



UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

BACHELOR THESIS

Models for Credit Risk in Static Portfolios

Authors:

Joel Johansson

Anton Engblom

Supervisor:

Ph.D. Alexander Herbertsson

Department of Economics and Statistics

Spring 2015

Abstract

Models for Credit Risk in Static Portfolios

In this thesis we investigate models for credit risk in static portfolios. We study Vasicek's closed form approximation for large portfolios with the mixed binomial model using the beta distribution and a two-factor model inspired by Merton as mixing distributions. For the mixed binomial model we estimate Value-at-Risk using Monte-Carlo simulations and for the one-factor model inspired by Merton we analytically calculate Value-at-Risk, using Vasicek's large portfolio approximation. We find that the mixed binomial beta model and Vasicek's large portfolio approximation yields similar results. Furthermore, we find that Value-at-Risk is lower in the two-factor model than in the one-factor model, but when the loss given default depends on the factors the results are mixed. However, when the factors are positively correlated, Value-at-Risk is higher in the two-factor model than in Vasicek's large portfolio approximation.

Acknowledgements

We would like to thank our supervisor Alexander Herbertsson for his invaluable support and expertise. His contribution of both time and enthusiasm has really been valuable, making this a truly joyful process for both of us.

Contents

Abstract	ii
Acknowledgements	iii
List of Figures	vi
List of Tables	vii
1 Introduction	1
2 What is Risk	2
2.1 Different types of risk	2
2.2 Credit risk measurement and management	4
2.3 Financial crisis	6
2.4 The Basel Regulations	7
2.5 Measures for Risk Management	8
2.5.1 Value-at-Risk	8
2.5.2 Expected Shortfall	10
3 Static Credit Risk Models	12
3.1 The Binomial Model	12
3.2 The Binomial Mixture Model	15
3.2.1 Large Portfolio Approximation	17
3.3 Value-at-Risk and Expected Shortfall in Credit Risk	19
3.4 Various Mixing Distributions	20
3.5 The Binomial Mixture Model Inspired by Merton	22
3.5.1 The Merton Framework	22
3.5.2 A One-factor Model Inspired by the Merton Framework	23
3.6 Large Portfolio Approximation in the Mixed Binomial Merton Model	26
3.7 A Multi-factor Model Inspired by Merton	27
3.7.1 A Two-factor Model Inspired by Merton	28
4 Simulations and Numerical Examples	31
4.1 Simulation Algorithm	31
4.2 Examples of Large Portfolio Approximation	32
4.3 Simulations of Mertons One-factor Model and the Mixed Binomial Beta Model	36

4.4	Comparison and Simulations of a two-factor Model	38
4.4.1	Simulations of a two-factor Model when the Loss Given Default Depend on the Factors	40
5	Conclusion	43
A	Appendix	46
A.1	Calibration of parameters	46
A.2	The Wiener Process	48
A.2.1	A Generalized Wiener process	48

List of Figures

3.1	Binomial distribution with 100 obligors and the individual default probability 10%.	15
3.2	Two different beta densities, but the parameters calibrated such that $\mathbb{E}[p(Z)]$ is the same.	21
4.1	LPA and Merton's one-factor model for 20 obligors	33
4.2	LPA and Merton's one-factor model for 100 obligors	33
4.3	The mixed binomial beta model and the beta distribution, for 10 and 50 obligors	34
4.4	The mixed binomial beta model and the beta distribution, for 10 and 50 obligors	35
4.5	LPA for different ρ	35
4.6	$\text{VaR}_L(99.9\%)$ sensitivity to ρ	36
4.7	VaR in the Merton one-factor model and the mixed binomial beta model .	37
4.8	10^5 simulations of Value at Risk, $\alpha = 0.999$, for different weights w_1 and w_2 and correlations ρ	39
4.9	10^5 simulations of the one-factor model and the two-factor model. In the two-factor model the correlation between the factors is set to 0.5, and the weights 50% on each factor. Furthermore, $\rho = 0.3$ and the individual default probability is 5%.	39
4.10	Value-at-Risk for a two-factor model with constant loss given default and dependent loss given default.	41
4.11	The difference in Figure 4.10 increases exponentially.	41
4.12	10^5 simulations of the Two-factor model with simulated loss given default and LPA	42

List of Tables

4.1	Value-at-Risk for different α	37
4.2	How $\text{VaR}_L(99.9\%)$ is affected in different models	40
A.1	Correlations for given p and ρ	47
A.2	Parameters used in the beta-distribution	47

Chapter 1

Introduction

Financial institutions are forced by regulators to keep sufficient amount of capital in relation to its risk exposure. There are also strong incentives for the institutions themselves to keep capital to cover for unexpected losses. Therefore there is a need to model losses and to estimate risk. In 1974 Robert C. Merton developed a model on pricing corporate debt, see Merton (1974). From this framework a one-factor model for measuring credit risk was developed which many of the today used models are based on. In 1991, Oldrich Vasicek developed a large portfolio approximation-formula based on the one-factor model inspired by Merton, see Vasicek (1991). Because of its simplicity, Vasicek's large portfolio approximation is widely used and implemented in financial credit risk management. In this thesis we study the one-factor model and perform sensitivity tests of Vasicek's closed-form expression. We will then add a second factor to the one-factor model which forces us to perform Monte-Carlo simulations to observe the loss-distribution. After that, we compare the models using Value-at-Risk and finally we relax the assumption of constant loss given default, and let it depend on the outcome of the common factors.

The outline of this thesis is as follows. In Chapter 2 we discuss risk, risk management and give a brief description of the recent financial crisis. We also define Value-at-Risk, the risk measure we will work with to compare the models. In Chapter 3 we begin with a simple model of loss distributions and theoretically work our way to a one-factor model and Vasicek's large portfolio approximation. In Chapter 4 we perform simulations and compare the models explained in Chapter 3. In Chapter 5 we present our conclusions.

Chapter 2

What is Risk

In this chapter we discuss risk in general. In Section 2.1 we describe some different types of risks. Then in Section 2.2 we discuss credit risk measurement and management, and the difference between the two. Because of its major impact on risk management, Section 2.3 give a brief description of the recent financial crisis. In Section 2.4 we discuss the Basel Regulations which is a regulatory framework for the financial industry. Finally, In Section 2.5 we describe the risk-measures Value-at-Risk and Expected Shortfall.

2.1 Different types of risk

In this section we have a introducing discussion on the field of risk.

An old definition of risk comes from Oxford English Dictionary from the year of 1655. It defines risk as the exposure to:

the possibility of loss, injury, or other adverse or unwelcome circumstance; a chance or situation involving such a possibility. (Oxford English Dictionary)

When people talk about risk most often the downside is mentioned and rarely a possible upside,like potential gain. There are many types of risk and there is not a one-and-only definition that perfectly captures the elements of risk in all contexts. Financial risk which is the focus of this thesis could be defined as any event or action that may adversely affect an organizations ability to achieve its objectives and execute its strategies or the quantifiable likelihood of loss or less-than-expected returns (McNeil et al. 2005).

In finance, risk is one of the main elements to understand and to handle in order to stay competitive. Most situations in any business involves some sort of risk that things do not go the way it was planned and one of the ultimate goals in finance is to evaluate

and quantify this risk. An increased risk means a more uncertain future. In financial context this could be a more uncertain future value of an asset i.e a stock or a bond. Uncertainty also brings us to the notion of randomness. In order to measure, evaluate and better understand the concept of risk i.e., the level of uncertainty and exposure to randomness, probability and statistics are used as we will also see examples of in this thesis.

A common view among laymen is that risk is something bad and that needs to be avoided. A better word to use would instead be compensated. Lets put up an easy example: If asset A is risk free with an annual rate of 5% and asset B is significantly more risky than asset A, but have the same rate of 5%, a rational investor will have no incentive to chose B unless he is compensated some way. In order to make B a sensible alternative as an investment a compensation i.e., a risk premium in form of a higher rate will be demanded. The difference between the price of the risk free asset and the price of the risky, the spread, is a central idea in Harry Markowitz famous paper on portfolio selection, see Markowitz (1952), and the Capital Asset Pricing Theory developed 12 years later by William Sharpe, John Lintner and Jan Mossin. These much debated papers is setting a framework for how to compensate for an increased financial risk and evaluate assets on a risk adjusted basis. There are different types of risks that a firm might face and need to handle. They can be classified into three types: Business risk, Non- Business risk and Financial risk which is the focus of this thesis.

Business risk

This type of risk is taken by business enterprises themselves in order to maximise shareholder value and profits. Examples of Business risks could be a company that undertake high costs for a commercial in order to launch a new product or decides to produce higher volumes in order to gain more profit. The higher costs and the uncertain result in the future leads to the risk of not meet its cost obligations of paying rent, salaries etc. Non- business risk: This type of risk is out of the control of the firms but affects its performance.

Financial risk

It is appropriate to specify the context when we discuss risk. In this thesis we consider risk in the field of finance and insurance. Our perspective will mainly be from a banks point of view or in other words from a lenders perspective. As we have mentioned before, a definition of financial risk in this context could be the quantifiable likelihood of loss or less-than-expected returns (McNeil et al. 2005) and this definition seems quite suitable

for the focus of this thesis. In banking industry there are a number of risks and we will now briefly describe some of the more important ones. Widely, financial risk in the financial industry can be divided into three main categories of risks; Market risk, operational risk and Credit risk, which is the focus of this thesis. In addition to these three categories there are also notions of risk that could be found in almost every category we have mentioned such as model and liquidity risk. Model risk is associated with the risk of using inappropriate models to measure and manage risk. It could be argued that model risk always is present to some degree. (McNeil et al. 2005). A good example of this that relates directly to our study is the use of inappropriate distributions for underlying macro factors that affect dependence among obligors default probabilities. Liquidity risk is the risk of negative consequences due to lack of marketability of an investment that cannot be sold or bought quickly enough to prevent or minimize loss (McNeil et al. 2005). *Market risk* is the risk associated with changes in value of a financial position due to changes in the value of the underlying components on which that position depends. This can be affected by factors such as changes in stock prices, bond prices, rates and exchange rates. Since market risk is a quite wide definition of risk it is often subdivided into more specific parts such as equity risk, interest rate risk, exchangers risk. *Operational risk* is the risk of losses due to something that directly affects the firms operations such as failure of systems, machines or internal processes and people. It can also be the risk of external events that affects the operations and performance of the firm. *Credit risk*, which is the focus of this study is the risk of not getting payments for investments such as loans due to the risk of default of the counterpart. It is probably the most important type of risk for a bank since it relates its core business as a lender. This important category of financial risk has got even more attention after the credit crunch of 2008 that started in the US housing market and spread over the world. The misjudgement of credit risks in the housing loan market were one important driving factor of the crisis.

2.2 Credit risk measurement and management

The focus of this thesis is measuring credit risk in a portfolio consisting of loans. From a banks perspective, who's core business is in lending, it would be crucial to know the probability that some of the obligors will default and thus impose credit loss in the portfolio. Throughout this text credit risk will therefore only concern future losses since no possible gains are possible in this context. We will let L be a random variable defined as the loss in a portfolio consisting of loans. The distribution of L , called the *loss distribution*, gives information about loss probabilities in the portfolio. Most modern credit risk models use the loss distribution to find statistical measures of the credit

risk. The regulatory framework Basel II introduced the use of Value-at-Risk (VaR) and Expected Shortfall (ES) which are examples of common measures derived from the loss distribution. We will further discuss VaR for credit risk in Chapter 3. One of the challenging tasks when using loss distribution models is to find a model that generates an appropriate and realistic loss. In this thesis we will discuss some well known models and their different loss distributions and how these differences affects VaR in a portfolio of loans.

We measure risk through probability statistics in order to understand our expected losses under different scenarios. Suppose we have a portfolio consisting of 1000 loans. It would then be in our interest to know the probability of how many of the obligors that will default on payments and also how much the amount of expected defaults likely could deviate from our expected numbers. With probability theory and statistics we can measure the risk of possible scenarios such as the risk of the losses of the portfolio to be more than x defaults or y dollars. The expected losses and the risk of deviations from these expected losses in the loan portfolio affects the amount of capital a bank needs to hold as capital requirements.

Management of risk means how the risk is acted on by the one facing the risk. In a bank for example, the risk managers actively and willingly takes on risks since risk is crucial for getting returns. Since higher risks means higher potential returns the risk managers role is to evaluate and manage the risk to a level that fit the firm's risk profile. Different types of banks and firms can take on different levels of risk and the actual level is set with regulatory framework as the Basel accords. We will further discuss the Basel accords later on in this chapter. There are different perspectives of risk management such as for stock holders, management, creditors and society as a whole. There are often conflicts of interest between these parties and their view of ultimate risk exposure. For instance, stockholders may be more willing to take on higher risk by leveraging the the company i. e. increasing financial risk, in order to get a higher return on equity on their investment whereas the creditors might be more interested in limit the level of financial risk in order to preserve the companys creditability in order to get their loans repayed.

The societys interest of well-functioning financial institutions has led to the need of supervision and regulations. Some of the most important regulations is connected to credit risk and is focused on capital requirements. In the aftermath of the credit crunch of 2008-2009, even more regulations is being implemented which consider capital requirements in financial institutions. One lesson taught from the recent financial crisis is that the negative impacts on society due to failure in the financial sector caused by errors in risk measures, and therefore also in risk management, can be devastating.

2.3 Financial crisis

In this section we will give a brief description of the financial crisis 2008-2009 because of its obvious connection to the field of credit risk.

The financial crisis that started in 2007 had its origin on the U.S housing market. It had major impact on markets all over the world and its aftermath is still evident to this day. We will here give a brief description of some of the factors that caused this crisis in order get an brief overview since it is closely related to the topic for this thesis which is credit risk.

The years prior to the crisis the world experienced an increase in exposure to credit risk. This was accomplished by an increase in the quantity of consumer and commercial debt and at the same time a decline in in the quality of that debt. The U.S housing market had for a long time been associated with rising prices and was considered as stable. There was an over-confidence within the American financial institutions and U.S government which led to an increased use of so called subprime mortgages. These loans were constructed to make it possible for people with low income and bad credit history to get the opportunity to buy their own house or apartment. This brought a new segment of costumers to the housing market which increased the demand and thus further pushing up prices. In the 1960s U.S banks found that they could not keep pace with the demand for mortgages which led to the development of securitization. This means packaging many loans or assets into securities that can be sold to investors. Securitization played a role in the crisis and in particular the securitization of subprime mortgages into so called CDOs. The investors of these products take on the default risk which transfers the risk from the bank, that is the issuer, to the investor. This makes it possible to increase the lending faster than the deposits grow. It also means that the banks incentives for correctly screening the riskiness in these assets is reduced since it is no longer the holder of the loan. Instead of the traditionally bank setting of keeping loans and their risk on the balance sheet the strategies was influenced by their knowledge that mortgages would be securitised and sold. As Hull (2012) writes:

When considering new mortgage applications, question was not Is this a credit we want to assume? Instead it was Is this a mortgage we can make money on by selling to someone else?

Saunders and Allen (2010) describes this change in the nature of banking as one contributing factor to set the stage for the emerging bubble. Prior to the crisis, this setting of holding loans and its risks changed successively to more of a originate-to-distribute setting. During the period 2001 to 2006 there was a huge increase in subprime mortgage lending and securities as CDOs (Hull, 2012). This fact together with historically low

interest rates around the year of 2000 and relaxed lending standards fuelled the emerging bubble.

With structured financial products such as CDOs banks could sell its risk to investors and had weak incentives to truly scan the credit risk in these assets. This led to a deterioration in credit quality. At the same time there was also a significant increase in both consumer and corporate leverage and, as we mentioned, an increasing quantity of loans. These circumstances led to higher systemic risk and were not detected by regulators. Few people sensed that a price-bubble was building up.

In 2007 many of the subprime borrowers found that they could no longer afford their mortgages. This led to foreclosures and a large number of houses coming on the market. This led to a decline in house prices. Many peoples and speculators with high leverage found that they had negative equity on their mortgage. This led to even more foreclosures which further added to the downward pressure on house prices. Financial institutions around the worlds had huge positions in mortgage backed securities which had served them well up until 2007 with relatively high returns. With foreclosures much higher than predicted by the banks and rating institutes, values of assets as CDOs dropped significantly. Some CDOs that was originally rated AAA lost about 80% of their value by the end of 2007 and was essentially worthless by mid-2009. This incurred huge losses for financial institutions as UBS, Merrill Lynch and Citigroup. (Hull 2012) The capital of many banks had been badly eroded which made them much more risk averse and reluctant to lend. In 2006, banks were well capitalized and loans were easy to obtain. By 2008, creditworthy individuals and corporations found it difficult to borrow which led to severe impacts of real economy due to liquidity problems leading to the worst recession of our time.

Clearly credit risk management played a central role in the crisis. The models used for measuring risk failed to incorporate these extreme events and in particular contagion effects. However, there were many other determining factors that set the scene for the crisis. While credit risk measurement models always can be improved, we cannot place all of the blame for the crisis on these models. Models are only as good as their assumptions and assumptions are driven by market conditions and incentives (Saunders & Allen, 2010).

2.4 The Basel Regulations

In this section we will give a short description of regulations for credit risk.

In the 1980s, bank activities were becoming more global which led to a need for an international regulatory framework. This led to the formation of the Basel Committee on Banking Supervision. In the year of 1988 the committee published a set of rules for the capital that banks were required to keep to compensate for credit risk. These requirements became known as Basel I. In 1999 there were significant changes proposed for the calculation of capital requirements for credit risk and capital requirements for operational risk were also introduced. These updated rules became known as Basel II and were finally implemented in 2007, just around the time of the credit crisis. The crisis 2008 led to further changes of regulations which resulted in Basel III which will be fully implemented in 2019. A commonly used measure in the Basel regulations for calculating capital requirements is the so called Value-at-Risk (VaR) which we will describe in detail in Section 2.5. VaR gives a single dollar number that summarize the total risk in a portfolio of assets. In the Basel accords this number is used for determining the required capital in order to cover for the risk the bank is bearing.

2.5 Measures for Risk Management

In this section we will introduce two risk measures that can be used for all types of losses in a portfolio. It can be applied to equity-portfolios as well as credit-portfolios. In Subsection 2.5.1 we will define and have a short discussion of Value-at-Risk and in Subsection 2.5.2 we will discuss the related measure Expected Shortfall.

2.5.1 Value-at-Risk

Value-at-Risk is defined as, see e.g in McNeil et al. (2005)

Definition 2.1. Value at Risk

Given a loss L and a confidence level $\alpha \in (0, 1)$, then $VaR_\alpha(L)$ is given by the smallest number y such that the probability that the loss L exceeds y is no larger than $1 - \alpha$, that is

$$\begin{aligned} VaR_\alpha(L) &= \inf\{y \in \mathbb{R} : \mathbb{P}[L > y] \leq 1 - \alpha\} \\ &= \inf\{y \in \mathbb{R} : 1 - \mathbb{P}[L \leq y] \leq 1 - \alpha\} \\ &= \inf\{y \in \mathbb{R} : F_L(y) \geq \alpha\} \end{aligned}$$

where $F_L(x)$ is the loss distribution.

If $F(x)$ is an continuous strictly increasing function, we have (McNeil et al. 2005)

$$VaR_\alpha(L) = F_L^{-1}(\alpha) = q_\alpha(F_L) \quad (2.1)$$

where $q_\alpha(F_L)$ is the α -quantile of the loss distribution $F_L(x) = \mathbb{P}[L \leq x]$.

VaR is thus the α -quantile of the loss distribution. An interpretation of Value at Risk, abbreviated VaR, is as follows: *We are $\alpha\%$ certain that our loss L will not be bigger than $VaR_\alpha(L)$ dollars ut to time T .* In practice, typical values of α are $\alpha = 0.95$ or $\alpha = 0.99$ and the time period T is usually one year for credit risk management (McNeil et al. 2005).

If $F(x)$ is strictly increasing the following is true

1. $F^{-1}(F(x)) = x$ for all x in its domain
2. $F(F^{-1}(y)) = y$ for all y in its range.

This means that if we find an expression for F_L^{-1} , we have an expression for $VaR_\alpha(L)$. If the inverse function F_L^{-1} is well defined we have that (McNeil et al. 2005)

$$\begin{aligned} VaR_\alpha(L) &= F_L^{-1}(\alpha) \\ F_L(VaR_\alpha(L)) &= F_L(F_L^{-1}(\alpha)) = \alpha \\ \mathbb{P}[L \leq VaR_\alpha(L)] &= \alpha. \end{aligned}$$

Estimating Value-at-Risk

Most models will not lead to an explicit formula for the loss distribution $F_L(x) = \mathbb{P}[L \leq x]$ and hence no formula for the inverse $F_L^{-1}(x)$ which is used to compute $VaR_\alpha(L)$. Therefore, VaR has to be computed in another way. When simulations are performed the sampled loss distribution will not be continuous, it will have "flat" regions where we can't find a unique x for $F(x)$. Therefore, the inverse function has to be generalized as

$$F^{\leftarrow} = \inf\{x \in \mathbb{R} : F(x) \geq y\}.$$

Practically in our simulations, this is done as follows. We perform n number of simulations and from each simulation we will have a value of the losses, so we have $X_1, X_2 \dots X_n$,

which we sort as $X_1 \geq X_2 \geq \dots X_n$. We then pick the $([n(1 - \alpha)] + 1)$:th sample, where $[y]$ is the integer part of y , and get the following estimation of $q_\alpha(F_L)$

$$q_\alpha(F_L) = X_{[n(1-\alpha)]+1}. \quad (2.2)$$

For more details on this, see e.g Mcneil et al. (2005).

Drawbacks of Value-at-Risk

Value-at-Risk is *non-subadditive*. In the context of credit-risk non-subadditivity can be interpreted as that the VaR of the sum of two portfolios is not less than or equal to the sum of the VaR of the individual portfolios. That is, for two portfolios L_1 and L_2 the following inequality does not necessary hold

$$VaR_\alpha(L_1 + L_2) \leq VaR_\alpha(L_1) + VaR_\alpha(L_2).$$

Value-at-Risk has been criticized as a risk-measure because of this aggregation property since the risk of two assets together shouldn't be higher than the summed risk of the two assets individually, known as diversification and is fundamental in finance. Since the inequality above doesn't hold, risk managers can't be sure to have an upper bound of VaR when summing up different portfolios.

The second drawback of VaR is that the measure doesn't give any information about the severity of the loss, VaR only produce a "least dollar amount". From VaR we only know that our loss is in the $1 - \alpha$ area.

Further, the statement "we can be $\alpha\%$ sure on that we won't lose more than VaR_α in one year" can be very misleading since the process involve estimations. For example, model risk is always a concern. Interpreting Value at Risk is of little value if a model is used that don't fit the real world close enough. This could be the case if we are using a probability distribution with to thin tails. As we will see next, expected shortfall takes some of these drawbacks into account.

2.5.2 Expected Shortfall

In this section we define Expected Shortfall and show the connection to Value-at-Risk. We also explain why Expected Shortfall sometimes is preferred instead of Value-at-Risk.

Definition 2.2. Expected shortfall

For a loss L with $\mathbb{E}(|L|) < \infty$ and a confidence level $\alpha \in (0, 1)$ the Expected shortfall is defined as

$$ES_{\alpha}(L) = \frac{1}{1 - \alpha} \int_{\alpha}^1 VaR_u(L) du. \quad (2.3)$$

This is taking the average of VaR for all confidence levels $u \geq \alpha$ in the loss distribution. If L is a continuous random variable, we have (see McNeil et al. 2005)

$$ES_{\alpha}(L) = \mathbb{E}[L | L \geq VaR_{\alpha}(L)]. \quad (2.4)$$

Unlike Value-at-Risk, Expected Shortfall is a coherent measure, meaning it is sub-additive. That is, for two portfolios L_1 and L_2 the following inequality holds

$$ES_{\alpha}(L_1 + L_2) \leq ES_{\alpha}(L_1) + ES_{\alpha}(L_2).$$

Except for the sub-additive property of Expected Shortfall, Expected Shortfall can, as seen in Equation (2.4), be interpreted as "If things go bad and VaR is exceeded, how much can we expect to lose?". A risk-measure with these properties are clearly useful for risk-managers and are easily understood in boardrooms.

Chapter 3

Static Credit Risk Models

In this chapter we will discuss models for loss distributions in credit portfolios. First, in Section 3.1 we present the simple binomial model, which is shown to have unrealistically thin tails because of independence between obligors. Then in Section 3.2 we introduce the mixed binomial model which is a generalization of the binomial model and can be used to create "thicker tails". In Subsection 3.2.1 we use the law of large numbers to show that the fraction of defaults in our portfolio can be approximated by the so called mixing distribution when the portfolio is large. In Section 3.3 we use the large portfolio approximation to get an approximation of Value-at-Risk and Expected Shortfall in a general binomial mixture model. Then in Section 3.4 we discuss some commonly used mixing distributions used in the mixed binomial model. Section 3.5 describes the Merton model and derive a one-factor model inspired by Merton. From this we spend Section 3.6 to derive Vasicek's large portfolio approximation of the one-factor model inspired by Merton which gives us a closed expression widely used in the finance industry and implemented in the Basel Accords. Finally, in Section 3.7 we expand the one-factor model into a multi-factor model. This theoretical preparation will be used in the simulations in Chapter 4. The main ideas and notations in this chapter comes from Herbertsson (2014), Hull (2012) and McNeil et al. (2005).

3.1 The Binomial Model

In this section we present the binomial model and its problems if used as a credit loss model.

Because of its simplicity, a starting point for credit loss models is the Binomial model. Since we are considering a *static* credit portfolio with m obligors we study the probability

of default during a given time period T , ignoring exactly when during this time period the default occurred, as opposed to a *dynamic* portfolio where the exact time of the default is considered. Here, T is typically one year. We define X_i as a random variable such that:

$$X_i = \begin{cases} 1 & \text{If obligor } i \text{ defaults before time } T \\ 0 & \text{Otherwise} \end{cases} \quad (3.1)$$

where $i = 1, 2 \dots m$ labels the m obligors.

We assume that the obligors are *independent* and *identical distributed* variables, *i.i.d.*, meaning that the obligors have identical default probabilities and that they are independent of each other. Furthermore, let $\mathbb{P}[X_i = 1] = p$ and thus $\mathbb{P}[X_i = 0] = 1 - p$ for each i .

Each obligor has identical credit loss at default, ℓ , since we are considering a homogeneous portfolio. If an obligor defaults, the obligor will not be able to pay back all of its debts, but only some of it - given by ℓ (Bluhm, Overbeck & Wagner, 2003). Credit loss at default is the dollar amount that a lender will not get back if the obligor defaults. The total credit loss L_m at time T is then given by:

$$L_m = \sum_{i=1}^m \ell X_i = \ell \sum_{i=1}^m X_i = \ell N_m$$

where $N_m = \sum_{i=1}^m X_i$ is the number of default in the portfolio up to time T .

Because ℓ is a constant, it is enough to study the distribution of N_m . Since $X_1, X_2 \dots X_m$ are *i.i.d* Bernoulli random variables, see Equation (3.1), then by construction N_m must be binomially distributed random variables. That is, $N_m \sim \text{Bin}(m, p)$. Hence, the probability that exactly " k " number of obligors will default is given by

$$\mathbb{P}[N_m = k] = \binom{m}{k} p^k (1 - p)^{m-k}.$$

The expected value of the number of defaults is the number of obligors multiplied by the individual default probability, $\mathbb{E}[N_m] = mp$. Since $X_1, X_2, \dots X_m$ are independent, we also have that the variance of N_m is given by

$$\text{Var}(N_m) = \text{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \text{var}(X_i) = mp(1 - p). \quad (3.2)$$

We can use Chebyshev's theorem (see e.g. Wackerly, Mendenhall & Scheaffer, 2007) together with Equation (3.2) to analytically show that the binomial model produces very thin tails. For example, for $p = 10\%$ and $m = 100$ the $Var(N_m) = 100p(1-p) = 9$ and $\mathbb{E}[N_m] = 100p = 10$. Then, according to the binomial model, the probability of having 10 more, or less defaults than expected is smaller than or equal to:

$$\mathbb{P}[|N_m - 10| \geq 10] \leq \frac{9}{10^2} = 9\%.$$

So having defaults outside of the interval $[0, 20]$ is smaller than or equal to 9%. This is actually a quite large overestimation, as seen in Figure 3.1 the probability of having more than 20 defaults is $1 - 0.992 = 0.008$, i.e. 0.8%.

Chebyshev's theorem can be used to further explain why the binomial model is unrealistic as a model to predict credit losses. As random variable we use the average number of defaults in the portfolio, $\frac{N_m}{m}$, and use Equation (3.2).

$$\mathbb{P}\left[\left|\frac{N_m}{m} - p\right| \geq c\right] \leq \frac{Var\left(\frac{N_m}{m}\right)}{c^2} = \frac{\frac{1}{m^2}Var(N_m)}{c^2} = \frac{mp(1-p)}{m^2c^2} = \frac{mp(1-p)}{m^2c^2} = \frac{p(1-p)}{mc^2}.$$

From this we conclude that $\mathbb{P}\left[\left|\frac{N_m}{m} - p\right| \geq c\right] \rightarrow 0$ as $m \rightarrow \infty$. So $\frac{N_m}{m}$, the fraction of defaults in the portfolio, converge to the constant p which is the individual default probability. In reality $\frac{N_m}{m}$ tends to have much bigger values than the individual default probability p , hence we once again conclude that the Binomial model is not very useful for predicting loss distributions in a portfolio (Herbertsson, 2014). The underlying reason is the thin tails of the binomial model, and we will therefore in the next section proceed with models that produces "fatter" tails than the binomial model, which creates more realistic loss distributions.

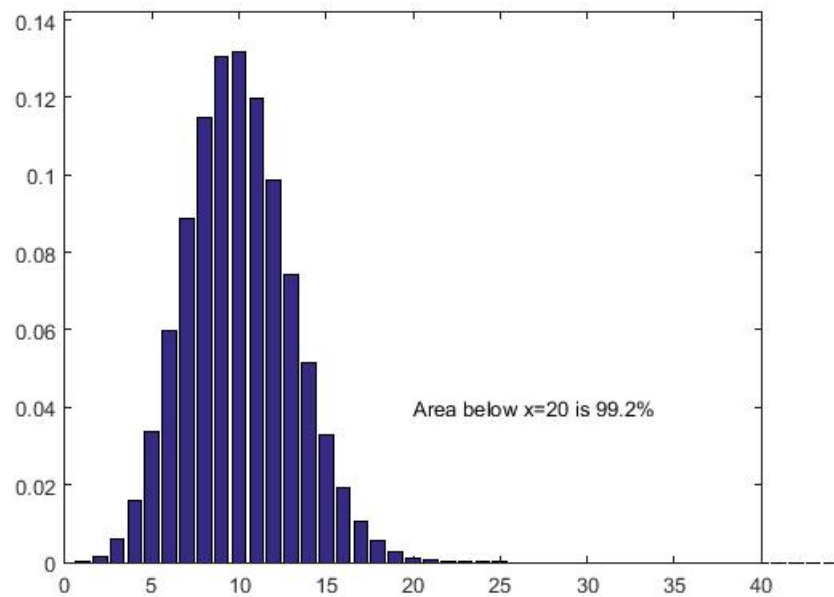


FIGURE 3.1: Binomial distribution with 100 obligors and the individual default probability 10%.

3.2 The Binomial Mixture Model

In this section we present the mixed binomial model, which randomizes the default probability in a static credit portfolio. First we give an example of how correlation can be created, which hopefully gives the reader a more intuitive understanding to subsequent parts of the chapter. We also introduce conditional independence. Then in Subsection 3.2.1 we discuss how to use the law of large numbers to find an approximation of losses in large portfolios.

To randomize the default probability we introduce a factor variable that is common to all obligors in our portfolio and therefore creates a dependence among the obligors. Intuitively this can be seen as some macroeconomic factor, e.g. interest rates, affecting all the obligors in our portfolio. To understand how a factor can create a dependence among two obligors, imagine the following scenario. Two companies, say NCC and Peab, are positively correlated. Their up's and down's occur at about the same time. NCC and Peab are correlated because they are both affected by the same factors in the economy. Possible factors would be the construction industry and probably some other factors for Sweden. If the construction industry gets under pressure it's likely that NCC and Peab also gets under pressure, they correlate because they are both correlated to the underlying factor of the construction industry.

The main idea is to randomize the default probability p so that it depends on the outcome of a factor Z for each obligor (McNeil et al. 2005). Therefore, the default probability for each obligor is a function of Z , $p(Z)$, where $p(\cdot) \in [0, 1]$ and $p(Z)$ is often referred to as the mixing distribution. Given an outcome of the random variable Z , the obligors are independent and identically distributed. To reconnect with our NCC- and Peab-example this means that NCC and Peab are independent of each other if we condition on the underlying factor Z . The economic future of Peab is not in any way decided by the firm specific risk of NCC. In mathematical terms, let Z be a continuous random variable on \mathbb{R} with density $f_Z(z)$ and let $p(Z) \in [0, 1]$ be a random variable with distribution $F(x)$ and mean \bar{p} , that is:

$$F(x) = \mathbb{P}[p(Z) \leq x] \quad \text{and} \quad \mathbb{E}[p(Z)] = \bar{p} \quad (3.3)$$

where

$$\mathbb{E}[p(Z)] = \int_{-\infty}^{\infty} p(z)f_z(z)dz = \bar{p}. \quad (3.4)$$

As before, for $i = 1, 2, \dots, m$, we define X_i as

$$X_i = \begin{cases} 1 & \text{If obligor } i \text{ defaults before time } T \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore we assume that conditional on Z the random variables X_1, X_2, \dots, X_m are independent with default probability $p(Z)$, that is: $\mathbb{P}[X_i = 1|Z] = p(Z)$. Here $p(Z)$ is referred to as the mixing distribution. The conditional probability above implies that the unconditional probability is the expected value of the random variable $p(Z)$

$$\mathbb{P}[X_i = 1] = \mathbb{E}[p(Z)]. \quad (3.5)$$

Recall that we want to find the loss distribution in the credit portfolio so that we can use risk-measures such as Value-at-Risk and Expected Shortfall. As in the binomial model all losses are constant and same, given by ℓ . The total credit loss in the portfolio at time T , called L_m , is then given by

$$L_m = \sum_{i=1}^m \ell X_i = \ell \sum_{i=1}^m X_i = \ell N_m$$

where $N_m = \sum_{i=1}^m X_i$ is the number of defaults in the portfolio up to time T .

Thus it is enough to study the distribution of N_m , i.e. $\mathbb{P}[N_m = k]$ for $k = 0, 1 \dots m$.

If we start with the conditional probability $\mathbb{P}[N_m = k|Z]$, we know that conditional on Z the random variables X_1, X_2, \dots, X_m are independent. They are by construction bernoulli variables and their sum, N_m , follows a binomial distribution, that is

$$\mathbb{P}[N_m = k|Z] = \binom{m}{k} p(Z)^k (1 - p(Z))^{m-k}. \quad (3.6)$$

Now we go on the unconditional probability. From Equation (3.6) we conclude that

$$\mathbb{P}[N_m = k] = \mathbb{E}[\mathbb{P}[N_m = k|Z]] = \mathbb{E}\left[\binom{m}{k} p(Z)^k (1 - p(Z))^{m-k}\right]. \quad (3.7)$$

If Z is a continuous random variable on \mathbb{R} with density $f_Z(z)$, then Equation (3.7) can be rewritten as

$$\mathbb{E}\left[\binom{m}{k} p(Z)^k (1 - p(Z))^{m-k}\right] = \int_{-\infty}^{\infty} \binom{m}{k} p(Z)^k (1 - p(Z))^{m-k} f_Z(z) dz. \quad (3.8)$$

In view of Equation (3.7), $\mathbb{P}[N_m = k]$ is given by the right hand side of Equation (3.8), in particular the density $f_Z(z)$ of Z . The random variable Z that creates the dependence among our variables can be distributed in various ways, for example the *beta distribution* or the *logit-normal distribution*. These will be briefly covered in this thesis and further discussed in Section 3.4, but our main focus will be a factor-model developed from Robert C. Merton's award-winning option pricing theory, see Merton (1974).

Equation (3.8) is the loss distribution for a given function of Z and at this point one would think that we are done. But Equation (3.8) will actually fail for large numbers. For some numbers the binomial coefficient will be so large that a computer can not precisely determine it, and $p(Z)^k$ will be so small for large numbers that a computer would determine it as zero. This is why we have to use the law of large numbers to find an approximation of the loss distribution.

3.2.1 Large Portfolio Approximation

In this subsection we show that the fraction of defaults, $\frac{N_m}{m}$, in our portfolio can be approximated by the mixing distribution $p(Z)$ when the portfolio is large.

We will now motivate why the distribution of $\frac{N_m}{m}$ will have fatter tails in a mixed binomial model. First, one can show that (see McNeil et al. 2005)

$$\text{Var}(X_i) = \bar{p}(1 - \bar{p}) \quad \text{and} \quad \text{Cov}(X_i, X_j) = \mathbb{E}[p(Z)^2] - \bar{p}^2 = \text{Var}(p(Z)). \quad (3.9)$$

Furthermore X_1, X_2, \dots, X_m are no longer independent and by homogeneity in the portfolio we get (see e.g. Wackerly et al. 2007)

$$\text{Var}(N_m) = m\text{Var}(X_i) + m(m-1)\text{Cov}(X_i, X_j). \quad (3.10)$$

So inserting Equation (3.9) in Equation (3.10) yields

$$\text{Var}(N_m) = m\bar{p}(1-\bar{p}) + m(m-1)(\mathbb{E}[p(Z)^2] - \bar{p}^2). \quad (3.11)$$

From Equation (3.11) we can study how $\text{Var}(\frac{N_m}{m})$ behave when $m \rightarrow \infty$. Recall from Section 3.1 that the variance of the average number of defaults in the binomial model converged to 0 as $m \rightarrow \infty$. Using that $\text{Var}(aX) = a^2\text{Var}(X)$ and Equation (3.11), we get

$$\text{Var}\left(\frac{N_m}{m}\right) = \frac{\text{Var}(N_m)}{m^2} = \frac{\bar{p}(1-\bar{p})}{m} + \frac{(m-1)(\mathbb{E}[p(Z)^2] - \bar{p}^2)}{m} \quad (3.12)$$

and from this we conclude that $\text{Var}(\frac{N_m}{m}) \rightarrow \mathbb{E}[p(Z)^2] - \bar{p}^2$ as $m \rightarrow \infty$. As seen, when we use the binomial mixture model the variance of the average number of defaults in the portfolio does not converge to 0, and as a consequence, the average number of defaults in the portfolio does not converge to a constant. This is due to the fact that the random variables X_1, X_2, \dots, X_m are no longer independent, they are all affected by the outcome of the factor Z and a dependence between them is created from this common factor. However, for a given outcome of Z the random variables X_1, X_2, \dots, X_m are conditional independent and we can use the conditional version of the law of large numbers, that is

$$\text{given a fixed outcome of } Z, \text{ then } \frac{N_m}{m} \rightarrow p(Z) \text{ as } m \rightarrow \infty. \quad (3.13)$$

This result implies that for any $x \in [0, 1]$ we have

$$\mathbb{P}\left[\frac{N_m}{m} \leq x | Z\right] \rightarrow 1_{(p(Z) \leq x)} \text{ as } m \rightarrow \infty. \quad (3.14)$$

Hence, when $m \rightarrow \infty$ the conditional probability of having a fractional loss, $\frac{N_m}{m}$, less than or equal to x given an outcome of Z is equal to 1, if $p(Z) \leq x$ and zero otherwise. Thus, we get

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \rightarrow \mathbb{P}[p(Z) \leq x] = F(x) \text{ when } m \rightarrow \infty. \quad (3.15)$$

So Equation (3.15) means that the fraction of defaults in our portfolio, $\frac{N_m}{m}$, can be approximated by the mixing distribution $p(Z)$ when the portfolio size, m , is large enough. The importance behind this result is that if the mixing distribution, $p(Z)$ in our case, has "fat" tails, then $\frac{N_m}{m}$ and the loss distribution L_m will also have "fat" tails as $m \rightarrow \infty$. Also, we do not have to use the loss distribution given by Equation (3.8), which fails for large numbers of m , but can simply use the mixing distribution $p(Z)$ to approximate the fraction of defaults in our portfolio.

Recall that $F_{L_m}(x) = \mathbb{P}[L_m \leq x] = \mathbb{P}[\ell N_m \leq x] = \mathbb{P}[\frac{N_m}{m} \leq \frac{x}{\ell m}]$, so if m is large Equation (3.15) therefore implies that $F_{L_m}(x) \approx F_L(\frac{x}{\ell m})$ when $F(x) = \mathbb{P}[p(Z) \leq x]$. This will be important when we compute $VaR_\alpha(L)$ in this model.

3.3 Value-at-Risk and Expected Shortfall in Credit Risk

In this section we discuss how Value-at-Risk and Expected Shortfall can be applied in credit risk using the large portfolio approximation derived in the previous Subsection. Our exact loss distribution $F_{L_m}(x)$ is given by

$$F_{L_m}(x) = \sum_{k=0}^{\lfloor \frac{x}{\ell} \rfloor} \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1-p(z))^{k-m} f_z(z) dz. \quad (3.16)$$

To calculate the $VaR_\alpha(L)$ we first have to take the inverse of this function and then calculate it numerically, which will fail for large numbers of m . Fortunately, when m is large the formula boils down to

$$VaR_\alpha(L) \approx \ell \cdot m \cdot F^{-1}(\alpha) \quad (3.17)$$

where $F(x) = \mathbb{P}[p(Z) \leq x]$. This makes Value-at-Risk really easy to calculate if one has a closed formula for the distribution function $F(x)$ to the mixing distribution $p(Z)$.

To understand Equation (3.17), recall from Equation (3.15) that

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \rightarrow \mathbb{P}[p(Z) \leq x] = F(x) \quad \text{when } m \rightarrow \infty$$

and from this we conclude that

$$\begin{aligned}
VaR_\alpha(L) &= \inf\{y \in \mathbb{R} : \mathbb{P}[L \leq y] \geq \alpha\} \\
&= \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[\frac{L}{\ell m} \leq \frac{y}{\ell m}\right] \geq \alpha\right\} \\
&= \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[\frac{N_m}{m} \leq \frac{y}{\ell m}\right] \geq \alpha\right\} \\
&\rightarrow \inf\left\{y \in \mathbb{R} : F\left(\frac{y}{\ell m}\right) \geq \alpha\right\} \quad \text{as } m \rightarrow \infty \\
&= \left\{\text{let } x = \frac{y}{\ell m}\right\} = \inf\left\{\ell \cdot m \cdot x \in \mathbb{R} : F(x) \geq \alpha\right\} \\
&= \ell \cdot m \cdot \inf\left\{x \in \mathbb{R} : F(x) \geq \alpha\right\} = \ell \cdot m \cdot F(\alpha)^{-1}.
\end{aligned}$$

An approximation of Expected Shortfall follows from the approximation of Value-at-Risk.

$$\begin{aligned}
ES_\alpha(L) &= \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L) du \\
&\approx \frac{1}{1-\alpha} \int_\alpha^1 \ell \cdot m \cdot F^{-1}(u) du \\
&= \frac{\ell \cdot m}{1-\alpha} \int_\alpha^1 F^{-1}(u) du.
\end{aligned}$$

3.4 Various Mixing Distributions

In this subsection we discuss the beta distribution and the logit-normal distribution which can be used as mixing distributions in the binomial model. We also introduce the binomial mixture model inspired by the Merton framework.

The Beta distribution

The Beta-distribution is intuitively a good distribution to use for loss distributions since if we let $p(x) = x$ and $Z \sim \text{Beta}(a, b)$, then $p(Z) = Z \in [0, 1]$, that is, $p(Z)$ is a distribution of probabilities. The density function for a variable $Z \sim \text{Beta}(a, b)$ is

$$f_z(z) = \frac{1}{\beta(a, b)} z^{a-1} (1-z)^{b-1} \quad z \in [0, 1] \quad (3.18)$$

where $\beta(a, b)$ is the beta function.

$$\beta(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz \quad (3.19)$$

(see e.g. Wackerly et al. 2007).

One can use the Beta-distribution in our loss distribution given by Equation (3.6) and show that

$$\mathbb{P}[N_m = k] = \binom{m}{k} \frac{\beta(a+k, b+m-k)}{\beta(a, b)}. \quad (3.20)$$

However, if m and k are large, this formula will fail since $\binom{m}{k}$ is too big to be numerically correctly handled in a computer. In Figure 3.2 we display $\mathbb{P}[N_m = k]$ for $k = 0, 1, \dots, m$ and for two different pairs of a and b , for $\bar{p} = 0.05$.

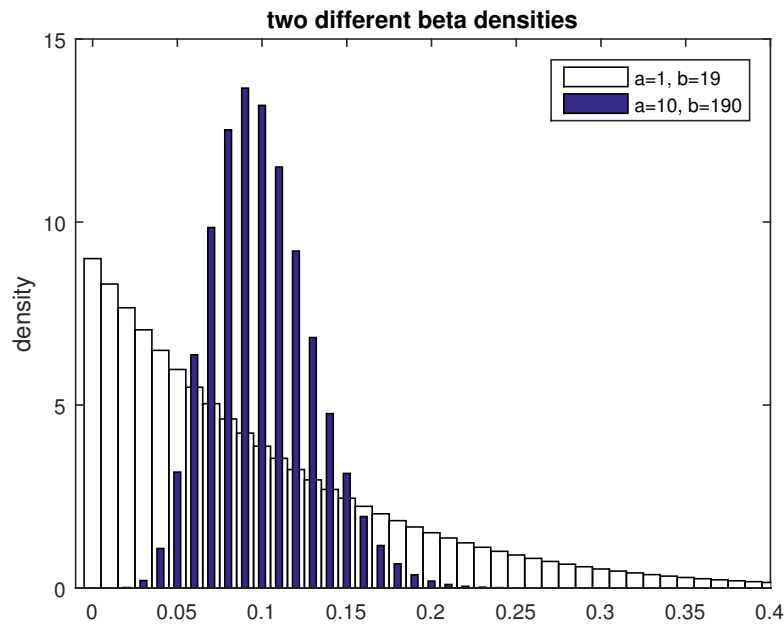


FIGURE 3.2: Two different beta densities, but the parameters calibrated such that $\mathbb{E}[p(Z)]$ is the same.

Logit-normal distribution

If we use the logistic function, $\frac{1}{1+e^x}$ for $p(X)$, on a random variable which is standard normal distributed, we get the logit-normal distribution, that is,

$$p(Z) = \frac{1}{1 + e^{-(\mu + \sigma Z)}} \quad (3.21)$$

where $\sigma > 0$ and Z is standard normal. Then it is easy to see that $p(Z) \in [0, 1]$ as we want it to be to be able to use in our binomial model.

Because $p(Z) \in [0, 1]$ the function is well defined and the cumulative mixing distribution is easily computed as $F(x) = \mathbb{P}[p(Z) \leq x] = \mathbb{P}[Z \leq p(x)^{-1}] = N(p^{-1}(x))$, where $N(x)$ is the distribution function of a standard normal distribution.

The binomial mixture model inspired by the Merton model

In the binomial mixture model inspired by the Merton model $p(Z)$ is given by

$$p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \quad (3.22)$$

where $\rho \in [0, 1]$, $N(x)$ is the distribution function of a standard normal distribution and $\bar{p} = \mathbb{P}[X_i = 1]$. Importantly, $p(Z) \in [0, 1]$. Here, the choice of $p(Z)$ is based on economic theory and each parameter have an economic interpretation. For example, the beta distribution and its parameters have no direct economic meaning. In contrast, each parameter in Equation (3.22) can be estimated from real data.

The mixed binomial Merton model is a big part of this thesis and will be rigorously derived in the next section.

3.5 The Binomial Mixture Model Inspired by Merton

In this section we discuss Merton's model, see Merton (1974) and Black-Scholes (1973), and show a one-factor model based on the Merton framework.

3.5.1 The Merton Framework

In this subsection we discuss Merton's asset value model, see Merton (1974). Notations and explanations are gathered from Herberthsson (2014) and Lando (2004).

The value of a firms assets is the sum of its equity and its debt. In the Merton model the debt of the firm is considered a single zero-coupon bond with face value D and maturity T . At time t the value of the firms equity is S_t and the value of its debt is D_t , $t \leq T$. The value of the obligors asset, V_t at time t , is

$$V_t = E_t + D_t.$$

If the value of the obligor's assets is less than its debt at time T , that is $V_T \leq D_T$ the firm defaults, the shareholders receives nothing and the bondholders receives what's left. If the value of the obligor's assets is greater than its debt at time T , $V_T > D_T$, the firm is alive, the shareholders receives $V_T - D_T$ and the bondholders receives D_T .

In the Merton Model, V_t is a stochastic process which follows a Geometric Brownian motion, that is,

$$dV_t = \mu V_t dt + \sigma V_t dB_t$$

where μ is the expected drift of the firms asset and σ is a measure of its variance. This is a stochastic differential equation which can be solved by using Itos lemma, see Hull (2012), and the solution is

$$V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}. \quad (3.23)$$

3.5.2 A One-factor Model Inspired by the Merton Framework

In this subsection we use Merton's model to derive a one-factor binomial mixed model for credit portfolios.

Consider a credit portfolio with m obligors and where each obligor can default up to a fixed time point T . Assume that each obligor i is a firm with asset $V_{t,i}$ which follows Merton's model so that

$$dV_{t,i} = \mu_i V_{t,i} dt + \sigma_i V_{t,i} dB_{t,i}$$

where μ_i is the expected increase of the firms asset and σ_i is a measure of its variance, and $B_{t,i}$ is a stochastic process defined as

$$B_{t,i} = \sqrt{\rho} W_{t,0} + \sqrt{1 - \rho} W_{t,i}. \quad (3.24)$$

One can show that $B_{t,i}$ is a Brownian motion so Equation (3.23) then implies that $V_{t,i}$ is given by

$$V_{t,i} = V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i B_{t,i}}. \quad (3.25)$$

So from Equation (3.24) we see that $V_{t,i}$ in Equation (3.25) creates dependence among the obligors via the parameter ρ . More specific, the $W_{t,0}$ term in Equation (3.24) can be

interpreted as a common economic variable that affects all of the obligors while " $W_{t,i}$ " is some firm specific process. The common economic variable is weighted by $\sqrt{\rho}$ and the firm specific risk is weighted by $\sqrt{1-\rho}$. If the correlation between $B_{t,i}$ and $B_{t,j}$ for two obligors i and j is computed, we get ρ , so the dependence among the obligors, which is created by the common economic variable, is the asset-correlation ρ . Recall that in Merton's model, obligor i defaults if $V_{t,i} \leq D_i$, or, by using Equation (3.25), if

$$V_{0,i}e^{(\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma B_{T,i}} < D_i. \quad (3.26)$$

Taking ln of both sides of Equation (3.26) and replacing W_i with its definition yields

$$\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho}W_{T,0} + \sqrt{1-\rho}W_{T,i} \right) < 0. \quad (3.27)$$

For each obligor i we have that $W_{i,T} \sim N(0, T)$. Dividing by \sqrt{T} creates a new variable $Y_i \sim N(0, 1)$, so $Y_i\sqrt{T}$ has the same distribution as $W_{i,T}$. Now define $Z = Y_0$ (the common economic variable) and insert in Equation (3.27)

$$\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho}\sqrt{T}Z + \sqrt{1-\rho}\sqrt{T}Y_i \right) < 0. \quad (3.28)$$

Dividing with $\sigma_i\sqrt{T}$,

$$\frac{\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} + \sqrt{\rho}Z + \sqrt{1-\rho}Y_i < 0 \quad (3.29)$$

and solving for Y_i gives

$$Y_i < \frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}} \quad (3.30)$$

where C_i is

$$C_i = \frac{\ln \left(\frac{V_{0,i}}{D_i} \right) + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}. \quad (3.31)$$

Hence, from our equations we find that

$$V_{T,i} < D_i \Leftrightarrow Y_i < \frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}. \quad (3.32)$$

Inspired by Merton's model we define X_i as

$$X_i = \begin{cases} 1 & \text{If } V_{T,i} < D_i \\ 0 & \text{If } V_{T,i} > D_i \end{cases} \quad (3.33)$$

and because of the independency between Y_i and Z and that $Y_i \sim N(0, 1)$ we have

$$\mathbb{P}[X_i = 1|Z] = \mathbb{P}[V_{T,i} < D_i = 1|Z] = \mathbb{P}\left[Y_i < \frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}} \middle| Z\right] = N\left(\frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \quad (3.34)$$

where $N(x)$ is the distribution function of a standard normal distribution. To use Equation (3.34) in our mixed binomial model, we assume a homogeneous portfolio, so that all m obligors are identical with $V_{0,i} = V_0$, $D_i = D$, $\mu_i = \mu$ and $C_i = C$ for all obligors.

Let us again think of Z as an economic background variable, and define $p(Z)$ as

$$p(Z) = N\left(\frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \quad (3.35)$$

where we remind the reader that $p(Z) = \mathbb{P}[X = 1|Z]$. From Equation (3.32) we can re-arrange for C and get

$$-C \geq \sqrt{\rho}Z + \sqrt{1-\rho}Y_i \quad (3.36)$$

where $C = C_i$ is given by Equation (3.31) with $V_{0,i} = V_0$, $D_i = D$, $\mu_i = \mu$. Furthermore, since Z and Y_i are standard normals, then $\sqrt{\rho}Z + \sqrt{1-\rho}Y_i$ will also be standard normal. Hence, $\mathbb{P}[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C] = N(-C)$ therefore,

$$\bar{p} = \mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D] = N(-C) \quad (3.37)$$

so that

$$C = -N^{-1}(\bar{p}) \quad (3.38)$$

which means that we can ignore C and instead directly work with the default probability \bar{p} which indirectly will solve for the variables in C . Now we can write Equation (3.35) as

$$p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right). \quad (3.39)$$

If we use this in the mixed binomial model, we have arrived at the mixed binomial Merton model. If we want to add more factors, more economic variables effecting our portfolio of obligors, Z is replaced as the sum of weighted factors, which will be discussed in Chapter 5. Equation (3.39) can be used in our simulations. To find the loss distribution we perform a large number of simulations, sum up the number of defaults in each simulation and divide by the number of simulations. But as discussed in Subsection 3.2.1 we can find an approximation of the loss distribution as a closed expression when the number of obligors in our portfolio are large. That is what Oldrich Vasicek did in his famous paper, see Vasicek (1991), and is derived in the next section.

3.6 Large Portfolio Approximation in the Mixed Binomial Merton Model

In this section we derive Vasicek's large portfolio approximation of the one-factor model inspired by Merton. This gives us a closed form expression for the loss distribution which is widely used in the finance industry. We also present the approximation of Value-at-Risk in a large portfolio. The derivation of Vasicek's large portfolio approximation can be found in e.g. Vasicek (1991) or McNeil et al. (2005).

We know that $p(Z) = N\left(\frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}\right)$ so

$$F(x) = \mathbb{P}[p(Z) \leq x] = \mathbb{P}\left[N\left(\frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \leq x\right] \quad (3.40)$$

and thus

$$\begin{aligned} \mathbb{P}\left[N\left(\frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \leq x\right] &= \mathbb{P}\left[\left(\frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \leq N^{-1}(x)\right] \\ &= \mathbb{P}\left[-Z \leq \frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}N^{-1}(x) + C)\right] \\ &= N\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}N^{-1}(x) + C)\right) \end{aligned}$$

that is, $F(x) = N\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}N^{-1}(x) + C)\right)$. Finally, by using that $C = -N^{-1}(\bar{p})$ we conclude that

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}N^{-1}(x) - N^{-1}(\bar{p}))\right) \quad (3.41)$$

where $F(x) = \mathbb{P}[p(Z) \leq x]$. Hence, in the Merton mixed binomial model we arrive at the following approximation of the loss distribution

$$\mathbb{P}[L_m \leq x] \approx N\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}N^{-1}\left(\frac{x}{\ell m}\right) - N^{-1}(\bar{p}))\right) \quad (3.42)$$

where $\bar{p} = \mathbb{P}[X_i = 1]$ is the individual default probability for each obligor and ρ is the asset correlation specified in Equation (3.24).

The expression in Equation (3.42) is Vasicek's large portfolio approximation of the one-factor model inspired by Merton. It is widely used by financial institutions to manage risk in large credit portfolios and implemented in the Basel accords, see e.g. McNeil et al. (2005). Now that we have a closed expression of the loss distribution in Equation (3.42), we can use Equation (3.17) with Equation (3.42) to derive a closed-form expression for Value-at-Risk in the mixed binomial Merton model and then get

$$VaR_\alpha(L) = \ell \cdot m \cdot N\left(\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right). \quad (3.43)$$

3.7 A Multi-factor Model Inspired by Merton

In this section we expand the one-factor model inspired by Merton into a multi-factor model.

In multi-factor models inspired by the Merton framework, we extend the one-factor model by adding more factors. When we add another factor to the model, it can be seen as adding another common economic variable affecting all obligors. In Section 3.2 we gave an example of Peab and NCC and said that they were both affected by how the construction industry developed. We can expand this example and say that NCC and Peab are affected by 75% on how well the construction industry develops and by 25% on changes in the interest level in Sweden. Hence, they are affected by a weighted sum of factors. In the general case we can say that for a homogeneous portfolio of obligors, each obligor is affected by a composite factor Ψ which is a weighted sum of factors, that is

$$\Psi = \sum_{k=1}^K w_k \psi_k$$

where $k = 1, 2, \dots, K$ are factor indices (Bluhm et al. 2002).

As in Subsection 3.5.2 the value of the obligors assets are driven by a systematic risk and a firm-specific risk, Ψ can here be seen as the systematic risk (Bluhm et al. 2002). It is assumed that the weights on the factors can not be negative and needs to sum up to 100%, that is

$$\sum_{k=1}^K w_k = 1, \quad w_k \geq 0.$$

It is both intuitively clear and essential for our model to work that the weights sum up to 1. Intuitively, obligors are affected by both a systematic risk and a firm specific risk and of course the factors representing the systematic risk has to be 100% of the systematic risk. Also, if the weights do not sum up to one the property $\mathbb{E}[p(Z)] = \bar{p}$ does not hold, which is essential to do for our model to work. The conditional default probability in the multi-factor model looks the same as in Equation (3.39), but the factor Z is substituted for a composite factor Ψ . The multi-factor model inspired by Merton for a homogeneous portfolio looks like this

$$p(\Psi) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}\Psi}{\sqrt{1-\rho}}\right) \quad (3.44)$$

where \bar{p} is the individual default probability, ρ is the asset correlation between the obligors, Ψ is the composite factor affecting all obligors and $N(x)$ is the cumulative distribution function of a standard normal distribution which makes $p(Z) \in [0, 1]$.

3.7.1 A Two-factor Model Inspired by Merton

In this subsection we will discuss a two-factor model inspired by Merton which simply is a special case of the multi-factor model. In Chapter 4 we will use this two-factor model in our simulations, therefore we dedicate a subsection to explain its outline.

In Section 3.7 Ψ consisted of K factors, we will now limit the model to two factors so that $\Psi = w_1 Z + w_2 Y$. As in the multi-factor model the weights of the factors are assumed not to be negative and needs to sum up to 1. For a two-factor model we have $w_1 + w_2 = 1$ and therefore $w_2 = 1 - w_1$.

$$p(Z, Y) = N\left(\frac{N^{-1}(p) - \sqrt{\rho}(w_1 Z + w_2 Y)}{\sqrt{1 - \rho}}\right). \quad (3.45)$$

This is just a special case of the multi-factor model inspired by Merton as described in Bluhm et al. (2002) and McNeil et al. (2005) where the composite factor consists of only two factors. Further, we will introduce some correlation between the factors Z and Y . This is intuitively appealing, in our previous example the outcome of the construction industry is probably to some extent affected by the interest level in Sweden. In our model the correlation between the factors are created as follows. The variables Z and X are *i.i.d* and $Z \sim N(0, 1)$ and $Y = a_1 Z + a_2 X$, which makes $Y \sim N(0, a_1^2 + a_2^2)$. This implies that

$$\text{Corr}(Z, Y) = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \quad (3.46)$$

and in the appendix we show how to calculate the correlation between Z and Y . Because both the variance and the correlation is determined by a_1 and a_2 we can determine a_1 and a_2 in terms of the correlation and the variance. This is crucial, since the loss distribution will look different depending on the variance (which is a measure of risk) and we will not be able to isolate the effect of adding another factor to our model if we do not hold the variance or the correlation fixed. If v is the variance and $\text{Corr}(Z, Y) = \rho_Z$ we solve the system of equations,

$$a_1^2 + a_2^2 = v \quad (3.47)$$

$$\frac{a_1}{\sqrt{a_1^2 + a_2^2}} = \rho_Z \quad (3.48)$$

and get

$$a_2 = \sqrt{v - v\rho_Z^2}$$

$$a_1 = \rho_Z \sqrt{v}.$$

There are two special scenarios worth mentioning in the two-factor model. One is that if the correlation is -1 and the weights are set to 50%, the systematic risk, i.e. the dependence on the factors, disappears. This is because if the correlation is -1 and the variance is 1 then $a_1 = -1$ from Equation (3.48) and hence $Y = -Z$. If the weights are

50% each, we have $0.5 \cdot Z - 0.5 \cdot Z$ and then the expression $\sqrt{\rho}(w_1 Z + w_2 Y)$ in Equation (4.2) becomes zero, and what is left is

$$p(Z) = N\left(\frac{N^{-1}(p)}{\sqrt{1-\rho}}\right)$$

and if the asset-correlation, ρ , is set to zero, we are actually back in the binomial model. Second, if the correlation between Z and Y is 1, we are back in the one-factor model. That is, if $\rho_z = 1$ and $v = 1$ we have $a_1 = 1$ and $a_2 = 0$ which generates $Y = 1 \cdot Z$ and because the weights sum up to 1 the weighted sum will simply be Z .

Chapter 4

Simulations and Numerical Examples

In this chapter we will present simulations and numerical examples. In Section 4.1 we explain the algorithm used in our simulations of the mixed binomial model and mixed binomial Merton model. Section 4.2 shows some examples on how the simulations of the mixed binomial model converges to $p(Z)$ i.e. the large portfolio approximation. Then in Section 4.3 we compare the one-factor model inspired by Merton and the mixed binomial beta model. In Section 4.4 we perform simulations of the two-factor model and compare Value-at-Risk to the large portfolio approximation of the one-factor model. Finally in Subsection 4.4.1 we let the loss given default, ℓ , depend on the outcome of the factors.

Throughout this chapter we use a portfolio containing 1000 obligors (i.e. $m = 1000$), the individual default probability, $\bar{p} = \mathbb{E}[p(Z)]$ is set to 5% and ℓ , the loss rate given default, set to 60%. The software we use for the simulations is Matlab. When we compare the models we sometimes fix the α of Value-at-Risk, when this is done we continuously use $\alpha = 0.999$ which is common in the industry (Herbertsson, 2014).

4.1 Simulation Algorithm

In this section we present the algorithm used in our simulations. The algorithm is in its full form gathered from Herbertsson (2012).

We use Monte Carlo simulations to estimate $\mathbb{P}[N_m \leq k]$ where $k = 0, 1, \dots, m$.

We simulate N_m and for each simulation check if N_m is less than or equal to k . The accuracy of the approximation increases as the number of simulations increases, and by the law of large numbers the loss distribution given by the simulation is very close to the

true loss distribution when the number of simulations are large. When the simulations are done we sum up the number of times that $N_m \leq k$ and divide by the number of simulations. In this way we get an approximation of $\mathbb{P}[N_m \leq k]$. Herbertsson (2012) describes the algorithm as follows

1. To simulate X_1, X_2, \dots, X_m and $N_m^{(j)}$ for $j = 1, 2, \dots, n$ do as follows
 - 1.1. Simulate the random variable Z and compute $p(Z) \in [0, 1]$.
 - 1.2. Simulate the *i.i.d* sequence U_1, U_2, \dots, U_m where U_i is uniformly distributed on $[0, 1]$ and independent of Z .
 - 1.3. For each $i = 1, 2, \dots, m$ define X_i as

$$X_i = \begin{cases} 1 & \text{if } U_i \leq p(Z) \\ 0 & \text{if } U_i > p(Z) \end{cases}$$

- 1.4. Compute $N_m^{(j)} = \sum_{i=1}^m X_i$.

2. Define n random variables $Y_1^{(k)}, Y_2^{(k)}, \dots, Y_n^{(k)}$ as

$$Y_j^{(k)} = \begin{cases} 1 & \text{if } N_m^{(j)} \leq k \\ 0 & \text{if } N_m^{(j)} > k \end{cases}$$

3. Compute $\frac{1}{n} \sum_{j=1}^n Y_j^{(k)}$
4. Let $\frac{1}{n} \sum_{j=1}^n Y_j^{(k)}$ be an estimate of the probability $\mathbb{P}[N_m \leq k]$.

For more details on this algorithm and its consistency, see Herbertsson (2012).

4.2 Examples of Large Portfolio Approximation

In this section we show some examples on how the simulations of the mixed binomial model converges to $p(Z)$ i.e. the large portfolio approximation. We do this for both the one-factor model inspired by Merton and the mixed binomial beta model.

In Chapter 3 we analytically showed that the loss distribution in the Merton model, for large portfolios, could be approximated as

$$\mathbb{P}[L_m \leq x] \approx N\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}N^{-1}\left(\frac{x}{\ell m}\right) + N^{-1}(\bar{p}))\right). \quad (4.1)$$

Following the algorithm in Section 4.1 we simulate the one-factor model inspired by Merton and graphically assure ourselves that the approximation holds. As seen in Figures 4.1 and 4.2, the Large Portfolio Approximation (LPA) for Merton's one-factor model becomes more and more accurate when the number of obligors increases.

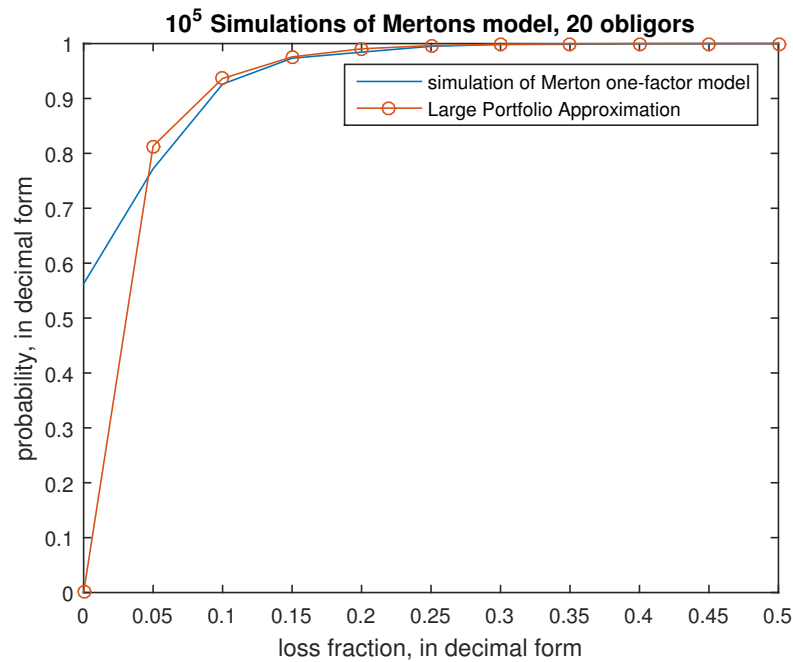


FIGURE 4.1: LPA and Merton's one-factor model for 20 obligors

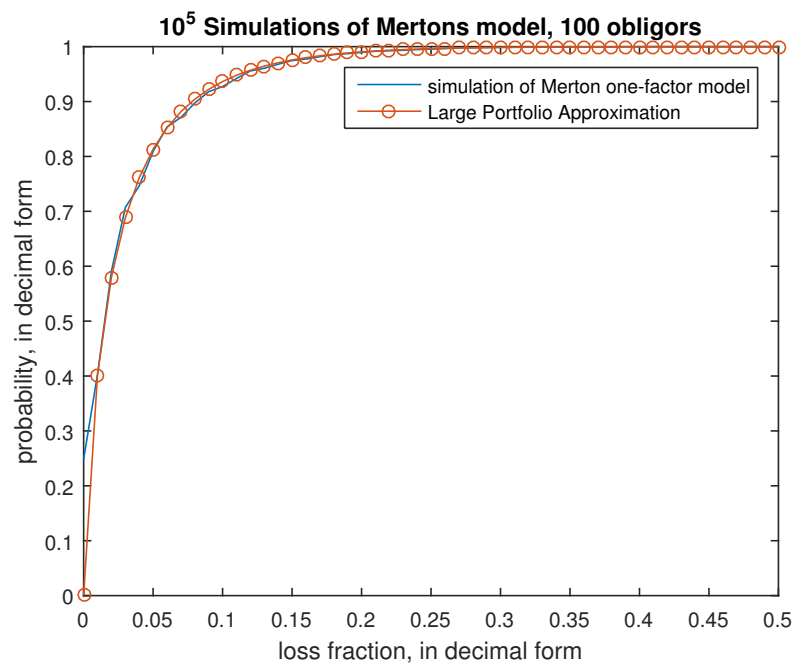


FIGURE 4.2: LPA and Merton's one-factor model for 100 obligors

In Chapter 3 we argued that the distribution of the average number of defaults in the portfolio converges to the distribution of the mixing variable $p(Z)$ as $m \rightarrow \infty$. In Figures

4.1 and 4.2 the random probability $p(Z)$ came from Merton's one-factor model, but recall that $p(Z)$ can come from any function that take values in $[0, 1]$. One possible function is the beta distribution described in Chapter 3. We now perform simulations of the mixed binomial beta model for increasing number of obligors and plot the result together with the beta distribution to again show that the simulation converges to $p(Z)$, in this case the beta-distribution. This is done in Figures 4.3 and 4.4 and again we see that the LPA-distribution is a good approximation when m increases.

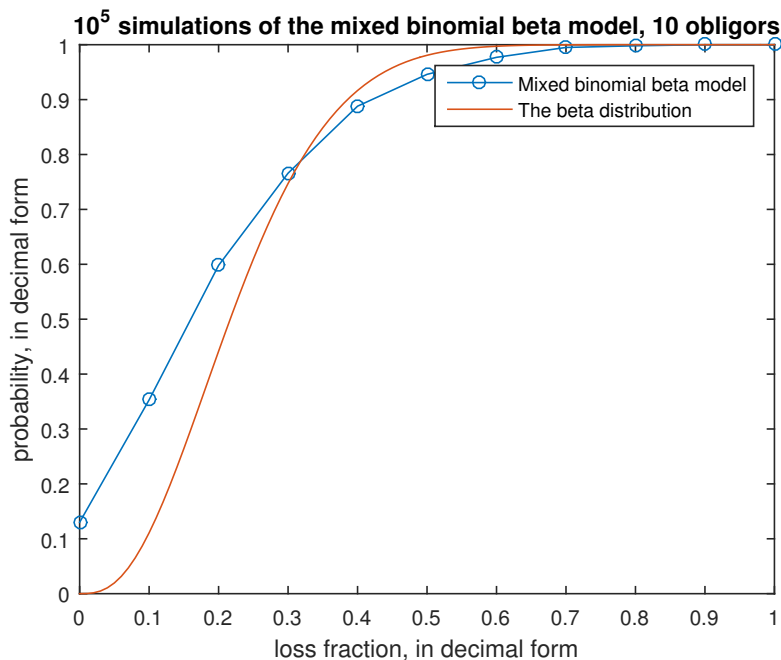


FIGURE 4.3: The mixed binomial beta model and the beta distribution, for 10 and 50 obligors

We will now proceed to investigate how the large portfolio approximation of the one-factor model inspired by Merton behaves when we hold p constant and change ρ .

In Figure 4.5 we have performed sensitivity tests on Vasicek's large portfolio approximation, holding everything constant except for ρ , the asset correlation. When we increase ρ , LPA becomes more flat. For $\rho = 0.01$ LPA is almost vertical and we can be almost 100% sure not to lose more than 5% (which is the individual default probability). When ρ increases LPA becomes more flat, meaning that the difference in loss fraction increases more for a small change in probability. For example we can compare LPA for $\rho = 0.3$ and $\rho = 0.95$. First look at the probability to lose at most 5% of our portfolio i.e. 0.05 on the x-axis. For $\rho = 0.3$ we can be about 80% sure on that we will not lose more than 5% and for $\rho = 0.95$ we can be about 90% sure on that we will not lose more than 5%. That means that we can be more sure not to lose more than 5% of our portfolio if the asset correlation is 0.95 compared to 0.3. Now go along the x-axis and look at the probability to lose at most 20% of our portfolio. For $\rho = 0.95$ the probability is still about

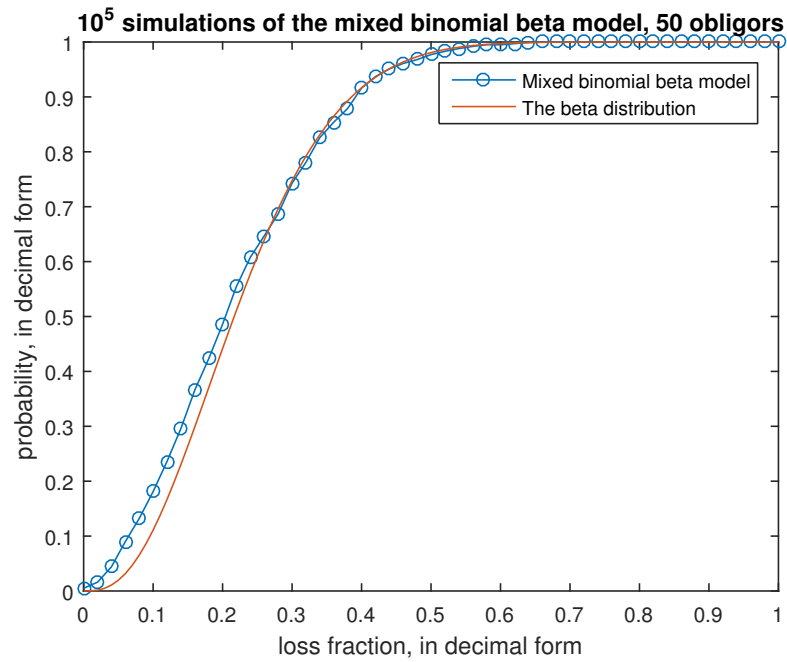


FIGURE 4.4: The mixed binomial beta model and the beta distribution, for 10 and 50 obligors

90% but for $\rho = 0.3$ the probability has increased to almost 100%. The difference lay in that if one obligor default when the asset correlation is high, the higher the probability that many of the obligors default. If we lose, we lose a lot. For lower values of the asset correlation the obligors are more independent which means that if one obligor default the others still may survive.

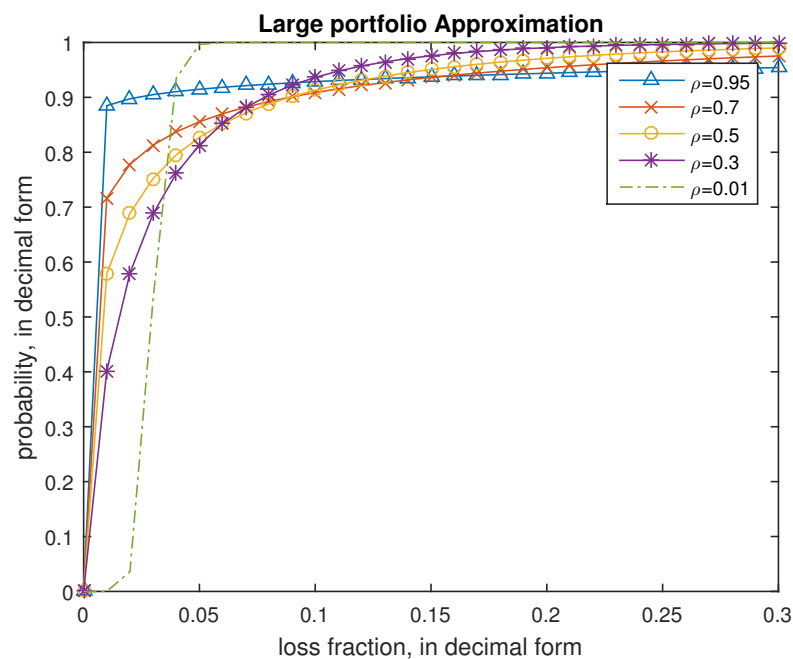


FIGURE 4.5: LPA for different ρ

In Figure 4.6 we fix α and study Value-at-Risk for different asset correlations, that is, we pick the 99.9:th percentile of the tail and look at how large Value-at-Risk is for different values of ρ . As expected from Figure 4.5, Value-at-Risk increases as ρ increases. When ρ is large VaR flatten out at 60% of our portfolio. This is because the loss given default is 60%, so that if every obligor defaults in the portfolio we lose 60% of it. Also, if the asset correlation is 0 Value-at-Risk is the individual default probability, 5%.

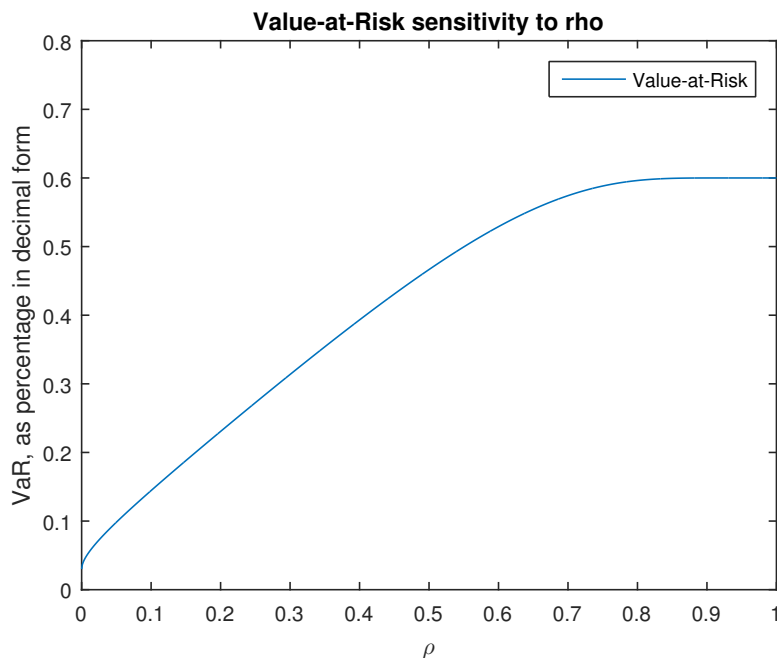


FIGURE 4.6: $\text{VaR}_L(99.9\%)$ sensitivity to ρ .

4.3 Simulations of Mertons One-factor Model and the Mixed Binomial Beta Model

In this section we compare the one-factor model inspired by Merton and the mixed binomial beta model. Such a study has previously also been done in Frey and McNeil (2003) with similar results.

It is only relevant to compare the mixed binomial model inspired by Merton and the mixed binomial beta model if the individual default probability and the default correlation are the same in both models. We need to calibrate ρ in the Merton model and a and b in the beta model to match the same default correlation, given the individual default probability p . An explanation on how this is done and which corresponding parameters are used in the examples can be found in the appendix.

Once we calibrate the parameters in each model, we can compare VaR in the two different models. In Figure 4.7 Value-at-Risk in the two models are compared, the parameters calibrated to a 2.43% default correlation and \bar{p} is set to 5%.

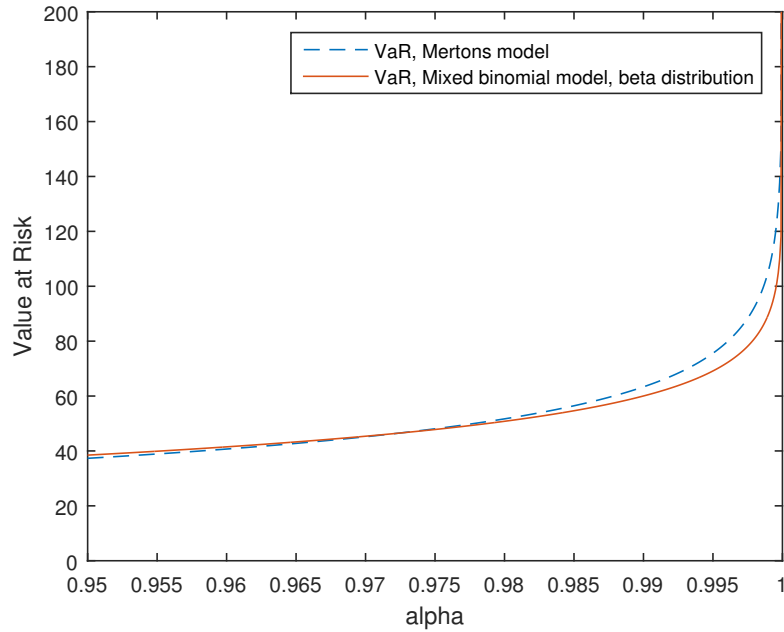


FIGURE 4.7: VaR in the Merton one-factor model and the mixed binomial beta model

There is no significant difference between VaR in the models up to the 99-th percentile where VaR in the Merton model starts to increase faster than VaR in the beta model. This can also be seen in Table 4.2, which is a numerical example of some values from Figure 4.7

TABLE 4.1: Value-at-Risk for different α

	$\alpha = 0.999$	$\alpha = 0.99$	$\alpha = 0.95$
Merton model	105	63	37
Mixed binomial beta model	89	60	38

4.4 Comparison and Simulations of a two-factor Model

In this section we perform simulations of the two-factor model and compare Value-at-Risk to the large portfolio approximation of the one-factor model. In Subsection 4.4.1 we let the loss given default ℓ depend on the outcome of the factors.

In multi-factor models inspired by the Merton framework, we use the model with one factor and add more factors. In this section we perform simulations and comparison of a two-factor model and hence add one more factor to the one-factor model, that is

$$p(Z, Y) = N\left(\frac{(N^{-1}(p) - \sqrt{\rho}(w_1 Z + w_2 Y))}{\sqrt{1 - \rho}}\right) \quad (4.2)$$

where Z and Y are defined as in Chapter 3. As before we have a homogeneous portfolio with 1000 obligors (i.e. $m = 1000$), the individual default probability $\bar{p} = \mathbb{E}[p(Z)] = 5\%$, $\rho = 30\%$ and the loss given default, $\ell = 60\%$.

In Figure 4.8 we have performed 10^5 simulations of Value at Risk for an $\alpha = 0.999$ in the two-factor model for different factor-correlations and weights. From the figure it is clear that Value-at-Risk is reduced when the correlation between the factors is low, and when the weights are close to 50%. As analytically showed in Chapter 3 a perfect hedge can be achieved when the correlation is -1 and the weights w_1 and w_2 are set to 50%. That means that the systematic risk is zero and the only thing determining if the obligor will default or not is the firm-specific risk. In Chapter 3 we showed that if the correlation between Z and Y is 1 we are back in the one-factor model. This can be verified from Figure 4.8, for correlation=1 Value-at-Risk stays the same for all weights. Value-at-Risk in LPA for the same parameters is 313. We can also conclude that VaR for the two-factor model always is lower than or equal to VaR in the two-factor model. This is intuitively clear and follows economic theory, the two-factor model can be seen as a more diversified portfolio and hence the risk should be lower. Comparing the loss distributions between the one-factor model and the two-factor model is straightforward since the parameters are the same in the models and will therefore create equal default correlations between the obligors. This is done in Figure 4.9. As expected the two-factor model have thinner tails than the one-factor model.

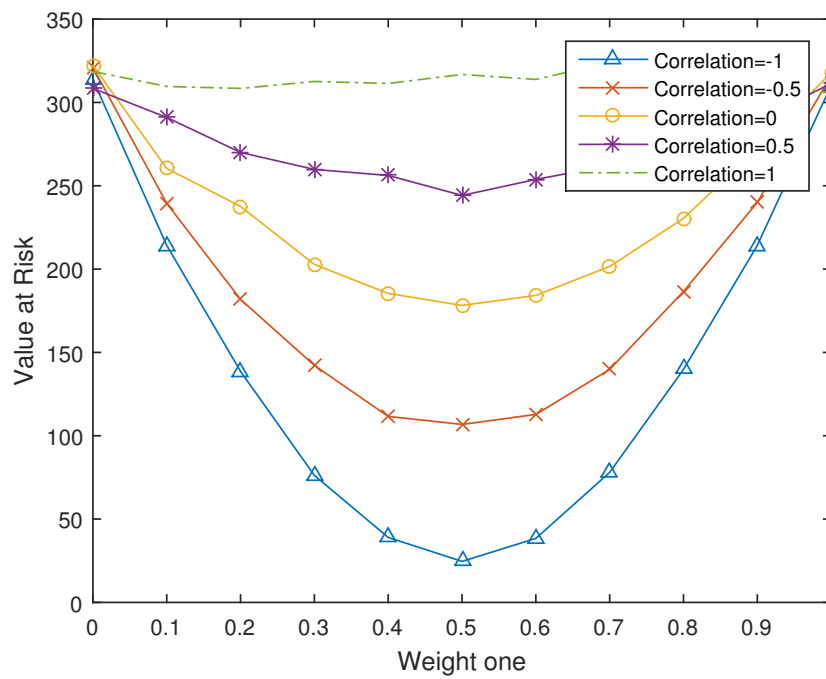


FIGURE 4.8: 10^5 simulations of Value at Risk, $\alpha = 0.999$, for different weights w_1 and w_2 and correlations ρ .

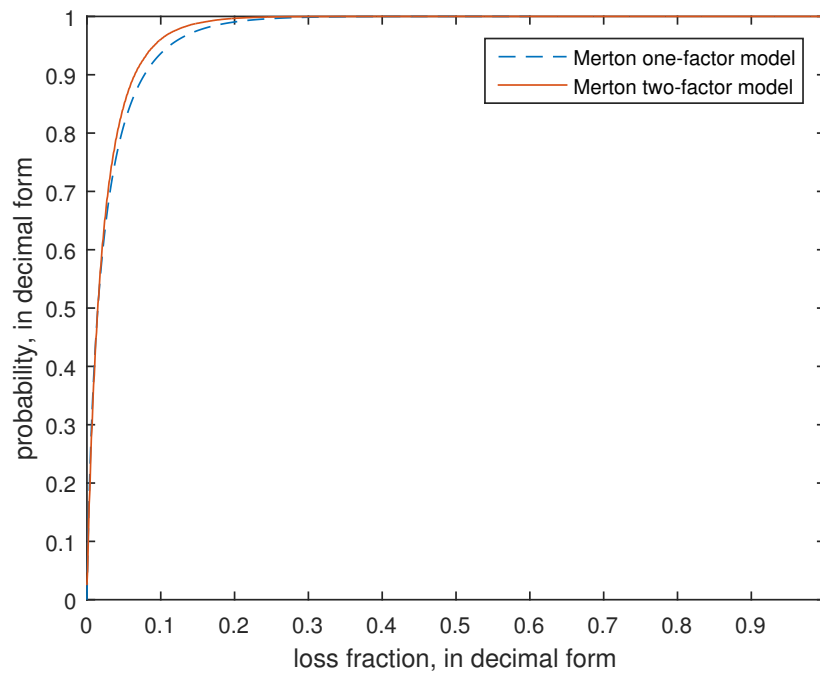


FIGURE 4.9: 10^5 simulations of the one-factor model and the two-factor model. In the two-factor model the correlation between the factors is set to 0.5, and the weights 50% on each factor. Furthermore, $\rho = 0.3$ and the individual default probability is 5%.

4.4.1 Simulations of a two-factor Model when the Loss Given Default Depend on the Factors

In this subsection we let the loss given default, ℓ , depend on the outcome of the factors. First we explain the intuition behind that ℓ should depend on the outcome of the factors. Then we explain how this is done in the simulations and present the simulations. We also compare the two-factor model when the loss given default depend on the factors, against LPA of the one-factor model.

In Figures 4.10 and 4.11 we let ℓ , the loss given default, depend on the outcome of the factors instead of being constant to 60% as before. The intuition is that a creditor is able to receive more from an obligors assets if the obligor defaults in a time when the overall economy is doing well than if the obligor defaults in an economic downturn (Schuermann, 2004). This is done as follows. In Equation (4.2), $p(Z)$ becomes smaller when the outcome of Z and Y is high, and higher when the outcome of Z and Y is low. In our simulations (see Section 4.1), a high $p(Z)$ generates more defaults than a low $p(Z)$. This means that a "low" weighted outcome of Z and Y represents a worse outcome of some economic factors than a high outcome of Z and Y . Therefore we simulate ℓ to take higher values (between 0 and 1) when the outcome of Z and Y is low and lower values when the outcome of Z and Y is high. That is, $\ell = 1.2 \cdot N(-(w_1 Z + w_2 Y))$ where $N(\cdot)$ is the cumulative normal distribution and 1.2 is a constant so that $\mathbb{E}[\ell] = 0.6$. Note that to get a reasonable comparison between the models we let the expected value of ℓ be 60%. As seen in Figure 4.10 Value-at-Risk is higher when ℓ depends on the outcome of the economic factors. Especially, the difference in VaR becomes greater when α increases. This is seen in Figure 4.11, where the difference in VaR increases exponentially. This is because if we have an extreme outcome of our factors, the creditor will loose close to 100% of the obligors assets. Not as clear is the fact that Value-at-Risk is lower when ℓ is dependent on the factors than when it is not, for low α . But a closer look at Figure 4.11 reveals that the difference indeed is negative.

So far, we have examined how Value-at-Risk is affected by different models. This is summarized in Table 4.2.

TABLE 4.2: How $\text{VaR}_L(99.9\%)$ is affected in different models

	Simulated Loss Given Default	Two factors, $\text{Corr} \neq 1$	$\uparrow \rho$
$\text{VaR}_L(99.9\%)$	\uparrow	\downarrow	\uparrow

When we added a factor to the one-factor model, $\text{VaR}_L(99.9\%)$ decreased and we said that the two-factor model *always* generated values of VaR that was lower than or equal to the one-factor model. But as seen in Table 4.2 $\text{VaR}_L(99.9\%)$ increases when we let the

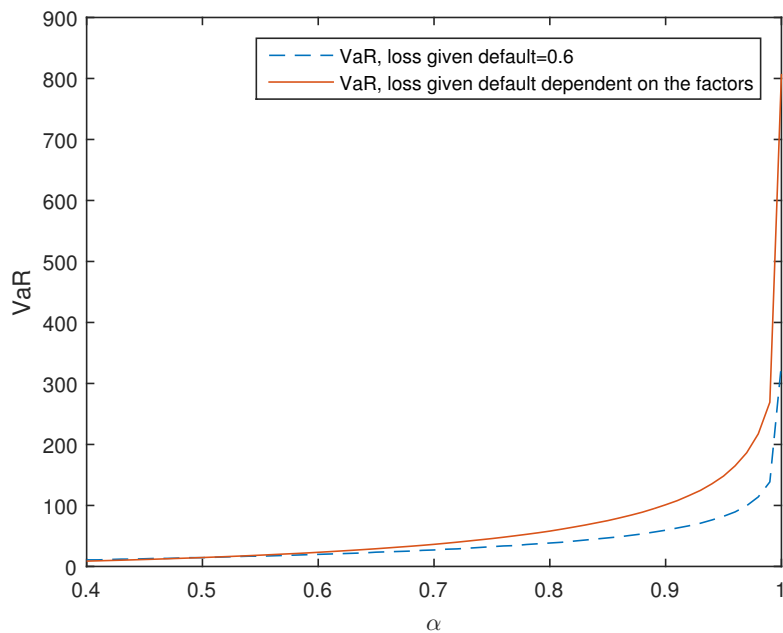


FIGURE 4.10: Value-at-Risk for a two-factor model with constant loss given default and dependent loss given default.

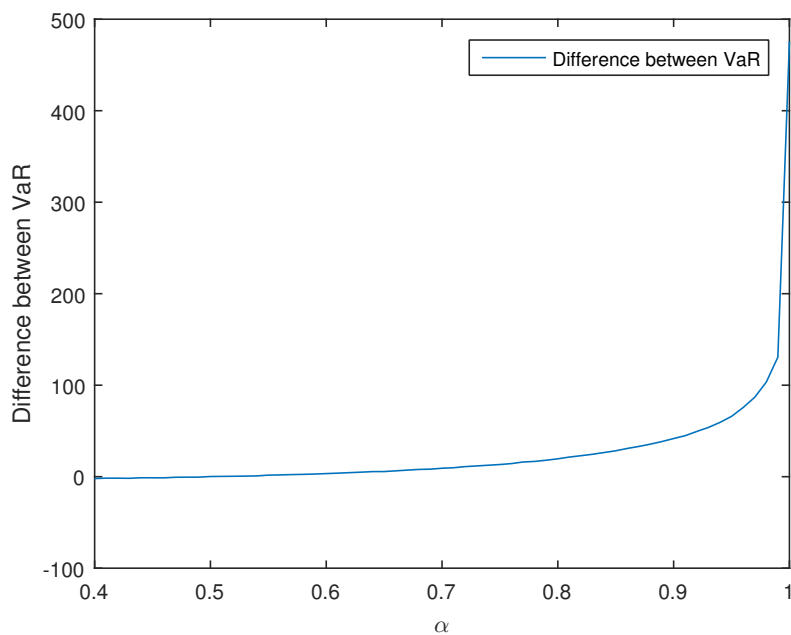


FIGURE 4.11: The difference in Figure 4.10 increases exponentially.

loss given default depend on the factors. The question arises, will the two-factor model with dependent loss given default generate lower or equal VaR than the one-factor model without dependent loss given default, i.e. LPA. This is a motivated question, since the Large Portfolio Approximation of the one-factor model is widely used in the industry, and we are studying the effect of using a more complex model.

Figure 4.12 is a simulation as in Figure 4.8, except for that ℓ , the loss given default, depends on the outcome of the two factors. Value-at-Risk for the two-factor model is sometimes lower than LPA and sometimes higher than LPA. The relationship between VaR, correlation and weights are the same as in Figure 4.8 which means that a higher correlation and more uneven weights increases Value at Risk and hence the possibility of higher Value-at-Risk than LPA.

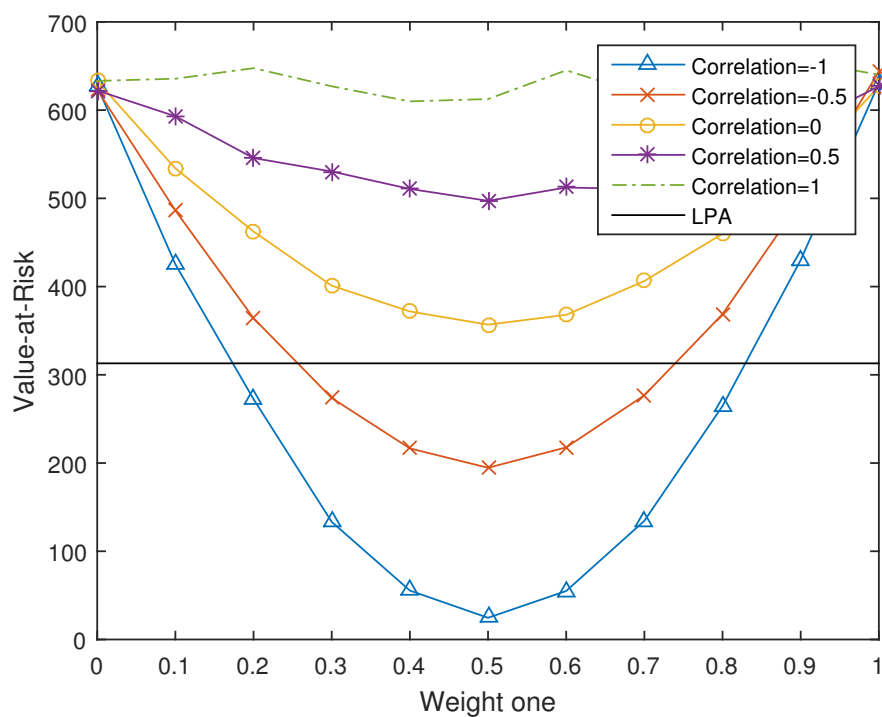


FIGURE 4.12: 10^5 simulations of the Two-factor model with simulated loss given default and LPA

Chapter 5

Conclusion

In this thesis we have compared different static credit portfolio models measuring credit risk. Since the LPA is a popular model, widely used and is a foundation for more complex models, we have studied how the mixed binomial model performs compared to LPA using both a beta-distribution and a two-factor model as our mixing distributions. In our simulations we got several results of Value-at-Risk which for the different models is presented below. A higher Value-at-Risk basically means that more capital is required for the credit risk that the financial institution bears.

By using the LPA formulas we conclude that VaR in the mixed binomial beta model does not differ substantially from Mertons LPA model. A small deviation was shown in the 99:th percentile of the loss distribution where Mertons LPA model had higher VaR than the mixed binomial beta model. Even if there is a difference, its significance is small compared to the impact of choosing and estimating parameters.

VaR in the two-factor model with a constant loss given default is always lower than VaR in the one-factor model, except for when the correlation between the factors was set to one. At that point the two-factor model gives the same results as the one-factor model.

We interpret the lower VaR in the two-factor model as that the model captures a diversification effect in the portfolio. Except for the individual default probability and the asset correlation, VaR in the two-factor model is a function of the correlation between the factors and the weights in the portfolio.

Finally, we simulated the loss given default in the two-factor model as dependent on the outcome of the factors. We found that VaR in this two-factor model can take values both higher and lower than Mertons LPA model. However, given that the factor-correlation is positive, this two-factor model always gives higher VaR than LPA. This means that when choosing between the two-factor model with dependent loss given default, which has to

be simulated, and the more easy to use closed form expression of LPA, one should be aware of that as long as the factor-correlation is positive LPA gives lower Value-at-Risk than the two-factor model. That is, if we know that the factor-correlation is positive and we rather want to overestimate than to underestimate our risk, we should use the two-factor model. This relies on the assumption that the loss given default depends on the factors. If it does not, we are back in the before-mentioned two-factor model with constant loss given default. In that case we should choose LPA prior to the two-factor model if we rather want to overestimate our risk than to underestimate it.

Suggestions for later research that builds on this topic would be to examine the distribution of the loss given default and the outcomes of the factors. We have assumed that these are normally distributed.

Bibliography

- Bluhm, C., Overbeck, L., & Wagner, C. (2003). *An Introduction to Credit Risk Modelling*. Crc Press.
- Herbertsson, A. (2012) *Some Complementary Material to the Course Financial Risk*
- Herbertsson, A (2014) *Lecture notes in the course "Financial Risk"*.
- Hull, J. C. (2012). *Options, futures, and other derivatives*. Pearson Education India.
- Lando, D. (2004). *Credit Risk Modeling: Theory and Applications*. Princeton University Press.
- Markowitz, H. (1952). *Portfolio selection*. The journal of finance, 7(1), 77-91.
- McNeil, A., Embrechts, P. & Frey, R. (2005). *Quantitative risk management*. Princeton Series in Finance, Princeton, 10.
- McNeil, A., & J. Frey, R.(2003). *Dependent defaults in models of portfolio credit risk*. Journal of Risk, 6, 59-92.
- Merton, R. C. (1974). *On the pricing of corporate debt: The risk structure of interest rates*. The Journal of Finance, 29(2), 449-470.
- Saunders, A., & Allen, L. (2010). *Credit risk management in and out of the financial crisis: New approaches to value at risk and other paradigms* (Vol. 528). John Wiley & Sons.
- Schuermann, T. (2004) *What Do We Know About Loss Given Default*, Working paper
- Vasicek, O.A. (1991) *Limiting Loan Loss Distribution* Working paper, KMV Corporation, available at kmv.com.
- Wackerly, D., Mendenhall, W., & Scheaffer, R. (2007). *Mathematical statistics with applications*. Cengage Learning.

Appendix A

Appendix

A.1 Calibration of parameters

In this thesis we among other distributions use the beta-distribution to create "thicker" tails. This can easily be done by changing the parameters a and b . See Figure 3.2 in Subsection 3.4 for an example.

The question remains, how "thick" tails should we create to reflect reality in the best way? This is done by calibrating a and b to some default correlation, estimated or observed, given a individual default probability.

The default correlation between two obligors, ρ_X , is given by

$$\rho_x = \frac{\mathbb{E}[p(Z)^2] - \mathbb{E}[p(Z)]^2}{\mathbb{E}[p(Z)] - \mathbb{E}[p(Z)]^2} \quad (\text{A.1})$$

Also, for a beta-distribution we have the following properties

$$\begin{aligned} \mathbb{E}[p(Z)] &= \frac{a}{a+b} = \bar{p} \\ \mathbb{E}[p(Z)^2] &= \frac{a(a+1)}{(a+b)(a+b+1)} \\ \rho_x^{(B)} &= \frac{1}{a+b+1} \end{aligned}$$

This means that we can solve for a and b in terms of the individual default probability and the correlation between the obligors Solving the system of equations gives us

$$a = \bar{p} \frac{1 - \rho_x^{(B)}}{\rho_x^{(B)}}$$

$$b = (1 - \bar{p}) \frac{1 - \rho_x^{(B)}}{\rho_x^{(B)}}$$

In Table A.1 we see the default correlation between obligors in each cell given some p and ρ in the merton one-factor model. Those correlations all correspond to some estimated value of $\mathbb{E}[p(Z)^2]$. In Table A.2 we have calibrated a and b in the beta distribution to match each correlation in Table A.1.

TABLE A.1: Correlations for given p and ρ

corresponding probability	$\rho = 15\%$	$\rho = 45\%$	$\rho = 70\%$
p=2%	0.0243	0.1260	0.2998
p=2%	0.0578	0.2159	0.4087
p=2%	0.0774	0.2597	0.4556

TABLE A.2: Parameters used in the beta-distribution

Correlation	(a,b)
0.0243	(0.8030,39.3492)
0.0578	(1.6301,14.6709)
0.0774	(2.3840,9.5359)
0.1260	(0.1387,6.7978)
0.2159	(0.3632,3.2686)
0.2597	(0.5701,2.2805)
0.2998	(0.0467,2.2888)
0.4087	(0.1447,1.3021)
0.4556	(0.2390,0.9559)

A.2 The Wiener Process

In this subsection we introduce the Brownian motion, also called a Wiener process.

The Brownian motion is a way of describing a random walk and has been used in physics to describe the random motion of small particles that collides with molecules. As it turns out, the Brownian motion is greatly used in finance and often used to describe stock prices (Hull, 2009).

The Brownian motion is a stochastic process which has a mean change of zero and a variance rate of 1.0 per time unit, T . More specific, a stochastic process W_t follows a brownian motion if the following two properties are satisfied:

1. The change $\Delta W_t = W_{t+\Delta t} - W_t$ during a small period of time Δt has the distribution $X\sqrt{\Delta t}$, where X is a standard normal random variable, i.e. $X \sim N(0, 1)$.
2. The values of ΔW_t for any two different intervals of time, Δt are independent.

From the first property it follows that ΔW_t itself has a normal distribution with mean of 0 and variance Δt , and thus a standard deviation of $\sqrt{\Delta t}$.

From the second property we have that the difference of W_t on two intervals of time in a wiener process are independent. This means that if we sum up those changes in W , the mean will still be zero. The variance of W will, again because of the independence property, be the sum of the variances. That is, $W(T) - W(0)$ is normally distributed with

$$\begin{aligned} \text{mean of } [W(T) - W(0)] &= 0 \\ \text{variance of } [W(T) - W(0)] &= N\Delta t = T \\ \text{standard deviation of } [W(T) - W(0)] &= \sqrt{T} \end{aligned}$$

where $N = \frac{T}{\Delta t}$. We can break the time interval T into smaller and smaller pieces, letting $\Delta \rightarrow 0$ and use the notation dW_t when we refer to a wiener process with the properties above when $\Delta t \rightarrow 0$.

A.2.1 A Generalized Wiener process

In this subsection the Generalized Wiener Process is explained.

What we have described so far is a basic wiener process, we now move on to a generalized wiener process. As shown above, the basic wiener process has a drift rate of zero. This

means that the expected value of a variable W_t will be zero, no matter how long our time period T is. A generalized Wiener process has a *drift rate* as a measure on the mean change of the variable W_t . The drift rate in the basic wiener process is zero. A general wiener process also has a variance rate, which is 1 in the basic wiener process. Hence, a generalized wiener process for a variable W can be described as

$$dW_t =adt + bdX_t \tag{A.2}$$

where a and b are constants and dX_t is a basic Wiener process (Hull, 2009).

Now, relating the subject somehow to the title of this thesis, we extend the wiener process and analyze two or more variables following correlated stochastic processes. For two variables $W_t^{(1)}$ and $W_t^{(2)}$ and two wiener processes $dX_t^{(1)}$ and $dX_t^{(2)}$, we have

$$dW_t^{(1)} = a_1dt + b_1dX_t^{(1)} \quad \text{and} \quad dW_t^{(2)} = a_2dt + b_2dX_t^{(2)}$$

For simplified analysis and perhaps a more intuitive view, this is the discrete version of the processes

$$\Delta W_t^{(1)} = a_1\Delta t + b_1X^{(1)}\sqrt{\Delta t} \quad \text{and} \quad \Delta W_t^{(2)} = a_2\Delta t + b_2X^{(2)}\sqrt{\Delta t}$$

where $X^{(1)}$ and $X^{(2)}$ are standard normal distributed. To create a correlation ρ we set

$$X^{(1)} = \mu \quad \text{and} \quad X^{(2)} = \rho\mu + \sqrt{1 - \rho^2}v$$

where μ and ρ are uncorrelated variables with standard normal distributions.