MASTER'S THESIS

Minimal Surfaces
A proof of Bernstein's theorem

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The surface on the previous page is called the Enneper minimal surface.

Thesis for the Degree of Master of Science

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Abstract

This thesis is meant as an introduction to the subject of minimal surfaces, i.e. surfaces having mean curvature zero everywhere. In a physical sense, minimal surfaces can be thought of as soap films spanning a given wire frame.

The main object will be to prove Bernstein's theorem, which states that a minimal surface in $\mathbb{R}^3$ which is defined in the whole parameter plane is linear, meaning it is a plane. We will give two proofs of this theorem, both involving methods from complex analysis, and relying on a proposition stating that we can always reparametrize the surface into so called isothermal parameters.

Acknowledgements

First of all I would like to thank my supervisor Bo Berndtsson for introducing me to this fascinating subject and for providing the help and guidance I needed to write this thesis.

I would also like to thank Åsa Lideström and Christian Leppänen for helping me with the writing process, Emma Wijkmark for making sure I get up in time most mornings, and Ester Sandström for being my emotional support as per agreement.

Jenny Larsson, Gothenburg, June 2015

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1

Introduction

T HE STUDY OF minimal surfaces is a fascinating subject combining many the areas of mathematics, mostly differential geometry, calculus of variation, complex analysis and geometric measure theory. Relating to the physical world it is the mathematics of soap films spanned by wire frames, which makes it a field full of aesthetically pleasing objects. The close connection to concrete objects makes it possible even for people without much mathematical background to admire the beauty and complexity which minimal surfaces give rise to.

1.1   History

The first person to investigate minimal surfaces was Joseph-Louis Lagrange (1736-1813) in the 1760's. He wanted to find the surface with the least area given some Jordan curve, which is a closed continuous curve without self-intersections, as its boundary. Although Lagrange was the first to consider this problem mathematically, it is known as the problem of Plateau after the physicist Joseph Plateau (1801-1883) who did numerous experiments with soap films investigating this problem.

Plateau's problem can be divided into two parts, to prove the existence of a minimal surface given some boundary, and to have some way of constructing such surfaces when we know that they exists. The problem of Plateau is considered to be part of calculus of variations and is concerned with finding minima and stationary points of functionals.

For some special cases the existence part of the problem of Plateau was solved during the 1800's, but it took until the 1930's until both Radó and Douglas solved it in some generality, independently of one another. Douglas' proof was more general and proved the existence of a minimal surface for any Jordan curve, while Radó only proved it for Jordan curves of finite length. This was considered such an achievement that it earned Douglas the Fields medal in 1936, the first year the Fields medal was awarded.

Plateau's problem is restricted to study minimal surfaces with a single Jordan curve

as its boundary, but this can be generalized considerably by allowing boundaries to be more complicated, for example having multiple curves. The subject of minimal surfaces also contains many more areas of study in addition to this problem.

The two most important contributions to the theory of minimal surfaces during the 1900's are the existence proof by Douglas mentioned above and the theorem of Bernstein which is the main objective of this thesis to prove. The theorem by Bernstein says that the only minimal surface in $\mathbb{R}^3$ which is defined for the whole parameter plane is itself a plane. We will prove this by mostly following the path done by Osserman [8].

1.2   Surfaces with minimal area

In the beginning of the study of minimal surfaces they were seen mostly as solutions to a special partial differential equation, and later it was realized that solving this was equivalent to having mean curvature zero everywhere. Intuitively this means that each point on the surface is a saddle point having largest and smallest normal curvature of equal magnitude but with opposite signs. The exact definition of this requires some work and will not be given until section 2.3.

Any surface which has the smallest area, at least locally, will also satisfy having mean curvature zero, i.e. be a minimal surface. The opposite is not necessarily true, there are minimal surfaces which are not surface with smallest area. A minimal surface which is not locally a surface of minimal area is called an unstable minimal surface. For unstable minimal surfaces, like saddle points, there exists arbitrarily small perturbations which will alter the surface to having a smaller area but without perturbations it will not change. If the surface on the other hand is stable, then small perturbations will only lead to the surface going back to its stable shape.

Because of the physical properties of soap solutions, the surface spanned by a soap film on a wire frame will be a minimal surface having, at least locally, minimal area. The unstable minimal surfaces will never exist as soap films for any length of time, since there will always be small perturbations disturbing the surface which will make it change to a smaller and stable surface. The stable minimal surfaces are the ones Plateau studied, and their geometric beauty is one of the reasons why so many mathematicians have become intrigued by this problem.

The reasons for soap film behaving like minimal surfaces has to do with the effects of soap on the surface tension of water, and the minimal possible thickness a soap film can achieve. For more information about the physics behind this see the article by Almgren and Taylor [2].

Soap bubbles on the other hand will not satisfy the minimal surface equation since they have constant non-zero mean curvature. This is because there is a difference in pressure on the inside and outside of the bubble, something which is not the case with soap films on a given boundary. The study of surfaces with constant mean curvature is a more general problem (since constant equal to zero is just a special case) than the study of minimal surfaces which, although interesting, will not be discussed in this thesis.

Minimal surfaces are studied in several other disciplines besides mathematics, for

example they arise in molecular engineering, material science, architecture and the study of black holes.

1.3 Examples of minimal surfaces

Not counting the trivial case of a plane, there are three classical examples of minimal surfaces in $\mathbb{R}^3$ which we are going to introduce here to let the reader get an intuition for the subject. None of these three surfaces are defined for the whole x,y-plane in non-parametric form, which is no coincidence but, as we will see later, a consequence of Bernstein's theorem.

Catenoid

The catenoid was the first (non-trivial) minimal surface to be found, and it was discovered and shown minimal by Leonhard Euler in 1744 [7]. You can get this surface by dipping two parallel circles of wire into a soap solution and holding them not too far away from each other, see fig. 1.1. The boundary of this minimal surface is thus two separated circles. It is the only minimal surface which is also a surface of revolution, which means that it can be obtained by rotating a plane curve around a straight line in that plane, for proof see [6]. The equations for the catenoid are

$$
\begin{aligned}
x(u,v) &= a \cosh\left(\frac{v}{a}\right)\cos(u), \\
y(u,v) &= a \cosh\left(\frac{v}{a}\right)\sin(u), \\
z(u,v) &= v,
\end{aligned}
\tag{1.1}
$$

where $u,v \in \mathbb{R}$ and a is some non-zero constant related to its size. It can also be written in non-parametric form as $f(x,y) = \operatorname{arccosh}\left(\sqrt{x^2 + y^2}\right)$.

The area of a catenoid obtained from two circles having distance $2h$ from each other and radius $r = \cosh(h)$ is $2\pi(h + \sinh(h)\cosh(h))$, for calculation see example 2.11. Another possible minimal surface, or pair of surfaces to be exact, which can form on this boundary is the pair of flat discs each in one of the circles. These discs would together have a total area of $2\pi \cosh^2(h)$. So, depending on the distance and radius the surface of least area will be the catenoid if $h + \sinh(h)\cosh(h) < \cosh^2(h)$, and the discs if the reverse inequality holds.

In addition to the catenoid and the two separated discs there is in fact one more minimal surface spanned by the two circles. It is usually called the *inner catenoid*, since it looks like a catenoid with a smaller waist, and it is an example of an unstable minimal surface [4]. Since it is an unstable minimal surface it is not locally a surface of least area.

Helicoid

The second minimal surface to be discovered was the helicoid, found by Jean Baptiste Meusnier in 1774 [7]. The helicoid has gotten its name from its similarity to the helix, a
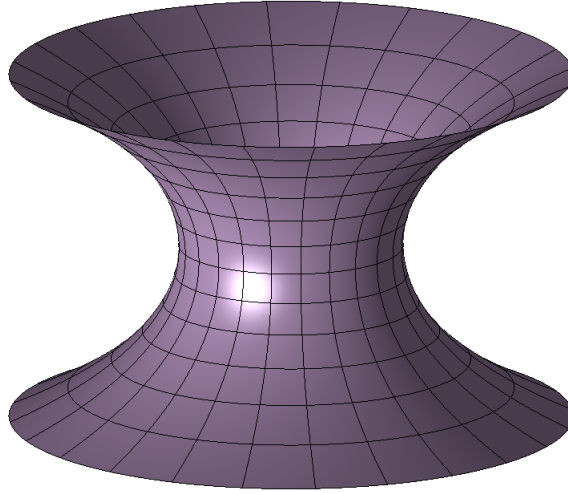
Figure 1.1: Catenoid

spiral circling an axis with constant rotation around the axis and constant speed parallel to the axis. In fact, at each point of this surface there is a helix contained in the surface going through that point. It is the only non-trivial minimal surface which is also a ruled surface, meaning that for each point we can find a straight line contained in the surface and going through that point, for a proof see [9].

In the sense of soap films, the helicoid can be obtained by dipping a wire frame in the shape of either a helix with center axis or a double helix into a soap solution. Figure 1.2 can be seen as having the boundary of a double helix, i.e. two helices circling the same axis, but on opposite sides of it.

It can be defined by the equations

$$
\begin{aligned}
x(u,v) &= v\cos(au),\\
y(u,v) &= v\sin(au),\\
z(u,v) &= u,
\end{aligned}
\tag{1.2}
$$

where a is a constant related to the rotation. It gives a right-handed helicoid if $a > 0$, left-handed if $a < 0$ and a plane if $a = 0$. In non-parametric form the defining equation becomes $f(x,y) = \arctan(x/y)$.

There exists an isometric deformation between the helicoid and the catenoid. Isometric means that the deformation preserves distances between points. The deformation
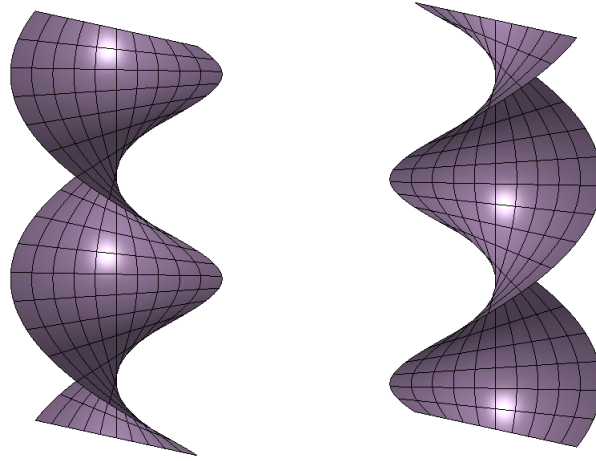
Figure 1.2: One left-handed and one right-handed helicoid.

can be written as

$$
\begin{aligned}
x &= \sin(\theta)\cosh(v)\cos(u) + \cos(\theta)\sinh(v)\sin(u), \\
y &= \sin(\theta)\cosh(v)\sin(u) - \cos(\theta)\sinh(v)\cos(u), \\
z &= u\cos(\theta) + v\sin(\theta),
\end{aligned}
\tag{1.3}
$$

where $(u,v) \in (-\pi,\pi] \times (-\infty,\infty)$ and θ is the deformation parameter in the interval $(-\pi,\pi]$. For $\theta = \pi$ we have a right handed helicoid, $\theta = 0$ a left handed helicoid, and for $\theta = \pm\pi/2$ we have catenoids. Every member of this family of surfaces is in fact also a minimal surface.

Scherk's surface

Scherk's surface was the third minimal surface to be discovered and this was done by Heinrich Scherk in 1834 [7]. Scherk actually discovered several minimal surfaces, but the one usually referred to as his surface is sometimes also known as Scherk's first minimal surface, and is the one having the simplest equation.

Scherk's surface is the shape of a soap film having the boundary of a square which is bent upward on two opposing sides and downward on the other two sides as in fig. 1.3.

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This surface has the defining equation in non-parametric form

$$f(x,y) = \ln\left(\frac{\cos(x)}{\cos(y)}\right). \tag{1.4}$$
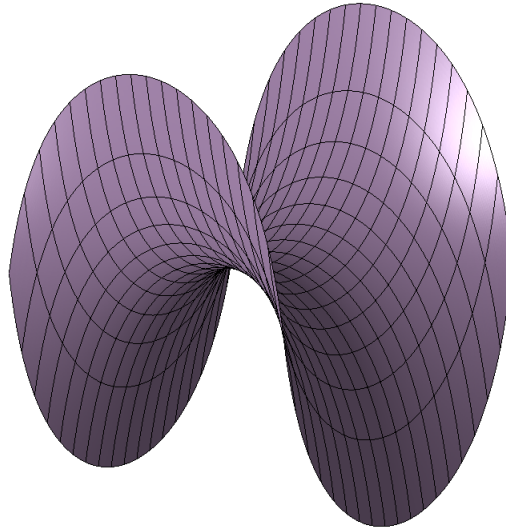
Figure 1.3: Scherk's surface

It is defined only when $\cos(u)$ and $\cos(v)$ have the same sign, which is on every other square on a chessboard like pattern throughout $\mathbb{R}^2$. At the edges of these squares, the surface goes to plus or minus infinity. The squares have their vertices in the points $(\pi/2 + m\pi, \pi/2 + n\pi)$ for $m,n \in \mathbb{Z}$ and centers in $(m\pi, n\pi)$, which gives that the surface is defined when $m + n$ is even.

Scherk's surface is the only minimal surface of translation, which means that it can be written as a sum of two functions each depending on only one variable. This reformulation is done trivially as $\ln\left(\frac{\cos(x)}{\cos(y)}\right) = \ln(\cos(x)) - \ln(\cos(y))$. That it is a surface of translation means that it looks the same on each square on which it is defined.

2

Preliminaries

T HE GOAL of this chapter is to properly define what a minimal surface is. We will start off by defining a k-dimensional manifold and some concepts related to them, and then restrict us to the 2-dimensional case with surfaces. Intuitively, a k-dimensional manifold is just a set in $\mathbb{R}^n$ which locally around each point looks like a piece of $\mathbb{R}^k$.

First we recall the inverse function theorem. Since this will be used later without much consideration, a reminder of its statement could be useful. We will not prove it, but a proof can be found in [11] for example.

THEOREM 2.1: *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ where $f \in C^1$ in some open set $D \subset \mathbb{R}^n$ containing a point a such that $f'(a) \neq 0$. Then there exists open sets $V \ni a$, $W \ni f(a)$ such that $f : V \to W$ has an inverse $f^{-1} : W \to V$ which is also C^1 and satisfies $(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}$.*

DEFINITION 2.2: A function which is a differentiable bijection with differentiable inverse is called a *diffeomorphism.*

Note that in order for this theorem to be applicable we need the function to go to a space of the same dimension as its domain of definition. When we later use this theorem we will usually have a function mapping to a higher dimensional space, but then we will first restrict it to only consider a part which maps into a subspace of the same dimension as the domain of definition.
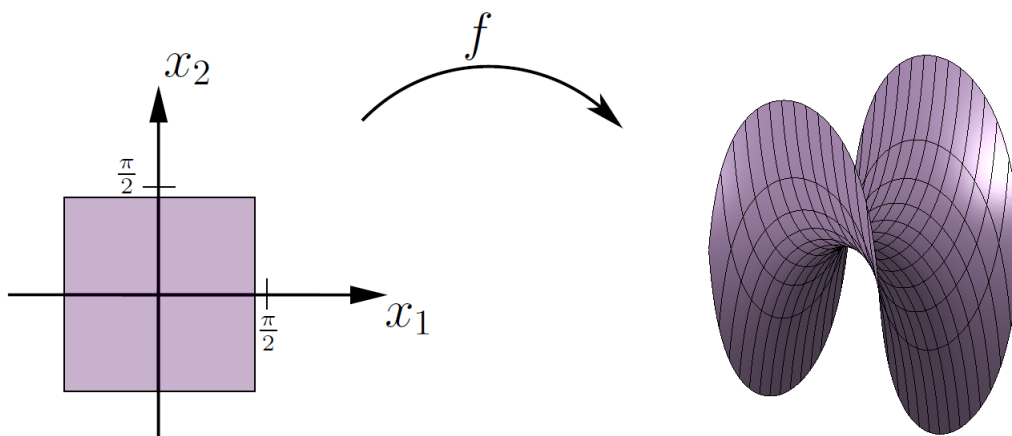
2.1 Manifolds

There are several equivalent ways of defining a manifold, but since we will work with parametrizations, we will choose the one related to local coordinate systems. It is possible to define manifolds without differentiability but in this thesis we will always have it, so for simplicity we include it in the definition. This section will be based on Spivak [11].

DEFINITION 2.3: A subset $\mathcal{M} \subset \mathbb{R}^n$ is a *k-dimensional (differentiable) manifold* if $\forall p \in \mathcal{M}$ there exists an open neighbourhood $U \ni p$, an open set $W \subset \mathbb{R}^k$ and a one-to-one differentiable function $f : W \to \mathbb{R}^n$ such that

i) $f(W) = \mathcal{M} \cap U$

ii) The Jacobian $f'(x)$ has rank k, $\quad \forall x \in W$

iii) $f^{-1} : f(W) \to W$ is continuous.

This function f is called a *local coordinate map* around p, and for $x = (x_1, \ldots, x_k) \in W$ we say that the x_i's are *local parameters*.

As a consequence of the inverse function theorem 2.1, the function f is locally a diffeomorphism onto a k-dimensional subspace of $\mathbb{R}^n$. This also gives that if $f_1 : W_1 \to \mathbb{R}^n$ and $f_2 : W_2 \to \mathbb{R}^n$ are two local coordinate maps around the same point p on the manifold $\mathcal{M}$, the composition $f_2^{-1} \circ f_1 : f_1^{-1}(f_2(W_2)) \to \mathbb{R}^k$ is a local diffeomorphism, given that we have chosen the open sets W_1 and W_2 such that $f_2(W_2) \subseteq f_1(W_1)$. This composition maps one set of local parameters to another and is called a *reparametrization*.

Figure 2.1: A map f defining the part of Scherk's surface seen before.

Manifolds can have the property of being oriented or not. An oriented manifold can be defined to have an "interior" and an "exterior", which is not the case if it is non-orientable. The classical example of a non-orientable surface is the Möbius strip, which is a 2-dimensional manifold with only one side. There are in fact Möbius strip-like minimal surfaces, see fig. 2.2, so non-orientable manifolds are indeed important. But any non-orientable manifold corresponds to an oriented manifold through a local diffeomorphism for which the inverse image of any point on the non-orientable manifold consists of two

points on the orientable [8]. In this thesis this correspondence will be enough, so we can restrict ourselves to only consider orientable manifolds.
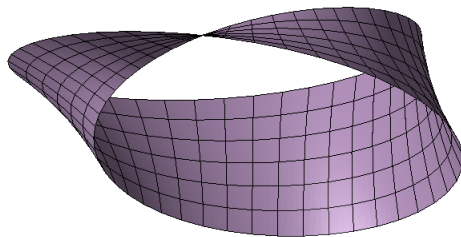
Figure 2.2: A Möbius strip-like minimal surface, an example of a non-orientable 2-manifold. The equation defining this surface can be found in [1].

Before we are able to define volume of a manifold we have to make some other definitions. First, we need to introduce the notation $\mathbb{R}_p^n$, which for $p \in \mathbb{R}^n$ means the set of all pairs (p,v) such that $v \in \mathbb{R}^n$, usually (p,v) is denoted v_p and is called the vector v at p. $\mathbb{R}_p^n$ is called the tangent space of $\mathbb{R}^n$ at p and induces an inner product as $\langle v_p, w_p \rangle_p = \langle v, w \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$.

Next, we need to define the tangent space of a manifold and also find an inner product on this space. To do so we need to define the *pushforward* $f_* : \mathbb{R}_p^k \to \mathbb{R}_{f(p)}^n$ of a function f as the linear transformation taking a vector v_p, transforming it by the Jacobian matrix of f at p and associate it to the point $f(p)$, i.e.

$$f_*(v_p) = (f'(p)(v))_{f(p)}. \tag{2.1}$$

This transformation is one-to-one since $f'(p)$ has rank k and thus the image $f_*(\mathbb{R}_p^k)$ is a k-dimensional vector space.

DEFINITION 2.4: Let $\mathcal{M} \in \mathbb{R}^n$ be a k-dimensional manifold with coordinate map f around a point $p \in \mathcal{M}$. Then the k-dimensional space $f_*(\mathbb{R}_p^k)$ is called the *tangent space* of $\mathcal{M}$ at p and will be denoted $T_p(\mathcal{M})$.

Examples of a coordinate map and a tangent space can be found in figures 2.1 and 2.3 respectively.
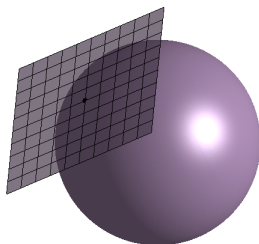
Figure 2.3: The sphere is an example of a 2-manifold, here illustrated with its tangent space at one point.

On this tangent space we can define a natural inner product as in the case of the tangent space for $\mathbb{R}^n$ above. Let v_p, $w_p \in T_p(\mathcal{M})$, then the inner product is as before $\langle v_p, w_p \rangle_p = \langle v, w \rangle$. This inner product defines a metric on the manifold which for each point $p \in T_p(\mathcal{M})$ and pair of tangent vectors associates the real value $\langle v_p, w_p \rangle_p$.

Since the tangent space is a k-dimensional vector space it has a basis consisting of k elements, and the standard basis to choose is $\left\{ \frac{\partial f}{\partial x_i} \right\}$, which is known as the coordinate basis. Thus it is enough to know the value associated to all pairs of vectors from this basis, and this gives that the metric is a $k \times k$ matrix G consisting of

$$g_{ij}(p) = \left\langle \frac{\partial f}{\partial x_i}(p), \frac{\partial f}{\partial x_j}(p) \right\rangle_p. \tag{2.2}$$

DEFINITION 2.5: Let $\mathcal{M}$ be an oriented k-dimensional manifold in $\mathbb{R}^n$. For a point $p \in \mathcal{M}$ we have that the orientation and the inner product at p determine a *volume element* $dV = \sqrt{\det G}\, dx_1 \wedge \cdots \wedge dx_k$, where G is the metric and $\wedge$ is the wedge product. The *volume* of the manifold $\mathcal{M}$ is then defined as $\int_{\mathcal{M}} dV$.

When $k = 2$ we call this the *area element*, usually denoted as dA or dS, and becomes just $\sqrt{\det G}\, dx_1 dx_2$ where G then is a 2×2 matrix.

2.2 Surfaces and parametrizations

Here we will define a regular surface and see that it is essentially the same as a 2-dimensional manifold. The concept of a surface should be something that locally looks

like a piece of $\mathbb{R}^2$. It will be useful to change some notations here to avoid confusion in the later part of the thesis. This section is based on [8].

An open and connected set is called a *domain* and we will only be working with domains in $\mathbb{R}^2$.

DEFINITION 2.6: Let $u_1, u_2 \in \mathbb{R}$ be parameters, D a domain in the u_1, u_2-plane, and let $x(u)$ be a differentiable transformation of D into $\mathbb{R}^n$. Then we say that the map $\mathcal{S} = x(D)$ is a *surface* in $\mathbb{R}^n$.

If the map x is r times continuously differentiable for some $r \in \mathbb{N}$, denoted $x \in C^r$, we say that $\mathcal{S}$ is a C^r-surface.

DEFINITION 2.7: For such a transformation, we define the *metric* matrix $G = (g_{ij})$ as

$$g_{ij} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} = \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}. \tag{2.3}$$

In fact $G = J^T J$, where J is the Jacobian matrix of x. Note that this gives the same metric as defined in (2.2).

LEMMA 2.8: *Let $x(u) : D \to \mathbb{R}^n$ be a differentiable map for $D \in \mathbb{R}^2$. For each point in D the following are equivalent:*

   i) the vectors $\frac{\partial x}{\partial u_1}$, $\frac{\partial x}{\partial u_2}$ are independent,

   ii) the Jacobian matrix of x has rank 2,

   iii) $\exists i, j$ such that $\det\left(\frac{\partial(x_i, x_j)}{\partial(u_1, u_2)}\right) \neq 0$, i.e. some subdeterminant of the Jacobian is non-zero,

   iv) $\det G > 0$.

PROOF. By using facts from linear algebra, we obtain the equivalences fairly straightforward.

$i) \iff ii)$ That the vectors $\frac{\partial x}{\partial u_1}$, $\frac{\partial x}{\partial u_2}$ are linearly dependent means that the Jacobian matrix has 2 independent rows. This is equivalent to it having 2 independent columns, i.e. rank 2.

$ii) \iff iii)$ If the Jacobian has two independent columns then the subdeterminant of these will be non-zero. Similarly, any submatrix with non-zero determinant will consist of two independent columns.

$i) \iff iv)$ If the two vectors are linearly dependent then $\frac{\partial x}{\partial u_2} = c \frac{\partial x}{\partial u_1}$ for some constant c. But then

$$\det(G) = \left(\frac{\partial x}{\partial u_1}\right)^2 \left(c \frac{\partial x}{\partial u_1}\right)^2 - \left(\frac{\partial x}{\partial u_1} \cdot c \frac{\partial x}{\partial u_1}\right)^2 = 0. \tag{2.4}$$

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Conversely, if $\det(G) = 0$, then $\left(\frac{\partial x}{\partial u_1}\right)^2 \left(\frac{\partial x}{\partial u_2}\right)^2 = \left(\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2}\right)^2$. This is the Cauchy-Schwarz inequality with equality, which can only occur when the vectors are dependent.

$\square$

DEFINITION 2.9: A surface is *regular at a point* if lemma 2.8 holds at that point, and hence in a neighbourhood of that point. If a surface is regular at every point we say that we have a *regular surface*. If a surface is not regular at a point, we say that it is *singular* there.

For the rest of the thesis we will assume that the surface is at least C^1. If we for some $r \geq 1$ have a surface $\mathcal{S}$ defined by $x(u) \in C^r(D)$, and $u(\tilde{u}) \in C^r(\widetilde{D})$ is a diffeomorphism of a domain $\widetilde{D}$ onto D, then the surface $\widetilde{\mathcal{S}}$ defined by $x(u(\tilde{u}))$ is said to be obtained from $\mathcal{S}$ by a *change of parameters*. If a property of $\mathcal{S}$ also holds for corresponding points of all surfaces $\widetilde{\mathcal{S}}$ obtained by a change of parameters, we say that it is *independent of parameters*. That some properties of a surface are unchanged by reparametrizations is very useful since we then can choose parameters with good properties as we will see later.

Let $U = \left(\frac{\partial(u_1, u_2)}{\partial(\tilde{u}_1, \tilde{u}_2)}\right)$ be the Jacobian matrix for a change of parameters as above and J, $\widetilde{J}$ the Jacobian matrices for $x(u)$ and $x(u(\tilde{u}))$ respectively. Then the determinant of U is clearly non-zero in the domain $\widetilde{D}$, and by the chain rule

$$\frac{\partial x_i}{\partial \tilde{u}_k} = \sum_{j=1}^{2} \frac{\partial x_i}{\partial u_j} \frac{\partial u_j}{\partial \tilde{u}_k}. \tag{2.5}$$

We also get an expression for the new metric $\widetilde{G}$ in terms of the old one G as

$$\widetilde{G} = \widetilde{J}^T \widetilde{J} = (JU)^T (JU) = U^T J^T J U = U^T G U. \tag{2.6}$$

Now since $\det \widetilde{G} = \det G (\det U)^2$ and $(\det U)^2 > 0$ we get that $\det G > 0$ if and only if $\det \widetilde{G} > 0$. This means that the regularity of a surface is independent of parameters. Consequently, if a surface is regular for some parameters it will be regular for whichever parameters we choose to represent it with, as long as there is a diffeomorphic transformation between the pairs of parameters.

DEFINITION 2.10: Suppose that $\Omega \subset D$ such that the closure $\overline{\Omega} \subset D$ for a domain D. Let Σ be the restriction of the surface $x(u)$ to $u \in \Omega$. The *area* of Σ is defined as

$$A(\Sigma) = \iint\limits_{\Omega} \sqrt{\det G} \, du_1 \, du_2. \tag{2.7}$$

As remarked before, this is the 2-dimensional volume element from definition 2.5.

EXAMPLE 2.11: Find the area of the catenoid defined in 1.1 (note that we have changed notation). We first need the vectors

$$\frac{\partial x}{\partial u_1} = \left(-\cosh(u_2)\sin(u_1),\ \cosh(u_2)\cos(u_1), 0\right),$$
$$\frac{\partial x}{\partial u_2} = \left(\sinh(u_2)\cos(u_1),\ \sinh(u_2)\sin(u_1), 1\right),$$

(2.8)

which gives the metric

$$G = \begin{pmatrix} \cosh^2(u_2) & 0 \\ 0 & \cosh^2(u_2) \end{pmatrix}.$$

(2.9)

The area of a catenoid bounded by $|z| < h$ is thus

$$\int_0^{2\pi} \int_{-h}^{h} \sqrt{\cosh^4(u)}\, du_1 du_2 = 2\pi(h + \sinh(h)\cosh(h)).$$

(2.10)

The area can easily be shown to be independent of parameters: Assume that $x(u)$ is a surface and $u(\tilde{u}) : \widetilde{D} \to D$ a transformation of parameters which maps $\widetilde{\Omega}$ onto Ω with Jacobian matrix U. Then using standard variable substitution rules for integration we obtain

$$A(\widetilde{\Sigma}) = \iint_{\widetilde{\Omega}} \sqrt{\det \widetilde{G}}\, d\tilde{u}_1\, d\tilde{u}_2 = \iint_{\widetilde{\Omega}} \sqrt{\det G}\, |\det U|\, d\tilde{u}_1\, d\tilde{u}_2$$
$$= \iint_{\Omega} \sqrt{\det G}\, du_1\, du_2 = A(\Sigma).$$

(2.11)

So the area does not change under reparametrization.

Now we introduce one of the more useful parametrizations and prove that for any regular surface it is possible to reparametrize to this form.

DEFINITION 2.12: Let $x_1 = u_1$, $x_2 = u_2$ and D be a domain in the (x_1,x_2)-plane, then a surface $\mathcal{S}$ defined by

$$x_k = f_k(x_1,x_2), \quad k = 3,\ldots,n,$$

(2.12)

for $(x_1,x_2) \in D$ and f_k differentiable function $f : D \to \mathbb{R}$, is said to be in *non-parametric form*.

Note that if the domain of definition instead is the (x_i,x_j)-plane where $i \neq j$ and $i,j \neq 1,2$, we can always rename them to get $i = 1$ and $j = 2$ anyway.

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PROPOSITION 2.13: *Let S be a surface defined by $x(u) \in C^r$ and let $\mathcal{S}$ be regular at a point a. Then there exists a neighbourhood $\Omega \ni a$ such that the surface Σ obtained by restricting $x(u)$ to Ω has a reparametrization $\widetilde{\Sigma}$ in non-parametric form.*

PROOF. By regularity condition iii) of lemma 2.8 there exists indices i,j such that $\det\left(\frac{\partial(x_i,x_j)}{\partial(u_1,u_2)}\right) \neq 0$, so we can use the inverse function theorem 2.1 to get that there exists a neighbourhood $\Omega \ni a$ where the map $(u_1,u_2) \mapsto (x_i,x_j)$ is a diffeomorphism.

Since $x(u) \in C^r$, we also have that the inverse map $(x_i,x_j) \mapsto (u_1,u_2)$ is C^r, and thus the same holds for the composition $(x_i,x_j) \mapsto (u_1,u_2) \mapsto (x_1,\ldots,x_n)$. Renaming the indices if necessary, this defines our surface in non-parametric form.

□

We will now prove that the definition of surface is just a special case of the definition of manifold.

PROPOSITION 2.14: *Assume $x(u) \in C^r$, where $r \geq 1$. If x defines a regular surface $\mathcal{S}$ for some domain $D \subset \mathbb{R}^2$, then $x(u)$ is a 2-dimensional manifold. Conversely, if $\mathcal{S}$ is a 2-dimensional manifold, then locally at each point it defines a surface.*

PROOF. Assuming that a surface $\mathcal{S}$ defined by $x(u) \in C^r$ is regular on a set $D \in \mathbb{R}^2$ and has a Jacobian matrix with rank 2. Then by the inverse function theorem there exists for any point $p \in \mathcal{S}$ two open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^n$ such that $p \in V$, $x(p) \in U$ and the restricted map $x : U \to V$ is a diffeomorphism with C^r-inverse $x^{-1} : V \to U$, for which $x(U) = \mathcal{S} \cap V$. Thus it satisfies the definition of being a manifold.

For the converse, let $x(u)$ be a local coordinate map defining a manifold as in definition 2.3. Then we get immediately that for each point there exists neigbourhoods U, W as in the definition such that $\mathcal{S} \cap U$ defined by $x(W)$ is a regular surface.

□

For generalization to the theory of minimal surfaces it is often necessary to use the more general concept of manifolds from definition 2.3, but for the purposes of this thesis it is enough to have this simpler definition of surfaces. This is due to the fact that almost all of the concepts we will be working with are of local nature.

2.3 Curvatures

To study surfaces properly we need to have some way of knowing how it bends in the surrounding space, we need some measure of curvature. There are in fact several different types of curvatures, as we will see in this section, but for this thesis we are mostly interested in the so-called principal curvatures. The principal curvatures are measures of how much the directions with the biggest and smallest curvature bends at a given point. In order to find these values, we first need to look at curves on our surfaces.

All surfaces in this section are assumed to be regular, at least C^1 and in non-parametric form. This section is based on Osserman [8].

DEFINITION 2.15: A *curve* $\mathcal{C}$ is a function $\gamma(t) = (\gamma_1(t),\ldots,\gamma_n(t)) \in C^1(I)$ for some open interval $I = (\alpha,\beta) \subset \mathbb{R}$.

Since all surfaces will be C^1 it is enough to consider curves which are also C^1, so for simplicity we have included it in the definition. The *tangent vector* of a curve $\mathcal{C}$ at a point $t_0 \in I$ is denoted $\gamma'(t_0) = (\gamma_1'(t_0),\ldots,\gamma_n'(t_0))$, and γ is said to be *regular* at t_0 if $\gamma'(t_0) \neq 0$. The curve *lies on the surface* $\mathcal{S}$ if $\gamma(I) \subset \mathcal{S}$. Note that a regular curve is a 1-dimensional manifold.

Similarly to the case of a surface, a reparametrization $\tilde{\gamma}$ of a regular curve γ is a diffeomorphism $\phi : (\tilde{\alpha},\tilde{\beta}) \to (\alpha,\beta)$ such that $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for all $\tilde{t} \in (\tilde{\alpha},\tilde{\beta})$.

Now, for any regular point p on a given C^1-surface $\mathcal{S}$, consider the set of all curves on the surface going through this point. We can assume for simplicity that all such curves go through this point when $t = t_0$, i.e. $\gamma(t_0) = p$ for all such curves γ.

Let a be the point in the parameter plane D for which $x(a) = p$. Since $\mathcal{S}$ is in non-parametric form there is clearly a one-to-one correspondence between curves $\gamma(t)$ going through p on the surface and curves $u(t) = (u_1(t),u_2(t))$ going through a in the parameter plane. That is $\gamma(t) \subset \mathcal{S}$ corresponds to $u(t) \subset D$, where $\gamma(t_0) = p$, $u(t_0) = a$ and $\gamma(t) = x(u(t))$.

Applying the chain rule gives us the expression $\gamma'(t) = x'(t) = \frac{\partial x}{\partial u_1} u_1'(t) + \frac{\partial x}{\partial u_2} u_2'(t)$, and by regularity of $\mathcal{S}$, that $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ are linearly independent. Considering all curves γ through p corresponds to considering all curves u through a, so u_1' and u_2' can have any real values. Thus the set

$$\{\gamma'(t_0)\} = \left\{ v_1 \frac{\partial x}{\partial u_1} + v_2 \frac{\partial x}{\partial u_2}; \ v_1,v_2 \in \mathbb{R} \right\} \tag{2.13}$$

of tangent vectors of $\mathcal{S}$ at p is a 2-dimensional vector space.

The set above is called the *tangent plane*. It is independent of parameters since a reparametrization $u(\tilde{u})$ gives that

$$\gamma'(t) = \frac{\partial x}{\partial u_1} \left(\frac{\partial u_1}{\partial \tilde{u}_1} \tilde{u}_1' + \frac{\partial u_1}{\partial \tilde{u}_2} \tilde{u}_2' \right) + \frac{\partial x}{\partial u_2} \left(\frac{\partial u_2}{\partial \tilde{u}_1} \tilde{u}_1' + \frac{\partial u_2}{\partial \tilde{u}_2} \tilde{u}_2' \right), \tag{2.14}$$

which is still just a linear combination of $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ for which the coefficients can take any values in $\mathbb{R}$. As we will see in the following lemma, this corresponds to the tangent space defined before for $k = 2$.

LEMMA 2.16: *For a surface $\mathcal{S}$ defined by $x(u)$, and a point $p = x(a) = \gamma(t_0) \in \mathcal{S}$, the definitions of tangent plane (2.13) and 2-dimensional tangent space in definition 2.4 are equivalent, i.e. $x_*(\mathbb{R}_a^2) = \{\gamma'(t_0)\}$.*

PROOF. For an arbitrary vector $v_p \in \mathbb{R}_p^2$ we have

$$x_*(v_a) = (x'(a)(v))_p = \left(\frac{\partial x}{\partial u_1} v_1 + \frac{\partial x}{\partial u_2} v_2 \right)_p. \tag{2.15}$$

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This means that at each point $p \in \mathcal{S}$ the set $T_p(\mathcal{S}) = \{x_*(v_a)\}$ is all linear combinations of $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ since v_1, v_2 can be any values in $\mathbb{R}$. Thus $T_p(\mathcal{S})$ is by definition the same as $\{\gamma'(t_0)\}$.

$\square$

The square of the length of a tangent vector is

$$|x'(t)|^2 = \sum_{i,j=1}^{2} g_{ij} u_i'(t) u_j'(t), \tag{2.16}$$

which is called the *first fundamental form*. This expression is often written as $g_{11} du_1^2 + 2g_{12} du_1 du_2 + g_{22} du_2^2$. Note the connection between the determinant of this expression and the area element from definition 2.10.

EXAMPLE 2.17: Finding the metric and first fundamental form for the helicoid.
First, we calculate the tangent vectors

$$\frac{\partial x}{\partial u_1} = (-au_2 \sin(au_1), au_2 \cos(au_1), 1), \quad \frac{\partial x}{\partial u_2} = (\cos(au_1), sin(au_1), 0). \tag{2.17}$$

This gives that the matrix for the metric is

$$G = \begin{pmatrix} 1 + (au_2)^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.18}$$

and we get the expression $du_1^2 + (1 + (au_2)^2) du_2^2$ for the first fundamental form.

In order to get unit tangent vectors we will reparametrize the curves $\mathcal{C}$ with respect to arclength. This means that we want to find a parametrization for which the tangent vector has length one at all points on the curve.

For any regular curve $\gamma(t)$, where $t \in [\alpha, \beta]$, we define a function

$$s(t_0) = \int_{\alpha}^{t_0} |\gamma'(t)| dt, \quad \text{for } t_0 \in [\alpha, \beta]. \tag{2.19}$$

Then $s(\beta) = L$ is the length of the curve, and $s'(t_0) = |\gamma'(t_0)| > 0$ since γ is regular. This gives that s has a differentiable inverse $t(s)$, and we can define the composite function

$$\gamma(s) : [0, L] \xrightarrow{t(s)} [\alpha, \beta] \xrightarrow{\gamma(t)} \mathcal{C}. \tag{2.20}$$

This is a reparametrization of $\mathcal{C}$ with respect to arclength since at each point we have the tangent vector $T = \frac{d\gamma}{ds} = \frac{dx}{ds}$, where $\left|\frac{dx}{ds}\right| = 1$ since $s'(t_0) = |\gamma'(t_0)|$.

In this next part we are going to look at second order effects, so from now on we are going to assume that all surfaces and curves are at least C^2. The derivative of the unit tangent vector with respect to s is the curvature vector

$$\frac{dT}{ds} = \frac{d^2 x}{ds^2} = \sum_{i=1}^{2} \left(\frac{d^2 u_i}{ds^2} \frac{\partial x}{\partial u_i} + \sum_{j=1}^{2} \frac{du_i}{ds} \frac{du_j}{ds} \frac{\partial^2 x}{\partial u_i \partial u_j} \right). \tag{2.21}$$

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Let the space $N_p(\mathcal{S})$ denote the normal space, i.e. the set of all vectors in $\mathbb{R}^n$ orthogonal to the tangentspace $T_p(\mathcal{S})$. This space $N_p(\mathcal{S})$ is of course an $(n-2)$-dimensional space in $\mathbb{R}^n$ and together with the tangentspace we have $T_p(\mathcal{S}) \times N_p(\mathcal{S}) = \mathbb{R}^n$. This also means that any vector in $\mathbb{R}^n$ is uniquely determined by its projection into these two subspaces, and therefore any vector $N \in N_p(\mathcal{S})$ is orthogonal to both $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$. Multiplying (2.21) with N gives therefore a function

$$
\begin{aligned}
k(N,T) :&= \frac{d^2 x}{ds^2} \cdot N = \sum_{i,j=1}^{2} b_{ij}(N) \frac{du_i}{ds} \frac{du_j}{ds}, \quad \text{where} \\
b_{ij}(N) &= \frac{\partial^2 x}{\partial u_i \partial u_j} \cdot N, \quad N \in N_p(\mathcal{S}), T \in T_p(\mathcal{S}).
\end{aligned}
\tag{2.22}
$$

We will rewrite this by noting that $\left(\frac{ds}{dt}\right)^2 = |x'(t)|^2$ which is the first fundamental form, and that $\frac{du_i}{ds} = \frac{du_i}{dt} \frac{dt}{ds}$. So, at the point $p \in \mathcal{S}$,

$$
\begin{aligned}
k(N,T) &= \sum_{i,j=1}^{2} b_{ij}(N) \left(\frac{du_i}{dt} \frac{dt}{ds} \right) \left(\frac{du_j}{dt} \frac{dt}{ds} \right) \\
&= \left(\sum_{ij=1}^{2} b_{i,j}(N) \frac{du_i}{dt} \frac{du_j}{dt} \right) \left(\frac{dt}{ds} \right)^2 = \frac{\sum_{i,j=1}^{2} b_{ij}(N) u_i' u_j'}{\sum_{i,j=1}^{2} g_{ij} u_i' u_j'},
\end{aligned}
\tag{2.23}
$$

which is a quotient of two quadratic forms. The numerator of this is called the *second fundamental form*, which depends linearly on the normal vector N. Thus the function $k(N,T)$ also depends linearly on N, and it depends only on the tangent vector T in the sense of its direction. The denominator is strictly positive as the metric G is positive definite.

There are two values of the $k(N,T)$ that are of special interest, namely $k_1(N) = \max_T k(N,T)$ and $k_2(N) = \min_T k(N,T)$. To find these we need to do some linear algebra, and therefore we write (2.23) in matrix form

$$
k(N,T) = \frac{v^T B v}{v^T G v}, \quad \text{where } B = (b_{ij}(N)), \, v = \frac{du}{dt}.
\tag{2.24}
$$

Recall that any real symmetric matrix can be transformed into the identity matrix I by some change of basis. Let P be a transformation $v = Py$ such that $P^T G P = I$ (note that $\det(P) \neq 0$), then we obtain the more useful expression

$$
k(N,T) = \frac{y^T P^T B P y}{y^T y}.
\tag{2.25}
$$

Moreover, we need the following proposition, a proof of which can be found in [5].

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PROPOSITION 2.18: *For a real symmetric matrix A we have*

$$\min \frac{v^T A v}{v^T v} = \lambda_{\min} \quad and \quad \max \frac{v^T A v}{v^T v} = \lambda_{\max}, \tag{2.26}$$

where $\lambda_{\min}$, $\lambda_{\max}$ are the smallest and biggest eigenvalues of A respectively.

Thus, the maximum and minimum of the function $k(N,T)$ are the biggest and smallest eigenvalues of the matrix $P^T B P$. But we will not even have to find this matrix because the eigenvalues of $P^T B P$ are the solutions to $\det(P^T B P - \lambda I) = 0$ which are the same as the solutions to $\det(B - \lambda G) = 0$. This is because $\det(P) \neq 0$ and

$$\det(P^T B P - \lambda I) = \det(P^T B P - \lambda P^T G P) = \det(P^T) \det(B - \lambda G) \det(P). \tag{2.27}$$

When expanded, this determinant equation becomes

$$\begin{aligned} 0 &= \det(b_{ij}(N) - \lambda g_{ij}) \\ &= \det(g_{ij})\lambda^2 - (g_{22}b_{11}(N) - 2g_{12}b_{12}(N) + g_{11}b_{22}(N))\lambda + \det(b_{ij}(N)), \end{aligned} \tag{2.28}$$

and by an elementary fact about the roots of second degree polynomials we get that the sum of the two solutions to the above equation is

$$k_1(N) + k_2(N) = \frac{g_{22}b_{11}(N) - 2g_{12}b_{12}(N) + g_{11}b_{22}(N)}{\det(g_{ij})}. \tag{2.29}$$

DEFINITION 2.19: We say that $k(N,T)$ is the *normal curvature* of a surface $\mathcal{S}$ in the direction T with respect to the normal N. The two values $k_1(N)$ and $k_2(N)$ are called the *principal curvatures* and $H(N) = (k_1(N) + k_2(N))/2$ is the *mean curvature*.

Remark 2.20: Note that $H(N)$ is linear in N and therefore there exists a unique vector $H \in N_p(\mathcal{S})$ such that $H(N) = H \cdot N$, $\forall N \in N_p(\mathcal{S})$. Such H will then be called the *mean curvature vector* of $\mathcal{S}$ at p, and is equal to zero exactly when

$$g_{22}b_{11}(N) - 2g_{12}b_{12}(N) + g_{11}b_{22}(N) = 0 \quad \forall N \in N_p(\mathcal{S}). \tag{2.30}$$

Now, we are finally ready to define what a minimal surface is.

DEFINITION 2.21: A surface $\mathcal{S}$ for which the mean curvature is zero at all points $p \in \mathcal{S}$ is called a *minimal surface*.

2.4 Theorems from complex analysis

Since the reader is assumed to have some knowledge of complex analysis this section will not contain any proofs, but will only serve as a reminder of the theory needed for the remaining chapters. For proofs and further theory one can consult textbooks in introductory complex analysis, for example [10].

DEFINITION 2.22: A *conformal map* is a map which preserves angles, and orientation, between curves.

This means that for two curves in the domain of definition going through some point a and intersecting each other with an angle α, then their corresponding images under the map f will meet at an angle α at the point $f(a)$. If a map preserves angles but changes their orientation it is called *anti-conformal*.

DEFINITION 2.23: A function is called *real (or complex) analytic* at a point if it is expressible as a convergent power series in some neighbourhood in $\mathbb{R}$ (or $\mathbb{C}$) of that point. It is analytic in an open set if it is analytic at all points of that set.

Some properties of analytic functions which we will use later are that sums, products and compositions of analytic functions are analytic, and if its derivative is non-zero, the inverse is also analytic.

DEFINITION 2.24: If the Laplacian of a C^2-function $f(x,y)$ vanishes everywhere, $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, on some open set, then f is said to be *harmonic* on that set.

DEFINITION 2.25: If a function is complex differentiable, i.e. the limit $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists, then it is called *holomorphic*. A function which is holomorphic in the whole complex plane is called *entire*.

The most important theorems of complex analysis for this thesis are the following five.

THEOREM 2.26: *A function is complex analytic if and only if it is holomorphic.*

THEOREM 2.27: *If a complex valued function $f(x,y) = u(x,y) + iv(x,y)$ is holomorphic then it satisfies the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.31}$$

Conversely, if $u(x,y)$, $v(x,y) \in C^1$ satisfy (2.31), then $f = u + iv$ is holomorphic.

The equation (2.31) can be written as the shorter expression

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right) f = 0, \tag{2.32}$$

which is the one we will use later on.

THEOREM 2.28: *Any harmonic function is real analytic.*

THEOREM 2.29 (Liouville): *Any bounded entire function is constant.*

If a function f has negative imaginary part, i.e. $\operatorname{Im} f < 0$, then $|e^{-if}| = e^{\operatorname{Im} f}$ is bounded, so e^{if} is constant by theorem 2.29, but then f need also be constant. So Liouville's theorem gives thus that any entire function with negative imaginary part is constant.

3

Important lemmas

I N THIS chapter we will define the minimal surface equation for surfaces in non-parametric form and find some consequences of this representation. Furthermore, we will show that we can always find a reparametrization into something called isothermal parameters which is a crucial step of proving Bernstein's theorem. We will obtain a pair of isothermal parameters which have especially useful properties, as will be seen in section 3.3.

3.1 The minimal surface equation

In proposition 2.13 we proved that for any regular point of a surface $\mathcal{S}$ we can find a neighbourhood in which we can reparametrize the surface into non-parametric form. So if we assume that the surface is in this form, i.e. $x_1 = u_1$, $x_2 = u_2$, and $x_k = f_k(u_1, u_2)$ for $k = 3, \ldots, n$, where $f_k \in C^1$, we obtain the tangent vectors

$$
\begin{aligned}
\frac{\partial x}{\partial u_1} &= \left(1, 0, \frac{\partial f_3}{\partial u_1}, \ldots, \frac{\partial f_n}{\partial u_1}\right), \\
\frac{\partial x}{\partial u_2} &= \left(0, 1, \frac{\partial f_3}{\partial u_2}, \ldots, \frac{\partial f_n}{\partial u_2}\right).
\end{aligned}
\tag{3.1}
$$

Note that if a surface is in non-parametric form it clearly follows that it must be regular. We also obtain the following expressions for the elements of the metric G

$$
\begin{aligned}
g_{11} &= 1 + \sum_{k=3}^{n} \left(\frac{\partial f_k}{\partial u_1}\right)^2, \quad g_{22} = 1 + \sum_{k=3}^{n} \left(\frac{\partial f_k}{\partial u_2}\right)^2, \\
g_{12} &= g_{21} = \sum_{k=3}^{n} \left(\frac{\partial f_k}{\partial u_1} \frac{\partial f_k}{\partial u_2}\right).
\end{aligned}
\tag{3.2}
$$

If we assume that all $f_k \in C^2$ we can also consider

$$\frac{\partial^2 x}{\partial u_i \partial u_j} = \left(0, 0, \frac{\partial^2 f_3}{\partial u_i \partial u_j}, \ldots, \frac{\partial^2 f_n}{\partial u_i \partial u_j}\right). \tag{3.3}$$

which by (2.22) gives, for any $N = (N_1, \ldots, N_k) \in N_p(\mathcal{S})$,

$$b_{ij}(N) = \sum_{k=3}^{n} \frac{\partial^2 f_k}{\partial u_i \partial u_j} \cdot N_k. \tag{3.4}$$

Inserting this into the mean curvature equation (2.30) yields

$$\sum_{k=3}^{n} \left(\left(1 + \sum_{l=3}^{n} \left(\frac{\partial f_l}{\partial u_2}\right)^2\right) \frac{\partial^2 f_k}{\partial u_1^2} - 2 \left(\sum_{l=3}^{n} \frac{\partial f_l}{\partial u_1} \frac{\partial f_l}{\partial u_2}\right) \frac{\partial^2 f_k}{\partial u_1 \partial u_2} \right.$$
$$\left. + \left(1 + \sum_{l=3}^{n} \left(\frac{\partial f_l}{\partial u_1}\right)^2\right) \frac{\partial^2 f_k}{\partial u_2^2}\right) \cdot N_k = 0. \tag{3.5}$$

To improve this equation further we are going to use the fact that for arbitrary $N_3, \ldots, N_n$ there are unique N_1, N_2 such that $N \in N_p(\mathcal{S})$. This follows directly from the fact that $N \in N_p(\mathcal{S})$ if and only if N is perpendicular to all tangent vectors, i.e. $N \cdot \frac{\partial x}{\partial u_i} = 0$ for $i = 1, 2$. This in turn gives the equation $N_i = -\sum_{k=3}^{n} N_k \frac{\partial f_k}{\partial u_i}$ for $i = 1, 2$. Since (3.5) holds for all normal vectors N, we must have that the coefficients of all N_k, $k = 3, \ldots, n$, are equal to zero, which gives the following equation.

DEFINITION 3.1: The *minimal surface equation* for non-parametric surfaces in $\mathbb{R}^n$, where $f = (f_3, \ldots, f_n)$, is

$$\left(1 + \left|\frac{\partial f}{\partial x_2}\right|^2\right) \frac{\partial^2 f}{\partial x_1^2} - 2 \left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2}\right) \frac{\partial^2 f}{\partial x_1 \partial x_2} + \left(1 + \left|\frac{\partial f}{\partial x_1}\right|^2\right) \frac{\partial^2 f}{\partial x_2^2} = 0. \tag{3.6}$$

Since the surface is in non-parametric form, there is no difference between differentiating f with respect to x_1 or u_1 (neither to x_2 or u_2), and from now on we will use them interchangeably. Note that if f is linear in x_1 and x_2, then it obviously satisfies (3.6) for any $n \geq 3$. The examples in section 1.3 also satisfies the equation, and since it is easy to verify this, we will only do it for one of those surfaces.

EXAMPLE 3.2: The helicoid is a minimal surface.

In non-parametric form the equation for the helicoid is $f(x_1, x_2) = \arctan(x_2/x_1)$. Differentiating yields

$$\frac{\partial f}{\partial x_1} = -\frac{x_2}{x_1^2 + x_2^2}, \quad \frac{\partial f}{\partial x_2} = \frac{x_1}{x_1^2 + x_2^2},$$
$$\frac{\partial^2 f}{\partial x_1^2} = \frac{2x_1 x_2}{(x_1^2 + x_2^2)^2}, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}, \quad \frac{\partial^2 f}{\partial x_2^2} = -\frac{2x_1 x_2}{(x_1^2 + x_2^2)^2},$$

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and plugging this into the minimal surface equation (3.6) gives

$$\frac{2x_1x_2}{(x_1^2 + x_2^2)^2}\left(1 + \frac{x_1^2}{(x_1^2 + x_2^2)^2} + \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2} - 1 - \frac{x_2^2}{(x_1^2 + x_2^2)^2}\right) = 0.$$

Hence it satisfies the equation and is therefore a minimal surface.

For simplification, we introduce the following notation

$$p = \frac{\partial f}{\partial x_1}, \quad q = \frac{\partial f}{\partial x_2}, \quad r = \frac{\partial^2 f}{\partial x_1^2}, \quad s = \frac{\partial^2 f}{\partial x_1 \partial x_2}, \quad t = \frac{\partial^2 f}{\partial x_2^2},$$
$$W = \sqrt{1 + |p|^2 + |q|^2 + |p|^2|q|^2 - (pq)^2}. \tag{3.7}$$

This is standard notation when dealing with minimal surfaces. It gives the shorter expressions $g_{11} = 1 + |p|^2$, $g_{12} = pq$, $g_{22} = 1 + |q|^2$ for the elements of G, $\det(g_{ij}) = W^2$ for its determinant, and the minimal surface equation (3.6) becomes

$$(1 + |q|^2)r - 2(pq)s + (1 + |p|^2)t = 0. \tag{3.8}$$

The following lemma and its implications will be crucial later.

LEMMA 3.3: *Any solution to the minimal surface equation* (3.8) *also satisfy*

$$\frac{\partial}{\partial x_1}\left(\frac{1 + |q|^2}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{pq}{W}\right), \quad \frac{\partial}{\partial x_1}\left(\frac{pq}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{1 + |p|^2}{W}\right). \tag{3.9}$$

PROOF. We will only prove the first equation since the second then follows by symmetry. Starting with the left hand side we get

$$\frac{\partial}{\partial x_1}\left(\frac{1 + |q|^2}{W}\right) = \frac{1}{W^2}\left(W\frac{\partial}{\partial x_1}\left(1 + |q|^2\right) - (1 + |q|^2)\frac{\partial W}{\partial x_1}\right), \tag{3.10}$$

which after differentiating and separating into parts containing r or s is equal to

$$\frac{1}{W^2}\left(2qsW - (1 + |q|^2)\frac{1}{2W}\left(2qs + 2rp + 2rp|q|^2 + 2qs|p|^2 - 2pq(rq + ps)\right)\right)$$
$$= \frac{1}{W^3}\Big(\left((2W^2 - (1 + |q|^2)(1 + |p|^2))\, q + (pq)(1 + |q|^2)p\right)s \tag{3.11}$$
$$+ \left((pq)(1 + |q|^2)q - (1 + |q|^2)(1 + |q|^2)p\right)r\Big).$$

Similarly for the right hand side.

$$\frac{\partial}{\partial x_2}\left(\frac{pq}{W}\right) = \frac{1}{W^2}\left(W\frac{\partial}{\partial x_2}(pq) - (pq)\frac{\partial W}{\partial x_2}\right), \tag{3.12}$$

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which can be separated into parts containing s or t

$$
\frac{1}{W^2}\left((sq-pt)W-(pq)\frac{1}{2W}\left(2tq+2sp+2sp|q|^2+2tq|p|^2-2pq(sq+pt)\right)\right)
$$
$$
=\frac{1}{W^3}\Big(\left((W^2+(pq)^2)q-(pq)(1+|q|^2)p\right)s \tag{3.13}
$$
$$
+\left((W^2+(pq)^2)p-(pq)(1+|p|^2)q\right)t\Big).
$$

Using that $W^2=(1+|p|^2)(1+|q|^2)-(pq)^2$, we obtain that

$$
\frac{\partial}{\partial x_1}\left(\frac{1+|q|^2}{W}\right)-\frac{\partial}{\partial x_2}\left(\frac{pq}{W}\right)
$$
$$
=\frac{1}{W^3}\Big(\left((pq)q-(1+|q|^2)p\right)(1+|q|^2)r
$$
$$
+\left(2(pq)(1+|q|^2)p-2(pq)^2q\right)s \tag{3.14}
$$
$$
-\left((1+|p|^2)(1+|q|^2)p-(pq)(1+|p|^2)q\right)t\Big)
$$
$$
=\frac{1}{W^3}\left((pq)q-(1+|q|^2)p\right)\left((1+|q|^2)r-2(pq)s+(1+|p|^2)t\right).
$$

The last factor is equal to zero since it is the minimal surface equation (3.8), and hence the wanted equation holds.

$\square$

Lemma 3.3 also implies the existence of two C^1-functions F_1, F_2 defined in the same domain as the minimal surface equation, for which

$$
\begin{aligned}
\frac{\partial F_1}{\partial x_1} &= \frac{1+|p|^2}{W}, & \frac{\partial F_1}{\partial x_2} &= \frac{pq}{W}, \\
\frac{\partial F_2}{\partial x_1} &= \frac{pq}{W}, & \frac{\partial F_2}{\partial x_2} &= \frac{1+|q|^2}{W}.
\end{aligned} \tag{3.15}
$$

In turn, F_1, F_2 imply the existence of another function $E \in C^2$, still defined on the same domain, with the property

$$
\frac{\partial E}{\partial x_1}=F_1,\quad \frac{\partial E}{\partial x_2}=F_2. \tag{3.16}
$$

This function E has thus a Hessian which is closely related to the metric g_{ij} of a minimal surface in non-parametric form, namely

$$
\frac{\partial^2 E}{\partial x_i \partial x_j}=\frac{g_{ij}}{W},\quad \text{for } i,j=1,2. \tag{3.17}
$$

We conclude this section by noting the fact that

$$
\det\left(\frac{\partial^2 E}{\partial x_i \partial x_j}\right)=\det\begin{pmatrix}\frac{1+|p|^2}{W} & \frac{pq}{W} \\ \frac{pq}{W} & \frac{1+|q|^2}{W}\end{pmatrix}\equiv 1. \tag{3.18}
$$

3.2   Isothermal parameters

DEFINITION 3.4: If the metric of a surface satisfies $g_{ij} = \lambda^2 \delta_{ij}$ for some $\lambda = \lambda(u) > 0$, we say that the parameters u_1, u_2 are *isothermal.*

An equivalent way of expressing that u_1, u_2 are isothermal is to write it as $g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$, i.e

$$\left(\frac{\partial x}{\partial u_1}\right)^2 = \left(\frac{\partial x}{\partial u_2}\right)^2 \quad \text{and} \quad \left(\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2}\right) = 0. \tag{3.19}$$

The advantage of these parameters are that the preserve some of the geometric properties between the parameter plane and the surface. For example, the make the map defining the surface to be conformal, that it preserves angles.

If we have isothermal parameters then we obviously have $\det(g_{ij}) = \lambda^4$, which gives a shorter expression for the mean curvature, definition 2.19, as

$$H(N) = \frac{b_{11}(N) + b_{22}(N)}{2\lambda^2}. \tag{3.20}$$

Remark 3.5: If u_1, u_2 are isothermal parameters for a metric g_{ij}, then they are also isothermal for the metric $\lambda' g_{ij}$ where λ' can be a function depending on u. This should be clear from observing that the equations $\lambda' g_{11} = \lambda' g_{22}$ and $\lambda' g_{12} = \lambda' g_{21} = 0$ still hold. Note in particular that parameters are isothermal for a metric g_{ij} if and only if they are isothermal with respect to the normalized metric g_{ij}/W.

We are going to show the existence of isothermal parameters, but in order to simplify the proof we are going to use the following three lemmas.

LEMMA 3.6: *If we for two pairs of vectors u_1, u_2 and v_1, v_2 in a 2-dimensional vector space have that $u_1 = R(u_2)$ and $v_1 = R(v_2)$, where R is a, say counter-clockwise, rotation by $\pi/2$. Then $u_1 + v_1 = R(u_2 + v_2)$, and $|u_1 + v_1| = |u_2 + v_2|$.*

PROOF. Rotation is a linear operation, so we immediately get that $R(u_2 + v_2) = R(u_2) + R(v_2) = u_1 + v_1$. Rotation does not change the length of a vector, hence by the first part we get trivially $|u_1 + v_1| = |R(u_2 + v_2)| = |u_2 + v_2|$.

$\square$

Note that we need the rotation to be in the same direction for both pairs, either clockwise or counter-clockwise. If not, we can always add a minus sign to one of the vectors to get the right rotation in order to use the lemma. The lemma only works in 2-dimensional vector space and cannot easily be generalized to higher dimensions.

For the following lemmas we need a few definitions first. The differential form $d\eta$ for a real valued function η is defined as $d\eta = \frac{\partial \eta}{\partial x_1} dx_1 + \frac{\partial \eta}{\partial x_2} dx_2$. Moreover, we can define a scalar product between two such differential forms $d\eta_1 = \eta_1^1 dx_1 + \eta_1^2 dx_2$ and

$d\eta_2 = \eta_2^1 dx_1 + \eta_2^2 dx_2$ as

$$\langle d\eta_1, d\eta_2 \rangle_{G^{-1}} = \sum_{i,j=1}^{2} g^{ij} \eta_1^i \eta_2^j, \tag{3.21}$$

and the *wedge product*, denoted $\wedge$, for them as $d\eta_1 \wedge d\eta_2 = (\eta_1^1 \eta_2^2 - \eta_1^2 \eta_2^1) dx_1 \wedge dx_2$.

LEMMA 3.7: *The parameters ξ_1, ξ_2 are isothermal if and only if the corresponding 1-forms $d\xi_1, d\xi_2$ are perpendicular and of equal length.*

PROOF. Any reparametrization satisfy equation (2.6), so if ξ_1, ξ_2 is a reparametrization of u_1, u_2 we have $\widetilde{G} = U^T G U$ as the new metric where $U = \left(\frac{\partial(u_1, u_2)}{\partial(\xi_1, \xi_2)} \right)$. Consider $\widetilde{G}^{-1} = U^{-1} G^{-1} (U^{-1})^T$, where $U^{-1} = \left(\frac{\partial(\xi_1, \xi_2)}{\partial(u_1, u_2)} \right)$ and $G^{-1} = (g^{ij})$. By multiplying these matrices together we will obtain

$$\widetilde{G}^{-1} = \left(\sum_{k,l=1}^{2} g^{kl} \frac{\partial \xi_i}{\partial u_k} \frac{\partial \xi_j}{\partial u_l} \right) = \left(\langle d\xi_i, d\xi_j \rangle_{G^{-1}} \right) \tag{3.22}$$

If we have that ξ_1, ξ_2 are isothermal, we have that $\widetilde{G} = \lambda^2 I$ and thus $\widetilde{G}^{-1} = \lambda^{-2} I$. But by equation (3.22) this is equivalent to $\langle d\xi_1, d\xi_1 \rangle_{G^{-1}} = \langle d\xi_2, d\xi_2 \rangle_{G^{-1}}$ and $\langle d\xi_1, d\xi_2 \rangle_{G^{-1}} = 0$, which was what we wanted to prove.

$\square$

LEMMA 3.8: *If the metric is Hessian with $\det(g_{ij}) \equiv 1$, then both pairs $\eta_1 = x_1$, $\eta_2 = \frac{\partial E}{\partial x_2}$ and $\nu_1 = \frac{\partial E}{\partial x_1}$, $\nu_2 = x_2$ respectively are isothermal parameters.*

PROOF. Since $\det(g_{ij}) \equiv 1$, the inverse matrix for g_{ij} is just

$$(g^{ij}) = \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}. \tag{3.23}$$

First we want to show that $\eta_1 = x_1$, $\eta_2 = \frac{\partial E}{\partial x_2}$ are isothermal. We have

$$\begin{aligned} d\eta_1 &= dx_1 \\ d\eta_2 &= \frac{\partial^2 E}{\partial x_1 \partial x_2} dx_1 + \frac{\partial^2 E}{\partial x_2^2} dx_2 = g_{12} dx_1 + g_{22} dx_2. \end{aligned} \tag{3.24}$$

Showing that η_1, η_2 are isothermal is the same as showing that $d\eta_1$ and $d\eta_2$ are perpendicular and of equal length by lemma 3.7. If we denote $d\eta_i = \eta_i^1 dx_1 + \eta_i^2 dx_2$, then

$$\langle d\eta_1, d\eta_2 \rangle_{G^{-1}} = \sum_{i,j=1}^{2} g^{ij} \eta_1^i \eta_2^j = g_{22} g_{12} - g_{12} g_{22} = 0, \tag{3.25}$$

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which means that they are perpendicular. They are of equal length since

$$|d\eta_1|^2_{G^{-1}} = \sum_{i,j=1}^{2} g^{ij}\eta_1^i\eta_1^j = g_{22},$$

$$|d\eta_2|^2_{G^{-1}} = \sum_{i,j=1}^{2} g^{ij}\eta_2^i\eta_2^j = g_{22}g_{12}^2 - 2g_{12}^2g_{22} + g_{11}g_{22}^2 = g_{22}\det(g_{ij}) = g_{22}.$$

(3.26)

Hence, the parameters η_1, η_2 are isothermal. By symmetry, the same calculations for ν_1, ν_2 gives that they too are isothermal.

$\square$

PROPOSITION 3.9: *If the metric is Hessian, that is $g_{ij} = \frac{\partial^2 E}{\partial x_i \partial x_j}$ for some function E, and $\det(g_{ij}) \equiv 1$ then*

$$\xi_1 = x_1 + \frac{\partial E}{\partial x_1}, \quad \xi_2 = x_2 + \frac{\partial E}{\partial x_2}$$

(3.27)

are isothermal parameters.

PROOF. Using the above lemmas, the only thing left to prove is that the signs are correct. But since

$$d\eta_1 \wedge d\eta_2 = dx_1 \wedge (g_{12}dx_1 + g_{22}dx_2) = g_{22}dx_1 \wedge dx_2,$$

$$d\nu_1 \wedge d\nu_2 = (g_{11}dx_1 + g_{12}dx_2) \wedge dx_2 = g_{11}dx_1 \wedge dx_2,$$

(3.28)

and that both g_{11} and g_{22} are positive, they are oriented in the same way. Thus $d\eta_2$, and $d\nu_2$, are $\pi/2$-rotations in the positive direction of $d\nu_1$, and $d\eta_1$ respectively.

By lemma 3.6 we will thus have that $\xi_1 = \eta_1 + \nu_1$ and $\xi_2 = \eta_2 + \nu_2$ are isothermal parameters.

$\square$

COROLLARY 3.10: *The parameters $\xi_1 = x_1 + F_1$, $\xi_2 = x_2 + F_2$, with F_1,F_2 as in (3.15), are isothermal for the metric g_{ij} of a minimal surface in non-parametric form.*

PROOF. Since (3.18) the normalized metric can be written as the Hessian of a C^2-function with determinant constantly equal to 1. By proposition 3.9 we know that they are isothermal for this normalized matrix g_{ij}/W, and by remark 3.5 this must also hold for g_{ij}.

$\square$

Now that we have made sure that isothermal parameters always exists, at least locally, we can prove some useful properties related to them.

LEMMA 3.11: *Let a regular surface $\mathcal{S}$ be defined by $x(u) \in C^2$ where u_1, u_2 are isothermal parameters. Then $\Delta x = 2\lambda^2 H$ where H is the mean curvature vector.*

PROOF. We will prove this by first showing that Δx is perpendicular to the tangent plane. In order to do so, we are going to differentiate the left equation of (3.19) with respect to u_1 and the right with respect to u_2.

$$
\begin{aligned}
\frac{\partial}{\partial u_1}\left(\left(\frac{\partial x}{\partial u_1}\right)^2\right) &= 2\frac{\partial x}{\partial u_1} \cdot \frac{\partial^2 x}{\partial u_1^2} \\
\frac{\partial}{\partial u_1}\left(\left(\frac{\partial x}{\partial u_2}\right)^2\right) &= 2\frac{\partial x}{\partial u_2} \cdot \frac{\partial^2 x}{\partial u_1 \partial u_2} \\
\frac{\partial}{\partial u_2}\left(\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2}\right) &= \frac{\partial^2 x}{\partial u_1 \partial u_2} \cdot \frac{\partial x}{\partial u_2} + \frac{\partial x}{\partial u_1} \cdot \frac{\partial^2 x}{\partial u_2^2} = 0.
\end{aligned}
\tag{3.29}
$$

Since we have isothermal parameters the first two equations above are equal and we obtain

$$
\begin{aligned}
\frac{\partial x}{\partial u_1} \cdot \frac{\partial^2 x}{\partial u_1^2} &= \frac{\partial x}{\partial u_2} \cdot \frac{\partial^2 x}{\partial u_1 \partial u_2} = -\frac{\partial x}{\partial u_1} \cdot \frac{\partial^2 x}{\partial u_2^2} \\
&\implies \frac{\partial x}{\partial u_1} \cdot \left(\frac{\partial^2 x}{\partial u_1^2} + \frac{\partial^2 x}{\partial u_2^2}\right) = 0.
\end{aligned}
\tag{3.30}
$$

Similarly by differentiating the left equation of (3.19) with respect to u_2 and the right with respect to u_1 we obtain

$$
\frac{\partial x}{\partial u_2} \cdot \left(\frac{\partial^2 x}{\partial u_1^2} + \frac{\partial^2 x}{\partial u_2^2}\right) = 0.
\tag{3.31}
$$

Hence, Δx is perpendicular to the tangent plane, i.e. a normal vector. Now we need to show that $\Delta x \cdot N = 2\lambda^2 H(N)$ for each $N \in N_p(\mathcal{S})$, but

$$
\frac{\Delta x \cdot N}{2\lambda^2} = \frac{1}{2\lambda^2}\left(\frac{\partial^2 x}{\partial u_1^2} \cdot N + \frac{\partial^2 x}{\partial u_2^2} \cdot N\right) = \frac{b_{11}(N) + b_{22}(N)}{2\lambda^2} = H(N).
\tag{3.32}
$$

By remark 2.20 the mean curvature vector is the unique vector $H \in N_p(\mathcal{S})$ which satisfies this, so we must have that $\Delta x = 2\lambda^2 H$.

$\square$

LEMMA 3.12: *Let $x(u) \in C^2$ define a regular surface $\mathcal{S}$ in isothermal parameters. Then $\mathcal{S}$ is a minimal surface if and only if the coordinate functions $x_k(u_1, u_2)$ are harmonic.*

PROOF. Using that $\Delta x = 2\lambda^2 H$, this is trivial.

$\square$

COROLLARY 3.13: *Let $\mathcal{S}$ be a minimal surface in non-parametric form defined by $x_k = f_k(x_1,x_2)$, then f_k are real analytic functions of x_1,x_2 for $k \geq 3$.*

PROOF. By proposition 3.9 we can reparametrize the surface into isothermal co-ordinates ξ_1,ξ_2, and by lemma 3.12 we know that $x_k(\xi_1,\xi_2)$ are harmonic functions for $k = 1,\ldots,n$. This implies that all x_k are real analytic functions of ξ_1,ξ_2.

Using that $x_1(\xi_1,\xi_2)$ and $x_2(\xi_1,\xi_2)$ are real analytic, and that they have non-zero derivatives, the inverse functions $\xi_1(x_1,x_2)$ and $\xi_2(x_1,x_2)$ must also be real analytic. By composition this gives that $x_k(x_1,x_2)$ are real analytic for $k = 3,\ldots,n$.

$\square$

For $k = 1,\ldots,n$ we introduce the C^1-functions

$$\phi_k(\zeta) = \left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\right)x_k = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}, \quad \text{where } \zeta = u_1 + iu_2, \tag{3.33}$$

which will give us access to some of the powerful tools from complex analysis. We have

$$\sum_{k=1}^{n}|\phi_k(\zeta)|^2 = \sum_{k=1}^{n}\left(\left(\frac{\partial x_k}{\partial u_1}\right)^2 + \left(\frac{\partial x_k}{\partial u_2}\right)^2\right) = g_{11} + g_{22} \quad \text{and} \tag{3.34}$$

$$\sum_{k=1}^{n}(\phi_k(\zeta))^2 = \sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}\right)^2 = \sum_{k=1}^{n}\left(\left(\frac{\partial x_k}{\partial u_1}\right)^2 - \left(\frac{\partial x_k}{\partial u_2}\right)^2 - 2i\frac{\partial x_k}{\partial u_1}\frac{\partial x_k}{\partial u_2}\right)$$
$$= g_{11} - g_{22} - 2ig_{12}. \tag{3.35}$$

LEMMA 3.14: *Using the above notation, we get*

i) *$\phi_k(\zeta)$ is analytic $\iff x_k$ is harmonic in u_1,u_2.*

ii) *u_1,u_2 are isothermal parameters $\iff \sum_{k=1}^{n}(\phi_k(\zeta))^2 \equiv 0$.*

iii) *If u_1,u_2 are isothermal parameters, then $\sum_{k=1}^{n}|\phi_k(\zeta)|^2 \neq 0 \iff \mathcal{S}$ is regular.*

PROOF.

i) From complex analysis we know that a function being complex analytic in an open set is equivalent to it satisfying the Cauchy-Riemann equations and having continuous partial derivatives. The functions ϕ_k for $k = 1,\ldots,n$ are continuously differentiable by definition and clearly they satisfy Cauchy-Riemann (2.32) if and only if $\Delta x_k = 0$, since

$$\left(\frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2}\right)\phi_k = \frac{\partial^2 x_k}{\partial u_1^2} - i\frac{\partial^2 x_k}{\partial u_1 \partial u_2} + i\frac{\partial^2 x_k}{\partial u_1 \partial u_2} + \frac{\partial^2 x_k}{\partial u_2^2} = \Delta x_k. \tag{3.36}$$

ii) That $\sum_{k=1}^{n}(\phi_k(\zeta))^2 \equiv 0$ is equivalent to $g_{11} - g_{22} = 0$ and $g_{12} = 0$ by (3.35), which is the definition of u_1,u_2 being isothermal parameters.

iii) We have $g_{11} = g_{22}$, so using the equation (3.34) gives $g_{11} + g_{22} = (\frac{\partial x}{\partial u_1})^2 + (\frac{\partial x}{\partial u_2})^2$ which by lemma 2.8 is non-zero exactly when the surface is regular.

$\square$

LEMMA 3.15: *Let a surface be defined by $x(u)$ where u_1,u_2 are isothermal parameters, and let $u(\tilde{u})$ be a reparametrization. Then $\tilde{u}_1,\tilde{u}_2$ are isothermal if and only if $u(\tilde{u})$ is conformal or anti-conformal.*

PROOF. Since u_1,u_2 are isothermal $g_{ij} = \lambda^2 \delta_{ij}$. If we assume that $\tilde{u}_1,\tilde{u}_2$ are also isothermal then $\tilde{g}_{ij} = \tilde{\lambda}^2 \delta_{ij}$. Since $\tilde{\lambda}^2 \delta_{ij} = \widetilde{G} = U^T G U = \lambda^2 U^T U$, where U is the Jacobian matrix of the parameter transformation, we get that $\left(\tilde{\lambda}/\lambda\right)^2 \delta_{ij} = U^T U$ which is equivalent to saying that $u(\tilde{u})$ is conformal or anti-conformal.

$\square$

3.3 Reparametrizing into isothermal coordinates

Up until now, all results are local, but the theorem we want to prove is of global type. The aim of the following section is to make sure that when parametrization into the isothermal coordinates in (3.27) we will still have parameters defined in a domain which is not smaller than the original parameter domain. This will in particular prove that if the surface is defined for the whole x_1,x_2-plane in non-parametric form, then it will be defined in the whole ξ_1,ξ_2-plane for the isothermal parameters.

Remember that in equation (3.16) we found that for any minimal surface there is some function E which has a Hessian matrix equal to the normalized metric, this will be used when proving lemma 3.19.

LEMMA 3.16: *Let $E(x_1,x_2) \in C^2$ in a convex domain D, and suppose that the Hessian matrix of E is positive definite. Define a mapping $(x_1,x_2) \mapsto (u_1,u_2)$, where $u_i = \frac{\partial E}{\partial x_i}$, and let $x \neq y$ be two points in D.*

i) *If $x \mapsto u$ and $y \mapsto v$ using this map, then $(v - u) \cdot (y - x) > 0$.*

ii) *Define the map $(x_1,x_2) \mapsto (\xi_1,\xi_2)$ by*

$$
\begin{aligned}
\xi_1(x_1,x_2) &= x_1 + u_1(x_1,x_2), \\
\xi_2(x_1,x_2) &= x_2 + u_2(x_1,x_2).
\end{aligned}
\tag{3.37}
$$

Then if ξ, η are the respective images of two points x, y, we have that the following inequality holds

$$
(\eta - \xi) \cdot (y - x) > |y - x|^2.
\tag{3.38}
$$

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PROOF.

i) Introduce the function $\varphi(t) = E(ty + (1-t)x)$ for $0 \leq t \leq 1$. Then

$$\varphi'(t) = \sum_{i=1}^{2} \left(\frac{\partial E}{\partial x_i}(ty + (1-t)x) \right)(y_i - x_i),$$

$$\varphi''(t) = \sum_{i,j=1}^{2} \left(\frac{\partial^2 E}{\partial x_i \partial x_j}(ty + (1-t)x) \right)(y_i - x_i)(y_j - x_j) > 0, \tag{3.39}$$

since the Hessian of E is positive definite, so $\varphi'(1) > \varphi'(0)$. But

$$\varphi'(0) = \sum_{i=1}^{2} \left(\frac{\partial E}{\partial x_i}(y) \right)(y_i - x_i) = \sum_{i=1}^{2} u_i(y_i - x_i),$$

$$\varphi'(1) = \sum_{i=1}^{2} \frac{\partial E(x)}{\partial x_i}(y_i - x_i) = \sum_{i=1}^{2} v_i(y_i - x_i), \tag{3.40}$$

which implies that that

$$\sum_{i=1}^{2} v_i(y_i - x_i) > \sum_{i=1}^{2} u_i(y_i - x_i), \tag{3.41}$$

and after moving everything to the same side, this is the wanted inequality.

ii) Since $\eta - \xi = (y - x) + (u - v)$, it follows from i) that

$$(\eta - \xi) \cdot (y - x) = (y - x) \cdot (y - x) + (y - x) \cdot (u - v) > |y - x|^2. \tag{3.42}$$

$\square$

Note also that ii) together with Cauchy-Schwarz inequality implies

$$|\eta - \xi| > |y - x|. \tag{3.43}$$

LEMMA 3.17: *A function h is convex if and only if its Hessian matrix is positive semi-definite.*

In particular this implies that the function E defined above is convex. It is also useful to recall that if a matrix satisfies the stronger condition of being positive definite, then the determinant is strictly greater than zero.

LEMMA 3.18: *Let $D = D_R(0)$, i.e the disc of radius R with 0 as center, then the map (3.37) is a diffeomorphism of D onto a domain which includes a disc $D_R(\xi(0))$ with the same radius R.*

PROOF. The map is clearly continuously differentiable since $E \in C^2$. We also have that since the Hessian H of E is positive definite, E is convex and

$$\det \left(\frac{\partial \xi_i}{\partial x_j} \right) = \det \begin{pmatrix} 1 + \frac{\partial^2 E}{\partial x_1} & \frac{\partial^2 E}{\partial x_1 \partial x_2} \\ \frac{\partial^2 E}{\partial x_1 \partial x_2} & 1 + \frac{\partial^2 E}{\partial x_2} \end{pmatrix} = 1 + \Delta E + \det(H) > 0. \tag{3.44}$$

Therefore the map is injective in all of D onto some domain Ω, and the inverse is also continuously differentiable.

Next, we need to show that all ξ such that $|\xi - \xi(0)| < R$ lies in Ω. We may assume that Ω is not the whole $\mathbb{R}^2$-plane, since otherwise it would be trivially true. Let $\mu = \min_{\xi \in \Omega^c} d(\xi, \xi(0))$, i.e. the point outside Ω which minimizes the distance to $\xi(0)$. Also let $\{\mu_k\} \subset \Omega$ be a sequence of points such that μ_k goes to μ as k goes to infinity, and let y_k be their corresponding points in D, so $\xi(y_k) = \mu_k$.

If $\{y_k\}$ has a limit point in D, then the image of the limit point would be μ since ξ is continuous. But $\mu \notin \Omega$ which is a contradiction, so we must have that $y \in D^c$, i.e that $|y_k| \geq R$. But then

$$|\mu - \xi(0)| = \lim_{k \to \infty} |\mu_k - \xi(0)| > \lim_{k \to \infty} |y_k - 0| = |y| \geq R. \tag{3.45}$$

This means that there are no points outside Ω at a closer distance to $\xi(0)$ than R, which was to be proven.

$\square$

Now we are going to connect these lemmas to the isothermal coordinates found before in 3.27.

LEMMA 3.19: *Let $f(x_1, x_2)$ be a solution to the minimal surface equation (3.6) for a disc $D = D_R(0)$. Then, using F_1 and F_2 as in (3.15), the map*

$$\xi_1 = x_1 + F_1(x_1, x_2), \quad \xi_2 = x_2 + F_2(x_1, x_2) \tag{3.46}$$

is a diffeomorphism onto a domain Ω which includes a disc $D_R(\xi(0))$.

PROOF. From (3.16), there exists a function $E(x_1, x_2) \in C^2$ defined in D for which $\frac{\partial E}{\partial x_1} = F_1$ and $\frac{\partial E}{\partial x_2} = F_2$. This function has positive definite Hessian since

$$\frac{\partial^2 E}{\partial x_1^2} = \frac{1 + |p|^2}{W} > 0,$$
$$\det \left(\frac{\partial^2 E}{\partial x_i \partial x_j} \right) \equiv 1 > 0. \tag{3.47}$$

Hence the conditions for both lemmas 3.16 and 3.18 are satisfied for this E, where (3.46) is of the form specified in (3.37), which thus has the wanted properties.

$\square$

To summarize, if we have a minimal surface in non-parametric form defined in a disc around the origin, then we can reparametrize it into the isothermal parameters defined in corollary 3.10, which are defined in a disc with at least the same radius as the first one. Note that the center of the disc where the isothermal parameters are defined need not be at origin.

4

Bernstein's theorem

W<small>E WILL</small> present two different proofs of Bernstein's theorem. First the proof by Osserman [8] which proves the existence of a non-singular linear transformation, and second the proof by Chipot [3] which uses Jörgens theorem. Both proofs involve bounding analytic functions defined in the whole complex plane to show that they are constant.

Now, we state the theorem.

T<small>HEOREM</small> 4.1 (Bernstein): *The only solution to the minimal surface equation* (3.6) *for $n = 3$ which is defined in the whole x_1, x_2-plane is the trivial solution, i.e. that f is linear in x_1 and x_2.*

4.1 Osserman's proof

This first lemma will assume that we are in $\mathbb{R}^3$, while the second makes no restriction on the dimension.

L<small>EMMA</small> 4.2: *Let $f(x_1, x_2) \in C^1$ in a domain D, where f is real-valued. The surface $\mathcal{S}$ defined by $x_3 = f(x_1, x_2)$ lie on a plane if and only if there exists a non-singular linear transformation $(u_1, u_2) \mapsto (x_1, x_2)$ such that u_1, u_2 are isothermal parameters on $\mathcal{S}$.*

P<small>ROOF.</small>

$\Leftarrow$) Suppose we have such a transformation. Let $\phi_k(\zeta)$ be as in (3.33). Since x_1, x_2 are linear in u_1, u_2, we have that ϕ_1, ϕ_2 are constant, and by (3.35) we must also have that $\phi_3 = \frac{\partial x_3}{\partial u_1} - i \frac{\partial x_3}{\partial u_2}$ is constant. But then the gradient ∇x_3 with respect to u_1, u_2 is constant and, again by the linearity of the transformation, it is also constant with respect to x_1, x_2. So f can be written as $f = ax_1 + bx_2 + c$.

$\Rightarrow$) If f is of this form, that is $f = ax_1 + bx_2 + c$, we can explicitly write down such a linear transformation. Let $x_1 = \lambda a u_1 + b u_2$, $x_2 = \lambda b u_1 - a u_2$ where $\lambda^2 = (1 + a^2 + b^2)^{-1}$.

This would give $f = \lambda(a^2 + b^2)u_1 + c$, and

$$g_{11} = \lambda^2 a^2 + \lambda^2 b^2 + \lambda^2(a^2 + b^2)^2 = \lambda^2(1 + a^2 + b^2)(a^2 + b^2) = a^2 + b^2$$
$$g_{22} = b^2 + a^2 \tag{4.1}$$
$$g_{12} = g_{21} = \lambda ab - \lambda ab = 0.$$

Hence, u_1, u_2 are isothermal coordinates.

$\square$

LEMMA 4.3: *Let $f(x_1, x_2)$ be a solution to the minimal surface equation* (3.6) *in the whole (x_1, x_2)-plane. Then there exists a nonsingular linear transformation*

$$x_1 = u_1, \quad x_2 = au_1 + bu_2, \quad b > 0, \tag{4.2}$$

such that (u_1, u_2) are (global) isothermal parameters for the surface $\mathcal{S}$ defined by

$$x_k = f_k(x_1, x_2), \quad k = 3, \dots, n.$$

PROOF. By lemma 3.19 the map (3.46) is a diffeomorphism of the x_1, x_2-plane onto the entire ξ_1, ξ_2-plane. We also know by proposition 3.9 that ξ_1, ξ_2 are isothermal on $\mathcal{S}$.

We want to show that $u_1 + iu_2$ is a conformal map of $\xi_1 + i\xi_2$ because then by lemma 3.15 we will have that u_1, u_2 are isothermal.

Since ξ_1, ξ_2 are isothermal we have by lemma 3.14 that the ϕ_k's are analytic, and since $\phi_1 \neq 0$ we have also that ϕ_2/ϕ_1 is analytic. Furthermore

$$\text{Im}\left(\frac{\phi_2}{\phi_1}\right) = \frac{1}{|\phi_2|^2}\text{Im}(\overline{\phi_1}\phi_2) = -\frac{1}{|\phi_2|^2}\det\left(\frac{\partial(x_1, x_2)}{\partial(u_1, u_2)}\right) < 0. \tag{4.3}$$

So ϕ_2/ϕ_1 is an analytic function with negative imaginary part, and by Liouville's theorem it is therefore constant. So $\phi_2 = c\phi_1$ for some complex number $c = a - ib$ with $b > 0$, i.e.

$$\frac{\partial x_2}{\partial \xi_1} + i\frac{\partial x_2}{\partial \xi_2} = a\left(\frac{\partial x_1}{\partial \xi_1} - i\frac{\partial x_1}{\partial \xi_2}\right) - b\left(\frac{\partial x_1}{\partial \xi_2} + i\frac{\partial x_1}{\partial \xi_1}\right), \tag{4.4}$$

which after matching real and imaginary parts becomes

$$\frac{\partial x_2}{\partial \xi_1} = a\frac{\partial x_1}{\partial \xi_1} - b\frac{\partial x_1}{\partial \xi_2}, \quad \frac{\partial x_2}{\partial \xi_2} = b\frac{\partial x_1}{\partial \xi_1} + a\frac{\partial x_1}{\partial \xi_2}. \tag{4.5}$$

Transforming these by (4.2) will give

$$\frac{\partial u_1}{\partial \xi_1} = \frac{\partial u_2}{\partial \xi_2}, \quad \frac{\partial u_2}{\partial \xi_1} = -\frac{\partial u_1}{\partial \xi_2} \tag{4.6}$$

which is the Cauchy-Riemann equations. We also know that the map is C^1, which implies that $u_1 + iu_2$ is a complex analytic function of $\xi_1 + i\xi_2$, and since any complex analytic function is conformal, lemma 3.15 gives that also u_1, u_2 are isothermal parameters.

$\square$

From these two lemmas we obtain Bernstein's theorem directly.

4.2 Proof using Jörgens' theorem

The alternate proof of Bernstein's theorem is done by using Jörgens theorem, which states that the only solutions to some special differential equation are quadratic polynomials. The proof given here follows the one done by Chipot [3], which in turn is based on one by Nitsche [6].

THEOREM 4.4 (Jörgens): *Let $h \in C^2$ such that $h : \mathbb{R}^2 \to \mathbb{R}$ is a solution to $\det(H) \equiv 1$ in $\mathbb{R}^2$, where $H = \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)$ is the Hessian matrix of h. Then h is a quadratic polynomial.*

PROOF. The function h must be convex by lemma 3.17. By lemma 3.18, the map $(x_1, x_2) \to (\xi_1, \xi_2)$ defined in (3.37), with h as the function E, is a diffeomorphism of $\mathbb{R}^2$ onto itself since h is defined for the whole of $\mathbb{R}^2$. The Jacobian matrices for this map and its inverse are

$$J = \begin{pmatrix} 1 + \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & 1 + \frac{\partial^2 h}{\partial x_2^2} \end{pmatrix}, \quad J^{-1} = \frac{1}{\det J} \begin{pmatrix} 1 + \frac{\partial^2 h}{\partial x_2^2} & -\frac{\partial^2 h}{\partial x_1 \partial x_2} \\ -\frac{\partial^2 h}{\partial x_1 \partial x_2} & 1 + \frac{\partial^2 h}{\partial x_1^2} \end{pmatrix}. \tag{4.7}$$

We have that $\det(J) = 1 + \Delta h + \det(H) = 2 + \Delta h$, which is strictly positive since h is convex, so the inverse exists. Now, define a function g as

$$g(\zeta) = x_1 - \frac{\partial h}{\partial x_1} - i \left(x_2 - \frac{\partial h}{\partial x_2} \right), \quad \text{where } \zeta = \xi_1 + i\xi_2. \tag{4.8}$$

We want to show that $g(\zeta)$ is analytic, so that its derivative is analytic, and then show that the derivative is bounded and thus constant. The function g is clearly continuously differentiable for all $\zeta \in \mathbb{C}$, and by noting that $\frac{\partial h}{\partial x_i} = \xi_i - x_i$ for $i = 1,2$, we obtain

$$\frac{\partial}{\partial \xi_1}(\text{Re}(g(\zeta)) = \frac{\partial}{\partial \xi_1} \left(x_1 - \frac{\partial h}{\partial x_1} \right) = \frac{\partial}{\partial \xi_1}(x_1 - (\xi_1 - x_1)) = 2\frac{\partial x_1}{\partial \xi_1} - 1$$
$$= \frac{1}{\det(J)} \left(2 + 2\frac{\partial^2 h}{\partial x_1^2} - \left(2 + \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial h^2}{\partial x_2^2} \right) \right) = \frac{1}{\det(J)} \left(\frac{\partial^2 h}{\partial x_2^2} - \frac{\partial^2 h}{\partial x_1^2} \right). \tag{4.9}$$

We also have $\frac{\partial \xi_1}{\partial \xi_2} = 0$, since ξ_1 and ξ_2 are independent, so

$$\frac{\partial}{\partial \xi_2}(\text{Re}(g(\zeta)) = \frac{\partial}{\partial \xi_2} \left(x_1 - \frac{\partial h}{\partial x_1} \right) = \frac{\partial}{\partial \xi_2}(x_1 - (\xi_1 - x_1)) = 2\frac{\partial x_2}{\partial \xi_1} - \frac{\partial \xi_1}{\partial \xi_2}$$
$$= -\frac{2}{\det(J)} \left(\frac{\partial^2 h}{\partial x_1 \partial x_2} \right). \tag{4.10}$$

By symmetry of the real and imaginary parts of g, differentiation of $\text{Im}(g)$ will be similar and gives

$$\left(\frac{\partial}{\partial \xi_1} + i\frac{\partial}{\partial \xi_2} \right) g = 0, \tag{4.11}$$

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so g satisfies the Cauchy-Riemann equations and is therefore analytic in the whole ζ-plane with

$$g'(\zeta) = \frac{1}{\det(J)} \left(\frac{\partial^2 h}{\partial x_2^2} - \frac{\partial^2 h}{\partial x_1^2} + 2i \frac{\partial^2 h}{\partial x_1 \partial x_2} \right). \tag{4.12}$$

The function g' is then by definition also analytic, and

$$|g'(\zeta)|^2 = \frac{1}{\det(J)^2} \left(\left(\frac{\partial^2 h}{\partial x_2} - \frac{\partial^2 h}{\partial x_1} \right)^2 - 4 \left(\frac{\partial^2 h}{\partial x_1 \partial x_2} \right)^2 \right) = \frac{(\Delta h)^2 - 4 \det(H)}{(2 + \Delta h)^2}$$
$$= \frac{(\Delta h - 2)(\Delta h + 2)}{(\Delta h + 2)^2} = \frac{(\Delta h - 2)}{(\Delta h + 2)} < 1, \tag{4.13}$$

since $\Delta h \geq 0$. By Liouville's theorem any bounded analytic function must be constant, and therefore the real and imaginary parts of g' are constant. But then all $\frac{\partial h^2}{\partial x_i \partial x_j}$, where $i,j = 1,2$, also have to be constant. Hence h is a polynomial of degree 2.

$\square$

Let f be a solution to the minimal surface equation (3.6) for the whole $\mathbb{R}^2$-plane, then similarly to the proof of lemma 3.19, there exists a C^2-function E whose Hessian is the normalized metric for the surface which is therefore constant equal to 1. By Jörgens' theorem this function must be a quadratic polynomial, and therefore

$$\frac{\partial^2 E}{\partial x_1^2} = \frac{1 + |p|^2}{W}, \quad \frac{\partial^2 E}{\partial x_2^2} = \frac{1 + |q|^2}{W}, \quad \text{and} \quad \frac{\partial^2 E}{\partial x_1 \partial x_2} = \frac{pq}{W} \tag{4.14}$$

are constant.

For $n = 3$ we have that $|p|^2 |q|^2 = (pq)^2$, so $W = \sqrt{1 + |p|^2 + |q|^2}$. This gives that p and q are bounded, which we prove in the following lemma. It is however not true if $n > 3$.

LEMMA 4.5: *Assume that p, q are real-valued continuous functions for which*

$$\frac{1 + |p|^2}{\sqrt{1 + |p|^2 + |q|^2}} = c_1 \quad and \quad \frac{1 + |q|^2}{\sqrt{1 + |p|^2 + |q|^2}} = c_2, \tag{4.15}$$

where c_1, c_2 are constants. Then p and q are bounded.

PROOF. If p is unbounded, then

$$c_1 = \lim_{|p| \to \infty} \frac{1 + |p|^2}{\sqrt{1 + |p|^2 + |q|^2}} = \lim_{|p| \to \infty} \frac{|p|}{\sqrt{\frac{1}{|p|^2} + 1 + \frac{|q|^2}{|p|^2}}} \tag{4.16}$$

which can only happen if $\frac{|q|}{|p|} \to \infty$ as $|p| \to \infty$. In particular this gives that $|q|$ is unbounded, and since

$$c_2 = \lim_{|q| \to \infty} \frac{1 + |q|^2}{\sqrt{1 + |p|^2 + |q|^2}} = \lim_{|q| \to \infty} \frac{|q|}{\sqrt{\frac{1}{|q|^2} + 1 + \frac{|p|^2}{|q|^2}}}, \tag{4.17}$$

we have that $\frac{|p|}{|q|} \to \infty$ as $|q| \to \infty$. But this means that when $|p| \to \infty$ we would have both that $\frac{|p|}{|q|} \to \infty$ and $\frac{|p|}{|q|} \to \infty$, which is a contradiction. We can make the same argument starting with q, so we must have that both p and q are bounded.

$\square$

LEMMA 4.6: *If $\mathcal{S}$ is a minimal surface defined in the whole x_1, x_2-plane by $f(x_1, x_2)$ for which $p = \frac{\partial f}{\partial x_1}$ and $q = \frac{\partial f}{\partial x_2}$ are bounded, then p and q are constant.*

PROOF. We can reparametrize the surface into the isothermal coordinates ξ_1, ξ_2 from proposition 3.9. Since E from (3.16) is a polynomial of degree 2 by Jörgens theorem, we have that $\frac{\partial E}{\partial x_1}$ and $\frac{\partial E}{\partial x_2}$ are both linear, so the transformation into isothermal parameters $\xi_i = x_i + \frac{\partial E}{\partial x_i}$ for $i = 1, 2$, is linear.

By using the lemmas 3.12 and 3.14 we then have that $f(\xi_1, \xi_2)$ is harmonic and $\phi = \frac{\partial f}{\partial \xi_1} - i\frac{\partial f}{\partial \xi_2}$ is analytic, in fact it is entire. Since both $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ are bounded, and ξ_1, ξ_2 is a linear transformation of x_1, x_2, then $\frac{\partial f}{\partial \xi_1}$ and $\frac{\partial f}{\partial \xi_2}$ are also bounded. This gives that the entire function ϕ is bounded, and by theorem 2.29 it is constant.

Again by the linearity of the transformation we obtain that $p = \frac{\partial f}{\partial x_1}$ and $q = \frac{\partial f}{\partial x_2}$ are also constant.

$\square$

This gives directly that f is linear in x_1, x_2 and hence it defines a plane, and thus Bernstein's theorem follows from these lemmas.

4.3 Consequences

Now when we have proved Bernstein's theorem, we would like to know what it can be used for. We shall prove a few corollaries which actually follows from lemma 4.3, the first two valid for any n and the third describing all possible solutions to the minimal surface equation defined in the whole plane for the case $n = 4$. All of them can be found in Osserman [8].

COROLLARY 4.7: *A bounded solution to the minimal surface equation* (3.6) *in the whole plane must be constant.*

PROOF. By lemma 3.14 we know that each x_k is harmonic in u_1, u_2. So we have that x_k is a bounded harmonic function defined in the whole parameter plane, and by Liouville's theorem must thus be constant.

$\square$

COROLLARY 4.8: *Suppose that f is a solution to the minimal surface equation* (3.6) *in the whole x_1, x_2-plane and $\widetilde{S}$ is the surface $x_k = \tilde{f}_k(u_1, u_2)$ obtained by referring to S in*

the isothermal coordinates (4.2). *Then*

$$\tilde{\phi}_k = \frac{\partial \tilde{f}_k}{\partial u_1} - \frac{\partial \tilde{f}_k}{\partial u_2} \quad \textit{for } k = 3,\ldots,n \tag{4.18}$$

are analytic in the whole complex plane of $u_1 + iu_2$ *and*

$$\sum_{k=3}^{n} \tilde{\phi}_k^2 \equiv -1 - c^2, \quad \textit{where } c = a - ib. \tag{4.19}$$

Conversely, suppose that $c = a - ib$ *is any complex number with* $b > 0$, *and that we have entire functions* $\phi_3, \ldots, \phi_n$ *of* $u_1 + iu_2$ *satisfying* (4.19). *Then we can define harmonic functions* $\tilde{f}_k(u_1, u_2)$ *from* (4.18), *and using the substitution* (4.2) *will give a solution to the minimal surface equation* (3.6) *defined in the whole plane.*

PROOF.

$\Rightarrow$) Using lemma 3.14 we have that the functions ϕ_k are analytic in the whole plane and that $\sum_{k=1}^{n} \phi_k^2 = 0$. The transformation (4.2) gives

$$\tilde{\phi}_1 = \frac{\partial x_1}{\partial u_1} - i\frac{\partial x_1}{\partial u_2} = 1, \quad \tilde{\phi}_2 = \frac{\partial x_2}{\partial u_1} - i\frac{\partial x_2}{\partial u_2} = a - ib, \tag{4.20}$$

which implies

$$\sum_{k=3}^{n} \phi_k^2 = 0 - 1^2 - (a - ib)^2 = -1 - c^2 \tag{4.21}$$

$\Leftarrow$) If we define $\tilde{\phi}_k = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$ for $k = 1,2$, where $x_1 = u_1$ and $x_2 = au_1 + bu_2$, then

$$\sum_{k=1}^{n} \tilde{\phi}_k^2 = 1 + (a - ib)^2 - 1 - c^2 = 0 \quad \text{and} \tag{4.22}$$

$$\sum_{k=1}^{n} |\tilde{\phi}_k|^2 \geq 1 + \sum_{k=2}^{n} |\phi_k|^2 \geq 1 > 0. \tag{4.23}$$

By defining $x_k = \text{Re}(\int \phi_k(\zeta)d\zeta)$ for $k = 3,\ldots,n$ and using equation (4.2) we have that these x_k define a minimal surface for the whole plane.

$\square$

COROLLARY 4.9: *Any solution* $f = (f_3, f_4)$ *to the minimal surface equation for the whole* (x_1, x_2)-*plane when* $n = 4$ *can be described in one of the following two forms.*

i) *An entire function* $g(z) = f_3 \pm if_4$, *where* $z = x_1 + ix_2$.

ii) Functions $f_k = \mathrm{Re} \int \tilde{\phi}_k(w)dw$ where $k = 3,4$, obtained from an arbitrary transformation of the form (4.2), and $d = 1 + (a - ib)^2$ such that

$$\tilde{\phi}_3 = \frac{1}{2}\left(e^{H(w)} - de^{-H(w)}\right), \quad \tilde{\phi}_4 = \frac{i}{2}\left(e^{H(w)} - de^{-H(w)}\right), \quad (4.24)$$

where $H(w)$ is an arbitrary entire function.

PROOF. To every global solution f_3, f_4 to the minimal surface equation (3.6) there is by lemma 4.3 a transformation $x_1 = u_1$, $x_2 = au_1 + bu_2$ with $b > 0$ into isothermal parameters, and by corollary 4.8 there are then entire functions $\tilde{\phi}_3$, $\tilde{\phi}_4$ such that $\tilde{\phi}_3^2 + \tilde{\phi}_4^2 = -d$ where $d = 1 + c^2$. This gives two cases corresponding to the two possible descriptions of the solution.

Case 1: $c = \pm i$.

Then $\tilde{\phi}_3^2 + \tilde{\phi}_4^2 = 0$, so $\tilde{\phi}_4 = \pm i\tilde{\phi}_3$. This gives that $f_3 + if_4$ is an analytic function of z or $\bar{z}$ where $z = x_1 + ix_2$.

Case 2: $c \neq \pm i$.

Factorizing will then give $(\tilde{\phi}_3 + i\tilde{\phi}_4)(\tilde{\phi}_3 - i\tilde{\phi}_4) = -d$, where $d \neq 0$. Since none of the factors can be zero anywhere but both are entire, we will have that $\tilde{\phi}_3 - i\tilde{\phi}_4 = e^{H(w)}$ and $\tilde{\phi}_3 + i\tilde{\phi}_4 = -de^{-H(w)}$ for some entire function $H(w)$, which gives the wanted formulas.

$\square$

Remark 4.10: Part *i*) of Corollary 4.9 says that the graph of any complex analytic curve viewed as surface in real Euclidean space is always a minimal surface.

The way of finding global solutions by letting $f_3 + if_4$ be defined as an entire function is possible to generalize quite easily if the dimension is even.

For n even, let $z = x_1 + ix_2$ and $g_1, \ldots, g_m$ be complex analytic functions of z where $n = 2m + 2$. If we for $j = 1, \ldots, m$ let

$$f_k = \begin{cases} \mathrm{Re}\, g_j(z), & k = 2j + 1 \\ \mathrm{Im}\, g_j(z), & k = 2j + 2 \end{cases} \quad (4.25)$$

then these equations will define a solution to the minimal surface equation (3.6). If all the functions g_j are entire, the surface will be defined for the whole parameter plane.

To see that these f_k above satisfy (3.6), we start by noting that $g_j = f_l + if_{l+1}$ where $l = 2j + 1$ and $j = 1, \ldots, m$. Since g_j is analytic f_l and f_{l+1} will satisfy

$$\frac{\partial f_l}{\partial x_1} = \frac{\partial f_{l+1}}{\partial x_2}, \quad \frac{\partial f_l}{\partial x_2} = -\frac{\partial f_{l+1}}{\partial x_1}$$

$$\frac{\partial^2 f_l}{\partial x_1^2} = \frac{\partial^2 f_{l+1}}{\partial x_1 \partial x_2} = -\frac{\partial^2 f_l}{\partial x_2^2}, \quad \frac{\partial^2 f_l}{\partial x_1 \partial x_2} = -\frac{\partial^2 f_{l+1}}{\partial x_1^2} = \frac{\partial^2 f_{l+1}}{\partial x_2^2}, \quad (4.26)$$

for l odd. Using these equations we will obtain that

$$\frac{\partial^2 f}{\partial x_1^2} = -\frac{\partial^2 f}{\partial x_2^2}, \quad \left|\frac{\partial f}{\partial x_1}\right|^2 = \left|\frac{\partial f}{\partial x_2}\right|^2, \quad \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} = 0. \tag{4.27}$$

Plugging these into the minimal surface equation gives

$$\begin{aligned}
&\left(1 + \left|\frac{\partial f}{\partial x_2}\right|^2\right)\frac{\partial^2 f}{\partial x_1^2} - 2\left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2}\right)\frac{\partial^2 f}{\partial x_1 \partial x_2} + \left(1 + \left|\frac{\partial f}{\partial x_1}\right|^2\right)\frac{\partial^2 f}{\partial x_2^2} \\
&= \left(1 + \left|\frac{\partial f}{\partial x_1}\right|^2\right)\frac{\partial^2 f}{\partial x_1^2} - 2\cdot 0 \cdot \frac{\partial^2 f}{\partial x_1 \partial x_2} - \left(1 + \left|\frac{\partial f}{\partial x_1}\right|^2\right)\frac{\partial^2 f}{\partial x_1^2} = 0,
\end{aligned} \tag{4.28}$$

which proves that these functions f_k defines a minimal surface.

To conclude this thesis and to give a hint of what else there is to know about Bernstein's theorem and minimal surfaces we will state its generalization to hypersurfaces in higher dimensions. The theorem then says that minimal hypersurfaces in dimension $n = 3,\ldots,8$ which is defined in the entire $(x_1,\ldots,x_{n-1})$-space must be linear. This has also been proven to be false when $n > 8$.

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