THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Exercising Mathematical Competence
Practising Representation Theory and Representing Mathematical Practice

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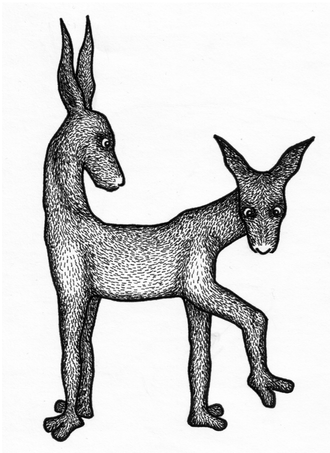

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ABSTRACT

This thesis assembles two papers in mathematics and two papers in mathematics education. In the mathematics part, representation theory is practised. Two Clebsch-Gordan type problems are addressed. The original problem concerns the decomposition of the tensor product of two finite dimensional, irreducible highest way representations of $GL_{\mathbb{C}}(n)$. This problem is known to be equivalent with the characterisation of the eigenvalues of the sum of two Hermitian matrices. In this thesis, the method of moment maps and coadjoint orbits are used to find equivalence between the eigenvalue problem for skew-symmetric matrices and the tensor product decomposition in the case of $SO_{\mathbb{C}}(2k)$. In addition, some irreducible, infinite dimensional, unitary highest weight representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$ are determined.

In the mathematics education part a framework is developed, offering a language and graphical tool for representing the exercising of competence in mathematical practices. The development sets out from another framework, where competence is defined in terms of mastery. Adjustments are made in order to increase the coherence of the framework, to relate the constructs to contemporary research and to enable analysis of the exercising of competence. These modifications result in two orthogonal sets of essential aspects of mathematical competence: five competencies and two aspects. The five competencies reflect different constituents of mathematical practice: representations, procedures, connections, reasoning and communication. The two aspects evince two different modes of the competencies: the productive and the analytic. The operationalisation of the framework gives rise to an analysis guide and a competency graph.

The framework is applied to two sets of empirical data. In the first study, young children's exercising of competencies in handling whole numbers is analysed. The results show that the analytical tools are able to explain this mathematical practice from several angles: in relation to a specific concept, in a certain activity and how different representations may pervade procedures and interaction. The second study describes university students' exercising of competencies in a proving activity. The findings reveal that, while reasoning and the analytic aspect are significant in proving, the other competencies and the productive aspect play important roles as well. Combined, the two studies show that the framework have explanatory power for various mathematical practices. In light of this framework, this thesis exercises both aspects of mathematical competence: the productive aspect in representation theory and the analytic aspect in the development of the framework.

Keywords: Mathematical competence, exercising competencies, young children, whole number arithmetic, tertiary level, proving, highest weight representation, tensor product decomposition, skew-symmetric matrix, moment map, infinite dimensional unitary representation.

Contents

Preface

När nu arbetet är utfört och texten skriven, infinner sig naturligtvis en stor tacksamhet till alla de som varit med under den här tiden. Det finaste är förstås att omsätta sin tacksamhet i handling, men just nu får ni nöja er med ord.

Mitt första tack går till mina handledare. Mats Andersson, som tog på sig huvudhandledarskapet då jag ännu var ett oprövat kort i utbildningsvetenskaplig forskning. Han har varit ett viktigt stöd genom processen och många gånger hjälpt mig att formulera mina tankar tydligare. Dessutom såg han till att jag fick en enastrålande biträdande handledare i Jesper Boesen. Jesper har hjälpt mig på allehanda sätt, med kontakter, läsning och diskussioner. Han har alltid funnits till hands och alltid varit noggrann med, inte minst positiv, återkoppling, vilket har betytt oerhört mycket. Tack också till Genkai Zhang, som handledde min licentiatavhandling.

Jag vill även tacka Kerstin Pettersson, för att hon givit mig tillgång till hennes data och för samarbetet med vår artikel, samt personalen och barnen på förskolan där intervjuerna till den första studien genomfördes. Ett särskilt tack riktas också till Moa Säfström, som har skrivit in nästan all text i LaTeX. Utan henne hade detta inte varit en avhandling, bara 400 sidor klotter.

De fem år och åtta månader som har gått har varit en kamp, ja, ett riktigt slagsmål, som åsamkat sår jag än idag pillar skorporna av. Men det ska inte glömmas att den här tiden även har bjudit på mycket glädje och lärdomar som inte kostat någonting. Mycket av detta har jag att tacka FLUM för. Cecilia, för att hon bjöd in mig till gruppen och förklarade hur saker låg till. Åse, för hennes alltid utmanande, men aldrig dömande frågor. Rimma och Hoda för allt. Förutom att de är de finaste av kollegor, har de tjänat som arbetsmiljöombud och sjuksystrar och varit frikostigare med sina böcker och sitt kontorsmaterial än jag någonsin kommer att bli. Dessutom alla andra i FLUM för allt de bidragit med på seminarier, träffar och med korrekturläsning, samt inte minst i form av arbetsmiljö och sammanhang.

Slutligen vill jag tacka min familj och mina vänner för att de har följt mig under det här arbetet och stöttat mig med råd och kärlek. Mina föräldrar, som alltid trott på mig och låtit mig hållas i allt jag hittat på. Mina fantastiska syskon (svågrar och svägerskor inräknade!) för att de alltid lyssnar, hjälper och ifrågasätter. Sara, Sverker, Emma, Adam, Helene och Rebecka; vad skulle jag ha gjort utan dem? Antagligen arbetat mig fördärvad.

Jag tror inte att någon bestrider att jag under detta arbete har gjort skäl för mitt namn. Nu är det dock tid för fliten att stå åt sidan och för att vila i den nåd det är att avhandlingen är klar.

Anna Ida Säfström, mars 2013

Chapter 1

Introduction

The work which lies before you assembles two papers in mathematics and two papers in mathematics education. This means that this thesis is a two-headed creature, with two minds with quite disparate aims and dispositions. One of them is the result of *doing* mathematics, the other one is the result of *thinking about* mathematics. In this sense, the two parts exercise two different aspects of competence in relation to mathematical practice. The first two papers are produced within the practice. They are examples of practising mathematics or, more specifically, representation theory, which is a branch of mathematics. The latter two papers have an external, analytic perspective on mathematical practice. They offer a language and a graphical tool for representing mathematical practice by describing the phenomena and processes involved. These two aspects, the productive and the analytic, comprise one dimension of competence as described in the framework used in Chapter 3. Using one part of this thesis to depict the entirety may seem to cause an imbalance between the parts, but there are good reasons for this.

Mathematics can be used to *do* mathematics. To some extent, it can also be used to speak about itself, e.g. about provability of mathematical statements with respect to an axiomatic system (for the general audience, see Davis and Hersh (1998) or Hofstadter (1999)). Nevertheless, other epistemological issues such as the nature of mathematical knowledge and how such knowledge is acquired, cannot be discussed in the language of mathematics (Berts & Solin, 2011). Despite the recursive character of the construction of mathematics, it cannot speak about itself as a field of knowledge. For such purposes one may turn to the philosophy of mathematics. Neither can it speak about itself as a practice, but that is what theories in mathematics education can do. Moreover, as far as such theories have general perspectives on knowledge and the teaching and learning thereof, they can also depict themselves. Consequently, if one of the two heads of this creature is to eye the other, it is educational research which is to eye mathematics.

But do they have enough in common to form a whole? At first sight, mathematics and mathematics education are two very different scientific fields. The mathematics in the first two papers included here deals with things such as Lie algebras, flag manifolds and infinite dimensional representations, which do not have a material existence. The choice of the problems addressed and the methods used are not elaborated upon. Actually, the method by which the proofs were constructed, how the work was done, is not mentioned. The theorems and propositions are not proved in order to understand the perceived world, even if we define the perceived world in very narrow terms and intend a very specific kind of understanding. In contrast, the last two articles deal with the utterances and actions of eight preschool children and four university students, who exist in the regular sense of the word. The design and the analytical tools used play a central role in the papers and the results and the eligibility of the papers are ultimately justified by their usefulness with respect to the understanding of a phenomenon, which in turn could advance teaching and learning.

On the other hand, it would be too extreme to say that methods do not matter in mathematics research. Regarding Horn's conjecture (cf. Chapter 2 and Paper I) and equivalent problems concerning invariant factors, highest weights and Schubert calculus, Fulton (2000, p. 245) states that: "It would be interesting to find more direct relations between the subjects ... that would give better explanation of why questions in each subject have the same answers", since the current proofs do nothing of the sort. New proofs of established theorems have led to new areas, e.g. in the case of the Briançon–Skoda theorem (Sznajdman, 2012). Furthermore, in both mathematical and educational research it is a polished version of the work which is presented. The attempts which led to dead ends are disregarded and the descriptions given are simplified accounts of events. The constructs formed in theorising mathematics education do not exist in the same material way as the objects studied. They must be argued for, both in terms of intrinsic meaning and consistency between them, much like new mathematical objects. In both mathematical and educational research there are studies with more direct applications and relevance for practice and studies of more theoretical nature. Fundamental research in all scientific fields is obliged to justify its importance and is faced with criticism for pursuing knowledge for knowledge sake (e.g. Wiliam & Lester, 2008).

By looking more closely at these two areas of research, from the perspective of the other, the intention is to go beyond the obvious and give some substantial answers to the question: What can mathematics education do for mathematics and what can mathematics do for mathematics education?

1.1 Mathematics from the perspective of mathematics education

To begin with, the issue of a perspective on mathematics would not normally have been dealt with in a pure mathematics thesis. A mathematician does not talk about mathematics, she *does* mathematics. Reflecting on the nature of her practice was not part of her education and is not part of her daily work. To obtain means for discussing these issues she needs to go outside of mathematics. But why should she bother? Could she not be a proficient professional without considering these issues? Well, I would maintain that it depends on the mathematician and on her conception of her trade. Perhaps she equates her work with mathematics and mathematics with the products of research: written artefacts such as textbooks and scientific articles. She might claim, therefore, that within her profession there is no doubt and no choices to be considered, there is no other matter than whether lines of arguments are correct or erroneous. However, this view is suspected to be attributed to others more often than personally held.

Every mathematician has experienced the discovery of a mistake in a proof, which the day before seemed convincing. As Davis and Hersh (1998, p. 57) state: "This happens to the best of us every day of the week. When the error is pointed out, one recognizes it as an error and acknowledges it. This situation is dealt with routinely". It might even happen that the argument that convinced you yesterday, which you found flawed today, is the one convincing you yet again tomorrow. A proof is thus a line of arguments without detected errors. The more people who have gone through the steps of the proof, the higher is the certainly of its validity. When mathematicians speak of a proof of a theorem, they refer to a line of arguments accepted by the community of mathematicians, which gives firm enough grounds for taking the theorem as true. The criteria for acceptance, e.g. with regard to the level of detail, are not made explicit. Furthermore, even accepted and transparent proofs sometimes lead to contradictions or undecidability, bringing the ontological and epistemological stances of mathematicians to light. One example is the naive set theory definition of sets, implying Russell's paradox:

> A set is a well-defined collection of objects. The objects included in

a set are called the elements of the set. We can define sets of sets, i.e. sets where the elements in themselves are sets. Now, consider a set X, which is not an element in itself. Define the set Y as the set of all such sets. Is Y an element in Y? The answer is that Y can neither be nor not be an element in Y.

One can address this paradox in a number of ways. One may argue that the definition of sets needs to be changed, or one may propose that the principle of bivalence does not hold for sets, or one may even conclude that set theory is a futile business altogether. If one settles on the first option, and accepts the Zermelo–Fraenkel set theory, one ends up with the issue of whether to add the Axiom of choice. If adopted, it allows for unconstructive proofs and implies such fantasticalities as the Banach–Tarski theorem. This theorem states that a three-dimensional ball can be decomposed into a finite number of parts and then put together into two balls, identical to the first one. With a view of mathematics as pure formalism, such results are just part of the game. However, it might be hard to argue for the worthwhileness of mathematics by its virtues as a tool for understanding reality.

Different epistemologies of mathematics are possible, and they matter in the choice of which branch of mathematics to pursue, and they matter in the discussions with other mathematicians, with students and with people outside academia. Being aware of possible differences can ease these discussions. Birgisson (2004) suggests a model, based on interviews with university students, including both problem solving and philosophical dialogue. The model is depicted in Figure 1.1.

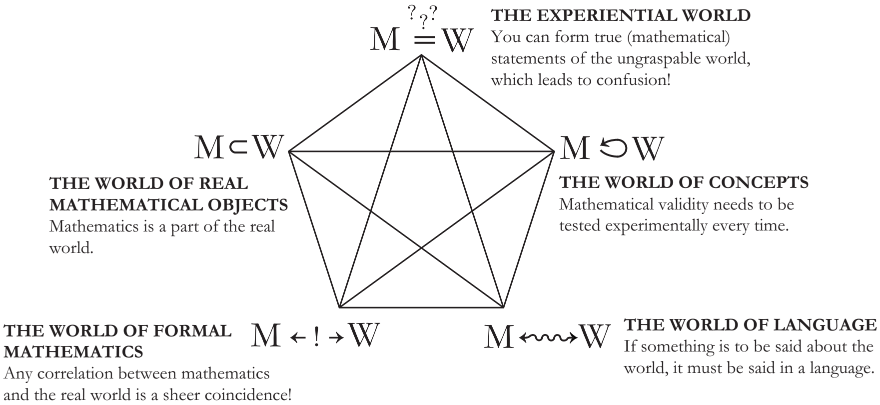


FIGURE 1.1: A model of mathematical epistemologies, adapted from Birgisson (2004, p. 277). The epistemologies reflect different beliefs regarding the relationship between mathematics (M) and the experiential world (W).

This model allows for travelling between worlds, providing a tool for understanding cognitive conflicts among students. In addition, it is in line with Davis and Hersh's (1998) description of mathematicians' handling of epistemological issues: believing in the reality of mathematical objects when doing mathematics, but hiding behind formalism when attacked by philosophers.

However, it is not necessary to dive into deep philosophical issues to find examples where research in mathematics education can aid the practice of mathematics. Many mathematicians are teachers and, as such, know that a perfect demonstration of a proof or a method does not automatically lead to understanding of the theorem or ability to apply the method on a task. Two mathematically equivalent presentations may have different pedagogical qualities. The heuristic methods used in problem solving, in class or in research, and the manners of presenting mathematics, the style in which mathematicians speak and write, cannot be described by mathematics. Neither can the significance of emotions and intuition connected to the research process be described, nor the perception of aesthetic values in mathematics. Nonetheless, the interviews with 70 mathematicians conducted by Burton (2004) show that these aspects matter in mathematical research and they matter in mathematical education, since, as one of the informants states:

> Whether what you are thinking about is new, research, known things or not, for you it is all new. When you understand a new proof, it becomes your own. Internally it is as though you did it (Burton, 2004, p. 290).

However, this is not merely a concern for teachers, since as a mathematician you regularly learn new mathematics and you need to teach yourself and newcomers to your field, as well as explain your work to colleagues. Needless to say, there is the possibility for unconscious assimilation of knowledge about mathematical practice. In fact, this is usually what happens. In a mathematician's own words:

> I am a mathematician because I have absorbed all those habits of mind that my lecturers encouraged me to absorb when I was an undergraduate. So I feel as if I have joined the club because I know what it is all about (Burton, 2004, p. 289).

This quote evinces awareness of the learning process, but if this process is undergone unconsciously, it will fail to generate means for reflection on the practice, e.g. identification of the strengths and weaknesses in the personal competence. In terms of the framework used in the latter part of this thesis: the productive aspect will be developed, whereas the analytic aspect will not. The knowledge and skill required to *do* is not the same knowledge and skill required to explain, evaluate and reflect upon what is done. Educational research can provide tools for advancing this second type of knowledge, thus adding to the apprehension of mathematics as a scientific field. In summary: research in mathematics education says something about mathematics and, since mathematicians use, produce, communicate, learn and teach mathematics, it should be in their interest to hear what is said.

1.2 Mathematics education with a mathematician's sight

> The best way to sensitise yourself to learners is to experience parallel phenomena yourself.
>
> John Mason

A first, modest reflection is that to be able to research teaching and learning of a subject, you need to know that subject yourself. That does not, however, mean that you need to be a mathematician to study mathematics education. As an example, to study children learning whole number arithmetic, you need to be well acquainted with whole number arithmetic. You could also benefit from experience with algebraic structures, where the properties of arithmetic are made explicit, but you would not have any use for the latest research in number theory. With this in mind, it seems most reasonable for a mathematician to research the education of advanced mathematics. Not because a mathematician necessarily can do it, but since a non-mathematician cannot. As explained by Davis and Hersh (1998, pp. 43-44), it is

> a too obvious and therefore easily forgotten fact that mathematical work ... is a mysterious, almost inexplicable phenomenon from the point of view of an outsider. In this case, the outsider could be a layman, a fellow academic, or even a scientist who uses mathematics in his own work...
>
> If such a person accepts our discipline, and goes through two or three years of graduate study in mathematics, he absorbs our way of thinking, and is no longer the critical outsider he once was. In the same way, a critic of Scientology who underwent several years of 'study' under 'recognized authorities' in Scientology might well emerge a believer instead of a critic...

> Of course, none of this proves that we are not correct in our self perception that we have a reliable method for discovering objective truths. But we must pause to realize that, outside our coterie, much of what we do is incomprehensible. There is no way we could convince a self-confident sceptic that the thing we are talking about make sense, let alone 'exist'.

The position taken in this thesis is not quite as extreme: it is believed that the practice of mathematics can be explained and justified without use of mathematical language. Nonetheless, it would be hard to fully apprehend and describe mathematical practice without mathematical knowledge. However, the assertion that mathematicians should exclusively research education in advanced mathematics overlooks an important fact: a mathematician has not merely experience with advanced mathematics, but with all levels in the educational system. If this experience is kept in recent memory, assembled to a full picture and reflected upon, it forms a unique growing ground for ideas for research theorising mathematics across social settings and subject matter. Such research is necessary to understand the notion of mathematics, harbouring disparate phenomena: from the child's conception of fair sharing to the mathematician's perception of beauty in a proof, from the counting of material objects to the investigation of group actions on manifolds.

The comprehension of mathematics, which we conceive of as the common denominator for the above examples, and the specification of its components and characteristics are not merely valuable in their own right. On the one hand, they may serve as a tool for studying what opportunities students are given, as done by E. Bergqvist et al. (2010a, 2010b), or how mathematical knowledge is exercised in a specific situation, as in Paper IV. On the other hand, they may form a background for highlighting differences between situations sharing mathematical content. This is not necessarily restricted to differences in which or how components of mathematical knowledge are presented, but may concern various pedagogical and social issues.

The experience of mathematicians could give further contributions to research aiming at improving teaching and learning. Mathematicians are successful in learning mathematics, because of and despite all the circumstances and events they have encountered. Since they have such rich backgrounds, they have gathered a collection of both more and less fruitful environments and practices for mathematical learning. Clearly, this experience could be used to create better learning environments, helping teachers to stimulate their students' development and appreciation of mathematical knowledge. Furthermore, it could offer counterbalance to the 'myth of the mathematical life course', described by Burton (2004, p. 288), quoting Murray (2000):

> She explained that, "according to the myth, mathematical talent and creative potential emerge very early in childhood ... and the student proceeds from college to an elite graduate high school without a break" (p.16) and so on through a rewarded career. As she made clear, "the myth of the mathematical life course was never a viable model" (p. 231) despite its strong influence, especially on the general public and on the teachers, who tend to believe that mathematicians are born not made.

In her own research, Burton (2004) found that mathematicians form a motley crew:

> The course of their lives, as mathematicians, was also very different from one to another, some only choosing mathematics as their discipline when they were well into their studies at university thus offering counter-evidence to the myth of the mathematical life course (Burton, 2004, p. 288).

Altogether, there are reasons for entering educational research for mathematicians so inclined, as there are benefits for educational research if they do. The rich experience of presentation and learning mathematics, which mathematicians carry, can provide fertile grounds for research in mathematics education.

1.3 Structure of the thesis

After the above reflections on the relationship between the two parts, the two areas of research are introduced separately. In Chapter 2 the Classical Clebsch-Gordan problem is described, which is the common origin of the two problems addressed in the mathematical papers. Chapter 3 introduces mathematical competence as a perspective on mathematical knowledge and practice. Section 3.1 begins with a background discussing how competence has been treated in the Swedish school mathematics context, as well as in educational research. This section also includes a description of the original framework and an overview of related research. In Section 3.2, the aims and the research questions are stated, while Section 3.3 deals with the methods used. Section 3.3.1 describes the development of the new framework and Section 3.3.2 accounts for the collection and presentation of the data used. In the beginning of Section 3.3.3 the new framework is formulated, followed by a explanation of its analytical tools. After summarising the results of the latter two papers, these results and their implications are discussed in Section 3.5. At the end of the thesis, the four papers are included.

Chapter 2

Clebsch-Gordan type problems

This chapter, together with the first two papers, constitute the mathematics part of this thesis. This part is a reorganised and revised version of a previously published licentiate thesis (Säfström, 2010). Below, a brief introduction to Paper I and II is offered. Since the papers make use of results from diverse fields, it is found reasonable to assume little more of the reader, than a general knowledge of Lie and representation theory, as well as algebraic and Riemannian geometry.[1] While this chapter focuses on highest weight representations, the prerequisites needed for Paper I is therefore included within the paper.

The two papers address questions which, at first sight, seem vastly different. Paper I, investigating a skew version of Horn's conjecture from 1962, asks for a classification of triples of eigenvalues belonging to skew-symmetric matrices A, B and C, with $A+B=C$, whereas Paper II obtains a set of irreducible factors in a tensor product of infinite dimensional polynomial representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$. Though, the amount of concepts needed to formulate each problem give no hint of the methods needed to solve them: Paper I travels through a number of deep results from various fields of mathematics, whereas Paper II mostly consists of, perhaps snarling but rather elementary, calculations. The connection between the two papers is that they spring from a common, classical problem, which will be discussed here. For more details, see Olver (1999).

Preceding abstract algebra was the study of transformations on vector spaces. Considering some class of functions on a vector space X and what happens to them under the action of a group of transformations G, is known as **classical invariant**

[1]In need of reviving these subjects, confer Fulton and Harris (1991) or Knapp (1986) on representation theory, Griffiths and Harris (1994) on algebraic geometry and Boothby (2003) on Riemannian geometry.

theory. More precisely, a G-invariant function on the vector space X, on which the group G acts, is a function

$$I : X \longrightarrow \mathbb{R} : I(g.x) = I(x) \quad \forall g \in G.$$

Oftentimes one considers vector spaces of polynomials of certain degree, e.g. binary forms Q. This was what Alfred Clebsch and Paul Gordan were occupied by in the late 19th century Germany, giving name to problems as well as methods for solving them.

If G is a linear group, i.e. a group which is isomorphic to a matrix group over some field, the action of G on X is nowadays called a representation of G. Any invariant I will give rise to an invariant subspace $\{x : I(x) = 0\} \subseteq X$. This subspace is also a representation of G, and called a subrepresentation of X. Representations without subrepresentations besides the empty set and itself, are called irreducible. Finding some sort of basis for the invariants, e.g. a Hilbert basis $I_1, \ldots, I_m$ for which every other invariant can be written as a polynomial $p(I_1, \ldots, I_m)$, is then closely related to finding irreducible subrepresentations of X for G.

Once all irreducible representations are found, the attention is shifted to figuring out which ones occur in more complicated representations and a natural starting point is to attack the tensor product of two irreducibles $V_{\overline{\alpha}}$ and $V_{\overline{\beta}}$. This problem is known as the **Clebsch-Gordan problem**, and concerns the determination of the coefficients $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}}$ in the equation

$$V_{\overline{\alpha}} \otimes V_{\overline{\beta}} = \bigoplus_{\overline{\gamma}} c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} V_{\overline{\gamma}}.$$

They are given by

$$c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} = \dim \operatorname{Hom}_G(V_{\overline{\gamma}}, V_{\overline{\alpha}} \otimes V_{\overline{\beta}}),$$

the dimension of the space of G-equivariant linear transformations, and are related to the **Clebsch-Gordan coefficients**, which appear in formulas for writing products of spherical harmonics as linear sums. A similiar question is asking for when the triple tensor product contains a G-invariant factor, i.e. for which $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$

$$(V_{\overline{\alpha}} \otimes V_{\overline{\beta}} \otimes V_{\overline{\gamma}})^G > 0 \quad \text{or when} \quad c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} \neq 0.$$

This is the formulation preferred by Allen Knutson and Terrence Tao (Knutson &

Tao, 1999). This triple tensor product in Paper I, where it shows up in the Kirwan-Ness theorem, belongs to **geometrical invariant theory** (see further Mumford, Fogarty & Kirwan, 1994). This will be the end point of Paper I: the reformulation of the eigenvalue problem as a Clebsch-Gordan type problem.

This type of problems are obviously of varying difficulty, depending on for which Lie algebra to solve it. A popular exercise in introductory Lie theory textbooks (e.g. Humphreys, 1972), is to consider the $\mathfrak{sl}_{\mathbb{C}}(2)$ case, by noting that every representation of this algebra can be realised as the space of homogeneous polynomials in two variables, x and y, of degree equalling the highest weight of the representation. If, for example, there is a representations of highest weight α, there is a single diagonal element H, acting on the variables as $x \mapsto x$ and $y \mapsto -y$. This yields x^α as a highest weight vector, and the action on general monomials

$$H(x^{\alpha-k}y^k) = (\alpha - 2k)x^{\alpha-k}y^k.$$

Tensoring two representations V_α and V_β will correspond to multiplying polynomials, and it is then easily seen that

$$V_\alpha \otimes V_\beta = V_{\alpha+\beta} \oplus V_{\alpha+\beta-2} \oplus \cdots \oplus V_{|\alpha-\beta|}.$$

The reason for lingering on this small example is that it will be referred to in the end of Paper I – getting there through the general setting requires quite some work.

Another well-known example is the $GL_{\mathbb{C}}(n)$ case, where the coefficients are usually called **Littlewood-Richardson coefficients**. The usual way to define them is in terms of partitions and Young diagrams. A partition of a positive integer is a way of writing n as a sum of other positive integers. While each positive integer has a finite number of partitions, it is also possible to consider all partitions up to a specific length n, regardless of the integers which are partitioned, which are infinitely many. If the parts are listed in decreasing order, partitions can be described as weakly decreasing n-tuples. Below are the partitions of the integer 3 and some partitions of length 3.

$$3 \qquad 2+1 \qquad 1+1+1$$

$$(1,0,0) \quad (1,1,0) \quad (1,1,1) \quad (2,0,0) \quad (2,1,0) \quad (2,1,1) \quad \ldots$$

A weakly decreasing n-tuple, $\overline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n)$, also gives a Young diagram, consisting of $\sum \alpha_i$ boxes, arranged in n left-justified rows, with α_i boxes on the ith row. As an example, the Young diagram for the 3-tuple $(4,2,2)$ is depicted in Figure 2.1.

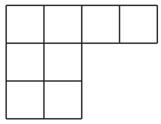


FIGURE 2.1: The Young diagram for the partition $(4,2,2)$ of 8.

For three partitions, $\overline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n)$, $\overline{\beta} = (\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n)$ and $\overline{\gamma} = (\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n)$, with

$$\sum \gamma_i = \sum \alpha_i + \sum \beta_i,$$

it is possible to define the Young diagram $\overline{\gamma} \setminus \overline{\alpha}$. The Littlewood-Richardson coefficient, $c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}}$, is then the number of ways to fill $\overline{\gamma} \setminus \overline{\alpha}$ with β_i i:s, so that every row is weakly decreasing from left to right and every column is strictly decreasing from top to bottom, and so that, when filled, the boxes read a lattice word [2]. When filled, the diagram is called a Young tableau.

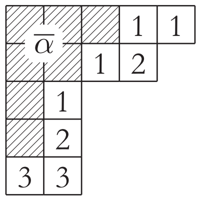


FIGURE 2.2: A possible Young tableau for $\overline{\alpha} = (3,2,1,1,0)$, $\overline{\beta} = (4,2,2,0,0)$ and $\overline{\gamma} = (5,4,2,2,2)$. The lattice word in this case is 11121233.

The Littlewood-Richardson coefficients show up in the proof of Horn's conjecture, the Hermitian eigenvalue problem, and they will also appear in an example in the end of Paper I. A special case will also play a role in the construction of factors of the infinite representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$ in Paper II.

The classical problem concerns the decomposition of the tensor product of two finite dimensional representations of $GL_{\mathbb{C}}(n)$. The two papers find results for problems related in two directions: finite dimensional, irreducible representations for $SO_{\mathbb{C}}(2k)$ in Paper I, and infinite dimensional, irreducible representations of $GL_{\mathbb{C}}(n)$ in Paper II. The classical case can be formulated in terms of eigenvalues for sums of Hermitian matrices and the $SO_{\mathbb{C}}(2k)$-case in terms of eigenvalues for sums

[2] This means that for any k, $1 \leq k \leq \sum \beta_i$ and i, $1 \leq i \leq n$, the i:s in the k first boxes are at least as many as the $i+1$:s.

of skew-Hermitian matrices. Therefore, it would be interesting to see whether the infinite-dimensional case of $GL_{\mathbb{C}}(n)$ is related to an eigenvalue problem, but this is yet to be investigated.

2.1 Summary of Paper I and II

2.1.1 Paper I: Skew symmetric matrix equations $A+B+C=0$

The first paper offers some results on the way to determine the possible eigenvalues of a skew-symmetric matrix C, which is the sum of two other skew-symmetric matrices, A and B, given that their eigenvalues are known. The uniform version of the problem concerns finding triples of eigenvalues for A, B and C when $A+B+C=0$. For skew-symmetric matrices of dimension $2k$, the eigenvalues are on the form $(\pm i\alpha_1,\ldots,\pm i\alpha_k)$, where the α_i:s are real and positive. The eigenvalues are then given by the eigen-k-tuple $\overline{\alpha}=(\alpha_1,\ldots,\alpha_k)$. Therefore, it is the set of triples of eigen-k-tuples,

$$\{(\overline{\alpha},\overline{\beta},\overline{\gamma}) : \pm i\overline{\alpha},\pm i\overline{\beta},\pm i\overline{\gamma} \text{ are eigenvalues of } A,B,C, \text{ respectively}\},$$

which is studied. This problem is identified with a Clebsch-Gordan type problem, i.e. a problem concerning invariant factors in tensor products of irreducible representations, through a number of steps. The main ingredients are the following:

1. (Theorem 3.2 of Paper I) Identifying the map from a matrix in the set of skew-symmetric matrices with eigen-k-tuple $\overline{\alpha}$ to the upper right elements of its diagonal 2×2-blocks, $A=(a_{ij})_{i,j=1}^{2k}\mapsto(a_{12},a_{34},\ldots,a_{2k-1,2k})$ as a moment map. In the same time, determining the map's image as the union of two convex polytopes: the convex hulls of permutations of $\{\pm\alpha_1,\ldots,\pm\alpha_k\}$ with either an even or an odd number of minus signs.

2. (Theorem 3.3 of Paper I) Showing that given two matrices, A and B, with given eigen-k-tuples, the set of possible eigen-k-tuples for $C=A+B$ is a union of four convex polytopes.

3. (Proposition 3.2 of Paper I) By successive embeddings of the polytope, showing that there are skew-symmetric matrices, $A+B+C=0$, with eigen-k-tuples $(\overline{\alpha},\overline{\beta},\overline{\gamma})$ whenever there is a $SO_{\mathbb{C}}(2k)$-invariant vector in $\mathcal{V}_{\overline{\alpha}}\otimes\mathcal{V}_{\overline{\beta}}\otimes\mathcal{V}_{\overline{\gamma}}$ and that there is a $SO_{\mathbb{C}}(2k)$-invariant vector in $\mathcal{V}_{N\overline{\alpha}}\otimes\mathcal{V}_{N\overline{\beta}}\otimes\mathcal{V}_{N\overline{\gamma}}$, for some

$N \geq 0$ whenever there are skew-symmetric matrices $A+B+C=0$ with $(\overline{\alpha},\overline{\beta},\overline{\gamma})$ as eigen-k-tuples.

In addition, the set of possible eigen-k-tuples is concretely determined in two cases of low dimension, $2k=4$ and $2k=6$, where Lie algebra isomorphisms can be employed.

2.1.2 Paper II: Unitary highest weight representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$

The second paper considers tensor products of infinite dimensional, irreducible polynomial representations of $\mathfrak{gl}_{\mathbb{C}}(n+1)$. Such representations form a certain class of infinite dimensional representations, which arise from finite dimensional representations of $U(n)$ on polynomial spaces. The aim is to find irreducible subrepresentations in the tensor product of two infinite dimensional polynomial representations. Below the main result in the paper is stated, giving a set of such subrepresentations.

Theorem. (Theorem 2.1 in Paper II) Let $\mathscr{P}_k^{\varkappa}$ and $\mathscr{P}_l^{\lambda}$ be two infinite dimensional polynomial representations of highest weight $\underline{k}=(0,\ldots,0,k)$ and $\underline{l}=(0,\ldots,0,l)$, respectively, and consider the tensor product $\mathscr{P}_k^{\varkappa}\otimes\mathscr{P}_l^{\lambda}$. Then, for every s_1,s_2,s_3,s,t such that

$$\begin{cases} s_1+s_2+s_3=s \\ 0\leq s_1 \\ 0\leq t\leq \min\{k,l\} \\ 0\leq s_3\leq t \\ 0\leq s_2\leq k+l-2t, \end{cases}$$

the polynomial space

$$\mathscr{P}^{\varkappa+\lambda}\otimes\mathscr{P}_{s_1}\otimes\bigodot^{k+l-s_2-2t}(\mathbb{C}^n)'\otimes\bigodot^{s_2-s_3+t}((\mathbb{C}^n)'\wedge(\mathbb{C}^n)')\otimes\bigodot^{s_3}((\mathbb{C}^n)'\wedge(\mathbb{C}^n)'\wedge(\mathbb{C}^n)')$$

is an irreducible subrepresentation with heighest weight

$$\underline{m}_{s_1,s_2,s_3,t}:=(0,\ldots,0,-s_3,-(s_2+t),-(k+l+s_1-t),-(\varkappa+\lambda+k+l+s)).$$

Chapter 3

Mathematical competence

3.1 Background and theoretical frame

Since the establishment of the compulsory elementary school in 1962-72, Swedish mathematics teachers have been requested to adjust to two opposing curriculum trends. On the one hand, there is the change in the description of content knowledge. In the three first curricula, Lgr62, Lgr69 and Lgr80 (Skolöverstyrelsen, 1962, 1969, 1980), the words *concept* (begrepp) and *computation* (räkning) are highly frequent, while words such as connection, discussion and interpretation are scarce. Even if *concept* maintains a strong position in Lpo94 and Lgr11 (Utbildningsdepartementet, 1994; Skolverket, 2011b), it is now joined with meaning, value, confidence, importance, language, symbols, expressions, properties, limitations, relevance and relationships. Even more so, *computation* has given way for an abundance of terms: develop, use, explain, reason, draw conclusions, argue, critically review, compare, estimate, describe, present, communicate, analyse and choose. From this point of view, it seems as though mathematics is a more diverse practice nowadays. The interpretation of self-evaluation as a mere matter of correctness in Lgr69, is replaced by more serious requirements of reflections in Lgr11, as seen in the following two excerpts:

> During the studies in mathematics the pupils should get used to accuracy, critical thinking and self-control. The knowledge acquired will be greater and longer lasting, if the pupils can establish immediately if the reached result is correct or erroneous. Since the teacher can not have time to control every written task immediately after it has been carried out, the pupils should subsequently get used to their own use of

> answer keys and solution books. (Skolöverstyrelsen, 1969, p. 140, my translation)

> Pupils can solve simple problems in familiar situations in a basically functional way by choosing and applying strategies and methods with some adaptation to the type of problem. Pupils describe their approach in a basically functional way and apply simple and to some extent informed reasoning about the plausibility of results in relation to the problem situation, and can also contribute to making some proposals on alternative approaches. (Skolverket, 2011a, p. 65)

On the other hand, the curricula have turned from descriptions of school practice in terms of what should be done in the classrooms, to descriptions of the goals pupils are supposed to attain. Rather than describing the teaching and activities pupils should be engaged in, Lpo94 and Lgr11 describe what pupils should be able to do. In this sense, there has been a shift from process-oriented to goal-oriented curricula. While the description of mathematics has become less focused on results and more focused on processes, in line with international reform movements (Kilpatrick, Swafford & Findell, 2001; NCTM, 2000; Niss & Jensen, 2002), the description of school practice has thus become more focused on results and less focused on processes. These conflicting trends may be one of the reasons why teachers find it hard to interpret the new curriculum goals and put them into practice. National quality reviews, conducted between the introductions of Lpo94 and Lgr11 state that:

> many of the teachers [show] great insecurity when it comes to the purpose of the different parts of the syllabus (except the goals to attain) and their role in the teaching... The majority of the teachers perceive that the syllabus does affect their teaching, even if many of them find it difficult to specify what this effect looks like, and whether it is the whole, some parts or some aspects of the syllabus which is affecting them. (E. Bergqvist et al., 2010a, p. 28, my translation)

> Many teachers find it difficult to find working methods where the general competencies present in the syllabus can be exercised and become lasting abilities among the pupils. (Skolinspektionen, 2009, p.16 , my translation)

As a consequence, it is found that:

> at large, the teaching gives the pupils limited opportunities to develop five of the six competencies. The competencies, which are included in the goals to strive for in the syllabus, are thus given limited space in the teaching. (Skolinspektionen, 2009, p. 18, my translation)

As conveyed by E. Bergqvist et al. (2010a, p. 35, my translation), "one possible conclusion is that the teachers have evaded through reflection on the relationship between goals and methods, e.g. by not describing how their methods are connected to their goals". Another explanation could be that such reflection is hindered by the goal-orientation of the curriculum, where descriptions of actual classroom practices are absent. When the teachers are compelled to look for aids for the arrangement of teaching and student activities elsewhere, the result is a heavy reliance on textbooks.

> Many of the teachers, who display great insecurity when it comes to the purpose of the different parts of the syllabus and their role in the teaching, except when it comes to the goals to attain, regard the textbook as a guide and support. They trust the textbook to interpret the syllabus in a sensible manner. (Skolinspektionen, 2009, p. 17, my translation)

As a result, the curriculum reforms do not have a large effect on the teaching practice. Instead, it tends to follow a standard model starting each lesson with a short introduction, followed by individual pace learning, i.e. working with textbook tasks in your own pace (E. Bergqvist et al., 2010a; Hansson, 2011; Löwing, 2004; Skolverket, 2003). In such a practice, the teacher functions as a tutor, helping students who request help, leaving the main responsibility for learning to the pupils (Hansson, 2011; Löwing, 2004). This causes a differentiation of the given education, which is further enhanced by the common practice of ability grouping (Skolverket, 2010, 2012). Giving pupils at different ability levels different teaching may have caused additional harm to the development of competencies, due to some teachers' interpretation of the curriculum Lpo94 (Utbildningsdepartementet, 1994). The expressions 'goals to attain', including content knowledge, and 'goals to strive for', including competencies, have contributed to the view that competencies can be developed only *after* the basic content knowledge is learned. In the latest curriculum, Lgr11 (Skolverket, 2011b), these terms are changed to 'core content' and 'knowledge aquirements'.

Nevertheless, the intention of the reformulation of mathematical knowledge in terms of competence in the latter curricula, is still obscured by the formulation of education in terms of goals and requirements rather than learning processes. There is a need for a clarification of the concept of competence and for descriptions of how competence could be achieved.

3.1.1 Knowledge as competence – premises and consequences

Competence has been used to speak about what students are accounted for mathematically and what is valued in classroom discourse (Cobb, Gresalfi & Hodge, 2009; Gresalfi, Martin, Hand & Greeno, 2009; Horn, 2008), without any beforehand favour of mathematical competence. Gresalfi et al. (2009, p. 50):

> propose a concept of competence as an attribute of participation in an activity system such as a classroom. In this perspective, what counts as 'competent' gets constructed in particular classrooms, and can therefore look very different from setting to setting... As a consequence, whether or not a student is deemed to be 'competent' is no longer seen as a trait of that student, but rather an interaction between the opportunities that a student has to participate completely and the ways that individuals take up those opportunities.

This definition to some extent parallels Gee's definition of Discourse, as being:

> composed of distinctive ways of speaking/listening and often, too, writing/reading *coupled* with distinctive ways of acting, interacting, valuing, feeling, dressing, thinking, believing, with other people and with various objects, tools, and technologies, so as to enact specific socially recognizable identities engaged in specific recognizable activities. (Gee, 2008, p. 155, emphasis in original)

Niss (2003, p. 6) presents a more condense version:

> To possess a competence (to be competent) in some domain of personal, professional or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain.

This definition, despite being as general as the previous two, directs focus to mathematical competence through its associated idea that it is possible to obtain a generic set of aspects which range over mathematical domains. This withstanding that mathematics constitutes a large variety of domains divided by content and social setting, where the aspects take different expressions. Regarding mathematics as a superdomain and defining its essential aspects will give a means for describing local mathematical practices. The Danish KOM-project (Niss & Jensen, 2002) provides one such generic set, which is composed of eight so-called competencies in two groups. These are, on the whole, the same as the mathematical competencies in the OECD/PISA mathematical literacy framework (OECD, 1999). Two other descriptions are given by the US government's Adding it up (Kilpatrick et al., 2001) identifying five strands of mathematical proficiency, and the NCTM's five process standards (NCTM, 2000).

All four have a common objective: to give advice and recommendations to teachers and curriculum writers to improve mathematics education, more or less based on results of research. Since they are not scientific texts, they do not have scientific requirements to account for the links and possible conflicts between different threads of research used. However, the benefit of building on the notion of mathematical competence is that it arose within the field of mathematics education and tends to the specific characteristics of mathematics as a field of knowledge. To begin with, the essential aspects are not seen as mutually exclusive and independent, but as both seeping into and supporting each other. Kilpatrick et al. (2001, p. 5) formulates it as follows: "Mathematical proficiency, as we see it has five strands... The most important observation we make about these five strands is that they are interwoven and interdependent." Niss (2003, p. 9) describes the competencies as: "closely related – they form a continuum of overlapping clusters – yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint". This is important with respect to previous debates within research in mathematics education, e.g. regarding perceived conflicts between conceptual and procedural knowledge. Furthermore the existing categorisations describe features of mathematical practice, such as proof making, modelling and handling procedures and representation, which if not meaningless at least would take very different meanings in non-mathematical contexts.

There are strands within research which have similar objectives – to somehow capture and categorise the complexity of mathematical knowledge and practice. One is taking on the widened idea of literacy, which in the case of mathematics is also known as numeracy. This is done by Johanning (2008) who studies fraction literacy, and defines literacy, with the words of Scribner and Cole (1981, p. 236):

> as a set of socially organized practices which make use of a symbol system and a technology for producing and disseminating it. Literacy is ... applying ... knowledge for specific purposes in specific context of

use.

However, sometimes literacy still carries its original connotations, as in Temple and Doerr (2012, p. 287): "developing content area literacy consists of learning to read, write and speak in accordance with the discursive norms of the disciplines". Accordingly, there is no unified agreement upon what mathematical literacy consists of.

Another strand developed as an answer to voices criticising mathematical literacy for being too abstract and reflecting a conservative view of mathematics and coined the term number sense (McIntosh, Reys & Reys, 1992). Greeno (1991, p. 170) proclaims that number sense "requires theoretical analysis, rather than a definition", and both Greeno (1991) and McIntosh et al. (1992) state that it is an elusive term. Viewing number sense as knowing in the conceptual domain of numbers and quantities, Greeno (1991) places it as an example of mathematical competence. Often though, it is associated with an inclination to use numbers and an expectation that numbers are useful (McIntosh et al., 1992), which is not a component of being competent and may obstruct considerations of critical thinking as part of competence.

In this thesis, there is a wish to avoid conceptualising mathematical competence as an idealised goal for education. Instead, the competence is meant to give the idea of a shifting spectrum where participants in mathematical practice position themselves in ways which are not necessarily linearly ordered. Therefore, a modified version of Niss' definition will be used, where 'master' is replaced by 'handle':

> **Competence** in some domain of personal, professional, or social life is ability to handle essential aspects of life in that domain.

Effects on the perspective on learning and teaching

Seeing knowledge as competence means that learning is about acquiring ways to handle essential aspects of life in a domain, and may include apprehending and reflecting upon these essential aspects. Furthermore Niss (2003, p. 10) states that "a mathematical competency can only be developed and exercised in dealing with [mathematical] subject matter", which parallels Hiebert's baseline conclusion, "students learn what they have opportunity to learn" (Hiebert, 2003, p. 10), or at least says that they do not learn what they are not given opportunity to. In other words, to develop competence one needs to take part in life in the domain of issue; one needs to participate in the practice of this domain. Competence is therefore seen as something under continual development, rather than a final goal, and the domain specific expressions of competence are formed in practice. In this sense, I see competencies as constructed in settings, rather than as attributes of individuals, in agreement with Gresalfi et al. (2009). However, the object of study here is the exercising, rather than the construction, of competencies (Gresalfi et al., 2009) or students' sense of competence (Horn, 2008).

Exercising is to be understood in a twofold way: both as applying and using existent competence as a means towards an end and as exerting competence in order to sustain and improve it. When it takes place in interaction with others it functions in a third way: as co-construction of and feedback on the conception of competence within the domain.

When seeing knowledge as competence, learning means exercising competence and teaching will mainly be about creating opportunities to exercise competence. This perspective is similar to the one of Lave and Wenger (1991), where learning is seen as participation, which is legitimised by old-timers or full participants. In both cases, one can speak of an existing practice, or domain, which both shapes and is shaped by the participants. In the practice, learning takes place as knowledge is used. However, in Lave and Wenger's work the learning is four-grounded, as the key term 'participation' reflects learning, while knowledge – what is learnt – is 'how to participate'. In comparison, 'competence' foregrounds what is learnt, while learning is seen as 'exercising competence'. Though both perspectives could be used to study the aspects of mathematical practice, once more, the aptness to the characteristics of mathematical practice suggests the use of a competence perspective.

3.1.2 The mathematical competency research framework

The theoretical framework, which serves as the starting point for the research project presented in this thesis, is the Mathematical Competency Research Framework (Lithner et al., 2010), abbreviated MCRF. As of today, it is the sole research framework for studying mathematical competence, i.e. analysing empirical data with a competence perspective. The framework was developed within the project *National tests in mathematics as a catalyst for implementing educational reforms* at Umeå Research Centre for Mathematics Education, where it was used to analyse tests, teacher interviews and classroom activities. The purpose of the analysis was to describe the opportunities students are given to develop mathematical competence, by investigating the competence requirements in tasks and other activities. The actual work of students was not considered.

The framework incorporates two orthogonal sets of constructs: the six competencies and the three competency related activities (CRA:s). The competencies are the constituents of general mathematical competence, whereas the CRA:s are the aspects of mastering each competency. All the definitions of the constructs are in terms of mastery. The CRA definitions are found in Table 3.1, while the competency definitions are summarised in Table 3.2.

The two competencies representations and connections rely on the notion of mathematical entity. According to Lithner et al. (2010, p. 162) entities include a large variety of mathematical concepts, such as “numbers, functions, geometrical objects, tasks, methods, principles, concepts, phenomena, and ideas, and their properties”. As mathematical objects are abstract, it is often necessary to deal with them through more concrete representations. In order to organise and structure one’s mathematical knowledge, connections between entities and representations, or parts of them, are crucial. The relations between entities, representations and connections are depicted in Figure 3.1.

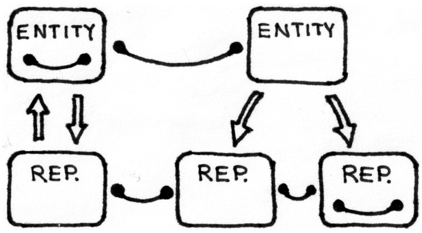


FIGURE 3.1: The relations between entities, representations and connections. Representation mappings are illustrated by white arrows, and connections by black connectors. The picture is an adaptation from Lithner et al. (2010, p. 164).

The CRA:s are meant to capture the dual nature of competencies, having both a productive and an analytic aspect (Niss, 2003). The productive aspect is operationalised in the activity *do and use*, which comes in two main versions: imitate and construct, brought in from the framework for creative and imitative reasoning (Lithner, 2008). The analytic aspect is distributed over two activities *interpret* and *judge*. *Interpreting* concerns taking in information with respect to the competencies, while *judging* covers meta-level considerations, such as evaluation, reflection and monitoring.

As seen above, the definitions of and the relations between the constructs within the MCRF are fairly explicit and elaborated. However, as Lithner et al. (2010,

p. 165-166) themselves state:

> the categorization of empirical data does by no means follow directly from the framework. Its operationalization in the research project mentioned above is based on sub-frameworks that depend on both the specific research tasks and the types of data... As the research project proceeds new aspects need to be included and new related frameworks need to be developed.

CRA	**DEFINITION**
CRA I: Interpret	[build knowledge, understand, interpret, identify, recognize] This CRA concerns *taking in information* in relation to the competencies. Since one purpose is to form specific definitions useful in characterizing data, the general and vague term 'knowledge building' is not included in the activity definitions.
CRA II: Do and use	[engage in task, pose, solve, use, respond, develop, argue, select, create, support, specify, apply, adapt, estimate] This CRA is about using one's knowledge in order to solve tasks (in a wide sense). Our interpretation of the distinction between 'do' and 'use' is that 'do' concerns developing our knowledge within mathematics as a scientific discipline, and 'use' concerns applying this within and outside mathematics. We also consider two main versions of this activity: a) imitate and b) construct (See (Lithner, 2008)).
CRA III: Judge	[evaluate, monitor, reflect] This CRA includes meta-level considerations and concerns evaluating, reflecting, and forming opinions and conclusions on mathematics and on the activities related to learning, understanding, doing and using mathematics.

TABLE 3.1: The MCRF definitions of the competency related activites (CRA's), taken verbatim from Lithner et al. (2010, p. 160). They include the words in square brackets to show how other international frameworks speak of the activities.

COMPETENCY	DEFINITION OF MASTERY
Problem solving	Engaging in a task for which the solution method is not known in advance. Understanding methods, tools and goals of problem solving. Applying and adapting various appropriate strategies and methods. Posing and specifying problems. Evaluating, monitoring and reflecting on solutions.
Reasoning	Explicitly justifying choices and conclusions by mathematical arguments. Select, use and create informal and formal arguments. Interpreting and evaluating one's own and others' reasoning. Reflecting on the role of reasoning. Knowing what a proof is.
Applying procedures	Using a sequence of mathematical actions that is an accepted way of solving a task. Understanding and interpreting procedures. Selecting and using procedures fluently. Evaluating applications and outcomes. Reflecting on the function of procedures.
Representation	Handling concrete replacements (substitutes), mental or real, of abstract mathematical entities. Interpreting and evaluating representations. Selecting, using and creating representations to organise, record and model phenomena, and communicate ideas.
Connection	Connecting between mathematical entities or representations of mathematical entities. Understanding how mathematical ideas interconnect and build on one another. Selecting, using and creating connections to organise, solve problems and model phenomena.
Communication	Engaging in a process where information is exchanged through a common system of symbols, signs and behaviour. Interpreting and formulating written, oral and visual mathematical statements at different levels of theoretical and technical precision.

TABLE 3.2: A summary of the MCRF definitions of the competencies.

3.1.3 A research context for the Mathematical Competency Research Framework

The MCRF is inspired by the Danish KOM-project and the NCTM standards and, to some extent, 'Adding it up' and the TIMSS and PISA studies (Lithner et al., 2010). These are not themselves frameworks, but rather aim to reform mathematics education or measure results in school mathematics. They do, however, base their categorisations on results from research, often conclusions from syntheses of various studies in each field. While such a process corroborates compatibility with the overall currents and directions of contemporary research, it also obstructs a consideration of deeper connections and contradictions, since epistemological issues are kept concealed. This issue is not resolved by the research companion (Kilpatrick, Martin & Schifter, 2003) which gives additional, more careful, relations between the standards and research, but sheds no light upon the path from initial stances and assumptions of research, through analysis and results, eventually leading to the standards. This story is also left untold by Lithner et al. (2010), where the principle 'opportunity to learn' from Hiebert (2003) and the concepts of competence and competencies from Niss (2003) are put together with the process goals in the NCTM standards. The definitions of the six competencies are in some cases linked to research: *problem solving* to the work of Schoenfeld (1985) and *reasoning* to Lithner's reasoning framework (Lithner, 2008), in some cases only to the reform frameworks and in the case of connections and communication not to any other source. Nevertheless, there is extensive research made on problem solving, procedures, representations, connections, reasoning and communication in mathematics educations. Relating the MCRF to such studies would further the delimitation and clarification of the constructs within the framework. Therefore, this is precisely what is done in the sections hereunder.

Representations

Though present in any human practice, representations have a particularly important role in mathematics and have been studied from various perspectives (Smith, 2003; Goldin, 2003). The representation definitions range from being restricted to very specific forms, such as notations, pictures and structured materials (Brinker, 1996), to more inclusive definitions, as "a configuration of signs, characters, icons, or objects that can somehow stand for, or 'represent' something else" (Goldin, 2003, p. 276). Respecting the multi-modality of human practice leads to more comprehensive views on representations, but also runs the risk of loosing precision. By adopting more general semiotic theories, as the ones originating from the work of Ferdinand de Saussure or Charles S. Pierce (see D. Chandler (1995) for an introduction), the focus on the specifics of mathematics may be lost. Furthermore, as in other areas of mathematics education, research on representations has given rise to local theories for analysis of specific representations, leaving the overall perspective somewhat scattered. This was the situation summarised by Kaput (1987) and, looking at articles published in journals within the field during the last five years, the situation remains. Many studies regarding specific representations do not draw on general theories, linking their findings to the comprehension of representations at large (e.g. C. C. Chandler & Kamii, 2009; David & Tomaz, 2012; Lee, Brown & Orrill, 2011; Monk, 2003; Murata, 2008).

In these studies, representations are often thought of as some class of artefacts used in the practice of mathematics teaching and learning, e.g. graphs, diagrams or concrete materials. However, in the theorising about representations the word can be restricted to inner mathematical representations, like isomorphisms (Kaput, 1987) or to representations of mathematical knowledge constructed in the analysis of data (Vergnaud, 1998). Neither one of these two extremes is the sole intended meaning of the term within this framework, thus such theories are expected to have little impact on the MCRF. Research discussing certain aspects of representations are more valuable, since they can help delineate the meaning of 'representation' with respect to learning and exercising mathematical competence.

Smith (2003) speaks of idiosyncratic and conventional representations, highlighting the importance of the creator of the representation. He states that "individually designed representations ... are invested with meaning by an individual (or a small group) for localized use, for solution of particular problems" (Smith, 2003, p. 269). In order to progress to the general and abstract levels of mathematics, teachers need to help students bridge the discrepancy between idiosyncratic and conventional representation. Otherwise, the students' own representations, being local and concrete, may constrain learning. In the creation and use of a representation, the creator finds her own meaning and value, while it may cause an "alternative use or interpretation of the same representation by another individual" (Smith, 2003, p. 266). Cooperative and reflective use may therefore lead to shared representations, which is one step towards conventional mathematical representations. As Smith (2003, p. 273) emphasise:

> How children understand the role of the process of sharing and analyzing representations in learning mathematics is an important aspect of learning. Children need to view analyzing one another's solutions as a

tool for deepening their mathematical understanding.

Kilhamn (2011) discusses representations in a wide sense, comprising both internal and external ones, including verbal expressions, symbols, symbolic artefacts, pictures, sounds and gestures. Following Damerow (2007), she separates first order and second order representations. Abstract objects can be first order representations of real objects or, if of meta-cognitive nature, second order representations of other abstract objects. Kilhamn (2011) also shares the view of Sfard (2008), that mathematical objects cannot be separated from their representations, but rather consist of their representations and the links between them.

> Mathematical objects very often have multiple representations and no one representation is consummate. Increased knowledge about mathematical objects is often linked to new ways of representing that object, and making connections between them... it is a mathematical object along with all the external representations that make up to essence of the object, as well as its place in a mathematical structure (Kilhamn, 2011, p. 62-63)

Kilhamn (2011) also speaks about metaphors and models, which are representations of larger systems of objects, e.g. motion along a path as a metaphor for basic arithmetic. Models can be seen as going in the opposite direction of the first order representations, representing abstract mathematics by real processes and objects. The mathematics and the model becomes a two-way metaphor, where it is possible both to see mathematics as represented by the model and the model as represented by mathematics.

As seen above, Goldin (2003) also offers an inclusive definition of representation and, just as Kilhamn, he acknowledges the existence of both internal and external representations. Moreover, he puts representations in relation to problem solving, by focusing on how they function in mathematical practice. He exemplifies the action 'represent' by "correspond to, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize" (Goldin, 2003, p. 276), where 'denote' is close to the first order representation used by Kilhamn (2011). An observation is made, that "representation relationships are often two-way, with the depiction or symbolization going in either direction to the context" (Goldin, 2003, p.276). He also stresses the connectedness of representations, introducing the notion 'representational systems'. Examples of external systems are natural language, formal mathematical notations and computer-based micro-worlds. Five types of eternal systems are identified, which also include planning and monitoring problem solving and affective aspects. Both internal and external representational systems can represent other systems, both internal and external. Goldin (2003, p. 278) concludes that:

> An overarching goal of school mathematics, then, must be to develop in students powerful internal representational systems of different kinds and to learn how to infer these systems from their observable, external manifestations.

Stylianou (2011) goes into further detail regarding the functions of representations in mathematical problem solving. As an outset for her study, she reflects on the complexity of representation use. On the one hand, mathematical objects can only be accessed through their representations. On the other hand, no representation conveys all aspects of an object. The use of multiple representations has been found to facilitate more complete understandings of objects, but also to hinder understanding by causing confusion. In addition, there are both social and content related complexities "involved in negotiating individually constructured representations in the shared space of the group or a classroom as well as the teacher's role in facilitating these interactions" (Stylianou, 2011, p. 268). To describe the role played by representations in the problem-solving process, Stylianou (2011) interviewed mathematicians solving non-standard calculus problems. Also, a class of sixth-graders were videotaped during lessons and interviewed. The problem solving of both the mathematicians and the students was analysed for finding examples of how representations were used. In both sets of data, four roles of representations were found: as a means to understand information, as a recording tool, as tools that facilitate exploration and as monitoring and evaluating devices. In the first two roles, representations have a static function as a product of one's acting, while in the last two roles, they function dynamically as a process within problem solving. Representations play different roles as the problem solving proceeds, and different representations may be more or less suited for different roles. Being able to talk about the roles of representation is not a mere issue for researchers, as Stylianou (2011, p. 278) states:

> An awareness of the different roles that representations play and the ways the representations can be manipulated or used for the purpose at hand can de-mystify for students the request for use of multiple representations and can, ultimately, improve students' appreciation of the role of representations in mathematics.

Altogether, the research provides various examples of representations, both in terms of their expressive and their functional form. Representations can be verbal, symbolic, graphical, audial or gestural and they can represent by corresponding to, denoting, depicting, labelling or meaning. They can be more or less formal, as well as internal or external. All these examples reflect the inner dimensions of representations, which span the conception of 'representation'. Research is also found to support the view of representations as two way both explicitly and implicitly, by revealing an ambiguity regarding what to see as the representation and what to see as the represented. It is possible to view the abstract as a representation of the real, but also to view the concrete as a representation of the abstract. In relation to the analytic aspect, the importance of reflection on one's own and other's use of representations is stressed, in particular for the development of shared meaning. It is also noted that representations can be used to monitor and reflect on other competencies.

Connections

Recognition of patterns and construction of connections are generally seen as key elements of mathematical understanding (Baroody, Feil & Johnson, 2007; Hiebert & Carpenter, 1992; Jacobs, Franke, Carpenter, Levi & Battey, 2007; Johanning, 2008; Simon, 2006). Jacobs et al. (2007) call this relational thinking and link it to both representations and reasoning. Vergnaud (1988, p. 141-142) states that: "A single concept usually develops not in isolation but in relationship with other concepts". It is safe to say that there is no conflict regarding the connectivity of neither knowledge in general, nor mathematical knowledge in particular, though one's personal connections may vary.

Hiebert and Carpenter (1992) discuss two metaphors for networks of knowledge, hierarchies and webs, and suggest a mixed perspective on understanding. Seeing knowledge network as some kind of partially ordered sets allows for many types of connections, e.g. those based on comparing similarities and differences and those based on inclusion. Other aspects are also considered, as the difference between internal and external connections and how they can connect within or between representations and procedures. Rather than the building of networks, they think of learning as reorganisation, acknowledging that learners do not come as blank slates. An increase in understanding is to be seen as an increase of the network, both in terms of size and level of organisation. A well-organised network does not merely lead to understanding, but facilitates remembering and transfer as well. In their own words:

> The growth of mathematical knowledge can be viewed as a process of constructing internal representations of information and, in turn, connecting the representations to form organized networks. An issue of special importance for mathematics educators is the issue of acquiring knowledge with understanding. Within the context of representing and connecting knowledge, understanding can be viewed as a process of making connections, or establishing relationships, either between knowledge already internally represented or between existing networks and new information (Hiebert & Carpenter, 1992, p. 80).

Johanning (2008) stresses the importance of connections reaching across contexts. Stepping outside one mathematical domain into another, comparing the two, may lead to deeper understanding of both domains. Such connections are called reflected, in contrast to the primary connections within a context (Hiebert & Lefevre, 1986). Furthermore, the development of a disposition to look for and use connections is argued for. According to Johanning (2008), students need to take responsibility for making sense of what they learn, to eventually view mathematics as connected and usable.

Though connections are associated with understanding and the importance of connected knowledge stressed, there are few research studies with connections as their primary focus. General remarks on the need to construct and use connections are common, while more detailed descriptions of the nature of connections and their structure in mathematical practice are scarce. Hiebert and Carpenter (1992) found two types of connections: inclusion and comparison. Ellis (2007a, 2007b) goes into more detail with her taxonomy for categorising generalisations. Though not claiming to present of definitive scheme, Ellis formulates a model for analysing generalisations based on two empirical settings, which are considered from two different angles in the two articles. The first one sets out of from the kinship between generalisations and justifications. Connections are assumed to play an important role in reasoning and proof:

> the ways in which students generalize will influence the tools that they can bring to bear when justifying their general statements. If a student creates a generalization based solely on empirical patterns, it would not be surprising if her proof were limited to the use of specific examples. In contrast, a student who generalizes from attending to the quantitative relationships ... in a problem might have a better chance

of producing a more general argument in her justification (Ellis, 2007a, p. 195).

In the second article generalisations are seen from the actor-oriented transfer perspective (Lobato, 2006, 2012). Students' generalising actions are found to be of three types: relating, searching and extending. Each type has several subcategories. *Relating objects*, by similarities in property of form, corresponds to the comparison connection mentioned by Hiebert and Carpenter (1992), while their inclusion connection is captured in Ellis' extending actions, *expanding the range of applicability* and *removing particulars*. Extending further comprises generating new cases by *operating* on an object and *continuing* a pattern. *Relating situations*, either by connecting back to previous situations or by creating new situations similar to former ones, parallels Johanning's reflective connections between contexts. In addition, Ellis' taxonomy includes searching actions. These are in the form of repetition, in order to detect relationships and patterns, or sameness in procedures or solutions. Repeating an action on different objects can thus be a way to construct a connection between these objects.

Despite the importance of connectivity stressed by many researchers, studies investigating the nature of connections are scarce. Connections between concepts, representations and procedures are related to understanding and they play a significant role in reasoning. Besides comparison and inclusion, connections can be constructed by repetition of actions or continuation of patterns.

Problem solving and applying procedures

These two competencies jointly aim to capture the solving of tasks and are therefore treated concomitantly. Problem solving is a common term in research, though often used merely to refer to the general activity informants are engaged in (see e.g. Lithner, 2008; Maher, 2005; Morris & Speiser, 2010; Mueller, 2009; Mueller, Yankelewitz & Maher, 2012; Smith, 2003; Speiser, Walter & Sullivan, 2007; Stylianou, 2011; Voutsina, 2012). The term indicates some qualities in the task, e.g. that it is demanding, rich, non-standard or requires successive steps in the solving process. However, there are also examples of studies of such activities where problem solving is not highlighted, e.g. David and Tomaz (2012) and Superfine, Canty and Marshall (2009). When problem solving is used in the general sense, it becomes the context, or scene, where the data is collected and analysed. The main part is played by other constructs, which are identified as tools or sub-processes in problem solving. When these sub-processes are linked to other competencies, such as representations (Goldin, 2003; Stylianou, 2011), reasoning (Maher, 2005; Mueller, 2009; Mueller et al., 2012) or both (Morris & Speiser, 2010; Speiser et al., 2007), problem solving is positioned as an overarching activity, describing mathematical practice at large. Such a view on problem solving obstructs the separation of competencies and complicates the relationship between them.

In contrast, procedures are seldom studied and historically the term carries negative connotations, as being in conflict with conceptual knowledge and understanding (Hiebert & Carpenter, 1992). In recent years, procedural knowledge has been re-conceptualised as intertwined with conceptual knowledge. Rather than seeing procedures as mechanical and isolated and conceptions as connected systems, deep understanding its thought to be reached by interconnecting both kinds of knowledge (Baroody et al., 2007; Star, 2005, 2007). Voutsina (2012, p. 195) describes the development of procedural and conceptional knowledge as an iterative process:

> The 'iterative model' suggests that during development, procedural and conceptual knowledge develop in a gradual process and each type of knowledge interacts with, and influences, the other.

With this view, procedures may also fill other purposes, besides reaching a result on a task.

> First, the application of a procedure may strengthen the initial conceptual knowledge upon which the procedure is based. Second, improved application of procedures may release cognitive resources that can then be devoted to reflection of mathematical relations and concepts. Third, improvement in the application of procedures may help children recognise and address previous misconceptions and, fourth, encouraging children to reflect upon and explain their procedures can lead to improved understanding of the underlying concepts (Voutsina, 2012, p. 196).

Voutsina stresses the importance of explaining and justifying procedures, both for the development of more effective procedures and the conceptual understanding connected to them. Nevertheless, Sáenz (2009, p. 126) points out that:

> although there is broad agreement on the importance of conceptual knowledge, it is less clear how this knowledge can be elicited and made to work in an integrated fashion with contextual and procedural knowledge when solving problems.

Though some examples in this direction are offered by Voutsina (2012), the interplay between procedures and other competencies is mostly absent in current research.

Two examples of considerations of other aspects of procedures are provided by Gravemeijer and Van Galen (2003) and Handa (2012). Gravemeijer and Van Galen (2003, p. 114) advocate the learning of procedures by a constructive process, "instead of concretizing mathematical algorithms for the students, the teacher can let students develop or reinvent the algorithm themselves". Facts and algorithms are compared and are both found to be associated to the acquisition metaphor for learning, seeing knowledge as a commodity which can be transmitted (Sfard, 1991). However, when seeing learning as a form of activity, facts and algorithms can be the products of this activity, with the production process providing a framework of reference giving meaning to the outcomes. Though personal, semi-formal procedures have value in their own right, Gravemeijer and Van Galen do not purpose that students should take on the whole responsibility for learning. Teachers need to provide learning environments for development of procedures through guided reinvention.

> The core idea is that students develop mathematical concepts, notations, and procedures as organizing tools when solving problems. In such a process, informal algorithms may come to the fore as forms of well-organized routines for solving certain types of problems. With guidance from the teacher, these informal algorithms can be developed into conventional algorithms (Gravemeijer & Van Galen, 2003, p. 117)

A key element in the process from personal to conventional procedures is the reflection on one's own solutions.

Handa (2012) makes a distinction between repetition and rote. While rote learning is mechanical by definition, systematic repetition can lead to deepened personal attachment to the topic and the procedure.

> Actions that are performed (including thinking) as a means to some other end – such as doing mathematics drills is not to learn but for one's grades – generally lead to low personal involvement and investment or else to a form of involvement typified by boredom. They become, in essence, rote. Yet, if the task being repeated is an end per se for the individual, this *task* – even the math drills, for example, for the *person intent on mastering the topic* at hand – can generate positive personal involvement that would overshadow any rote facet of the repetitive act.

> In turn, the kind of 'oneness' or intimacy as described by the musicians can begin to emerge (Handa, 2012, p. 268, emphasis in original).

Working with an organised set of quick calculations makes it possible for students to see the underlying structure while solving tasks. If the tasks are carefully constructed with respect to systematic variation and invariance, mathematical properties and deepened understanding of the topic "are surfaced in a form of instruction reliant on repetition" (Handa, 2012, p. 269). He emphasises that he is:

> not sounding the conventionally claimed virtues of repetition for the sake of automaticity but for that someone else ... referred to as a sense of personal involvement, intimacy, or 'relationship' with mathematics – something otherwise overlooked in most educational discourse (Handa, 2012, p. 271).

He thus joins one of the most objective parts of mathematics, applying procedures, with the most subjective, the individual relation to the subject. However, he admits that without a will to learn, repetition easily deteriorate into rote. Once again, awareness of one's actions and reflection upon the process are put forward. The issue of how to promote such will and awareness in students is though left unresolved.

Due to the vague use of 'problem solving' in contemporary research, this section has paid the most attention to the small numbers of studies or procedures. It was found that procedures are not merely related to rote application of algorithms. Some researchers advocate student construction of procedures, while others consider the development of attachment in relation to repetition. In addition, analytic activities, as reflection and explanation of procedures, are generally highly regarded.

Reasoning

The MCRF definition of reasoning is based on the framework for creative and imitative reasoning (Lithner, 2008), which also lends to the imitative and constructive versions of the competency related activity *do and use*. Therefore, it is reasonable to start by considering this framework and the research applying it (Boesen, Lithner & Palm, 2010; T. Bergqvist & Lithner, 2012; Palm, Boesen & Lithner, 2011). Lithner (2008, p. 257) defines reasoning as "the line of thought adopted to produce assertions and reach conclusion in task solving". Formal logic and proofs are seen as one form of reasoning, which also includes incorrect arguments, as long as they are sensible to the reasoner. Before application to a task, the choice of strategy may be supported by predicative reasoning, while the implementation of the strategy can be argued for afterwards, by verificative argumentation. Lithner makes a distinction between creative mathematical reasoning and imitative reasoning, in order to categorise the requirements on different assessments and textbook tasks. Creative mathematical reasoning is defined by three conditions: novelty, plausibility and mathematical foundation. Plausibility lies within the notion of reasoning, since it consists of arguments and motivations. Novelty is reflected in 'creative' and seen with respect to the reasoner. As long as it is an invention of the reasoner, the arguments can be well-established mathematical facts. Finally, the reasoning must be anchored in mathematical objects and properties, in order to be mathematical.

Imitative reasoning is divided into four types: memorised reasoning and familiar, delimiting and guided algorithmic reasoning. Memorised reasoning requires no reasoning at all, consisting of pure recollection, whereas the three types of algorithmic reasoning (AR) include choice of which algorithm to apply. In familiar AR the choice is based on recognition of the task type, in delimiting AR there is a choice between algorithms based on surface properties of the task and in guided AR it is some outside source which influences the decision, e.g. a teacher or a textbook. As imitative reasoning is more superficial than creative reasoning, it is often found to result in incorrect choices and solutions. However, the need for learning imitative reasoning is acknowledged, as routine tasks are part of mathematical practice (T. Bergqvist & Lithner, 2012).

Reasoning, as defined within the framework for creative and imitative reasoning, is thus strongly connected to the solving of tasks. However, widening the perspective to other studies gives a more fragmented image of the concept. Reasoning is often used in a general sense, much like problem solving, and rarely defined (Yackel & Hanna, 2003). It is both conceived as the general process of learning and specifically linked to proof. This second relationship is elaborated further than by Lithner (2008), not merely thinking of proof as a formal type of reasoning, but expressing a mutual dependence between them. Though reasoning plays a key role in proof making, proofs may also support reasoning:

> chains of logical arguments do not function as satisfactory proofs unless they serve explanatory and communicative functions for an interpreting individual... mathematicians would like good proofs to make them wiser. In this view, a good proof is one that also helps one understand the meaning of what is being proved: to see not only that it is true but also why it is true (Yackel & Hanna, 2003, p. 228).

Nevertheless, these benefits are not always acknowledged in school mathematics, as noted by e.g. Yankelewitz, Mueller and Maher (2010, p. 77):

> many students view proof as a procedure or a series of steps to follow that lack meaning and they construct proof without using logical reasoning or attempting to provide justification for their ideas.

To capture the exercising of reasoning, it may therefore be more suitable to speak of proving, than of proof, which could refer to the static product of someone else's reasoning. As argued by Martinez, Brizuela and Superfine (2011, p. 46): "conjecturing and proving are interrelated mathematical processes and they should not be taught separately".

The practice of proving and reasoning is claimed to belong to certain classroom norms, or classroom participation structures (Cobb, Wood, Yackel & McNeal, 1992; Lampert, 1990; Maher & Martino, 1996; Nickerson & Whitacre, 2010; Yackel & Cobb, 1996; Yackel & Hanna, 2003). In trying to introduce school practices more alike the practices of mathematicians, Lampert (1990) promoted social interaction where students were involved in a zig-zag path of discovery. Initial assumptions were tested and revised by means of counterexamples and refutations, eventually leading to formal arguments of proof. As these practices developed, the words 'knowing', 'thinking', 'explaining' and 'answer' took on new meanings, and a culture of modesty and courage emerged. Tentative conjectures were put forward and justifications and revisions followed naturally.

> Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of determining what is true. Students learn about how the truth of a mathematical assertion gets established in mathematical discourse as they zig-zag between their own observations and generalizations – their own proofs and refutations – revealing and testing their own definitions and assumptions as they go along. At the same time, they are introduced to the tools and conventions used in the discipline, which have been refined over the centuries to enable the solution of theoretical and practical problems (Lampert, 1990, p. 42).

The social norms, which need to be developed in order to promote reasoning, include the expectation to explain and justify choices and conclusions, as well as making sense of the reasoning of others (Nickerson & Whitacre, 2010). It requires an environment of interaction and communication among students, but also the use of open-ended, complex and related tasks, allowing for collaborative inquiry and choices of strategies (Yankelewitz et al., 2010). In such settings:

> the teacher's role becomes central in promoting student exploration. Indeed, the role of the teacher shifts from conveyer of information to one of moderator and observer of children's thinking. As teachers monitor the thinking of their students, they are better able to pose timely questions that encourages students to build deeper mathematical understanding and access the progress of their reasoning (Maher & Martino, 1996, p. 197).

Teaching the practice of reasoning is "like teaching someone to dance, it requires some telling, some showing, and some doing it with them along with regular rehearsals" (Lampert, 1990, p. 58). Though it may take time to develop these norms and practices, there is no need to restrict reasoning to the later school years. As found by Maher and Martino (1996), it is possible to foster explaining and justifying from early ages, by giving opportunities to extend and reflect on one's ideas.

Taking this social perspective on reasoning, as a "communal activity in which learners participate as they interact with one another" (Yackel & Hanna, 2003, p. 228), the two extreme cases, imitative and constructive reasoning (Lithner, 2008), are unlikely to capture all reasoning activities. In collaboration, one builds on each others ideas, with various degrees of novel contribution. Mueller (2009) provides three actions used in co-construction of ideas: reiterating, redefining and expanding. Reiteration is a mere restatement, possibly with minor alterations, such as leaving out unnecessary words. When redefining an idea it is expressed in other terms, which requires more thought and may add new perspectives, e.g. redefining 'A is half of B' as 'B is twice as much as A'. Offering further explanation or detail is a way to expand an idea. These three types of building on ideas are accompanied by two other collaborative actions, questioning and correcting, and the individual construction of initial ideas (Mueller et al., 2012). These actions are combined in co-construction, integration and modification of reasoning. The analyses by Mueller (2009) and Mueller et al. (2012) show that ideas and arguments seldom emerge independently of the input of others. Examples of novel ideas are scarce, while most ideas are built collaboratively, by redefinition and expansion.

When considering research related to reasoning, it is clear that the concept covers a wide range of activities. While suitable for analysis of the solving of tasks, the types of creative and imitative reasoning offered by Lithner cannot fully reflect the reasoning in a social context. In such contexts, the presence of reasoning is closely connected to the norms within the practice. When justification, questioning and connections are natural elements of the discourse, co-construction of ideas follows. In terms of the MCRF, this means that the analytic aspect of reasoning catalyses the productive aspect. When this is the case, there is a spectrum from imitative to constructive activities: reiteration, redefinition and expansion.

Communication

Among the six competencies, communication is the most difficult one to handle. No other competency is subject to such different strands of research related to such many different theories and perspectives. In the same time, the framework definition gives no references to other studies or research traditions. Generally, the term 'communication' is not as commonly used as 'language', 'conversation', 'interaction' and 'discourse'. These terms are in turn associated to various theoretical perspectives and carry no unambiguous meanings (see e.g. Ryve, 2011). When attempting to find support for the delimitation of communication with respect to the other competencies it is soon realised that distinctions between the communication and the communicated message are seldom made. In their overview of research on communication and language, Lampert and Cobb (2003) refer to research where the focus lies on the reflection on representations (Stevens & Hall, 1998; Walkerdine, 1988), procedures (Cobb, Boufi, McClain & Whitenack, 1997; Walkerdine, 1988) or connections (McClain & Cobb, 1998), or on reasoning (Cobb, Yackel & Wood, 1989; O'Connor, 1996). When the difference between *talking about* and *talking about talking* (Cobb et al., 1989) is discussed, it is not clear whether it concerns communicating about communication or reasoning about reasoning. While actions and symbols do not make sense apart from the talk describing them (Lampert & Cobb, 2003; Walkerdine, 1988), it also seems difficult to make sense of communication without referring to the content.

Some studies stress the need for learning conventional mathematical language, for example by advocating genre instruction (Lampert & Cobb, 2003) or the development of the mathematical register (Temple & Doerr, 2012). A possible interpretation of imitative use of communication, in line with such research, would be as conventional use, while informal communication would be the constructive use. This, however, would imply that constructive communication should be taught to be avoided. In contrast, Wagner (2007) maintains that transformational freedom, being able to easily shift between levels of formality, is a mark of a successful mathematics student. Moreover, focusing on the correct use of symbols and words deprives communication of its interactive nature. Communication requires both a communicator and a listener, since apart from own reflection, communication is about reaching shared meaning.

Truxaw and DeFranco (2008) suggest that communication constitutes a spectrum with two extremes: uni-vocal and dialogic. Uni-vocal communication aims to convey meaning by producing an accurate transmission of a message. In a dialogic communication meaning is constructed by dialogue, where there may be conflict among voices. Uni-vocal communication can be monologic, without expectation of response, but also interactive forms can be uni-vocal if one participant controls the communication, leading the others to a certain point of view. Exploratory and accountable talk are examples of dialogic communication, allowing for corrections and requiring justifications. Longer episodes of interaction often go through multiple shifts between uni-vocal and dialogic phases. Inductive processes of exploratory talk are alternated with leading or monological talk where the frame of reference is established. Nonetheless, the overall weight may lie more towards uni-vocal or dialogic communication.

> The most commonly identified pattern of classroom discourse follows the three-part exchange of teacher initiation, student response, and teacher evaluation or follow-up ... within triadic exchanges, the teacher's verbal moves influence the function of the discourse. For example, when follow-up moves are used to evaluate a student's response, the intention of the discourse is likely to trend toward transmitting meaning (i.e. , uni-vocal). In contrast, questions that invite student to contribute ideas that might change or modify a discussion are more likely to be associated with dialogic discourse (Truxaw & DeFranco, 2008, p. 491).

Truxaw and DeFranco's categories could provide a different perspective on the imitative and constructive versions of doing and using communication, which could separate the competency from representations and reasoning. If dialogic communication is seen as constructive, as it functions as co-construction of meaning, and uni-vocal communication is seen as imitative, since it conveys an already consisting meaning, a difference between the use of communication and the other competencies emerges. Using a constructed representation or confirming reasoning by restatement of an argument are examples of imitative use of representations and reasoning, but functions as co-construction of meaning and are hence examples of constructive communication. Shein (2012) shows that re-voicing, i.e. verbatim or modified repetition or other's utterance, can have co-constructive effects.

> Studies show that re-voicing can be used to facilitate students' participation in three ways. First, it opens up opportunities expanded and iterative evaluations between the teacher and student on an original student response in order to achieve mutual understanding and minimize misinterpretation...
>
> Second, it can be used by mathematics teachers to set different student ideas against one another to formulate a mathematical argument that can be fruitful for achieving deeper conceptual understanding (Shein, 2012, p. 191-192)

Re-voicing can take non-verbal forms, such as return gestures.

> Through re-voicing both words and actions, the teacher was able to construct meanings of and assign ownership to a student-invented and student-adopted strategy ... ratification and adoption of students' gestures not only provided the kinds of opportunities for students to communicate about them and understand the content but also provided a way for students to identify themselves as someone who contributes to the shared repertoire of practices and who communicates effectively in mathematics (Shein, 2012, p. 216).

Communication has proven to be a difficult notion and since the framework offers little guidance, it is hard to navigate within the diverse research on the subject. An attempt was made to make sense of the imitative–constructive spectrum in terms of the function of communication: to convey or to co-construct meaning. This contributed to the demarcation of communication from representations and reasoning. Nevertheless, the role of 'communication' within the framework remains less definite than the roles of the other competencies.

The framework has been presented in its original form and the constructs have been related to other research studies. While the MCRF definitions are expressed in terms of what it means to master the competencies, the research review has provided examples of tentative and exploratory use. Regarding representation and reasoning, research was found to inform the constructs by giving further examples of the inner dimensions of the competencies. To some extent, this was also the case for producers and connections. The situation with the two remaining competencies, problem solving and communication, is more complicated. Problem solving is mostly used in a general, undefined sense, as part of the context for the studies. The term 'communication' is treated similarly. When communication is focused upon it is more common to speak of e.g. discourse or conversation, which are notions built into larger theoretical constructions. Knowing this to be the lie of the land, it is now time to state the aim of the research on mathematical competencies in this thesis.

3.2 Aims and research questions

The aim of the mathematics education part of this thesis is to modify and expand the Mathematical Competency Research Framework in order to include methods for the analysis of mathematical practice and the exercising of mathematical competencies. Additionally, the aim is to provide examples of how competencies are exercised in various mathematical activities. To each of the two aims there are related research questions.

First aim

1. How well do the constructs of the framework suit the analysis of data on mathematical practice?
2. How can the framework be related to current research of mathematical activities and practice?
3. What tools need to be developed in order to operationalise the framework for the analysis of mathematical practice?

Second aim

4. How are competencies exercised by young children in the domain of whole number arithmetic?
5. How are competencies exercised in university students' proving?

The first aim is mainly dealt with in Paper III, while the second aim is the focus of Paper IV. However, while Question 1-3 are primarily addressed in Paper III, Question 1 and 3 are also dealt with in Section 3.3.1 and the research review in Section 3.1.3 responds to Question 2. Some answers to Question 4 emerge as secondary results of the study presented in Paper III. Question 5 is the major concern of Paper IV.

3.3 Method

In order to answer the research questions related to the two different aims, two methods were applied, where the outcome of the first method is part of the second one. In the first of the three following sections, the development of the framework is described. To be able to apply the framework to empirical data of mathematical practices, describing how competencies are exercised, adjustments were made in three phases. As a start, the definitions of the constructs and their interrelations were analysed. Next, an overview of related contemporary research was made, in order to inform the definitions and to further clarify them. Finally, an analytic toolbox was built as the framework was applied to a first set of empirical data.

In the second section, the collection and presentation of data will be described. The first set of data was collected by the author and for this an activity was designed, consisting of tasks and questions. Four interviews with pairs of five-years-old children were conducted. Due to the extensive use of non-verbal manipulation of materials during the activity, invention of data presenting techniques was needed. The second set of data was taken from a previous case study. The case was chosen since it was not specifically designed to elicit competencies, but still provides rich data. In addition, it differs from the first data set, by consisting of a group work session with university students. The two forms of data in this case, video recordings and copies of students' notes, entailed some special techniques for transcription and presentation.

The third section opens with a description of the adapted framework, which was the result of the adjustments discussed in Section 3.3.1. Then, the analysis process as well as the analytical tools, the analysis guide and the competency graph, are explained.

3.3.1 Developing the framework

As discussed in previous sections, the MCRF construct definitions are rather brief and do not prescribe an explicated method of analysis. Furthermore, the analysis of competencies exercised in practice require wider definitions of competencies than merely in terms of mastery. Altogether, adaptation of the framework was needed, but as the framework is rather new and not elsewhere expatiated upon, initial interpretations and analysis of the definitions of and relations between the constructs were necessary.

Analysis of the framework: the conflation and demarcation of the constructs

The analysis was conducted in order to justify the constructs and their distinctions, as well as to reach an understanding of the relations between them. An initial graphical representation of the framework was constructed, to visualise the current circumstances. In the first phase, overlaps and correlations were identified and remedied, gradually combining constructs to form a smaller set. To begin with, words in the description of the competency related activities which related to competencies, such as solve and respond, were removed. Next, the competency related activity *interpret* was questioned, as well as the distinction between problem solving and procedures, between the analytic aspect of representations and the productive aspect of connections, between communication and the combination of representations and reasoning, as well as between reasoning and the analytic aspect in general. This process was conducted to pull the constructs together as far as possible, finding potential problems with the definitions.

The result served as an outset for the reverse process, being the second phase. As the distinction problems were acknowledged, the work proceeded by clarifying and arguing for the distinctions as far as possible. Communication was argued for by means of the difference between soundness and understandability; it is possible to reason soundly, but still not communicate one's reasoning, while the most immaculate argument may not always be the most communicative. Connections were separated from the analytic aspect of representations, since connections may link abstract entities as well as representations and since representations may be evaluated with regard to other qualities than their connectivity. In addition, in a merger of connections and the analytic aspect of representations, the analytic aspect of connections would be lost. The difference between reasoning and the analytic aspect was explained in a similar fashion, partly by the fact that the analytic aspect includes monitoring and evaluation, which not necessarily come in the form of reasoning and partly due to the observation that the analytic aspect of reasoning otherwise would be lost.

Nevertheless, there were two issues which could not be resolved. The first one concern the competency related activity *interpret* and the second one concerns the two competencies *applying procedures* and *problem solving*. Interpreting is defined as taking in information, but also related to building knowledge, understanding, identification and recognition. These concepts seem to reside at two opposite ends of a process, starting with initial interpretations, which may be revised, eventually leading to increased knowledge and understanding. The definition, taking in information, directs thought to the first end, while the descriptions of the competencies in the terms of mastery is more associated to the second. However, apart from this problematic ambiguity, there is still a need to consider this activity's existence outside of the other two, *do and use* and *judge*. These actions rely on interpretations and cannot be performed without them. The other way around, interpretations are manifested in these activities. Unconscious interpretations can only be spotted in doing or using, while conscious interpretations can be seen as evaluative. Consequently, interpreting cannot be separated from the other two activities. This results in the removing of CRA I from the set of constructs and it is henceforth more convenient to refer to *do and use* as the *productive aspect* and to *judge* as the *analytic aspect*.

The second adjustment concerns the two competencies applying procedures and problem solving, as well as the two main versions of the productive aspect: imitate and construct. These two versions are not highlighted in the present formulation of the framework, but are nevertheless mentioned in the definition of *do and use*. Since applying procedures is defined as using an accepted way of solving a task, while problem solving is defined as engaging in a task where the solution method is not known in advance, it raises the question of how to define the constructive version of applying procedures and the imitative version of problem solving. The imitative–constructive spectrum seems to capture both competencies, ranging from mechanical application, through more or less need for adaptation, to novel construction. Reflection and monitoring are parts of the analytic aspect, and justification of choices and conclusion belong to the reasoning competency. In addition, the two categories of tasks, problems and non-problems, mentioned in the definition of problem solving, are less interesting when studying exercising of competencies rather than opportunities provided by the task. It is the handling of the task which is studied and this may not necessarily coincide with the researcher's classification of the task.

Besides these two adjustments, the analysis of the framework made clear that two of the competencies rely on another construct, which is not in itself a competency or a competency related activity. In the definition of representations and connections, *abstract mathematical entities* play a crucial role. These entities are not given a precise definition in the framework, but are exemplified by numbers, functions, geometrical objects, tasks, methods, principles, concepts, phenomena, ideas and properties. While some of the these notions are easily conceived as things, others are more vague. Abstract mathematical entities are also implicitly defined by the two related competencies. They can be given mental or real concrete replacements, i.e. be represented, and they are related to each other through connections between them, as well as between representation and parts of them. To speak of representations of, parts of and connections between 'numbers', 'functions' and 'geometrical objects' seems uncontroversial and these entities are not present anywhere else in the framework. However, 'methods' rather belong to the procedural competency, while 'tasks', 'principles', 'phenomena' and 'ideas' are larger structures, possibly consisting of multiple entities, representations and connections. 'Concept' is a vague term. It could refer to an entity, but also to a representation or a connection. For the above reasons, a more restrictive description of abstract mathematical entities was decided upon. A choice was made to attend to the denotation of the word *entity*, as a thing with a distinct and independent existence leading to the following definition:

> In a mathematical practice, an **abstract mathematical entity** is a self-sufficient concept which is treated as an object by the participators.

The definition implies that a concept may be an entity in some instances of mathematical practice and not in others. It further demarcates representation and connections, as a representations rely on some abstract entity, which may carry qualities overlooked by the representation and a connection lacks meaning without the things it connects. In addition, the definition excludes 'properties', which always belong to something, possibly an entity.

Altogether, the Mathematical Competency Research Framework has been modified in three ways: the competency related activity *interpret* was removed, the two competencies *applying procedures* and *problem solving* were combined and the definition of *abstract mathematical entity* was restricted and clarified. With those reflections and adjustments made, it is possible to take in influences from other studies.

Informing the constructs by contemporary research

Since the original formulation of the framework is rarely linked to studies of particular competencies, an overview of contemporary research was made. The issues of four international journals, ranging from 2007–2012, were surveyed: Educational Studies in Mathematics, The Journal of Mathematical Behaviour, Journal for Research in Mathematics Education and Mathematical Thinking and Learning. As a start, articles related to general competence or specific competencies were singled out. The selection was based on the title, the abstract and the keywords of the articles and the words looked for are displayed in Table 3.3.

Competence	competency, proficiency, ability
Problem solving	problem posing
Reasoning	proof, argumentation
Applying procedures	algorithms, strategies
Representation	drawings, concrete materials
Connection	relations, relationships
Communication	talk, conversation, discourse, gestures

TABLE 3.3: The words associated to competencies used in the overview of contemporary research.

The result was a set of about 400 articles, where about half concerned either problem solving or reasoning. These concepts were often handled as descriptions of mathematical activity in general. From the 400 articles, 25 were found to have specific relevance for the framework. These were then used to find further references in a snowball sampling manner resulting in articles, books, book chapters and doctoral theses from 1986 to today. The result of the overview is presented in Paper III and Section 3.1.3. In general, studies which provided examples of how the aspects of each competency can be exercised were included. These examples helped explicating the competencies and building definitions of the competencies in terms of exercising, rather than mastery. These descriptions were in turn part of the development of the analytical tools for analysing competencies in mathematical practice.

Building the analytical toolbox

The analysis of the framework constructs and the overview of contemporary research both contributed to the tools for analysis of empirical data. The first tool, the analysis guide, was created to meet the need to identify and separate actions as expressions of different aspects of the competencies. After testing various models, it took the form of a series of questions with auxiliary explanations and sub-questions. Though the use of the guide is an iterative process, there is a purpose behind the ordering of the questions. The competencies were found to be more easily delineated in this order, as they relate to each other pairwise differently. The analysis guide is included in full, in Section 3.3.3.

The second tool, the competency graph, serves the purpose of relating the competencies to each other and depicting the interplay between them. It both aids the understanding of the actions analysed and of the general competencies, as it reflects their conceptual nature. The graph builds on the relations between entities, representations and connections (Figure 3.1 on p. 34). Depictions of procedures, reasoning and communication were chosen. The representation mappings were assigned double pointed arrows in order to reflect their two way nature, which the research review revealed. The visualisations of the competencies were gradually refined and chosen to separate between competencies and carrying natural associations to their essence, but also for the combined graph to be comprehensible and suitable for monochromatic printing. As a consequence, the grey scope of communication used in earlier versions (Paper III) was replaced by a dotted line. However, the construction of each competency graph is a creative process, and the construction of the specific graphs in the course of analysis is described in Section 3.3.3. Before going into analysis, the empirical data will be discussed.

3.3.2 Data collection and presentation

The framework was applied to two sets of data. In both studies, video recordings were used, but in other respects the two data sets were very different. The first study, presented in Paper III, consisted of four semi-structured interviews with pairs of preschool children. The interviews were structured around an activity specifically designed for this study. The frequent use of non-verbal actions required multiple data presentation styles. In the second study, presented in Paper IV, the data come from a previous case study, were four university students work on a proof by induction. The data consisted of the video recording, a full transcript of the session and copies of the students' notes. These data also required special approaches in order to conduct the competency analysis.

The first set of data: preschoolers handling whole number arithmetic

The study was conducted in integration with the development of the analytical tools and considered preschoolers' whole number arithmetic. The mathematical content and the age group were chosen for several reasons. Firstly, young children's arithmetic is a rather well-researched field, which was beneficial for the construction of the activity. Secondly, though mathematical activities are becoming more and more prominent in Swedish preschools, children of this age are not yet subject to the practices of school mathematics. As a consequence, more varied examples of the exercising of competencies, than among school children, could be expected. Thirdly, whole number arithmetic is a rich, but still not too complicated, area of mathematics. Since it was the first application of the framework, it seemed wise to choose a content as straightforward as possible.

The activity items, consisting of tasks, questions and follow-ups, were constructed to give as rich opportunities for the exercising of competencies as possible. In the first interview, conducted in the summer 2011, the items were divided into four topics: attending to studs and quantity, representing numbers by bricks, representing numbers by binary bricks and a mind reading game. The first two topics used a mixed set of uni-coloured bricks, while the last two used a special set of four types of bricks, seen in Figure 3.2. After this interview, which lasted 1h 15 min, the items were revised in order to shorten the activity and to balance among the competencies. The new activity comprised three topics: written numbers, written numbers associated with bricks and written numbers associated with binary bricks. The first topic used cardboard cards with numbers 1 – 15. Along with the second topic, the mixed, unicoloured bricks were introduced, which were then put aside, as the special set was presented with the third topic. The interviews using the revised activity lasted between 25 and 40 min.

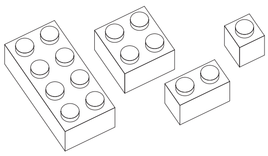


FIGURE 3.2: The four types of lego bricks used to represent numbers in binary form. Each type had a colour of its own.

Before the competency analysis, the data were viewed several times. Sequences of non-mathematical activities were set aside from the data. In two separate rounds, the verbal utterances and the handling of concrete materials were transcribed. Due to the frequent handling of bricks and number cards, pure text transcripts became difficult to apprehend. Therefore, a mixture of three presenting styles were used. The first is in the form of regular transcripts with pictures of the bricks referred to at the right hand side. This was suitable for sequences with much talk, but less manipulation of objects. When the situation was the opposite, with no verbal acts and specific constructions or procedures were to be displayed, single pictures were used. When both talk and manipulations were frequent and the positions and movements of objects were important, a combination of schematic pictures and transcripts were used, similar to comic strips. The three styles of data presentation are exemplified below.

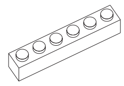
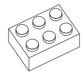
Nina:	Hm. Six again. One two three. One two three. Six.	
	‘Cause if you break it apart, it becomes like this ((claps hands together and points at))	

FIGURE 3.3: The first presentation technique: excerpts accompanied by the bricks referred to.

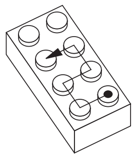


FIGURE 3.4: The second presentations technique: single pictures without transcripts.

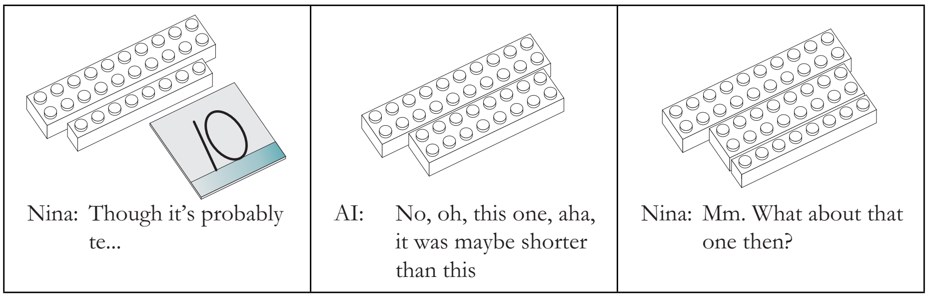


FIGURE 3.5: The third presentations technique: schematic pictures of the arrangement of bricks and cards, accompanied by transcripts of utterances.

The second set of data: University students' proving activity

This case study has been previously analysed by Pettersson (2004, 2008b) and Ryve, Nilsson and Pettersson (2013). Four university students attempt to solve a calculus task by constructing a proof by induction. The session is 115 min, and the data consists of a video recording and a transcript of the session, as well as copies of the students' own notes. Since the data has been thoroughly described elsewhere (Pettersson, 2004), it will not be done here. However, it is worth remarking on one difficulty which emerged while handling the data. Since the students often referred to drawings and writings in their notes, it was necessary to link the notes to their utterances and gestures. Unfortunately, the notes are not visible on the video recording, though the papers are. To allow for determination of what the students referred to during the session, the papers were attended to while viewing the video. Time codes for the students position on their pens were noted on a transparent sheet placed over a copy of their notes. By this method it was possible to follow the writing and pointing of the students.

3.3.3 Analysis of data

In this section, the adapted framework and its operationalisation for analysis of empirical data will be described. Although the constructs naturally have a large impact on the analytical tools, the tools also reflect back on the constructs. The formulation of the framework is affected by the research presented in the research review (Section 3.1.3). In particular, the representation competency was influenced by Goldin (2003) and Smith (2003), the procedure competency by Gravemeijer and Van Galen (2003), Handa (2012) and Voutsina (2012), the connection competency by Ellis (2007a, 2007b) and Hiebert and Carpenter (1992), the reasoning competency by Mueller (2009) and Mueller et al. (2012) and, finally, the connection competency by Truxaw and DeFranco (2008).

This framework aims to describe the exercising of mathematical competence. Competence in some domain of personal, professional or social life is the ability to handle essential aspects of life in that domain. In mathematical domains this is called mathematical competence. The essential aspects of mathematical competence are found to run along two dimensions and therefore the aspects along one dimension are given a new name: competencies. The framework comprises five competencies: representations, procedures, connections, reasoning and communication. The five competencies contribute to mathematical competence in different ways and they relate differently to one another. However, the competencies all relate to an additional construct, which can be thought of as the foundational building blocks of mathematics: *abstract mathematical entities* (AMEs). An AME is defined as a self-sufficient concept which is treated as an object within a mathematical practice. The definition implies that the issue of whether a concept is an AME depends on the context. A concept which may be considered as a separate whole in one activity, may be seen as a part of something bigger in another.

The competencies are successively defined by relating them to the AMEs and to the previous competencies. Each competency is also specified in terms of its productive and analytic aspect. The *productive aspect* reflects use, application, adaptation and development, forming a spectrum from imitative to constructive activities. The *analytic aspect* is expressed in meta-level considerations, such as monitoring, evaluation and reflection.

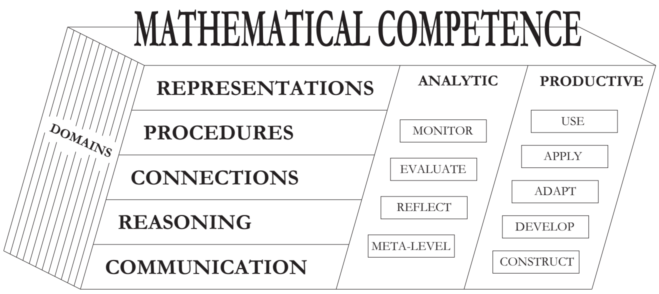


FIGURE 3.6: A graphical summary of the adaptation of the MCRF. Mathematical competence varies along several dimensions, with respect to domains, competencies and aspects.

The representation competency

AMEs can be represented in various ways. A representation is both a concrete replacement of the entity and a mapping between the entity and its replacement. The mapping is two-way, both carrying properties from the AME to the representation and allowing for access to the AME through the representation. Representations can be internal or external, verbal, gestural or in forms of artefacts and more or less formal. There may be multiple levels of representations, where the representation of one entity in turn is the entity represented by another mapping. The construction, use and evaluation of representations constitute the representation competence. The use of representations can be denotational, depicting, referential or symbolising, among other things.

The procedure competency

A procedure is a sequence of actions aiming at solving a mathematical problem or reaching a result on a mathematical task. A procedure is always based on one or more AMEs and often relies on specific representations. While the procedure has meaning only with respect to the essence of the related AMEs, the representations allow for actions utilizing certain properties of the AMEs. Applying a known algorithm, modifying procedures for adaptation to new settings and constructing one's own procedures are all parts of the productive aspect of procedures. The analytic aspect is exercised in the monitoring and the evaluation of solving processes, but also in the detection of subtle variations in repetitive application of an algorithm.

The connection competency

Connections form the texture within each class of concepts: AMEs, representations and procedures. They can be intra-connections, relating parts within an AME, a representation or a procedure. They can be inter-connections, linking two AMEs, two representations or two procedures. However, the relations between different kinds of concepts, such as the representation mappings and the links between procedures and AMEs, are not seen as connections, but as part of the representations and procedures, respectively. A connection can be constructed by acknowledging similarities or differences between the concepts or forming a hierarchy between them, e.g. in terms of generality. The use of connections can play various roles in mathematical practice, ultimately building a network of AMEs, representations procedures and their properties. In this process, exercising the analytic aspect means to monitor and evaluate the use and construction of connections, as well as to reflect on the purpose and implications of these activities.

The reasoning competency

Reasoning is the explicit justification of mathematical choices and conclusions. It may justify the choice of a representation of an AME or the use of a procedure. It may also justify the conclusion that two AMEs are connected or the result of a procedure. It can be used in the exercising in the analytic aspect of other competencies, as justification can be part of evaluation. Reasoning can be imitative, repeating or reiterating previous arguments, but often builds on the ideas of others. This can be done by answering questions concerning one's choices and conclusions and by redefining or expanding others' arguments. Occasionally, reasoning is also purely constructive, as novel ideas are introduced. As the productive aspect of reasoning plays a role in the analytic aspect in general, it can be difficult to separate the two. While monitoring reasoning can be done by other means, evaluation of reasoning may exercise both aspects.

The communication competency

All the previous competencies can be the topic for communication. The intention to communicate is required for an action to be communicative and therefore the communicator has a receiver for the message in mind. While some concepts and reasoning are communicated in an activity, others may be implicitly used. Communication can take the form of speech, gestures or symbols and may have the intention to convey or to construct meaning. To convey meaning is seen as imitative use of communication. The monitoring and evaluation of communication should not be confused with the monitoring and evaluation of the communicated. The analytic aspect of communication concerns the qualities of communication with the other competencies of benchmarks.

These are the five competencies in the adapted framework, which is summarised in Figure 3.6. The two phases of the analysis of competencies exercised will now be described.

The analysis guide

In order to aid the identification and description of the competencies, an analysis guide was developed. The guide consists of a number of questions with auxiliary explanations. To enhance the description of the inner dimensions of the competencies, each question is given a number of sub-questions. Rather than requiring explicit answers, the questions are meant to guide the thinking about the data in relation to the competencies, structuring the description of the exercising of competencies. If the answer to a question is ‘yes’, it calls for a description and an explanation of the actions involved. The outcome of the process of following the guide should be a narrative of the data in competency terms.

Guide to
Analysis of Competencies Exercised

A. CONCEPTS – Which concepts are handled in the sequence?

Pick a sequence of data which forms an entirety, but is short enough to apprehend. List all mathematical concepts present in the dialogue and the activities in the sequence. They can be nouns, verbs and adjectives. It will later be determined if these concepts are abstract mathematical entities, representations, procedures, connections or parts of these.

1. Abstract mathematical entities (AMEs)

Which concepts are self-sufficient and treated as objects by the participants? Mark these as AMEs. Concepts which are not AMEs could be representations, procedures, connections, or parts of these.

2. Representations

A representation is a mental or real replacement of an AME. A representation consists of both an object and a mapping between the AME and that object. Search the list of concepts for representations and be aware of implicit representations used. Consider the AMEs and how they are handled, looking for representations. Consider the representations, determining what AMEs they represent.

- ▹ Are all AMEs represented and do all representations have AMEs?
- ▹ Are there multiple representations of AMEs?
- ▹ Are there subsequent representations?

The Productive Aspect

- ▹ How are the representations used?
- ▹ Who introduces the representations?
- ▹ If the representations are given, do they require adaptation?

The Analytic Aspect

- ▹ Is the use or construction of representations monitored?
- ▹ Are representations evaluated or discussed?

Concepts cont.

3. Procedures

This competency is exercised in relation to tasks: small routine tasks, as well as complicated problems.

- What procedures are used, constructed or otherwise considered?
- How do they relate to AMEs and representations?

The Productive Aspect

- Do the participants pose any problems or questions?
- Do they address problems raised by someone else?
- Do they use any procedure which, in line with previous research or sequences, can be argued to be well-known?
- Do they adapt a well-known or previously used method to a new setting?
- Do they find new methods or procedures for solving a task?

The Analytic Aspect

- Do the participants question or argue for any solution or method?
- Do they evaluate or reflect upon any solution or method?
- Do they step outside a procedure to monitor the process or deciding on upcoming steps?

4. Connections

Connections are the relationships within one class of concepts. In other words, they connect two AMEs, two representations, two procedures or two parts of an AME, a representation or a procedure. Relate the connections found to the lists of concepts, AMEs and representations.

- What connections are used or discussed?

The Productive Aspect

- Who introduces the connections?
- Are connections constructed?
- How are connections used?
- Are some (previously defined) connections used implicitly?

Connections cont.

The Analytic Aspect

- ▹ Is the use or construction of connections monitored?
- ▹ Are connections evaluated, e.g. by questions or explanations?
- ▹ Is there a discussion about connections in general?

Return to the list of concepts.

- ▹ Are they all classified as AMEs, representations, procedures or connections?
- ▹ Is there any doubt regarding, whether something should be an AME in itself or a part of another?
- ▹ Employ the participants use of the concepts in order to justify your choices.

B. REASONING – Is there arguments or justification present?

Reasoning is the explicit act of justifying mathematical choices and conclusions by arguments. Though reasoning needs to be explicit – avoid guessing at implicit reasoning in the analysis – it does not have to be verbal, but could take e.g. gestural or written form. The choices and conclusions which are justified should be related to the competencies above. Reasoning could be for a choice of a representation or a procedure. It could also be for the conclusion of a connection or a result of a procedure.

- ▹ What is justified? Relate to previous results.

The Productive Aspect

- ▹ Are new arguments introduced?
- ▹ Do participants build on each other's ideas by means of restatements, redefinitions or expansions?
- ▹ Are justification of choices or conclusions asked for and are they given?

The Analytic Aspect

- ▹ Do participants question or evaluate statements or arguments?
- ▹ Do they monitor the line of arguments?
- ▹ Do they consider reasoning in general?

C. COMMUNICATION – Is there intentional communication present?
Communication comes in many forms: as speech, body language and production of artefacts. The exercising of communication requires an intention to communicate, therefore actions need to address someone in order to be communicative.

- ▹ What is communicated and what is taken for granted?

The Productive Aspect

- ▹ Is there communication conveying a specific message or meaning?
- ▹ Is there communication constructing a shared meaning?
- ▹ What forms of communication are used?

The Analytic Aspect

- ▹ Is the forms of communication discussed?
- ▹ Is the relation between means and message discussed?

The competency graph

To further clarify the use of and interplay between competencies, one part of the analysis is to construct a competency graph. Though tentative sketches accompany the use of the analysis guide, the final version of the graph is possible to construct only after that phase of the analysis. Structuring the concepts in a graph facilitates both the demarcation of AMEs, representations and procedures and the relation between them in terms of connections, representation mappings and procedure links. The iterative construction of graphs provides a space for exploring different interpretations of concepts as AMEs, representations, procedures, connections and parts of these, which in turn gives the necessary grounds for pinpointing reasoning and communication. The process continues until interpretative choices with the power to explain the data are reached. The graphical elements are then arranged in order to make the graph as straightforward and comprehensible as possible. The final graph is successively built up, starting with the most abstract AMEs on the top. One or more levels of representations may follow below. The representation mappings are drawn as white double arrows. Procedures are illustrated as clouds, indicating their relationship to AMEs and representations by thin black lines. Connections are depicted as black connectors. Reasoning is marked around the justified by inverse cloud lines. Finally, the scope of communication is visualised as a dotted line, enclosing the communicated AMEs, representations, procedures, connections and reasoning, excluding the ones implicitly used. In Figure 3.7 a generic graph is shown, while an example of an authentic competency graph is given in Figure 3.8.

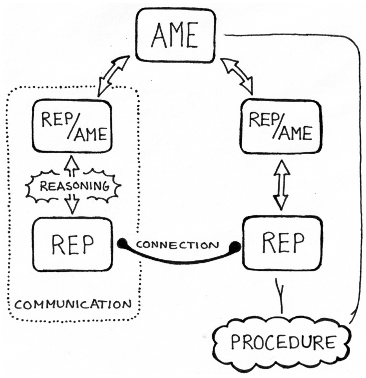


FIGURE 3.7: A generic competency graph showing how the competencies and their relations are depicted.

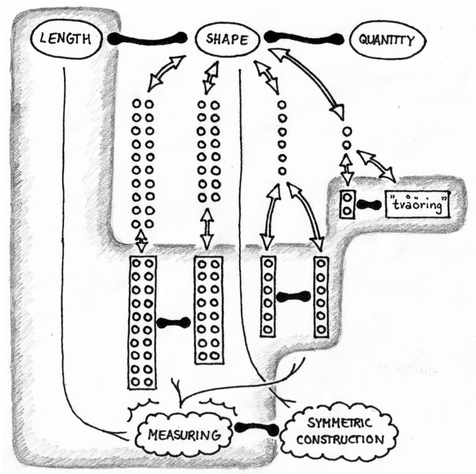


FIGURE 3.8: An authentic example of a competency graph, taken from Paper III, showing the competencies exercised by Olle. The communicative scope is here depicted in grey.

Once the final graph is constructed, the explanation and discussion of it may further the narrative. It provides tighter correspondence to the constructs and lifts the analytical qualities of the results.

3.4 Summary of Paper III and IV

In this section, the two studies included in the education research part, Paper III and IV, are presented. The summaries focus on their respective results. The primary aim of Paper III is to modify and expand the Mathematical Competency Research Framework to include methods for analysis of mathematical practice and the exercising of mathematical competence. In addition, it gives some examples of young children's exercising of competencies in the domain of whole number arithmetic. The primary aim of Paper IV is to describe university students' exercising of competencies in proving. As a consequence, the application of the framework to this case study reflected back on the constructs and analytical tools, developing them further.

Paper III: Developing a framework for competencies exercised

This paper has a methodological focus. It describes the process of developing a framework and associated tools for analysis of the exercising of mathematical competencies. The results are of three types: revisions of the framework, analytical tools and examples of obtainable results from analysis of empirical data.

The revisions of the framework include the exclusion of some constructs and the redefinition of others. First, the competency related activity *interpret* was removed, since it was not found to be empirically and theoretically distinguishable from the other two activities. As a second result, the two remaining activities were downplayed in favour of the two aspects. Second, a conflict was detected between the imitative–constructive spectrum of the productive aspect and the two competencies procedures and problem solving. This conflict was resolved by incorporating problem solving in procedures. Third, abstract mathematical entities were treated in a restricted sense, thinking of them as objects. This perspective also affected the representation and connection competencies, as they build upon the notion of entities. However, the empirical results uncovered a need for allowing connections between procedures. Forth, the competency definitions were discussed in general. A particular emphasis was put on the research studies which informed the revisions of the framework. These studies provided more nuances to the constructs.

The operationalisation of the framework gave rise to two analytical tools: the analysis guide and the competency graph. The analysis guide is described in the paper, but not given in full (it is included in Section 3.3.3). The competency graph is both described and exemplified by both a generic graph and authentic graphs within the empirical results. The empirical results give three examples of possible aims and highlights of a competency analysis. The first example compares two children interplay of competencies around double and doubling. Excerpts were chosen in order to describe how the children treat and use this concept at different occasions during the interviews. As the two narratives and the two graphs are contrasted, they add to the variety of ways to exercise competencies in relation to doubling. The second example shows the exercising of competencies during a session of free play. In his building activity, a child makes use of representations and procedures related to both shape and quantity. He also evaluates and justify his choices. The third and last example focuses of a short sequence of interaction between two children. In this sequence they question each other's representations and justify their own. Their choices are then linked to their procedures later in the interview.

Though all analyses result in narratives and graphs describing the interplay between competencies, they vary in angel. The first one sets out from a concept, while the second one focuses on an activity. The third one describes an interesting interaction, which could be used as an explanation for the differences in the procedures used.

Paper IV: Competencies exercised in the process of proving

This paper represents a case study, where a group of four university students work on a complex calculus task, involving a proof by induction. The task was stated as follows:

> Let f be a function defined on all of $\mathbb{R}$.
>
> 1. How many zeroes at most can the function have if $f'(x) \neq 0$ for all x?
> 2. If instead $f''(x) \neq 0$, what can you say about the number of zeroes of the function?
> 3. If we have $f^{(n)}(x) \neq 0$, what can be said about the number of zeroes of the function? Use induction to prove your statement.

In order to conduct a competency analysis of the students' work, a pre-analysis was needed. This pre-analysis revealed three different modes in the students' proving, which were also found in the different generality levels of the task. The most detailed level concerned functions, derivatives and their properties, while the most general level concerned the over-all idea of structure and induction. In the middle there is a level of adaptation of the specific and the general to each other, finding a way to prove the statement about functions and derivatives by induction. In the analysis, a number of features in the students' exercising of competencies stood out. First, the students formed new entities, representations and connections. For example, a statement in form of a connection between two entities was treated as a distinct object. This object could then be connected to other statements also seen as entities.

Second, representations were used in several ways. Denotational use of verbal representations was common, but there were also examples of graphical representations aiding reasoning and construction of connections. Some verbal representations were metonymic, as they spoke of the general in terms of the specific or the other way around. Third, the students made extensive use of connections. Both similarities and differences were found to be important in the proving process, as they play a key role in reasoning. Fourth, one reason for the students' dissatisfaction with their solution were found in the imbalance between the productive and the analytic aspect. When they alternate between production and meta-level discussion they do reach conclusions. However, they often 'got stuck on the analytic level', discussing what should be done rather than actually doing it.

This study called for some further clarifications of the framework. To begin with, the definition of abstract mathematical entities needed to become more solid. In addition, the relations between the competencies were strengthened. The frequent use of 'entitification' necessitated a consistent way to depict parts of entities or representations in the competency graph. In relation to previous research, this study supports the importance of reasoning in proving activities. However, the roles of other competencies are also highlighted.

3.5 Discussion and conclusion

The aim of this part of the thesis, this chapter together with Paper III and IV, was to develop and apply a framework for the analysis of the exercising of competencies in mathematical practices. This framework originates from the Mathematical Competency Research Framework (Lithner et al., 2010), which was modified in three phases: analysis of the constructs, letting research inform the framework and developing and testing analytical tools by analysis of data. These three phases were not completely separated, as the findings in one phase often led to new considerations in other phases. Nevertheless, they served different purposes. The analysis of the constructs was necessary for the coherence of the framework. The definitions of the construct were scrutinised and related to each other, so that the elements of the framework were balanced and created a whole. Since the original definitions were brief and rarely linked to research, the research review added more flesh to the constructs, providing nuances within the definitions. The development of the analytical tools were useful in concretising the constructs. By application to empirical data, the framework was operationalised and its explanatory power tested. While the initial analysis of the framework gave grounds for the latter phases, the research review presented new relations between constructs, as well as gave concrete examples which aided the early formulations of the analysis guide. The construction of the competency graph highlighted and problematised the different natures of the constructs and their interrelations.

The generic competency graph (Figure 3.7, p. 70) is the second of two visualisations of the framework, whereas the mathematical competence parallelepiped (Figure 3.6, p. 63) is the first one. The parallelepiped was constructed in the first phase, as a summary of the key constructs. It relates the domains and the aspects of competence to competence in general. However, it does not include AMEs and it does not capture the relationships between competencies. The competency graph, on the other hand, does precisely this. The representations of the competencies are chosen to reflect their essence and to facilitate the comprehension of their interdependence. Since the graph is meant for use when the domain is already settled, the domain is not visualised in the graph. In addition, the graph fails to visualise the productive and analytic aspects, which would have been preferable. As it was required to maintain the graph clear and monochrome, no means were found to include these aspects in the current version of the graph. As these two visualisations serve different purposes, they are not seen as conflicting, but rather as complementary.

The alterations of the MCRF, resulting in the new framework, were both of theoretical and practical nature. This was due to the fact that besides making the framework more coherent, the new framework was attended for application to another kind of data. In the MCRF, the competencies are defined in terms of what it means to master them, but in regular mathematical practices mastery excludes a considerable part of the activities. In addition, defining competence in terms of mastery reflects a view of domains as static and negotiations of competence as inexistent. Competence then become a dichotomous notion, something you either have or not. In the new framework, competence is seen as being under continual development. Participation in mathematical activities evolves both one's competence and one's conception of competence. Therefore, the competencies are defined by what it means to exercise, rather than master, them. The term 'exercise' was chosen since it covers both use and improvement.

This perspective on competence also affected the usage of the word 'mathematical' in the description of the framework. Rather than speaking of mathematical means, means are required to have mathematical ends. For reasoning this meant that the choices and conclusions are supposed to be mathematical, rather than the justification. For procedures it is the task which is mathematical, rather than the actions aiming to solve it. This is not to imply that mathematical actions and justifications are not desirable, but to obtain a framework which has the power to describe various mathematical practices in an adequate manner. As competencies are exercised they will eventually become more and more developed and mathematical. All the same, an attempt to solve a task or justify a choice by any means is more mathematical than no attempt at all.

The theoretical alterations comprise a refined definition of abstract mathematical entities and the removal of one competency and one competency related activity. In the MCRF, abstract mathematical entities were implicitly defined by exemplifications and relations to the competencies. Nonetheless, the demarcation of entities from representations, connections and procedures were not specified. The new definition is more restrictive, and the word 'concept' is now used when referring to entities, representations, procedures and connections all together. Since procedures are no longer included among entities, this adjustment had consequences for the definition of connections. Though it is not obvious whether the MCRF allows for connections between procedures or not, such connections were on record in both previous research and the empirical data. Therefore, connections between procedures are now explicitly permitted.

The two constructs which were omitted, the competency *problem solving* and the competency related activity *interpret*, are not to be conceived as absent in mathematical practice. Their exclusion is due to the wish for coherence and the detection of too large overlaps with other constructs. While interpretations play important roles in mathematical activities, they are not found to be empirically separable from use and evaluation. As the imitative – constructive spectrum was emphasised in the new framework procedures and problem solving seemed to be two parts of one competency rather than two separate ones. The choice to call this combined competency *procedures* was supported by the use of 'problem solving' in contemporary research. Though problem solving is seen as a valuable mathematical activity, it is often thought of as a perspective on mathematics. In this sense, it seems to lie on a different generality level than the other competencies; problem solving is a practice signified by certain ways of exercising competencies rather than an aspect of practice in general.

After these modifications, the framework is closer to becoming a theory. There are several descriptions of what a theory is. Radford (2008) suggests that a theory consists of basic principles, a methodology supported by these principles and a set of pragmatic research questions. Niss (2007) proposes that a theory is a system of concepts and claims forming an organised network. The concepts and the claims must be linked in connected hierarchies, where basic concepts are the building blocks of others and the higher order claims are derived from the fundamental ones. This is true for the concepts and claims in the new framework. The competencies are successively built up from AMEs and, together with the productive and analytic aspects, they constitute mathematical competence. The basic stance, to view knowledge as competence within a domain, leads to claims concerning teaching. In relation to empirical data, the analytical tools and their accompanying instructions do form a substantial component of a methodology. In addition, the research review reinforced the intertextuality of the framework. This is an important quality of a theory, as "mathematics education is discursive in nature and can only be understood in reference to previous research" (Jablonka & Bergsten, 2010, p. 37). Though the framework lacks a set of research questions, this is probably not the major obstacle on the way to becoming a theory. The elements which are present need further elaboration and sharpening. In particular, the methodology would benefit from acquaintance with more empirical data, as well as other researchers.

Though the constructs within the framework are discussed in various places of this thesis and one aim was to provide clearer definitions of and relationships between the competencies, it was a conscious choice not to settle for one ultimate description. Though successive reformulations of the framework have led to a deeper and more lucid understanding of the framework for the author, that understanding is not necessarily fully captured in the last, most immaculate version of the adapted framework. Instead, this understanding has been expressed in slightly different ways, not to complicate, but to explicate the framework for the reader. If it is interpreted and applied by other researchers in the future, it will be further elucidated.

Nevertheless, the immaturity of the framework is offset by its qualities. As it is not restricted to a specific content domain, as for example number sense, but offers a vocabulary for speaking about any mathematical practice, it can be a tool for synthesising and comparing different areas of research. Rather than competing with research fields such as mathematical literacy, the analytical tools developed here could be used for describing specific content literacies. As Johanning (2008, p. 308) notes:

> Research that looks at how students learn to use other mathematical content is needed to develop a clearer picture of what is involved in helping students become literate users of mathematics.

However, if such research is made using local theories, there is a risk of the picture not becoming clearer, but rather more confusing. Moreover, mathematical domains are not disjunct. The boundaries of conceptual fields are not well-defined, and filiations between them are common (Vergnaud, 1988). Therefore, it may be a hindrance to restrict research to conceptual domains. This framework allows for inclusion of various entities and provide ways to describe the relations between concepts. This is evident in the analysis in Paper III, where the case of Olle gives examples of the use of connections between quantity and shape. Thus, the analysis of this case bridges between number sense and spatial sense (Van Nes & Van Eerde, 2010).

Though the results from the empirical data are not the focus of Paper III and the limited cases do not allow for any far drawn conclusions, the framework was shown to have explanatory potential. Besides supporting previous research concerning the richness of children's mathematical activities (Clements & Sarama, 2007), the analysis produced more structured results than, for example, studies of number sense. The components of number sense form a less organised set than the aspects of mathematical competence in the domain of young children's arithmetic. The research on early childhood mathematics is a diverse field and, needless to say, the framework can not capture all its features. For instance, the framework does not have the same level of detail as studies of specific procedures. However, it can serve to contextualise the specifics, as seen in Paper III. The empirical results show that while some procedures, such as doubling and reshaping, where used to obtain results, other slower, though more certain, procedures functioned as means for reasoning and evaluation.

In Paper IV, the framework is applied to empirical data from a proving activity. Research on proving often deals with reasoning, but studies of the roles of other competencies in proving are scarce. Some relevant results in this direction can be found in studies on problem solving processes. However, though it might often be the case, proving activities are not always characterised by problem solving approaches. In addition, the strong association between reasoning and proving makes it especially interesting to see how other competencies come into play. The results in Paper IV show that the reasoning is aided by representations and connections and that the lack of procedures hinders the students to reach their goals.

The combined experience from the two empirical studies supports the view that the framework reflects mathematical practice rather well. As a start, it is worth noting that the competency analysis was able to account for two very different mathematical activities. Besides the functionality of the system of constructs, this is largely due to the flexibility of the analytical tools. In Paper III, schematic pictures of the artefacts were extensively used in both the narratives and the competency graphs. The graphs in Paper IV, on the other hand, merely included some drawings from the students' notes. While already the study of whole number arithmetic made use of multiple representational levels, the second study revealed the need for another kind of recursive representations. The students were found to 'entitify' whole systems of AMEs, representations and connections and to represent these in full.

A comparison of the two studies uncovers both differences and similarities. In both studies, representations were found to support reasoning. In the first study, Nina makes use of hand gestures to justify the conclusion that the 1x6 and 2x3-bricks are both representations of six. In the student group work, Diana constructs a graph of the derivative, which is then used to construct a graph of the function, showing the number of zeroes. The same holds for connections. Nina uses the connection between ten and twenty, that twenty is two times ten, as well as the connection between the two bricks 1x8 and 2x8 to justify her representation. The students use a chain of connections, from the non-zeroes of the second order derivative to the number of zeroes of the function.

In the case of the student group work, no examples of the productive aspects of procedures are found. In the first study, however, the use of procedures is intertwined with representations and connections. For example, the connection between length and shape plays a crucial role in Olle's symmetric construction and in its sub-procedures measuring and balancing. Tom's initial representation of numbers on one row of studs influences his counting procedures, even if the representations have changed. In addition, the procedure verbal counting is often used as an evaluation of representation justifying its correctness. In this way, procedures are related to both the analytic aspect of representations and the productive aspect of reasoning. In both studies, there seems to be a correlation between communication and the analytic aspect of other competencies. Though it seems reasonable that communication can elicit evaluation, this may also be a sign of the difficulty to analytically seize private monitoring.

Altogether, the analysis has provided many examples of how competencies are exercised in mathematical practice. In visualising the interplay between competencies, the competency graph becomes a powerful tool, as it forms a structured image of the content of the activities and how it is addressed. Nevertheless, it does not illustrate the two aspects of the competencies, which could be an issue for the improvement.

3.5.1 Implications for practice

As described in the background, there has been a change in the formulation of Swedish mathematics syllabi. Accounts of the organisation of teaching have yielded, in favour of goals in terms of the abilities students are expected to attain and strive for. Since the competencies correspond to the syllabus goals to a large extent, results from the competency analyses can provide ideas for school practices. Viewing the competencies as essential aspects of mathematical competence, these ideas have bearing outside the Swedish school context. In order to develop competencies, students need to exercise competencies and therefore, teachers need to provide opportunities for this. As seen in the two studies, competencies rely on each other and are not exercised in isolation. This implies that school practice should aim at providing rich opportunities for exercising various combinations of competencies.

In addition, both the productive and the analytic aspect should be exercised. The whole spectrum of the productive aspect, from imitative to constructive activities, is necessary. Own construction gives a natural context to the product, which can enhance meaning. Modifications and adaptations tend to inner qualities, as well as to the relations to other concepts. Repetition advances skill and memory, but it can also lead to attachment to the subject. If repetitive activities are carefully designed, they can also facilitate meta-level considerations, as they allow for detection of variations.

Besides the meta-level, which provides an outside perspective on one's actions, the analytic aspects include monitoring and evaluation. Monitoring is important in keeping track of one's actions and intentions. Evaluation plays a key role, both in the establishment of justification norms and in the development of mathematical autonomy. Participation in mathematical practices, in which questioning and justification are customary, is also likely to foster such standards for one's own, private thinking.

If, on the other hand, merely a subset of the competencies and aspects are allowed to be exercised, also the ones exercised will suffer. The most common example is when the imitative part of the productive aspect of procedures thrive on the expense of other activities. In such cases, learning can deteriorate to rote, obstructing both creativity and understanding. However, the second study showed that there is danger in putting too much weight on the analytic aspect as well. Regardless of their quality, hypothetical reflections do not lead to concrete results. Discussions on the process and product of a procedure need to be grounded in actual application of that procedure. Otherwise, in other respects, sound reasoning about the procedure, may leave a feeling of dissatisfaction.

These implications do not merely hold for the upper levels of mathematics education. The data presented in Paper III show that young children can exercise all mathematical competencies in assigned tasks, as well as in their own play. Nonetheless, it is just as important to supply young children with rich opportunities for the exercising of competencies. Creating an environment for exploration of various representations and procedures, asking them to make connections and to justify their choices and conclusions, as well as promoting peer and teacher–child communication, increases the chances of children entering school with a balance between and an awareness of mathematical competencies. It seems reasonable that such a start would ease the continued movement on the same path.

Though the competencies and aspects in the framework are chosen to reflect mathematical competence, some claims might hold beyond the mathematical domains. For instance, a balance between the productive and the analytic aspect could be desirable also in the practice of teaching. In order to produce a learning environment with the potential for exercising competencies, it can be beneficial to reflect upon and evaluate one's actions. The framework, with its description of the competencies and their interrelations, could be used as a tool for monitoring the design of tasks and teaching in mathematics. The competency analysis and the competency graph, or some modifications thereof, could then aid the evaluation of one's practice. Such use of the framework and the analytical tools would perhaps also further the elaboration and refinement of the constructs.

Altogether, it is the diverse opportunities to exercise all aspects of competence which is called for. This applies to students on all educational levels, as well as to teachers.

3.5.2 Suggestions for future research

The first, obvious proposition for future studies is application of the framework to other mathematical domains. Though the two sets of data analysed in Paper III and

IV, respectively, differ in substantial aspects, they by no means cover all mathematical activities. To begin with, none of them are placed in a regular school setting. There are also numerous other mathematical areas, which may offer new examples of the exercising of competencies. Results from other domains may in turn reflect back on the framework, giving rise to increased refinement of its components. Application of the analytical tools on new activities may strengthen the appreciation of their capacity. In addition, it may lead to further developments and new contributions to the tools, expanding the analytical toolbox.

Alongside empirical studies, additional theoretical research could enhance the framework's coherence and its explanatory power. Even though a considerable effort has been made to clarify the constructs and to explicate their interrelationships, there are some vaguenesses remaining. First of all, the definitions of the competencies vary slightly in form and precision. Though some differences may be due to dissimilarities in nature, there is probably room for improvement. In particular, the communication competency caused some trouble, partly since it is less specific to mathematics than the other competencies and partly because it was especially difficult to make sense of the two aspects in this case. Although the question was raised whether communication should rather be seen as a third aspect of the other competencies, the choice was made to keep it as it was. Even so, its position in the framework remains uncertain. This issue can only be resolved by further theoretical analysis.

Furthermore, the current competency graph fails to capture the two aspects of competencies, which is perceived as a limitation. Though this is remedied by the narrative, it is a deficiency in the visualisation of the results, no less. It is therefore desirable to find a way to either include the aspects in the graph or to construct another tool for their depiction. The presence or absence of the aspects are just as important as the competencies.

3.5.3 Conclusion

In light of the contemporary changes in descriptions of mathematical knowledge, in both Swedish curricula and international reform frameworks, the aim was to develop a research framework for the analysis of the exercising of mathematical competencies. The Mathematical Competency Research Framework was taken as a starting point for this process. Though the main part of this framework was found to be suitable for the purpose, the constructs were refined and in two cases removed. Research focusing on single competencies additionally informed the construct definitions. The operationalisation of the framework for application of empirical data entailed the development of two analytical tools: the analysis guide and the competency graph. In the two empirical studies, these tools were used to obtain results in form of narrative and graphical descriptions on how competencies were exercised. Differences, as well as similarities, were shown between young children's arithmetic and university students' proving activities. The studies demonstrate the explanatory power of the new framework, suggesting future development and improvement to be worthwhile.

Kapitel 4

Sammanfattning på svenska

4.1 Inledning

Denna avhandling består av två studier i matematik och två studier i matematik med utbildningsvetenskaplig inriktning. Därmed är den en tvåhövdad varelse, med två olika syften och upplägg. Studierna som hör hemma i matematisk vetenskap visar exempel på hur matematik, eller mer specifikt representationsteori, praktiseras. De utbildningsvetenskapliga studierna antar ett utifrånperspektiv på matematik. De syftar till att beskriva hur matematisk kompetens utövas och därmed representera matematisk praktik. För detta ändamål utvecklas ett ramverk som belyser olika dimensioner av matematisk kompetens. Längs en dimension löper en produktiv och en analytisk aspekt. Den produktiva aspekten innefattar utveckling och användning av matematik, medan den analytiska aspekten innebär reflektion och utvärdering. På så sätt kan avhandlingen sägas utöva båda aspekterna: den produktiva i arbetet bakom och utformandet av de två studierna i representationsteori och den analytiska i utvecklandet och användandet av ramverket.

Följande text är upplagd så att de matematiska studierna introduceras och sammanfattas i avsnitt 4.2, medan de utbildningsvetenskapliga studierna behandlas i avsnitt 4.3. Förutom att sammanfatta studiernas resultat, tar detta avsnitt upp bakgrund, teoretiska utgångspunkter, syfte, forskningsfrågor, metod och diskussion.

4.2 Problem av Clebsch-Gordantyp

De två studierna I och II ger några resultat i representationsteori för Liegrupper och Liealgebror. En grupp G kallas linjär om den är isomorf med en grupp av matriser över någon kropp K. En representation av G består av ett vektorrum X och en verkan av G på X. Med andra ord ska det finnas en avbildning $G \times X \to X$ som respekterar grupp- och vektorrumsoperationerna. Ett delrum $V \subseteq X$ som är invariant med avseende på gruppverkan kallas för en delrepresentation av X. En representation som saknar äkta delrepresentationer kallas irreducibel.

När det handlar om en Liegrupp G och dess Liealgebra $\mathfrak{g}$ spelar vikter och högstaviktrepresentationer en avgörande roll. En vikt är en endimensionell representation av $\mathfrak{g}$ på kroppen K, $\lambda : \mathfrak{g} \to K$. Om V är en godtycklig representation av $\mathfrak{g}$ definieras viktrummet av V med vikt λ som

$$V_\lambda = \{\mathbb{v} \in V : \forall G \in \mathfrak{g} : G\mathbb{v} = \lambda(G)\mathbb{v}\}.$$

Om $V_\lambda \neq 0$ sägs λ vara en vikt för V. Det går att definiera en ordning på dessa vikter och således att tala om högsta vikter. Om det existerar en högsta vikt för V, sägs V vara en högstaviktrepresentation. Givet två irreducibla högstaviktrepresentationer, $V_{\overline{\alpha}}$ och $V_{\overline{\beta}}$, är även tensorprodukten av dessa en representation av $\mathfrak{g}$. Det ursprungliga Clebsch-Gordanproblemet handlar om att bestämma de irreducibla högstaviktrepresentationer som är delrepresentationer till denna tensorprodukt i fallet $\mathfrak{g} = \mathfrak{gl}_{\mathbb{C}}(n)$. Med andra ord skall koefficienterna

$$c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} \quad \text{i} \quad \bigoplus_{\overline{\gamma}} c^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} V_{\overline{\gamma}} = V_{\overline{\alpha}} \otimes V_{\overline{\beta}}$$

beräknas. Vikter kan ses som en generalisering av egenvärden. Det är därför ingen överraskning att detta problem är ekvivalent med ett problem avseende bestämning av egenvärden till summor av hermiteska matriser. De två studierna som ingår i denna avhandling betraktar två relaterade problem.

4.2.1 Studie I

Givet två $2k \times 2k$-dimensionella, skevsymmetriska matriser A och B, med egenvärden $\pm i\overline{\alpha} = (\pm i\alpha_1, \dots, \pm i\alpha_k)$, respektive $\pm i\overline{\beta} = (\pm i\beta_1, \dots, \pm i\beta_k)$, är frågan vilka egenvärden som är möjliga för matrisen $C = A + B$. I denna studie visas att mängden av dessa egenvärden är en union av fyra konvexa polytoper. Det visas också att problemet är relaterat till frågan om $SO_{\mathbb{C}}(2k)$-invarianta faktorer i tensorprodukten av tre högstaviktrepresentationer. I fallen $k = 2, 3$ ges allmänna villkor för polytoperna, givet $\overline{\alpha}$ och $\overline{\beta}$.

4.2.2 Studie II

Här betraktas en specifik klass av oändligtdimensionella, irreducibla representationer av $\mathfrak{gl}_{\mathbb{C}}(n+1)$, som kommer från ändligtdimensionella representationer av $U(n)$ på vektorrum av polynom. Givet två sådana representationer $\mathscr{P}_k^{\varkappa}$ och $\mathscr{P}_l^{\lambda}$ av högsta vikter $\underline{k}=(0,\ldots,0,k)$ och $\underline{l}=(0,\ldots,0,l)$, bestäms ett antal irreducibla delrepresentationer till tensorprodukten $\mathscr{P}_k^{\varkappa}\otimes\mathscr{P}_l^{\lambda}$.

4.3 Matematisk kompetens

Sedan införandet av den svenska grundskolan har styrdokumentens beskrivningar av matematikämnet skiftat och utvecklats. När det gäller matematiskt kunnande och skolans praktik har utvecklingen gått åt två olika håll, som i viss mån motsäger varandra. Å ena sidan har det kunnande som eleverna ska uppnå blivit mer nyanserat genom att beskriva detta mer i termer av processer. Å andra sidan har läroplanen gått från att fokusera på de aktiviteter som eleverna ska ta del av, till att enbart beskriva de mål som ska uppnås. Medan omformuleringen av målen syftat till att bredda synen på matematiskt kunnande, kan avsaknaden av metodbeskrivningar bidra till en mer ensidig undervisning där eget arbete i boken tar stor plats.

I denna avhandling antas ett kompetensperspektiv på kunskap. Kompetens är alltid knutet till ett visst område, som kan vara avgränsat socialt, rumsligt och innehållsligt. Att vara kompetent inom ett område innebär att kunna hantera essentiella aspekter av livet i detta område. Kompetens under en matematiklektion är inte samma sak som under en svensklektion, och båda dessa skiljer sig ytterligare från kompetens i umgänget under rasten.

Begränsat till matematisk kompetens, det vill säga kompetens inom områden med matematiskt innehåll, går det att närma sig en generisk uppsättning av essentiella aspekter. Aspekterna kan ta sig olika uttryck i relation till innehåll och miljö, men har ändå en gemensam kärna. Förslag på sådana uppsättningar har givits av NCTM (2000), Kilpatrick et al. (2001) och i de två närbesläktade KOM-projektet (Niss & Jensen, 2002) och OECD/PISA-ramverket (OECD, 1999). Dessa ramverk är dock inte avsedda för forskningsanalyser. De kompetenser eller förmågor som presenteras är således inte tydligt definierade eller särskiljda.

'The Mathematical Competency Research Framework' (MCRF), som utformats av Lithner et al. (2010), syftar till att uppfylla dessa kriterier. Det utvecklades för att analysera vilka kompetenskrav som ställs i provuppgifter och under lektioner. Matematisk kompetens är här uppdelad i sex kompetenser och tre kompetens-relaterade aktiviteter. Begreppen definieras genom att beskriva vad det innebär att bemästra de tre kategorierna, i relation till varje kompetens. Deltagare befinner sig generellt någonstans emellan, inte i, de två extremerna total frånvaro och total bemästring av kompetenser. På grund av detta ses kompetens här som en form av kunskap som utvecklas samtidigt som och genom att den utövas. Detta ställningstagande leder fram till syftet och forskningsfrågorna.

4.3.1 Syfte

Det huvudsakliga syftet är att utveckla ett ramverk för att analysera hur matematiska kompetenser utövas i matematisk praktik. Detta görs med utgångspunkt i ramverket MCRF. Dessutom är syftet att ge exempel på utövandet av matematiska kompetenser i specifika områden.

Studie III fokuserar på det första syftet. Genom att begreppen i MCRF analyseras och knyts till annan forskning utvecklas ett nytt ramverk, som även omsätts i analytiska verktyg. I relation till det andra syftet ges även exempel på hur matematiska kompetenser utövas av barn. Studie IV är en empirisk studie av högskolestudenters bevisföring och behandlar således främst det andra syftet.

4.3.2 Metod

Med hänsyn till de två syftena har två metoder använts, där den ena är ett resultat av den andra. Utvecklingen av ramverket bestod av tre faser: analys av ramverkets begrepp och inre relationer, en genomgång av relaterad forskning, samt utveckling av verktyg för analys av empiriska data. Det resulterande ramverket och de analytiska verktygen bygger sedan upp metoden som använts i de två empiriska studierna.

I den första studien intervjuades fyra par av förskolebarn om hela tal och legobitar. Intervjuerna innefattade ett antal frågor och uppgifter tänkta att ge rika möjligheter att utöva kompetenser. Intervjuerna videofilmades. Den andra studien är en reanalys av en fallstudie, genomförd av Pettersson (2008a). En grupp av fyra förstaårsstudenter vid universitetet har fått i uppgift att formulera och bevisa påståenden om förhållandet mellan antalet nollställen hos en funktion och funktionens derivator. Data består av en videoupptagning från studenternas arbete och kopior av deras anteckningar.

Det nya ramverket beskriver matematisk kompetens med hjälp av fem kompetenser och två aspekter. Kompetenserna berör representationer, procedurer, samband, resonemang och kommunikation. De två aspekterna är produktiv och analytisk aspekt. I botten finns också ett ytterligare begrepp: *abstrakt matematisk enhet*

(AME). Dessa enheter kan ses som de grundläggande byggstenarna i matematik. En AME är definierad som ett självständigt begrepp som behandlas som ett objekt inom matematisk praktik. Kompetenserna är successivt definierade genom att knyta an till AME:r och tidigare kompetenser. Varje kompetens är också förklarad i termer av de två aspekterna. *Den produktiva aspekten* innefattar användning, tillämpning, anpassning och utveckling, vilket sammanlagt utgör ett spektrum från imitativa till konstruktiva aktiviteter. *Den analytiska aspekten* handlar om metaperspektivet på kompetenserna, i form av reflektion, utvärdering och övervakning av den produktiva aspekten.

AME:r kan representeras på olika sätt och *representationskompetensen* handlar om att hantera sådana representationer. I en representation innefattas både det konkreta objekt som ersätter AME:n och avbildningen mellan AME:n och objektet. *Procedurkompetensen* berör procedurer, det vill säga sekvenser av handlingar som syftar till att lösa matematiska uppgifter. En procedur är alltid grundad i en eller flera AME:r och är ofta kopplad till en specifik representationsform. Samband länkar samman AME:r, representationer eller procedurer och är ämnet för *sambandskompetensen*. *Resonemangskompetens* visas genom att argumentera för matematiska val och slutsatser. Det kan vara val av representationer eller procedurer och slutsatser i form av en lösning eller ett samband. Alla tidigare kompetenser kan utgöra ämnet för kommunikation. Representationer kan dessutom fungera som ett medel för att kommunicera. *Kommunikationskompetensen* handlar om kommunikationen i sig, medan budskapet har att göra med de andra kompetenserna.

De två analytiska verktyg som utvecklats är en analysguide och en kompetensgraf. Analysguiden består av frågor avsedda att skärpa fokuset på kompetenserna. Frågorna följer en viss ordning för att främja urskiljning och särskiljning. Det tänkta resultatet av att följa guiden är en kompetensskildring av data. Kompetensgrafen är menad att ytterligare tydliggöra samspelet mellan kompetenserna. Under analysprocessen stärker de två verktygen varandra, då grafen strukturerar relationen mellan kompetenserna och kompetensskildringen ger en fylligare förklaring. Slutprodukten ger både en grafisk och en berättande beskrivning av hur kompetenser utövas.

4.3.3 Studie III

Denna studie beskriver utvecklingen av ramverket. Begreppen kompetens och abstrakt matematisk enhet tydliggörs. Samtliga definitioner av och relationer mellan begrepp utreds. Somliga begrepp revideras och andra utesluts. Relaterad forskning tillåts nyansera och klargöra de återstående fem kompetenserna. Tre exempel från en intervjustudie med femåriga barn tas upp. Exemplen visar hur de analytiska verktygen kan användas för att förklara matematisk praktik och belysa hur kompetens utövas.

4.3.4 Studie IV

En reanalys av en fallstudie beskriver hur matematiska kompetenser utövas i bevisföring i en grupp av fyra universitetsstudenter. Den komplexa uppgift de fått skapar ett behov av att röra sig mellan flera generalitetsnivåer. Ett väntat resultat var att resonemang skulle vara framträdande, eftersom studenterna arbetar med ett bevis. Så var mycket riktigt fallet, men andra kompetenser spelade också in. Representationer och samband var viktiga för studenternas resonemang. Avsaknaden av procedurer sågs som en möjlig förklaring till att de inte känner sig säkra på sitt resultat i slutet av sessionen.

4.3.5 Diskussion och slutsats

Det huvudsakliga syftet var att utveckla ett ramverk för att analysera utövandet av kompetenser i matematisk praktik. Med utgångspunkt i MCRF gjordes detta i tre faser beskrivna ovan. Under processen skedde en rörelse mellan faserna, då de påverkade varandra. Varje fas tjänade dock ett eget syfte. Analysen av begreppen och deras relationer var viktiga för koherensen hos det nya ramverket. Kopplingarna till andra forskningsstudier var nödvändiga för att sätta ramverket i ett sammanhang och för förståelsen av de enskilda kompetenserna. Genom att använda ramverket för analys av empiri skapades verktyg som testade ramverkets förklaringskraft. I och med dessa ändringar och tillägg kan det nya ramverket anses ligga närmare en teori än det ursprungliga.

De två empiriska studierna visar att ramverket kan användas för att beskriva och förklara olika matematiska aktiviteter på olika nivåer av utbildningssystemet. Den främsta styrkan hos ramverket är hur det belyser samspelet mellan kompetenser; hur de stöttar och beror av varandra. I båda studierna ses exempel på hur representationer och samband används i resonemang. I den första studien ses även procedurer användas för att utvärdera och rättfärdiga slutsatser. Ett påfallande resultat från den andra studien är att system av AME:r, representationer och samband kunde 'enhetifieras' och bilda nya AME:r. I denna studie visade sig redan analysen av de tre första kompetenserna ge upphov till komplexa grafer. Samtidigt var resonemang och kommunikation mycket frekventa, och det ansågs därför bättre att behandla dessa kompetenser separat. Detta vittnar om en flexibilitet hos de analytiska verktygen som är en ytterligare styrka hos ramverket.

Kompetetensperspektivet medför idén om att utveckling av kompetens sker i samband med utövandet. Om målen i läroplanen är beskrivna som processer betyder detta att målen kan uppnås genom att delta i dessa processer. Lärarnas uppgift blir då att skapa så rika förutsättningar som möjligt för eleverna att utöva kompetenser. Resultaten från de empiriska studierna bidrar med exempel på aktiviteter där sådana förutsättningar finns.

Även om ramverket har utvecklats på flera sätt och visats kunna procudera resultat som förklarar matematisk praktik, finns det utrymme för förbättringar. Till att börja med skulle data från andra typer av matematiska aktiviteter säkerligen utmana begreppen och verktygen och skärpa dessa ytterligare. En annan punkt är kommunikationskompetensen, som i den nuvarande versionen av ramverket är något udda. Den skulle behöva utvärderas och specificeras ännu en gång. Dessutom finns det en begränsning hos kompetensgrafen, då den inte skiljer mellan den produktiva och den analytiska aspekten hos kompetenserna. Detta gör att denna dimension av kompetens inte visualiseras på samma sätt som den andra.

Sammantaget visar dock det nya ramverket både på större koherens och på förklaringskraft vad det gäller matematiska aktiviteter. Detta indikerar att det finns mening och värde i fortsatt utveckling och förbättring av detta ramverk.

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