ has rank 2κ on Y. By the Morse–Bott lemma there are real local coordinates $t_1, \dots, t_{2n}$ for X such that $|s|^2 = t_1^2 + \cdots + t_{2\kappa}^2$.

(c) Using that $|s|^2 = \sum_{j=1}^{2\kappa} t_j^2$ we have

$$\frac{\partial^2}{\partial t_j^2}|s|^{2(\lambda-k)} = (\lambda-k)(\lambda-k-1)|s|^{2(\lambda-k-2)}4t_j^2 + (\lambda-k)|s|^{2(\lambda-k-1)}2$$

and

$$\begin{aligned} \Delta_t |s|^{2(\lambda-k)} &= (\lambda-k)(\lambda-k-1)|s|^{2(\lambda-k-1)} + \kappa(\lambda-k)|s|^{2(\lambda-k-1)} \\ &= (\lambda-k)(\lambda-k-1+\kappa)|s|^{2(\lambda-k-1)}. \end{aligned}$$

Iterating this gives part (c) of the lemma. □

Proof of Theorem 1.2. First we suppose that $|s|^{2\kappa}\omega$ is smooth and let χ be a smooth function which is identically 1 in a neighbourhood of Y in which $\omega = \widetilde{\omega} \wedge \big(\frac{\bar\partial\partial|s|^2}{|s|^2}\big)^\kappa$. Then

$$\int_X |s|^{2\lambda}\omega\xi = \int_X |s|^{2\lambda}\omega\chi\xi + \int_X |s|^{2\lambda}\omega(1-\chi)\xi \tag{11}$$

and the second integral is holomorphic in λ close to the origin and therefore does not contribute to $C_1(\omega)$.

We need the following two identities, valid for $\mathrm{Re}(\lambda) \gg 1$;

$$\bar\partial|s|^{2\lambda} \wedge \partial\log|s|^2 \wedge \big(\bar\partial\partial\log|s|^2\big)^{\kappa-1} = \lambda|s|^{2\lambda}\frac{\bar\partial|s|^2}{|s|^2} \wedge \frac{\partial|s|^2}{|s|^2} \wedge \Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^{\kappa-1}, \tag{12}$$

$$\text{if } s \neq 0: \quad \bar\partial\Big(\partial\log|s|^2 \wedge \big(\bar\partial\partial\log|s|^2\big)^{\kappa-1}\Big) = \Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^\kappa - \kappa\frac{\bar\partial|s|^2}{|s|^2} \wedge \frac{\partial|s|^2}{|s|^2} \wedge \Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^{\kappa-1}. \tag{13}$$

They are straightforward to prove by applying the left-most $\bar\partial$ in both cases and then noting that

$$\frac{\bar\partial|s|^2}{|s|^2} \wedge \frac{\partial|s|^2}{|s|^2} \wedge \big(\bar\partial\partial\log|s|^2\big)^{\kappa-1} = \frac{\bar\partial|s|^2}{|s|^2} \wedge \frac{\partial|s|^2}{|s|^2} \wedge \Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^{\kappa-1}.$$

Multiplying the identity (13) with $|s|^{2\lambda}$ it holds, in the sense of currents, on all of X if $\mathrm{Re}(\lambda) \gg 1$ and it then says

$$|s|^{2\lambda}\bar\partial\Big(\partial\log|s|^2 \wedge \big(\bar\partial\partial\log|s|^2\big)^{\kappa-1}\Big) = |s|^{2\lambda}\Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^\kappa - \kappa|s|^{2\lambda}\frac{\bar\partial|s|^2}{|s|^2} \wedge \frac{\partial|s|^2}{|s|^2} \wedge \Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^{\kappa-1}. \tag{14}$$

Using (14) and (12) we get

$$\begin{aligned} \frac{\kappa}{\lambda}\bar\partial|s|^{2\lambda} \wedge \partial\log|s|^2 \wedge \big(\bar\partial\partial\log|s|^2\big)^{\kappa-1} &= \kappa|s|^{2\lambda}\frac{\bar\partial|s|^2}{|s|^2} \wedge \frac{\partial|s|^2}{|s|^2} \wedge \Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^{\kappa-1} \\ &= |s|^{2\lambda}\Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^\kappa - |s|^{2\lambda}\bar\partial\Big(\partial\log|s|^2 \wedge \big(\bar\partial\partial\log|s|^2\big)^{\kappa-1}\Big). \end{aligned} \tag{15}$$

By Lemma 2.1 (a) and by our choice of the function χ we can write $\omega = \big(\frac{\bar\partial\partial|s|^2}{|s|^2}\big)^\kappa \wedge \widetilde\omega$, with $\widetilde\omega$ smooth in some neighbourhood of Y. Using this and (15) gives

$$\begin{aligned}\int_X |s|^{2\lambda}\chi\omega\xi &= \int_X |s|^{2\lambda}\Big(\frac{\bar\partial\partial|s|^2}{|s|^2}\Big)^\kappa \wedge \chi\widetilde\omega\xi \\ &= \frac{\kappa}{\lambda}\int_X \bar\partial|s|^{2\lambda}\wedge\partial\log|s|^2\wedge\big(\bar\partial\partial\log|s|^2\big)^{\kappa-1}\wedge\chi\widetilde\omega\xi \qquad (16)\\ &+ \int_X |s|^{2\lambda}\bar\partial\Big(\partial\log|s|^2\wedge\big(\bar\partial\partial\log|s|^2\big)^{\kappa-1}\Big)\wedge\chi\widetilde\omega\xi. \qquad (17)\end{aligned}$$

The integral in (16) is studied in [**And**, Proposition 4.1 and 4.3]. It is holomorphic in λ for $\mathrm{Re}(\lambda) > -\varepsilon$, for some $\varepsilon > 0$, and its value at $\lambda = 0$ equals $\kappa(2\pi\mathrm{i})^\kappa\int_Y\widetilde\omega\xi$.

We claim that the integral (17) does not contribute to $C_1(\omega)$. To see this it suffices to prove that it is holomorphic in λ close to $\lambda = 0$. Let $\pi : \mathrm{Bl}_Y(X) \to X$ be the blow-up of X along Y as before. Locally, $\pi^*s = z_1 s'$ where $s' \neq 0$ and hence $\bar\partial\partial\log|\pi^*s|^2 = \bar\partial\partial\log|z_1|^2 + \bar\partial\partial\log|s'|^2 = \pi\mathrm{i}[z_1 = 0] + \bar\partial\partial\log|s'|^2$. The form $\bar\partial\partial\log|s'|$ is the first Chern form of the bundle π^*E, up to some constant, in particular it is a global form on $\mathrm{Bl}_Y(X)$. For $\mathrm{Re}(\lambda) \gg 1$ we have $|\pi^*s|^{2\lambda}[z_1 = 0] = 0$ and therefore

$$|\pi^*s|^{2\lambda}\bar\partial\Big(\partial\log|\pi^*s|^2\wedge\big(\bar\partial\partial\log|\pi^*s|^2\big)^{\kappa-1}\Big) = |\pi^*s|^{2\lambda}\big(\bar\partial\partial\log|s'|^2\big)^\kappa$$

which is integrable for $\mathrm{Re}(\lambda) > -1/2$. Hence, the integral (17) becomes

$$\int_{\mathrm{Bl}_Y(X)} |\pi^*s|^{2\lambda}\big(\bar\partial\partial\log|s'|^2\big)^\kappa\wedge\pi^*(\chi\widetilde\omega\xi) \qquad (18)$$

which is holomorphic in λ for close to $\lambda = 0$. Therefore the integral (17) is holomorphic in λ and does not contribute to $C_1(\omega)$. Summing up the only contribution to $C_1(\omega)$ comes from (16) and

$$C_1(\omega) = \kappa(2\pi\mathrm{i})^\kappa\int_Y\widetilde\omega\xi,$$

since $\chi \equiv 1$ on Y. This gives the first part of the theorem with $\mathrm{res}(\omega) = \widetilde\omega|_Y$.

Now we suppose $|s|^{2N}\omega$ is smooth with $N > \kappa$. By Lemma 2.1 (b) there are local coordinates $(t_1, \dots, t_{2\kappa}, \tau)$ around every point in Y such that $|s|^2 = t_1^2 + \cdots + t_{2\kappa}^2$. Let $\{\rho_j\}$ be a partition of unity chosen so that we may find such coordinates in every $\mathrm{supp}(\rho_j)$. We will use the local differential operator Δ_t defined in Lemma 2.1 (c). Recall that $\Delta_t^\ell|s|^{2(\lambda-\kappa)} = d(\lambda)|s|^{2(\lambda-\kappa-\ell)}$ with $d(0) \neq 0$. Letting $\ell = N - \kappa$ we get

$$\begin{aligned}\int_X |s|^{2\lambda}\omega\xi &= \sum_j\int_X |s|^{2(\lambda-\kappa-\ell)}|s|^{2N}\rho_j\omega\xi \\ &= \frac{1}{d(\lambda)}\sum_j\int_X \Delta_t^\ell\big(|s|^{2(\lambda-\kappa)}\big)|s|^{2N}\rho_j\omega\xi \\ &= \frac{1}{d(\lambda)}\sum_j\int_X |s|^{2(\lambda-\kappa)}\Delta_t^\ell\Big(|s|^{2N}\rho_j\omega\xi\Big).\end{aligned}$$

We define a global differential operator on (n,n)-forms ψ acting as Lie derivatives by

$$Q(\psi) = d(0)^{-1}\sum_j\Delta_t^\ell\big(\psi\rho_j\big)$$

which obviously depends on local coordinates and the partition of unity. Letting $\omega' = |s|^{-2\kappa}Q(|s|^{2N}\omega\xi)$ we have

$$\int_X |s|^{2\lambda}\omega\xi = \frac{d(0)}{d(\lambda)}\int_X |s|^{2\lambda}\omega'$$

and $|s|^{2\kappa}\omega'$ is smooth. Therefore the calculations in the proof of the first part of the theorem now gives

$$C_1(\omega).\xi = \kappa(2\pi \mathrm{i})^\kappa \int_Y \mathrm{res}\Big(|s|^{-2\kappa} Q(|s|^{2N}\omega\xi)\Big).$$

If Q' is another differential operator constructed in the above way we get

$$\int_Y \mathrm{res}\Big(|s|^{-2\kappa} Q(|s|^{2N}\omega\xi)\Big) - \mathrm{res}\Big(|s|^{-2\kappa} Q'(|s|^{2N}\omega\xi)\Big) = C_1(\omega).\xi - C_1(\omega).\xi = 0.$$

Since ξ is a test function both $\mathrm{res}\big(|s|^{-2\kappa} Q(|s|^{2N}\omega\xi)\big)$ and $\mathrm{res}\big(|s|^{-2\kappa} Q'(|s|^{2N}\omega\xi)\big)$ have compact support and thus by Poincaré duality $\big[\mathrm{res}\big(|s|^{-2\kappa} Q(|s|^{2N}\omega\xi)\big)\big]$ is a well-defined de Rham cohomology class on Y. This proves the second claim of the theorem with $\mathrm{Res}(\omega\xi) = \big[\mathrm{res}\big(|s|^{-2\kappa} Q(|s|^{2N}\omega\xi)\big)\big]$. $\square$

References

[**And**] M. Andersson. "Residues of holomorphic sections and Lelong currents". In: *Arkiv för matematik* 43 (2 2005), pp. 201–219.

[**FK**] G. Felder and D. Kazhdan. "Divergent integrals, Residues of Dolbeault Forms, and Asymptotic Riemann Mappings". In: *International Mathematics Research Notices* 2017 (19), pp. 5897–5918.

[**FK2**] G. Felder and D. Kazhdan. "Regularization of divergent integrals". In: *Selecta Mathematica* 24.1 (2018), pp. 157–186.

[**ML**] M. Lennartsson. "Residues and currents from singular forms on complex manifolds". In: *arXiv:2005.02888* (2020).

[**Wi**] E. Witten. "Perturbative String Theory Revisited". In: *arXiv:1209.5461* (2012).