CHALMERS | GÖTEBORGS UNIVERSITET

The Banach–Tarski paradox and its implications on the problem of measure

Banach–Tarski paradoxen och dess implikationer på måttproblemet

Examensarbete för kandidatexamen i matematik vid Göteborgs universitet
Kandidatarbete inom civilingenjörsutbildningen vid Chalmers

Lukas Enarsson
Oskar Johansson
Vincent Molin
Emil Timlin

Institutionen för Matematiska vetenskaper
CHALMERS TEKNISKA HÖGSKOLA
GÖTEBORGS UNIVERSITET
Göteborg, Sverige 2020

The Banach–Tarski paradox and its implications on the problem of measure

Examensarbete för kandidatexamen i matematik vid Göteborgs universitet

Vincent Molin

Kandidatarbete i matematik inom civilingenjärsprogrammet Teknisk matematik vid Chalmers

Lukas Enarsson Oskar Johansson Emil Timlin

Handledare: Andrew McKee

Institutionen för Matematiska vetenskaper
CHALMERS TEKNISKA HÖGSKOLA
GÖTEBORGS UNIVERSITET
Göteborg, Sverige 2020

Preface

This bachelor thesis presents the Banach–Tarski paradox and discusses the problem of defining a measure on $\mathbb{R}^n$. It was written at the department of Mathematical Sciences at Chalmers University of Technology in Gothenburg. Each group member has kept an individual log of their work on the project. The group has also kept a communal more general log describing the progress of the project.

Work process

In order to structure the work on the project, the group has each week met with their supervisor to discuss the reading material and to decide what should be read before and discussed at the next meeting. The main points for each meeting were then usually assigned for presentation internally to individual group members.

Report

All parts in the report has an individual group member as the responsible author.

- **Lukas Enarsson:** Popular science presentation, Section 1, Subsections 3.1, 4.2, 5.1-5.2.1, 5.3 (the part about amenable groups), 5.4 (The part about why the measure must extend Lebesgue measure), Introduction in section 5.

- **Oskar Johansson:** Subsections 2.1, 4.1, 5.3 (the part about Tarski's Theorem), Figure 2 in Appendix A.

- **Vincent Molin:** Preface, Subsections 2.2.4, 3.2, 4.4-4.6 up to and including Theorem 4.20 with proof, Appendix B, Biblatex-references, short introductions Sections 2,3 and 4.

- **Emil Timlin:** Abstract, subsections 2.2-2.2.3, 2.2.5, discussion after Theorem 4.20 and Theorem 4.21, subsection 5.2.2, section 5.4, Appendix C.

Acknowledgments

We would like to thank our supervisor Andrew McKee for his guidance, ideas and mathematical input; as well as Ellen, Åsa and Per Enarsson for proofreading the popular science presentation.

Popular Science presentation

The Banach–Tarski paradox, the art of cloning balls with rotations

If you have ever cooked a meal or baked something, you probably have had to measure volume at some point. Imagine how annoying it would be if you measured flour, and when you rotated the measure to pour it in, more flour would pour out of it than what you have measured. Of course, rotations or movement cannot affect the volume of an object, right?

In 1901, Henri Lebesgue described the Lebesgue measure, a way to mathematically determine the volume of objects, regardless of how many dimensions the object is in. The Lebesgue measure of two objects is the sum of their individual measurements, which remain the same when rotating or moving the objects, given that they can be Lebesgue measured. This means that as long as the object can be measured, its Lebesgue measure works exactly as a volume should work.

While at the time, it was assumed that every object could be Lebesgue measured, abstract objects with no defined Lebesgue measure were found by mathematicians over time, which created some problems with fully defining a volume. In particular, Stefan Banach and Alfred Tarski found in 1924 that a ball could be cut into a few of these objects which would form two balls identical to the original when rotated and moved. This has become known as the Banach–Tarski paradox.

While this might seem crazy at first, it should be noted that a ball contains an infinite number of points and infinities work in really strange ways. There are two types of infinity relevant to the Banach–Tarski paradox. The first is countable infinity, which is infinite, but can be ordered. One example is the amount of natural numbers, since they can be order, even if they would never end. The other is uncountable infinity, which is so large it can never be ordered. An example of this is the number of points on a ball. If you were to try to order every point, you could always find another point between the points that have been ordered.

Now, some strange things start happening when we use infinities. Imagine taking a disc, marking a point on it, and start to rotate the disc. Every time you rotated it at a chosen angle, you mark the point reached from the last point by the rotation. At some angles, for example $\sqrt{2}^{\circ}$, you would never mark the same point twice, since you would never go 360° around the disc after any number of $\sqrt{2}^{\circ}$ rotations. If you did this a countably infinite number of times and then rotate the disc $\sqrt{2}^{\circ}$ in the opposite direction, the point one rotation behind the location marked would now be marked, while every previously marked point would still be marked, since every point has a marked point one rotation in front of them to take its place when rotated back. We have somehow marked another point by simply rotating the disc. Because of this, you could fill in a single hole on the disc by rotating points that would be marked by this method into the hole. In fact, if you kept doing it an infinite number of times, you could fill in countably infinitely many holes.

Another oddity of infinity is that you can create two infinities out of one. Imagine an infinite labyrinth consisting of four-way crossroads going up, down, left and right, with only one path to each crossroad. To clarify, this means that going up and then right would lead to a different crossroad than going right and then up. Pick a starting point and write U when going up, D when going down, L when going left and R when going R. If you keep writing each step from right to left and remove any instance of LR, RL, UD and DU, the word corresponds to the simplest path from the starting point, and if you were to write every path from the starting point, it would cover every crossroad, with each crossroad corresponding to a word. If you were then to take every crossroad whose word starts with L, and go to the crossroad on the right of each of them, you would end up in every crossroad except the ones starting with R, since L and R cannot be next to each other. This means that the crossroads to the right of every crossroad starting with L, together with the crossroads starting with R, giving every crossroad in the labyrinth, using only part of it. By doing the same with U and D, we also get every crossroad. The key idea of the Banach–Tarski paradox is to find rotations that work like this labyrinth on the entire ball except for a countable number of points, then we can clone those points using this method and clone the countably many points by filling in them like a disc.

So let's now show the Banach–Tarski paradox on a ball by dividing the ball into groups of points. Start by marking a starting point on the surface and place it in a group called "S". From there, rotate the ball $\sqrt{2}^{\circ}$ to the left, right, up or down. Write the rotations as a word like the labyrinth example, and put each point accessed from these rotations into the group "U", "D", "L" or “R”, depending on the first letter of the rotations corresponding word. Once you have done this for every rotation, you have only covered a countably infinite number of points, so not all points have been put in a group. Pick a new starting point that hasn’t been marked yet and repeat the process an uncountably infinite number of times until the whole surface has been covered.

Now some of these points will have been put into multiple groups since every rotation corresponds to an axis, making the poles remain in place when you rotate around the axis. However, since the rotations correspond to a word, they are countable, meaning that there are countably many poles. We put these poles into a group called “P”. For the interior of the ball, put the center into its own group “C”, and for the rest, let them be in the same group as the surface point just above them. We have now divided the ball into seven pieces. S, U, D, L, R, P and C. Now, let’s start copying the sets. Since all the poles have been removed, each rotation from a point in S will be taken to a new point. This is like the labyrinth example, and like it, L rotated to the right, together with R, creates the groups S, U, D, L and R. The same thing can be done with U and D to get another copy of these groups. We then place P and C onto the first ball. After this, the second ball does not have P, or C on it, but since they are not uncountably infinite, we can deal with them easily. To get a copy of P on the second ball, observe that since P only contains countable points, they make up individual points on the ball. We can then take an axis that doesn’t correspond to any word, and fill in holes of our second ball using rotations like the disc. C is copied onto the ball in a similar manner. The end result is two balls identical to the first.

Now the reason why this result is interesting is that we increased the volume using only rotations on pieces of the ball. The reason why this happens is because the pieces that we cut the ball into are infinitely complex, leading to them having a different volume depending on which other pieces they are put together with. Since the rotations we used can also duplicate themselves, we use these to clone the pieces and thus clone the ball. It should be noted that rotations cannot duplicate themselves the same way in one or two dimensions. In fact, they have some nice properties that give the line and plane a nice measure. This means that while we cannot measure the volume of all objects in space, we can measure the area of every object on a plane and the length of every object on a line. This measure is also equal to the Lebesgue measure if the object is Lebesgue measurable, giving them the length or area that they intuitively should have.

Sammanfattning

Vi presenterar ett bevis av en sats av Stefan Banach och Alfred Tarski, som bygger på resultat av Felix Hausdorff: Det finns två ändliga samlingar av disjunkta delmängder av enhetsbollen i $\mathbb{R}^3$ sådana att varje samling kan transformeras till en ny enhetsboll under verkan av stela rörelser (ändliga kombinationer av translationer och rotationer). Detta resultat förlängs sedan till dess starka form: Om A, B är två begränsade delmängder av $\mathbb{R}^3$ med icke-tomt inre så finns två partitioner $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$ av A och B respektive, och stela rörelser $\rho_1, \rho_2, ..., \rho_n$ sådana att $\rho_i(A_i) = B_i$ för varje $i = 1, 2, ..., n$. Dessa satser kallas för Banach–Tarski paradoxen.

Måttproblemet ställer frågan huruvida man kan tilldela en volym till varje delmängd av $\mathbb{R}^n$ för $n \in \mathbb{N}$ så att volym bevaras under stela rörelser och partitionering. Vi visar att, som en konsekvens av Banach–Tarski paradoxen, kan man inte ge ett jakande svar till måttproblemet för $n > 2$. Vi diskuterar om detta kan ges i en och två dimensioner, och i allmänhet hur problemet att tilldela en volym till varje delmängd av en mängd X relaterar till existensen av dekompostitoner av delmängder av X liknande dem ovan, där elementen som transformerar dekompositionerna kan höra till vilken klass som helst av bijektioner av X.

Abstract

We present a proof of a theorem of Stefan Banach and Alfred Tarski, building on work by Felix Hausdorff: There exist two finite collections of disjoint subsets of the unit ball in $\mathbb{R}^3$ such that each collection is transformed to another unit ball when subject to rigid motions (finite combinations of translations and rotations). This result is extended into its strong form: For any two bounded subsets A, B of $\mathbb{R}^3$ with nonempty interior there exist partitions $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$ of A and B respectively, and rigid motions $\rho_1, \rho_2, ..., \rho_n$ such that $\rho_i(A_i) = B_i$ for each $i = 1, 2, ..., n$. These theorems are referred to as the Banach–Tarski paradox.

The problem of measure asks if one can assign a volume to every subset of $\mathbb{R}^n$ for $n \in \mathbb{N}$ in a way so that volume is preserved under rigid motion and partitioning. We show that, as a consequence of the Banach–Tarski paradox, one cannot give a positive answer to the problem of measure for $n > 2$. We discuss whether this can be done in one and two dimensions, and in general how the problem of defining a volume of every subset of a set X relates to the existence of decompositions of subsets of X similar to those above, where the elements transforming the decompositions can belong to any class of bijections of X.

Contents

1 Introduction

1.1 Background

The length of a line, the area of a surface and the volume of a shape in three or more dimensions, these are some of the most fundamental concepts encountered in Euclidean geometry [1], all based around measuring subsets of $\mathbb{R}^n$. But even then, there is still the question of how to define these measures in a way that also corresponds to intuitive parts of geometry, like the unit cube having a measure of one, having the measure of disjoint parts be equal to the sum of the measures of each part, and the measure being preserved by rigid motions like rotations and translations. Finding a way to define the measure of subsets of $\mathbb{R}^n$ is known as the problem of measure.

In 1901, Lebesgue [2] defined the Lebesgue measure, a countably additive, isometry-invariant measure on $\mathbb{R}^n$, as a solution to the problem of measure. However, it was shown by Vitali [3] in 1905 that by using the axiom of choice, countable additivity and the other properties of Lebesgue measure, one could find sets without a defined Lebesgue measure. Since countable additivity caused sets with no defined Lebesgue measure, mathematicians tried to find an extension of Lebesgue measure that solves the problem of measure but only required the measure to be finitely additive.

The search for a finitely additive measure eventually proved fruitless, as Hausdorff [4] proved that there was no finitely additive measure with measures preserved by rigid motions in three dimensions or higher. After Hausdorff's discovery, mathematicians looked for and discovered more geometrical paradoxes, the most striking being the Banach–Tarski paradox [5] [6] discovered by Banach and Tarski in 1924, a theorem that in its most general form says that given two bounded sets $A, B \subset \mathbb{R}^3$ with nonempty interiors, A can be transformed into B by cutting A into a finite number of parts and rearrange the parts with rigid motions. In particular, you could cut a ball into pieces and rearrange the pieces to become two new balls, each identical to the original ball. This shows the subtlety of problem of measure, since the Banach–Tarski paradox implies that objects change measure during rigid motions. In this report we aim to show the proof of the general Banach–Tarski paradox and explore some of its impact on measure theory.

1.2 The axiom of choice

The Banach–Tarski paradox relies on an axiom known as the *axiom of choice* (AC), which was formulated by Zermelo [7] in 1904. AC says that whenever we have a collection of nonempty sets, we can create a new nonempty set by picking one element from each of the previous sets. The reader might realise that this is obvious for finite sets regardless of AC since we can always pick the first element from each set but it might not be for infinite sets, thus requiring AC.

Note that the axiom of choice is essential to the Banach–Tarski paradox as it has been proven [8] that the Banach–Tarski paradox does not exist in set theories without it. Because of this, some mathematicians have been unsure whether AC is true or not. Tomkowicz and Wagon [9] write that Borel [10] objected to the use of AC in the proof of the Hausdorff paradox (a paradox we will later use to prove the Banach–Tarski paradox), since the proof uses AC to create a vague set. Meanwhile, Banach and Tarski [5] defended AC as there are theorems proven by it that are fully intuitive. We are not going to think too deeply into the philosophical aspects of AC and assume it to be true for the rest of the paper. We will however denote theorems that use it with AC.

2 Preliminaries

Before we can begin our exposition of the Banach–Tarski paradox we establish some basic definitions and facts about groups and cardinality. The more experienced reader can skip ahead to Section 3 where we start to discuss paradoxes.

2.1 Cardinality

It is tempting to say that the Banach–Tarski paradox is incorrect because seemingly one ball contains fewer points than two copies of the same ball. We need to be careful when talking about the number of elements in infinite sets. What exactly is meant with "fewer" in this case? The mathematical term we are looking for is cardinality. The cardinality of a set A is denoted $|A|$ and is a way of quantifying how many elements A contains. For finite sets, cardinality means precisely the number of elements. For example $|\{1, ..., n\}| = n$. However, cardinality extends beyond the finite case and lets us compare sets with an infinite number of elements.

Definition 2.1. Let A and B be two sets. We say that:

- $|A| = |B|$ if there is a bijective map $f : A \to B$,
- $|A| \leq |B|$ if there is an injective map $f : A \to B$,
- A is finite if $A = \emptyset$ or $|A| = |\{1, ..., n\}|$ for some[1] $n \in \mathbb{N}$, otherwise we say that A is infinite,
- A is countable if $|A| \leq |\mathbb{N}|$, otherwise we say that A is uncountable.

Proposition 2.2. *Let $A_1, A_2, ...$ be a countable collection of countable sets A_i, $i \in \mathcal{I} \subseteq \mathbb{N}$. Then the set $\bigcup_{i \in \mathcal{I}} A_i$ is countable.*

Proof. Assume that $A_i \cap A_j = \emptyset$ for all $i, j \in \mathcal{I}$, $i \neq j$. If not, consider the new sets $A'_1 = A_1$, $A'_2 = A_2 \setminus A_1$, $A'_3 = A_3 \setminus (A_1 \cup A_2)$ and so on. Since A_i is countable it makes sense to talk about the jth element of A_i for some ordering. Let $a_{i,j}$ denote the jth element of A_i. The map $a_{i,j} \mapsto 2^i 3^j$ is an injective map from $\bigcup_{i \in \mathcal{I}} A_i$ to $\mathbb{N}$. Thus $|\bigcup_{i \in \mathcal{I}} A_i| \leq |\mathbb{N}|$ and $\bigcup_{i \in \mathcal{I}} A_i$ is countable. □

Proposition 2.3. *Let A and B be two sets where A is uncountable and B is countable. Then $A \setminus B$ is uncountable*

Proof. Assume that $A \setminus B$ is countable. By Propostion 2.2 the set $(A \setminus B) \cup B$ is also countable. We have $A \subseteq (A \setminus B) \cup B$, thus A has to be countable. This is a contradiction, therefore $A \setminus B$ is uncountable. □

Example 2.4. Let $B = \{(r, \theta, \phi) : r \in (0, 1], \theta \in [0, 2\pi), \phi \in [0, \pi]\}$. In words, B is a unit ball missing the center point. The map

$$(r, \theta, \phi) \mapsto \begin{cases} (2r, \theta, \phi), & \text{if} \quad r \in (0, \frac{1}{2}] \\ (2r - 1, \theta, \phi) & \text{on a second copy of the same ball, if} \quad r \in (\frac{1}{2}, 1] \end{cases}$$

is a bijection from B to two identical copies of B. Therefore, B has the same cardinality as the set containing two copies of B.

Does Example 2.4 prove the Banach–Tarski paradox? No it does not: the Banach–Tarski paradox only relies on rigid motions; no stretching is required, which our bijection certainly uses.

2.2 Selected concepts in group theory

In this section we will build the basic group theory that we need. Great parts of a standard introduction to group theory are left out. A thorough introduction to group theory can be found in Modern Algebra – An Introduction, by Durbin [11].

2.2.1 Groups and subgroups

Definition 2.5. Let G be a set and let $*$ be a binary operation on G. The ordered pair $(G, *)$ is called a *group* if the following axioms are satisfied:

1. For all $a, b, c \in G$ it holds that $(a * b) * c = a * (b * c)$. We say that $*$ is *associative* on G.
2. There is an element $e \in G$ such that $e * g = g * e = g$ for all $g \in G$. The element e is called an *identity element*.
3. For every $g \in G$, there is an element $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. The element g^{-1} is called an *inverse* of g.

[1]In this paper $0 \notin \mathbb{N}$.

We will often write simply G to refer to the group $(G, *)$. Also, the term *binary* will always be implicit when we are talking about operations.

Example 2.6. The integers together with addition is a group, $(\mathbb{Z}, +)$. Addition is associative, 0 is an identity element and an inverse of $n \in \mathbb{Z}$ is $-n \in \mathbb{Z}$.

Note that the group axioms only assumes the existence of an identity and of inverses, not uniqueness. Of course, in the example above we know that 0 is the only identity and that $-n$ is the unique inverse of n. This is true in general.

Proposition 2.7. *Let G be a group. There is only one identity element $e \in G$. If $g \in G$, then $g^{-1} \in G$ is its unique inverse.*

Proof. Assume that e and f are identity elements of G. Then, $e = e * f = f$. In the first equality we used that f is an identity, and in the second equality that e is an identity.

Let $g \in G$ and assume that h is an inverse of a. Using the definition of an inverse and associativity we have that $g^{-1} = g^{-1} * e = g^{-1} * (g * h) = (g^{-1} * g) * h = e * h = h$. □

Here follows a few examples of groups, the first and last of which will be used in the paper.

Example 2.8. For any $n \in \mathbb{N}$, congruence modulo n is an equivalence relation on the integers. Let $\mathbb{Z}_n$ denote the set of equivalence classes. If m and k are integers, define $[m]_n + [k]_n := [m+k]_n$. This operation is independent of the choice of representatives, so it is well-defined. It is clearly associative since addition of integers is associative. The identity is $[0]_n$ and for any $[m]_n \in \mathbb{Z}_n$, $[-m]_n \in \mathbb{Z}_n$ is its (unique) inverse. Thus, $(\mathbb{Z}_n, +)$ is a group. We will omit the brackets when working with elements of $\mathbb{Z}_n$.

Example 2.9. Let V be a real vector space[2] and let $\mathrm{GL}(V)$ denote the set of all linear bijections on V. If S and T are linear bijections on V, then so is $T \circ S$, thus $\mathrm{GL}(V)$ is closed with respect to composition (which is always associative). The identity map on V is clearly in $\mathrm{GL}(V)$ and if $S \in \mathrm{GL}(V)$, then S^{-1} exists and is linear, so inverses are contained in $\mathrm{GL}(V)$. Thus, $\mathrm{GL}(V)$ with composition is a group.

Example 2.10. Let $\mathrm{GL}_n(\mathbb{R})$ denote the set of all real, invertible $n \times n$ matrices. The product of two invertible matrices is again invertible, so matrix multiplication is an (associative) operation on $\mathrm{GL}_n(\mathbb{R})$. The identity matrix $I \in \mathrm{GL}_n(\mathbb{R})$, and if $A \in \mathrm{GL}_n(\mathbb{R})$ then A^{-1} exists and is in $\mathrm{GL}_n(\mathbb{R})$. Thus, $\mathrm{GL}_n(\mathbb{R})$ with matrix multiplication is a group, called *the general linear group*. As we will make precise later on, $\mathrm{GL}_n(\mathbb{R})$ is basically the same group as $\mathrm{GL}(V)$ for any vector space V with $\dim(V) = n < \infty$.

Example 2.11. Let SO_n denote the set of all real, orthogonal $n \times n$ matrices with determinant equal to one. That is,

$$\mathrm{SO}_n = \{A \in \mathbb{R}^{n \times n} : \ AA^T = I, \ \det(A) = 1\}.$$

As we will see in Section 2.2.4, SO_n is a group together with matrix multiplication, called *the special orthogonal group*. Note that $\mathrm{SO}_n \subseteq \mathrm{GL}_n(\mathbb{R})$, the next definition will clarify their relationship.

Definition 2.12. Let $(G, *)$ be a group and let H be a subset of G. If $(H, *)$ is also a group, then we say that $(H, *)$ *is a subgroup of* $(G, *)$, or more simply that H *is a subgroup of* G.

Example 2.13. Since $\mathrm{SO}_n \subseteq \mathrm{GL}_n(\mathbb{R})$ and they share the same group operation, SO_n is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

At first glance, one may take for granted that a group and a subgroup share the same identity and inverse elements. While this is true and easy to prove, it is not completely trivial.

Lemma 2.14. *Let G be a group with identity e and let H be a subgroup of G. Then e is the identity of H, and if $g \in H$ then the inverse of g in G, g^{-1}, is the inverse of g in H.*

[2] V could be a vector space over any *field* K. But since fields are objects of abstract algebra which we have not defined, we will be content with talking about vector spaces over $\mathbb{R}$ only. This is also all we need in this paper.

Proof. Let e_H be the identity of H. Being the identity, e_H is its own inverse in H, and if we let f be the inverse of e_H in G we get that $e = e_H * f = (e_H * e_H) * f = e_H * (e_H * f) = e_H * e = e_H$.

Assume that $g \in H$. Let g_H^{-1} denote the inverse of g in H. Using the first part we have that $g_H^{-1} * g = e_H = e = g^{-1} * g$. Multiplying both sides from the right by g^{-1} and using associativity gives that $g_H^{-1} = g^{-1}$. □

Next, we will state and prove a few basic algebraic laws for elements of a group. We will now also make use of multiplicative notation, omitting the operator $*$ if G is a general group. Thus, if $g, h \in G$, we will write gh for the element $g * h \in G$. Also, we will occasionally omit parentheses in products of more than two group elements; we can do this because the associativity axiom in the definition of a group generalizes to arbitrary products of finitely many group elements, meaning that the order in which we multiply elements does not matter.

Proposition 2.15. *Let G be a group and let a, b, c be elements in G.*

1. *If $ab = ac$ or $ba = ca$, then $b = c$.*
2. *The equations $ax = b$ and $xa = b$ have unique solutions $x = a^{-1}b$ and $x = ba^{-1}$, respectively.*
3. *Finally, $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. Consider statement (1). If $ab = ac$, then by multiplying from the left by a^{-1} and using associativity and the definition of an inverse we get $b = c$. The other case is analogous. These laws are called the *left cancellation law* and the *right cancellation law*, respectively.

Consider the equation $ax = b$ in (2). If x solves this equation, then again we multiply from the left by a^{-1} to get $x = a^{-1}b$; so this is the only possible solution. Insert $a^{-1}b$ into $ax = b$ to see that it is in fact a solution. The other part of (2) is analogous.

Consider statement (3). If $a \in G$, then a has inverse a^{-1}, i.e. $aa^{-1} = a^{-1}a = e$. But this also shows that the inverse of a^{-1} is a. Thus $(a^{-1})^{-1} = a$. If $a, b \in G$, then $(ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$ by associativity. Thus, ab has the (unique) inverse $b^{-1}a^{-1}$. □

Now we will consider an important class of subgroups. Let $(G, *)$ be a group and S a nonempty subset of G. Define the set $\langle S \rangle$ by

$$\langle S \rangle = \{g_1 g_2 ... g_n \in G : n \in \mathbb{N} \text{ and } g_i \in S \text{ or } g_i^{-1} \in S \text{ for all } i = 1, 2, ...n\},$$

i.e. S is the set of all finite combinations of elements and inverses of elements in S.

Proposition 2.16. *The set $\langle S \rangle$ together with the operation $*$ is a subgroup of G. It is called the subgroup generated by S.*

Proof. Associativity is inherited from G. Since a product of two finite products is a finite product, $\langle S \rangle$ is closed. Inverses are contained in $\langle S \rangle$ by definition. Since S is nonempty we can take $g \in S$, then $e = gg^{-1} \in \langle S \rangle$. □

If $S = \{g_1, g_2, ..., g_n\}$ is finite, then we simply write $\langle g_1, g_2, ..., g_n \rangle$. The case when S contain only two elements is important in this paper.

We continue our quick exposition of group theory by defining integer powers of elements. This is done recursively. Let G be a group and let $g \in G$. Define $g^0 = e$, $g^n = g^{n-1}g$ and $g^{-n} = (g^{-1})^n$ for all positive integers n. We also have a few basic counting rules regarding powers of elements.

Proposition 2.17. *Let G be a group and let $g \in G$. Then,*

$$g^m g^n = g^{m+n} \quad \text{and} \quad (g^m)^n = g^{mn} \text{ for all integers } m \text{ and } n.$$

This result is proved using induction, but will not be done here. See Chapter 1.1 in [12] for proofs.

2.2.2 Homomorphisms and isomorphisms

The next part of this section regards certain functions between groups. These functions are fundamental in group theory and are just as important as groups themselves.

Definition 2.18. Let $(G, *)$ and $(H, \star)$ be groups. A *homomorphism* is a function $\theta : G \to H$ such that
$$\theta(g * h) = \theta(g) \star \theta(h) \text{ for all } g, h \in G.$$

Homomorphisms occur in other parts of algebra as well, so to be specific one can also call the function in the definition above a *group homomorphism.* Though, we will be content with saying just homomorphism. In words the definition says that it does not matter whether we multiply in G and then apply θ, or if we apply θ first and then multiply in H.

Example 2.19. Let $G = (\mathbb{R}, +)$ and $H = (\mathbb{R}_{>0}, \cdot)$ where $\mathbb{R}_{>0}$ denotes the positive real numbers, and define $\theta : G \to H$ by $\theta(x) = e^x$ for all $x \in G$. Then, θ is a homomorphism since
$$\theta(x + y) = e^{x+y} = e^x e^y = \theta(x)\theta(y) \text{ for all } x, y \in G.$$
It is easy to check that $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$ are in fact groups.

The following proposition states three properties of homomorphisms.

Proposition 2.20. *Let G and H be groups, let $g \in G$ and $\theta : G \to H$ be a homomorphism. Then,*

1. $\theta(e_G) = e_H$,
2. $\theta(g^{-1}) = \theta(g)^{-1}$,
3. $\theta(g^k) = \theta(g)^k$, $k \in \mathbb{Z}$.

Proof. Consider statement (1). We have $\theta(e_G) = \theta(e_G e_G) = \theta(e_G)\theta(e_G)$. Multiplying both sides by the inverse of $\theta(e_G)$, we get that $e_H = \theta(e_G)$.

Using the first part, we get that $e_H = \theta(gg^{-1}) = \theta(g)\theta(g^{-1})$ and similarly that $e_H = \theta(g^{-1})\theta(g)$. By definition, $\theta(g^{-1})$ is the inverse of $\theta(g)$, so $\theta(g^{-1}) = \theta(g)^{-1}$.

We can show the last statement by induction on k. Let $k = 0$, then $\theta(g^0) = \theta(e_G) = e_H = \theta(g)^0$, by definition of the zeroth power. Assume that property (3) holds for $k - 1 \geq 0$. Then $\theta(g^k) = \theta(g^{k-1}g) = \theta(g^{k-1})\theta(g) = \theta(g)^{k-1}\theta(g) = \theta(g)^k$. The case when $k < 0$ is done analogously. □

Theorem 2.20 says that the identity, inverses and powers of elements are preserved under θ. Homomorphisms preserves many other properties of the group G to its possibly smaller image $\theta(G) \subseteq H$ as well. For example, homomorphisms preserve subgroups.

Example 2.21. Let G and H be groups and let A be a subgroup of G. If $\theta : G \to H$ is a homomorphism, then $\theta(A)$ is a subgroup of H.

Proof. By Proposition 2.20 (1), $e_H = \theta(e_G)$. So $e_H \in \theta(A)$ since $e_G \in A$.

Let $u, v \in \theta(A)$, then there are $g, h \in A$ such that $u = \theta(g)$ and $v = \theta(h)$. Since $gh \in A$, $uv = \theta(g)\theta(h) = \theta(gh) \in \theta(A)$, so $\theta(A)$ is closed with respect to the operation of H.

Since $g^{-1} \in A$, Proposition 2.20 (2) gives that $u^{-1} = \theta(g)^{-1} = \theta(g^{-1}) \in \theta(A)$. Thus, inverses are contained in $\theta(A)$ and therefore $\theta(A)$ is a subgroup of H. □

In Example 2.10 we said that if V is an n-dimensional vector space then the groups $\mathrm{GL}(V)$ and $\mathrm{GL}_n(\mathbb{R})$ are basically the same. We can use homomorphisms to make this precise.

Definition 2.22. Let G and H be groups and let $\theta : G \to H$ be a homomorphism. If θ is also bijective, then θ is called an *isomorphism.* If there is an isomorphism from G to H, then G and H are said to be *isomorphic*, denoted by $G \approx H$.

Example 2.23. The exponential function from $\mathbb{R}$ to $\mathbb{R}_{>0}$ is bijective, thus the homomorphism θ from Example 2.19 is an isomorphism and the groups $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$ are isomorphic.

Example 2.24. Consider Example 2.10 and assume that $\dim(V) = n < \infty$. If we pick a basis of V, then for every $T \in \mathrm{GL}(V)$ we have a corresponding transformation matrix $A \in \mathrm{GL}_n(\mathbb{R})$ in this basis. The map $\theta : \mathrm{GL}(V) \to \mathrm{GL}_n(\mathbb{R})$ defined by $\theta(T) = A$ for every $T \in \mathrm{GL}(V)$ is bijective. Since we also know that the linear map ST has transformation matrix BA, θ is a homomorphism and thus an isomorphism, and $\mathrm{GL}(V)$ and $\mathrm{GL}_n(\mathbb{R})$ are isomorphic.

As we have seen examples of, a homomorphism $\theta : G \to H$ preserve properties of the group G in the images of its elements and subgroups. In the special case when the homomorphism is also an isomorphism, then all properties (from a group theoretic perspective) are preserved, so G and H differ only by the names of their elements and operations.

2.2.3 Group action

We will now introduce the last concept in group theory needed in this paper. Let X be a two-dimensional plane and let x_0 be a fixed point in X. The set of all rotations of X around the point x_0 with composition as operation is a group, call it G. Define a map $\cdot : G \times X \to X$ by $\cdot(g, x) = g(x)$, which we simply write as $g \cdot x = g(x)$. It would be natural say that the elements of G acts on the elements of X (by rotating them), or simply that G acts on X. There are many geometrical examples connecting groups and sets in this way, but we can also generalize this idea by letting G be any group and X any set.

Definition 2.25. Let G be a group with identity e and let X be a set. If $\cdot : G \times X \to X$ is a map satisfying:

1. $e \cdot x = x$ for all $x \in X$,
2. $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$,

then we say that *G acts on X by $\cdot$*. The map $\cdot$ is a called a *group action*.

One can view the group action $\cdot$ as a way of multiplying an element of G with an element of X to yield an element of X. As usual, we will omit $\cdot$ and simply write gx instead of $g \cdot x$. We also make the following definitions: If $g \in G$ and E is a subset of X, let $gE := \{gx \in X : x \in E\}$, and if A is a subset of G, let $AE := \{gx \in X : g \in A, x \in X\}$. We will make use of both of the following examples.

Example 2.26. Let G be a group and define a map $G \times G \to G$ by left multiplication. The properties (1) and (2) in Definition 2.25 is just the second and first group axiom. Thus, every group acts on itself by left multiplication (also called *left translation*).

Example 2.27. Let H be any subgroup of $\mathrm{GL}_n(\mathbb{R})$ and define a map $H \times \mathbb{R}^n \to \mathbb{R}^n$ by matrix multiplication. Obviously, $Ix = x$ for all $x \in \mathbb{R}^n$ and $A(Bx) = (AB)x$ for all $A, B \in \mathrm{GL}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. Thus, every group of invertible $n \times n$ matrices acts on $\mathbb{R}^n$. In particular, SO_3 acts on $\mathbb{R}^3$.

We will now see how group actions give rise to partitions.

Theorem 2.28. *Let G be a group acting on a set X and let $x, y \in X$. Define a relation $\sim$ on X by $x \sim y$ if and only if $gx = y$ for some $g \in G$. The relation $\sim$ is an equivalence relation.*

Proof. Let $x, y, z \in X$. If e is the identity of G, then $ex = x$. So, $\sim$ is reflexive. If $gx = y$, then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = x$. So, $\sim$ is symmetric. Finally, if $gx = y$ and $hy = z$, then $(hg)x = h(gx) = hy = z$. Thus, $\sim$ is also transitive and therefore an equivalence relation. □

The partition is given by the set of equivalence classes of $\sim$, which are called *G-orbits* or just *orbits*.

Example 2.29. Let X be the plane in the introduction to this subsection. The orbits induced by the group of rotations G are all the circles centered at x_0.

In the process of cutting the unit ball into pieces, orbits induced by a group action will be needed and next we will characterize the group SO_3 which is involved in this action.

2.2.4 The matrix group SO_3

We will now see that SO_3 is exactly the group of rotations of $\mathbb{R}^3$ about lines through the origin. In particular, the elements of SO_3 preserve and rotate the unit sphere $\mathbf{S}^2$. Thus, SO_3 not only acts on $\mathbb{R}^3$, but also on $\mathbf{S}^2$. There is only one orbit in $\mathbf{S}^2$ induced by SO_3, namely $\mathbf{S}^2$ itself (compare example 2.29). This orbit is not so fascinating but as we will see in Section 4.1 and 4.2 there is a subgroup of SO_3 that gives rise to interesting orbits of $\mathbf{S}^2$.

Proposition 2.30. *SO_3 forms a group under matrix multiplication. From Example 2.11 we have that $SO_3 = \{A \in \mathbb{R}^{3\times3} :\ AA^T = I,\ \det(A) = 1\}$.*

Proof. Let I denote the identity matrix in $\mathbb{R}^3$. Trivially, since $\det(I) = 1$ and $II^T = I$ we have that $I \in SO_3$. Let $A, B \in SO_3$. Then $AA^T = I$ yields $A^{-1} = A^T$. Further, $A^TA = (AA^T)^T = I^T = I$ and $\det(A^T) = \det(A) = 1$ so SO_3 is closed under inversion. Finally, since $(AB)(AB)^T = (AB)(B^TA^T) = A(BB^T)A^T = AIA^T = AA^T = I$ and $\det(AB) = \det(A)\det(B) = 1$ we have that SO_3 is also closed under multiplication, which verifies that it is indeed a group. □

For any real orthogonal matrix A it holds that both the columns of A and the rows of A form orthonormal sets. Given an orthonormal basis of the inner-product space $\mathbb{R}^n$, equipped with the usual dot product, any real orthogonal matrix A induces a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by $x \mapsto Ax$ satisfying the first two following properties. The third is unrelated to T but will also be of use to us.

$$\langle u, v\rangle = \langle Tu, Tv\rangle \quad \text{for all } u, v \in V, \tag{1}$$

$$\text{The matrix representation of } T, [T]_E \text{ is orthogonal for any ON-basis } E \text{ of } \mathbb{R}^n. \tag{2}$$

$$\text{Any orthogonal } 2\times 2\text{-matrix with determinant 1 corresponds to a rotation of } \mathbb{R}^2. \tag{3}$$

Proofs of these properties can be found in Appendix B.

Proposition 2.31. *For any orthonormal basis of $\mathbb{R}^3$, every element of SO_3 gives a rotation of $\mathbb{R}^3$ about some line through the origin. Conversely, any rotation of $\mathbb{R}^3$ about some line through the origin corresponds to an element of SO_3.*

Proof. Take any $A \in SO_3$. By our definition of SO_3, A is orthogonal. We first show that A has an eigenvector with corresponding eigenvalue 1. Let $p_A(\lambda) := \det(A - \lambda I)$ denote the characteristic polynomial of A, where I is the identity matrix in $\mathbb{R}^{3\times3}$. Then

$$\begin{aligned} p_A(1) &= \det(A - I) = \det(A - A^TA) = \det\big((I - A^T)A\big) = \det(I - A^T)\det(A) \\ &= \det(I - A)^T = \det(I - A) = (-1)^3\det(A - I) = -\det(A - I). \end{aligned}$$

So $p_A(1) = -p_A(1) = 0$ which shows that there is such an eigenvector of A. Let e_1 be a normalized such vector and let U be the orthogonal complement of the subspace spanned by e_1. Taking an ON-basis e_2, e_3 of U we get an ON-basis e_1, e_2, e_3 of $\mathbb{R}^3$. Since $U = \{u \in \mathbb{R}^3 \ :\ \langle u, e_1\rangle = 0\}$, (1) yields that if $u \in U$ then $Au \in U$ by $0 = \langle u, e_1\rangle = \langle Au, Ae_1\rangle = \langle Au, e_1\rangle$. Specifically we have that $Ae_2, Ae_3 \in U$. The matrix of A in the basis e_1, e_2, e_3 is therefore of the form

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Since the determinant is unaffected by a change of basis, we have that $1 = \det(A') = 1 \cdot \det(B) = \det(B)$. A' is also orthogonal by (2) and so by extension, B is orthogonal. Considering the linear transformation of $\mathbb{R}^3$ induced by A restricted to the subspace U, T_U, given by $T_Uu = Au$ for $u \in U$, we see that the matrix of T_U in the basis e_2, e_3 is B, an orthogonal 2×2-matrix. By (3) B is a rotation matrix. If e_2 and e_3 are picked such that the ON-basis e_1, e_2, e_3 is right-handed, this is a positive rotation of U of θ radians viewed from e_1. Thus the transformation $x \mapsto Ax$ of $\mathbb{R}^3$ fixes all points on the line spanned by e_1 and gives a rotation of the orthogonal complement to this line, a rotation about a line through the origin.

To show the converse, let l be a line through the origin spanned by some non-zero vector v and let θ be an angle. We want to show that the rotation about l of θ radians is a linear transformation T with $[T] \in \mathrm{SO}_3$. It is clear that the rotation is linear, we denote it by T. Let e_1 be v after normalization and let e_2, e_3 be an ON-basis of the orthogonal complement of l. In the ON-basis $E = \{e_1, e_2, e_3\}$ we have that

$$[T]_E = A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

an orthogonal matrix with $\det(A') = 1$. Since orthogonality is preserved under a change of basis by (2) we have that the matrix A of T with respect to the standard basis of $\mathbb{R}^3$ is orthogonal with determinant one, which shows that $A \in \mathrm{SO}_3$.

□

2.2.5 Isometries

An isometry of $\mathbb{R}^n$ is a bijection on $\mathbb{R}^n$ that preserves distance between points, i.e. if $x, y \in \mathbb{R}^n$ and ρ is an isometry, then $|x-y| = |\rho(x)-\rho(y)|$. We have already seen an example of isometries, namely rotations. Isometries can be further divided into translations and reflections. Let $\mathbb{E}^n$ denote set of all isometries.

Proposition 2.32. *The set $\mathbb{E}^n$ with composition is a group.*

Proof. Then identity map clearly preserves distance. Let $x, y \in \mathbb{R}^n$ and $\rho, \sigma \in \mathbb{E}^n$. Then $|\sigma(\rho(x)) - \sigma(\rho(y))| = |\rho(x) - \rho(y)| = |x - y|$, so $\mathbb{E}^n$ is closed. Inverses are in $\mathbb{E}^n$ since $|x - y| = |\rho(\rho^{-1}(x)) - \rho(\rho^{-1}(y))| = |\rho^{-1}(x) - \rho^{-1}(y)|$. Thus, E_n is a group. □

Definition 2.33. The group $\mathbb{E}^n$ is called *the Euclidean group.*

Rigid motions are finite combinations of translations and rotations, but not reflections. Let G_n denote the set of all rigid motions. It is trivial to see that G_n with composition is a group.

Definition 2.34. The group G_n is called *the special Euclidean group.*

We will discuss the problem of measure in the context of the Euclidean group and the special Euclidean group. Also, the special Euclidean group is needed in the very last step of the proof of the standard form of the Banach–Tarski paradox. We finally note the relationship between the isometry-groups that we have seen: $\mathrm{SO}_n \subset G_n \subset \mathbb{E}^n$.

3 Paradoxes

We are now ready to introduce two paradoxical constructions. First we will see how rotating a specific subset of the plane surprisingly yields the same set with additional points.

3.1 Spokes on a wheel paradox

In order to show how geometrical paradoxes arise, we will show a paradox in $\mathbb{R}^2$ known as the "Spokes on a wheel paradox" using the reasoning from [13]. Later on, we are going to use a version of the paradox in $\mathbb{R}^3$ but we will show it first in $\mathbb{R}^2$ to make it easier to understand.
We let L be the line $(0, 1)$ in $\mathbb{R}^2$ along the x-axis and $\rho(L)$ be the act of rotating L $\frac{1}{10}$ radians around the origin, though we could rotate L by any angle $\theta \in [0, 2\pi)$ where $n * k \neq 2m\pi$ for any positive n or m. Next, we define W_ρ as $\{\rho^n(L) : n \in \mathbb{N}\}$.

Since $\frac{n}{10} \neq 2m\pi$ for all $n, m \in \mathbb{N}$, $\rho^n(L) \neq L$ or more generally, $\rho^n(L) \neq \rho^m(L)$ for all $m, n \in \mathbb{N} : m \neq n$. Due to our definition of W_ρ and ρ, we can see that $\rho^{-1}W_\rho = \{\rho^n(L) : n \in \mathbb{N} \cup \{0\}\}$, where ρ^{-1} is the act of rotating $-\frac{1}{10}$ radians around the origin. Since ρ^{-1} returns $\rho(L)$ back to the line L, we get that

$$\rho^{-1}W_\rho = W_\rho \sqcup L,$$

where $\sqcup$ denotes that the sets are disjoint. We have generated an extra line by a rotation. The reason why this paradox is known as the "Spokes on a wheel paradox" is because W_ρ and the unit circle on $\mathbb{R}^2$ give the appearance of a wheel with an infinite number of spokes as seen in figure 1.

3.2 Free groups

Another paradoxical construction can be found by studying free groups. Our aim is to later transfer the paradoxical nature of free groups to the group of rotations of the unit sphere. The elements in a free group are called *reduced words* and we define them as follows.

Definition 3.1. Let G be a group. A *word w on $S \subseteq G$* is a finite product of elements in $S \cup S^{-1} = \{s \ : \ s \in S \text{ or } s^{-1} \in S\}$. That is,

$$w = s_1 s_2 \cdots s_n \quad s_i \in S \cup S^{-1},$$

where $n = 0, 1, 2, ...$ is called the length of w. The word of length 0 is called the *empty word.*

Definition 3.2. Let $w = s_1 \cdots s_n$ be a word of length $n > 0$. If it holds that $s_i \neq (s_{i+1})^{-1}$ for $0 < i < n$ where $(s_j^{-1})^{-1} = s_j$ for all j then w is called a *reduced word.* In other words, a reduced word is a string where no element is immediately adjacent to its inverse.

From any word we can find a unique corresponding reduced word by repeatedly cancelling all occurrences of elements next to their inverses. Since the length of any word is finite and the length of the word is reduced in every iteration this cancellation is unproblematic. We are now ready to define a free group.

Definition 3.3. The *free group* of order $n = 1, 2, ...$ is the group of all reduced words on $S = \{a_1, a_2, ..., a_n\}$ with concatenation (and possibly cancellation) as the group operation. It also satisfies $a \neq b^{-1}$ for all $a, b \in S$, and is denoted by $\mathbb{F}_n$. The identity element of a free group is the empty word, which we will denote by e.

Example 3.4. The free group of order 2, $\mathbb{F}_2$, is the group of all reduced words on $\{a, b\}$. More explicitly, if $w \in \mathbb{F}_2$ is a reduced word of length $n = 1, 2, ...$ then $w = s_1 \cdots s_n$ where $s_i \in \{a, b, a^{-1}, b^{-1}\}$. Examples of elements of $\mathbb{F}_2$ are $w_1 = a^2ba$ and $w_2 = a^{-1}b^{-1}$. We have that

$$w_1 w_2 = (a^2ba)(a^{-1}b^{-1}) = a^2baa^{-1}b^{-1} = a^2bb^{-1} = a^2,$$

a reduced word of length 2, and that $w_2 w_1 = (a^{-1}b^{-1})(a^2ba) = a^{-1}b^{-1}a^2ba$, a reduced word of length 6.

Proposition 3.5. *The free group of order n is countable.*

Proof. Let $W_m \subset \mathbb{F}_n$ be all reduced words of length $m = 0, 1, 2, ...$ For each word $w = s_1 \cdots s_m \in W_m$ there are at most $2n$ possibilities for each s_i so $|W_m| \leq (2n)^m$. Thus W_m is finite for all m and n. Since $\mathbb{F}_n = \cup_{i=0}^{\infty} W_i$ is the countable union of finite sets, it follows from Proposition 2.2 that $\mathbb{F}_n$ is countable. ☐

The reason we are introducing the notion of free groups is that they have a so called *paradoxical decomposition.* This is the property we are interested in transferring to SO_3.

Definition 3.6. Let G be a group acting on a set X. We say that X is *$G-$paradoxical* if there exist disjoint subsets $A_1, ..., A_n, B_1, ..., B_m$ of X and elements $g_1, ..., g_n, h_1, ..., h_m$ in G such that

$$\bigcup_{i=1}^{n} g_i A_i = X = \bigcup_{j=1}^{m} h_j B_j.$$

These subsets together with the group elements are called a *paradoxical decomposition* of X. In the case that the group G is acting on itself by left multiplication we simply say that G is *paradoxical.*

The definition above states that a set is paradoxical if we can find two disjoint subsets, partition them into a finite number of pieces and then by letting some elements from the group act on these pieces create two copies of the original set. At first glance it may seem unintuitive that there are any such sets, but the following theorem states that free groups have this property.

Theorem 3.7. *The group $\mathbb{F}_2$ is paradoxical.*

Proof. Any nonempty word w in $\mathbb{F}_2$ is a string of characters $s \in \{a, a^{-1}, b, b^{-1}\}$. Define for $c \in \{a, a^{-1}, b, b^{-1}\}$

$$W_c = \{\text{words beginning on the left with c}\}.$$

Since any word except the empty word necessarily begins with one of these four letters, we have that

$$\mathbb{F}_2 = W_a \sqcup W_{a^{-1}} \sqcup W_b \sqcup W_{b^{-1}} \sqcup \{e\}.$$

Since elements of $\mathbb{F}_2$ are reduced words, this is a partition of $\mathbb{F}_2$. Now, we claim that

$$a^{-1}W_a = W_a \sqcup W_b \sqcup W_{b^{-1}} \sqcup \{e\}.$$

Since $a \in W_a$ we have that $a^{-1}a = e \in a^{-1}W_a$. Now let w be a word beginning with a, b or b^{-1}. Then $aw \in W_a$ so $w \in a^{-1}W_a$. So far we have shown that $a^{-1}W_a \supseteq W_a \sqcup W_b \sqcup W_{b^{-1}} \sqcup \{e\}$.

Let w be a reduced word in $a^{-1}W_a$. If w is the empty word then trivially $w \in \{e\}$. Assume w is a nonempty word, that is $w = s_1 \cdots s_n$. Then $w = a^{-1}w_a$ for some reduced word $w_a = as_1 \cdots s_n \in W_a \setminus \{a\}$. Since w_a is a reduced word we have that $s_1 \neq a^{-1}$. It follows that $w \notin W_{a^{-1}}$, which shows that $a^{-1}W_a \subseteq \mathbb{F}_2 \setminus W_{a^{-1}} = W_a \sqcup W_b \sqcup W_{b^{-1}} \sqcup \{e\}$. Now, repeating the argument for $b^{-1}W_b$ we have shown that

$$\mathbb{F}_2 = a^{-1}W_a \sqcup W_{a^{-1}} = b^{-1}W_b \sqcup W_{b^{-1}},$$

which completes the proof. □

4 The Banach–Tarski paradox

The goal of this section is to show both the general and the standard form of the Banach–Tarski paradox. To do this, we first show how we can generate a paradoxical subgroup of SO_3.

4.1 A subgroup of SO_3 is isomorphic to $\mathbb{F}_2$

What really makes the Banach–Tarski paradox work is that we can transfer the paradoxical properties of $\mathbb{F}_2$ to the rotation group SO_3. The reason we can do that is because SO_3 contains a subgroup isomorphic to $\mathbb{F}_2$. Consider the rotations ϕ and ψ, where ϕ is a counterclockwise rotation by $\arccos(\frac{1}{3})$ around the x-axis and ψ is a counterclockwise rotation by $\arccos(\frac{1}{3})$ around the z-axis. In matrix form $\phi^{\pm 1}$ and $\psi^{\pm 1}$ has the form

$$\phi^{\pm 1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & \mp 2\sqrt{2} \\ 0 & \pm 2\sqrt{2} & 1 \end{pmatrix} \quad \psi^{\pm 1} = \frac{1}{3}\begin{pmatrix} 1 & \mp 2\sqrt{2} & 0 \\ \pm 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We define the group $\mathbb{F}_2(\phi, \psi)$ as the free group with $S = \{\phi, \psi\}$. This is not to be confused with the group $\langle \phi, \psi \rangle$, which is a subgroup of SO_3 . The difference between these groups is that in $\mathbb{F}_2(\phi, \psi)$ elements are words and in $\langle \phi, \psi \rangle$ elements are rotations. What we will prove in this section is that $\mathbb{F}_2(\phi, \psi)$ and $\langle \phi, \psi \rangle$ are isomorphic. Throughout this section we follow a proof given by Weston [13]. We know that the rotations in $\langle \phi, \psi \rangle$ are described by words in $\mathbb{F}_2(\phi, \psi)$. We can therefore construct a map $\theta : \mathbb{F}_2(\phi, \psi) \to \langle \phi, \psi \rangle$, where each word is mapped to the corresponding rotation it describes. All we need to verify is that θ is an isomorphism. The central part of the proof is showing that θ is a bijective map, meaning that each rotation is described by precisely one word. Another way of phrasing that is that only the empty word can correspond to the empty rotation. We prove this by showing that any rotation corresponding to a nonempty word will move (0,1,0) to a new location on the sphere. This proof is purely analytical and gives very little geometric insight. Therefore, before we move on we should consider a few examples, and think through geometrically why these particular rotations will not move (0,1,0) back to where it started.

Example 4.1. Consider the rotation $\psi^{-1}\phi^{-1}\psi\phi \in \langle\phi,\psi\rangle$ applied to (0,1,0). It is tempting to say that the rotations will cancel each other out and the point is moved back to where it started. Figure 2 in Appendix A illustrates what happens at each step. It is clear from the picture why, for example ψ and ψ^{-1} will not cancel each other out. The point covers different distances for the two rotations since the two circles it rotates around have different circumferences and the same angle of rotation. Thus the rotation as a whole is different from the identity rotation.

Example 4.2. Consider exponents of the rotation $\phi \in \langle\phi,\psi\rangle$. If $\phi^n((0,1,0)) = (0,1,0)$ then the point (0,1,0) must have rotated an integer number of times around the equator such that $\arccos(\frac{1}{3})n = 2\pi k$, $k \in \mathbb{Z}$. But this is impossible since $\frac{\arccos(\frac{1}{3})}{\pi} \notin \mathbb{Q}$.

Now that we at least have an idea of what some of these rotations look like, and why they will move (0,1,0) to a new location, we are ready to take a look at the formal proof. Throughout the proof, whenever we refer to a rotation of length n we mean a rotation described by a reduced word of length n.

Lemma 4.3. *Let $\rho \in \langle\phi,\psi\rangle$ be a rotation of length n, then $\rho((0,1,0)) = \frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})$ for some integers a, b, c.*

Proof. The proof is done by induction. The base case $n = 0$ is clear since $n = 0$ implies $\rho = e_{SO_3}$, hence $\rho((0,1,0)) = (0,1,0) = \frac{1}{3^0}(0\sqrt{2}, 1, 0\sqrt{2})$. Let ρ be a rotation of length $n > 0$, then ρ is on one of the forms $\rho = \phi^{\pm 1}\rho'$ or $\rho = \psi^{\pm 1}\rho'$ for some rotation ρ' of length $n-1$. By the induction hypothesis, $\rho((0,1,0))$ is on one of the forms

$$\phi^{\pm 1}\rho'((0,1,0)) = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & \mp 2\sqrt{2} \\ 0 & \pm 2\sqrt{2} & 1 \end{pmatrix} \frac{1}{3^{n-1}} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^n}\left(3a\sqrt{2}, b \mp 4c, (c \pm 2b)\sqrt{2}\right),$$

$$\psi^{\pm 1}\rho'((0,1,0)) = \frac{1}{3}\begin{pmatrix} 1 & \mp 2\sqrt{2} & 0 \\ \pm 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \frac{1}{3^{n-1}} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^n}\left((a \mp 2b)\sqrt{2}, b \pm 4a, 3c\sqrt{2}\right),$$

which all have the desired form. □

By Lemma 4.3 we have $\rho \in \langle\phi,\psi\rangle$ is of length $n \implies \rho((0,1,0)) = \frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})$ for integers a, b, c. The function $N : \langle\phi,\psi\rangle \to \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is defined by $N(\rho) = (a,b,c) \bmod 3$, for these integers.

Lemma 4.4. *Let $\rho \in \langle\phi,\psi\rangle$ with $N(\rho) = (a,b,c)$. Then for $n > 0$*

$$N(\phi^{\pm n}\rho) = \begin{cases} (0, b \mp c, c \mp b), & n \text{ odd} \\ (0, -b \pm c, -c \pm b), & n \text{ even} \end{cases}, \quad N(\psi^{\pm n}\rho) = \begin{cases} (a \pm b, b \pm a, 0), & n \text{ odd} \\ (-a \mp b, -b \mp a, 0), & n \text{ even} \end{cases}.$$

Proof. All cases are proven in the exact same way, therefore we will only do the proof for $N(\phi^n\rho)$, $n > 0$. Suppose that $\rho((0,1,0)) = \frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})$ meaning $N(\rho) = (a,b,c)$. By the calculations in Lemma 4.3 we have that $\phi\rho((0,1,0)) = \frac{1}{3^{n+1}}(3a\sqrt{2}, b - 4c, (2b + c)\sqrt{2})$, hence $N(\phi\rho) = (3a, b - 4c, 2b + c) \equiv (0, b - c, c - b)$ (since $3 \equiv 0$, $4 \equiv 1$ and $2 \equiv -1$). This proves the case when $n = 1$. For $n = 2$, note that $N(\phi^2\rho) = N(\phi(\phi\rho)) = (0, (b - c) - (c - b), (c - b) - (b - c))$ by applying the same calculations again. This reduces to $(0, 2b - 2c, 2c - 2b) \equiv (0, c - b, b - c)$, which proves $n = 2$.

For $n > 2$ we need to verify that N is independent of the parity of n. This can be done by induction on the odd and even numbers separately. For the odd numbers we want to show that $N(\phi^{2m-1}\rho) = (0, b - c, c - b)$ for all $m \in \mathbb{N}$. The base case $m = 1$ is already proven. Take some $m > 1$, then $N(\phi^{2m-1}\rho) = N(\phi^2(\phi^{2(m-1)-1}\rho))$ where $N(\phi^{2(m-1)-1}\rho) = (0, b - c, c - b)$ by the induction hypothesis. We can use the result for $n = 2$ and conclude that $N(\phi^{2m-1}\rho) = N(\phi^2(\phi^{2(m-1)-1}\rho)) = (0, (c - b) - (b - c), (b - c) - (c - b)) = (0, 2c - 2b, 2b - 2c) \equiv (0, b - c, c - b)$. The even case and the other rotations are proven in the exact same way. □

Proposition 4.5. *Let $\rho \in \langle \phi, \psi \rangle$ correspond to a nonempty word. Then $N(\rho)$ can only take on values in the set $N(\rho) \in N_\rho := \{(0,1,1),(0,1,2),(0,2,1),(0,2,2),(1,1,0),(1,2,0),(2,1,0),(2,2,0)\}$.*

Proof. We will show that if the leftmost rotation of ρ is ϕ then $N(\rho) \in \{(0,1,1),(0,1,2),(0,2,1),(0,2,2)\} \subset N_\rho$, and if the leftmost rotation of ρ is ψ then $N(\rho) \in \{(1,1,0),(1,2,0),(2,1,0),(2,2,0)\} \subset N_\rho$. Any rotation $\rho \in \langle \phi, \psi \rangle$ corresponding to a nonempty word must alternate in nonzero powers of ϕ and ψ. We will do the proof by induction in the number of alternations (not the same as the length of the word). The base cases are ϕ^{n_1} and ψ^{n_2}. Note that $\phi^{n_1} = \phi^{n_1} e_{\mathrm{SO}_3}$ with $N(e_{\mathrm{SO}_3}) = (0,1,0)$. Depending on if n_1 is positive or negative, even or odd, $N(\phi^{n_1})$ will take on one of the four values (0,1,1), (0,1,2), (0,2,1) or (0,2,2). This follows from Lemma 4.4. With exactly the same argument we must have $N(\psi^{n_2}) \in \{(1,1,0),(1,2,0),(2,1,0),(2,2,0)\}$.

Let now ρ be a word that alternates between powers of ϕ and ψ at least once. We then have $\rho = \phi^{n_{1,1}}\psi^{n_{1,2}}...$ or $\rho = \psi^{n_{2,1}}\phi^{n_{2,2}}...$ for nonzero powers $n_{i,j}$. In order to make the induction argument valid we can assume that both $\psi^{n_{1,2}}...$ and $\phi^{n_{2,2}}...$ alternates between powers of ϕ and ψ a total of k times. By the induction hypothesis we then have $N(\psi^{n_{1,2}}...) \in \{(1,1,0),(1,2,0),(2,1,0),(2,2,0)\}$. We can use Lemma 4.4 to check all 16 cases: four possible values for $N(\psi^{n_{1,2}}...)$, times two possible parities for $n_{1,1}$, times two possible signs for $n_{1,1}$ and conclude that $N(\phi^{n_{1,1}}\psi^{n_{1,2}}...)(\rho) \in \{(0,1,1),(0,1,2),(0,2,1),(0,2,2)\}$. Similarly we can use Lemma 4.4 together with the induction hypothesis $N(\phi^{n_{2,2}}...) \in \{(0,1,1),(0,1,2),(0,2,1),(0,2,2)\}$ to again check all 16 cases and conclude that $N(\psi^{n_{2,1}}\phi^{n_{2,2}}...) \in \{(1,1,0),(1,2,0),(2,1,0),(2,2,0)\}$. We will not write out all 32 cases since it is just basic arithmetic. This finishes the induction. □

Theorem 4.6. *Let $\rho \in \langle \phi, \psi \rangle$ be a rotation of length > 0. Then $\rho((0,1,0)) \neq (0,1,0)$*

Proof. Assume $\rho((0,1,0)) = (0,1,0)$. By Lemma 4.3 we have $\rho((0,1,0)) = \frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})$ for integers a, b, c and $n > 0$. Since $\rho((0,1,0)) = (0,1,0)$ we must have $a = 0$, $b = 3^n$, $c = 0$. Since $n > 0$ we have $3^n \equiv 0 \mod 3$. This implies $N(\rho) = (0,0,0)$. By Proposition 4.5 we must have $N(\rho) \in N_\rho$, but $(0,0,0) \notin N_\rho$. This is a contradiction. Therefore $\rho((0,1,0)) \neq (0,1,0)$. □

Theorem 4.7. *The free group $\mathbb{F}_2(\phi, \psi)$ is isomorphic to the rotation group $\langle \phi, \psi \rangle$ which is a subgroup of SO_3.*

Proof. The subgroup claim follows from Theorem 2.16. To conclude the isomorphism we use the map $\theta : \mathbb{F}_2(\phi, \psi) \to \langle \phi, \psi \rangle$ defined earlier in this section. The homomorphism criteria $\theta(w_1 * w_2) = \theta(w_1) * \theta(w_2)$ for all $w_1, w_2 \in \mathbb{F}_2(\phi, \psi)$ is trivial, both concatenation and composition essentially means putting the elements after each other and canceling any adjacent pairs where some element ends up right next to its inverse. Surjectivity of θ is also trivial since every rotation $\rho \in \langle \phi, \psi \rangle$ has at least one corresponding word in $\mathbb{F}_2(\phi, \psi)$. What remains to verify is injectivity.

Suppose that θ is not injective, meaning that there exists $w_1, w_2 \in \mathbb{F}_2(\phi, \psi)$ such that $w_1 \neq w_2$ and $\theta(w_1) = \theta(w_2)$. This implies $\theta(w_1) * \theta(w_2)^{-1} = e_{\mathrm{SO}_3}$, using that θ is a homomorphism we get $\theta(w_1 * w_2^{-1}) = e_{\mathrm{SO}_3}$. Since $w_1 \neq w_2$ the word $w_1 * w_2^{-1}$ is nonempty and the rotation $\theta(w_1 * w_2^{-1})$ is of length > 0. Hence $\theta(w_1 * w_2^{-1})((0,1,0)) \neq (0,1,0)$ by Theorem 4.6. Since the trivial rotation cannot move $(0,1,0)$ to a new location we get $\theta(w_1 * w_2^{-1}) \neq e_{\mathrm{SO}_3}$. This is a contradiction which implies that no such w_1 and w_2 exists. Therefore θ is injective and $\mathbb{F}_2(\phi, \psi) \approx \langle \phi, \psi \rangle$. □

4.2 Hausdorff paradox

In section 4.1, we proved that there is a subset $\langle \phi, \psi \rangle$ of SO_3 isomorphic with $\mathbb{F}_2$ and now we are going to let $\langle \phi, \psi \rangle$ act on $\mathbf{S}^2$ to try to transfer the paradoxical composition of $\langle \phi, \psi \rangle$ onto $\mathbf{S}^2$. However, there are parts of $\mathbf{S}^2$ that make the application a bit more complicated than directly applying $\langle \phi, \psi \rangle$ onto $\mathbf{S}^2$. Instead we get a result known as the Hausdorff paradox [4].

4.2.1 Why $\langle \phi, \psi \rangle$ cannot be directly applied to $\mathbf{S}^2$

We start off by making the observation that since $\langle \phi, \psi \rangle$ is countable, every $\langle \phi, \psi \rangle$-orbit on $\mathbf{S}^2$ countable, while $\mathbf{S}^2$ contains an uncountable number of point. This means that $\mathbf{S}^2$ consists of an uncountable number of disjoint $\langle\phi,\psi\rangle$-orbits. Because we assume that the axiom of choice holds, we can pick one representative from each orbit to create the set M, thus we can rewrite $\mathbf{S}^2$ as

$$\mathbf{S}^2 = \langle\phi,\psi\rangle M,$$

since we can access every point of an orbit using $\langle\phi,\psi\rangle$ on one point, and with M we have access to every orbit in $\mathbf{S}^2$. With this partition, we can divide $\mathbf{S}^2$ into the union of five pieces, much like we divided $\mathbb{F}_2$.

$$\mathbf{S}^2 = eM \cup W_\phi M \cup W_{\phi^{-1}} M \cup W_\psi M \cup W_{\psi^{-1}} M.$$

Unlike $\mathbb{F}_2$, these sets might not be disjoint because there are non-trivial fixed points on the sphere that cause parts of the sets to overlap, making us unable to use the same partitions with $\mathbf{S}^2$ as $\mathbb{F}_2$. An example would be the point $(1,0,0)$, which is kept in place by the rotation $\phi \in \langle\phi,\psi\rangle$. So what do we do about the non-trivial fixed points of $\mathbf{S}^2$? Well, each element of $\langle\phi,\psi\rangle$ corresponds to the rotation around an axis, making the ends of the axes fixed points. These points are also the only fixed points of $\mathbf{S}^2$, since the only way to fix a point on $\mathbf{S}^2$ using rotations is by rotating the sphere around the axis through that point.

Since $\langle\phi,\psi\rangle$ is countable, the poles of $\langle\phi,\psi\rangle$ are countable as well, meaning that most of the points on $\mathbf{S}^2$ are not a pole of $\langle\phi,\psi\rangle$. So instead of applying $\langle\phi,\psi\rangle$ on $\mathbf{S}^2$, we instead try to apply it on all non-poles and try to copy the poles later. We call D the set of $\langle\phi,\psi\rangle$-poles, defined as

$$D = \{p : \rho p = p \text{ for some } \rho \in \langle\phi,\psi\rangle \setminus \{e\}\}.$$

Then the set of all points not fixed by any non-trivial $\langle\phi,\psi\rangle$-rotations on $\mathbf{S}^2$ is $\mathbf{S}^2 \setminus D$.

4.2.2 The $\langle\phi,\psi\rangle$ action on $\mathbf{S}^2 \setminus D$

In order to apply $\langle\phi,\psi\rangle$ on $\mathbf{S}^2 \setminus D$, we first need to show that $\langle\phi,\psi\rangle$ is still a group action on $\mathbf{S}^2 \setminus D$.

Proposition 4.8. *The group $\langle\phi,\psi\rangle$ acts on $\mathbf{S}^2 \setminus D$ with no non-trivial fixed points.*

Proof. Let ρ be an arbitrary element in $\langle\phi,\psi\rangle$. We need to show that for all $p \in \mathbf{S}^2 \setminus D$, then $\rho p \in \mathbf{S}^2 \setminus D$. Since $\langle\phi,\psi\rangle$ maps $\mathbf{S}^2$ onto itself, we just need to show that $\rho p \in D$ only when $p \in D$. By our definition of D, there is a non-identity element $g \in \langle\phi,\psi\rangle$ where $g\rho p = \rho p$. If we multiply by ρ^{-1} from the left, the right-hand side cancels out so we get $\rho^{-1}g\rho p = p$, but since g is not the identity element, $\rho^{-1}g\rho$ is also a non-identity element, so $p \in D$, making $\langle\phi,\psi\rangle$ act on $\mathbf{S}^2 \setminus D$. By the definition of D, $\mathbf{S}^2$ has no non-trivial fixed points. □

Since $\langle\phi,\psi\rangle$ is a group action on $\mathbf{S}^2 \setminus D$ and $\mathbf{S}^2 \setminus D$ is an uncountable set, we will once again invoke AC to create a set M containing one representative from each orbit on $\mathbf{S}^2 \setminus D$ and divide $\mathbf{S}^2 \setminus D$ in the same way we did with $\mathbf{S}^2$. This time however, the sets will be disjoint. Suppose that there exists a ρ_1, ρ_2 such that $\rho_1 M \cap \rho_2 M \neq \varnothing$. Then there are points $p_1, p_2 \in M$ such that $\rho_1 p_1 = \rho_2 p_2$. But then $p_1 = \rho_1^{-1}\rho_2 p_2$ so p_1 and p_2 must be in the same orbit. Since M contains exactly one point from each orbit, $p_1 = p_2$. But since we also have no non-trivial fixed points in $\mathbf{S}^2 \setminus D$, we also have that $\rho_1^{-1}\rho_2 = e$ meaning that $\rho_1 = \rho_2$. This means that any two sets AM and BM are disjoint if A and B are disjoint subsets of $\langle\phi,\psi\rangle$. We can then safely make the partition

$$\mathbf{S}^2 \setminus D = eM \sqcup W_\phi M \sqcup W_{\phi^{-1}} M \sqcup W_\psi M \sqcup W_{\psi^{-1}} M,$$

and since $\langle\phi,\psi\rangle$ is isomorphic with $\mathbb{F}_2$, we can use $\mathbb{F}_2$'s paradoxical composition to get

$$\phi^{-1} W_\phi M \sqcup W_{\phi^{-1}} M = \mathbf{S}^2 \setminus D = \psi^{-1} W_\psi M \sqcup W_{\psi^{-1}} M,$$

leading to the Hausdorff Paradox.

Theorem 4.9. *(Hausdorff Paradox, AC) There is a countable set D such that $\mathbf{S}^2 \setminus D$ is SO_3-paradoxical.*

4.3 Equidecomposability

We introduce the concept of *equidecomposability* and derive some useful propositions which we later apply to finalize the proof of the Banach–Tarski paradox. Our presentation closely follows those of Wagon [9] and Knudby [14].

Definition 4.10. Let G be a group acting on a set X. We say that $A, B \subseteq X$ are (finitely) *G-equidecomposable* if there are finite partitions of A and B, $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$, with the same number of pieces and group elements $\{g_i\}_{i=1}^n$ such that

$$g_i A_i = B_i, \quad i = 1, ..., n.$$

We denote this relation by $A \sim_G B$ or simply $A \sim B$ if it is clear what groups action we are referring to. For the rest of this paper all mentions of equidecomposability will refer to finite equidecomposability.

Proposition 4.11. *Let G be a group acting on a set X. Then G-equidecomposability is an equivalence relation on all subsets of X.*

Proof. Suppose G acts on X and that $A, B, C \subseteq X$. Since $e \in G$ and $A = eA$ we have that $A \sim A$ so $\sim$ is reflexive. Assume $A \sim B$, witnessed by $A_1, ..., A_n, B_1, ..., B_n$ and $g_1, ..., g_n$. Defining $h_1 = (g_1)^{-1} \in G$ we have that $h_i B_i = A_i$ for $i = 1, ..., n$, i.e. $B \sim A$. Hence $\sim$ is symmetric. Finally we want to show that if $A \sim B$ and $B \sim C$ then $A \sim C$. Assume that A_i, B_i^0, B_j^1, C_j are finite partitions of A, B, C and that $g_i A_i = B_i^0,\ h_j B_j^1 = C_j$ holds for the appropriate group elements g_i, h_j. Defining a new partition of A and C by

$$A_{i,j} = g_i^{-1}\big(B_i^0 \cap B_j^1\big) \text{ and } C_{i,j} = h_j\big(B_i^0 \cap B_j^1\big),$$

ignoring possibly empty intersections, with group elements $g_{i,j} = h_j g_i$ we have that

$$g_{i,j} A_{i,j} = h_j g_i g_i^{-1}\big(B_i^0 \cap B_j^1\big) = h_j\big(B_i^0 \cap B_j^1\big) = C_{i,j}.$$

Thus $\sim$ is also transitive, so $\sim$ is an equivalence relation. □

We can now phrase G-paradoxicality in terms of G-equidecomposability; a set X is paradoxical if there are two disjoint subsets of X both equidecomposable with the whole of X. The following proposition shows that the converse also holds.

Proposition 4.12. *Let G be a group acting on a set X and let A be a subset of X. Then A is G-paradoxical if and only if there are disjoint subsets B, C of A such that $B \sim A \sim C$.*

Proof. If there are disjoint subsets B, C of A such that $B \sim A \sim C$ then A is G-paradoxical by definition. To show the other direction, assume that $B_1, ..., B_n, C_1, ..., C_m$ are disjoint subsets of A and $g_1, ..., g_n, h_1, ..., h_m$ are elements of G witnessing that A is G-paradoxical. While the subsets B_i and C_j are necessarily disjoint, after applying the group elements to them the sets $\{g_i B_i\}$ and $\{h_j C_j\}$ need not be. To remedy this we can shrink the B_i and C_j to ensure that no overlapping occurs. Let

$$B_1' = B_1, \quad \text{and inductively, } B_i' = B_i \setminus g_i^{-1}\left(\bigcup_{k=1}^{i-1} g_k B_k'\right).$$

Since $B_i' \subseteq B_i$ for $i = 1, .., n$ we have that the possibly smaller B_i' are disjoint. By definition of B_i' we are only removing elements that have already been covered by the preceding $\{g_k B_k'\}_{k=1}^{i-1}$, so it holds that $A = \sqcup_{k=1}^n g_n B_n'$. Defining C_j' analogously, we find that $\sqcup_{i=1}^n B_i' = B' \sim A \sim C' = \sqcup_{j=1}^m C_j'$. □

We can now show that G-paradoxicality is really a property of the equivalence classes of $\sim_G$, leading to the following useful proposition.

Proposition 4.13. *Let G be a group acting on a set X and assume that A, B are G-equidecomposable subsets of X. If A is G-paradoxical, so is B.*

Proof. Let C, D be disjoint subsets of A such that $C \sim A \sim D$. Since $A \sim B$ there is a bijection $f : A \to B$ defined by $a \mapsto g_i a$ for $a \in A_i$, where A_i, g_i are subsets and group elements witnessing that $A \sim B$. By bijectivity of f and $C \cap D = \emptyset$ we have that $C' = f(C)$ and $D' = f(D)$ are disjoint subsets of B. By definition of f, we also have that $C \sim C'$ and $D \sim D'$ which shows that $C' \sim B \sim D'$. □

4.4 The Banach–Tarski paradox for $\mathbf{S}^2$ and $\mathbf{B}^3$

To recap, we have so far shown that the unit sphere $\mathbf{S}^2$ minus a countable set D is SO_3–paradoxical. We will now use this to show that we can cover also copy points in D so that we end up with two full copies of the unit sphere. First we show that there exists a rotation in SO_3 which maps all points in D to points in $\mathbf{S}^2 \setminus D$.

Lemma 4.14. *Let $D \subset \mathbf{S}^2$ be a countable subset of $\mathbf{S}^2$. Then there exists a rotation $\sigma \in SO_3$ such that $\sigma^n D \cap D = \emptyset$ for $n = 1, 2, ...$*

Proof. Let l be a line through the origin such that it does not intersect D. There certainly is such a line since the set of lines through the origin is uncountable while the set of lines that intersect one of the countable number of points in D is countable.

Let $l_\theta \in \mathrm{SO}_3$ be the rotation about l of θ radians. We can identify all such rotations with the interval $I = [0, 2\pi) \subset \mathbb{R}$ under the bijection $l_\theta \mapsto \theta$. For each p in D, let I_p be the set of angles θ such that $l_{n\theta}(p) \in D$ for some $n = 1, 2, ...$ Each point in D contributes to a countable number of elements of I_p. Since countable unions of countable sets are countable by Proposition 2.2, I_p is countable for all p and it follows that

$$I_D = \bigcup_{p \in D} I_p$$

is countable. Thus $I \setminus I_D$ is nonempty by Proposition 2.3, so there exists an angle $\theta_0 \in I \setminus I_D$. By construction, the rotation $\sigma = l_{\theta_0}$ satisfies $\sigma^n D \cap D = \emptyset$ for all $n \geq 1$. □

With this rotation we are now able to recreate what resembles a three-dimensional analogue of the Spokes on a wheel paradox described in Section 3, this time yielding the countable set D instead of an additional line.

Theorem 4.15 (The Banach–Tarski paradox for $\mathbf{S}^2$, AC)**.** *The unit sphere $\mathbf{S}^2$ is SO_3–paradoxical.*

Proof. Let D be the countable subset of $\mathbf{S}^2$ in the Hausdorff Paradox and let σ be a rotation as in Lemma 4.14. Let $E = \cup_{n=0}^{\infty} \sigma^n D$. Then $\{E, \mathbf{S}^2 \setminus E\}$ is a partition of $\mathbf{S}^2$ and, by construction of σ, $\{\sigma E, \mathbf{S}^2 \setminus E\}$ is a partition of $\mathbf{S}^2 \setminus D$. Trivially, since $e(\mathbf{S}^2 \setminus E) = \mathbf{S}^2 \setminus E$ and $\sigma(E) = \sigma E$ we have that $\mathbf{S}^2$ and $\mathbf{S}^2 \setminus D$ are SO_3-equidecomposable. Since $\mathbf{S}^2 \setminus D$ is paradoxical by the Hausdorff Paradox, $\mathbf{S}^2$ is paradoxical by Proposition 4.13. □

Extending the Banach–Tarski paradox for $\mathbf{S}^2$ to the unit ball $\mathbf{B}^3$ without the origin is straightforward since we can extend each point on $\mathbf{S}^2$ radially towards the origin to get $\mathbf{B}^3 \setminus \{0\}$. Using this radial correspondence, the paradoxical decomposition of $\mathbf{S}^2$ yields one for $\mathbf{B}^3 \setminus \{0\}$.

Corollary 4.16 (AC)**.** *The unit ball in $\mathbb{R}^3$ with the origin removed is SO_3–paradoxical.*

Proof. Let $\{A_i\}_{i=1}^n$, $\{B_j\}_{j=1}^m$ and $\{g_i\}_{i=1}^n$, $\{h_j\}_{j=1}^m$ be subsets of $\mathbf{S}^2$ and elements of SO_3 witnessing that $\mathbf{S}^2$ is SO_3–paradoxical. Let A_i^C be the conical extension of A_i given by $A_i^C = \{rx \ : \ x \in A_i \text{ and } 0 < r \leq 1\}$. Then

$$\bigcup_{i=1}^{n} g_i A_i^C = \{rx \ : \ x \in \cup_{i=1}^n g_i A_i \text{ and } 0 < r \leq 1\} = \{rx \ : \ x \in \mathbf{S}^2 \text{ and } 0 < r \leq 1\} = \mathbf{B}^3 \setminus \{0\}.$$

Forming $\{B_j^C\}_{j=1}^m$ analogously, we see that the conical extensions of the subsets of $\mathbf{S}^2$ witnessing that $\mathbf{S}^2$ is paradoxical yields a paradoxical decomposition of $\mathbf{B}^3 \setminus \{0\}$. □

To get the paradox for the whole unit ball, the rotations of SO_3 are not sufficient since they all map the origin onto itself. We instead consider the larger group of rigid motions, G_3, introduced in Definition 2.34. As before we have to deal with a set of problematic points, this time only the origin. Once again we use the idea from the Spokes on a wheel paradox; we find a transformation that in some sense allows us to absorb this point.

Theorem 4.17 (The Banach–Tarski paradox for $\mathbf{B}^3$, AC)**.** *The unit ball in $\mathbb{R}^3$ is G_3-paradoxical.*

Proof. Since $\mathbf{B}^3 \setminus \{0\}$ is SO_3-paradoxical and since SO_3 is a subgroup of G_3 it follows that $\mathbf{B}^3 \setminus \{0\}$ is also G_3-paradoxical. Let l_θ be a rotation of θ radians about a line sufficiently close but not through the origin; i.e. a rotation such that $l_\theta(0) \in \mathbf{B}^3$ for all θ. Pick θ_0 such that $(l_{\theta_0})^n(0) \neq 0$ for all $n = 1, 2, ...$, this is certainly possible since there are only countably many rational multiples of 2π, and let $\sigma = l_{\theta_0} \in G_3$. We construct a subset E of $\mathbf{B}^3$ by $E = \{\sigma^n 0 \ : \ n = 0, 1, 2, ...\}$. Then, as usual, we have that $E \setminus \sigma E = \{0\}$. Thus $\{E, \mathbf{B}^3 \setminus E\}$ is a partition of $\mathbf{B}^3$ and $\{\sigma E, \mathbf{B}^3 \setminus E\}$ is a partition of $\mathbf{B}^3 \setminus \{0\}$ which shows that $\mathbf{B}^3 \sim_{G_3} \mathbf{B}^3 \setminus \{0\}$. The theorem now follows from Proposition 4.13. □

The paradox easily generalizes to any solid ball in $\mathbb{R}^3$ since we can translate any ball to have its center at the origin and since the radius of the ball is not important. A more rigorous proof is left out.

Corollary 4.18 (AC)**.** *Any solid ball in* $\mathbb{R}^3$ *is* G_3*-paradoxical.*

4.5 The general form of the Banach–Tarski paradox

We can define a new relation $\lesssim$ on the equivalence classes of $\sim_G$ by $A \lesssim B$ if and only if A is equidecomposable with a subset of B. The following theorem, due to Banach, is an adaptation of the proof of the set-theoretic Schröder–Bernstein Theorem. We follow Wagon's [9] presentation.

Theorem 4.19 (The Banach–Schröder–Bernstein Theorem)**.** *Let G be a group acting on a set X and let A, B be subsets of X. If $A \lesssim B$ and $B \lesssim A$, then $A \sim_G B$.*[3]

Proof. We begin by showing two properties of the relation $\sim_G$, denoted $\sim$ for brevity.

1. Assume that $A \sim B$. Then there is a bijection $g : A \to B$ such that if $C \subseteq A$ then $C \sim g(C)$, defined by $a \mapsto g_i a,\ a \in A_i$ where the g_i and A_i are as in the definition of $A \sim B$.
2. Assume that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, that $A_1 \sim B_1$ and that $A_2 \sim B_2$. Then $A_1 \sqcup A_2 \sim B_1 \sqcup B_2$. This is easily seen by letting the partition of $A_1 \sqcup A_2$ be the union of the induced partitions of A_1 and A_2, similarly for $B_1 \sqcup B_2$, and applying the corresponding group elements.

Assume that $A \lesssim B$ and $B \lesssim A$. Then $A \sim B_1 \subseteq B$ and $B \sim A_1 \subseteq A$. By property 1, there are two bijections $f : A \to B_1$ and $g : B \to A_1$. Let $C_0 = A \setminus A_1$ and define inductively $C_{n+1} = g(f(C_n))$. Let $C = \cup_{n=0}^{\infty} C_n$. We have that

$$g^{-1}(A \setminus C) = g^{-1}(A \setminus \cup_{n=0}^{\infty} C_n) = g^{-1}(A_1 \setminus \cup_{n=1}^{\infty} C_n) = B \setminus g^{-1}(\cup_{n=1}^{\infty} C_n) = B \setminus f(C),$$

so by the choice of g we have that $A \setminus C \sim B \setminus f(C)$. Then, again by property 1, we also have that $C \sim f(C)$ so property 2 yields $A = (A \setminus C) \sqcup C \sim (B \setminus f(C)) \sqcup f(C) = B$. □

We are now equipped to prove the strong form of the Banach–Tarski paradox, concluding this section.

Theorem 4.20 (The Banach–Tarski paradox, general form, AC)**.** *Any two bounded subsets of* $\mathbb{R}^3$ *with nonempty interior are* G_3*-equidecomposable.*

Proof. Let A and B be bounded subsets of $\mathbb{R}^3$ with nonempty interior. It is sufficient to show that $A \lesssim B$, since by the same argument we will also have that $B \lesssim A$ and thus by Theorem 4.19 $A \sim B$. Let K be a solid ball such that $A \subseteq K$, there certainly is such a ball since A is bounded. Since B has a nonempty interior we can also find a solid ball $L \subseteq B$. Let n be large enough such that K can be covered by n possibly overlapping copies of L, and let S be a set consisting of n disjoint copies of L. Then $K \lesssim S$, and by repeated use of Corollary 4.18 we also have that $S \lesssim L$ which yields $A \subseteq K \lesssim S \lesssim L \subseteq B$, so $A \lesssim B$. □

The proof of the Banach–Tarski paradox is quite long and technical and it is very valuable to identify and abstract the key steps. There are essentially two main ideas. Firstly, we realized that SO_3 has a subgroup $\langle \phi, \psi \rangle$ isomorphic to the free group of rank 2, which is paradoxical. Secondly, we considered the action of $\langle \phi, \psi \rangle$ on $\mathbf{S}^2$ and realized that if we removed the nontrivial

[3]This shows that $\lesssim$ is a *partial ordering* of the equivalence classes of $\sim_G$.

fixed points in $\mathbf{S}^2$, we could push the paradoxical decomposition of $\langle \phi, \psi \rangle$ onto $\mathbf{S}^2 \setminus D$. After having proved these steps it was a rather technical matter to transfer the paradoxical decomposition of $\mathbf{S}^2 \setminus D$ to $\mathbf{S}^2$, $\mathbf{B}^3 \setminus \{0\}$ and finally $\mathbf{B}^3$ (the strong form of the Banach–Tarski paradox required the additional machinery of equidecomposability and the Banach–Schröder–Bernstein Theorem). In fact, to further clarify the importance of these steps we can put these two ideas together and abstract them in the context of paradoxical decompositions, which is a generalization of the Hausdorff paradox.

Theorem 4.21 (AC)**.** *Let the group G act on the set X without nontrivial fixed points. If G is paradoxical, then X is G-paradoxical.*

5 The implications on the problem of measure

In this section, we will discuss the implications of the Banach–Tarski paradox on the problem of measure. We start by defining what a *measure* is with the definition from Cohn [15]. Then, we define what the problem of measure is and find that there is a measure known as *Lebesgue measure* that solves the problem of measure for sets known as *Lebesgue measurable sets.* Then, we discuss the Lebesgue measure of the sets we use to prove the Banach–Tarski paradox and how the paradox further affects the problem of measure.

5.1 Defining measures

In order to define measures, we must first define algebras, the domain of measures.

Definition 5.1. Let X be a set. A collection of subsets $\mathcal{A}$ is known as an *algebra on X* if

1. $X \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A^C \in \mathcal{A}$.
3. If $A_1, A_2, ..., A_n \in \mathcal{A}$, then $\bigcup_{k=1}^{n} A_k \in \mathcal{A}$.
4. If $A_1, A_2, ..., A_n \in \mathcal{A}$, then $\bigcap_{k=1}^{n} A_k \in \mathcal{A}$.

In other words, $\mathcal{A}$ is an algebra on X if X is in the algebra and the algebra is closed under complementation, finite unions and finite intersections. If $\mathcal{A}$ is also closed under countable unions and intersections, $\mathcal{A}$ is called a *σ-algebra.* Note that since $X^c = \emptyset$, the empty set is in every algebra.

A measure is a function that tries to give sensible volumes to sets in an algebra.

Definition 5.2. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. A function $m : \mathcal{A} \to [0, \infty]$ is called a *countably additive measure* on $\mathcal{A}$ if it satisfies the following properties:

1. $m(\emptyset) = 0$,
2. $m(\bigsqcup_{k=1}^{\infty}(E_k)) = \sum_{k=1}^{\infty} m(E_k)$ for all disjoint collections $\{E_k\}$, $E_k \in \mathcal{A}$.

If $m(E \sqcup F) = m(E) + m(F)$ for $E, F \in \mathcal{A}$ and $\mathcal{A}$ is an algebra, the function is called a *finitely additive measure.* If $\mathcal{A} = \mathcal{P}(X)$[4], we will call m a *finitely/countably additive measure on X.*

This definition agrees with our intuition of measure, as something empty should have a measure of zero and if a set can be separated into countably or finitely many disjoint parts, the measure should be the sum of its parts.

Example 5.3. Let X be a finite set. Then the function $\mu(A) = |A|$ is a finitely additive measure on X. This measure is known as the *counting measure.*

Since we often have groups interact on sets, we are interested in seeing how the groups interact with the measures on the set.

Definition 5.4. Let μ be a measure on X and let G be a group acting on X. We say that μ is *G-invariant on X* if for all $g \in G$ and $E \subseteq X$, $\mu(gE) = \mu(E)$. If $\mu(E) = 1$, we say that μ *normalizes E.*

[4]The collection of all subsets of X

5.2 The problem of measure

The problem of measure is about finding a measure m on $\mathbb{R}^n$ for $n \in \mathbb{N}$ which normalizes the n-dimensional unit cube, and where m is $\mathbb{E}^n$-invariant. Remember that this means that m is isometry-invariant. Essentially, we want a measure on $\mathbb{R}^n$ that generalizes the notion of length, area and volume, and these properties satisfy this. The volume is unaffected by any distance-preserving isometries, the unit cube has the right volume, and because of the finite additivity, smaller cubes will have the right volume, which through unions can at least approximate the most common geometric shapes.

5.2.1 Lebesgue measure

In order to find a suitable measure, let us start by constructing the length of sets of the form $[a, b], a \leq b$, an interval in $\mathbb{R}$, that is translation-invariant and normalizes $[0, 1]$. By our intuition, the length should be $b - a$, so let $len(I) = b - a$ denote the length of the interval. It is obvious from the definition that len normalizes $[0, 1]$ and is translation invariant since the set $I + x$ has length $(b + x) - (a + x) = b - a$ for all $x \in \mathbb{R}$. It should also be intuitive that it does not matter if the interval is open, closed or neither, since an individual point would have length 0. Since an n-dimensional box can be described as the Cartesian products of intervals, we can define the n-dimensional box as $C^n = \{(x_1, x_2, ..., x_n), x_k \in I_k\}$, where I_k are intervals. By our intuition, we should define the volume of a box as $vol(C^n) = \prod_{k=1}^{n} len(I_k)$ which similarly normalizes $[0, 1]^n$ and is translation-invariant. While $vol(E)$ is only a volume of boxes, and the collection of boxes on $\mathbb{R}^n$ is not an algebra on $\mathbb{R}^n$, it does provide the basis of several measures, including the Lebesgue measure.

Definition 5.5. The smallest σ-algebra on $\mathbb{R}^n$ that contains all boxes (Proposition 1.1.5 in [15]), is known as the algebra of *Borel sets*. We say that the Borel sets are *generated by boxes*.

Proposition 5.6. *There exists a, countably additive, translation-invariant measure known as Lebesgue measure [2] that normalizes the unit cube and is unique on Borel sets [9].*

Let E be a subset of $\mathbb{R}^n$. Since we already have the volume of n-dimensional boxes, we try to cover E with boxes and try to minimize the total volume of the boxes used to cover E. Then the infinum of the total volume is larger or equal to the logical measure. We call this upper bound the *Lebesgue outer measure*, which is defined as

$$\lambda^*(E) = \inf_{\bigcup_{k=1}^{\infty}(C_k^n) \supseteq E} \sum_{k=1}^{\infty} vol(C_k^n).$$

Since the volume of boxes is translation-invariant, λ must be translation-invariant.

Definition 5.7. If $E \subset \mathbb{R}^n$ partitions all $A \subset \mathbb{R}^n$ so that the Lebesgue outer measure is consistent, in other words if $\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$, E is called *Lebesgue measurable* with Lebesgue measure $\lambda(E) = \lambda^*(E)$ [15].

To show that Lebesgue measure is unique on Borel sets, let μ be a countably additive, translation-invariant measure normalizing the unit cube defined on Borel sets. Then the translation-invariance, countable additivity and $\mu(\emptyset) = 0$ gives that $\mu = \lambda$ on all boxes. Since these boxes generate the Borel sets, μ has to agree with λ on all Borel sets.

We will need that Lebesgue measure is isometry invariant but could not find a reference, so we will show a quick proof based on [16]. To show this, we will show that the measure of an isometry of a set fixing the origin is translation-invariant. We only need to show it for isometries fixing the origin since every isometry is the combination of a translation and an isometry fixing the origin. Let S be an isometry fixing the origin and E be a Lebesgue measurable subset of $\mathbb{R}^n$. Then

$$\lambda(S(E + x)) = \lambda(S(E) + S(x)) = \lambda(S(E)),$$

where the second equality comes from the translation-invariance of λ. Since Lebesgue measure is the only translation-invariant measure on Borel sets, it implies that $\lambda(S(E)) = c\lambda(E)$ for some constant c. Since isometries fixing the origin fixes the unit ball, $c = 1$ meaning that $\lambda(S(E)) = \lambda(E)$.

5.2.2 Non-measurable sets

Vitali [3] showed the existence of sets in $\mathbb{R}$ that are not measurable by any countably additive, $\mathbb{E}^n$-invariant measure where $m([0,1]) = 1$, which of course includes the Lebesgue measure. The method by which he found these sets can be used to find similar non-measurable sets in higher dimensions as well. The question then arises if there is a *finitely* additive, $\mathbb{E}^n$-invariant measure on $\mathbb{R}^n$ for which all subsets are measurable.

Let $\{A_i\}_{i=1}^n$ be any finite partition of the unit ball in $\mathbb{R}^3$ such that A_i is measurable for $i = 1, 2, ..., n$ and let $\rho_1, \rho_2, ..., \rho_n$ be arbitrary elements of G_3. Since Lebesgue measure is G_3-invariant and finitely additive it holds that

$$\lambda(\mathbf{B}^3) = \lambda\Big(\bigsqcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \lambda(A_i) = \sum_{i=1}^{n} \lambda(\rho_i A_i),$$

i.e. disassembling the unit ball into finitely many *measurable* pieces and moving these around in a rigid manner preserves the volume with respect to Lebesgue measure. Now, let $A_1, ..., A_n, B_1, ..., B_m$ and $g_1, ..., g_n, h_1, ..., h_m$ be sets and elements witnessing that $\mathbf{B}^3$ is G_3-paradoxical. Suppose that all the A_i:s and B_i:s are measurable, then we have

$$\begin{aligned} \lambda(\mathbf{B}^3) &\geq \lambda\left(\Big(\bigsqcup_{i=1}^{n} A_i\Big) \cup \Big(\bigsqcup_{i=1}^{m} B_i\Big)\right) = \sum_{i=1}^{n} \lambda(A_i) + \sum_{i=1}^{m} \lambda(B_i) = \sum_{i=1}^{n} \lambda(g_i A_i) + \sum_{i=1}^{m} \lambda(h_i B_i) \\ &\geq \lambda\Big(\bigcup_{i=1}^{n} g_i A_i\Big) + \lambda\Big(\bigcup_{i=1}^{m} h_i B_i\Big) = 2\lambda(\mathbf{B}^3). \end{aligned} \tag{4}$$

Since Lebesgue measure assigns the usual volume to the unit ball, i.e. $\lambda(\mathbf{B}^3) = \frac{4}{3}\pi$, we have that $\frac{4}{3}\pi \geq \frac{8}{3}\pi$. Thus, at least one of the sets $A_1, ..., A_n, B_1, ..., B_m$ is *not* Lebesgue measurable – this is interesting since the Lebesgue measurable sets cover a wide range of subsets of $\mathbb{R}^3$. Ultimately, the sets used in Corollary 4.16 only exist due to the axiom of choice, which hints at the subtle nature of AC – it gives rise to somewhat artificial sets.

Assume that μ is any finitely additive, $\mathbb{E}^3$-invariant measure on $\mathbb{R}^3$. By Theorem 4.20, the unit cube $[0,1]^3$ is G_3-equidecomposable with two copies of itself. With $\mathbf{B}^3$ replaced by $[0,1]^3$, an identical calculation as in equation (4) will give that $\mu([0,1]^3) \geq 2\mu([0,1]^3)$, which is only true if $\mu([0,1]^3)$ is equal to 0 or ∞. Thus, there cannot be such a measure on $\mathbb{R}^3$ normalizing the unit cube, giving a negative answer to the problem of measure on $\mathbb{R}^3$.

The paradoxical subgroup $\langle \phi, \psi \rangle$ of SO_3 can be identified as a subgroup of SO_n for $n > 3$. One might expect it, though it is not completely obvious, this implies that the Banach–Tarski paradox exists in higher dimensions as well (see Chapter 5 in [9] for a proof) and consequently giving a negative answer to the problem of measure for $n > 3$. We will return to answer the problem of measure in one and two dimensions in Section 5.4 after we have defined a few more concepts.

5.3 Tarski's Theorem and Amenable groups

The key properties of the Lebesgue measure is that it is isometry-invariant and that it assigns the unit cube a measure of 1. For other sets however, we might want to construct a measure that is invariant under some other group action, or that normalizes some other subset. If a measure μ is scaled by a constant factor the result is still a measure. Therefore any subset with finite, nonzero measure can always be normalized. We saw in the last section that because $\mathbf{B}^3$ is G_3-paradoxical, it can not have a finite nonzero measure, assuming that all subsets of $\mathbb{R}^n$ are measurable. This reasoning can be generalized to other measures as well.

Proposition 5.8. *Let μ be a finitely additive G-invariant measure on X, and let $E \subseteq X$. If E is G-paradoxical then $\mu(E) = 0$ or $\mu(E) = \infty$.*

Proof. Let $A_1, ..., A_n, B_1, ..., B_m$ and $g_1, ..., g_n, h_1, ..., h_m$ be sets and elements witnessing that E is G-paradoxical. Assume $\mu(E) < \infty$. Then $\mu(E) \geq \Sigma\mu(A_i) + \Sigma\mu(B_j) = \Sigma\mu(g_i A_i) + \Sigma\mu(h_j B_j) \geq \mu(\cup g_i A_i) + \mu(\cup h_j B_j) = \mu(E) + \mu(E) = 2\mu(E)$. Therefore, since $\mu(E) < \infty$ we have $\mu(E) = 0$. $\square$

Note that this proof is exactly analogous to the calculation in Equation 4. This is because all we used in Equation 4 is that $\mathbf{B}^3$ is G_3-paradoxical. Maybe this result is not so surprising. If it is possible to build two copies of E from E itself only by doing an action that does not change the measure, then E can not have a finite nonzero measure. The interesting part is that this goes in both directions. This is what is called Tarski's Theorem.

Theorem 5.9 (Tarski's Theorem, AC)**.** *Let G act on a set X and let $E \subseteq X$. Then there is a finitely additive, G-invariant measure μ on X with $\mu(E) = 1$ if and only if E is not G-paradoxical.*

Proof. Proposition 5.8 takes care of one direction. A complete proof is included in Wagon and Tomkowicz [9] book and is dependent on the axiom of choice. □

Since paradoxical groups can prove the absence of a finitely additive, G-invariant measure normalizing a set, we might want to look at the converse. Is there a property on some non-paradoxical group that, when transferred to a set, gives a finitely additive, G-invariant measure, and are $\mathbb{R}$ and $\mathbb{R}^2$ sets that the property can be transferred to?

Von Neumann [17], realized in 1929 that there were non-paradoxical groups with G-invariant normalizing measures on them that could be transferred over to other sets. While there are multiple ways of defining these groups [18], we are going to define them using measures.

Definition 5.10. Let G be a group. If there is a finitely additive, G-invariant measure μ on G that normalizes G, G is called an *amenable group.*

From Proposition 5.8, one finds that paradoxical groups are not amenable, since paradoxical groups have to satisfy $\mu(G) = 2\mu(G)$.

Example 5.11. Let G be a finite group. Then the normalized counting measure $\mu(E) = \frac{|E|}{|G|}$ is a finitely additive, G-invariant measure nomalizing G, meaning that G is amenable.

The isometry group $\mathbb{E}^n$ is amenable for $n = 1, 2$ [9] but is not amenable for $n \geq 3$. The first statement will not be proven due to it being outside the scope of the thesis. The second statement comes from $\mathbb{E}^n$ containing the paradoxical subgroup of SO_3.

These group measures can then be pushed onto any set through group actions.

Proposition 5.12. *Let G be a group acting on X. If G is amenable, there exists a finitely additive, G-invariant measure μ_X on X normalizing X.*

Proof. Let $x \in X$ be a fixed point. Then for $E \subseteq X$, the function $\mu_X(E) = \mu(\{g \in G : gx \in E\})$ is a finitely additive, G-invariant measure with $\mu_X(X) = 1$. □

Using this result on $\mathbb{R}^n$ using the group $\mathbb{E}^n$ for $n = 1, 2$ gives us a finitely additive isometry-invariant measure on the line and plane, though one that normalizes the whole set and not the unit cube. Thus, this still does not give an answer to the problem of measure for $\mathbb{R}$ and $\mathbb{R}^2$. With some more work however, we can find a measure that normalizes the unit cube.

5.4 Measure on $\mathbb{R}$ and $\mathbb{R}^2$

In section 5.2.2 we answered the problem of measure for dimensions higher than two. What about one and two dimensions? It turns out that we can get the measure we have hoped for, and moreover, this measure extends Lebesgue measure. In fact, the solution to the problem of measure must extend Lebesgue measure on Borel sets, since this is the unique $\mathbb{E}^n$-invariant measure on Borel sets.

Theorem 5.13 (AC)**.** *If G is an amenable group of isometries, then there is a finitely additive, G-invariant measure on $\mathbb{R}^n$ which extends Lebesgue measure.*

We give a sketch of a proof of this theorem in Appendix C. Since $\lambda([0, 1]^n) = 1$, this measure normalizes the unit cube. Choosing $G = \mathbb{E}^n$ for $n = 1, 2$ we get $\mathbb{E}^n$-invariant extensions of Lebesgue measure on $\mathbb{R}^1$ and $\mathbb{R}^2$, giving a positive answer to the problem of measure in one and two dimensions. Since bounded subsets with nonempty interior have positive, finite Lebesgue measure, this result together with Theorem 5.9 implies that all such sets are not $\mathbb{E}^n$-paradoxical for $n = 1, 2$. In particular, it denies the existence of the Banach–Tarski paradox in one and two dimensions.

References

[1] Terence Tao. *An Introduction to Measure Theory*. Graduate studies in mathematics. Los Angeles: American Mathematical Society, 2011, pp. 2–4. ISBN: 9780821869192. URL: https://books.google.se/books?id=HoGDAwAAQBAJ.

[2] Henri L. Lebesgue. "Intégrale, Longueur, Aire". fre. PhD Thesis. Université de Paris, 1902.

[3] Giuseppe Vitali. *Sul problema della misura dei gruppi di punti di una retta*. ita. Bologna: Tipografia Gamberini e Parmeggiani, 1905.

[4] Felix Hausdorff. "Bemerkung über den Inhalt von Punktmengen". ger. In: *Math. Ann.* 75 (1914), pp. 428–434. DOI: https://doi.org/10.1007/BF01563735.

[5] Stefan Banach and Alfred Tarski. "Sur la décomposition des ensembles de points en parties respectivement congruentes". fre. In: *Fund. Math.* 6 (1924), pp. 244–277.

[6] Andrew McFarland, Joanna McFarland, and James Smith. "On Decomposition of Point Sets into Respectively Congruent Parts (1924)". In: Translation of [5]. New York: Birkhäuser, July 2014, pp. 93–123. DOI: https://doi.org/10.1007/978-1-4939-1474-6_6.

[7] Ernst Zermelo. "Beweis, daß jede Menge wohlgeordnet werden kann. (Aus einem an Herrn Hilbert gerichteten Briefe)". ger. In: *Math. Ann.* 59 (1904), pp. 514–516. URL: http://eudml.org/doc/158167.

[8] Robert Solovay. "A Model of Set Theory in Which Every Set of Reals is Lebesgue Measurable". In: *Ann. Math.* 92.1 (1970), pp. 1–56. DOI: https://doi.org/10.2307/1970696.

[9] Grzegorz Tomkowicz and Stan Wagon. *The Banach–Tarski Paradox*. 2nd Edition. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 2016. DOI: https://doi.org/10.1017/CBO9781107337145.

[10] Émile Borel. *Lecons Sur la Theorie Des Fonctions*. fre. 3rd Edition. Paris: Gauthier-Villars, 1928.

[11] John R. Durbin. *Modern Algebra: An Introduction*. 6th Edition. Wiley, 2008. ISBN: 9780470384435.

[12] Geoff Smith and Olga Tabachnikova. *Topics in Group Theory*. Springer Undergraduate Mathematics Series. Springer-Verlag London, 2000. ISBN: 9781852332358. DOI: https://doi.org/10.1007/978-1-4471-0461-2.

[13] Tom Weston. "The Banach–Tarski Paradox". 2003. URL: https://people.math.umass.edu/~weston/oldpapers/banach.pdf.

[14] Søren Knudby. "The Banach–Tarski Paradox". B.S. Thesis. [Accessed 2020 May 10]. University of Copenhagen, 2009. URL: https://www.math.ku.dk/english/research/tfa/ncg/paststudents/bs-theses/SK_bsthesis.pdf.

[15] Donald L. Cohn. *Measure Theory: Second Edition*. Birkhäuser Advanced Texts Basler Lehrbücher. New York: Springer New York, 2013. ISBN: 9781461469568. DOI: https://doi.org/10.1007/978-1-4614-6956-8.

[16] Omar A. Camarena. *Lebesgue measure is invariant under isometries*. [Cited 2020 May 9]. URL: https://www.matem.unam.mx/~omar/notes/lebesgue.html.

[17] John von Neumann. "Zur allgemeinen Theorie des Masses". ger. In: *Fund. Math* 13.1 (1929), pp. 73–116. URL: http://eudml.org/doc/211921.

[18] Alan L.T. Paterson. *Amenability*. Mathematical surveys and monographs. Providence: American Mathematical Society, 1988, pp. 1–5. ISBN: 9780821874844. URL: http://dx.doi.org/10.1090/surv/029.

[19] Walter Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 2006. ISBN: 9780070619883. URL: https://books.google.se/books?id=l7XFfDmjp5IC.

A Figures

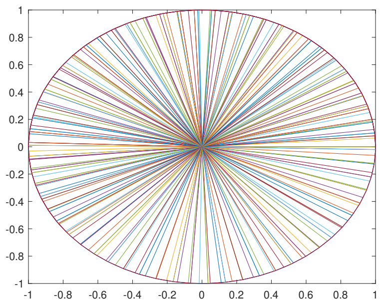

Figure 1: The unit circle with 100 spokes generated by rotating the line $(0, 1)$ $\frac{1}{10}$ radians, showing the appearance of the “Spokes on a wheel paradox”.

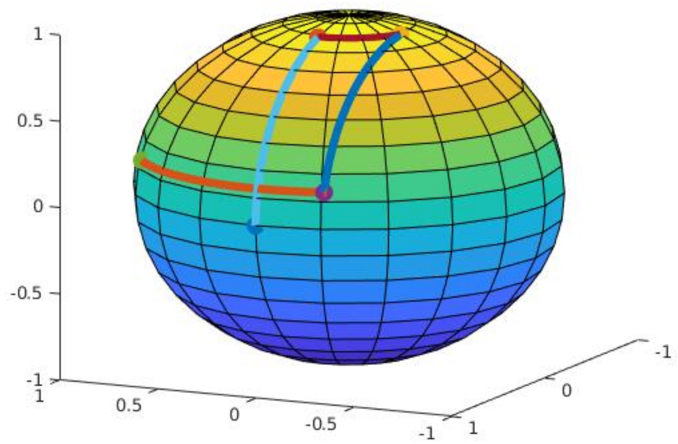

Figure 2: What happens to (0,1,0) at each step when $\psi^{-1}\phi^{-1}\psi\phi$ is applied. The blue point represents the starting point and the green point the ending point.

B Orthogonal Matrices

In Section 2.2.4 we list some properties of orthogonal matrices needed to show that the elements of SO_3 are in a one-to-one correspondence with the group of rotations about lines through the origin. We give proofs of these properties here.

Definition B.1. Let $\mathbb{R}^n$ be the inner-product space with the ordinary Euclidean dot product. A matrix A over $\mathbb{R}^n$ is *orthogonal* if $AA^T = I$.

The following is an equivalent definition.

Proposition B.2. *Both the columns and the rows of an orthogonal matrix A form orthonormal sets.*

Proof. Let $\{A_i\}_{i=1}^n$ be the rows of A. Then

$$I = AA^T = (\langle A_i, A_j \rangle)_{i,j}$$

which shows that $\langle A_i, A_j \rangle = 1$ if $i = j$ and 0 otherwise. For the columns, instead consider $I = I^T = (AA^T)^T = A^T A$. □

Proposition B.3. *Take any $A \in SO_n$ and let T be the linear transformation of $\mathbb{R}^n$ given by A under any orthonormal basis of $\mathbb{R}^n$. For any $u, v \in \mathbb{R}^n$ it holds that $\langle u, v \rangle = \langle Tu, Tv \rangle$.*

Proof. Given any orthonormal basis of $\mathbb{R}^n$, we have that T is given by $x \mapsto Ax$. Let $u, v \in \mathbb{R}^n$. Then $\langle Tu, Tv \rangle = \langle Au, Av \rangle = (Au)^T(Av) = u^T A^T Av = u^T v = \langle u, v \rangle$. □

Proposition B.4. *Let E be an orthonormal basis of $\mathbb{R}^n$ and let T be a linear transformation of $\mathbb{R}^n$ given by $[T]_E = A$ for some $A \in SO_n$. Then $[T]_{E'}$ is an orthogonal matrix for any orthonormal basis E' of $\mathbb{R}^n$.*

Proof. We only need to show that the change of basis matrix $B = [I]_{E'E}$ is orthogonal, since $[T]_{E'} = [I]_{E'E}[T]_E[I]_{EE'} = BAB^{-1}$. Let E and $E' = \{v_1, ..., v_n\}$ be orthonormal bases of $\mathbb{R}^n$. But then since

$$B = \begin{bmatrix} v_1, ..., v_n \end{bmatrix}$$

we have that

$$BB^T = (B^T B)^T = ((\langle v_i, v_j \rangle)_{i,j})^T = I.$$

This shows that $B^{-1} = B^T$ so

$$BAB^{-1}(BAB^{-1})^T = BAB^T(BAB^T)^T = BAB^T BA^T B^T = BAA^T B^T = ... = I.$$

□

Proposition B.5. *Let A be a real orthogonal 2×2-matrix with determinant 1. Then A can be written as*

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for some $\theta \in \mathbb{R}$ and therefore corresponds to a rotation of $\mathbb{R}^2$.

Proof. Since each row of A necessarily is a unit vector and that each unit vector can be written as $(\cos\theta, \sin\theta)$ for some θ, we have that for some θ, ϕ

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ \cos\phi & \sin\phi \end{pmatrix}.$$

The rows must also be perpendicular which together with $\det(B) = 1$ yields

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

which we recognize as a rotation matrix. □

C Sketching a proof of Theorem 5.13

We will only give a sketch of the proof – which closely follows the proof by Knudby [14] – and try to provide some intuition since there are several concepts involved that will not be constructed and which cannot be expected from the reader to have knowledge of; such as *measurable functions*, *Lebesgue integration* see Tao [1], and *integrals on groups defined through probability measures* see Knudby [14]. Also, we will need a central extension theorem from functional analysis called the *Hahn–Banach Theorem*, the proof of which is outside the scope of this paper. The theorem was proved independently by Hans Hahn and Stefan Banach in the late 1920s. See Rudin [19] for a proof. Before stating it we need some terminology.

Definition C.1. Let V be a real vector space. A *linear functional* on V is a linear map from V to $\mathbb{R}$. A *sublinear form* on V is a map $f : V \to \mathbb{R}$ such that

1. $f(\alpha x) = \alpha f(x)$ for all $\alpha \in [0, \infty)$ and $x \in V$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in V$.

Theorem C.2 (The Hahn-Banach Theorem, AC)**.** *Let V be a real vector space and $V_0 \subseteq V$ a subspace. If p is a sublinear form on V and F_0 a linear functional on V such that $F_0(x) \leq p(x)$ for all $x \in V_0$, then there exists a linear extension $F : V \to \mathbb{R}$ of F_0 to V such that*

$$-p(-x) \leq F(x) \leq p(x) \text{ for all } x \in V.$$

Theorem C.2 relies on the axiom of choice and is non-constructive, meaning that we only know that there exists an extension F but have no other information about it.

Let $\mathcal{L}$ be the set of Lebesgue measurable and integrable functions $f : \mathbb{R}^n \to \mathbb{R}$, and let V be the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f| \leq f_0$ for some $f_0 \in \mathcal{L}$. It is easy to see that these are both vector spaces under pointwise operations and that $\mathcal{L}$ is a subspace of V. Define $F_0 : \mathcal{L} \to \mathbb{R}$ by

$$F_0(f) = \int f d\lambda.$$

This is a linear functional on $\mathcal{L}$ and it can be proved that F_0 is dominated by some sublinear form on V. The proof of Theorem 5.13 builds upon the fact one can apply The Hahn-Banach Theorem on F_0 and consequently get an extension F of F_0 to V and thereby *almost* extending Lebesgue measure.

Let $E \subseteq \mathbb{R}^n$ be measurable with finite measure and denote its indicator function by 1_E. Since $1_E \in \mathcal{L}$ if and only if $\lambda(E) < \infty$ we get that

$$F_0(1_E) = \int 1_E d\lambda = \lambda(E). \tag{5}$$

It could be useful to keep this in mind to provide some motivation to why we are working with F_0 when trying to extend Lebesgue measure.

Let $L^\infty(G)$ be defined as the set of bounded, measurable functions from G to $\mathbb{R}$. Essential to the proof is that one can define an integral on $L^\infty(G)$ via a finitely additive, G-invariant measure ν on G (which exists since G is amenable). It can be constructed as to obey the usual properties of integrals as well as G-invariance and the analogous relationship of equation (5) (with λ replaced by ν).

Sketch of proof of Theorem 5.13. Let $\mathcal{L}, V$ and F_0 be defined as above. The map from $G \times V$ to V given by $g \cdot f = f \circ g^{-1}$ for all $g \in G$ and $f \in V$ is a *linear* [5] group action of G on V. The space $\mathcal{L}$ is invariant under this action so G acts on $\mathcal{L}$ as well. We need this action to eventually push the measure of G onto $\mathbb{R}^n$.

By definition of V, for any $f \in V$ there is a map $f_0 \in \mathcal{L}$ such that $|f| \leq f_0$. Therefore, if $h \in \mathcal{L}$ satisfy $f \leq h$, we have that $-f_0 \leq h$ and by monotonicity of Lebesgue integration that $F_0(-f_0) \leq F_0(h)$. The elements of the set $I_f = \{F_0(h) : h \in \mathcal{L} \text{ and } f \leq h\}$ are therefore bounded

[5] If $\alpha_1, \alpha_2 \in \mathbb{R}$, $f_1, f_2 \in V$ and $g \in G$, then $g(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 g f_1 + \alpha_2 g f_2$.

from below and since I_f is clearly nonempty we can define the sublinear form $p : V \to \mathbb{R}$ by $p(f) = \inf I_f$. It is not difficult to check that p is a sublinear form. Since Lebesgue integration is invariant under isometries, F_0 is G-invariant and consequently p is G-invariant. Moreover, p agrees with F_0 on $\mathcal{L}$, thus we can apply Theorem C.2 to extend F_0 to V through a linear functional $F : V \to \mathbb{R}$.

One could interpret the meaning of p when applied to a function $f \in V \setminus \mathcal{L}$ as assigning to f the best possible approximation (from above) of what we would have wanted its Lebesgue integral to be if f would actually have been Lebesgue measurable. Thus, because of the bounds on F given by Theorem C.2, F seems like a reasonable extension of Lebesgue integration to functions that are bounded by measurable and integrable functions.

The problem with F is that it is not necessarily G-invariant. To fix the invariancy we try to merge the properties of the amenable group with our extension. For each $f \in V$, define $\varphi_f : G \to \mathbb{R} \cup \{\infty\}$ by $\varphi_f(g) = F(g^{-1}f)$. For every $g \in G$, we get the following bounds on $\varphi_f(g)$:

$$\varphi_f(g) = F(g^{-1}f) \leq p(g^{-1}f) = p(f)$$
$$\varphi_f(g) = F(g^{-1}f) \geq -p(-g^{-1}f) = -p(g^{-1}(-f)) = -p(-f),$$

where we have used the bounds on $F(f)$, the G-invariancy of p and to attain the lower bound also that the action is linear. Thus, $\varphi_f \in L^\infty(G)$ and we may therefore define a function $\mu : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ by invoking the amenability of G and thereby acquiring a measure ν on G and an accompanying integral:

$$\mu(A) = \begin{cases} \int_G \varphi_{1_A}(g) d\nu & \text{if } 1_A \in V, \\ \infty & \text{otherwise.} \end{cases}$$

This function extends Lebesgue measure: Let $A \subset \mathbb{R}^n$ be measurable. If $\lambda(A) < \infty$, then $1_A \in \mathcal{L}$, so

$$\mu(A) = \int_G \varphi_{1_A}(g) d\nu = \int_G F(g^{-1}1_A) d\nu = \int_G F_0(g^{-1}1_A) d\nu = \int_G F_0(1_A) d\nu = \lambda(A) \int_G 1 d\nu = \lambda(A).$$

In the last equality we used that $\int_G 1 d\nu = \int_G 1_G d\nu = \nu(G) = 1$. If $\lambda(A) = \infty$, then $1_A \notin V$, so $\mu(A) = \infty = \lambda(A)$.

Using the bounds given by Theorem C.2, the linearity of the action, the linearity of F and the G-invariance of ν, we can show that μ is non-negative, finitely additive and G-invariant. □

Notice that the only functions we are really interested in are indicator functions, but we have to work in the general setting of $\mathcal{L}$ and V to be able to apply Theorem C.2.