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On the Significance of Capturing the Early Exercise Boundary for the American Put Price

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Abstract

I show that the three-piece exponential boundary by Ju (1998) accurately 'tracks' the early exercise boundary. This results in more accurate option pricing than other comparable methods. Numerical results obtained in this paper agree that a multipiece exponential function approximation yields very accurate prices for short as well as moderate maturity put options. These results are partially at odds with previous research.

Keywords: American put option ◇ Analytical approximation ◇ Early exercise boundary

1 Introduction

Derivatives are essential for financial markets. Over the past forty years there has been an explosive growth of financial derivatives underlining their importance for financial markets. For practitioners, this has entailed not only an increase in volume but also in terms of variety.

Black and Scholes (1973) proposed an ingenious closed form expression for the valuation of European options when the underlying asset follows geometric Brownian motion. Merton (1973) could not however reconcile the closed form expression to the valuation of American put options. The simple

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yet great difference between European and American put options is that the latter can be exercised at any time prior to maturity. This flexibility difference has great impact on the difficulty in pricing the American put option. More specifically, in order to price the put option one must first determine the early exercise boundary that is associated with the right to early exercise. Determining this boundary is at the heart of the problem considered in this paper. Focus is then on the related problems of determining the early exercise boundary and pricing the American put option.

To deal with the problem of pricing American put options, researchers employed mainly two types of approaches. Either implicitly solve the partial differential equation governing the option price subject to early exercise conditions or approximating the option price via the early exercise boundary. The former approach is often classified as numerical while the latter is primarily an analytical approximation approach. In this context the term 'analytical' is the deduction of a problem my smaller tangible steps and follows from James (1992, p.12) definition.

The classical numerical methods include Brennan *et al.* (1977) finite difference method, the binomial method of Cox *et al.* (1979), and the Monte Carlo simulation method of Grant *et al.* (1996). The Monte Carlo simulation approach of Grant *et al.* (1996) cannot adequately determine if an early exercise has occurred, therefore Longstaff and Schwartz (2001) proposed an improvement based on the insight that the decision of exercise or not can be imbedded in the simulated asset price movement. Zhu (2007) note that this approach is highly accurate but suffers from efficiency problems. The binomial method of Cox *et al.* (1979) makes an 'exercise or not' decision at each future time point. However, Zhu and Francis (2004) observed one important drawback, which is that it is not able to capture the early exercise boundary. The third numerical method that deserves mentioning is the finite difference method of Brennan *et al.* (1977). Several algorithms, such as the finite volume method (FVM) of Forsyth and Vetzal (2002), the finite element method (FEM) of Allegretto *et al.* (2001), are all extensions where the methodology is the same, all seek to solve the partial differential equation.

To adequately deal with the issue of computational efficiency, a series of approximation methods, such as Barone-Adesi and Whaley (1987) and MacMillan (1986), Kuske and Keller (1998), Bjerksund and Stensland (1993), emerged. In their extensive research, Cheng and Zhang (2012) notes that, despite being more efficient than numerical methods they suffer from pricing errors especially for long maturity options. Another drawback is that neither approach is convergent, since there are no parameters which can be altered in order to reduce pricing errors. Hence, their scope is limited.

A second wave of approximation methods capable of pricing long maturity options with convergence property emerged. This includes, the infinite series solution of Geske and Johnson (1984), the multipiece constant function approach of Huang *et al.* (1996) and the multipiece exponential function approach of Ju (1998). The shared methodology among the three being that time prior to maturity can be discretisize in order to approximate the early exercise boundary. Additionally, they all use Richardson extrapolation technique in order to minimize pricing errors. Past numerical studies by Ju (1998) and more recently Chung *et al.* (2010) have investigated their accuracy in pricing long maturity put options. For this reason then, an extensive numerical study comparing their accuracy in pricing moderate and short maturity put options will be presented in this paper.

In this numerical study it will be shown that for short and moderate maturity options Ju's (1998) three-point piecewise exponential function has the lowest pricing errors of the three. The accuracy is comparable to a 1000 time-step binomial method. This is inline with the numerical results presented by Ju (1998) focused on long maturity options. From Geske and Johnson (1984), were the American put option can be priced by a series of equivalent Bermuda options exercisable at discrete points in time, I find the largest pricing errors. Huang *et al.* (1996) provided the most efficient approximation method in this numerical study, by the four-point piecewise constant function. Also included is their six-point piecewise constant function which was proven to be the second most accurate.

Furthermore, it will be shown that Ju's (1998) three-piece exponential boundary is not substantially different from the 'true' early exercise boundary. By computing a more accurate approximation of the early exercise boundary using the idea of Hou *et al.* (2000), I was able to show in contradiction to Ju (1998) that his three-piece exponential boundary does is in fact 'track' the early exercise boundary rather well. A discussion regarding the implications of this result is also presented.

The next section develops the necessary mathematical framework upon which subsequent approximation methods evolve from. Section 3 presents a numerical study on those approximation methods, subsequently the results and significance of the early exercise boundary are discussed. Section 4 concludes the paper.

2 Theoretical Background

In this paper, we find ourself in the Black-Scholes framework, with perfect capital markets, continuous trading and no-arbitrage. Perfect capital markets imply that all information affecting stock prices are instantaneously incorporated, with infinite liquidity and where every transaction made is frictionless. Continuous trading, which follows from the previous assumption, simply means that the time between security prices being quoted tends to zero. Hence, we can follow security price movements continuously. Last, from a no-arbitrage assumption, we also assume all security prices (including dividends) to be regarded as martingales relative to some unique equivalent martingale measure.

The underlying stock is said to follow the stochastic differential equation

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t \tag{1}$$

Note if $\delta = 0$ the stochastic differential equation collapses to

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2}$$

Here we assume the following. Interest rate earned is positive $r > 0$, the continuous dividend rate is positive or equal to zero $\delta \geq 0$, the uniform volatility of the asset is positive $\sigma > 0$ and stock price S_t (at time t) is driven by the Brownian motion W_t. The terms r, δ, σ are assumed to be constant.

From this framework a broad variety of option types can be analysed. The focus of this paper is on American put options. The distinguishing feature of America options compared to European options is that the former can be exercised at any time before maturity. This distinguishing feature is at the heart of the problem considered in this paper. McDonald and Schroder (1998) showed that there exist a parity relationship such that American call options are conveniently priced if one knows the put option price. Focus is therefore on American put options.

From the seminal work of Black-Scholes (1973) and Merton (1973) we know that the function $P(S_t, t)$ represent the put price at time t, and is the solution to the 'free-boundary' problem. From the point B_t (hereafter optimal early exercise boundary) and below, for each $t \in [0, T]$ it is optimal to exercise the option early. From Merton (1973), the solution to 'free-boundary' problem is finding $P(S_t, t)$ that satisfies the partial differential equation

$$\frac{\partial P}{\partial t} = \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P}{\partial S^2} + (r-\delta) S_t \frac{\partial P}{\partial S} - rP, \tag{3}$$

subject to the conditions

$$\lim_{S_t \uparrow \infty} P(S_t, t) = 0, \tag{4}$$

$$\lim_{S_t \downarrow B_t} P(S_t, t) = K - B_t, \tag{5}$$

$$\lim_{S_t \downarrow B_t} \frac{\partial P(S_t, t)}{\partial S_t} = -1, \tag{6}$$

As expected the solution is non-trivial. In addition, note the following. The first condition implies that the option becomes worthless when the underlying stock price goes to infinity. The last conditions guarantee that the early exercise boundary 'smoothly' pastes on to the slope of the payoff-function, this ensures optimality in the case of early exercise.

Instead of solving for $P(S_t, t)$ via the partial differential equation route, Kim (1990), Jacka (1991) and Carr *et al.* (1992) derived an alternative expression. Let, P_0 be the current option price ($t = 0$), and S_0 be the current stock price ($t = 0$), then

$$\begin{aligned} P_0 =& K e^{-rT} N(-d_-(S_0, K, T)) - S_0 e^{-\delta T} N(-d_+(S_0, K, T)) \\ &+ \int_0^T \left[rKe^{-rt} N(-d_-(S_0, B_t, t)) - \delta S_0 e^{-\delta t} N(-d_+(S_0, B_t, t) \right] dt, \end{aligned} \tag{7}$$

where

$$d_\pm(\alpha, \beta, t) \equiv \frac{\log(\frac{\alpha}{\beta}) + (r - \delta \pm \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}. \tag{8}$$

The first part in (7) is simply the equivalent European put, followed by the integral (hereafter early exercise premium integral) containing the early exercise boundary as a function in the integrand. Note, $N(\cdot)$ is the cumulative normal distribution function where the early exercise boundary B_t appears as a logarithmic argument. Note also the economic choice facing the holder of the option. More specifically, the first term in the integrand is the discounted income received from an early exercise whereas the second term is the cost associated with the early sell of a dividend yielding stock. In the absence of dividend ($\delta = 0$) equation (7) collapses to,

$$\begin{aligned} P_0 =& K e^{-rT} N(-d_-(S_0, K, T)) - S_0 N(-d_+(S_0, K, T)) \\ &+ \int_0^T rK e^{-rt} N(-d_-(S_0, B_t, t)) ds, \end{aligned} \tag{9}$$

where again

$$d_\pm(\alpha, \beta, t) \equiv \frac{\log(\frac{\alpha}{\beta}) + (r \pm \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}. \tag{10}$$

Due to the early exercise premium integral in (9), it might be optimal to exercise early even in the absence of dividends if the stock price falls low enough. This is a fundamental property of American put options, not shared with other types of options (Zhu (2007, p.2)). However, prior to making an early exercise decision we need to solve for B_t, upon which, using (7) or (9), the put option price can be attained.

The following are true for B_t. From Kim (1990) we have that B_t is a continuously decreasing function of t on the interval $[0, \infty)$ however differentiable only on the interval $[x, \infty), x > 0$. Kim (1990) also established that B_t is

$$\min\left(K, \frac{rK}{\delta}\right). \tag{11}$$

Using the fact that we capture the payoff according to (5) in the event of an early exercise, then it follows from (7) that B_t satisfies the following equation, setting $P_t = K - B_t$,

$$\begin{aligned} K - B_t =& K e^{-r(T-t)} N(-d_-(B_t, K, T-t)) - B_t e^{-\delta(T-t)} N(-d_+(B_t, K, T-t)) \\ &+ \int_t^T rK e^{-r(s-t)} N(-d_-(B_t, B_s, s-t)) ds \\ &- \delta B_t e^{-\delta(s-t)} N(-d_+(B_t, B_s, s-t)) ds, \end{aligned} \tag{12}$$

where in the absence of dividend ($\delta = 0$),

$$\begin{aligned} K - B_t =& K e^{-r(T-t)} N(-d_-(B_t, K, T-t)) - B_t N(-d_+(B_t, K, T-t)) \\ &+ \int_t^T \left[rK e^{-r(s-t)} N(-d_-(B_t, B_s, s-t))\right] ds. \end{aligned} \tag{13}$$

For every value $S_t \leq B_t$ when $S_t > 0$, equations (12) and (13) holds. From equation (12) and (13), where $N(\cdot)$ is the cumulative normal distribution function, it follows that in integral form:

$$\begin{aligned} K - B_t =& Ke^{-r(T-t)}N(-d_-(B_t, K, T-t)) - B_t e^{-\delta(T-t)}N(-d_+(B_t, K, T-t)) \\ &+ \frac{rK}{\sqrt{2\pi}} \int_t^T \int_{-\infty}^{-d_-(B_t, B_s, s-t)} e^{-r(s-t)} e^{-\frac{1}{2}w^2} dw ds \\ &- \frac{\delta B_t}{\sqrt{2\pi}} \int_t^T \int_{-\infty}^{-d_+(B_t, B_s, s-t)} e^{-\delta(s-t)} e^{-\frac{1}{2}w^2} dw ds, \end{aligned} \tag{14}$$

where in the absence of dividend ($\delta = 0$),

$$\begin{aligned} K - B_t =& Ke^{-r(T-t)}N(-d_-(B_t, K, T-t)) - B_t N(-d_+(B_t, K, T-t)) \\ &+ \frac{rK}{\sqrt{2\pi}} \int_t^T \int_{-\infty}^{-d_-(B_t, B_s, s-t)} e^{-r(s-t)} e^{-\frac{1}{2}w^2} dw ds. \end{aligned} \tag{15}$$

When viewed in an integral representation one can see the difficulty in solving for B_t in (14) and (15). More specifically equation (14) requires solving two bivariate integrals numerically over two dimensions, and as noted in Press *et al.* (1996), integrals solved for N-dimensions requires evaluating a growing series of functions. To circumvent this, an approximation of the cumulative normal distribution functions $N(\cdot)$ is used to keep the attractive univariate integral form. However, Hou *et al.* (2000) notes that approximating $N(\cdot)$ may give rise to large numerical errors when solving (14) or (15). Such a solution would therefore be sensitive to the approximation accuracy of $N(\cdot)$. For this reason Hou *et al.* (2000)'s new integral representation (carefully derived in Appendix A) of the early exercise boundary does not include the cumulative normal distribution functions. According to Hou *et al.* (2000), literature has largely neglected the entire region $S < B_t$, and simply focused on the point $S_t = B_t$. Hou *et al.* (2000) used this fact to construct a new integral representation of the early exercise boundary.

If the stock price S_t, where $S_t \in (0, B_t]$, drops below or equal to the early exercise boundary ($S_t \leq B_t$) the option is exercised early. If we let $S_t = \varepsilon B_t$ with $\varepsilon \in (0, 1]$, then B_t is differentiable everywhere with respect to ε. Using this, equation (12) can be written as

$$
\begin{aligned}
K - \varepsilon B_t = {} & K e^{-r(T-t)} N(-d_-(\varepsilon B_t, K, T-t)) \\
& - \varepsilon B_t e^{-\delta(T-t)} N(-d_+(\varepsilon B_t, K, T-t)) \\
& + \int_t^T r K e^{-r(s-t)} N(-d_-(\varepsilon B_t, B_s, s-t)) ds \\
& - \delta \varepsilon B_t e^{-\delta(s-t)} N(-d_+(\varepsilon B_t, B_s, s-t)) ds, \qquad (16)
\end{aligned}
$$

rearranging terms we have that

$$
\begin{aligned}
& \varepsilon B_t \left\{ 1 - e^{-\delta(T-t)} N\big(-d_+(\varepsilon B_t, K, T-t)\big) - \delta \int_t^T e^{-\delta(s-t)} N\big(-d_+(\varepsilon B_t, B_s, s-t)\big)\, ds \right\} \\
& = K \left\{ 1 - e^{-r(T-t)} N\big(-d_-(\varepsilon B_t, K, T-t)\big) - r \int_t^T e^{-\delta(s-t)} N\big(-d_-(\varepsilon B_t, B_s, s-t)\big)\, ds \right\}. \qquad (17)
\end{aligned}
$$

Hou *et al.* (2000) was able to show that equation (17) can be represented without the cumbersome $N(\cdot)$ as,

$$
\begin{aligned}
B_t & \left\{ \sigma e^{-\delta(T-t) - \frac{1}{2} d_+^2(B_t, K, T-t)} + \delta \sqrt{2\pi(T-t)} \right\} = K r \sqrt{2\pi(T-t)} \\
& + \delta B_t \sqrt{T-t} \int_t^T e^{-\delta s - \frac{1}{2} d_+^2(B_t, B_s, s-t)} \left(\frac{d_-(B_t, B_s, s-t)}{s} \right) ds \\
& - K r \sqrt{t} \int_t^T e^{-r(s-t) - \frac{1}{2} \mu_-^2(B_t, B_s, s-t)} \left(\frac{d_+(B_t, B_s, s-t)}{s} \right) ds, \qquad (18)
\end{aligned}
$$

where in the absence of dividend ($\delta = 0$) equation (18) collapses to

$$
\begin{aligned}
B_t = {} & K r e^{\frac{1}{2} d_+^2(B_t, K, T-t)} \sqrt{2\pi(T-t)} \\
& - K r e^{\frac{1}{2} d_+^2(B_t, K, T-t)} \sqrt{T-t} \times \\
& \int_t^T e^{-r(s-t) - \frac{1}{2} \mu_-^2(B_t, B_s, s-t)} \left(\frac{d_+(B_t, B_s, s-t)}{s} \right) ds. \qquad (19)
\end{aligned}
$$

Hou *et al.* (2000) were able to show that their equation (18) is not prone to oscillations associated with equations (14) and (15) where the standard cumulative normal distribution function is present. This then, led them to believe that "our representation is better suited for use in any numerical implementation requiring an estimate of the exercise boundary" (Hou *et al.* (2000, p.11)). From here we shall now see how the analytical methods of Geske and Johnson (1984), Huang *et al.* (1996), and Ju (1998) approximate the American put price and tackle the problem of determining the early exercise boundary.

2.1 Approximation by a Series of Bermuda Options

Geske and Johnson (1984) was the first to apply the logic of an infinite series of European put options as representation for the American put price. Their approach is an extension of an earlier paper by Geske (1979) which originally showed how to price compound options. Geske and Johnson (1984) began by noting that this approach would require calculating an infinite series of put options, nonetheless in the limit it is an exact representation of the true option. A more feasible approach using fewer put options, each associated with different dates (prior to maturity) was also proposed in Geske and Johnson (1984). Combining these put options using Richardson extrapolation Geske and Johnson (1984) are able to approximate the price of an otherwise equivalent American put. Much of the intuition and methodology behind their approximation method is straightforward and applicable to the other approximation methods in this paper.

More specifically, at each discrete date prior to maturity the following considerations are made; the put will be exercised (i) if it is still alive and (ii) the payoff exceeds the intrinsic price of the put. At each date then, an optimal boundary B_t divides the holding region from the exercise region. This exercise region is bound by (5) whenever $S_t \leq B_t$ and thus is independent of the current stock price S_0.

Geske and Johnson (1984) considers the following, a European put has no probability of early exercise, hence the price can be easily calculated using the closed form solution of Black and Scholes (1973) and Merton (1973). In order to price an equivalent put option exercisable at dates $T/2$ and T, requires checking for early exercise at $T/2$. Similarly, going backwards two time steps from maturity T, correctly pricing such a put option requires checking for early exercise at $T/3$ and $2T/3$. The key insight follows from the intuition that the put was not exercised at earlier dates since S_t was always above B_t. From this insight, Geske and Johnson (1984) derived the following equation,

$$P = Kw_2 - Sw_1 \tag{20}$$

where

$$\begin{aligned} w_1 =& \Big\{ N_1 \Big(-d_+(S_{dt}, dt) \\ &+ N_2(d_+(S_{dt}, dt)), -d_+(S_{2dt}, dt); -\rho_{12} \Big) \\ &+ N_3 \Big(d_+(S_{dt}, dt), d_+(S_{2dt}, 2dt), -d_+(S_{3dt}, 3dt); \rho_{12}, -\rho_{13}, -\rho_{23} \Big) \ldots \Big\}, \end{aligned} \tag{21}$$

$$\begin{aligned} w_2 =& \Big\{ N_1 \Big(-d_-(S_{dt}, dt) \\ &+ N_2(d_-(S_{dt}, dt)), -d_-(S_{2dt}, dt); -\rho_{12} \Big) \\ &+ N_3 \Big(d_-(S_{dt}, dt), d_-(S_{2dt}, 2dt), -d_-(S_{3dt}, 3dt); \rho_{12}, -\rho_{13}, -\rho_{23} \Big) \ldots \Big\}, \end{aligned} \tag{22}$$

and the correlation coefficients ρ_{12} and ρ_{13} are

$$\begin{aligned} \rho_{12} =& 1/\sqrt{2}, \\ \rho_{13} =& 1/\sqrt{2}. \end{aligned} \tag{23}$$

Note some important observations. The equation contains an infinite series of options subject to an infinite number of exercise boundaries. Hence, in the limit equation (20) is regarded as an exact solution to the 'free-boundary' problem, but requires solving an infinite series of options each containing an infinite series of multivariate normal distribution functions.

For this reason, Geske and Johnson (1984) proposed a more practical implementation that could be comparable to numerical procedures such as Cox *et al.* (1979) binomial method, and Brennan *et al.* (1977) finite difference method. Assume that P_1 is the price of a European put option which can only be exercised at time T, then

$$P_1 = p_0. \tag{24}$$

Let P_2 be the price of an equivalent Bermuda option[1] exercisable at time $T/2$ and T, then

[1] A Bermuda option is defined as a limited American option, only exercisable at some pre-determined dates prior to maturity, for more information see Wilmott (2013, p.41).

$$
\begin{aligned}
P_2 =& Ke^{-r\frac{T}{2}} N_1[-d_-(B_{T/2}, T/2)] - S_{T/2} N_1[-d_+(B_{T/2}, T/2)] \\
&+ Ke^{-rT} N_2[d_-(B_{T/2}, T/2), -d_-(K,T); \frac{-1}{\sqrt{2}}] \\
&- S_{T/2} N_2[d_+(B_{T/2}, T/2), -d_+(K,T); \frac{-1}{\sqrt{2}}].
\end{aligned} \tag{25}
$$

The optimal exercise boundary $B_{T/2}$ follows from (25) and is the solution to

$$
S_{T/2} = K - p(S, K, T/2, r, \sigma) = B_{T/2}. \tag{26}
$$

Similarly, let P_3 be the price of an equivalent Bermuda option that can be exercised at $T/3$, $2T/3$ and T, then

$$
\begin{aligned}
P_3 =& Ke^{-r\frac{T}{3}} N_1\Big[-d_-\Big(B_{T/3}, T/3\Big)\Big] - S_{T/3} N_1\Big[-d_+\Big(B_{T/3}, T/3\Big)\Big] \\
&+ Ke^{-2rT/3} N_2\Big[d_-\Big(B_{T/3}, T/3\Big), -d_-\Big(B_{2T/3}, 2T/3\Big); -\frac{1}{\sqrt{2}}\Big] \\
&- S_{\frac{T}{3}, \frac{2T}{3}} N_2\Big[d_+\Big(B_{T/3}, T/3\Big), -d_+\Big(B_{2T/3}, 2T/3\Big); -\frac{1}{\sqrt{2}}\Big] \\
&+ Ke^{-rT} N_3\left[d_+\Big(B_{T/3}, T/3\Big), -d_+\Big(B_{2T/3}, 2T/3\Big), -d_+(K,T); \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right] \\
&- S_{\frac{T}{3}, \frac{2T}{3}} N_3\left[d_-\Big(B_{T/3}, T/3\Big), d_-(B_{2T/3}, 2T/3), -d_-(K,T); \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right],
\end{aligned} \tag{27}
$$

and the optimal exercise boundary $B_{T/3}$ and $B_{2T/3}$ follow from (27) and are the solutions to,

$$
\begin{aligned}
S_{T/3} =& K - P_2(S, K, 2T/3, r, \sigma) = B_{T/3}, &(28)\\
S_{2T/3} =& K - p(S, K, T/3, r, \sigma) = B_{2T/3}, &(29)
\end{aligned}
$$

respectively. The sequence of P_1, P_2, and P_3 are then combined to give a more accurate American put price P, by the following three-point Richardson extrapolation,

$$
P = P_3 + 7/2(P_3 - P_2) - 1/2(P_2 - P_1). \tag{30}
$$

However in their numerical representation they use a less efficient but more accurate four-point Richardson extrapolation[2]. Similar to the three-point but more accurate due to inclusion of a fourth Bermuda option P_4 exercisable at the following dates prior to maturity T/4, 2T/4, 3T/4, and T. The four-point Richardson extrapolation then looks like,

$$P = P_4 + \frac{29}{3}(P_4 - P_3) - \frac{23}{6}(P_4 - P_3) + \frac{1}{6}(P_2 - P_1). \tag{31}$$

This was the first paper utilizing an extrapolation technique (see Geske and Johnson (1984)). The improvement of accuracy, by the use of extrapolation technique, is a shared theme among Geske and Johnson (1984) and the other approximation methods, Huang *et al.* (1996) and Ju (1998), which now follow.

2.2 A Piecewise Constant Approximation

Huang *et al.* (1996) solution for the early exercise boundary problem described above is by the following two-step procedure. First, they begin by discretising the entire interval $[0, T]$ into n equal partitioned subintervals (or pieces). This enables Huang *et al.* (1996) to estimate the entire early exercise boundary by four-piece constant functions, combined by a four-point Richardson extrapolation yielding a put option price approximation.

Following a similar path as Geske and Johnson (1984) they began by acknowledging the limitations of expression (20) which involves calculating several multivariate normal distribution functions. Particularly, as P_n grows ($n \uparrow \infty$) expression (20) involves two univariate $N_1(\cdot)$ integrals, two bivariate $N_2(\cdot)$ integrals, two trivarate $N_3(\cdot)$ integrals, and two n-variate $N_n(\cdot)$ integrals, upon which the put option is priced. Huang *et al.* (1996) notes that the computational cost involved with expressions (20), (24), (25), and (27) would be very high, due to the multivariate normal distribution functions. Therefore, Huang *et al.* (1996) approximated the put price using only univariate normal integrals.

Huang *et al.* (1996) starts with equation (7) which include the cumbersome convolution type integral that needs to be solved over a region with two dimensions. Huang *et al.* (1996) circumvent this by approximating the

[2] Higher-point extrapolation schemes are less efficient but more accurate, see Geske and Johnson (1984, p.1518) Appendix 1, for more details on the implementation of Richardson extrapolation.

early exercise premium integral and the cumulative normal density function in (7) with piecewise time invariant (or constant) functions. For example, if P_n, where $n = 1$ denotes the price of an one-time exercisable put option at maturity (i.e. European put) and $n = 1, 2$ denotes the price P_2 of an two-times exercisable put option at maturity and halfway to maturity, for $n = 3$ we would have P_3, denoting a three-times exercisable option at maturity, 1/3 from maturity, and 2/3 from maturity. Expressed as three-piece constant function, we would have for P_1, P_2, and P_3, respectively,

$$P_1 = Ke^{-rT} N(-d_-(S_0, K, T)) - S_0 e^{-\delta T} N(-d_+(S_0, K, T)) \equiv p_0. \tag{32}$$

$$\begin{aligned} P_2 =& p_0 + \frac{rKT}{2} e^{-\frac{rT}{2} rT} N(-d_-(S_0, B_{\frac{T}{2}}, T/2)) \\ &- \frac{\delta S_0 T}{2} e^{-\frac{\delta T}{2}} N(-d_+(S_0, B_{\frac{T}{2}}, T/2)). \end{aligned} \tag{33}$$

$$\begin{aligned} P_3 =& p_0 \\ &+ \frac{rKT}{3} \left[e^{-\frac{rT}{3}} N(-d_-(S_0, B_{\frac{T}{3}}, T/3)) + e^{-\frac{2rT}{3}} N(-d_+(S_0, B_{\frac{2T}{3}}, 2T/3)) \right] \\ &- \frac{S_0 T}{3} \left[e^{-\frac{\delta T}{3}} N(-d_+(S_0, B_{\frac{T}{3}}, T/3)) + e^{-\frac{2aT}{3}} N(-d_+(S_0, B_{\frac{2T}{3}}, 2T/3)) \right], \end{aligned} \tag{34}$$

where other P_n, (as $n \uparrow \infty$) follow a similar pattern. Note in the case of no continuous dividend yield ($\delta = 0$), expressions P_1, P_2, and P_3, collapse respectively to,

$$P_1 = p_0. \tag{35}$$

$$\begin{aligned} P_2 =& p_0 + \frac{rKT}{2} e^{-\frac{rT}{2} rT} N(-d_-(S_0, B_{\frac{T}{2}}, T/2)) \\ &- \frac{\delta S_0 T}{2} e^{-\frac{\delta T}{2}} N(-d_+(S_0, B_{\frac{T}{2}}, T/2)). \end{aligned} \tag{36}$$

$$\begin{aligned} P_3 =& p_0 \\ &+ \frac{rKT}{3} \left[e^{-\frac{rT}{3}} N(-d_-(S_0, B_{\frac{T}{3}}, T/3)) + e^{-\frac{2rT}{3}} N(-d_+(S_0, B_{\frac{2T}{3}}, 2T/3)) \right] \\ &- \frac{S_0 T}{3} \left[e^{-\frac{\delta T}{3}} N(-d_+(S_0, B_{\frac{T}{3}}, T/3)) + e^{-\frac{2aT}{3}} N(-d_+(S_0, B_{\frac{2T}{3}}, 2T/3)) \right]. \end{aligned} \tag{37}$$

The sequence of approximate values P_1, P_2, P_3 are then combined via a three-point Richardson extrapolation method to yield a price of the put option P with greater accuracy. Huang *et al.* (1996) propose the following three-point Richardson extrapolation for equations (32)-(34), or (35)-(37), for a more accurate approximation of the American put option price P,

$$P = \frac{(P_1 - 8P_2 + 9P_3)}{2}. \tag{38}$$

However, in their Table 1 (Huang *et al.* (1996, p.292)) use the following four-point Richardson extrapolation scheme,

$$P = \frac{32P_4}{3} - 13.5P_3 + 4P_2 - \frac{P_1}{6}, \tag{39}$$

rather than the three-point extrapolation scheme. Huang *et al.* (1996) argues that their method is efficient enough that it is comparable to Geske and Johnson (1984), despite using a less efficient extrapolation scheme.

Some important observations. First, note that the P_1, P_2, and P_3 only involve the univariate normal distribution function. Second, the integrands in P_1, P_2, and P_3 are assumed to be time invariant between each successive boundary point. Third, only three boundary points $B_{\frac{T}{3}}$,$B_{\frac{T}{2}}$, and $B_{\frac{2T}{3}}$ need to be determined in order to calculate P. Huang *et al.* (1996) borrowed the ide from Kim (1990) that equation (12) can be numerically solved if one divides the entire interval $[0, T]$ into n subintervals $[t_{i-1}, t_i]$ with length $\triangle = t_i - t_{i-1}$, $i = 1, ..., n$, $t_n = T$. Recalling that the early exercise boundary is governed by (11), hence equation (12) can be solved recursively going one time step backwards $B_{t_{n-1}}$, creating a system of nonlinear equations,

$$\begin{aligned} K - B_{t_{n-1}} &= p_0(B_{t_{n-1}}, K, \triangle) \\ &\quad + \int_{t_{n-1}}^{t_n} rKe^{-r(t_{n-1}-s)}(d_-(B_{t_{n-1}}, B_s, s - t_{n-1}))ds \\ &\quad - \int_{t_{n-1}}^{t_n} rB_{t_{n-1}}e^{-\delta(t_{n-1}-s)}(d_+(B_{t_{n-1}}, B_s, s - t_{n-1}))ds. \end{aligned} \tag{40}$$

One gets $B_{t_{n-1}}$ by approximating the integral using the trapezoid rule (see Press *et al.* (1996)). From this approach one can time discretisize three (or more) points on the early exercise boundary, for equations (32)-(34) and equations (35)-(37) specifically $B_{\frac{T}{3}}$,$B_{\frac{T}{2}}$, and $B_{\frac{2T}{3}}$.

2.3 A Piecewise Exponential Approximation

Instead of approximating the integral and integrand in (7) by multipiece constant functions, Ju (1998) proposed a method of approximating the early exercise boundary by multipiece exponential functions, which can be evaluated in closed form. Ju (1998) motivates this by acknowledging that approximating the early exercise boundary by multipiece constant functions where the integrands are univariate normal distribution functions is indeed efficient however not very accurate. With this in mind, Ju (1998) proposed instead an approximation method based on equation (7) where the optimal boundary argument B_t is replaced by the exponential function Be^{bt}, which then permits closed form integral equations. Ju (1998) felt that exponential functions are able to better capture the nature of the early exercise boundary than constant functions.

In order to incorporate exponential functions as the boundary arguments Ju (1998) uses the following set of integrals

$$I_1 = \int_{t_1}^{t_2} re^{-rt} N(d_-(S_t, Be^{bt}, t))dt, \tag{41}$$

$$I_2 = \int_{t_1}^{t_2} \delta e^{-r\delta} N(d_+(S_t, Be^{bt}, t))dt, \tag{42}$$

where Be^{bt} is the exponential function with bases B and exponents b that need to be determined *a priori* (we will return to this point later). Using, $x_1 = (r - \delta - b - \sigma^2/2)/\sigma$, $x_2 = \log(S_t/B)/\sigma$ and $x_3 = \sqrt{x_1^2 + 2r}$ the I_1 integral and normal distribution function $N(d_-(S_t, Be^{bt}, t))$ can be evaluated in closed form by (for further details Ju (1998, p.631-632)),

$$\begin{aligned} I_1 =& e^{-rt_1} N\left(x_1\sqrt{t_1} + \frac{x_2}{\sqrt{t_1}}\right) - e^{-rt_2} N\left(x_1\sqrt{t_2} + \frac{x_2}{\sqrt{t_2}}\right) \\ &+ \frac{1}{2}\left(\frac{x_1}{x_3} + 1\right) e^{x_2(x_3 - x_1)} \left(N\left(x_3\sqrt{t_2} + \frac{x_2}{\sqrt{t_2}}\right)\right) \\ &- \left(N\left(x_3\sqrt{t_1} + \frac{x_2}{\sqrt{t_1}}\right)\right) \\ &+ \frac{1}{2}\left(\frac{x_1}{x_3} + 1\right) e^{-x_2(x_3 - x_1)} \left(N\left(x_3\sqrt{t_2} - \frac{x_2}{\sqrt{t_2}}\right)\right) \\ &- \left(N\left(x_3\sqrt{t_1} + \frac{x_2}{\sqrt{t_1}}\right)\right). \end{aligned} \tag{43}$$

Similarly, using $y_1 = (r - \delta - b - \sigma^2/2)/\sigma$, $y_2 = \log(S_t/B)/\sigma$ and $y_3 = \sqrt{y_1^2 + 2\delta}$ the I_2 integral and normal distribution function $N(d_+(S_t, Be^{bt}, t))$ can be evaluated in closed form by

$$\begin{aligned} I_2 =& e^{-rt_1} N\left(y_1\sqrt{t_1} + \frac{y_2}{\sqrt{t_1}}\right) - e^{-rt_2} N\left(y_1\sqrt{t_2} + \frac{y_2}{\sqrt{t_2}}\right) \\ &+ \frac{1}{2}\left(\frac{y_1}{y_3} + 1\right) e^{y_2(y_3 - y_1)} \left(N\left(y_3\sqrt{t_2} + \frac{y_2}{\sqrt{t_2}}\right)\right) \\ &- \left(N\left(y_3\sqrt{t_1} + \frac{y_2}{\sqrt{t_1}}\right)\right) \\ &+ \frac{1}{2}\left(\frac{y_1}{y_3} + 1\right) e^{-y_2(y_3 - y_1)} \left(N\left(y_3\sqrt{t_2} - \frac{y_2}{\sqrt{t_2}}\right)\right) \\ &- \left(N\left(y_3\sqrt{t_1} + \frac{y_2}{\sqrt{t_1}}\right)\right). \end{aligned} \tag{44}$$

To simplify the notations (41) and (42) can be expressed as

$$I_1 = I(t_1, t_2, S_t, B, b, -1, r), \tag{45}$$
$$I_2 = I(t_1, t_2, S_t, B, b, 1, \delta), \tag{46}$$

respectively. The possibility of evaluating the premium integral, for which the integrand contains Be^{bt}, in closed form is key.

In what follows the methodology is similar to that of Huang *et al.* (1996). Ju (1998) assumes, P_n, where $n = 1, 2, 3$, to be the approximate put option prices which are combined by a three-point Richardson extrapolation technique. The theme is such that $B_{11}e^{b_{11}t}$ corresponds to the one-piece exponential function, $B_{21}e^{b_{21}t}$, $B_{21}e^{b_{21}t}$, corresponds to the two-piece exponential function, $B_{31}e^{b_{31}t}$, $B_{32}e^{b_{32}t}$, $B_{33}e^{b_{33}t}$, corresponds to the three-piece exponential function. Subsequent put options would follow similar patterns.

Ju (1998) defines P_1, P_2, P_3, as

$$P_1 = \begin{cases} P_E + K(1-e^{-rT}) - S_t(1-e^{-\delta T}) \\ -KI(0, T, S_t, B_{11}, b_{11}, -1, r) \\ +S_t I(0, T, S_t, B_{11}, b_{11}, 1, \delta) \quad \text{if } S_t > B_{11} \\ K - S_t \quad \text{if } S_t \leq B_{11}. \end{cases} \tag{47}$$

$$P_2 = \begin{cases} P_E + K(1-e^{-rT}) - S_t(1-e^{-\delta T}) \\ -KI(0, T/2, S_t, B_{21}, b_{21}, -1, r) \\ +S_t I(0, T/2, S_t, B_{21}, b_{21}, 1, \delta) \\ -KI(T/2, T, S_t, B_{21}, b_{21}, -1, r) \\ +S_t I(T/2, T, S_t, B_{21}, b_{21}, 1, \delta) \quad \text{if } S_t > B_{22} \\ K - S_t \quad \text{if } S_t \leq B_{22}. \end{cases} \tag{48}$$

$$P_3 = \begin{cases} P_E + K(1-e^{-rT}) - S_t(1-e^{-\delta T}) \\ -KI(0, T/3, S_t, B_{33}, b_3, -1, r) \\ +SI(0, T/3, S_t, B_{33}, b_3, 1, \delta) \\ -KI(T/3, 2T/2, S_t, B_{32}, b_{32}, -1, r) \\ +SI(T/3, 2T/3, S_t, B_{32}, b_{32}, 1, \delta) \\ -KI(2T/3, T, S_t, B_{31}, b_{31}, -1, r) \\ +SI(2T/3, T, S_t, B_{31}, b_{31}, 1, \delta) \quad \text{if } S_t > B_{33} \\ K - S_t \quad \text{if } S_t \leq B_{33}. \end{cases} \tag{49}$$

Some important observations. The first argument in each approximate put option represent a shorthand notation for an equivalent European put. Note also that each approximate put option, is coarsely partitioned into equally spaced subintervals, for which t_1, and t_2 follows from equation (37) and (38). Lastly, it should be noted that the arguments (K, S_t, P_E, δ, r, T) are known *a priori* except for the bases B_m, and exponents b_m where $m = 11, 21, ..., 33$. For this, Ju (1998) used an ingenious 'bottom-up' approach to appropriately determine B_m and b_m for each exponential function respectively.

Ju (1998) starts with B_{11} and b_{11}, since they are the initial coefficients for the one-piece exponential function. Ju (1998) uses the approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987) as starting guesses for (B_{11}, b_{11}), from which to initialise the procedure. For an option with divided ($\delta = 0.12$), volatility ($\sigma = 0.2$), maturity ($T = 3.0$), current stock price ($S_0 = \$80$), strike price ($K = \$100$), and interest ($r = 0.08$), using MacMillan (1986) and Barone-Adesi and Whaley (1987), B_{11} is 52.452. Hence, assuming b_{11} is 0, Ju's (1998) initial guesses for (B_{11}, b_{11}) are (52.452, 0), respectively. Initializing the process by (54.452, 0) using equation,

$$\begin{aligned} K - B_{11}e^{b_{11}T} =& P_E(B_{11}, K, T) + K(1 - e^{-rT}) - B_{11}(1 - e^{-\delta T}) \\ &- KI(0, T, B_{11}, B_{11}, b_{11}, -1, r) \\ &+ B_{11}I(0, T, B_{11}, B_{11}, b_{11}, 1, \delta), \end{aligned} \tag{50}$$

and differentiating with respect to B_{11},

$$\begin{aligned} -1 =& - e^{-\delta T}N(-d_+(B_{11}, K, T)) - (1 - e^{-\delta T}) \\ &- KI_S(0, , B_{11}, B_{11}, b_{11}, -1, r) \\ &+ I(0, T, B_{11}, B_{11}, b_{11}, 1, \delta) \\ &+ B_{11}I_S(0, T, B_{11}, B_{11}, b_{11}, 1, \delta), \end{aligned} \tag{51}$$

(B_{11}, b_{11}) are found to be (54.457, 0.036), respectively. Ju (1998) gradually move up using (B_{11}, b_{11}) as initial guesses for (B_{21}, B_{22}) from which Ju can initialize the process using equation,

$$\begin{aligned} K - B_{21}e^{b_{21}T/2} =& P_E(B_{21}e^{b_{21}T/2}, K, T/2) + K(1 - e^{-rT/2}) \\ &- B_{21}e^{b_{21}T/2}(1 - e^{-\delta T/2}) \\ &- KI(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &+ B_{21}e^{b_{21}T/2}I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta), \end{aligned} \tag{52}$$

and differentiating with respect to $B_{21}e^{b_{21}T/2}$,

$$\begin{aligned} -1 =& - e^{-\delta T/2}N(-d_1(B_{21}e^{b_{21}T/2}, K, T/2)) - (1 - e^{-\delta T/2}) \\ &- KI_S(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &+ I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \\ &+ B_{21}e^{b_{21}T/2}I_S(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta), \end{aligned} \tag{53}$$

(B_{21}, b_{21}) are found to be (52.389, 0.0036), respectively. Finally, Ju (1998) use (B_{21}, b_{21}) as initial guesses for (B_{22}, b_{22}) from which the process can be initialized using equation,

$$
\begin{aligned}
K - B_{22} =& P_E(B_{22}, K, T) + K(1 - e^{-rT}) - B_{22}(1 - e^{-\delta T}) \\
& - KI(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\
& + B_{22}I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\
& - KI(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\
& + B_{22}I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta),
\end{aligned} \tag{54}
$$

and differentiating with respect to $B_{22}e^{b_{22}t}$,

$$
\begin{aligned}
-1 =& - e^{-\delta T/2}N(-d_1(B_{22}, K, T) - (1 - e^{-\delta T}) \\
& - KI_S(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\
& + I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\
& + B_{22}I_S(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\
& - KI_S(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\
& + I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\
& + B_{22}I_S(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta),
\end{aligned} \tag{55}
$$

(B_{22}, b_{22}) are found to be to be (54.453, 0.0307), respectively. Similarly, the bases and exponents of a third piece exponential function can be initialized using (B_{21}, b_{22}) as initial guesses for (B_{31}, b_{31}) where the follow bases and exponents are determined in a similar pattern. Although, the corresponding bases and exponents (B_m, b_m) are discontinuous from one piece to another, they are determined from an elaborate system.

To combine each approximate option prices P_1, P_2, P_3, Ju (1998) proposes the following tree-point Richardson extrapolation,

$$
P = 4.5P_3 - 4P_2 + 0.5P_1, \tag{56}
$$

in order for a more accurate approximation of the American put option price P, to be attained.

3 Numerical Results and Discussion

In this section I present the numerical results, efficiency and accuracy of the approximation methods included herein. Ju's (1998) one-, two-, three-point piecewise exponential function method (hereafter EP1, EP2 and EP3), the three-point extrapolation by Geske and Johnson (1984) (hereafter GJ3), the four- and six-point piecewise constant functions of Huang *et al.* (1996) (hereafter H4, H6). The option prices from the aforementioned methods are compared against the benchmark true (or 'exact') option prices from Cox *et al.* (1979) 10000 time-step binomial method. For intermediate reference, a similar binomial method with 1000 time-step (hereafter BT1000) is also included.

Tables 1-3 reports the prices for the moderate maturity options (def. $T = 0.50, 1.0, 1.50$) in Muthuraman (2008). Tables 5 and 7 reports the prices for the short maturity options (def. $T = 0.25$) considered in Bjerksund and Stensland (1993). Table 4 reports the pricing errors due to Tables 1-3. Table 6 and 8 reports the pricing errors from preceding table, respectively. In Table 9, I also compare the convergence of the unextrapolated prices from EP1, EP2, EP3 against the unextrapolated prices from H2, H4 and H6. A convergence property is displayed from respective method if an increasing number of parameters (or 'pieces') yield smaller pricing errors. Therefore, we must 'unextrapolate' in order to fully appreciate the pricing errors from each 'piece'.

The following considerations are made in all the Tables considered in this paper. The current stock price (S_0) starts from \$80.0, \$90.0, \$100.0, \$110.0, and \$120.0, respectively. For all options considered the strike price is constant ($K = \$100$). Volatility associated with stock prices in Tables 1-3 is 20 percent ($\sigma = 0.2$) and 30 percent ($\sigma = 0.3$), for stock prices considered in Table 5 volatility is 20 percent ($\sigma = 0.2$), and 40 percent ($\sigma = 0.4$) for stock prices considered in Table 7. From our Black-Scholes framework a risk-free interest rate that varies from 4 percent ($r = 0.04$), 5 percent ($r = 0.05$), and 6 percent ($r = 0.06$) is considered in Tables 1-3. Tables 5 and 7 considers risk-free interest rates of 4 percent ($r = 0.04$) and 8 percent ($r = 0.08$), respectively. From the stock a continuous dividend yield of 8 percent ($\delta = 0.08$) is paid in Tables 1-3, 12 percent ($\delta = 0.012$) and 4 percent ($\delta = 0.04$) in Tables 5 and 7. The 'cost of carry' (b), defined as $b = r - \delta$, varies from positive ($b > 0$) to negative ($b < 0$) in Tables 1-3, whereas in Tables 5 and 7 a cost of carry equal to zero ($b = 0$) is also considered. These considerations are made in order for a more 'complete' assessment of accuracy and efficiency of the aforementioned methods. Numerical results from Tables 1-9 are summarised as following.

From Table 9, the quick convergence of Ju's (1998) multipiece exponential functions is evident. The inclusion of EP2, EP3 substantially lowers the pricing errors with respect to EP1, although Ju (1998) argues that in many applications the accuracy of EP1 is still acceptable. From Table 9, the convergence of Huang *et al.* (1996) multipiece constant functions is also evident, however the convergence is not as quick as for the multipiece exponential functions. For example, the inclusion of H6 lowers the root mean squared error (hereafter RMSE) of H2 from 0.3520 to 0.0409 cents, higher still than the comparable EP3 with RMSE of 0.0026 cents. Although GJ3 shares many similarities to Huang *et al.* (1996) and Ju (1998) it critically depends on extrapolation technique for its pricing ability, it is therefore not included in Table 9.

For the moderate maturity options considered in Tables 1-3, from Table 4 I find that EP3 has the lowest pricing errors. This result is unanimous among the pricing error measurement methods included in Table 4, and highlighted by the fact that the pricing errors of EP3 are similar to that of BT1000 despite being considerably more efficient. Also shown in Table 4 is the reduction in pricing errors from H6 with respect to H4, for example mean absolute deviation (hereafter MAD) drops from 0.01991 to 0.0065. While the stand alone GJ3 is shown to have the largest pricing errors with a MAD value of 0.3519.

For the short maturity options considered in Tables 5, and 7, the results are similar. I find that EP3 has the lowest pricing errors, this result is also unanimous among the pricing error measurement methods in Tables 6, and 8, respectively. EP3 achieves a remarkable RMSE equal to 0.001 cents, similarly a MAD equal to 0.001, superior to that of BT1000. Perhaps even more surprising is that both EP3 and H6 produce a mean absolute percentage error (hereafter MAPE) of less than 3 percent. Here, GJ3 fairs better than for the options considered in Tables 1-3, the results of tables 5 and 7 shows an RMSE of less than 0.1168 cents.

Concerning the efficiency of respective approximation method, two patterns emerge. First, BT1000 is the most inefficient, as we would expect since it is a numerical approach whereas the others are approximations essentially. Second, H4 is always more efficient than H6, due to the higher order extrapolation which is less efficient. Third, EP3 and GJ3 are rather similar with respect to efficiency, however both are always more efficient than H6.

Concerning the accuracy of respective approximation method, the following picture emerges. The accuracy of EP3 is superior to the other methods, with an RMSE of 0.0013 cents in Table 4, 0.0001 in Table 5, and 0.0002 in Table 7. Remarkably, in much of the Tables considered in this paper the accuracy of EP3 is akin to that of BT1000. The success of EP3 is further highlighted in Table 9, where even the unextrapolated prices by EP3 show considerably low pricing errors. However, the least accurate approximation methods are GJ3 followed by H4.

From the results above it is therefore reasonable to argue that H6 which is the second most accurate approximation, is more accurate than H4 because it sacrifices efficiency for accuracy. However, generally this may not be the case, as I have shown the more accurate method need not to be the least efficient as is the case with EP3. In summary, the results of the numerical study conducted in this paper agree with the results of the numerical study conducted by Ju (1998) focused on long maturity options (def. $T = 3.0$). The accuracy improvement of EP3 to those of Geske and Johnson (1984) and Huang *et al.* (1996) was previously shown by Ju (1998) to be substantial for long term options, and from this numerical study the same conclusion can be drawn for moderate and short term options as well.

This substantial improvement led Ju (1998) to also study the 'tracking' ability of his three-piece exponential function with respect to the early exercise boundary. Using a finite difference scheme (hereafter FDM) to solve for the partial differential equation (3), Ju (1998) was able to obtain boundary values which he assumed to be the most accurate approximations to the true (or 'exact') early exercise boundary. Surprisingly, Ju (1998) found that the boundary of his three-piece exponential boundary differs substantially from that of the approximated early exercise boundary. Ju (1998) illustrates this in Figure 1, where the considered early exercise boundaries corresponds to the following put option, with $S_0 = \$100$, $K = \$100$, $\delta = 0.04$, $r = 0.08$, $\sigma = 0.2$, and $T = 3$ years. The numerical results of Ju (1998) found that errors associated with pricing the same option (by EP3) were less than 0.0036 cents. In Figure 1 then, the continuous bold line represents Ju's (1998) approximation to the 'true' early exercise boundary, the discontinuous dotted line from left to right represents the multipiece exponential functions by Ju (1998). The first plot represent the one-piece exponential boundary, the second plot represents the two-piece exponential boundary, and the third plot represents the three-piece exponential boundary. Unlike the 'true' boundary, Ju's (1998) multipiece exponential boundary is discontinuous between each 'piece' owing to the fact that the bases and exponents (B_m, b_m) are determined separately for each exponential piece.

From Figure 1 it is evident that more 'pieces' included yield a more accurate representation of the 'true' early exercise boundary. Even so, the three-piece exponential boundary in the third plot differs considerably from the 'true' early exercise boundary. This is surprising especially since the numerical results would indicate otherwise with respect to pricing the same option. This seemingly contradictory result led Ju (1998) to simply state that "the true values do not depend on the exact values of the early exercise boundary critically". Based on Ju's (**?**) own conclusion that "the multipiece exponential boundaries are not very close to the approximation boundary".

In this paper it will be shown however, that the early exercise boundary is indeed very well approximated by the three-piece exponential boundary in Ju (1998). To show this, I use the improved boundary representation formula (18) of Hou *et al.* (2000) in order to obtain more accurate boundary values than those used by Ju (1998) in his representation of the 'true' early exercise boundary in Figure 1. Using the same put option, in Figure 2, the more accurate 'true' early exercise boundary is plotted against the EP3 and H4 boundaries. In Figure 2 then, the bold continuous line depicts the improved 'true' early exercise boundary, the discontinuous multidotted line depicts the 'three-piece' exponential (EP3) boundary of Ju (1998), the dashed line represents the four-piece constant boundary (H4) of Huang *et al.* (1996).

From Figure 2 one can see that the three-piece exponential (EP3) boundary of Ju (1998) 'tracks' the 'true' early exercise boundary rather well, whereas the four-piece constant boundary (H4) of Huang *et al.* (1996) does not. Figure 2 then mainly shows that the three-piece exponential boundary of Ju (1998) 'track' the improved 'true' early exercise boundary more accurately than was shown in plot three in Figure 1. Especially for the two latter 'pieces' starting from $T = 1$, all of whom are evenly partitioned in time, the improvement in 'tracking' is significant compared to the same two latter 'pieces' shown in plot three in Figure 1. Also shown in Figure 2 is that, approximating the integrand in (7) using four 'pieces' (H4), all of whom are constant functions, results in large deviations from the 'true' early exercise boundary. Note here as well that additional pieces, as time to maturity of the option goes to zero ($T \to 3$), always improve the 'tracking' ability of the previous 'piece'. This is a characteristic, featured in both the three-piece exponential boundary and the four-piece boundary.

Figure 2 shows that assuming a more accurate 'true' early exercise boundary may very well alter the conclusion previously made by Ju (1998). Not only may the conclusions by Ju (1998) with regard to the three-piece exponential boundary's inability to 'track' the early exercise boundary change, so to the conclusion regarding the critical importance of precise 'tracking' for the option price. A consequence of the ability to accurately 'track' the early exercise boundary may be the explanation to why some approximations, such as the three-piece exponential function (EP3), outperform other approximations such as the four-piece (H4) constant function of Huang *et al.* (1996).

4 Conclusion

In this paper I have conducted a numerical study of several important approximation methods utilising a time discretisation methodology in order to price American put options. This numerical study has focused on short and moderate maturity options, unlike previous numerical studies involving the same approximation methods. Conclusively, the numerical results have shown the three-piece exponential function by Ju (1998) to yield the smallest pricing errors. This result is in accordance with the numerical study on long maturity options presented in Ju (1998). The pricing accuracy of the three-piece exponential function is akin to that of a 1000 time-step binomial method, despite being many times more efficient. Furthermore, the results show that the three-point extrapolated method of Geske and Johnson (1984) to produce the largest pricing errors. The natural trade-off between accuracy and efficiency have also been displayed by the four-point, and six-point piecewise constant functions of Huang *et al.* (1996).

In this paper I noticed primarily two important ingredients for the success of the multipiece exponential functions by Ju (1998) to price American put options. First, in Section 2.3 the ingenious 'bottom-up' approach adopted by Ju (1998) results in very elaborate, and shown in the numerical study, to be very accurate starting values for his multipiece exponential function. Secondly, the graphical comparison in Figure 2 has shown the three-piece exponential (EP3) boundary of Ju (1998) to accurately 'track' the 'true' early exercise boundary rather well. More importantly, I was able to give an alternative explanation to the success in pricing of EP3 by showing that the corresponding three-piece exponential boundary can 'track' the early exercise boundary accurately, more so than the comparable four-piece constant boundary. This 'improved' tracking ability of the three-piece (EP3) boundary is only noticeable after adopting a more accurate early exercise boundary representation then that of Figure 1. Thus the conclusions reached in this paper are partially in contradiction to the conclusions reached by Ju (1998).

Appendix A

In this appendix the details considering the boundary representation of Hou *et al.* (2000) omitted in the main text are presented. For further details see Hou *et al.* (2000).

If the stock price S_t, where $S_t \in (0, B_t]$, drops below or equal to the early exercise boundary ($S_t \leq B_t$) the option is exercised early. If we let $S_t = \varepsilon B_t$ with $\varepsilon \in (0, 1]$, then B_t is differentiable everywhere with respect to ε. Using this, equation (12) can be written as

$$\begin{aligned} K - \varepsilon B_t = & K e^{-r(T-t)} N(-d_-(\varepsilon B_t, K, T-t)) \\ & - \varepsilon B_t e^{-\delta(T-t)} N(-d_+(\varepsilon B_t, K, T-t)) \\ & + \int_t^T r K e^{-r(s-t)} N(-d_-(\varepsilon B_t, B_s, s-t)) ds \\ & - \delta \varepsilon B_t e^{-\delta(s-t)} N(-d_+(\varepsilon B_t, B_s, s-t)) ds, \end{aligned} \tag{A1}$$

rearranging terms we have that

$$\begin{aligned} & \varepsilon B_t \left\{ 1 - e^{-\delta(T-t)} N\big(-d_+(\varepsilon B_t, K, T-t)\big) - \delta \int_t^T e^{-\delta(s-t)} N\big(-d_+(\varepsilon B_t, B_s, s-t)\big)\, ds \right\} \\ & = K \left\{ 1 - e^{-r(T-t)} N\big(-d_-(\varepsilon B_t, K, T-t)\big) - r \int_t^T e^{-\delta(s-t)} N\big(-d_-(\varepsilon B_t, B_s, s-t)\big)\, ds \right\}. \end{aligned} \tag{A2}$$

Using the identity $d_+(\varepsilon B_t, B_s, s-t) - d_-(\varepsilon B_t, B_s, s-t) = \sigma\sqrt{s-t}$, we can express the relation in (A2) as,

$$A(\varepsilon) B_t = B(\varepsilon) \tag{A3}$$

where

$$\lim_{\varepsilon\to 1} A(\varepsilon) = 1 + \frac{\delta\sqrt{T-t}}{\sigma} e^{\delta(T-t)+\frac{1}{2}d_+^2(B_t,K,T-t)} \times \left\{ \sqrt{2\pi} - \int_t^T e^{-\delta(s-t)-\frac{1}{2}\mu_+^2(B_t,B_s,s-t)} \left(\frac{d_+(B_t,B_s,s-t)}{s} \right) \, ds \right\}, \quad \text{(A4)}$$

and

$$\lim_{\varepsilon\to 1} B(\varepsilon) = \frac{Kr}{\sigma}\sqrt{T-t} e^{\delta(T-t)+\frac{1}{2}d_+^2(B_t,K,T-t)} \times \left\{ \sqrt{2\pi} - \int_t^T e^{-\delta(s-t)-\frac{1}{2}\mu_-^2(B_t,B_s,s-t)} \left(\frac{d_+(B_t,B_s,s-t)}{s} \right) \, ds \right\}. \quad \text{(A5)}$$

The boundary can now be represented as an integral for any $(T-t) > 0$,

$$\begin{aligned} & B_t + B_t \frac{\delta\sqrt{T-t}}{\sigma} e^{\delta(T-t)+\frac{1}{2}d_+^2(B_t,K,T-t)} \times \\ & \left\{ \sqrt{2\pi} - \int_t^T e^{-\delta(s-t)-\frac{1}{2}\mu_-^2(B_t,B_s,s-t)} \left(\frac{d_+(B_t,B_s,s-t)}{s} \right) \, ds \right\} \\ & = \frac{Kr}{\sigma}(\sqrt{T-t}) e^{\delta(T-t)+\frac{1}{2}d_+^2(B_t,K,T-t)} \times \\ & \left\{ \sqrt{2\pi} - \int_t^T e^{-r(s-t)-\frac{1}{2}\mu_+^2(B_t,B_s,s-t)} \left(\frac{d_-(B_t,B_s,s-t)}{s} \right) \, ds \right\}, \end{aligned} \quad \text{(A6)}$$

rewriting equation (A6) we arrive at the new early exercises boundary

$$\begin{aligned} B_t & \left\{ \sigma e^{-\delta(T-t)-\frac{1}{2}d_+^2(B_t,K,T-t)} + \delta\sqrt{2\pi(T-t)} \right\} = Kr\sqrt{2\pi(T-t)} \\ & + \delta B_t \sqrt{T-t} \int_t^T e^{-\delta(s-t)-\frac{1}{2}d_+^2(B_t,B_s,s-t)} \left(\frac{d_-(B_t,B_s,s-t)}{s} \right) ds \\ & - Kr\sqrt{t} \int_t^T e^{-r(s-t)-\frac{1}{2}\mu_-^2(B_t,B_s,s-t)} \left(\frac{d_+(B_t,B_s,s-t)}{s} \right) ds, \end{aligned} \quad \text{(A7)}$$

where in the absence of dividend ($\delta = 0$) equation (A7) collapses to

$$
\begin{aligned}
B_t =& Kre^{\frac{1}{2}d_+^2(B_t,K,T-t)}\sqrt{2\pi(T-t)} \\
&- Kre^{\frac{1}{2}d_+^2(B_t,K,T-t)}\sqrt{T-t}\times \\
&\int_t^T e^{-r(s-t)-\frac{1}{2}\mu_-^2(B_t,B_s,s-t)}\left(\frac{d_+(B_t,B_s,s-t)}{s}\right)ds.
\end{aligned}
\tag{A8}
$$

Appendix B

In this appendix the details considering the pricing error measurement methods omitted in the main text are presented. The pricing error measurement methods considered in this paper are as follows, the root mean squared error (RMSE), mean absolute deviation (MAD), and mean absolute percentage error (MAPE). For further details see Chris (2008, p.292-294).

The root mean squared error approach is considered in this paper due to the possibility of aggregating pricing errors (RMSE) without the positive and negative errors canceling each other out. From the root mean squared error large pricing errors are disproportionately treated more severe than smaller errors. If the n-step ahead option price 'forecast' error at time t is defined as, $f_{t,n}$ and the true option price at time t as $P_{t,n}$, then the root mean squared error is defined as

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(P_{t+n} - f_{t,n})^2}, \tag{B1}$$

where T is the entire set of options. Alternatively, the mean absolute deviation (MAD) pricing error measurement considers large pricing errors similar to small pricing errors. Similarly, if the n-step ahead option price 'forecast' error at time t is defined as, $f_{t,n}$ and the true option price at time t as $P_{t,n}$, then the mean absolute deviation is defined as

$$MAD = \frac{1}{T}\sum_{t=1}^{T}|P_{t+n} - f_{t,n}|. \tag{B2}$$

However, from Makridakis (1993, p.528) I also included the mean absolute deviation percentage error, which incorporates the distinguishing features of the two aforementioned pricing error measurements. Hence, the absolute difference in pricing is divided by the true option price before dividing with the entire set of options T,

$$MAPE = \frac{\sum_{t=1}^{T} \left| \frac{P_{t+n} - f_{t,n}}{P_{t+n}} \right|}{T} \times 100. \tag{B3}$$

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Table 1. American put option prices with strike price ($K = \$100$) and fixed maturity ($T = 0.5$ years). The following inputs are abbreviated as initial stock price (S_0), risk-free interest rate (r), volatility of the stock price (σ), and dividend yield (δ).

(S_0, r, σ, δ)	TRUE	BT1000	GJ3	H4	H6	EP3
(80,0.04,0.2,0.00)	20.0000	20.0000	19.4441	20.0146	19.9938	20.0000
(120,0.04,0.2,0.08)	0.9363	0.9364	0.9363	0.9363	0.9363	0.9363
(100,0.05,0.3,0.08)	8.9238	8.9220	8.9237	8.9242	8.9241	8.9241
(90,0.06,0.2,0.00)	10.5221	10.5221	10.2567	10.5171	10.5105	10.5223
(110,0.06,0.3,0.08)	4.9237	4.9228	4.9223	4.9234	4.9236	4.9235
CPU time (seconds)		15.85	0.491	0.279	0.492	0.462

Column 1 represents the chosen input variation. Column 2 represents the true option prices from Cox *et al.* (1979) $N = 10000$ time-step binomial method (TRUE). Column 3 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 4 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 5-6 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 7 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Final row represents CPU time (measured in seconds) it took by respective methods to price the entire set of options.

TABLE 2. American put option prices with strike price (K = \$100) and fixed maturity ($T = 1.0$ years). The following inputs are abbreviated as initial stock price (S_0), risk-free interest rate (r), volatility of the stock price (σ), and dividend yield (δ).

(S_0, r, σ, δ)	TRUE	BT1000	GJ3	H4	H6	EP3
(80,0.04,0.2,0.00)	20.0108	20.0103	19.2910	20.0766	20.0193	20.0109
(120,0.04,0.2,0.08)	2.8824	2.8831	2.8823	2.8823	2.8823	2.8823
(100,0.05,0.3,0.08)	12.6472	12.6447	12.6421	12.6490	12.6486	12.6477
(90,0.06,0.2,0.00)	11.2167	11.2171	10.7744	11.1776	11.2002	11.2171
(110,0.06,0.3,0.08)	8.3730	8.3705	8.3611	8.3744	8.3751	8.3727
CPU time (seconds)		16.88	0.583	0.367	0.625	0.471

Column 1 represents the chosen input variation. Column 2 represents the true option prices from Cox *et al.* (1979) $N = 10000$ time-step binomial method (TRUE). Column 3 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 4 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 5-6 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 7 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Final row represents CPU time (measured in seconds) it took by respective methods to price the entire set of options.

Table 3. American put option prices with strike price ($K = \$100$) and fixed maturity ($T = 1.5$ years). The following inputs are abbreviated as initial stock price (S_0), risk-free interest rate (r), volatility of the stock price (σ), and dividend yield (δ).

(S_0, r, σ, δ)	TRUE	BT1000	GJ3	H4	H6	EP3
(80,0.04,0.2,0.00)	20.1388	20.1385	19.2899	20.2251	20.1535	20.1390
(120,0.04,0.2,0.08)	4.8368	4.8370	4.8368	4.8369	4.8369	4.8369
(100,0.05,0.3,0.08)	15.4118	15.4088	15.3933	15.4196	15.4164	15.4123
(90,0.06,0.2,0.00)	11.7292	11.7289	11.1100	11.6698	11.7077	11.7297
(110,0.06,0.3,0.08)	10.9810	10.9801	10.9467	10.9926	10.9871	10.9809
CPU time (seconds)		18.75	0.611	0.421	0.695	0.537

Column 1 represents the chosen input variation. Column 2 represents the true option prices from Cox *et al.* (1979) $N = 10000$ time-step binomial method (TRUE). Column 3 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 4 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 5-6 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 7 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Final row represents CPU time (measured in seconds) it took by respective methods to price the entire set of options.

TABLE 4. Summary of pricing errors due to put option prices considered in Tables 1-3.

Methods	BT1000	GJ3	H4	H6	EP3
RMSE	0.0018	0.3821	0.0347	0.0097	0.0013
MAD	0.0012	0.23519	0.01991	0.0065	0.0005
MAPE	0.0124	1.5386	0.1307	0.0496	0.0040

Column 1 represents the pricing error measurement methods. Column 2 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 3 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 4-5 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 6 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Row 1 represents the root mean squared error (RMSE). Row 2 represents the mean absolute deviation (MAD). Row 3 represents the mean absolute percentage error (MAPE). For further details on the pricing error measurement methods see Appendix B.

TABLE 5. American put option prices with strike price ($K = \$100$) fixed volatility ($\sigma = 0.2$) and maturity ($T = 0.25$ years). The following inputs are abbreviated as dividend yield (δ), risk-free interest rate (r), and initial stock price (S_0).

(δ, r, S_0,)	TRUE	BT1000	GJ3	H4	H6	EP3
(0.12,0.08,80)	20.4140	20.4140	20.4142	20.4141	20.4141	20.4140
(0.12,0.08,90)	11.2498	11.2500	11.2498	11.2498	11.2498	11.2498
(0.12,0.08,100)	4.3963	4.3954	4.3964	4.3964	4.3964	4.3964
(0.12,0.08,110)	1.1178	1.1172	1.1178	1.1178	1.1178	1.1178
(0.12,0.08,120)	0.1844	0.1841	0.1844	0.1844	0.1844	0.1844
(0.04,0.08,80)	20.0000	20.0000	19.9931	20.0078	20.0001	20.0000
(0.04,0.08,90)	10.2237	10.2239	10.2475	10.2304	10.2164	10.2237
(0.04,0.08,100)	3.5480	3.5475	3.5430	3.5628	3.5471	3.5484
(0.04,0.08,110)	0.7898	0.7894	0.7889	0.7900	0.7904	0.7900
(0.04,0.08,120)	0.1135	0.1134	0.1128	0.1139	0.1137	0.1136
(0.04,0.04,80)	20.0008	20.0008	20.0008	20.0112	20.0027	20.0008
(0.04,0.04,90)	10.6436	10.6438	10.6411	10.6449	10.6444	10.6435
(0.04,0.04,100)	3.9545	3.9537	3.9556	3.9550	3.9551	3.9547
(0.04,0.04,110)	0.9453	0.9447	0.9450	0.9454	0.9454	0.9453
(0.04,0.04,120)	0.1459	0.1457	0.1459	0.1459	0.1459	0.1460
CPU time (seconds)		114.68	4.12	2.37	4.42	4.36

Column 1 represents the chosen input variation. Column 2 represents the true option prices from Cox *et al.* (1979) $N = 10000$ time-step binomial method (TRUE). Column 3 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 4 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 5-6 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 7 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Final row represents CPU time (measured in seconds) it took by respective methods to price the entire set of options.

TABLE 6. Summary of pricing errors due to put option prices considered in Table 5.

Methods	BT1000	GJ3	H4	H6	EP3
RMSE	0.0004	0.0065	0.0053	0.0019	0.0001
MAD	0.0003	0.0027	0.0028	0.0008	0.0001
MAPE	0.0410	0.0816	0.0659	0.02633	0.0134

Column 1 represents the pricing error measurement methods. Column 2 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 3 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 4-5 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 6 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Row 1 represents the root mean squared error (RMSE). Row 2 represents the mean absolute deviation (MAD). Row 3 represents the mean absolute percentage error (MAPE). For further details on the pricing error measurement methods see Appendix B.

TABLE 7. American put option prices with strike price ($K = \$100$) fixed volatility ($\sigma = 0.4$) and maturity ($T = 0.25$ years). The following inputs are abbreviated as dividend yield (δ), risk-free interest rate (r), volatility (σ), and initial stock price (S_0).

(δ, r, S_0)	TRUE	BT1000	GJ3	H4	H6	EP3
(0.12,0.08,80)	21.4445	21.4449	21.4439	21.4470	21.4444	21.4445
(0.12,0.08,90)	13.9153	13.9162	13.9160	13.9158	13.9156	13.9152
(0.12,0.08,100)	8.2668	8.26511	8.2673	8.2671	8.2671	8.2671
(0.12,0.08,110)	4.5190	4.5206	4.5190	4.5190	4.5190	4.5190
(0.12,0.08,120)	2.2944	2.2952	2.2943	2.2943	2.2943	2.2943
(0.04,0.08,80)	20.5875	20.5879	20.6014	20.5817	20.5832	20.5876
(0.04,0.08,90)	12.9581	12.9592	12.9435	12.9663	12.9615	12.9582
(0.04,0.08,100)	7.4596	7.4583	7.4591	7.4765	7.4621	7.4603
(0.04,0.08,110)	3.9501	3.9519	3.9509	3.9524	3.9530	3.9505
(0.04,0.08,120)	1.9434	1.9443	1.9415	1.9439	1.9440	1.9436
(0.04,0.04,80)	21.0534	21.0539	20.6014	21.0545	21.0560	21.0534
(0.04,0.04,90)	13.4867	13.4878	13.4883	13.4923	13.4873	13.4868
(0.04,0.04,100)	7.8992	7.8975	7.9013	7.9001	7.9004	7.8996
(0.04,0.04,110)	4.2544	4.2561	4.2543	4.2546	4.2546	4.2545
(0.04,0.04,120)	2.1279	2.1287	2.1272	2.1282	2.1280	2.1279
CPU time (seconds)		134.68	5.01	3.98	5.03	4.88

Column 1 represents the chosen input variation. Column 2 represents the true option prices from Cox *et al.* (1979) $N = 10000$ time-step binomial method (TRUE). Column 3 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 4 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 5-6 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 7 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Final row represents CPU time (measured in seconds) it took by respective methods to price the entire set of options.

Table 8. Summary of pricing errors due to put option prices considered in Table 6.

Methods	BT1000	GJ3	H4	H6	EP3
RMSE	0.0012	0.1168	0.0053	0.0019	0.0002
MAD	0.0011	0.0326	0.003	0.0012	0.0002
MAPE	0.0218	0.1659	0.0334	0.0160	0.0032

Column 1 represents the pricing error measurement methods. Column 2 represents the option prices from Cox *et al.* (1979) $N = 1000$ time-step binomial method (BT1000). Column 3 represents the option prices from Geske and Johnson (1984) three-point extrapolation (GJ3). Columns 4-5 represents the option prices from Huang *et al.* (1996) four-point extrapolation of the four-piece constant function (H4), and six-point extrapolation of the six-piece constant function (H6). Column 6 represents the option prices from Ju (1998) three-point extrapolation of the three-piece exponential function (EP3). Row 1 represents the root mean squared error (RMSE). Row 2 represents the mean absolute deviation (MAD). Row 3 represents the mean absolute percentage error (MAPE). For further details on the pricing error measurement methods see Appendix B.

TABLE 9. Unextrapolated American put option prices with fixed strike price ($K = \$100$) and maturity ($T = 1.5$ years). The following inputs are abbreviated as initial stock price (S_0), risk-free interest rate (r), volatility of the stock price (σ), and dividend yield (δ).

(S_0, r, σ, δ)	TRUE	H2	H4	H6	EP1	EP2	EP3
(80,0.04,0.2,0.00)	20.1388	19.3770	20.0322	20.1500	20.1353	20.1368	20.1374
(120,0.04,0.2,0.08)	4.8368	4.8369	4.8369	4.8369	4.8369	4.8369	4.8369
(100,0.05,0.3,0.08)	15.4118	15.3839	15.4002	15.4059	15.4118	15.4120	15.4121
(90,0.06,0.2,0.00)	11.7292	11.5190	11.9029	11.9035	11.7162	11.7219	11.7244
(110,0.06,0.3,0.08)	10.9810	10.9374	10.9642	10.9727	10.9787	10.9803	10.9808
RMSE	0.0000	0.3520	0.0904	0.0784	0.0059	0.0035	0.0026
MAD	0.0000	0.20772	0.0607	0.0409	0.0034	0.0022	0.0018
MAPE	0.0000	1.2262	0.4431	0.3364	0.0282	0.0173	0.0129

Column 1 represents the chosen input variation. Column 2 represents the true option price from Cox *et al.* (1979) $N = 10000$ time-step binomial method (TRUE). Columns 3-5 represents Huang *et al.* (1996) two-piece unextrapolated constant function (H2), four-piece unextrapolated constant function (H4), and six-piece unextrapolated constant function (H6). Column 6-8 represents the option prices from Ju (1998) one-piece unextrapolated exponential function (EP1), two-piece unextrapolated exponential function (EP2), and three-piece unextrapolated exponential function (EP3). Row 1 represents the root mean squared error (RMSE). Row 2 represents the mean absolute deviation (MAD). Row 3 represents the mean absolute percentage error (MAPE). For further details on the pricing error measurement methods see Appendix B.

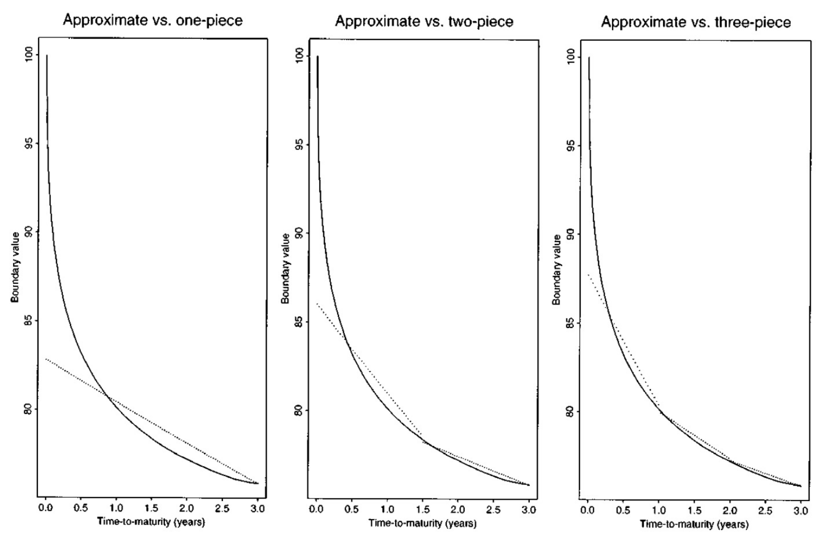

FIGURE 1. Approximations to the early exercise boundary by multipiece exponential function boundaries. The continuous bold line represents the 'true' early exercise boundary, plotted against the dotted one-piece exponential boundary, two-piece exponential boundary, and three-piece exponential boundary, respectively. The early exercise boundaries considered correspond to the following put option, with initial stock price (S_0 = \$100), fixed strike price (K = \$100), dividend yield ($\delta = 0.04$), risk-free interest rate ($r = 0.08$), volatility of the stock price ($\sigma = 0.2$) and time to maturity ($T = 3$ years). This is Figure 1 in Ju (1998, p.645).

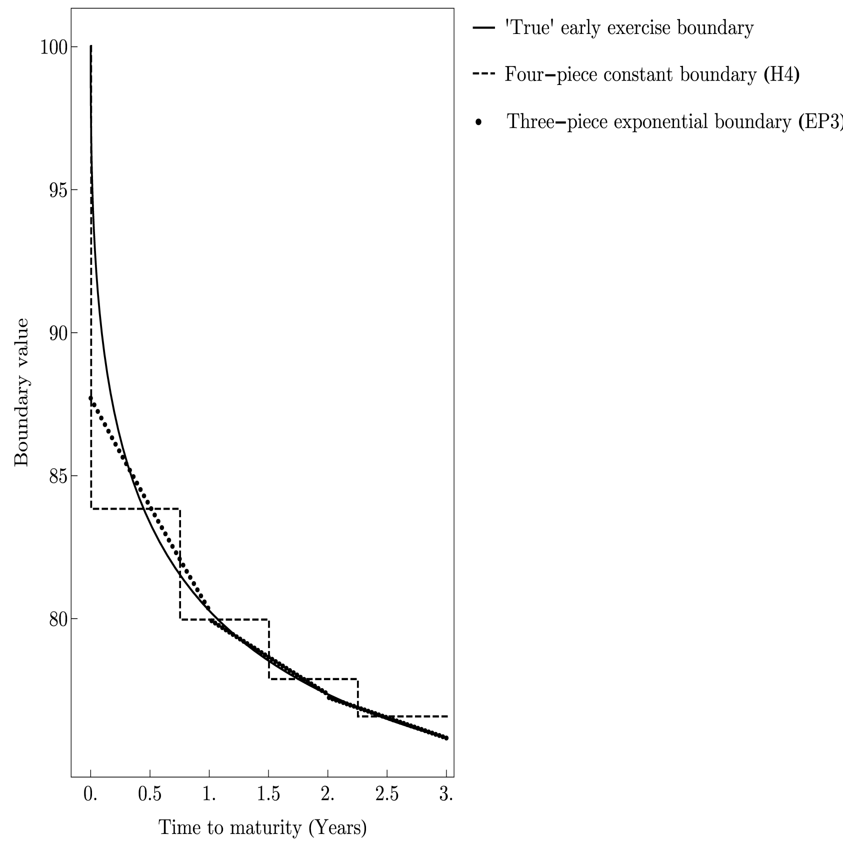

FIGURE 2. Approximations to the 'true' early exercise boundary derived from Hou *et al.* (2000). A continuous bold line represents the 'true' early exercise boundary plotted against the dotted three-piece exponential boundary (EP3) and the dashed four-point constant function boundary (H4). The early exercise boundaries considered correspond to the following put option, with initial stock price ($S_0 = \$100$), fixed strike price ($K = \$100$), dividend yield ($\delta = 0.04$), risk-free interest rate ($r = 0.08$), volatility of the stock price ($\sigma = 0.2$) and time to maturity ($T = 3$ years).