UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

Master Degree in Finance

Pricing Interest Rate Derivatives in a Negative Yield Environment

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Contents

Abstract

The main purpose of this thesis is to price interest rate derivatives in the today negative yield environment. The plain vanilla interest rate derivatives have now negative strikes and negative values of the underlying asset, the forward rate. The Black'76 model fails because of its assumption of log-normal distribution of the underlying that does not allow the underlying to be negative.

The normal model gives a solutions to this problem since it assumes the underlying being normally distributed and then it can takes every value also negative. The shifted Black model has the same hypothesis of the Black-Scholes model but it adds a shift value in order to overcome the issue generated by the negativity of the strike values and of the current forward rate, with the only restrictions that the sum of the shift and the strike and the sum between the underlying value and the strike are positive. The shifted SABR model is used to find the shifted black volatilities for different strikes to plug later on the shifted Black formula to price interest rate derivatives. A comparison between the models and a brief analysis on delta hedge strategies are made.

Keywords: Interest rate derivatives, negative strikes, negative yield, normal model, Bachelier model, shifted Black model, shifted SABR model.

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Acronyms

- ATM - At The Money
- BM - Brownian motion
- BS - Black and Scholes
- CEV - Constant Elasticity of Variance
- ECB - European Central Banks
- IR - Interest Rate
- ITM - In The Money
- IV - Implied Volatility
- OTC - Over The Counter
- OTM - Out of The Money
- SB - Shifted Black

Symbols

- F - Forward rate
- K - Strike price
- T - Time to exercise
- r - Interest rate
- σ - Volatility
- σ_{SB} - Shifted Black Volatility
- σ_N - Normal or Bachelier Volatility
- C^B - Call price under Black
- P^B - Put price under Black
- Θ - Shift
- C^B_{shift} - Call price under Shifted Black
- P^B_{shift} - Put price under Shifted Black
- σ_B - Implied Black volatility
- $\hat{F}$ - SABR forward rate
- f - SABR forward rate starting value
- X - Brownian motion of the forward price in Black
- W - Brownian motion of the forward price in SABR
- Z - Brownian motion of the volatility in SABR
- H - Brownian motion of the volatility in Normal model

- ρ - Correlation between forward value and volatility of SABR
- $\hat{\alpha}$ - SABR volatility
- α - SABR volatility starting value
- β - SABR constant elasticity of variance (CEV) exponent
- ν - SABR volatility of volatility
- $\sigma_B(K,f) \equiv \sigma_B(K,f,\alpha,\rho,\beta,\nu) \equiv \sigma_B^{SABR}$ - Implied SABR volatility for Black model
- σ_{SB}^{SABR} - Implied shifted SABR volatility for shifted Black model
- $\sigma_{ATM} \equiv \sigma_B(f,f)$ - Implied SABR ATM volatility
- Δ - Delta risk
- Δ_p - Delta portfolio's risk
- w - Portfolio weights
- Λ - Vega risk
- S - Underlying price
- V - Value of the held position
- Π - Portfolio value
- $C(t,\star)$ - CEV coefficient
- σ_{loc} - Local volatility
- V_C - Call value
- V_P - Put value
- V_{call}^{SABR} - SABR call value
- δ - Variation term
- δ_α - Variation caused by changes in α
- δ_f - Variation caused by changes in f

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Introduction

Negative yields affect the pricing formula of interest rate derivatives. The popular Black '76 model is not suitable anymore to price these financial instruments because of its assumption of log-normality of the underlying asset, the forward rate. In fact, this model allows the forward rates to take only positive values and it is not coherent with the today environment where they are negative.

Initially, the thesis gives a general overview of the evolution of interest rates in the last few years and its impact on the interest rate derivatives.

The thesis is then mainly focuses on the literature and the methodologies to price and hedge plain vanilla interest rate derivatives with negative strikes. The normal model gives a solution to the pricing problem since it assumes the underlying to be normally distributed. Thus, the forward rate can assume all the possible values from minus infinity to plus infinity. The analysis goes on with the shifted Black model. It is an alternative solution to the normal model and it has the same structure of the Black '76 model but now there is a shift parameter. The latter solves the negativity of the forward rate allowing it to be partially negative. Indeed, the forward rate can be negative but it is limited by the size of the shift. An issue deriving from the shifted Black model is amenable to the fact that shifted Black volatilities are not always available on the market.

An interest rate derivative is an over the counter agreement and, for this reason, the counter-parties can agree on whatever strike rate. Each interest rate derivative has its own strike rate and each strike rate has its own shifted Black volatility. Sometimes, the market shifted Black volatilities are available just for some strike rate values. This means that if the counter-parties agree on one strike rate that does not have its complementary market shifted Black volatility, then a model that estimates this volatility is needed.

That's why I introduced the shifted SABR model in my analysis. It captures the market volatility dynamic with the right shape and skew. The shfted SABR model it is used to estimate the shifted Black volatilities for a larger strike rate grid, obtaining then also shifted Black volatilities that are not available on the market. The model allows the implementation of the shifted Black pricing formula for those interest rate derivatives that do not have a market volatility. Also the shifted SABR model has the shift parameter for the same reasons of the shifted Black model: to solve the negativity of the forward rate.

Delta and vega of interest rate derivatives are considered in order to have a response on which of the two models between the shifted Black and the normal gives a better estimate and interpretation of the interest rate derivative premiums. Delta measures the sensibility of the option value to changes in the value of the underlying asset. Meanwhile, vega represents the sensibility of the option value with respect to changes in the underlying volatility. A hedged portfolio is obtained when the two greeks are zero. From the experiments, it is not clear which of the two models is the best. The obtained prices from the normal and shifted Black model seem to be both good estimates of the market premiums. Analyzing then the delta of the interest rate derivatives is a way to define which model is the most suitable. The normal model predicts the right deltas of the interest rate derivatives and it does not distort these values as the shifted Black model does. The existence of the shift in the shifted Black model distorts the true delta values compromising the delta-hedge strategies. Obviously, if a delta-hedge strategy is made with wrong delta values, it is not accurate.

The extension of the sensitivity analysis proposed by Bartlett is examined. Under the SABR model, the adjusted values for delta and vega both contain an extra term that is added to the original formulations. These terms come out from the reciprocal influence between the forward rate and its volatility. While the original formula of delta depends only on the forward rate, since delta is the change in the interest rate derivative value due to changes in the underlying, now if the forward rate changes its volatility changes as well. It is then advisable to consider this indirect effect, adjustment, in the calculation of delta. The same is for the vega: the adjusted vega considers the effects of changes in the forward rate due to changes in the volatility.

The experimental part contains comparisons between the normal and the shifted Black model for pricing swaptions and graphical illustrations of how some variables behave with respect to others are reported to better understand the existing relations and the interconnections within these important variables. For instance, I studied how the normal volatility reacts to changes in the shifted Black volatility or what happen if the shifted Black volatility is seen as function of the shift.

Delta hedge strategies are constructed on different portfolios of swaption and the difference within the normal deltas and shifted Black deltas is developed. A last experiment on a cap option through the shifted SABR model is made to test the SABR ability to capture the right dynamics of market volatility and its ability on price the cap combined it with the shifted Black model.

Chapter 1 gives an historical and economic background on how interest rates developed in last few years and it analyzes the impact of this evolution on interest rate derivatives. A short explanation of the most popular interest rate derivatives is provided. Chapter 2 and 3 summarize all the main theoretical concepts that are needed for the comprehension of the experimental part. Chapter 2 explains in details the models used to price interest rate derivatives. Chapter 3 is focuses on the hedging and its purpose is to give a further measure of comparison between pricing models in terms of model ability to capture the correct interest rate derivative sensibilities to forward rate and volatility. The fourth chapter is dedicated to the experiments and chapter five sums up the conclusions.

My analysis is based on some formulas from: ”The Pricing of Commodity Contracts” Black 1976 for what concerns the Black ’76 model and its formulation, ”New volatility conventions in a negative interest environment” d-fine December 2012 that contains some literature on the normal model, ”Managing the smile risk” Hagan 2000 and ”Hedging under SABR model” Bartlett 2006 which analyze the SABR model and the hedging.

Chapter 1

Negative Yield

1.1 The evolution of interest rates into a negative domain

The recent financial crisis that started in 2007 destabilized the financial markets damaging the trust in the financial institutions and in the market itself. In fact, the increased credit-risk and default-risk lead the market in an irreversible stagnant phase.

The need of an intervention arose. Some exceptional measures were adopted by Central Banks and in particular by the European Central Bank (ECB). Starting from 2008 until 2011, ECB lowered the interest-rates on deposits guaranteeing a cheaper cost of financing (Deloitte 2016). In 2012, the central bank of Denmark cut the interest-rates below zero (Blanke 2016). In June 2014, the ECB lowered for the first time the interest-rates on deposit to -0.10%. During the months later, other cuts were made by ECB: in September 2014 the rate was set to -0.20% and in 2015 and 2016 to -0.40% and -0.60%, respectively (European Central Bank, 2017). Also Switzerland, Japan and Sweden now have negative yield. Figure 1.1 shows the evolution of the interest-rates from 2012 to today.

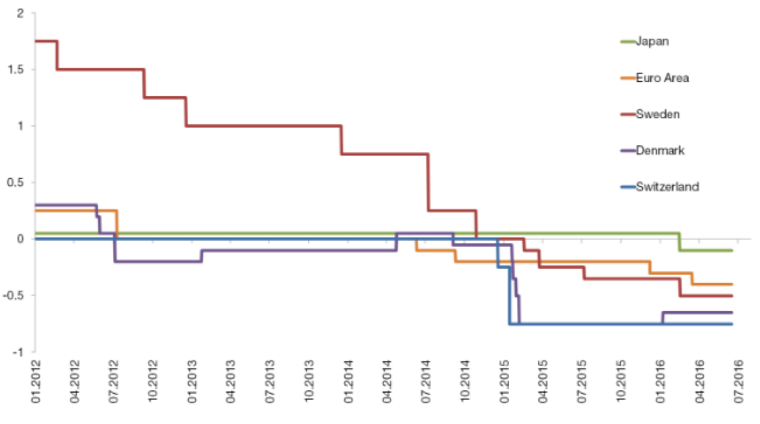


Figure 1.1: Evolution of interest-rates (World Economic Forum, 2017)

1.2 Implications

A negative interest-rate means that a depositor has to pay in order to have his cash and savings held by a bank. This seems illogical, since it is supposed to exist a return for those agents that lend money to whom who need.

The intention of the central banks and in particular of ECB is to spur the agents in the economy to invest in the market instead of retain their liquidity in safe deposits. The negativity of the interest-rates should trigger a mechanism of increased investments and therefore increased social welfare.

Unfortunately, for very risk-averse agents this trick does not work. In fact, a risk-averse agent prefers to put his money in the safest project of investment among the whole alternative choices, that is still buy government bonds, also if he has to pay for it.

Figure 1.2 shows the term structure for interest rate over 20 years for different countries.

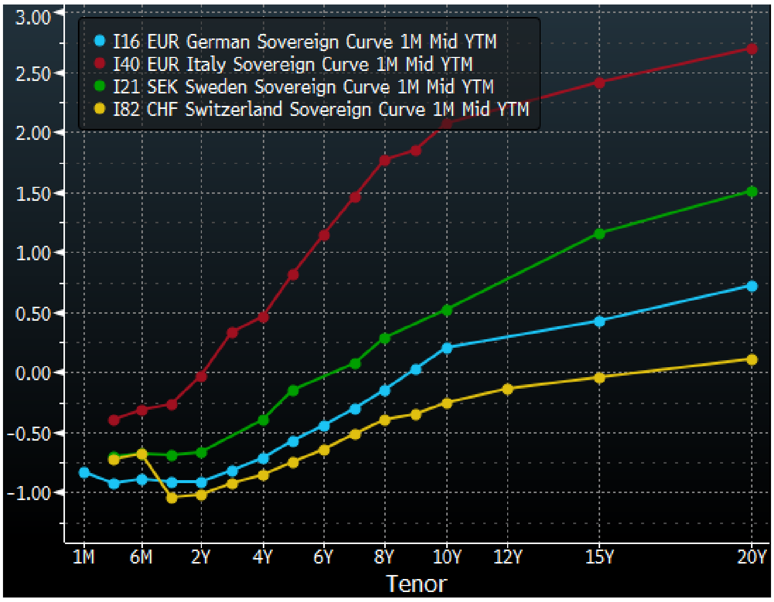


Figure 1.2: Term Structure of Interest of Interest-Rates for Germany, Italy, Sweden and Switzerland (Bloomberg, data of 31 Marc. 2017)

The return for a government bond of Germany starts to be *positive* only for tenor greater than 2 years, meanwhile in the extreme case of the Switzerland after 15 years.

1.3 How the negative yields affect interest-rates derivatives

An interest-rate below zero means a negative strike price for plain-vanilla interest-rate option, such as swaptions, caplets or floorlets.

These IR derivatives have as underlying the Euribor or Libor rate on 3 months or 6, *i.e* many interest-rate swaptions in Switzerland have as underlying the 6 months Libor - SF0006M - and it is negative since the beginnings of 2015 as reported in figure 1.3 below.

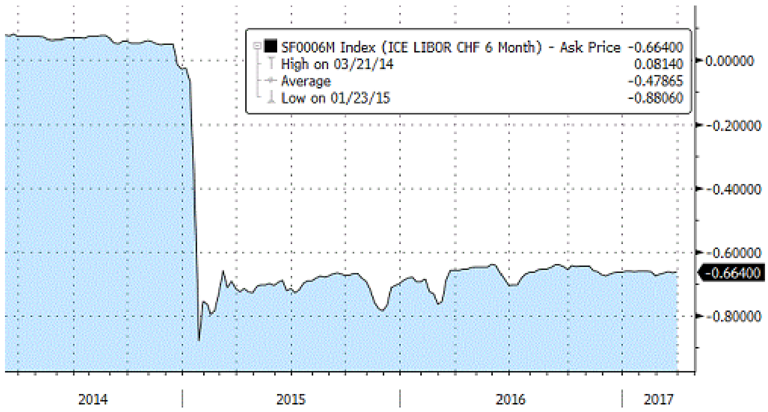


Figure 1.3: CHF Vs. 6 Month LIBOR (Bloomberg, data of 31 Marc. 2017)

1.4 Swaptions, Caplets, Floorlets

A swaption is an OTC contract between two counter-parties. The buyer of the option pays a premium to the seller in order to have the faculty, but not the obligation, to enter in an interest-rate swap contract in a prefixed -by the contract- future date, namely in the exercise date. *A priori*, it is determined if the buyer will be a *payer* or a *receiver*. In the first case if he or she decides to enter in the swap agreement will pay the fix rate and receive the floating one, meanwhile the second case indicates the other way around. The duration of the swap is called *tenor* and it is known at the settle date. At the end, the winner part, the one that generates a profit, will receive from the other the difference in rate applied on a notional amount, so called cash settlement. The notional is in some sense fictitious since it is used just to calculate the final payoff. The swaptions are very common because they can help agent to indirectly change the type of other contracts they have in their portfolios, such as mortgages. In fact, thanks to swaptions a floating contract could be changed to a fixed one mixing the previous contract held by the agent with the swaption.

The interest-rate caplet is a derivative contract in which the buyer, after the payment of a premium, obtains the right to receive by the writer, in a period of time and in prefixed date, the positive difference between the the market interest-rate, such as the LIBOR, and the interest-rate fixed by the contract -strike rate. Generally the payoff of a caplet is the following:

$$(\text{market interest rate} - \text{caplet strike rate}) \cdot \text{principal} \cdot \left(\frac{\text{number of days to maturity}}{360}\right)$$

if the difference between the interest-rates is negative the caplet does not generate any cash flow.

The covenants of a floorlet contract are the same of the caplet one, what changes is in the reason why an agent prefers to buy a caplet instead of a floorlet and *viceversa*. Namely, an agent with a floating rate debt prefers to buy a caplet since, thanks to this one, he can fix the maximum cost of the indebtedness, hedging himself against raising of the market interest-rates and taking advantages from the bear spread of interest-rates in the market. On the other hand, an agent that has a floating rate investment should buy a floorlet in order to set a minimum return of the investment.

Chapter 2

Pricing of interest-rate derivatives

2.1 The Black (1976) model

The Black model is a popular model used to price future options and interest-rate derivatives and more in general it applies to both investment and consumption assets (Hull, 2012).

The Black '76 model is an extension of the Black-Scholes (1973). In fact the only difference between the two is amenable to the fact that the Black model considers the forward rate of the underlying instead of the spot rate to price options.

Let's consider an European interest rate call option with strike K and maturity T, then the option price is given by the formula:

$$C^B = e^{-rT} \cdot [FN(d_1) - KN(d_2)] \tag{2.1}$$

meanwhile for the put option is

$$P^B = e^{-rT} \cdot [KN(-d_2) - FN(-d_1)] \tag{2.2}$$

$$d_1 = \frac{\log\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{2.3}$$

$$d_2 = \frac{\log\left(\frac{F}{K}\right) - \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where F is the forward rate and K the strike.

The model is valid and true if and only if the underlying is log-normally distributed. In fact, this is one of the most important assumption of Black'76 inherited by the BSM. Furthermore, the assumption of a constant interest rate is relaxed. The Black model considers the interest-rate that follows a stochastic process with constant volatility σ.

The assumption of constant volatility should no be appropriate in hedging American options since they have different exercise dates and the volatility. In the markets σ changes over time. Thus, the assumption of constant σ becomes unreal and restrictive but at the same time practical for simpling the formulation of the model.

On the other hand the log-normality of the underlying characterizes a great limitation of the model in todays environment. The negativity of the interest-rates causes a miss-specification of the Black model. The term $\log(\frac{F}{K})$ is no longer defined. There is a need to find alternative solutions able to price interest-rate options with negative strikes.

2.1.1 Shifted Black Model

The shifted Black model is a variation of the Black model and it is able to price interest-rate vanilla options in case of negative forward rates. Indeed, a shift is added to the forward rate in order to have a well-defined logarithm, respecting the only condition that the sum of the forward rate and the shift must be a positive value (the shift is added also to K). The overall distribution is now shifted to the positive domain and it is possible to continue to use the Black's setting. The price for a call option becomes

$$C^B_{shift} = e^{-rT} \cdot [(F-\Theta)N(d_1) - (K-\Theta)N(d_2)] \tag{2.4}$$

and for the put option is

$$P^B_{shift} = e^{-rT} \cdot [(K-\Theta)N(-d_2) - (F-\Theta)N(-d_1)], \tag{2.5}$$

with

$$d_1 = \frac{\log\left(\frac{(F-\Theta)}{(K-\Theta)}\right) + \left(\frac{\sigma^2_{SB}}{2}\right)T}{\sigma_{SB}\sqrt{T}},$$

$$d_2 = \frac{\log\left(\frac{(F-\Theta)}{(K-\Theta)}\right) - \left(\frac{\sigma^2_{SB}}{2}\right)T}{\sigma_{SB}\sqrt{T}} = d_1 - \sigma_{SB}\sqrt{T}.$$

The shift term is determined by Θ and it is negative. This model is also known as displaced diffusion because now the forward rate is described by a geometric Brownian Motion (diffusion) where the trend is displaced by a shift (Deloitte., 2016). The forward rate in time t, F_t, follows the stochastic process:

$$dF_t = \sigma_{SB}(F_t - \Theta)dX_t \tag{2.6}$$

$$\Theta < 0, \quad F_t - \Theta > 0, \quad \sigma_{SB} > 0$$

where X_t is a Brownian motion and $F_t - \Theta$ is a drift-less geometric Brownian motion with volatility σ_{SB}. The possible values the forward price can assume belong to the interval (Θ, ∞) (Lee & Wang, 2012). Thus, $F - \Theta$ and $K - \Theta$ must be positive and this is not true, for instance, for values of the forward rate lower then Θ. For example, if the forward rate is -0.075, a shift with absolute value at least greater than 0.075 is needed.

The strength of this model is its ability to find a closed-formula to price interest-rate derivatives, even if the only allowed negative values are those above the value of the shift ($F > \Theta$, $\Theta < 0$). Then the model allows the strikes to be negative but at the same time it puts a boundary on them.

The disadvantages reside in the fact that the volatility could not be available on the market and then it is necessary to estimate it trough other models. One of these is the SABR model.

2.2 SABR model

The SABR model was introduced by Hagan (2000) and it is a two factor stochastic volatility model. In fact, since the market is always affected by alternation of quiescence and chaotic periods is better to add a second factor to create the model (Hagan, 2000). The best factor that capture this intrinsic randomness of financial markets is obviously the volatility.

The SABR model considers both the forward price and its volatility to be stochastic following two different stochastic processes of the form:

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta}dW, \qquad \hat{F}(0) = f \tag{2.7}$$

$$d\hat{\alpha} = \nu\hat{\alpha}dZ, \qquad \hat{\alpha}(0) = \alpha \tag{2.8}$$

$$E^{Q}(dW, dZ) = \rho dW dZ \tag{2.9}$$

W and Z are two different Brownian Motions correlated with each other by the coefficient ρ.

Since the volatility of the forward price, α, is itself stochastic, it has its own volatility ν. Thus, ν is the volatility of the volatility. Obviously, in $t = 0$, today, the values of the forward price and its volatility are known and, as consequence, they are not random but rather constants f and α.

β is a special parameter: it is the constant elasticity of variance exponent and in the formulation of the SABR model it can take values between 0 and 1 included. Furthermore, according to which of the possible values β assumes, the model reacts in different ways. β allows the model to switch between a normal model or a log-normal one. In fact, if β is equal to 1 the model is supposed to be log-normal, meanwhile if β is 0 the model is normal. The latter expressions, normal and log-normal, refer to the fact that the forward rate is normally distributed or log-normally distributed. There is also the special case in which β assumes a value equal to 0.5. In this case the hypothesis of a CIR stochastic model is assumed. β connects the future spot with the at-the-money volatility: if $\beta \approx 1$ then the user believes that there is not a co-movement between the movements of the market and the behavior of at-the-money volatility. Namely, if the market moves down or up the ATM volatility remains almost significantly not affected by this change. Conversely, a really β lower than 1, $\beta << 1$, indicates a negative correlation between the movements of the market and the ATM volatility. Finally, a value of β close to 0 is translated in the belief that this negative correlation is extremely pronounced (West, 2005).

Under the SABR model the price of an European option is given by the Black's model formula and the implied volatility is:

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{\frac{(1-\beta)}{2}}\left[1+\frac{(1-\beta)^2}{24}log^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}log^4\left(\frac{f}{K}\right)+...\right]}\cdot\left(\frac{z}{x(z)}\right)$$
$$\cdot\left[1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{(1-\beta)}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{\frac{(1-\beta)}{2}}}+\frac{2-3\rho^2}{24}\nu^2\right]T+...\right]. \tag{2.10}$$

Here

$$z = \frac{\nu}{\alpha}(fK)^{\frac{(1-\beta)}{2}}log\left(\frac{f}{K}\right) \tag{2.11}$$

and $x(z)$ is

$$x(z) = log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right). \tag{2.12}$$

For the special case in which the forward rate is equal to the strike price and then the option is at-the-money with $f = K$, the reduced formula is

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f^{(1-\beta)}}\left[1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{(2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right)T+...\right]. \tag{2.13}$$

2.2.1 Calibration

The Stochastic α-β-ρ model (SABR) is a four parameters model. The parameters are α, β, ρ and ν and they have each different impact on the model.

α affects the height of the curve produced by the model, meanwhile ρ and ν control the curve's skewness and how pronounced the smile is, respectively. Calibrating all four parameters is difficult. Indeed, it is usual to let one of them be fixed and just let the other three vary. The forward price and the

ATM volatility are inputs.

The specification of β could be derived from:

1. a historical observation of the *backbone*;

2. aesthetic considerations.

In the first way, the exponent β is extracted from a log log plot of historical observation of $(f, \sigma_{ATM}$ pairs of the *backbone*. The *backbone* indicates the pattern of the changes in ATM volatility due to the changes in the forward rate.

The logarithm of the implied volatility for ATM options is

$$\log \sigma_B(f,f) = \log\alpha - (1-\beta)\log f$$
$$+\log\left[1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right)T+\ldots\right] \tag{2.14}$$

Although the term [...] is ignored in the calculation of β since its contribution is infinitesimal, the implication of this formula remains quite noisy (Hagan, 2000).

An alternative is given by the second way of determination of β *i.e.* according to some aesthetic considerations. In fact, β is arbitrary chosen before the calibration procedure for the other three parameters starts.

Recalling that β belongs to $[0,1]$, it can be set *a priori* equal to one of the numbers of the interval. The reason why that's exactly β the only one parameter fixed is amenable to the fact that α, ρ and ν can easily replicate its influence on the model. Then, it is unnecessary and time consuming to estimate also β.

The other parameters can be estimated directly solving an ordinary optimization problem. Furthermore, the values of the parameters are chosen in order to minimize the sum of squared residuals, *i.e.* the best values are those who minimize the squared error between the market volatilities and the volatilities.

$$\min_{\rho,\nu,\alpha}\sum_i(\sigma_{B_{mkt}} - \sigma_B(\nu,\alpha_0,\rho,\beta,K,f)) \tag{2.15}$$

It is also possible to directly estimate ρ and ν as before (minimizing the SSR), but considering now a different value for α: the implied by the market at-the-money volatility. In general, the goal of the calibration is to obtain an almost perfect match between the at-the-money implied volatilities produced by the calibrated model and the real ones, *i.e.* the implied volatilities quoted on the market. The closer the two volatilities are, the higher the goodness of the calibration is. Now α is the smallest positive real root that solves the following cubic polynomial

$$\frac{(1-\beta)^2 T}{24 f^{(2-2\beta)}}\alpha^3 + \frac{\rho\beta\nu T}{4 f^{(1-\beta)}}\alpha^2 + \left[1 + \frac{2-3\rho^2}{24}\nu^2 T\right]\alpha - \sigma_{ATM_{Market}} f^{(1-\beta)} = 0. \tag{2.16}$$

In this formula ρ and ν are already estimated. Specifically, first the error in function of ρ and ν, $\varepsilon_{\rho,\nu}$, is determined by keeping the pair input of the parameters. $\varepsilon_{\rho,\nu}$ explains the existing distance between the traded volatilities and the relative implied skew. The Nelder-Mead algorithm is a technique to minimize this error in a very robust way taking values for ρ between -1 and 1 included (ρ a coefficient of correlation this hypothesis is reasonable) and ν must be positive and generally a value bounded at 0.01 is sufficient (Nelder, 1965).

2.2.2 Limit of the SABR model

As every model also the SABR model has some weaknesses. The main one is that as the strike price goes near to the 0 axis the probability density function of the SABR model becomes negative. This implies that the model will fail for strikes that tend to zero.

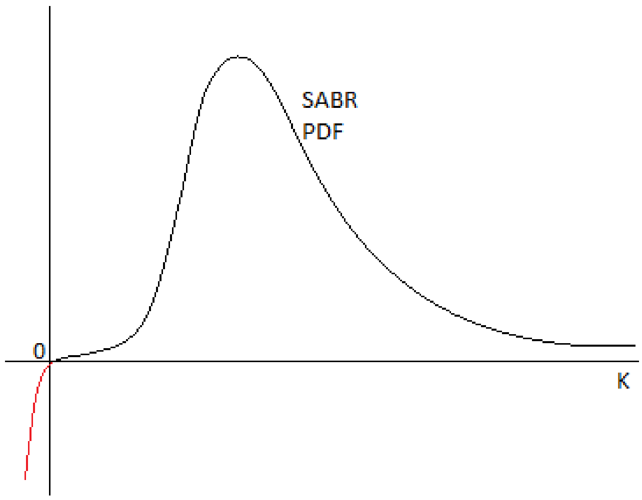


Figure 2.1: Probability density function of the SABR

2.2.3 Shifted SABR model

To implement the shifted Black pricing formula, the shifted Black volatilities are needed as inputs. However, sometimes the market shifted Black volatilities are not available for all the strike rates. For instance, there are market shifted Black volatilities just for a limited strike grid. Figure 2.2 shows an example of the existing issue related with the shifted Black model. Market gives shifted Black volatilities for some levels of strike rate.

Figure 2.2: Example of shifted Black volatilities available on the market

Since interest rate derivatives are over the counter agreements, the counter-parties can negotiate on the contractual terms and they can decide for a strike rate that does not have any market shifted Black volatility related. How can this kind of derivative be priced? If there is not a market value for the shifted Black volatility needed to implement the shifted Black pricing formula, a model that estimates it is needed. Figure 2.3 represents the case in which there are not market shifted Black volatilities for some intermediate strike rates for which we need a value in order to price derivatives with these strikes. The shifted SABR model gives the shifted Black volatility values for these intermediate strike rates allowing the pricing of the financial instruments.

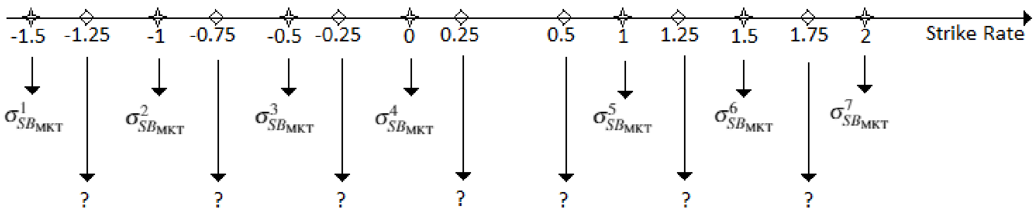


Figure 2.3: Absence of market shifted Black volatilities for some strike rates

The shifted SABR model is an extension of the stochastic $\alpha\beta\rho$ model. Also in this model the forward rate and the volatility are stochastic, while β, ρ and ν are not. A shift term Θ is added to the forward rate causing a shift also in the strike price overcoming the problem of their negativity.

$$d\hat{F} = \hat{\alpha}(\hat{F} - \Theta)^{\beta} dW, \qquad \hat{F}(0) = f, \qquad \Theta < 0 \tag{2.17}$$

the other two formulas that specify the model hold

$$d\hat{\alpha} = \nu\hat{\alpha}dZ, \qquad \hat{\alpha}(0) = \alpha \tag{2.18}$$

$$E^{Q}(dW, dZ) = \rho dW dZ. \tag{2.19}$$

Now the implied volatility to use in shifted Black model by SABR becomes

$$\sigma_{SB}^{SABR}(K-\Theta, f-\Theta) = \frac{\alpha}{((f-\Theta)(K-\Theta))^{\frac{(1-\beta}{2}}\left[1+\frac{(1-\beta)^2}{24}log^2\left(\frac{f-\Theta}{K-\Theta}\right)+\frac{(1-\beta)^4}{1920}log^4\left(\frac{f-\Theta}{K-\Theta}\right)+...\right]}\cdot\left(\frac{z}{x(z)}\right)$$

$$\cdot\left[1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{((f-\Theta)(K-\Theta))^{(1-\beta)}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{((f-\Theta)(K-\Theta))^{\frac{(1-\beta)}{2}}}+\frac{2-3\rho^2}{24}\nu^2\right]T+...\right]. \tag{2.20}$$

Here

$$z = \frac{\nu}{\alpha}((f-\Theta)(K-\Theta))^{\frac{(1-\beta)}{2}}log\left(\frac{f-\Theta}{K-\Theta}\right) \tag{2.21}$$

and $x(z)$ is

$$x(z) = log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right). \tag{2.22}$$

The ATM implied volatility is

$$\sigma_{ATM_{SB}}^{SABR} = \sigma_B(f-\Theta, f-\Theta) = \frac{\alpha}{(f-\Theta)^{(1-\beta)}}\left[1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{(2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{(f-\Theta)^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right)T+...\right]. \tag{2.23}$$

2.3 Bachelier or Normal Model

The normal model, or Bachelier model, is a model for option pricing that assumes the underlying asset to be normally distributed, where the forward rate is modeled as follow

$$dF_t = \sigma_N dH_t, \tag{2.24}$$

where H_t is the Brownian Motion and σ_N is the volatility of the forward rate. In contrast with the Black'76 model where the underlying is assumed to be log-normal distributed and then the standard deviation just represents the relative deflection, here the volatility of the model determines the absolute magnitude of the dispersion around the starting value

$$F_t = F_0 + \sigma_N H_t. \tag{2.25}$$

The pay-off at maturity T of an European interest rate call option C is given by the random variable

$$C_T = (F_T - K)^+. \tag{2.26}$$

To obtain the price today, we consider that the underlying F_T is normally distributed and then according to the no-arbitrage pricing theorem

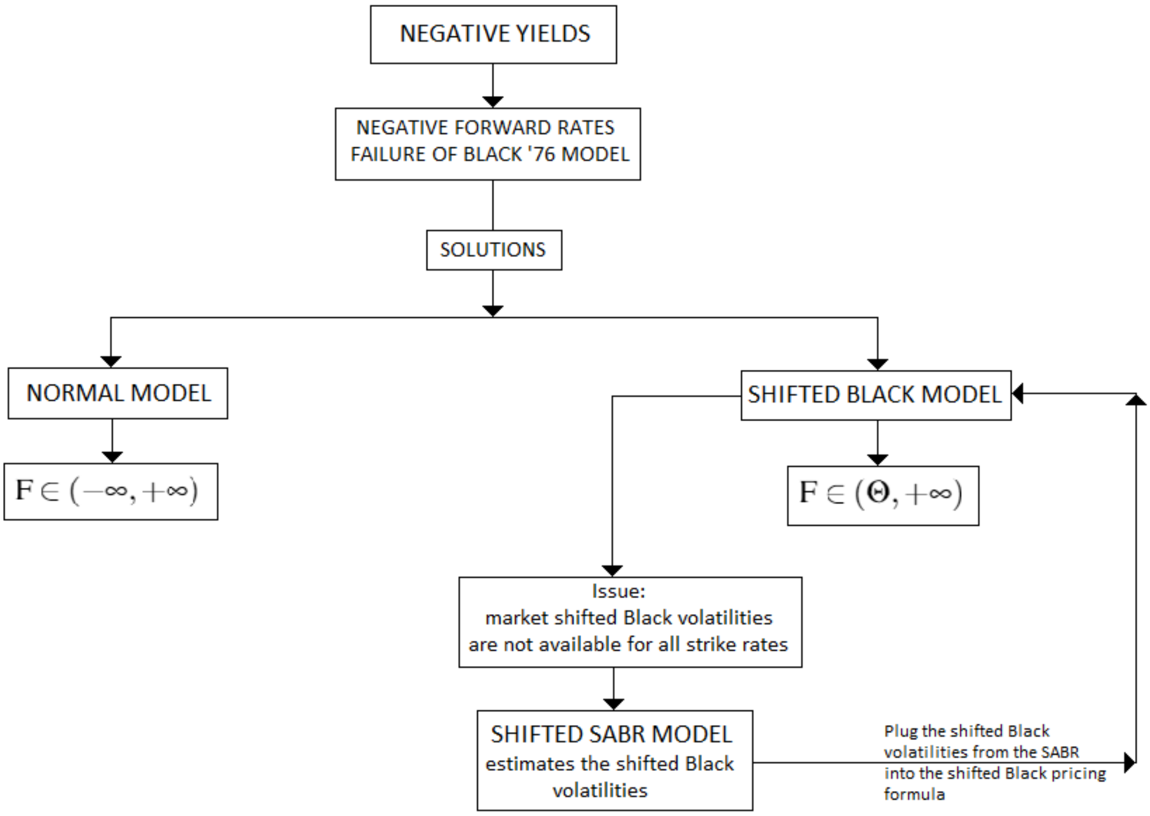
$$C_0 = E[(F_T - K)^+] = D(0,T)\left[(F_0 - K)\cdot N\left(\frac{F_0 - K}{\sigma_N\sqrt{T}} + \sigma_N\sqrt{T}\left(\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\left(\frac{F_0-K}{\sigma_N\sqrt{T}}\right)^2}{2}}\right)\right)\right] \tag{2.27}$$

where $D(0,T)$ is the discount factor. For the ATM options, where the strike price equals the underlying value in $t = 0$, the normal formula reduces to

$$C_0 = \sigma\sqrt{(\frac{T}{2\pi})}. \tag{2.28}$$

This model gives nice closed-formulas for pricing interest rate plain vanilla options and it is suitable especially in the today negative yield environment, since the forward rate (underlying) can assume all the possible values, positive or negative on the whole real line. In this scenario it then allows the underlying of IR derivatives, namely the forward rate, to be negative.

Summary



NEGATIVE YIELDS
NEGATIVE FORWARD RATES
FAILURE OF BLACK '76 MODEL
SOLUTIONS
NORMAL MODEL
$F \in (-\infty, +\infty)$
SHIFTED BLACK MODEL
$F \in (\Theta, +\infty)$
Issue:
market shifted Black volatilities
are not available for all strike rates
SHIFTED SABR MODEL
estimates the shifted Black
volatilities
Plug the shifted Black
volatilities from the SABR
into the shifted Black pricing
formula

Chapter 3

Hedging

A good hedging strategy is possible if the sensibilities one wants to hedge are truthful. This means that if deltas or vegas calculated for a single interest rate derivative or for a portfolio of these financial instruments are wrong, then also the corresponding hedging strategies are erroneous. Indeed, an analysis on the delta and vega greeks is made in order to check, later on with the experiments, if a specific model is able to capture the right sensibility of an interest rate derivative to changes in the forward rate or in the volatility. If this is not possible, even if one model gives good estimates of IR derivative premiums, it may not be the best one. As will be discovered, shifted models distort the true greeks because of the shift parameter.

Hence, this chapter goes further the pricing problem where the main purpose is to get premium estimates near to the market ones. It provides the basic concepts to test the power of a model from a different point of view: the model ability to correctly understand and replicate the reactions of interest rate derivative value to changes in some economical variables. The aim of this chapter is then to provide a further measure of comparison between pricing models.

3.1 The Greek letters

3.1.1 Delta

Delta, Δ, of an option measures how large is the change in value of the option due to changes of the underlying asset. Hence, it is the derivative of the option value with respect to the price of the underlying asset

$$\Delta = \frac{\partial V}{\partial F} \tag{3.1}$$

where V is the value of the option.

In other words, the delta of an option relates the option itself with F playing the role of function slope in the space R^2 where V is in the y-axis meanwhile F is in the x-axis. Only if the the slope of this curve is zero a position can be said to be *delta neutral*. In fact, a 0 slope means that even if the price of the underlying asset changes it does not affect the value of the derivative contract (this concept hold also for the next types of risk). In contrast, if the delta is for example 0.4 this means that a small change of the underlying price causes a variation of about 40% of the value of the option. In general, an agent can neutralize the risks to changes in F by buying or selling specific amount of the underlying (Hagan 2000). In fact with the right portfolio's composition the delta of the stock position and the one of the option position offset each others.

Since the forward rate varies frequently over time, the delta hedging should be re-balanced each short arbitrary interval of time in order to have an accurate and efficient hedging that persists in time. The *dynamic delta hedging* is a procedure in which the delta risk is adjusted on a regular basis. Meanwhile in the *static hedge* the delta risk is neutralized just once and then never adjusted (Hull, 2009).

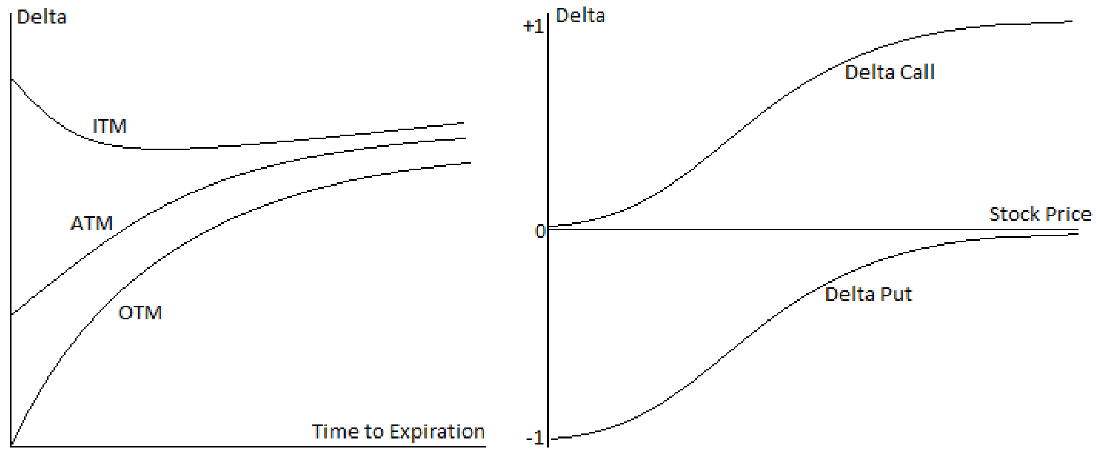


Figure 3.1: Delta in relation with time to expiration (left side) and the stock price(=forward rate, in our case) (right side)

Portfolio's delta The delta of a portfolio is calculated as the weighted sum of each delta associated to each option held

$$\Delta_P = \sum_{i=1}^{n} w_i \Delta_i, \quad i = 1...n. \tag{3.2}$$

Setting this sum equal to zero guarantees the derivation of the amount of underlying asset necessary to hedge the portfolio.

3.1.2 Vega

Vega risk measures the option's sensitivity to changes in the volatility of the underlying asset. Higher volatility means higher price for options because the probability that the options can move into favor in a smaller period of time and specially before the maturity of the option increase. Therefore, Λ represents the rate of change of the value of the option due to movements in the volatility $\Lambda = \frac{\partial V}{\partial \sigma}$.

The vega of a portfolio of derivatives is

$$\Lambda_p = \frac{\partial \Pi}{\partial \sigma} = \sum_{i=1}^{n} w_i \Lambda_i, \quad i = 1, \cdots n \tag{3.3}$$

where Π is the value of the portfolio.

Vega only affects the temporal value of the option premium and it is not constant. Vega thus changes as the maturity of the option approaches. Generally, the options with their non-linear payoffs tend to amplify the effects of an increase in volatility. Although this is a common phenomenon among options, the at-the-money and out-of-the-money ones are more sensitive to small changes in the implied volatility of the underlying. Therefore, the vega in these particular options plays a more important role. This is reasonable since vega affects just the portion of price dictated by the time value and ATM options are the ones with higher time value part. ITM options also have intrinsic value and then volatility changes affect them in a lower degree and with lower magnitude than it does for OTM and ATM options. In the figure below it is clear the behavior of Λ in relation of the volatility (left part) and stock price(=forward rate in our case) (right part).

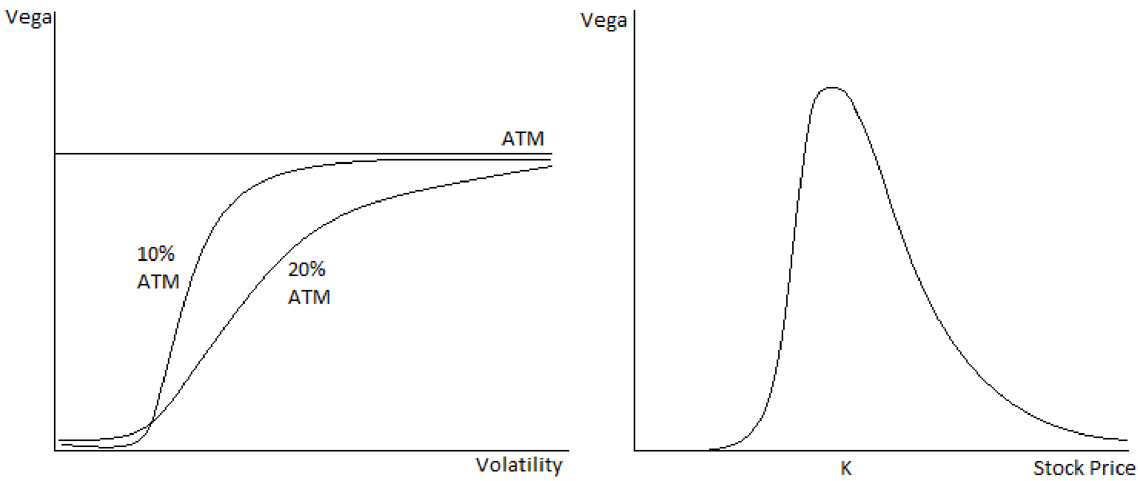


Figure 3.2: Λ as a function of the volatility σ, on the left, and S, on the right
The left part shows that more away-from-at-the-money options have a lower reaction to increases in volatility. Vega reaches more quickly the limit (highest vega is for ATM options) if the option is more near at ATM strikes instead of other. The right side shows that ITM options are less sensitive to σ changes.

Vega is function of the option position held. Namely, the vega of a long position (short position) is always positive (negative) and under the same conditions it assumes the same values both for the call options and the put options, it increases (decreases) as the underlying price volatility increases (decreases).

3.2 Hedging under the shifted SABR model

The shifted SABR model, equations 2.17-2.19, chooses $C(t,\star) = \hat{\alpha} F \hat{-} \Theta^{\beta}$ (see Appendix C.). A stochastic volatility $\hat{\alpha}$ function of time is considered.

The IV of the SABR model can be rewritten as

$$\sigma_{SB}(K-\Theta, f-\Theta) = \frac{\alpha}{(f-\Theta)^{(1-\beta)}} \left\{ \ 1 - \tfrac{1}{2}(1-\beta-\rho\lambda)\log\left(\tfrac{f-\Theta}{K-\Theta}\right) + \tfrac{1}{12}[(1-\beta^2)+(2-3\rho^2)\lambda^2]\log^2\left(\tfrac{f-\Theta}{K-\Theta}\right) + \cdots \right. \tag{3.4}$$

where

$$\lambda = \frac{\nu}{\alpha}(f-\Theta)^{(1-\beta)} \tag{3.5}$$

relates the volatility of volatility with the local volatility $\frac{\alpha}{(f-\Theta)^{(1-\beta)}}$.

This formulation makes more evident the characteristics and the behavior of $\sigma_B(K-\Theta, f-\Theta)$ that was quite hidden in the too abstruse original version, equations 2.20-2.22. Analyzing all the terms of $\sigma_B(K-\Theta, f-\Theta)$ arises that:

1. $\frac{\alpha}{(f-\Theta)^{(1-\beta)}}$ is the IV for ATM options, where β determines the slope of the backbone. If $\beta = 0$ the backbone is downward sloping while if $\beta = 1$ the backbone is flat;
2. $-\frac{1}{2}(1-\beta-\rho\lambda)\log\left(\frac{f-\Theta}{K-\Theta}\right)$ represents the slope of the IV with respect to K, where
   (a) $-\frac{1}{2}(1-\beta)\log\left(\frac{f-\Theta}{K-\Theta}\right)$ is the beta skew;
   (b) $\frac{1}{2}\rho\lambda\log\left(\frac{f-\Theta}{K-\Theta}\right)$ is the vanna skew.

The SABR model accurately fits the IV curves of the market for each exercise date and, moreover, it predicts the dynamics of the IV curves producing the usual dynamics of skews and smiles. The consequence is that the calibration of ρ and ν are stable and there is no need to recalibrate them frequently. Conversely, α should be adjust every few hours.

The SABR models permit precise hedges because the risk calculated for one strike is consistent with the others (Hagan 2000). The SABR value for a call option is given by the shifted Black's formulas 2.4-2.5 plugging σ_{SB}^{SS} the one resulting from the SABR model as shifted Black volatility. Hence

$$V_{call}^{SABR} = C^B(f, K, \sigma_{SB}^{SS}, T) \tag{3.6}$$

Differentiating the value of the call with respect to the volatility α, vega Λ risk is obtained.

$$\Lambda = \frac{\partial V_{call}^{SABR}}{\partial \alpha} \tag{3.7}$$

Vega risk is calculated by applying a unit variation to the volatility value letting f fixed

$$\begin{aligned} f &\to f, \\ \alpha &\to \alpha + \delta\alpha. \end{aligned} \tag{3.8}$$

Delta risk is calculate in the opposite way of the vega risk, namely

$$\begin{aligned} f &\to f + \delta f, \\ \alpha &\to \alpha \end{aligned} \tag{3.9}$$

and the change in the option value is $\left(\frac{\partial C_{Shift}^B}{\partial f} + \frac{\partial C_{Shift}^B}{\partial \sigma_{SB}}\frac{\partial \sigma_{SB}}{\partial f}\right) \cdot \delta f$, then

$$\Delta = \frac{\partial V_{call}^{SABR}}{\partial f} = \overbrace{\frac{\partial C_{Shift}^B}{\partial f}}^{\text{Black delta}} + \underbrace{\frac{\partial C_{Shift}^B}{\partial \sigma_{SB}}\frac{\partial \sigma_{SB}}{\partial f}}_{\text{systematic change in IV as f changes}} . \tag{3.10}$$

3.2.1 Delta and vega risk adjustments

Because of the existing correlation in the SABR model between the forward rate f and the α volatility (equation 2.19), there is an ulterior influence that can not be ignored in their changes. In fact, if the underlying changes also the volatility will change at least on average (Bartlett, 2006).

$$\begin{aligned} f &\to f + \delta f, \\ \alpha &\to \alpha + \delta_f \alpha \end{aligned} \tag{3.11}$$

We need to define δ_f, that's the change in α due to changes in f. Expressing the SABR model with independent Brownian motions as

$$d\hat{F}_t = \hat{\alpha}_t \hat{F}^\beta dW_t \tag{3.12}$$

$$d\hat{\alpha}_t = \nu\alpha_t(\rho dW_t + \sqrt{1-\rho^2}dZ_t) \tag{3.13}$$

thus the value of α over time can be decomposed as

$$d\hat{\alpha}_t = \frac{\rho\nu}{\hat{F}_t^{\beta}} d\hat{F}_t + \nu\hat{\alpha}_t\sqrt{1-\rho^2}dZ_t \tag{3.14}$$

where the first value after the equality on the right represents the change in α due to the changes in f and the second term shows the idiosyncratic risk of α. Then

$$\delta_f \alpha = \frac{\rho\nu}{\hat{F}^{\beta}}\delta f. \tag{3.15}$$

The new delta risk is

$$\Delta = \frac{\partial C^B}{\partial f} + \frac{\partial C^B}{\partial \sigma_B}\left(\frac{\partial \sigma_B}{\partial f} + \frac{\partial \sigma_B}{\partial \alpha}\frac{\rho\nu}{f^{\beta}}\right). \tag{3.16}$$

The addition of the the new term

$$\overbrace{\frac{\partial C^B}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial \alpha}}^{\text{Classic vega risk}} \frac{\rho\nu}{f^{\beta}} \tag{3.17}$$

guarantees a more efficient delta hedging since now delta is less sensitive to β and more sensitive to the slope of implied volatility curve.

The new change in the value of the option is $\Delta V = \left[\frac{\partial C^B}{\partial f} + \frac{C^B}{\partial \sigma_B}\left(\frac{\partial \sigma_B}{\partial f} + \frac{\partial \sigma_B}{\partial \alpha}\frac{\rho\nu}{f^{\beta}}\right)\right]\delta f$.

Following a similar patter as before, the calculation for the vega risk becomes

$$\begin{aligned} f &\to f + \delta_\alpha f, \\ \alpha &\to \alpha + \delta\alpha \end{aligned} \tag{3.18}$$

where $\delta_\alpha f = \frac{\rho f^{\beta}}{\nu}\delta\alpha$.

Hence, the new vega risk is

$$\Lambda = \frac{\partial C^B}{\partial \sigma_B}\left(\frac{\partial \sigma_B}{\partial \alpha} + \frac{\partial \sigma_B}{\partial f}\frac{\rho f^{\beta}}{\nu}\right) \tag{3.19}$$

where the change in the option value is $\frac{\partial C^B}{\partial \sigma_B}\left(\frac{\partial \sigma_B}{\partial \alpha} + \frac{\partial \sigma_B}{\partial f}\frac{\rho f^{\beta}}{\nu}\right)\delta\alpha$.

Chapter 4

Test and Experiments

4.1 Data

The data used in this thesis are all from Bloomberg: EUR call swaptions, CHF call swaptions, SEK call swaptions with different tenors and terms and CHF cap options. Concerning the notation EUR 1M-1Y characterizes a one-month option to enter in a one-year swap, namely a swaption with term one-month and tenor one-year. EUR stays for differentiate the derivatives from, for instance, CHF (Swiss swaptions) and SEK (Swedish swaptions).

Normal model and shifted Black model A first procedure adopted to price swaptions with negative strikes (EUR, SEK, CHF) is to apply the Bachelier formula for pricing options (equation 2.27) This model allows the underlying (in case of IR derivatives it is the forward rate) to be negative. In table 4.1, the normal formula is applied for EUR swaptions at-the-money with different terms and tenors one-year and two-years. An ATM IR derivative has current swap rate F equals to the strike price K. In fact, in the table all F are equal to their respective K. I selected for the analysis derivatives focusing on the ones with negative strikes. The current swap rates, strikes and normal volatilities are from Bloomberg and the market prices and the prices computed by the Bachelier formula are on a notional amount of one and without the adjustments for the number of payment per-year. The payment per-year are semi-annual payments, this means that each price should be multiplied for 2 or 4 according to a tenor of one-year or two-years respectively. I applied the normal formula

$$C = D(0,T)[(F-K)\cdot N\left(\frac{F-K}{\sigma_N}\sqrt{T}+\sigma_N\sqrt{T}N'(d)\right)], \tag{4.1}$$

where

$$d = \frac{F-K}{\sigma_N\sqrt{T}}, \tag{4.2}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \tag{4.3}$$

and $D(0,T)$ is the discount factor. The prices I obtained are reported on the table below and they are close to the market ones. There is an increase of error as we go far on time. The higher the term and tenor are, the higher the gap between the prices estimated by the model and the observed ones. The last column of the table indicates the error term increases in percentage terms between each difference of prices with the first one, to capture how in time the prices obtained from the normal model deviates from the market ones. Just for this time, I included also two swaptions with positive strikes EUR 9M-2Y and EUR 1Y-2Y.

Table 4.1: EUR Swapion Prices under the normal model

	F ATM	K ATM	σ_N	Mkt Premium	Premium Normal Model	Difference	Increment of error
EUR 1M-1Y	-0.2137%	-0.2137%	14.61%	0.01703	0.0168	0.00023	0%
EUR 3M-1Y	-0.1883%	-0.1883%	16.57%	0.03344	0.0331	0.00034	0.011%
EUR 6M-1Y	-0.1391%	-0.1391%	19.45%	0.05525	0.0549	0.00035	0.012%
EUR 9M-1Y	-0.0928%	-0.0928%	22.90%	0.07965	0.0791	0.00055	0.032%
EUR 1Y-1Y	-0.0404%	-0.0404%	26.48%	0.10605	0.1056	0.00045	0.022%
EUR 1M-2Y	-0.1165%	-0.1165%	17.67%	0.04128	0.0407	0.00058	0%
EUR 2M-2Y	-0.083%	-0.083%	23.01%	0.09285	0.0918	0.001	0.042%
EUR 6M-2Y	-0.025%	-0.025%	25.67%	0.14566	0.1448	0.00086	0.0854%
EUR 9M-2Y	0.0341%	0.0341%	28.34%	0.19741	0.1958	0.0016	0.102%
EUR 1Y-2Y	0.0971%	0.0971%	33.03%	0.26482	0.2635	0.0013	0.072%

F ATM is the forward rate at-the-money, K ATM is the strike rate at-the-money, σ_N is the normal volatility. "Difference" is (Mkt premium - Premium Normal Model). Increment of error is made as $(\text{Difference}_i - \text{Difference}_0) \cdot 100$, where $i = 1, 2, \cdots 5$ for 1Y tenor European swaptions and $i = 1, 2, \cdots 5$ for 2Y tenor European swaptions. *i.e* 0.0854%, 8th row and last column, is $(\text{Difference}_{\text{EUR 6M-2Y}} - \text{Difference}_{\text{EUR 1M-2Y}}) \cdot 100$.

Table 4.2 shows the price estimates under the normal model for CHF swaptions with different terms and constant tenor equals to one-year. Here, again, the notional is one and there are not adjustments for the payments per-year. Also in this case, it is visible how the prices from the Bachelier model become less accurate as the term increase.

Table 4.2: CHF Swaption Prices under the Normal model

	F ATM	K ATM	σ_N	Mkt Premium	Premium Normal Model	Difference	Increment of error
CHF 1M-1Y	-0.6604%	-0.6604%	51.59%	0.06041	0.0594	0.001	0%
CHF 3M-1Y	-0.6506%	-0.6506%	49.78%	0.10108	0.0993	0.0018	0.08%
CHF 6M-1Y	-0.6138%	-0.6138%	45.75%	0.13084	0.1291	0.0017	0.07%
CHF 9M-1Y	-0.5754%	-0.5754%	43.71%	0.15330	0.1510	0.0023	0.13%
CHF 1Y-1Y	-0.5272%	-0.5272%	41.85%	0.16920	0.1670	0.0022	0.12%
CHF 2Y-1Y	-0.3255%	-0.3255%	52.26%	0.29939	0.2948	0.0046	0.36%
CHF 3Y-1Y	-0.0574%	-0.0574%	62.81%	0.44355	0.4340	0.0096	0.86%

F ATM is the forward rate at-the-money, K ATM is the strike rate at-the-money, σ_N is the normal volatility. "Difference" is (Mkt premium - Premium Normal Model). Increment of error is made as $(\text{Difference}_i - \text{Difference}_0) \cdot 100$, where $i = 1, 2, \cdots 7$ for 1Y tenor Swiss swaptions.
i.e 0.036%, 6th row and last column, is $(\text{Difference}_{\text{CHF 2Y-1Y}} - \text{Difference}_{\text{CHF 1M-1Y}}) \cdot 100$.

Now let's consider another setting to price derivatives with negative strikes: the shifted Black model. Now we go back to BS model assumptions but assume the forward rate as underlying and the addition of a shift. The shift values are from Bloomberg and they are equal to -3% for EUR and SEK options, meanwhile for the CHF -2%. In table 4.3 the ATM swaptions are on a notional amount of 100 considering the payments per-year. Also in this example, the more accurate prices are the ones with lower terms and tenors.

In table 4.4 and 4.5, I compared the prices from the normal model and the ones from the shifted Black model with the ones of the market to better see which of the two models is the best one. It appears to be a non immediately response about which of the two models is the most accurate. For instance, in table 4.4 the shifted Black model 'wins' three times out of five, meanwhile in the case of CHF swaptions just once.

To better understand the existing relation between the models, let's see how the volatility of shifted Black varies in function of the normal volatility (figure 4.1). Consider that with the shifted Black formula we get a price P

$$D(0,T)\left[[(F-\Theta)N\left(\frac{\log\left(\frac{(F-\Theta)}{(K-\Theta)}\right)+\left(\frac{\sigma_{SB}^2}{2}\right)T}{\sigma_{SB}\sqrt{T}}\right)-(K-\Theta)N\left(\frac{\log\left(\frac{(F-\Theta)}{(K-\Theta)}\right)-\left(\frac{\sigma_{SB}^2}{2}\right)T}{\sigma_{SB}\sqrt{T}}\right)]\right]=P. \tag{4.4}$$

If we want to obtain the same price with the normal model we need to find the values for the volatility that make this equation satisfied

$$D(0,T)\left[(F-K)\cdot N\left(\frac{F-K}{\sigma_N\sqrt{T}}+\sigma_N\sqrt{T}\left(\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\left(\frac{F-K}{\sigma_N\sqrt{T}}\right)^2}{2}}\right)\right)\right]=P. \tag{4.5}$$

We now equate the two formulas above and we let as unknown σ_N, using fsolve in MATLAB we get the results. From figure 4.1 it is clear that the normal volatility is an increasing function of the shifted Black one and it moves up and it has a more pronounced slope for bigger shift Θ in magnitude (remember that the shift is consider negative but then in the formula it is subtracted).

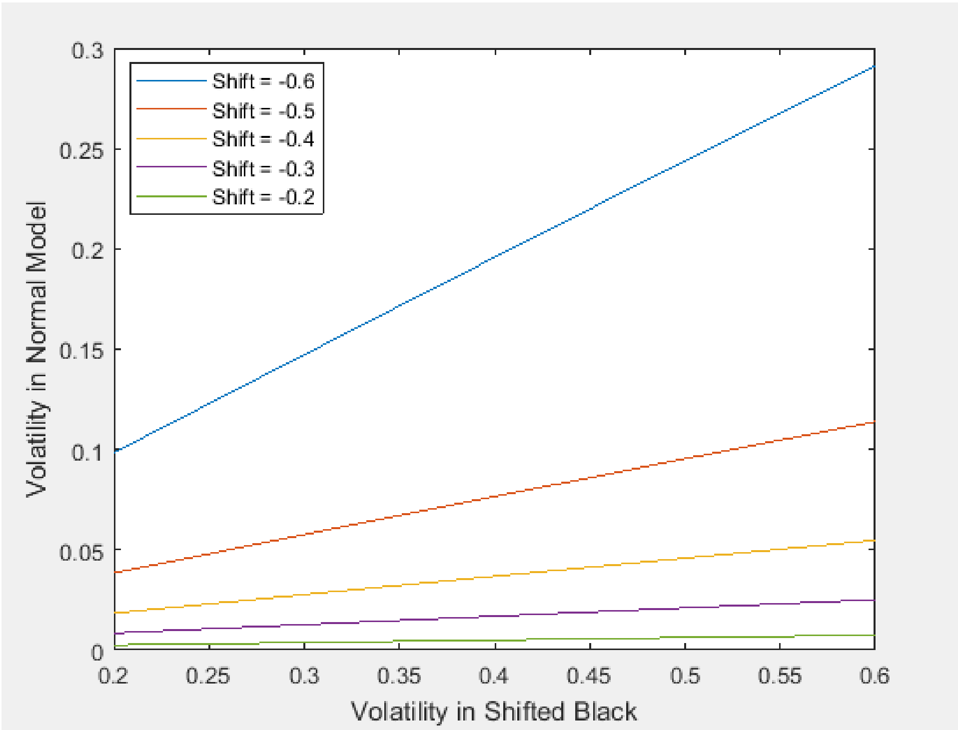


Figure 4.1: Shifted Black volatility as function of Normal Volatility

Table 4.3: Shifted Black Premiums

	F ATM	K ATM	Θ	σ_{SB}	Mkt Premium	Premium Shifted Black Model	Difference
EUR 1M-1Y	-0.2965%	-0.2965%	-3%	5.65%	3.50	3.4871	0.0129
EUR 2M-1Y	-0.2857%	-0.2857%	-3%	5.85	5.00	5.1279	-0.1279
EUR 3M-1Y	-0.2739%	-0.2739%	-3%	5.7%	6.50	6.1989	0.3011
EUR 6M-1Y	-0.2326%	-0.2326%	-3%	7.1%	11.00	11.0843	-0.0843
EUR 9M-1Y	-0.1848%	-0.1848%	-3%	8.2%	16.00	15.9479	0.0521
EUR 1Y-1Y	-0.1292%	-0.1292%	-3%	9.19%	21.00	21.0429	-0.0429
SEK 1M-1Y	-0.4367%	-0.4367%	-3%	7.65%	4.50	4.5165	-0.0165
SEK 3M-1Y	-0.399%	-0.399%	-3%	7.5%	8.00	7.7819	0.2181
SEK 6M-1Y	-0.3122%	-0.3122%	-3%	10.7%	16.50	16.2219	0.2781
SEK 1Y-1Y	-0.0932%	-0.000932	-3%	12.9%	30.50	29.8981	0.6019
CHF 1M-1Y	-0.6575%	-0.6575%	-2%	42.72%	13.00	13.1952	-0.1952
CHF 3M-1Y	-0.6506%	-0.6506%	-2%	38%	20.00	20.4259	-0.4259
CHF 6M-1Y	-0.6008%	-0.6008%	-2%	34.8%	26.00	27.4025	-1.4025
CHF 1Y-1Y	-0.5273%	-0.5273%	-2%	31.3%	34.00	36.6293	-2.6293
CHF 2Y-1Y	-0.3252%	-0.3252%	-2%	34.1%	60.00	63.8235	-3.8235

F ATM is the forward rate at-the-money, K ATM is the strike rate at-the-money, Θ is the shift parameter, σ_{SB} is the shifted black volatility. ”Difference” is (Mkt premium - Premium Normal Model).

Table 4.4: Normal Vs. Shifted Black (EUR)

	EUR 1M-1Y	EUR 3M-1Y	EUR 6M-1Y	EUR 9M-1Y	EUR 1Y-1Y
σ_N	14.61%	16.57%	19.45%	22.90%	26.48%
Θ Shift for Shifted Black	-3%	-3%	-3%	-3%	-3%
σ_{SB}	5.65%	5.7%	7.1%	8.2%	9.19%
Mkt Premium	3.50	6.50	11.00	16.00	21.00
Premium Normal Model	3.3651	6.6105	10.9735	15.8236	21.1280
Premium Shifted Black Model	3.4871	6.1989	11.0843	15.9479	21.0429
Abs. diff. Mkt and Normal	0.1349	0.1105	0.0265	0.1764	0.1280
Abs. diff.Mkt and Shifted Black	0.0129	0.3011	0.0843	0.0521	0.0429
Best fit	Shifted Black	Normal	Normal	Shifted Black	Shifted Black

σ_N is the normal volatility, Θ is the shift parameter, σ_{SB} s the shifted Black volatility and "Abs." stays for absolute.

Table 4.5: Normal Vs. Shifted Black (CHF)

	CHF 1M-1Y	CHF 3M-1Y	CHF 6M-1Y	CHF 1Y-1Y	CHF 2Y-1Y
σ_N	51.59%	49.78%	45.78%	41.85%	52.26%
Θ Shift for Shifted Black	-2%	-2%	-2%	-2%	-2%
σ_{SB}	42.72%	38%	34.48%	31.3%	34.1%
Mkt Premium	13.00	20.00	26.00	34.00	60.00
Premium Normal Model	11.88	19.86	25.82	33.40	58.96
Premium Shifted Black Model	13.20	20.43	27.39	36.73	63.81
Abs. diff. Mkt and Normal	1.12	0.14	0.18	0.60	1.04
Abs. diff. Mkt andShifted Black	0.20	0.43	1.39	2.73	3.81
Best fit	Shifted Black	Normal	Normal	Normal	Normal

σ_N is the normal volatility, Θ is the shift parameter, σ_{SB} s the shifted Black volatility and "Abs." stays for absolute.

Following the same idea used for the relation between normal and shifted Black volatilities, we now analyze how the shifted Black volatility must vary according to the shift to obtain the right price. We equate two SB formulas that give the same premium letting as unknown σ_{SB_2}, all the other variables are input and then known and they are the same between the two formulas

$$D(0,T)\left[(F-\Theta_1)N\left(\frac{\log\left(\frac{(F-\Theta_1)}{(K-\Theta_1)}\right)+\left(\frac{\sigma_{SB_1}^2}{2}\right)T}{\sigma_{SB_1}\sqrt{T}}\right)-(K-\Theta_1)N\left(\frac{\log\left(\frac{(F-\Theta_1)}{(K-\Theta_1)}\right)-\left(\frac{\sigma_{SB_1}^2}{2}\right)T}{\sigma_{SB_1}\sqrt{T}}\right)\right]=$$
$$D(0,T)\left[(F-\Theta_2)N\left(\frac{\log\left(\frac{(F-\Theta_2)}{(K-\Theta_2)}\right)+\left(\frac{\sigma_{SB_2}^2}{2}\right)T}{\sigma_{SB_2}\sqrt{T}}\right)-(K-\Theta_2)N\left(\frac{\log\left(\frac{(F-\Theta_2)}{(K-\Theta_2)}\right)-\left(\frac{\sigma_{SB_2}^2}{2}\right)T}{\sigma_{SB_2}\sqrt{T}}\right)\right]. \tag{4.6}$$

Figure 4.2 shows the behavior of σ_{SB_2} with respect to Θ for different level of initial σ_{BS_1}. Generally, the shifted Black volatility increases as the magnitude of the shift decreases. As the shift reaches the boundary of the forward rate and strike (recall: $F-\Theta$ and $K-\Theta$ must be positive) the shifted Black volatility reacts differently from the increasing trend in figure 4.2 showing some anomalies. If we overpass the limit that guarantees the sum of the forward rate with the shift and the strike with the shift is positive (*i.e.* $F=-0.02$ and $K=-0.02$ if we set $\Theta=-0.01$, then we have $F-\Theta=-0.02+0.01=-0.1$ and $K-\Theta=-0.1$) we get completely wrong results for the volatility since in this way we don't solve the problem of negative strikes and forward rate. Indeed, we can not have a direct connection between the shifted Black model and the Black model, that's a special case of the shifted Black where the shift is equal to 0. To give credibility to my approach is the fact that in the market there are not quotation of swaptions with negative strikes in terms of Black volatilities, meaning that there exist no a Black volatility for those derivatives. Another point to highlight is that, if we set $\Theta_2=\Theta_1$, $\sigma_{SB_2}=\sigma_{SB_1}$ since there are no more differences between the two formulas.

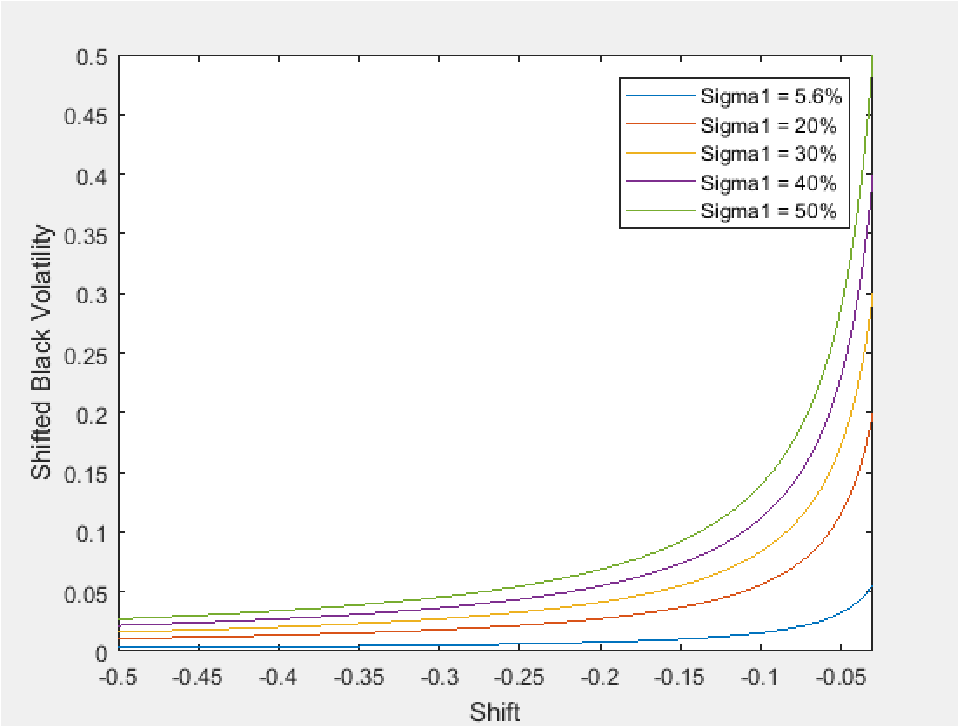


Figure 4.2: Shifted Black volatility as function of the shift Θ

Table 4.6 summarizes the results of figure 4.2 taking as example the EUR 1M-1Y swaption with market price 3.50. If we plug into the shifted Black pricing formula the official shift value and relative shifted Black volatilities we collected from Bloomberg we get an estimated price of about 3.4871. Now we want to see which are the right volatilities to use to obtain the same price changing the values of Θ, this is done because the shift value are arbitrary and they well work (they reproduce a price close to the market one) if and only if they are associated to the right level of volatility. In table 4.6 are then available all the shifted Black volatilities over a range of -50% shift and -1% shift that combines together give a premium of 3.4871. The bold row indicates the results applying the market shift of -3% and the market σ_{SB} of 5.6%. For bigger absolute values of the shift the volatility decreases consistently. This is because it must compensate the big level of arbitrary shift chosen that heavily distorts the pricing formula.

Table 4.6: EUR 1M-1Y with different shifts and adjustments for the volatilities

Mkt Premium	Premium SB	Θ	σ_{SB}
3.5000	3.4871	-50%	0.3046%
3.5000	3.4871	-40%	0.3813%
3.5000	3.4871	-30%	0.5097%
3.5000	3.4871	-20%	0.7684%
3.5000	3.4871	-10%	1.5602%
3.5000	3.4871	-5%	3.2188%
3.5000	3.4871	-4%	4.0879%
3.5000	**3.4871**	**-3%**	**5.6000%**
3.5000	3.4871	-2%	8.8875%
3.5000	3.4871	-1%	21.5236%

Premium SB is the premium estimate under the shifted Black model, Θ is the shift parameter, σ_{SB} is the shifted Black volatility.

A graphical view of how the premium changes to changes in the shift, letting all the other parameters fixed is given in figure 4.3.

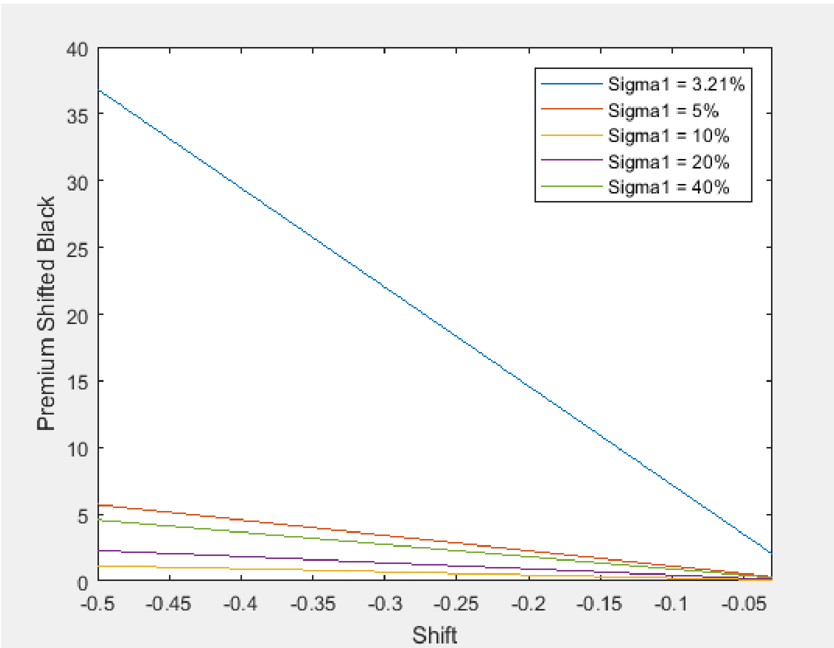


Figure 4.3: Premium as a function of the shift

If we change the shift without any adjustments to the level of volatility the prices from the shifted Black become heavily erroneous. In table 4.7 it is shown how wrong we can go without the right adjustments in σ_{SB}. In fact, here σ_{SB} is assumed to remain constant over the different Θ-s.

Table 4.7: EUR 1M-1Y with different shifts **without** adjustments for the volatilities

Mkt premium	Premium SB	Θ	σ_{SB}
3.50	64.11	-50%	5.60%
3.50	51.21	-40%	5.60%
3.50	38.31	-30%	5.60%
3.50	25.41	-20%	5.60%
3.50	12.52	-10%	5.60%
3.50	6.07	-5%	5.60%
3.50	4.78	-4%	5.60%
3.50	3.49	-3%	5.60%
3.50	2.20	-2%	5.60%
3.50	0.91	-1%	5.60%

Premium SB is the premium estimate under the shifted Black model, Θ is the shift parameter, σ_{SB} is the shifted Black volatility.

The only right price is the one with the right relation between the shift -3% and its volatility 5.6%.

Let's now pass to another analysis. We before focused our attention on ATM swaptions. Now we focus on one swaption with different strike prices (ATM-10, ATM-9,...,ATM, ATM+1,...,ATM+10). The swaption chosen is the CHF 1M-1Y and the pricing formula used is from the normal model. All the market premiums, K-s and F-s are again from Bloomberg.

Now we want to see the strength of the model across strikes. ATM-10 means that we start from the ATM value of the swaption and we go 10 basis point below. ATM+10 means that we add 10 basis point to the ATM value. The forward swap rate at-the-money for the CHF 1M-1Y swaption is equal to 0.0049% and, of course, K is equal to 0.0049% as well. But, if we move one basis point below the ATM we see that F and K change. In fact, the right value for F is given by the sum between the ATM-1 market value of F and the deviation from the ATM: $F_{ATM-1} = F_{Mkt\,ATM-1} + \text{Change in bp from ATM} = -0.0050 + (-0.01) = -0.0051$, see 10^{th} row in table 4.7 (this calculation is in percentage term where then -0.01 is 1 bp, because 1 bp is 0.01%).

Table 4.8: Normal premiums for CHF 1M-3Y with different strikes

	Mkt Premium	Normal Premium	K	F	σ_N
ATM-10	13.48	13.46	-0.0059	-0.1059	51.74
ATM-9	15.05	15.03	-0.0058	-0.0958	51.74
ATM-8	16.76	16.69	-0.0057	-0.0857	51.74
ATM-7	18.61	18.52	-0.0056	-0.0756	51.74
ATM-6	20.60	20.50	-0.0055	-0.0655	51.74
ATM-5	22.74	22.67	-0.0554	-0.0554	51.74
ATM-4	25.03	25.02	-0.0053	-0.0453	51.74
ATM-3	27.47	27.45	-0.0052	-0.0352	51.74
ATM-2	30.07	30.04	-0.0051	-0.0251	51.74
ATM-1	32.83	33.53	-0.0050	-0.0150	51.74
ATM	**35.75**	**36.48**	**-0.0049**	**-0.0049**	**51.74**
ATM+1	38.83	39.36	-0.0048	0.0052	51.74
ATM+2	42.09	42.64	-0.0047	0.0153	51.76
ATM+3	45.54	46.12	-0.0046	0.0254	51.85
ATM+4	49.17	49.76	-0.0045	0.0355	51.95
ATM+5	52.93	53.55	-0.0044	0.0456	52.04
ATM+6	56.85	57.49	-0.0043	0.0557	52.13
ATM+7	60.91	66.16	-0.0042	0.0658	52.22
ATM+8	65.10	65.80	-0.0041	0.0759	52.31
ATM+9	69.44	70.39	-0.0040	0.0860	52.41
ATM+10	73.90	74.75	-0.0039	0.0961	52.50

CHF 1M-3Y is a Swiss swaption with term 1 month and tenor 3 years. K is the strike rate of the swpation, F is the forward rate, σ_N is the normal volatility. F, K and σ_N are in percentage terms. ATM-10 means at-the-money minus 10 basis points valuation, ATM-9 means at-the-money minus 9 basis points valuation and so on. ATM means an at-the-money valuation. ATM+1 is an at-the-money plus 1 basis point valuation and so on.

Obviously, the premium for the OTM options are lower than the ones for the ITM options. The premiums from the normal model are close to the market ones. Indeed we can say that the normal model gives a consistent estimate of prices across strikes. Another aspect to analyze concerns the volatility: it is constant for all ATM- and equals to the at-the-money one. Meanwhile it changes a little bit for the ATM+ (this causes a vega almost equals to 0).

Deltas of ATM options under the normal model must be equal to 0.5 since

$$\Delta = N\left(\frac{F-K}{\sigma_N\sqrt{T}}\right), \tag{4.7}$$

for $F = K$ we have $\Delta = N(0) = 0.5$. This does not hold for ATM options under the shifted Black framework since the additional parameter Θ causes deviation of delta from its true value. Let's see how Δ_{SB} deviate from 0.5 in table 4.9. This table also presents two delta-hedging strategies. One with respect to the deltas normal and one with respect to the deltas shifted Black. Setting an arbitrary value of the portfolio to 1000, we want to discover the optimal composition that let the portfolio value equal 1000 and the portfolio delta equal to 0. The solution for the normal model is to go long on CHF 2Y-1Y of 18 and short on EUR 1M-1Y of 18, meanwhile for the shifted black it is to buy 16 of CHF 2Y-1Y and sell $\approx$ 20 of EUR 1M-1Y. If we consider the deltas of the shifted Black as fictitious and we ignore them, because we already know that the true deltas for at-the-money options are 0.5 each, we can develop an alternative strategy considering the deltas *a priori* equal to 0.5 each. In this way, the optimal quantity of options that we can buy and sell changes: -16.57 of CHF 1M-1Y and +16.57 of CHF 2Y-1Y.

A delta-hedging strategy for the results of table 4.8 is to sell 70 quantity of CHF 1M-3Y ATM-2 and buy 42 quantity of CHF 1M-3Y ATM+10, where

deltas= 0.25, 0.27, 0.30, 0.32, 0.34, 0.37, 0.39, 0.42, 0.45, 0.47, 0.50, 0.53,
0.55, 0.58, 0.61, 0.63, 0.65, 0.68, 0.70, 0.72, 0.75,

these values make sense since the delta of a call option is between 0 and 1 and it increase from 0 as the option becomes closer to at-the-money.

Table 4.9: Deltas and hedging strategies for normal and shifted Black model

	Δ_N	Δ_{SB}	Hedging strategy normal	Hedging strategy SB
EUR 1M-1Y	0.50	0.50	-18.01	-19.81
EUR 3M-1Y	0.50	0.51	0	0
EUR 6M-1Y	0.50	0.51	0	0
EUR 9M-1Y	0.50	0.51	0	0
EUR 3Y-1Y	0.50	0.52	0	0
CHF 1M-1Y	0.50	0.52	0	0
CHF 3M-1Y	0.50	0.54	0	0
CHF 6M-1Y	0.50	0.55	0	0
CHF 1Y-1Y	0.50	0.56	0	0
CHF 2Y-1Y	0.50	0.60	18.01	16.75

EUR 1M-1Y is an European swaption with 1 month term and 1 year tenor, CHF 2Y-1Y is a Swiss swaption with 2 years term and 1 year tenor and so on. Δ_N is the normal delta, Δ_{SB} is the shifted Black delta, SB stays for shifted Black model.

Shifted SABR model A last analysis is made trough the shifted SABR model. Taking the market implied shifted Black volatilities with a -2% shift from Bloomberg for CHF cap option with maturity one year and strike at-the-money -0.007 with market price 5, we try to replicate the market premium with the shifted SABR model.

Table 4.10: Market implied shifted Black volatilities for different strikes CHF cap with maturity 1Y

K%	-1	-0.5	0	0.5	1	1.5	2	3	4
$\sigma_{SB_{mkt}}$	33.74	30.27	28.85	28.41	28.50	28.88	29.45	30.86	32.42

K% is the strike rate in percentage, $\sigma_{SB_{mkt}}$ is the market shifted Black volatility.

The first step the shifted SABR model does is to calibrate itself trough the minimization of the SSR between the market implied volatilities and the ones estimated from the shifted SABR. We see just the calibration method in which ρ, α and ν are directly estimated from

$$\min_{\rho,\nu,\alpha} \sum_i (\sigma_{SB_{mkt}} - \sigma_{SB}(\nu, \alpha_0, \rho, \beta, K, F_0)) \tag{4.8}$$

we set $\beta = 1$. The calibrate parameters are the following

Table 4.11: Calibrated SABR parameters

α	β	ρ	ν
0.28	1	-0.09	0.21

α is the estimated parameter of the forward rate volatility, β is the CEV exponent, ρ is the estimated parameter of the correlation between the forward rate and its volatility, ν is the estimated parameter of the volatility of the forward rate volatility.

remember that ν is the volatility of the volatility of the underlying, β is the CEV exponent, ρ is the correlation between the underlying and its volatility, meanwhile α is the volatility of the underlying. According to this calibration the fitted volatilities across a strike grid [-1.8,4] with jumps of 0.01 is presented in figure 4.4.

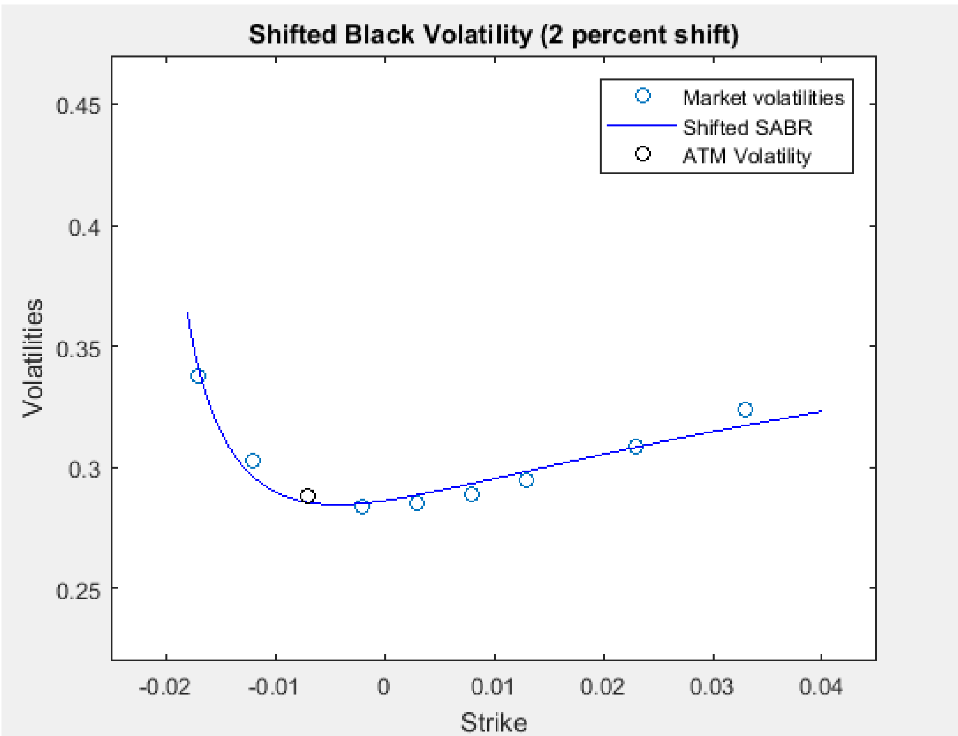


Figure 4.4: SABR shifted Black volatility

It is visible that the shifted SABR model well fits the structure and the smile of the volatility. The idea of the shifted SABR is to interpolate from the market volatilities and the relative limited strike grid (*i.e.* see table 4.10, strikes from -1 to 4) the volatilities of a bigger range (limited minimum dictated by the shift level, our example $|shift| = 2\%$ and, consequently strikes$\in [-1.8, 4]$) range of strikes with more intermediate values to get then the prices relative to these strikes through the shifted black formula (plug into the shifted Black formula the SABR volatilities).

Indeed, the prices from the combination of SABR ans SB model for the strikes from -1.8 to 4 are presented in figure 4.5.

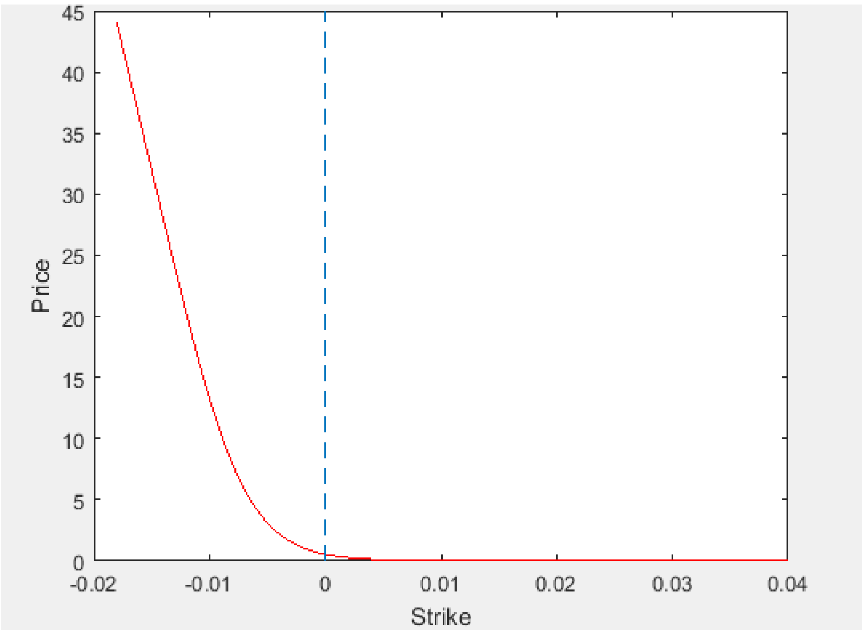


Figure 4.5: SABR prices for Cap option

Unfortunately, there are no market quotation for all the strikes. A faster way to check if the shifted SABR model price estimates are coherent with the market one is to look at the price for the ATM cap. Hence, the ATM cap price from shifted SABR at the current strike ATM -0.007 is 5.90, when the market one is 5. Therefore, the model gives an almost accurate price for the cap option over the strikes -1.8 to 4, since it replicates the dynamics of the volatility. The deltas given by the shifted SABR are shown in the figure 4.6

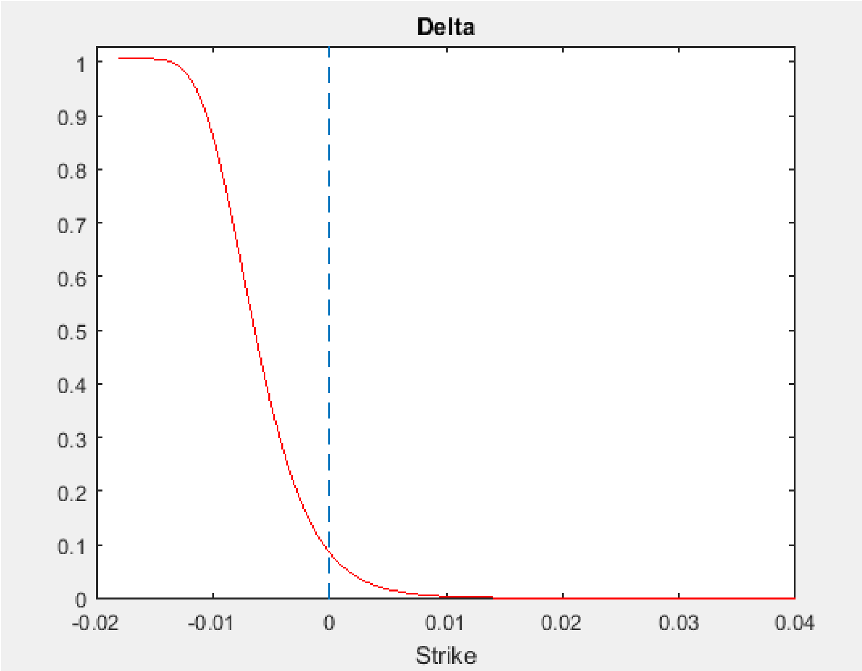


Figure 4.6: SABR deltas

Delta of the cap option ATM is 0.56, again we can see how the addition of the shift distorts the true one. Since the vega of options with negative strikes is 0, the adjustments for delta risk proposed by Bartlett are meaningless in this setting. In fact, the adjusted delta is

$$\Delta_{adj} = \Delta_{SB} + \Lambda_{SB}\left(\frac{\partial\sigma_{SB}}{\partial(f-\Theta)} + \frac{\partial\sigma_{SB}}{\partial\alpha}\frac{\rho\nu}{f^{\beta}}\right) = \Delta_{SB} + 0\cdot\left(\frac{\partial\sigma_{SB}}{\partial(f-\Theta)} + \frac{\partial\sigma_{SB}}{\partial\alpha}\frac{\rho\nu}{(f-\Theta)^{\beta}}\right) = \Delta_{SB}. \quad (4.9)$$

Hence the difference between the deltas and the modified deltas is almost 0 as figure 4.7 shows

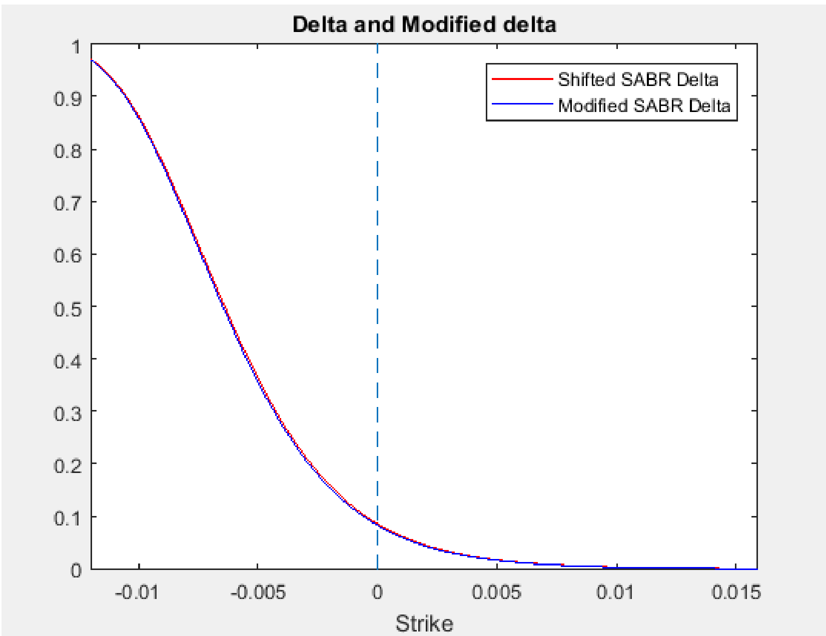


Figure 4.7: Delta Vs Modified Delta

Finally, let's take a look to the SABR probability density function

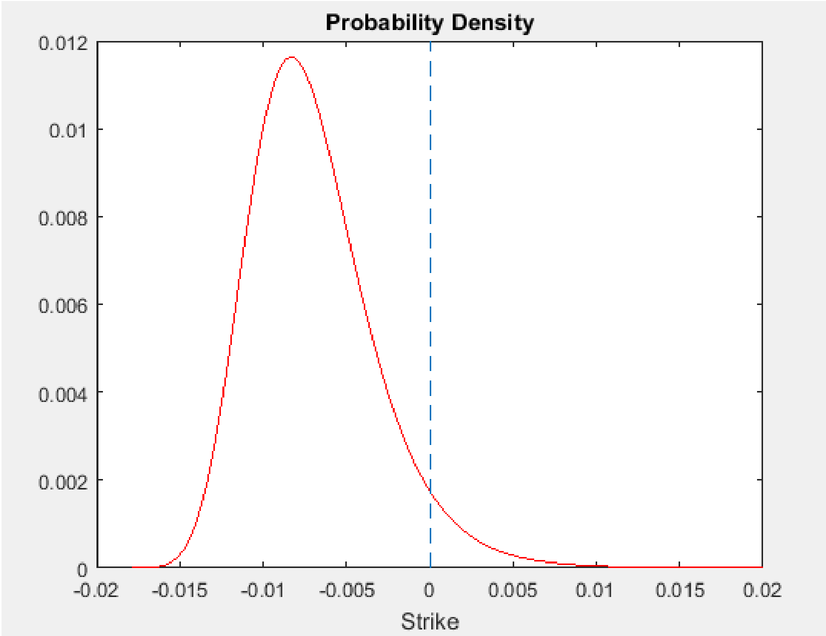


Figure 4.8: SABR probability density function

Positive prices are allowed by the SABR model.

Chapter 5

Conclusion

Three models to price interest rate plain vanilla options with negative strikes have been considered. The experiments shown that the normal model gives good estimates of prices for at-the-money swaptions with small term and tenor and it becomes less accurate as the term and the tenor increase. The normal model also gives prices closer to the market ones for a single swpation across different strikes, then for OTM and ITM options. The shifted Black model, if used with the right combination of shifts and volatilities, gives almost perfect prices with respect to the market premiums. However, if the wrong combination of shifts and volatilities is chosen, we obtain completely wrong prices. Furthermore, the shift parameter distorts the delta of the swaptions making a delta-hedge strategy slightly wrong, meanwhile the Bachelier model gives the right delta. The shifted SABR model with the shifted Black model gives less accurate prices in comparison with the previous models. However, the SABR model almost perfectly fits the dynamics of the market volatilities allowing, also with the assumption that the volatility follows a stochastic process, better hedge strategies. In conclusion, in order to price interest rate derivatives with negative strikes is better to use the normal model because of its simple implementation and because it does not include any arbitrary shift parameter that could bring to erroneous estimates.

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Appendix A. Itô Lemma

G is a continuous and differential function of the variable x and Δx is a small variation of x that changes G in ΔG as

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x \tag{5.1}$$

Since the error involves terms of second order we can use a Taylor series expansion for ΔG

$$\Delta G = \frac{\partial G}{dx}\Delta x + \frac{1}{2!}\frac{\partial^2 G}{\partial x^2}\Delta^2 x + \frac{1}{3!}\frac{\partial^3 G}{\partial x^3}\Delta^3 x + \ldots \tag{5.2}$$

If G is a continuous differential function of two variables,x and y, we have

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y \tag{5.3}$$

where the Taylor series expansion is

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2!}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial y}\Delta x \Delta y + \frac{1}{2!}\frac{\partial^2 G}{\partial y}\Delta y^2 + \ldots \tag{5.4}$$

$$\lim_{\Delta x, \Delta y \to 0} \Delta G = dG = \frac{\partial G}{\partial x}\Delta X + \frac{\partial G}{\partial y}\Delta y \tag{5.5}$$

Since a derivative is a function of a variable that follows a stochastic process we consider that this stochastic process is the Itô Lemma one

$$dx = a(x,t)dt + b(x,t)dz \tag{5.6}$$

Rewriting equation 5.4

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots \tag{5.7}$$

Discretizing 5.6

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t} \tag{5.8}$$

or

$$\Delta x = a\Delta t + b \in \Delta\varepsilon\sqrt{\Delta t} \tag{5.9}$$

where $\varepsilon \sim N(0,1)$. Using the results in 5.5 we get the Itô Lemma formula

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt \tag{5.10}$$

as Δt tends to 0 $\Delta x^2 = b^2\varepsilon^2\Delta t$ becomes $\Delta t = b^2$ and then non-stochastic.

Plugging dx from 5.6 we obtain

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz. \tag{5.11}$$

Appendix B. Black's formula

To prove the Black's formula we start from the dynamics of the forward rate

$$dF(t) = \sigma F(t)dW(t). \tag{5.12}$$

Let's consider in t a call option with time to maturity T, strike K and delivery date S, with $t < T < S$. The price of the cal is function of the strike price, time to maturity, interest rate, delivery date, time of today and the forward rate. Furthermore we can easily write it as $C(t,F(t),r,T,S,K) = C(t,F(t))$ since the T, r, S and K are fixed. Then by Itô Formula

$$\begin{aligned} dC(t,F(t)) &= C_t dt + C_F dF(t) + \frac{1}{2}C_{FF} d^2F(t) \\ &= C_t dt + \sigma C_F F(t)dW(t) + \frac{1}{2}C_{FF}\sigma^2 F^2(t)dt. \end{aligned} \tag{5.13}$$

In t we sell C_F futures contract and then we can eliminate the random part with $dW(t)$. The cost to enter in a future contract is 0, $u(t) = 0$, but in $t+\tilde{t}$ the futures contract will value $F(t+\tilde{t}) - F(t)$.

Indeed

$$du(t) = dF(t) = \sigma F(t)dW(t). \tag{5.14}$$

We can create a riskless position buying a call and selling C_F futures

$$d[C(t,F(t)) - u(t)C_F] = C_t dt + \frac{1}{2}C_{FF}\sigma^2 F^2(t)dt = r[C(t,F(t)) - u(t)C_F]. \tag{5.15}$$

Each risk-free portfolio Π must generates the same profit of any other risk-free alternative project, then $d\Pi = r\Pi dt$. Hence

$$C_t dt + \frac{1}{2}C_{FF}\sigma^2 F^2(t)dt = r[C(t,F(t))] \tag{5.16}$$

.

Solving equation 5.16 we end up with the Black's formula (Nekrasov, 2013)

$$C(t,F(t),r,T,S,K) = F(t)e^{-r(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{5.17}$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{F(t)}{K} + \frac{\sigma^2}{2}T\right]$$

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{F(t)}{K} - \frac{\sigma^2}{2}T\right] = d_1 - \sigma\sqrt{T}$$

Appendix C. SABR model

In order to price European options under the SABR model, singular perturbation techniques are used. The volatility $\hat{\alpha}$ and the volatility of the volatility ν are re-written, respectively, as $\varepsilon\hat{\alpha}$ and $\varepsilon\nu$ since we want to focus on small volatility expansion. As a consequence, ε by hypothesis bust be much much smaller than one, $\varepsilon << 1$

$$d\hat{F} = \varepsilon\hat{\alpha}C(\hat{F})dW \tag{5.18}$$

$$d\hat{\alpha} = \varepsilon\nu\hat{\alpha}dZ \tag{5.19}$$

$$dWdZ = \rho dt \tag{5.20}$$

The SABR model is a special case of this more general formulation of the model. In our case $C(\hat{F}) = \hat{F}^{\beta}$:

$$d\hat{F} = \varepsilon\hat{\alpha}\hat{F}^{\beta}dW, \quad \hat{F}(0) = f \tag{5.21}$$

$$d\hat{\alpha} = \varepsilon\nu\hat{\alpha}dZ, \quad \hat{\alpha}(0) = \alpha \tag{5.22}$$

$$dWdZ = \rho dt. \tag{5.23}$$

The implied normal volatility for this model is

$$\sigma_N(K) = \frac{\varepsilon\alpha(1-\beta)(f-K)}{f^{(1-\beta)} - K^{(1-\beta)}} \cdot \left(\frac{\xi}{\hat{\chi}}(\xi)\right)$$
$$\cdot \left[1 + \left(\frac{-\beta(2-\beta)\alpha^2}{24 f_{av}^{(2-2\beta)}} + \frac{\rho\alpha\nu\beta}{4 f_{av}^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right)\varepsilon^2 T + \ldots\right] \tag{5.24}$$

where $f_{av} = \sqrt{fK}$ and

$$\xi = \frac{\nu}{\alpha}\frac{f-K}{f_{av}^{\beta}}, \qquad \hat{\chi}(\xi) = \log\left(\frac{\sqrt{1-2\rho\xi+\xi^2}-\rho+\xi}{1-\rho}\right). \tag{5.25}$$

Expanding the formula without taking into account the terms higher than 4^{th} order

$$f-K = \sqrt{fK}\log\frac{f}{K}\left[1+\frac{\log^2(\frac{f}{K})}{24}+\frac{\log^4(\frac{f}{K})}{1920}+...\right] \tag{5.26}$$

$$f^{(1-\beta)}-K^{(1-\beta)} = (1-\beta)(fK^{\frac{(1-\beta)}{2}})\log(\frac{f}{K})\cdot\left[1+\frac{(1-\beta)^2}{24}\log^2(\frac{f}{K})+\frac{(1-\beta)^4}{1920}\log^4(\frac{f}{K})+...\right] \tag{5.27}$$

the formula reduces the implied normal volatility to

$$\sigma_N(K) = \varepsilon\alpha(fK)^{\frac{\beta}{2}}\cdot\frac{1+\frac{\log^2(\frac{f}{K})}{24}+\frac{\log^4(\frac{f}{K})}{1920}+...}{1+\frac{(1-\beta)^2\log^2(\frac{f}{K})}{24}+\frac{(1-\beta)^4\log^4(\frac{f}{K})}{1920}+...}\cdot\frac{\xi}{\hat{\chi}(\xi)}$$
$$\cdot\left[1+\left(\frac{-\beta(2-\beta)\alpha^2}{24(fK)^{(1-\beta)}}+\frac{\rho\alpha\nu\beta}{4(fK)^{\frac{(1-\beta)}{2}}}+\frac{2-3\rho^2}{24}\nu^2\right)\varepsilon^2T+...\right] \tag{5.28}$$

where

$$\xi = \frac{\nu}{\alpha}(fK)^{\frac{(1-\beta)}{2}}\log(\frac{f}{K}),$$
$$\hat{\chi}(\xi) = \log\frac{\sqrt{1-2\rho\xi+\xi^2}-\rho+\xi}{1-\rho}. \tag{5.29}$$

To obtain the implied Black volatility we need to equate the normal volatility obtained by formulas 5.21 and 5.22 to the implied normal volatility for Black's model, that is

$$\sigma_N^{\text{Implied Black}}(K) = \frac{\varepsilon\sigma_B(f-K)}{\log(\frac{f}{K})}\left[1-\frac{\varepsilon^2\sigma_{\text{B}}^2T}{24}+...\right]. \tag{5.30}$$

Then the implied Black volatility for SABR model is

$$\sigma_B(K) = \frac{\varepsilon\alpha}{(fK)^{\frac{(1-\beta)}{2}}} \cdot \left(\frac{\xi}{\hat{\chi}(\xi)}\right) \cdot \frac{1}{1+\frac{(1-\beta)^2\log^2(\frac{f}{K})}{24}+\frac{(1-\beta)^4\log^4(\frac{f}{K})}{1920}+...}$$
$$\cdot\left[1+\left(\frac{(1-\beta)^2\alpha^2}{24(fK)^{(1-\beta)}}+\frac{\rho\alpha\nu\beta}{4(fK)^{\frac{(1-\beta)}{2}}}+\frac{2-3\rho^2}{24}\nu\right)\varepsilon^2 T+...\right] \tag{5.31}$$

where ξ and $\hat{\chi}(\xi)$ are the same of equation 5.29.

Special case: $\beta = 0$ For the stochastic normal model the implied volatilities for European option are

$$\sigma_N(K) = \varepsilon\alpha\left[1+\frac{2-3\rho^2}{24}\varepsilon^2\nu^2 T+...\right] \tag{5.32}$$

$$\sigma_B(K) = \varepsilon\alpha\frac{\log(\frac{f}{K})}{f-K}\cdot\left(\frac{\xi}{\hat{\chi}(\xi)}\right)\cdot\left[1+\left(\frac{\alpha^2}{24fK}+\frac{2-3\rho^2}{24}\nu^2\right)\varepsilon^2 T+...\right] \tag{5.33}$$

where

$$\xi = \frac{\nu}{\alpha}\sqrt{fK}\log\frac{f}{K}, \qquad \hat{\chi}(\xi) = \log\left(\frac{\sqrt{1-2\rho\xi+\xi^2}-\rho+\xi}{1-\rho}\right). \tag{5.34}$$

Special case: $\beta = 1$ For the stochastic log-normal model the implied volatilities for European calls and puts are

$$\sigma_N(K) = \varepsilon\alpha\frac{f-K}{\log\frac{f}{K}}\cdot\left(\frac{\xi}{\hat{\chi}(\xi)}\right)\cdot\left[1+\left(-\frac{\alpha^2}{24}+\frac{\rho\alpha\nu}{4}+\frac{2-3\rho^2}{24}\nu^2\right)\varepsilon^2 T+...\right] \tag{5.35}$$

$$\sigma_B(K) = \varepsilon\alpha\cdot\left(\frac{\xi}{\hat{\chi}(\xi)}\right)\cdot\left[1+\left(\frac{\rho\alpha\nu}{4}+\frac{2-3\rho^2}{24}\nu^2\right)\varepsilon^2 T+...\right] \tag{5.36}$$

where

$$\xi = \frac{\nu}{\alpha}\log\frac{f}{K}, \qquad \hat{\chi}(\xi) = \log\left(\frac{\sqrt{1-2\rho\varepsilon+\varepsilon^2}-\rho+\varepsilon}{1-\rho}\right). \tag{5.37}$$