Minimality-preserving Ribaucour transformations

UNIVERSITY OF GOTHENBURG

Aksel Bergfeldt

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Aksel Bergfeldt 2nd July 2016

Degree project for Master of Science in Mathematics,
Department of Mathematics, University of Gothenburg.

Supervisor:
Jens Hoppe, KTH Royal Institute of Technology

Examiner:
Bo Berndtsson, Chalmers University of Technology

Abstract

The Ribaucour transformation classically relates surfaces via a sphere congruence that preserves lines of curvature. In this report, we generalise the concept to submanifolds of arbitrary dimension and codimension, and formulate a condition for minimality conservation. In addition, the transformation is carried out on an example surface.

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Chapter 1

Introduction

1.1 Motivation

Minimality conditions is a classical area of interest in mathematics, with many applications in physics. Of particular popularity is Lagrangians proportional tosubmanifold volume, such as in string and membrane theory , whose extreme corresponds to vanishing mean curvature. The main motivation for this project is to construct more tools to produce minimal submanifolds in higher dimensions.

1.2 Notation and conventions

We will be concerned with Riemannian manifolds and their tangent bundles. In this setting, it will be convenient to always suppress the point in the manifold that we are taking the tangent space of, and hence write TM for the tangent space of the manifold M at the current point. Because of this, we write the bundle of smooth tangent fields of M as $\mathrm{T}M$. Likewise, for maps defined on M, the function name is given to represent its image when there is no risk of confusion.

The Levi-Civita connection is always denoted ∇, even when the manifold is flat, and it acts solely on the adjacent object to the right. When no vector is given as a subscript, the combined object has a covectorial part. When more than one manifold is present, the context shows which connection is used.

Our convention is that tensors are vectors to the left and covectors to the right, and act on the adjacent object when possible. In the rare case of ambiguous configurations, we adopt a simple notation that indicates what objects has a non-acting vectorial part with a dot above, respectively covectorial part, indicated with a dot below. For example, if V is a vector and $\varPhi$ is an operator, the expressions $\nabla V\varPhi$ and $\varPhi\nabla V$ are unambiguous, while

$\nabla\Phi V$ has the interpretations

$$\underset{\cdot}{\nabla}\dot{\Phi}V \quad \text{and} \quad \underset{\cdot}{\nabla}\dot{\Phi}V = \nabla_V\Phi.$$

The reader familiar with index notation could view this as an indication of free indices.

We shall make extensive use of the canonical isomorphism between the tangent space and cotangent space, defined by the metric; for the metric g and $X, Y \in TM$, $X^\flat Y \doteq g(X,Y)$. Again, the index-oriented reader can relate to the application of $\flat$ as lowering of a higher index. While a similar functionality is obtained with the two-way adjoint, these one-way isomorphisms are more useful in a tensor context; on a tensor, $\flat$ simply acts on the vectorial part.

For a linear map $f : M \to E$ between Riemannian manifolds (M, g), $(E, \langle\rangle)$, the push forward $f_* : TM \to T_f E$ can have a well-defined adjoint in relation to the metrics g, $\langle\rangle$, which must satisfy:

$$\langle f_* X, V\rangle \doteq g(X, f_*^\dagger V), \quad \forall X \in TM,\ \forall V \in T_f N.$$

When f is an isometry, one then finds that $f_*^\dagger = f_*^{-1}$, which may be taken as a definition of isometry.

As we will be working in a real context, we shall take the expression *(totally) symmetric* to mean the same as self-adjoint for an operator, and for a general tensor that when $\flat$ is applied to all its vectorial parts, the result linear map is symmetric under interchange of (all) arguments.

In particular, we shall make use of the following:

Definition 1.1. A symmetric tensor T is *Codazzi* if ∇T is totally symmetric.

We will also make use of the following property of the torsion-free connection: Let d denote the exterior derivative. When acting on a covector u, we then have:

$$\begin{aligned} \mathrm{d}u(X,Y) &= Xu(Y) - Yu(X) - u([X,Y]) \\ &= Xu(Y) - Yu(X) - u(\nabla_X Y) + u(\nabla_Y X) \\ &= \nabla u(X,Y) - \nabla u(Y,X), \end{aligned} \tag{1.1}$$

for vectors X, Y and covector u, so d is the antisymmetrisation of ∇ when acting on covectors.

1.3 Submanifolds

Let $f : M \to E$ be an immersion of an m-dimensional manifold (M, g) into an $m+n$-dimensional manifold $(E, \langle\rangle)$. The tangent bundles are then related by $f_* : \mathrm{T}M \to f^*\mathrm{T}E \subset M \times \mathrm{T}E$, for $f^*\mathrm{T}E$ the pull back bundle of f, which is the points of $M \times \mathrm{T}E$ where the projection of $\mathrm{T}E$ hits the corresponding image of f. The pull back bundle hence has base space M and fibre $(f, T_f E)$, and we will abuse notation and identify the latter with $T_f E$.

We can then write the tangent space $T_f E$ as the sum $T_f E = T_f^\top E + T_f^\perp E$, where $T_f^\top E \doteq f_*(TM)$ and $T_f^\perp E \perp T_f^\top E$. These will be called the tangential and normal part of $T_f E$.

Likewise, we take tangential and normal parts of vectors. In particular, for the Levi-Civita connection acting on a smooth vector field $V : M \to T_f E$ (i. e. V is a section of $f^*\mathrm{T}E$), the corresponding parts are

$$\begin{aligned} \nabla V &= \nabla^\top V^\top + \nabla^\top V^\perp + \nabla^\perp V^\top + \nabla^\perp V^\perp \\ &= f_* \nabla f_*^{-1} V^\top - f_* \langle A, V^\perp \rangle + g(A, f_*^{-1} V^\top) + \nabla^\perp V^\perp. \end{aligned} \tag{1.2}$$

Here, f_*^{-1} is the inverse of f_*, which is well-defined on $T_f^\top E$ when f is an immersion, and A is the shape tensor, which is the smoothly varying linear map $TM \to TM \otimes T_f^\perp E$ defined by the above expression. The shape tensor is therefore related to the second fundamental form α as $A^\flat = \alpha$, and it is symmetric as an operator $TM \to TM$.

For a normal vector $\Xi \in T^\perp E$, the shape operator $A_\Xi \doteq \langle A, \Xi \rangle$ has as eigenvalues the principal curvatures related to Ξ. The corresponding eigenvectors are called principal directions (related to Ξ). Since each A_Ξ is symmetric and the eigenvalues necessarily are real, these directions can always be taken orthogonal.

We will be interested in the mean curvature, which in this general setting is the vector $\operatorname{tr} A$, where the trace is taken on the operator part. For each normal vector Ξ, the corresponding mean curvature is hence the sum of the eigenvalues of A_Ξ. A minimal submanifold has everywhere vanishing mean curvature.

1.4 Flat enveloping space

In the following analysis the enveloping manifold E will be Euclidean space or pseudo-Euclidean space. This makes it so that we can compare vectors of tangent spaces at different points in a natural way. In effect the point that the tangent space is attached to is no longer relevant, and we will adopt the notation TE and $\mathrm{T}_M E$ in stead of $T_f E$ respectively $f^*\mathrm{T}E$.

The case of indefinite metric shall not in general be treated separately, because few if any changes are needed to the reasoning. The most crucial problem in pseudo-Euclidean signature is normalisation of null vectors. In such situations the vector is implicitly required to not be null.

Chapter 2

Ribaucour transformation

2.1 Classical transformation

The classical Ribaucour transformation, studied extensively in the end of the 19th century by Bianchi [Bia99] and a few decades later more generally by Eisenhart [Eis23], relates a hypersurface with an orthogonal system of principal curves with an other such hypersurface. The submanifold is mostly taken to be a two-dimensional surface, enveloped in three-dimensional Euclidean space, but the extension to higher dimensional hypersurfaces is also well understood, although not all of the nice properties from the two-dimensional case carry over.

Definition 2.1. Let $f, \tilde{f} : M \to \boldsymbol{R}^{m+1}$ for M as above and f an isometry such that $f(M)$ is free from umbilic points, we say that $f, \tilde{f}$ are *related by a classical Ribaucour transformation* if

$$f + hN = \tilde{f} + h\tilde{N}, \quad h \in C^\infty(M) \tag{2.1a}$$

If f_*X, $X \in TM$, is a principal direction, then so is $\tilde{f}_*X$, (2.1b)

where $N, \tilde{N}$ are the normal vector fields corresponding to $f, \tilde{f}$.

The requirement to be free from umbilic points is needed in order for the principal directions to be uniquely defined; on a umbilic point the eigenspaces of A_N are not all one-dimensional.

The first condition on the transform is that the surfaces should be related by a *sphere congruence*. Specifically, this means that the surface and its transform should at all points tangent a common sphere, that varies smoothly along the surface. Furthermore, the central points of these spheres should also constitute a surface. The scalar field h hence represent the radius of the sphere.

Note that while in (pseudo-) Euclidean space we can carry out the addition in (2.1a) through naturally identifying the tangent space with the enveloping space, in a more general setting such as space forms, a similar condition can be achieved via geodesics.

Away from umbilic points, the second condition ensures that lines of curvature of the first surface are mapped onto such curves on the second. The importance of this condition is that it relates principal curvatures of the surfaces, and hence the mean curvature.

2.2 Generalisation

There are different notions of how to generalise the classic setting to higher codimension (e. g. [DT02; CFT04]), but the following viewpoint, presented in [DT03], has the additional benefit of a local correspondence with a simple geometric object.

Definition 2.2. We say that two immersions $f, \tilde{f} : M \to E = \boldsymbol{R}^{m+n}$ are *related by a Ribaucour transformation* if $|\tilde{f} - f| \neq 0$ everywhere, and there exists a smoothly varying symmetric operator $D : TM \to TM$, an smooth isometry $P : TE \to TE$ and a nowhere vanishing smooth vector field $H : M \to TE$, such that

$$PV - V = \langle V, H \rangle (f - \tilde{f}), \quad \forall V \in TE, \tag{2.2a}$$

$$\tilde{f}_* = P f_* D. \tag{2.2b}$$

We need a few tools in order to show that this definition generalises the classical one. In particular the following striking fact:

Any Ribaucour transformation is in local one-to-one correspondence with a symmetric operator on TM whose image in TE under f_* has a derivative which is symmetric on $TM \otimes TM$. Furthermore, this operator is fully described by a scalar field and a normal vector field satisfying a simple relation.

To prove this, we firstly deduce from (1.1) that in our setting with flat enveloping space, the condition for a symmetric operator $\Phi : TM \to TM$ to have a derivative symmetric on $TM \otimes TM$ in TE is

$$\mathrm{d} f_* \Phi = 0, \tag{2.3}$$

with $f_* \Phi$ regarded as a vector-valued one-form. We can therefore always locally find a vector field $F : M \to TE$ such that $f_* \Phi = \nabla F = F_*$. We note that such a field must necessarily satisfy

$$\nabla^\perp F = 0. \tag{2.4}$$

In particular, from the normal part of (2.3) we have that for arbitrary vectors $X, Y \in TM$, $0 = \nabla^\perp_X f_* \Phi Y - \nabla^\perp_Y f_* \Phi X = X^\flat A \Phi Y - X^\flat \Phi A Y$, using the symmetry of Φ and A. Hence,

$$[\Phi, A] = 0, \tag{2.5}$$

and this will be very important in the upcoming analysis.

Furthermore, we deduce from (1.2) that $\Phi^\flat = \nabla(f_*^{-1}F^\top)^\flat - \langle A, F^\perp\rangle^\flat$, and both Φ and A are symmetric, so antisymmetrisation gives $0 = \mathrm{d}(f_*^{-1}F^\top)^\flat$. Hence we can write $F = f_*\partial\omega + B$, for some $\omega \in C^\infty(M)$, $B \in \mathrm{T}_f^\perp E$ and ∂ the gradient. Equivalently,

$$\Phi = \nabla\partial\omega - A_B, \tag{2.6}$$

where $A_B = \langle B, A\rangle$. Note, however, that while this construction manifestly makes Φ symmetric, not all (ω, B) fulfil (2.4). We collect those that do in the set $\mathrm{Cmb}(f)$, and we note that this requirement is

$$\mathrm{d}\omega A + \nabla^\perp B = 0. \tag{2.7}$$

The name of this collection comes from *Combescure transform*, which is an other classical transform.

With these definitions, we are ready to state the theorem

Theorem 2.1 (Dajczer–Tojeiro). *Let $f, \tilde{f}$ be related by a Ribaucour transformation, as above. Then there exists locally $(\varphi, B) \in \mathrm{Cmb}(f)$ such that*

$$\tilde{f} = f - 2\omega F/F^2, \tag{2.8}$$

where $F = f_*\partial\omega + B$ and $F^2 = \langle F, F\rangle$. Moreover,

$$P = 1_{TE} - 2FF^\flat/F^2, \quad D = 1_{TM} - 2\omega\Phi/F^2, \quad H = -F/\omega, \tag{2.9}$$

where $f_*\Phi = dF$. Conversely, given $(\omega, B) \in \mathrm{Cmb}$ such that ω is nonvanishing, (2.8, 2.9) defines a Ribaucour transform of f on any open subset of M where D is invertible.

Proof. Let

$$\mu\hat{F} \doteq f - \tilde{f}, \tag{2.10}$$

for $\mu \in C^\infty(M)$, $\hat{F} \in \mathrm{T}_M E$ with $\hat{F}^2 = s = \pm 1$, so that $PV = V + \mu\langle H, V\rangle\hat{F}$. Since P is an isometry, we then have $\mu H^\flat = -2s\hat{F}^\flat$ and

$$P = 1_{TE} - 2s\hat{F}\hat{F}^\flat, \tag{2.11}$$

so we see that P is a reflection along $\hat{F}$. Moreover, $P^\dagger = P^{-1} = P$.

Now, $0 = \langle \tilde{f}_*, P\Xi\rangle$ for all $\Xi \in T_f^\perp E$, so (2.10) and (2.11) gives

$$\langle \Xi, \nabla\hat{F}\rangle = \langle \Xi, \hat{F}\rangle \frac{\mathrm{d}\mu - 2s\hat{F}^\flat f_*}{\mu} \doteq \langle \Xi, \hat{F}\rangle u. \tag{2.12}$$

Hence, a vector field $F \doteq \rho\hat{F}$, $\rho \in C^\infty(M)$, would have a normal derivative given by

$$\langle \Xi, \nabla F\rangle = \rho\langle \Xi, \hat{F}\rangle(\mathrm{d}\log\rho + u), \tag{2.13}$$

so in order to fulfil (2.4), we need a ρ that kills the above expression. Locally, this means that we must show that u is closed.

While the first term in u obviously is closed, for the second part, we look at the symmetric operator D. Using (2.11) and (2.2b) we then find

$$\begin{aligned} D &= f_*^{-1} P \tilde{f}_* = f_*^{-1} P(f_* - \mu \nabla \hat{F} - \hat{F} \mathrm{d}\mu) \\ &= 1_{TM} - 2s f_*^{-1} \hat{F} \hat{F}^\flat f_* - \mu^2 f_*^{-1} \nabla(\hat{F}/\mu). \end{aligned} \tag{2.14}$$

We conclude that the derivative of the second term of u is symmetric, and hence that the form is closed. We can therefore locally find a ρ that makes the normal derivative of F vanish.

Furthermore, letting $\omega = s\mu\rho/2$ makes $F^\flat f_* = \mathrm{d}\omega$, so that $F = f_* \partial\omega + B$, $B \in \mathrm{T}_f^\perp E$. Now the formulas (2.8) and (2.9) follow readily from the definitions, as does the converse of the theorem. □

It should be noted that this relationship was also proved using a different method in the special case of hypersurfaces in a curvature line-parametrisation. [CFT99]

In order to find all Ribaucour transformations of a given simply connected surface, one hence only has to solve (2.4), or (2.7). However, when the original surface is not simply connected, this correspondence is only true locally, and hence transformations found this way will not in general conserve topological quantities.

In what follows we will add a tilde to the object that corresponds to the transformed surface. Since we are interested in minimality, we shall make use of this next relation.

Corollary 2.2. *The shape operators transform as*

$$\tilde{A}_{P\Xi} = D^{-1}(A_\Xi + 2\langle \Xi, B\rangle \Phi / F^2), \quad \Xi \in T_f^\perp E, \tag{2.15}$$

Proof. Let $\Xi \in \mathrm{T}^\perp E$, and take the differential of $P\Xi$. Using (2.9) we then find

$$\nabla^{\tilde{\perp}} \Xi - \tilde{f}_* \tilde{A}_{P\Xi} = \nabla(P\Xi) = -P(A_\Xi + 2\langle \Xi, B\rangle \Phi/F^2) + P\nabla^\perp \Xi, \tag{2.16}$$

so that (2.2b) yields the result. □

In the next theorem for hypersurfaces, we will call the single normalised normal basis field N.

Theorem 2.3. *The generalised Ribaucour transformation coincides with the classical Ribaucour transformation under the classical assumptions.*

Proof. For condition (2.1a), we look at the restriction of P to the normal bundle. It follows from (2.9) that $\langle N, H\rangle$ is non-vanishing, and hence we can take $h \doteq \langle N, H\rangle^{-1}$, to achieve the same equation.

For (2.1b), the principal directions are eigenvectors of the shape operator, so the statement is equivalent to simultaneous diagonalisability of the old and the new shape operator, or equivalently that the operators commute.

These are defined locally, so we can make use of the theorem 2.1. From (2.15) and (2.9) we then find that it is enough to show that A_N commutes with Φ, but this is given in (2.5).

$\square$

Chapter 3

Minimality preservation

We will now utilise this elevated understanding of the Ribaucour transformation in order to formulate a simple condition for minimality-preservation. The following theorem is inspired by the work in [CFT03], although this setting is more general.

The analysis below will be taken in Euclidean signature. As mentioned earlier, pseudo-Euclidean signature restricts normalised vectors to not be null, and the square root below should be real.

Theorem 3.1. *A Ribaucour transformation with*

$$D = \frac{\omega}{\beta} A_{\hat{B}}, \quad \textit{where } \beta \doteq \sqrt{B^2},\ \hat{B} \doteq \frac{1}{\beta} B, \tag{3.1}$$

transform the shape operators as

$$\tilde{A}_{P\hat{B}} = \frac{\beta^2}{\omega^2} A_{\hat{B}}^{-1}, \tag{3.2}$$

$$\tilde{A}_{P\Xi_j} = \frac{\beta}{\omega} A_{\hat{B}}^{-1} A_{\Xi_j}, \quad 2 \leqslant j \leqslant n, \tag{3.3}$$

for $\{\hat{B}, \Xi_2, \dots \Xi_n\}$ an orthogonal basis of the normal bundle.

Proof. The second result follows readily from (2.15). The formula for D in (2.9) gives

$$\Phi = \frac{F^2}{2} \left(\frac{1}{\omega} - \frac{1}{\beta} A_{\hat{B}} \right), \tag{3.4}$$

yielding the first result. □

We note that in order to perform a transformation of this kind, all principal curvatures with respect to B must be non-vanishing.

The following corollary is immediate from (3.2) applied to a diagonal basis of principal curvatures:

Corollary 3.2. *For two-dimensional submanifolds, the condition (3.1) implies preservation of minimality.*

In higher dimension, we can obtain minimality-conservation from applying the transformation twice, with an extra condition, as is readily computed:

Corollary 3.3. *When applying two subsequent Ribaucour transformations, both of which individually satisfies (3.1) for the corresponding surface, and in addition*

$$B_2 \parallel P_1 B_1, \tag{3.5}$$

the shape operators transform as

$$\tilde{\tilde{A}}_{P_2 P_1 \hat{B}} = \frac{\beta_2^2 \omega_1^2}{\beta_1^2 \omega_2^2} A_{\hat{B}}, \tag{3.6}$$

$$\tilde{\tilde{A}}_{P_2 P_1 \Xi_l} = \frac{\beta_2 \omega_1}{\beta_1 \omega_2} A_{\Xi_l} \tag{3.7}$$

and the metric transform as

$$\tilde{\tilde{g}} = \frac{\beta_1^2 \omega_2^2}{\beta_2^2 \omega_1^2} g. \tag{3.8}$$

This transformation is hence conformal, and in particular minimality-preserving.

While (3.1) might seem like a strong condition, it simplifies when the submanifold to be transformed has a flat normal bundle. In particular this is trivially true in codimension one.

Theorem 3.4. *A Ribaucour transformation where the initial submanifold has flat normal bundle satisfies (3.1) if and only if*

$$F^2 = \frac{2}{k} \beta \omega, \quad k \in \boldsymbol{R}. \tag{3.9}$$

Proof. From (2.5) we know that there is basis of TM that diagonalises $A_{\hat{B}}$ and Φ. Let this basis be $\{E_i\}_{1 \leqslant i \leqslant m}$, so that $A_{\hat{B}} E_i = \lambda_i E_i$, $\Phi E_i = \varphi_i E_i$, $\lambda_i, \varphi_i \in C^\infty(M)$. Now, we can use that $\mathrm{d}F^2 = 2\langle F, \nabla F \rangle = 2\mathrm{d}\omega \Phi$, so that, assuming (3.9),

$$\varphi_i = \frac{\mathrm{d}F^2(E_i)}{2\mathrm{d}\omega(E_i)} = \frac{\beta \mathrm{d}\omega(E_i) + \mathrm{d}\beta(E_i)\omega}{k \mathrm{d}\omega(E_i)}. \tag{3.10}$$

Now when the normal bundle is flat, there exists an orthogonal basis $\{\Xi_l\}_{1 \leqslant l \leqslant n}$ such that $\nabla^\perp \Xi_l = 0, \Xi_l^2 = 1$. We can then write (2.7) as

$$0 = \mathrm{d}\omega A + \sum_l^n \Xi_l \mathrm{d}\langle B, \Xi_l \rangle = \sum_l^n \Xi_l (\mathrm{d}\omega A_{\Xi_l} + \mathrm{d}\langle B, \Xi_l \rangle), \tag{3.11}$$

so that

$$\mathrm{d}\beta = -\frac{1}{\beta}\sum_{l}^{n}\langle \Xi_l, B\rangle \mathrm{d}\omega A_{\Xi_l} = -\frac{1}{\beta}\mathrm{d}\omega A_B = -\mathrm{d}\omega A_{\hat{B}}, \qquad (3.12)$$

Using this in (3.10) we find $\varphi_i = (\beta - \omega\lambda_i)/k$, so that (2.9) and (3.9) gives $DE_i = \beta^{-1}\omega\lambda_i E_i$, and we deduce (3.1). For the converse, act with $\mathrm{d}\omega$ on (3.4) and use (3.12), to find

$$\mathrm{d}F^2 = F^2\left(\frac{\mathrm{d}\omega}{\omega} + \frac{\mathrm{d}\beta}{\beta}\right), \qquad (3.13)$$

giving (3.9) as solutions. □

The geometrical meaning of the constant k in (3.9) is seen from (2.8): It denotes the negative of the change in the normal direction.

3.1 Permutability property

Finding solutions to (2.4) and (3.9) for a complicated second surface is however not easily done in a general setting. Luckily, there is a permutability theorem, originally developed by Bianchi [Bia99], that relates two transformations of a surface with a transformation of the image surface.

More precisely, given transformations corresponding to some parameters F_1, F_2, the theorem states that there is a surface that is both achieved from taking the transformation corresponding to the parameter F_1 followed by F_2 as well as the transformations in the opposite order. Moreover, this combined transformation can be algebraically computed from the known transformations.

In our general setting, we can use a theorem developed in [DT03]. We shall not go in to the details at this time, but the main point is that when two transformations has $[\Phi_1, \Phi_2] = 0$, one can form a transformation from the image surface solely from algebraic manipulations.

Chapter 4

Applications

In practice when one wants to find a minimality-preserving transformation, one starts off by finding solutions to (2.7), or (3.11), from which it is straight forward to eliminate one of the fields. From a general solution one can then use (3.9) or (3.1) to find the minimality-preserving solutions.

4.1 Example

The application of corollaries 3.2 and 3.3 is under ongoing research, but as a proof of concept we shall make the transformation on a relatively simple surface.

Let $E = \boldsymbol{R}^3$ with the usual metric and $f(r, \theta) = (\theta, \sinh r\, \cos\theta, \sinh r\, \sin\theta)^\top$ the helicoid parametrised with $r, \theta \in \boldsymbol{R}$. The metric and shape operators are then

$$g = \cosh^2 r \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad A_N = -\frac{1}{\cosh^2 r} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{4.1}$$

In a hypersurface setting, (3.11) becomes

$$0 = \mathrm{d}\omega A_N + \mathrm{d}\beta. \tag{4.2}$$

Taking derivatives and antisymmetrising gives the following equation for β:

$$\partial_r(\cosh^2 r\, \beta_{,r}) = \cosh^2 r\, \beta_{,\theta\theta}, \tag{4.3}$$

and the ansatz $\beta \doteq \gamma / \cosh r$ yields the equation

$$\gamma_{,rr} - \gamma_{,\theta\theta} = \gamma. \tag{4.4}$$

A one-parameter family of solutions to (4.4) is

$$\gamma = \exp(\sinh a\, \theta \pm \cosh a\, r), a \in \boldsymbol{C}, \tag{4.5}$$

and it is easily checked that these solutions satisfy (4.2). The corresponding ω field is then

$$\omega = \frac{1}{\sinh a}(\pm \cosh a\, \cosh r - \sinh r) \exp(\sinh a\, \theta \pm \cosh a\, r), \tag{4.6}$$

and all these solutions does in fact satisfy the minimality condition (3.9) with $k = 1/(\sinh r \cosh r)$. To find the transformed surface, we use $F = N\beta + f_*\partial\omega$ with

$$N = \frac{1}{\cosh r}\begin{pmatrix} \sinh r \\ \sin\theta \\ -\cos\theta \end{pmatrix}, \quad f_* = \begin{pmatrix} 0 & 1 \\ \cosh r \cos\theta & -\sinh r \sin\theta \\ \cosh r \sin\theta & \sinh r \cos\theta \end{pmatrix}. \tag{4.7}$$

The image surface is then

$$f - \frac{k}{\beta}F = \begin{pmatrix} \theta - \dfrac{1}{\sinh a} \\ \dfrac{\cosh a \sinh r \mp \cosh r}{\cosh a \sinh a}\sin\theta + \dfrac{\cosh a \sinh a \sinh r \mp \sinh a \cosh r}{\cosh a \sinh a}\cos\theta \\ \dfrac{\mp \sinh a \cosh r + \cosh a \sinh a \sinh r}{\cosh a \sinh a}\sin\theta - \dfrac{\cosh a \sinh r \mp \cosh r}{\cosh a \sinh a}\cos\theta \end{pmatrix}, \tag{4.8}$$

which in this particular case appears as merely a reparametrisation of the helicoid. although this example does not show much of the merits of the transformation, there are several other examples, e. g. [CFT03; CFT04], that does.

Appendix A

Computer algebra script

The following Maxima-script was used to verify surfaces and compute the relevant equations.

```
/*
 * Computes mean curvature for parametric hypersurfaces
 * in a given (pseudo)-Riemannian ambient metric.
 */

load("nchrpl")$ /* Package needed for mattrace fn */
load("eigen")$

/*
 * The generalised cross product
 */
cross_product(et, dX) := block(
  [M,n,N,mino],
  n : matrix_size(dX)[1],
  M : addcol(dX, zeromatrix(n,1)),
  N : zeromatrix(n,1),
  for i:1 thru n do (
    mino : minor(M, i, n),
    N[i,1] : (-1)^(1+i)*determinant(mino)
    ),
  return(N)
  )
$

/*
 * Minkowskij metric
 */
minkowskij_metric(p,n) :=
block(
  [et],
  et : ident(p + n),
  for i : 1 thru n do et[p + i, p + i] : -1,
```

```
  return(et)
  )
$

/*
 * Christoffel symbol component
 */
cfl_component(g, vars, dn1, up, dn2) :=
  block(
    [r : 0],
    for nu:1 thru matrix_size(g)[1] do
    r : r + invert(g)[nu, up]*(diff(g[dn2, nu], vars[dn1])
      + diff(g[dn1, nu], vars[dn2])
      - diff(g[dn2, dn1], vars[nu])),
    return(r/2)
  )
$

/*
 * Christoffel matrices
 */
cfl_matrices(g, vars) := block(
  [cfl, n],
  n : matrix_size(g)[1],
  for dn1:1 thru n do (
    cfl[dn1] : zeromatrix(n,n),
    for dn2:1 thru n do
    for up:1 thru n do cfl[dn1][up,dn2] : cfl_component(g, vars, \
        dn1, up, dn2)
    ),
  return(listarray(cfl))
);

/*
 * Covariant derivatives
 */
cov_d_cov(g, vars, v, nu) := diff(v, vars[nu])
                             - v.cfl_matrices(g, vars)[nu]$
cov_d_v(g, vars, V, nu) := diff(V, vars[nu])
                           + cfl_matrices(g, vars)[nu].V$
/*
 * Hessians
 */
hess(g, vars, s) := block(
  [
  m,  /* dim of metric */
  v,  /* differential covector */
  M  /* matrix */
  ],
  m : length(vars),
```

```
  v : makelist(diff(s, vars[i]), i, m),
  M : cov_d_cov(g, vars, matrix(v), 1),
  for i:2 thru m do M : addrow(M, list_matrix_entries(
      cov_d_cov(g, vars, matrix(v), i)
      )),
  return(M)

)$
Hess(g, vars, s) := invert(g).hess(g, vars, s)$

/*
 * Metric, normal and mean curvature
 * of a hypersurface
 *
 * et = metric of enveloping mfd
 * X = coordinate of mfd
 * vars = parametrisation vars of mfd
 *
 * Global options:
 * normalise = (bool) if b and A should be normalised
 * ribaucour = (bool) if we should compute ribaucour condition
 */
hypersurface_mc(et, X, vars) := block(
  [
  warnings : "", /* warnings string */
  m,      /* dim of submfld */
  n,      /* dim of supmfld */
  dX,     /* tangent vector list/array */
  norm2_dX, /* norm^2 of dX */
  spacelike_normal : true, /* if the normal has positive square */
  N,      /* normal vector */
  norm2_N, /* norm^2 of N */
  g,      /* metric of submfd */
  ginv,   /* metric inverse */
  b,      /* second fund. form */
  A,      /* shape operator */
  A_eivals, /* principal curvatures */
  H,      /* mean curvature */
  RC,     /* Ribaucour--Combescure cond */
  mc,     /* minimality condition */
  mc2,    /* minimality condition */
  be,ph,k, /* dummy vars */
  a,M      /* tmp array, mtx */
  ],

  m : length(vars),
  n : m + 1,
  if n # matrix_size(et)[1] or n # matrix_size(et)[2]
  then return("ERROR: Dimension of fn and metric not equal."),
```

```
if normalise # true then normalise : false,
if ribaucour # true then ribaucour : false,

/* -- Display envel. metric and fn -- */
display(et, X, vars, normalise, ribaucour),

/* -- Tangent vectors -- */
dX : zeromatrix(n, m),
for row:1 thru n do for col:1 thru m do (
  dX[row,col] : diff(X[row], vars[col])
  ),
display(dX),

/* -- Submanifold metric -- */
g : trigsimp(ratsimp(transpose(dX).et.dX)),
display(g),
if determinant(g) = 0 then return("ERROR: Singular metric."),
ginv : trigsimp(ratsimp(invert(g))),

/* -- Normal -- */
N : ratsimp(trigsimp(cross_product(et,dX))),
norm2_N : trigsimp(ratsimp(transpose(N).et.N)),
if 0 >= norm2_N then (
  spacelike_normal : false,
  warnings : concat(warnings, "WARNING: Normal has nonpositive \
      square. "),
  if normalise then warnings : concat(warnings, "Not normalising \
      b/A. ")
  ),
display(N, norm2_N),

/* -- Second fund. form -- */
M : zeromatrix(m,n),
for row:1 thru m do for col:1 thru n do (
  M[row,col] : diff(N[col,1], vars[row])
  ),
b : -trigsimp(ratsimp(M.et.dX)),
if normalise then b : b/sqrt(norm2_N),

/* -- Shape operator w/ eivals -- */
A : ratsimp(ginv.b),
display(A),

/* -- Mean curvature -- */
H : trigsimp(ratsimp(mattrace(A))),
display(H),

/* Rib-Cmb equation */
if ribaucour then (
  if normalise # true then warnings : concat(warnings,
```

```
      "WARNING: RC-condition not correct w/o normalisation. "),
    depends([be,om],vars),
    RC : [],
    M : ratsimp(hess(g, vars, om).A - transpose(hess(g, vars, om).A\
        )),
    for i:1 thru m do for j:i+1 thru m do push(M[i,j], RC),
    M : makelist(diff(om, vars[i]),i,m).A,
    for i:1 thru m do push(ratsimp(diff(be,vars[i]) + M[1,i]), RC),
    RC : transpose(map(num, RC)),
    M : makelist(diff(om, vars[i]),i,m),
    mc : num(ratsimp(M.ginv.M + be^2 - 2*be*om/k)),
    mc2 : transpose(map(num,list_matrix_entries(ratsimp(
            be*ident(m) + (k*be - om)*A - k*Hess(g,vars,om)
            )))),
    disp("These should be zero: "),
    display(RC,mc,mc2)
    ),

  /* -- Return values -- */
  o_X : X,
  o_vars : vars,
  o_dX : dX,
  o_norm2_dX : norm2_dX,
  o_N : N,
  o_norm2_N : norm2_N,
  o_g : g,
  o_b : b,
  o_A : A,
  o_A_eivals : A_eivals,
  o_H : H,
  o_RC : RC,
  o_mc : mc,
  o_mc2 : mc2,

  return(concat(warnings, "Variables saved with o_ prefix."))
  )
$
```

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