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Asymptotics and dynamics in first-passage and continuum percolation

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Abstract

This thesis combines the study of asymptotic properties of percolation processes with various dynamical concepts. First-passage percolation is a model for the spatial propagation of a fluid on a discrete structure; the Shape Theorem describes its almost sure convergence towards an asymptotic shape, when considered on the square (or cubic) lattice. Asking how percolation structures are affected by simple dynamics or small perturbations presents a dynamical aspect. Such questions were previously studied for discrete processes; here, sensitivity to noise is studied in continuum percolation.

Paper I studies first-passage percolation on certain 1-dimensional graphs. It is found that when identifying a suitable renewal sequence, its asymptotic behaviour is much better understood compared to higher dimensional cases. Several analogues of classical 1-dimensional limit theorems are derived.

Paper II is dedicated to the Shape Theorem itself. It is shown that the convergence, apart from holding almost surely and in L^1, also holds completely. In addition, inspired by dynamical percolation and dynamical versions of classical limit theorems, the almost sure convergence is proved to be dynamically stable. Finally, a third generalization of the Shape Theorem shows that the above conclusions also hold for first-passage percolation on certain cone-like subgraphs of the lattice.

Paper III proves that percolation crossings in the Poisson Boolean model, also known as the Gilbert disc model, are noise sensitive. The approach taken generalizes a method introduced by Benjamini, Kalai and Schramm. A key ingredient in the argument is an extremal result on arbitrary hypergraphs, which is used to show that almost no information about the critical process is obtained when conditioning on a denser Poisson process.

Keywords: First-passage percolation, noise sensitivity, continuum percolation, Gilbert model, limit theorems, shape theorem, stopped random walks, large deviations, dynamical percolation.

List of papers

This thesis consists on an introduction to some asymptotical and dynamical aspects of percolation theory, followed by three research papers:

Paper I D. Ahlberg. *Asymptotics of first-passage percolation on 1-dimensional graphs.*

Paper II D. Ahlberg. *The asymptotic shape, large deviations and dynamical stability in first-passage percolation on cones.*

Paper III D. Ahlberg, E. Broman, S. Griffiths, and R. Morris. *Noise sensitivity in continuum percolation.*

Contents

Acknowledgements

Producing a doctoral thesis is an extensive process. The effort, discomfort and frustration can become greater than first imagined. I believe that I share this experience with many others. I considered myself well suited before my doctoral studies, but I soon realised that my attitude was a bit too naive. People who know me are probably not very surprised by this optimism. I realised that I had a lot to learn, and still have. However, I have learnt a lot, and my hope is that this thesis is a partial demonstration of that. Although many ’downs’ felt longer lasting than the ’ups’, it has been an exciting time. I am very grateful for the support I have been given from friends and colleagues over the past few years. There are many of you I wish to thank.

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Daniel Ahlberg
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Chapter 1

Introduction

The classical study of probability, before the 19th century, was limited to games of chance. Studies concerned trials, or sequences of trials, which could result in a finite number of equally probable outcomes. In the second half of the 19th century, probabilistic statements found its way into physics. Heat as consisting of molecular movements had recently become a leading theory. Ludwig Boltzmann's and James Clerk Maxwell's contributions resulted in the description of molecular movements in a gas in terms of a probability distribution. This laid the foundation of statistical mechanics.

As a part of classical physics, statistical mechanics aim to describe the macroscopic behaviour of a large number of molecules or particles, based on the properties on a microscopic level. Around the turn of the century, it became apparent that classical physics was unable to explain several empirical observations, such as heat radiation and radioactivity. Quantum mechanics was introduced to explain interaction at atomic scales. The first step towards a quantum theory was taken in 1900 by Max Planck when studying black-body radiation. Further contributions were made a few years later by Albert Einstein. A rapid development of quantum mechanics into an established theory took place between 1925 and 1927. It was led by Max Born, Werner Heisenberg and Erwin Schrödinger, and culminated with the derivation of Schrödinger's wave equation and Heisenberg's uncertainty principle. Via the wave equation, the position of a particle got a probabilistic interpretation.

The introduction of probabilistic statements as a mean to describe physical processes was by many contemporary scientists seen as a consequence of our ignorance, rather than as a belief of nature itself as being governed by chance. In classical physics, motion and interaction are caused by known or unknown forces. In quantum mechanics, however, the physical state of a system can only be given probabilistically. In particular, as a consequence of the uncertainty principle, on a microscopic scale, there is a bound for the precision by which the position and the motion of a particle can be determined simultaneously. With the progress of quantum mechanics, the world unavoidably had to be interpreted as indeterministic. At the same time, randomness and probability obtained a definite position in our understanding of the physical world.

In his book, von Plato (1994) gives a careful account of the creation of modern probability. The introduction of random processes in continuous time by Boltzmann and Maxwell, as well as Einstein's derivation of the mean displacement law of Brownian particles in 1905, called for a more rigorous mathematical framework. Measure theory was developed by Borel and Lebesgue around the turn of the century. Despite that, it would take until the early 1930's before measure theory would turn probability into a well-founded theory. von Plato explains further how the development of statistical mechanics, together with the rapid conceptual change towards an indeterministic view of the world, made important contributions for this change to take place.

Ever since probability theory received its position as a solid and respected branch of mathematics, probabilists have in their turn sought inspiration and motivation in various real-world phenomena. Inspiration was found in anything from physics and biology to finance and social science. An area of probability theory that has had a particularly fruitful exchange of ideas with, and motivation from, physics and physical phenomena is the area of percolation theory. Percolation models are examples of random spatial processes which aim to model physical phenomena via simple random rules. Common to percolation models is that rules are defined on a local scale. The effect small local changes have on the behaviour of the system on larges scales is thereafter studied. In this sense, there are clear connections to statistical mechanics. Percolation models generally allow many natural and intuitive problems to be posed with low effort, whereas giving satisfactory solutions to the problems often turn out far from trivial. This is of great appeal, since it often calls for creative development of new techniques in order to gain deeper understanding of the problem.

Before introducing the work of this thesis, I will first give a short introduction to percolation theory. I will begin with a rather informal description of a few models and concepts to give the reader a flavour of the field. The informal presentation will be sufficient to give a brief motivation behind, and description of, the content of the current thesis. After that, a more detailed presentation will be given of some relevant parts of the area, as well as a summary of the papers that build up the thesis. During the informal desctiption I have preferred not to burden the text with references. The reader will instead find all relevant references in the more detailed presentation in subsequent sections.

1.1 Percolation theory

The *bond percolation* model is arguably the simplest and most classical among percolation models. Simple here refers to the ease with which the model can be described. However, many natural questions regarding its behaviour pose great challenges and several of them remain unanswered until this day. Bond percolation was motivated as a model to describe the seemingly random structure of a porous material. It is a discrete model, where the discrete structure is provided by a suitably chosen graph. A *graph* consists of a set of vertices and a set of bonds between pairs of vertices. Each bond, also referred to as an edge, symbolizes a connection between the two vertices. The $\mathbb{Z}^d$ *lattice*, or the $\mathbb{Z}^d$ *nearest neighbour graph*, for $d \geq 2$, is the graph whose vertices are given by the points in $\mathbb{Z}^d$, and where two vertices are connected by an edge if they are at Euclidean distance one from each other.

The $\mathbb{Z}^d$ lattice is an infinite graph, and is used as an approximation of a large region. To obtain a random structure from the $\mathbb{Z}^d$ lattice, we proceed as follows. Go through each edge one by one, flip a coin, and decide to keep the edge if the coin turns up heads and remove the edge if the coin turns up tails. Thus, each edge is removed independently of all other edges. The resulting structure can be viewed as a representation of a large piece porous material if thinking of vertices as cells in the material, and edges symbolizing neighbouring cells having a reasonably large passage between them (as to allow a fluid to pass, say).

With this interpretation of the model, a fluid is able to flow from one cell to another if there is a path, that is, a sequence of edges between neighbouring cells, that connect the cells. To give a more specific definition of a path, a *path* between two points u and v of a graph refers to an alternating sequence of vertices and edges $u = v_0, e_1, v_1, \ldots, e_n, v_n = v$, starting and ending with a vertex, and such that the vertex v_k is the endpoint of the edges e_k and e_{k+1} preceding and succeeding v_k.

Studying the random structure obtained through coin tossing leads to questions concerning the existence of paths in the random structure. In particular, one may ask if the centre of a large piece of porous material will be wet when submerged in the fluid? This corresponds to the question of how far a fluid injected at the centre of the material will reach. Since the model is based on an infinite graph, is it possible for a fluid injected at the centre (the origin of the graph) to wet infinitely many cells? That the fluid will wet another cell corresponds to the existence of a path from the origin to that cell. Cells that are connected by paths form components of interconnected cells. What can be said about the size of these components?

In fact, the answers to these questions differs depending on the coin being fair or being biased. Consider some fixed dimension $d \geq 2$, and let $p \in [0,1]$ denote the probability that the coin tossed turns up heads. Thus, $p = 1/2$ corresponds to the coin being fair, and $p \neq 1/2$ to the coin being biased. For values of p close to 1, an infinite connected component of cells will exist, whereas for values of p close to 0, all components will be finite. As p ranges from 0 to 1, the system undergoes what physicists call a *phase transition*, that is, a sudden change in the qualitative behaviour of the model. An example of such a phenomena in nature is the structural transition that water experiences as temperature increases, going from solid to liquid to gas. In the case of bond percolation, the phase transition that occurs is that the random structure goes from having no infinite connected component of cells when p is close to 0 and to have one for p close to 1. In fact, there is a critical value $p_c(d)$ strictly between 0 and 1 such that for $p < p_c(d)$, there is no infinite connected component, but for $p > p_c(d)$ an infinite connected component does exist. The existence and non-existence of infinite components should be understood to hold with probability 1, or almost surely. When an infinite component exists, there is also positive probability for a fluid injected at the origin to reach infinitely far.

As a final remark, the restriction $d \geq 2$ was imposed in the above discussion to avoid the trivial case when $d = 1$. When $d = 1$ and $p < 1$, then only finite components will remain after edges have been removed in accordance with the result of the coin tosses. When $p = 1$, the graph will remain intact.

1.2 Alternative percolation models

It may seem naive to think that such a well structured graph as a lattice can be suitable to describe the seemingly irregular structure of a porous material. This is a relevant criticism. One should emphasise that from a probabilists point of view, the intention of the model was never to achieve a model that in a realistic way describes the local structure of the material. Rather, the objective was to find a reasonable model which on a large scale is plausible to have similar qualitative properties as the object it intends to describe. When a large portion of the material is resembled by a very fine grid, it seems reasonable to assume that the precise structure of the grid should have little influence on the qualitative behaviour of the model. However, there have been various reasons to introduce alternative models of similar flavour. Each model has its own advantages. It can offer easier computations, more symmetry, or enhanced generality. It is generally expected that small variations of a model on a local scale should not affect the global (qualitative) behaviour of the model. Morally, similar observations should hold for similar models. As physicists phrase it, models with similar behaviour belong to the same universality class.

As an alternative to bond percolation, *site percolation* is the model where vertices, instead of edges, are being removed. In bond and site percolation, a random structure is obtained form a fixed graph, such as a lattice. In order to achieve models that are homogeneous in space, and does not depend on an underlying discrete structure, certain continuum percolation models have been introduced. Continuum percolations models (in two dimensions) essentially amounts to constructing a random graph embedded in $\mathbb{R}^2$, which is accomplished in the following manner. A subset of points in $\mathbb{R}^2$ is chosen to constitute the vertex set of the graph. Next, pairs of vertices are joined by an edge depending on the local geometry around the two points. One such model that is studied further in this thesis is the *Poisson Boolean model*, also known as the *Gilbert disc model*. In this model, a Poisson point process with intensity λ is chosen to constitute the vertex set, and thereafter any two points are connected by an edge if their Euclidean distance is at most 2. An alternative way to visualize this is to at each Poisson point centre a disc of radius 1. The subset of the plane covered by the discs corresponds to the random graph. In particular, collections of overlapping discs corresponds to connected components in the graph. Questions such as size of connected components, existence of infinite connected components, and uniqueness of such, are questions that have similar qualitative answers as corresponding questions for bond percolation. In particular, the existence of an infinite component of overlapping discs depends on the intensity λ, for which there is a critical intensity $\lambda_c \in (0, \infty)$ such that $\lambda < \lambda_c$ implies non-existence of an infinite component, and $\lambda > \lambda_c$ implies existence, each with probability one.

1.3 A stochastic model for spatial growth

Another model that will be studied closer in this thesis is known by the name *first-passage percolation*. Similar to bond percolation, the model is defined on an underlying discrete structure, the typical such being the $\mathbb{Z}^d$ lattice. In contrast, bonds are not removed in first-passage percolation, but assigned random non-negative values according to some distribution. The values assigned to edges could be thought of as times associated with the crossing of the edges. In particular, if a fluid is injected at the origin, and is allowed to spread along the edges of the graph, then the passage of an edge is delayed the time indicated by its random value. With this picture in mind, one may ask how many vertices will be wet by the fluid during a fixed time period, and more precisely, how does the region of wet vertices evolve over time?

First-passage percolation can be viewed as a dynamic version of bond percolation. If an infinite value assigned to an edge symbolizes its absence, then the bond percolation model is retained in the case when edges are assigned values either 1 or ∞ with probability p and $1-p$. However, first-passage percolation should not be thought of as a mere generalization of bond percolation. Rather, it was introduced as a stochastic model for spatial growth, and the questions of interest differ from the ones posed for bond percolation. Here, the central object is not the component containing the origin, but the region of wet nodes evolving in time. An object that can be studied more directly is the time it takes the fluid to reach a distant vertex. Understanding such travel times is the key to describe the behaviour of the wet region. Since the fluid may advance along any path allowed by the underlying structure, the travel time to a specific vertex is not obtained by simply summing up random contributions. How does this influence the travel time? Is the time it takes the fluid to reach vertices far away proportional to their distance from the origin? Given a path from the origin to a vertex, the travel time to that vertex is at most the sum of the random times associated to the edges of the path. This is referred to as a subadditive behaviour, and led to the study of so called subadditive stochastic sequences.

1.4 Concepts of sensitivity in random structures

As the research literature in the area has grown, the perspective has widened to alternative questions and concepts. Investigations has concerned not only the structure of percolation clusters themselves, but also the behaviour of objects such as random walks on infinite percolation clusters. Stochastic growth models, such as first-passage percolation, have been employed to study the evolution of various objects competing for space. Other recent development in percolation theory have aimed to study how percolation models are affected by introducing simple dynamics, or when exposed to small perturbations. Both bond and site percolation are static models. A random structure is achieved through independent coin tosses. Depending on the bias of the coin, the resulting structure either contains or not an infinite connected component. *Dynamical (bond) percolation* is obtained when simple dynamics is introduced to invoke life to the model. Assume that each edge is assigned a Poisson clock, which is set independently of all other clocks. At each ring of the clock the edge changes its state, i.e., from absent to present and vice versa. Hence, is an edge was declared present from the start, then it will be removed at the first ring, and reappear when the clock rings again. At each fixed time point, the random structure that we observe corresponds to a bond percolation configuration obtained from independent coin flips. In particular, at each fixed time point we will have the same probability to observe an infinite component, and that probability is either 0 or 1. However, is it possible that there exists (random) times at which the presence of an infinite component is changed? When considering bond percolation away from criticality, the question can relatively easily be answered no. But, for bond percolation on the $\mathbb{Z}^2$ lattice at the critical probability $p = 1/2$, highly non-trivial techniques were needed in order to prove that, almost surely, there are exceptional times at which an infinite component appears, although is has probability 0 to occur at any fixed time point.

In the dynamical percolation model, an interesting question is how fast the information given by the initial configuration is lost as time elapses. To be able to quantify this in a suitable way, one investigate how a sequence of events of interest defined on an increasing sequence of subgraphs of the lattice correlates. The correlation is compared at time zero and at a small time δ. This correspond to comparing how the sequence of events correlates for a configuration, and a small perturbation of the same configuration. The perturbation is obtained by independently for each edge flip its state with very small probability. If the correlation of the sequence of events between the two configurations tends to zero as the region of the graph increases, this indicates that the information kept in the originating configuration is quickly lost. The sequence of events is judged sensitive to noise.

The connection between dynamical percolation and sensitivity to noise is apparent, but even more so, studying sensitivity of small perturbations of certain sequences of events renders the possibility to conclude that dynamical percolation experiences exceptional events that has zero probability of occurring at any fixed given time.

1.5 Thesis layout

The above rather loose introduction to percolation theory was meant to motivate further study of the field. Both first-passage percolation and the Poisson Boolean model are studied further in this thesis. A brief summary of the papers in this thesis is given next.

Paper I The behaviour of first-passage percolation in two and higher dimensions is still not well understood. In Paper I, first-passage percolation is considered on graphs that are essentially 1-dimensional. The 1-dimensional structure enables the analysis of the process to be simplified considerably, and its behaviour to be described more precisely.

Paper II One of the main results on first-passage percolation is the Shape Theorem. The result describes the almost sure evolution of the wet region on the $\mathbb{Z}^d$ lattice. Paper II generalizes this result to cone-like subgraphs of the lattice, and in addition discusses a few other modes of convergence. In particular, the effect of simple dynamics when introduced in a similar manner as in dynamical percolation is studied.

Paper III The Poisson Boolean model is studied from a perspective of how small perturbations affect the existence of connected components that intersect the sides of large boxes. This essentially amounts to generalizing techniques used to study similar phenomena in discrete cases. The perturbation that is intended can be visualized as follows. A configuration of discs in the plane of a predetermined density is assumed present before time is started. As time starts, new discs rain down from the sky, at the same time as discs on the ground disappear after spending a random time on the ground. The rate at which discs appear and disappear is balanced so that the density of discs at the ground is kept constant. Given the similarity between bond percolation and the Poisson Boolean model, one may expect that also the Poisson Boolean model will be sensitive to noise in the manner described above. That this is the case is proven in Paper III.

To prepare the reader further for the research papers in this thesis, I will dedicate the following pages to give a more detailed description of the percolation models already presented, those being bond percolation, the Poisson Boolean model and first-passage percolation. In order to get a feeling for what kind of means are taken to study percolation models, I will indicate, and sometimes outline, the proof of certain results. First, bond percolation and the Poisson Boolean model will be discussed. Thereafter, before proceeding to first-passage percolation, a detour will be taken to discuss certain random sequences. Although well-known objects to a probabilist, there are several reasons for this. Familiarity with large scale behaviour of sums of random variables builds up a pleasant framework to which more complicated systems, such as first-passage percolation, can be compared. A few words will be said about renewal sequences, since first-passage percolation can be thought of as a graph theoretical generalization of such. Moreover, the identification of a suitable 1-dimensional renewal sequence will in fact be the key to the analysis carried out in Paper I. Also subadditive sequences, which were fundamental in the early developments in first-passage percolation, will be discussed briefly. In fact, the original study of subadditive stochastic sequences was motivated by first-passage percolation. Several different aspects of first-passage percolation will later be discussed. The focus will be on its large scale behaviour, which is further studied in Paper I and II. In particular, some limitations in the understanding of the model in two and more dimensions will be indicated. Sensitivity to noise and dynamics will be discussed quite closely. The model of dynamical percolation will be introduced more formally as well as the concept of noise sensitivity. The link between them will also be explained in greater detail. The techniques used to study noise sensitivity are quite technical, and some time is therefore spent on putting up the correct framework. An overview of the already existing work on noise sensitivity and dynamical percolation is then presented, since parts of that is what Paper III is built on.

After a shorter summary of the three papers, the second part of this thesis follows, consisting of Paper I, II and III.

Chapter 2

Random spatial structures

The bond percolation model was introduced by Broadbent and Hammersley (1957). A brief description was given above, which I here will elaborate a bit further. Above, the presentation was intentionally a bit informal, but I will in what follows be more precise, with no intention of completeness. For a comprehensive introduction, I refer to Grimmett (1999), or alternatively to Bollobás and Riordan (2006). A more elementary source written in Swedish is Häggström (2004).

2.1 Bond percolation

Bond percolation on the $\mathbb{Z}^d$ lattice, where $d \geq 2$, is obtained of going through each edge of the graph and, independently of all other edges, declare it either 'open' or 'closed' with probability p and $1-p$, respectively, for some $p \in [0,1]$. The reason that $d = 1$ is excluded is that only trivial behaviour occurs. As a probabilist one is interested in the qualitative behaviour of the resulting random structure. A path between two vertices of the graph is, after the declaration of edges as open or closed, referred to as *open* if all its edges are open. Any two point in the graph are said to belong to the same *open component* if there is an open path between them. Hence, the declaration of edges as open or closed partitions the vertices of the graph into (connected) open components. Is it possible that the random structure contains an infinite open component? How many infinite components can there be?

Since the existence of an infinite open component cannot depend on the state (open or closed) of finitely many edges, is follows immediately from Kolmogorov's 0-1 law that for each $p \in [0,1]$, the probability that an infinite component exists is either 0 or 1. When an infinite open component exists we say the the system *percolates* at p. Let $\mathcal{C}$ denote the open component that contains the origin, or equivalently, the set of vertices that can be reached via open paths from the origin. Define the percolation function as

$$\theta_d(p) := \mathbb{P}_p\big(|\mathcal{C}| = \infty\big),$$

where $\mathbb{P}_p$ denotes the probability measure that independently for each edge declares it open or closed with probability p and $1 - p$, respectively. Due to lattice symmetry, there is no restriction in considering the open component at the origin as opposed to an open component positioned at any other vertex. Since each vertex of the graph is equally likely to be contained in an infinite open component, the almost sure existence of such coincides with the $\theta_d(p)$ being positive.

Given two values $p_1 < p_2$, can the system percolate at p_1, but not at p_2? This is not the case, which can be seen via a simple coupling argument. Couplings of random elements is a frequently used technique in the area. Coupling two random elements amounts to defining them on the same probability space in a way that their marginal distributions are unchanged, but enables them to be favourably compared for each realization. The argument runs as follows. Do not declare edges opened or closed, but assign to them independent uniformly distributed random variables on the interval $[0, 1]$. Let ξ_e denote the variable assigned to the edge e. Declare the edge p-open if $\xi_e \leq p$. Note that the set of p-open edges corresponds to the set of open edges when each edge independently has been declared open with probability p. Since each p_1-open edge also is p_2-open, we can conclude that the if an infinite open component exists almost surely at density p_1, then the same holds at density p_2. In fact, the argument implies the stronger statement that $\theta_d(p)$ is non-decreasing.

As already mentioned, the almost sure existence of an infinite open component coincides with the function $\theta_d(p)$ being positive. Clearly, $\theta_d(0) = 0$ and $\theta_d(1) = 1$. Since $\theta_d(p)$ was seen to be non-decreasing, there must exist a threshold $p_c(d) \in [0, 1]$ such that, almost surely, for $p < p_c(d)$ no infinite open component may exist, but for $p > p_c(d)$ it does. If $p_c(d)$ is either 0 or 1, then nothing interesting really happens. This is the case when $d = 1$, but not in higher dimensions.

Theorem 2.1. *For each $d \geq 2$, $0 < p_c(d) < 1$.*

In addition, several infinite open components cannot coexist.

Theorem 2.2 (Aizenman, Kesten, and Newman (1987))**.** *For any $d \geq 2$ and $p \in [0, 1]$, the number of infinite open components is either 0 or 1, almost surely.*

Non-triviality of the percolation threshold is a central and simple result in percolation theory, whose argument is instructive to see. There is a fairly elementary proof of the uniqueness of the infinite component which is due to Burton and Keane (1989). Since the uniqueness is not essential for the thesis, I omit the general proof, but will below present a short proof for $d = 2$ which is due to Harris (1960).

Proof Theorem 2.1, lower bound. A lower bound on $p_c(d)$ is given rather easily via a counting argument. Observe that if the open component at the origin is infinite, then there has to be a path starting at the origin consisting of n unique edges which are all open. There are at most $2d(2d-1)^{n-1}$ such paths, since from the origin we must take n steps, and cannot pass the same edge twice. The probability that all edges in one such path are declared open is p^n. Hence, the probability that there exists a path from the origin that contains n disjoint edges which are all open is at most $\frac{2d}{2d-1}[(2d-1)p]^n$. This holds for all $n \geq 1$. Hence, for all $p < (2d-1)^{-1}$

$$\theta_d(p) \leq \frac{2d}{2d-1}\big[(2d-1)p\big]^n \to 0, \quad \text{as } n \to \infty.$$

Thus, $\theta_d(p) = 0$ for small $p > 0$, which proves the lower bound in Theorem 2.1. $\square$

The $\mathbb{Z}^2$ lattice can be embedded in the $\mathbb{Z}^d$ lattice, for any $d \geq 3$. Hence, via a coupling argument similar to the above one, if there exists an infinite open component at p for $d = 2$, almost surely, then so must be the case for any $d \geq 2$. More than that,

$$p_c(2) \geq p_c(3) \geq p_c(4) \geq \dots.$$

Actually, the inequalities are strict, but that takes a greater effort to prove.

Since $p_c(d) \leq p_c(2)$ for all $d \geq 3$, in order to prove that $p_c(d) < 1$ for all $d \geq 2$, it suffices to do so for $d = 2$. To obtain an upper bound, a similar counting argument as the above one is carried out, but this time counting sets of closed edges blocking the existence of an infinite component at the origin. Doing so, a central rôle is played by a spatial duality of the two-dimensional lattice. This duality has far reaching consequences and has been of particular importance in the study of two-dimensional percolation models. It is equally important in discrete as in continuum percolation models, and will appear in the study of the two-dimensional Poisson Boolean model in Paper III. This calls for a proper presentation.

2.2 Duality of the square lattice and RSW techniques

The *dual* graph of the $\mathbb{Z}^2$ lattice is the graph obtained when centring a node on each facet of the lattice, and connecting each node with the nodes that belong to the neighbouring four facets. Note that each edge in the dual graph crosses precisely one edge in the original graph. Hence, the dual graph is identical to the $\mathbb{Z}^2$ lattice, only shifted in space by 1/2 in each coordinate direction. Let each bond in the dual lattice be declared open if the bond it crosses in the original lattice is declared closed, and vice versa. Sets of closed edges in the lattice that limit the open component at the origin corresponds to open paths in the dual. In particular, it is easily realized that if there is an open circuit, i.e., an open path with the same starting as endpoint, in the dual lattice that surrounds the origin of the original lattice, then the open component at the origin (of the lattice) can only consist of vertices on the inside of the dual circuit. Hence, the open component is finite. Moreover, absence of an open dual circuit surrounding the origin implies that the open component at the origin is infinite.

Proof of Theorem 2.1, upper bound. To derive an upper bound on $p_c(d)$, one can proceed as follows. Counting the number of dual circuits surrounding the origin (there are at most $n3^n$ of length n; the factor n is the number of choices of its rightmost point) one conclude that for $p > 2/3$ the expected number of open circuits surrounding the origin (at most $\sum_{n\geq 1} n3^n(1-p)^n$) is finite, and can be made arbitrarily small by picking p larger. Thus, for $p < 1$ sufficiently large the probability of an open dual circuit surrounding the origin is less than 1, which implies $\theta_2(p) > 0$. □

In two dimensions, a more complete and balanced picture of the critical phenomena is known. The duality is the key behind this.

Theorem 2.3 (Harris (1960) and Kesten (1980))**.** *For $d = 2$,*

$$\mathbb{P}_p\big(\exists \text{ an infinite open component}\big) = \begin{cases} 1, & \text{for } p > 1/2, \\ 0, & \text{for } p \leq 1/2. \end{cases}$$

In particular, the result says that $p_c(2) = 1/2$, and that at $p_c(2)$ all open component are almost surely finite. That $\theta_2(1/2) = 0$ is intuitively reasonable to believe, since the contrary would imply the coexistence of an infinite open component in the lattice with one in its dual. Also in higher dimensions it is believed that no infinite component should exist at the critical probability. However, this is known only for $d \geq 19$, due to Hara and Slade (1994). Which is the case for $d = 3$ is probably the most well-known open problem in percolation theory.

That $\theta_2(1/2) = 0$ was proved by Harris, which implies that $p_c(2) \geq 1/2$. Only much later could Kesten show that $p_c(2) \leq 1/2$ based on, at the time, recent work of Russo (1978) and Seymore and Welsh (1978). The techniques developed by Russo, Seymour and Welsh has proven to be a useful tool and provides additional knowledge about the spatial structure of the infinite component. In order to introduce parts of their work, I will turn attention to crossings of rectangles by open paths.

Let $H_{m\times n}$ denote the event that there exists an horizontal open crossing of the rectangle $[0, m] \times [0, n]$. That is, $H_{m\times n}$ denotes the event that there is an open path from some vertex in $\{0\} \times [0, n]$ to a vertex in $\{m\} \times [0, n]$, which is contained in the restriction of the $\mathbb{Z}^2$ lattice to the rectangle $[0, m] \times [0, n]$. In addition, let $V^*_{m\times n}$ denote the event that there is an open path in the dual lattice crossing the rectangle $[1/2, m-1/2] \times [-1/2, n+1/2]$ vertically. An important consequence of the duality is that $H_{m\times n}$ occurs if and only if $V^*_{m\times n}$ does *not* occur. In particular, an immediate consequence is that for any $p \in [0, 1]$, and integers $m, n \geq 1$

$$\mathbb{P}_p(H_{m\times n}) + \mathbb{P}_p(V^*_{m\times n}) = 1.$$

Furthermore, due to similarity between rectangles, and the fact that a bond in the dual graph is open if and only if the corresponding bond in the original graph is closed, one realizes that $\mathbb{P}_p(V^*_{m\times n}) = \mathbb{P}_{1-p}(H_{(n+1)\times(m-1)})$. For $p = 1/2$ and $m = n + 1$, it follows immediately that

$$\mathbb{P}_{1/2}\big(H_{(n+1)\times n}\big) = \mathbb{P}_{1/2}\big(V^*_{(n+1)\times n}\big) = \frac{1}{2}, \quad \text{for all } n \geq 1. \tag{2.1}$$

This demonstrates the balance between the $\mathbb{Z}^2$ lattice and its dual at $p = 1/2$. In fact, for any other value of $p \neq 1/2$, the probability of the event $H_{(n+1)\times n}$ tends to either 0 or 1 as n increases. The existence of crossings of arbitrarily large boxes at $p = 1/2$ may seem surprising in the light of Harris' result that $\theta_2(1/2) = 0$. However, since the existence of dual crossings is likewise implied, this is precisely what is needed to guarantee the existence of an open circuit in the dual lattice limiting the open component at the origin. Such circuits can be constructed based on the techniques due to Russo, Seymour and Welsh. The principal result can be stated as follows.

Theorem 2.4 (RSW Theorem)**.** *For every $\delta > 0$ there exists $\epsilon > 0$ such that for any $p \in (0, 1)$ and $n \geq 1$,*

$$\mathbb{P}_p(H_{n\times n}) \geq \delta \quad \textit{implies} \quad \mathbb{P}_p(H_{3n\times n}) \geq \epsilon.$$

Although it may seem easy to believe that having a reasonable probability of crossing a square would imply a reasonable probability of a crossing of a rectangle, the proof requires a fairly creative construction. For a proof, consult either Grimmett (1999) or Bollobás and Riordan (2006). Theorem 2.4 is itself not essential for the proof of Theorem 2.3 (see e.g. Grimmett (1999)). However, I will present a proof of Harris' part of Theorem 2.3 based thereon.

When $p = 1/2$, (2.1) and Theorem 2.4 implies that $\mathbb{P}_{1/2}(H_{3n\times n}) \geq c$ uniformly in n, for some $c > 0$. Let C_n denotes the event that there is an open circuit contained in the annuli $[-3n, 3n] \setminus [-n, n]$ that surrounds the origin. Crossings of rectangles are positively correlated events, according to Harris' inequality, also known as the FKG-inequality. In particular, this allows a lower bound on the event C_n in terms of the simultaneous occurrence of crossings of four rectangles. This is possible by tiling the annulus $[-3n, 3n]^2 \setminus [-n, n]^2$ by two rectangles of dimension $3n \times n$, and two of dimension $n \times 3n$. If each such rectangle contains an open crossing between its shorter sides, then the annulus contains an open circuit. Consequently, $\mathbb{P}_{1/2}(C_n) \geq c^4$ uniformly in $n \geq 1$. Let me sketch how $\theta_2(1/2) = 0$ can be obtained from this.

Proof of Theorem 2.3, part $\theta_2(1/2) = 0$. Choose a subsequence of the sequence $C_1, C_2, \ldots$ of events which are mutually independent. This will be the case e.g. when $n = 3^k$ for $k = 1, 2, \ldots$, since then the events are defined on disjoint parts of the lattice. Each event has the same (positive) probability to occur, so the Borel-Cantelli lemma assures that there will be infinitely many open circuits surrounding the origin, almost surely. This was in the original lattice. But, if the same argument is run in the dual, the existence of an (and even infinitely many) open dual circuit that surrounds the origin will follow analogously. This proves that $\theta_2(1/2) = 0$. □

As mentioned above, also Kesten's part of the proof that $p_c(2) = 1/2$ is based on the work of Russo, Seymour and Welsh. However, the argument is more involved and will not be presented here. Instead, observe that the argument used to prove that $\theta_2(1/2) = 0$ has more to say about the random structure at $p = 1/2$. It shows that around each point of the lattice there will be a nested sequence of open paths in the original lattice and in the dual, one containing the other. Moreover, each finite box centred at the origin will be surrounded by open circuits in both the lattice and its dual. This is the sufficient information we need in order to conclude uniqueness of the infinite open component in two dimensions.

Proof of Theorem 2.2, for $d = 2$. Note that if each finite box has probability one of being surrounded by an open circuit in the lattice at $p = 1/2$, then the existence of such an open circuit has probability one for all $p \geq 1/2$. For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}^2$, let $\Lambda(\mathbf{x}, \mathbf{y})$ denote the smallest box that contains $\mathbf{x}$ and $\mathbf{y}$. Observe that $\mathbf{x}$ and $\mathbf{y}$ can pertain to different infinite open clusters only if $\Lambda(\mathbf{x}, \mathbf{y})$ is not surrounded by an open circuit. As argued, this has probability zero. Summing over all pair of vertices in $\mathbb{Z}^2$ gives that

$$\mathbb{P}_p\big(\text{more than 1 infinite open component}\big) = 0, \quad \text{for all } p \in [0,1]. \qquad \square$$

If a little more care is taken when carrying out the above argument used to prove $\theta_2(1/2) = 0$, an upper bound on the so called 'one-arm' event is obtained. The *one-arm event* AE_n is the event that there exists an open path connecting the origin to the boundary of the box $[-n, n]^2$, i.e., $\{\mathbf{z} \in \mathbb{Z}^d : \|\mathbf{z}\|_\infty = n\}$. Note that AE_n fails to occur if there is an open circuit in the dual, surrounding the origin and contained entirely within $[-n, n]^2$. In turn, this occurs if there is an open dual circuit in an annuli of the form $[-3^k, 3^k]^2 \setminus [3^{k-1}, 3^{k-1}]^2$, for some $k \geq 1$ such that $3^k \leq n$. There are about $\log n / \log 3$ such annuli, each of which, independently of the other, has probability at least c to contain an open dual circuit (for some $c > 0$). Hence, if AE_n occurs, then each of these annuli has to fail to contain an open circuit. This leads to the upper bound

$$\mathbb{P}_{1/2}(\mathrm{AE}_n) \;\leq\; (1 - c)^{\log n / \log 3} \;=\; n^{-\alpha}, \tag{2.2}$$

for some $\alpha > 0$.

2.3 Poisson Boolean model

The Poisson Boolean model was introduced by Gilbert (1960) and can be seen as a continuum analogue to the bond (or rather site) percolation model. The behaviour of Gilbert's model is qualitatively similar to its discrete relatives. For this reason, I will keep the presentation concise and restricted to the two-dimensional case. It is in two dimensions the Poisson Boolean model will be studied in Paper III. In the two dimensional continuum model, $\mathbb{R}^2$ is partitioned into 'occupied' and 'vacant' space by randomly placing unit discs in the plane. Here, the randomness will come from the discs being placed in correspondance with the points of a Poisson point process. Rather informally, a *Poisson point process* η in $\mathbb{R}^2$ of intensity $\lambda \geq 0$ is a random subset of $\mathbb{R}^2$ such that

a) for disjoint Borel sets $B_1, \ldots, B_n \subseteq \mathbb{R}^2$, then $\eta \cap B_1, \ldots, \eta \cap B_n$ are independent.

b) for every Borel set $B \subseteq \mathbb{R}^2$ with Lebesgue measure $\nu(B) < \infty$,

$$\mathbb{P}\big(|\eta \cap B| = k\big) \;=\; e^{-\lambda\nu(B)} \frac{\lambda^k \nu(B)^k}{k!}, \quad \text{for } k = 1, 2, 3, \ldots$$

Alternatively, one can construct a Poisson point process in $\mathbb{R}^2$ by partitioning the plane into unit squares and, for each square independently, place a Poisson distributed number of points uniformly.

Let η be a Poisson point process in $\mathbb{R}^2$ of intensity $\lambda \geq 0$. Centre at each Poisson point a unit disc. Let $D(\eta)$ denote the union of these discs, that is

$$D(\eta) := \big\{x \in \mathbb{R}^2 : \mathrm{dist}(x, \eta) \leq 1\big\},$$

where $\mathrm{dist}(x, A) = \inf_{a \in A} |x - a|$. $D(\eta)$ is referred to as the occupied region, and the "Swiss cheese" $\mathbb{R}^2 \setminus D(\eta)$ as the vacant region. Equivalently, at least from a connectivity perspective, we can think of the occupied region as the random graph embedded in $\mathbb{R}^2$, with vertex set given by the Poisson point process and where any two vertices at distance at most 2 are joined by an edge.

Both the occupied and the vacant region will consist of connected components. Let $\mathcal{D}$ denote the connected component in the occupied region that contains the origin. If the origin lies in the vacant region, then $\mathcal{D} = \emptyset$. Define the percolation function

$$\theta_G(\lambda) := \mathbb{P}_\lambda(\mathcal{D} \text{ is unbounded}).$$

Similar to the percolation function for bond percolation, also $\theta_G(\lambda)$ is seen to be non-decreasing via a simple coupling argument. If $\lambda_1 < \lambda_2$, then the conclusion is drawn from the comparison of a Poisson process of intensity λ_1 with the super-positioning of that process with an independent Poisson process of intensity $\lambda_2 - \lambda_1$. It is well-known that the super-positioned process has a Poisson distribution of intensity λ_2. The critical density λ_c is defined as

$$\lambda_c := \inf\{\lambda \geq 0 : \theta_G(\lambda) > 0\}.$$

The critical density is known to be non-trivial, that is $0 < \lambda_c < \infty$. The upper bound on λ_c is easily obtained by comparing the continuum model to site percolation on the $\mathbb{Z}^2$ lattice. Site percolation was not discussed in this text, but behaves in a similar way as bond percolation. In particular, for p, the probability of a site being open, close to 1, the existence of an infinite open component of neighbouring sites has probability 1 to occur. Thus, to prove that λ_c is finite, discretize the plane into a square grid of side length $1/\sqrt{2}$. Note that if a square of the grid contains a Poisson point, then the entire square is contained in the occupied region. If the intensity of the Poisson process is sufficiently large, then each square will, independently of one another, contain a Poisson point with probability $p = p(\lambda)$ close to 1. Hence, almost surely, there exists an infinite component of neighbouring squares which each contain a Poisson point. But, the existence of such a component implies the existence of an infinite sequence of overlapping discs. Hence, an unbounded occupied component exists in the continuum model for sufficiently large λ.

To prove that for small λ the occupied component containing the origin is finite almost surely can be seen via a comparison of the Poisson points in $\mathcal{D}$ and a suitable branching process. The reader familiar with branching processes can easily complete the argument.

In the Poisson Boolean model, vacant space serves as dual to occupied space. Since the two regions have different geometry, the balance witnessed in (2.1) for the bond model will not hold here. Other than that, the duality can be used to derive a similar picture of the status of a possible infinite connected region.

Theorem 2.5. *The critical probability λ_c satisfies $0 < \lambda_c < \infty$ and distinguishes three regimes.*

a) In the subcritical regime $\lambda < \lambda_c$, there exists a unique unbounded vacant component, but no unbounded occupied component, almost surely.

b) In the supercritical regime $\lambda > \lambda_c$, there exists no unbounded vacant component, but a unique unbounded occupied component, almost surely.

c) At criticality, there is almost surely no unbounded occupied nor vacant component.

This result summarizes the state of affairs and is due to work of Hall, Roy, Meester and Alexander. Instead of presenting a full list of references, I refer to the works of Meester and Roy (1996) and Alexander (1996). The techniques used to prove this result are similar to those indicated above for percolation on the lattice. However, additional difficulties arise due to the random positioning of the vertices in the continuum. Further difficulties arise when considering discs with random radii. I have here restricted attention to discs of fixed radii. The more general case is treated in detail in the book by Meester and Roy (1996).

Chapter 3

Random sequences

Real-valued random sequences, and in particular sequences of i.i.d. random variables, have been extensively studied during the 20th century. Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables, set $S_0 := 0$ and denote its partial sums by $S_n := X_1 + X_2 + \ldots + X_n$, for $n \geq 1$. The sequence $\{S_n\}_{n\geq 0}$ of partial sums is often referred to as a random walk. Certain special cases of random walks are especially well known. A *simple random walk* is a random walk where the increments X_k, for $k \geq 1$, takes on the values -1 and 1 with equal probability. If the increments are non-negative, then the random walk is known as a *renewal sequence*. There are many classical results regarding random walks, and some of the most well-known concern the asymptotic behaviour of the sequence of partial sums. Let $\mu := \mathbb{E}[X_k]$ and $\sigma^2 := \mathrm{Var}(X_k)$.

Theorem 3.1 (Law of Large Numbers). *If $\mu < \infty$, then*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu, \quad \textit{almost surely.}$$

Theorem 3.2 (Central Limit Theorem). *If $\sigma^2 < \infty$, then*

$$\frac{S_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \chi, \quad \textit{in distribution,}$$

as $n \to \infty$, where χ has a standard normal distribution.

Theorem 3.3 (Law of the Iterated Logarithm). *If $\sigma^2 < \infty$, then*

$$\limsup_{n\to\infty} \frac{S_n - \mu n}{\sigma\sqrt{2\log\log n}} = 1, \quad \textit{almost surely.}$$

Loosely speaking, the Law of Large Numbers states that the average among the n first increments S_n/n is close to the mean μ when n is large, whereas the

Central Limit Theorem describes how S_n/n is distributed around the mean, and the Law of the Iterated Logarithm the magnitude of the fluctuations of S_n/n away from the mean. However, there are many situations in which it is not the sequence of partial sums itself, but rather some quantities that can be derived therefrom, that is the object of interest. A couple of such situations will be described next.

In renewal theory, for $k \geq 1$ the non-negative variables X_k are thought of as lifetimes, and S_k is referred to as renewal times. The main object of interest is the *renewal counting process* $\{N(t)\}_{t\geq 0}$ where $N(t)$ counts the number of renewals in the interval $(0, t]$, that is,

$$N(t) := \max\{n : S_n \leq t\}.$$

Renewal theory is concerned with the inverse problem of understanding the number of occurrences of events during certain time intervals. If the renewal sequence marks the arrival of customers to a queue, then $N(t)$ counts the number of arrivals until time t. Note that for a renewal sequence with exponentially distributed waiting times, the renewal counting process $\{N(t)\}_{t\geq 0}$ is a Poisson process on $[0, \infty)$.

Depending on the context, we may instead be interested in the position (value) of a random walk, not at fixed time point, but at the occurrence of certain events.

Example 3.4. To continue the example of customers in a queue, let $\{X_k\}_{k\geq 1}$ denote the inter-arrival times between the customers, and let $\{Y_k\}_{k\geq 1}$ denote their respective service times. For planing purposes, we may be interested in the service time required to serve all customers arriving in the interval $[0, t]$. As $N(t)$ counts the arrivals in the interval $[0, t]$, the quantity of interest is $Y_1 + Y_2 + \ldots + Y_{N(t)}$. □

As a second example, I will present a situation that will appear in Paper I of this thesis.

Example 3.5. Imagine we are interested in the asymptotic behaviour of the some random sequence $\{T_n\}_{n\geq 1}$, but which is not of the simple form a random walk is, i.e., does not have i.i.d. increments $T_k - T_{k-1}$. In some cases it is possible to identify a random subsequence $\{\rho_n\}_{n\geq 1}$ of the index set, for which the distribution of $\{T_{n+\rho_k} - T_{\rho_k}\}_{n\geq 1}$ does not depend on k, and the increments $\{T_{\rho_{n+1}} - T_{\rho_n}\}_{n\geq 1}$ are i.i.d. In this case, $\{T_{\rho_n}\}_{n\geq 1}$ is a random walk, and $\{T_n\}_{n\geq 1}$ is sometimes referred to as a *regenerative sequence*, as it starts anew at certain instances. One way to obtain such a sequence is to associate the sequence $\{\rho_n\}_{n\geq 1}$ to the occurrence of a suitably chosen event. In order for the identification of an embedded random walk to of any help, it has to provide information about the original sequence. In particular, if $\{T_n\}_{n\geq 1}$ has non-negative increments, then $\{T_{\rho_n}\}_{n\geq 1}$ is a renewal sequence, and for

$$\nu(n) := \min\{k \geq 1 : \rho_k \geq n\},$$

then $T_{\rho_{\nu(n)-1}} \leq T_n \leq T_{\rho_{\nu(n)}}$. □

In Paper I the approach in the above example is found favourable in the application to first-passage percolation, where both sequences $\{\rho_n\}_{n\geq 1}$ and $\{T_{\rho_n}\}_{n\geq 1}$ will be renewal sequences.

In general, this leads to the question, given some asymptotic property of a sequence $\{Y_n\}_{n\geq 1}$, what is required for $\{\lambda_n\}_{n\geq 1}$ in order to say something about $\{Y_{\lambda_n}\}_{n\geq 1}$? This will be discussed next.

3.1 Stopped random walks

The asymptotic properties of i.i.d. sequences are particularly well documented, and considerable efforts have been made to extend results concerning their partial sums to random subsequences thereof (see e.g. Gut (2009)). As above, let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables, and let $\{S_n\}_{n\geq 0}$ denote its partial sums. Moreover, let $\{\lambda_n\}_{n\geq 1}$ is a sequence of non-negative integer-valued random variables. The sequence $\{S_{\lambda_n}\}_{n\geq 1}$ is referred to as a *stopped random walk*, where the term 'stopped' comes from the fact that λ_n often is a stopping time, but this restriction is not necessary in general.

In some cases a result for stopped random walks is an easy consequences of the corresponding result for the sequence of partial sums. Assume that $\lambda_n \to \infty$ as $n \to \infty$ almost surely. Then, if $\{Y_n\}_{n\geq 1}$ is a sequence such that, almost surely, $Y_n \to Y$ as $n \to \infty$, then also $Y_{\lambda_n} \to Y$ as $n \to \infty$. In particular, as an immediate consequence of the Law of Large Numbers we obtain that

$$\lim_{n\to\infty} \frac{S_{\lambda_n}}{\lambda_n} = \mu, \quad \text{almost surely.}$$

The Central Limit Theorem does not extend as easily to random subsequences. The difficulty can be illustrated as follows. Assume that $\{S_n\}_{n\geq 1}$ is a simple random walk and let $\{\lambda_n\}_{n\geq 1}$ be the sequence of indices for which the random walk takes negative values. Hence, $S_{\lambda_n}/\sigma\sqrt{\lambda_n}$ is negative for all n, and cannot possibly converge to a normal distribution. Nevertheless, under some additional assumption, the Central Limit Theorem does extend to what is sometimes referred to as *Anscombe's theorem*. For a proof of this theorem I refer to either of two books by Gut (2005, 2009).

Theorem 3.6 (Anscombe's Theorem). *Let $\{X_k\}_{k\geq 1}$ be an i.i.d. sequence with mean μ, finite variance σ^2 and partial sums $\{S_n\}_{n\geq 1}$. Assume further that as $n \to \infty$*

$$\frac{\lambda_n}{n} \xrightarrow{p} \theta, \quad \textit{in probability.}$$

Then, as $n \to \infty$,

$$\frac{S_{\lambda_n} - \mu\lambda_n}{\sigma\sqrt{\lambda_n}} \xrightarrow{d} \chi, \quad \textit{in distribution,}$$

where χ has a standard normal distribution.

Also the Law of the Iterated Logarithm extends to a version for stopped random walks. As above, if $\{S_n\}_{n\geq 1}$ is a simple random walk and $\{S_{\lambda_n}\}_{n\geq 1}$ denotes the subsequence of which the partial sums are negative, then superior limit of $S_{\lambda_n}/\sigma\sqrt{2n\log\log n}$ cannot exceed 0. The necessary additional condition is that $\lim_{n\to\infty} \lambda_n/n \in \mathbb{R}$ exists almost surely.

3.2 Subadditive sequences

When studying more complex random objects, such as the model for spatial growth introduced earlier, one encounters situations where random sequences of more complicated structure need to be understood. This led Hammersley and Welsh (1965) to initiate the study of subadditive stochastic sequences.

Before I proceed, let me take a step back to consider real-valued sequences. A real-valued sequence $\{a_n\}_{n\geq 1}$ is called *subadditive* when

$$a_{m+n} \leq a_m + a_n, \quad \text{for all } m, n \geq 1.$$

Convergence of real-valued subadditive sequences was discovered already by Fekete (1923). In fact, given integers $1 \leq m \leq n$, choose $k \geq 1$ and $0 \leq \ell < m$ such that $n = km + \ell$. It follows from the subadditive property that

$$\inf_{m\geq 1} \frac{a_m}{m} \leq \frac{a_n}{n} \leq \frac{k \cdot a_m + a_\ell}{n} \leq \frac{a_m}{m} + \frac{a_\ell}{n}.$$

Sending $n \to \infty$, we immediately obtain that

$$\exists \lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}. \tag{3.1}$$

This result is commonly known as *Fekete's lemma.*

A collection of random variables $\{X_{m,n}\}_{0\leq m<n}$ is called *subadditive* if

$$X_{\ell,n} \leq X_{\ell,m} + X_{m,n}, \quad \text{for all } \ell < m < n. \tag{3.2}$$

Do subadditive stochastic sequences converge in a similar manner as in (3.1)? This question will be addressed shortly. First, I will present three examples that are suitable to keep in mind for the following discussion.

Example 3.7. Let X be a random variable and define $X_{m,n} := (n-m)X$. Then $\{X_{m,n}\}_{0\leq m<n}$ is subadditive. □

Example 3.8. Let $\{Y_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables. Then $\{S_{m,n}\}_{0\leq m<n}$ is a subadditive sequence, where $S_{m,n}$ denotes the partial sum $S_{m,n} := Y_{m+1} + Y_{m+2} + \ldots + Y_n$. □

Example 3.9. In first-passage percolation on the $\mathbb{Z}^2$ lattice, each edge of the graph is independently assigned a non-negative random variable. The variables are interpreted as the time it takes a fluid to traverse the edges. Denote the time it takes a fluid to reach the vertex $(n, 0)$ when started at $(m, 0)$ by $T_{m,n}$. Then $\{T_{m,n}\}_{0\leq m<n}$ is subadditive, since, intuitively, restricting the fluid to pass the vertex $(m, 0)$ on its way from $(\ell, 0)$ to $(n, 0)$ can only increase its travel time. □

The first two examples are in fact *additive*, meaning that equality holds in (3.2). The third example is the one that led Hammersley and Welsh to initiate the study of subadditive stochastic sequences. In the second example, when the sequence is assumed to have finite mean, the Law of Large Numbers implies that $\lim_{n\to\infty} S_{0,n}/n$ exists almost surely. In fact, for the convergence to hold, it suffices that the sequence $\{Y_n\}_{n\geq 1}$, instead of being i.i.d., is *stationary* in the sense that the distribution of $\{Y_{n+k}\}_{n\geq 1}$ does not depend on $k \geq 0$. This is a consequence of Birkhoff's more general *Ergodic Theorem*.

Under which additional assumptions does sequences satisfying (3.2) converge in a similar manner as in (3.1)? Typically, independence is a too strong assumption, and is not satisfied in Example 3.9. Stationarity is a more adequate assumption. Hammersley and Welsh (1965) worked with the following two additional assumptions.

$$\text{The distribution of } X_{m,n} \text{ depends only on the difference } n - m. \tag{3.3}$$

$$\text{There exists } c < \infty \text{ such that } -cn \leq \mathbb{E}[X_{0,n}] < \infty, \text{ for all } n \geq 1. \tag{3.4}$$

Each of the three examples presented above satisfy assumption (3.3) and (3.4), given that finite mean are assumed. For now, let $\{X_{m,n}\}_{0\leq m<n}$ be a sequence satisfying (3.2), (3.3) and (3.4). Note that $g_n := \mathbb{E}[X_{0,n}]$ is subadditive, so (3.1) directly gives that

$$\exists\, \gamma := \lim_{n\to\infty} \frac{\mathbb{E}[X_{0,n}]}{n}.$$

Further Hammersley and Welsh (1965) showed that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_{0,n}}{n} \leq \gamma\right) = 1 \tag{3.5}$$

is a sufficient condition to conclude that

$$\limsup_{n\to\infty} \frac{X_{0,n}}{n} = \gamma \quad \text{almost surely,} \quad \text{and} \quad \lim_{n\to\infty} \frac{X_{0,n}}{n} = \gamma \quad \text{in probability.}$$

Moreover, they showed that (3.5) is satisfied if for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ and an i.i.d. sequence $\{Y_n\}_{n\geq 1}$ such that $\mathbb{E}[Y_n] \leq k(\gamma + \epsilon)$ and

$$X_{0,kn} \;\leq\; Y_1 + Y_2 + \ldots + Y_n, \quad \text{for all } n \geq 1.$$

In Example 3.8 this condition is trivially met, and they managed to show that it is also met in Example 3.9. In Example 3.7 condition (3.5), and the following conclusions fail to hold, unless X is constant.

I will end this section with a comment on the fluctuations of a subadditive sequence. In Example 3.7 $\mathrm{Var}(X_{0,n}) = n^2\,\mathrm{Var}(X)$, whereas in Example 3.8 $\mathrm{Var}(S_{0,n}) = n\,\mathrm{Var}(Y_1)$. This indicates that the properties (3.2), (3.3) and (3.4) allows for quite different behaviour to occur. When $X_{m,n}$ is non-negative, and $\mathbb{E}[X_{0,1}^2] < \infty$, then $\mathrm{Var}(X_{0,n})$ can easily be bounded from above by $\mathbb{E}[X_{0,1}^2]n^2$. This is realized by squaring both sides and taking expectations in the inequality

$$X_{0,n} \;\leq\; X_{0,1} + X_{1,2} + \ldots, X_{n-1,n}.$$

In general, this cannot be improved significantly as Example 3.7 shows. Hammersley and Welsh showed that $\mathrm{Var}(X_{0,n})/n^2$ vanishes as $n \to \infty$, given that the sequence $\{X_{m,n}\}_{0\leq m<n}$ can be dominated by a certain less correlated sequence.

3.3 The Subadditive Ergodic Theorem

An important improvement upon the results of Hammersley and Welsh allows for almost sure and in L^1-convergence to be deduced. To obtain such a result (3.3) is exchanged for a stronger stationarity assumption. The result is due to Kingman (1968), also him motivated by first-passage percolation. Since then, other situations have appeared in which subadditive sequences do not meet Kingman's assumptions. An alternative formulation with somewhat relaxed conditions was later provided by Liggett (1985). Before presenting the precise result, it is necessary to introduce a few additional concepts.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\varphi_k : \mathbb{R}^{\mathbb{Z}_+} \to \mathbb{R}^{\mathbb{Z}_+}$ denote the shift operator that maps $(x_1, x_2, \ldots)$ to $(x_{k+1}, x_{k+2}, \ldots)$. Recall that a real-valued sequence of random variables $Y = \{Y_n\}_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called stationary if the distribution of $\varphi_k(Y) = \{Y_{n+k}\}_{n\geq 1}$ does not depend on $k \geq 0$. It is immediate that an i.i.d. sequence is stationary. An event $A \in \mathcal{F}$ is *invariant* with respect to Y if there exists a Borel set $B \subseteq \mathbb{R}^{\mathbb{Z}_+}$ such that $A = \{\omega \in \Omega : \varphi_k(Y) \in B\}$ for all $k \geq 0$. Finally, a stationary sequence Y is called *ergodic* if all invariant sets (with respect to Y) has measure either 0 or 1.

Example 3.10. Again, an i.i.d. sequence is a simple example of an ergodic stationary sequence. To see this, note that if A is invariant, then A is determined by $\varphi_k(Y)$ for each $k \geq 0$, i.e., $A \in \sigma(Y_{k+1}, Y_{k+2}, \ldots)$ for each $k \geq 0$. Hence, Kolmogorov's 0-1 law gives that A has measure either 0 or 1. □

An easy way to generate further ergodic stationary sequences is to pick an existing ergodic stationary sequence $Y = \{Y_n\}_{n\geq 1}$, and a measurable function $g : \mathbb{R}^{\mathbb{Z}_+} \to \mathbb{R}$; the sequence $\{Z_n\}_{n\geq 1}$ given by $Z_n := g\big(\varphi_n(Y)\big)$ is stationary and ergodic. I will come back to this below. First, I present Liggett's version of Kingman's *Subadditive Ergodic Theorem*.

Theorem 3.11 (Subadditive Ergodic Theorem)**.** *Let $\{X_{m,n}\}_{0\leq m<n}$ be a collection of random variables satisfying*

a) $X_{0,n} \leq X_{0,m} + X_{m,n}$, for all $0 < m < n$.

b) The distribution of the sequence $\{X_{m,m+k}\}_{k\geq 1}$ does not depend on $m \geq 0$.

c) The sequence $\{X_{km,(k+1)m}\}_{k\geq 1}$ is stationary for each $m \geq 0$.

d) For all n, $\mathbb{E}\big[|X_{0,n}|\big] < \infty$ and $\mathbb{E}[X_{0,n}] \geq -cn$, for some $c < \infty$.

Then, the following conclusions hold

e) $\exists\, \gamma := \lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}[X_{0,n}] = \inf_{n\geq 1} \frac{1}{n}\,\mathbb{E}[X_{0,n}]$.

f) $\exists\, X := \lim_{n\to\infty} \frac{X_{0,n}}{n}$, almost surely and in L^1, where $\mathbb{E}[X] = \gamma$.

Moreover, if all sequences in c) are ergodic, then $X = \gamma$ almost surely.

All three of Example 3.7 to 3.9 satisfy the conditions of the Subadditive Ergodic Theorem. The first two are immediate. Also the third, to which this theorem is of particular importance, is easily verified (see Proposition 4.1 below).

In percolation theory, one often deals with families of i.i.d. random variables indexed by the vertices or edges of a lattice. This is the case in both bond percolation and first-passage percolation. Arguments making use of ergodicity are common. The concepts of stationarity, invariance and ergodicity extends to families $Y = \{Y_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}$ of random elements, in terms of the shift operator $\varphi_{\mathbf{y}}$ that maps $\{x_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}$ to $\{x_{\mathbf{y}+\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}$, for $\mathbf{y} \in \mathbb{Z}^d$. Of course, an i.i.d. family is stationary, and can also be seen to be ergodic.

Let $\{Y_e\}_{e\in\mathcal{E}}$ be a family of random variables indexed by the edges $\mathcal{E}$ of the $\mathbb{Z}^d$ lattice. Let $\mathbf{Y}_\mathbf{z}$ denote the d-dimensional random vector consisting of the random variables associated with the d edges extending (in positive direction) from the vertex $\mathbf{z}$. In this way $\{Y_e\}_{e\in\mathcal{E}}$ corresponds to $\{\mathbf{Y}_\mathbf{z}\}_{\mathbf{z}\in\mathbb{Z}^d}$, and it is possible to talk about stationarity and ergodicity of the former family in terms of the latter. In particular, when the elements of $\{Y_e\}_{e\in\mathcal{E}}$ are i.i.d., also $\{\mathbf{Y}_\mathbf{z}\}_{\mathbf{z}\in\mathbb{Z}^d}$ is an i.i.d. family, and hence, both stationary and ergodic.

Example 3.12. Quite informally, an event A is invariant with respect to Y if from a realization of Y it is possible to decide whether A occurs or not, without knowing the position of the origin. In bond percolation, typical examples of such events are:

a) Existence of an infinite open component.

b) Existence of precisely $k \in \mathbb{N}$ infinite open components.

By ergodicity, both these events has measure either 0 or 1. $\square$

Additional ergodic stationary families can be constructed from known ones, as a consequence of the next simple result which is mentioned without proof. Although stated for families of real-valued random variables, the result holds also for more general random elements.

Proposition 3.13. *If $Y = \{Y_\mathbf{z}\}_{\mathbf{z}\in\mathbb{Z}^d}$ is stationary and ergodic, and $g : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ measurable, then the family $Z = \{Z_\mathbf{z}\}_{\mathbf{z}\in\mathbb{Z}^d}$ given by $Z_\mathbf{z} := g\big(\varphi_\mathbf{z}(Y)\big)$ is stationary and ergodic.*

Often, we are interested only in a sub-family of variables in of the family Z obtained from Y. Of course, the sub-family will as well be stationary. However, it is not necessarily ergodic. In the application of the Subadditive Ergodic Theorem to first-passage percolation, stationary sequences arise in this way. In this case, a more direct argument can be used to obtain ergodicity.

Chapter 4

First-passage percolation

It is time to describe the stochastic growth model known as first-passage percolation in greater detail. Attention will be restricted to the lattice case, i.e., the case where the discrete structure is taken to be the $\mathbb{Z}^d$ nearest neighbour graph, for some $d \geq 2$. This model has been extensively studied in the literature, and was introduced by Hammersley and Welsh (1965). Let $\mathcal{E}$ denote the set of edges of the $\mathbb{Z}^d$ lattice, and let $\{\tau_e\}_{e\in\mathcal{E}}$ denote a collection of i.i.d. non-negative random variables associated with the edges, referred to as *passage times*. Define the passage time of a path Γ as $T(\Gamma) := \sum_{e\in\Gamma} \tau_e$. (Here, and at other places, a path is identified with its set of edges.) In particular, we are interested in the travel time, also referred to as *passage time* or *first-passage time*, between two vertices $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}^d$, which is defined as

$$T(\mathbf{x}, \mathbf{y}) := \inf \big\{T(\Gamma) : \Gamma \text{ is a path from } \mathbf{x} \text{ to } \mathbf{y}\big\}.$$

As mentioned before, first-passage percolation is often motivated as a model for the spatial propagation of a fluid when injected at the origin of the lattice. The term passage time reflects the interpretation of the random variables as the time needed for a fluid to traverse the edge. Similarly, first-passage times (between two points) are commonly interpreted as the time it would take a fluid injected at one point to reach another. Relevant questions aim to understand the spatial growth of the fluid injected at the origin of the lattice. How far will the fluid reach in fixed time intervals? How does the number of wet sites grow in time? What can be said about the shape of the region of wet vertices? All these questions concern the central object defined as

$$\mathcal{W}_t := \{\mathbf{z} \in \mathbb{Z}^d : T(\mathbf{0}, \mathbf{z}) \leq t\}, \quad \text{for } t \geq 0,$$

and interpreted as the *wet region* at time t.

In renewal theory, the renewal counting process counts occurrences of the associated renewal sequence. Analogously, the wet region is the corresponding inverse quantity associated to the sequence of first-passage times $\big\{T(\mathbf{0},\mathbf{z})\big\}_{\mathbf{z}\in\mathbb{Z}^d}$. In this sense, first-passage percolation can be seen as a generalization of a renewal process to graphs. As in renewal theory, investigating how first-passage times behave is essential in order to understand how the wet region evolves in time. However, the known picture is still far from complete. My aim is to give a short introduction that is relevant for the contributions made in this thesis. A survey on the early developments in first-passage percolation is given by Smythe and Wierman (1978). Another extensive presentation is given by Kesten (1986), whereas a more recent reference is the survey by Howard (2004).

Particular efforts have been invested in studying the propagation of the fluid in coordinate directions. If $\mathbf{e}_1 \in \mathbb{Z}^d$ denotes the unit vector along the first coordinate axis, this corresponds to studying the sequence $\big\{T(\mathbf{0}, n\mathbf{e}_1)\big\}_{n\geq 1}$. Basic questions about first-passage times were studied already in Hammersley and Welsh (1965). Under which conditions, and in which sense does $T(\mathbf{0}, n\mathbf{e}_1)/n$ converge as $n \to \infty$? Is the expected travel time $\mathbb{E}\big[T(\mathbf{0}, n\mathbf{e}_1)\big]$ increasing in n? What can be said about $\operatorname{Var}\big(T(\mathbf{0}, n\mathbf{e}_1)\big)$? Does $T(\mathbf{0}, n\mathbf{e}_1)$ exhibit a central limiting behaviour when scaled properly?

First-passage times have a considerably more complex dependence structure than renewal sequences. However, as defined they are easily seen to be subadditive, i.e.,

$$T(\mathbf{x},\mathbf{y}) \leq T(\mathbf{x},\mathbf{z}) + T(\mathbf{z},\mathbf{y}), \quad \text{for any } \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{Z}^d. \tag{4.1}$$

Since the distribution of $T(\mathbf{x}, \mathbf{x}+\mathbf{y})$ is independent of $\mathbf{x} \in \mathbb{Z}^d$, it follows immediately from Fekete's lemma that

$$\exists\, \mu_{\mathbb{Z}^d} := \lim_{n\to\infty} \frac{\mathbb{E}\big[T(\mathbf{0}, n\mathbf{e}_1)\big]}{n}.$$

As a consequence of their study of subadditive stochastic sequences, Hammersley and Welsh (1965) were able to prove that whenever $\mathbb{E}[\tau_e] < \infty$, then

$$\begin{aligned} \limsup_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} &= \mu_{\mathbb{Z}^d} \quad \text{almost surely,} \\ \lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} &= \mu_{\mathbb{Z}^d} \quad \text{in probability.} \end{aligned} \tag{4.2}$$

In order to prove this they showed that (3.5) holds by dominating $T(n\mathbf{e}_1, m\mathbf{e}_1)$ by the passage time of paths between $n\mathbf{e}_1$ and $m\mathbf{e}_1$ restricted to cylinders of the form $\{\mathbf{z} \in \mathbb{Z}^d : n < z_1 \leq m\}$ (except for the first vertex of the path). An important advancement came with Kingman's already mentioned Subadditive

Ergodic Theorem (Theorem 3.11). When applied to the sequence of first-passage times $\{T(m\mathbf{e}_1, n\mathbf{e}_1)\}_{0\le m<n}$, we obtain the following, where

$$Y := \min(\tau_1, \tau_2, \dots, \tau_{2d}), \tag{4.3}$$

and $\tau_1, \tau_2, \dots, \tau_{2d}$ are independent and distributed as τ_e.

Proposition 4.1. *Whenever* $\mathbb{E}[Y] < \infty$*,*

$$\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} = \mu_{\mathbb{Z}^d}, \quad \textit{almost surely and in } L^1.$$

Proof. Conditions *a)*, *b)* and *c)* of the Subadditive Ergodic Theorem are immediate from (4.1) and translation invariance of the underlying i.i.d. structure of the lattice. Alternatively, stationarity of the sequeces in *c)* can be obtained as a consequence of Proposition 3.13, which gives that $\{T(\mathbf{y}, \mathbf{y}+\mathbf{z})\}_{\mathbf{y}\in\mathbb{Z}^d}$ is stationary and ergodic for every $\mathbf{z} \in \mathbb{Z}^d$. (That $T : \mathbb{R}^{\mathcal{E}} \to \mathbb{R}$ is measurable is easily seen; an argument was given by Hammersley and Welsh (1965).) However, this does not imply ergodicity of the sequences in *c)* directly. Instead, a more direct argument will be presented below.

First, let us see that *d)* holds. It suffice to show that $\mathbb{E}\left[T(\mathbf{0}, \mathbf{e}_1)\right] < \infty$, since $0 \le \mathbb{E}\left[T(\mathbf{0}, n\mathbf{e}_1)\right] \le n\,\mathbb{E}\left[T(\mathbf{0}, \mathbf{e}_1)\right]$, due to *a)* and *b)*. Between $\mathbf{0}$ and $\mathbf{e}_1$ there are $2d$ disjoint paths of length at most 9. Let Γ denote the longest of those paths. Then

$$\mathbb{P}\big(T(\mathbf{0}, \mathbf{e}_1) > s\big) \;\le\; \mathbb{P}\big(T(\Gamma) > s\big)^{2d} \;\le\; 9^{2d}\,\mathbb{P}(\tau_e > s/9)^{2d} \;=\; 9^{2d}\,\mathbb{P}(Y > s/9),$$

where the second inequality holds because if $T(\Gamma) > s$, then at least one of the 9 edges has $\tau_e > s/9$. Since $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\,dx$ for nonnegative random variables, $\mathbb{E}\left[T(\mathbf{0}, \mathbf{e}_1)\right] < \infty$ holds. Hence, the conditions of the Subadditive Ergodic Theorem are satisfied, and the limit $\lim_{n\to\infty} T(\mathbf{0}, n\mathbf{e}_1)/n$ exists almost surely and in L^1.

It remains to show that the limit is constant. Let Λ_k denote the box of side length $2k$ centered at the origin. With a slight abuse of notation, let $T(\Lambda_k, n\mathbf{e}_1) := \min_{\mathbf{z}\in\Lambda_k} T(\mathbf{z}, n\mathbf{e}_1)$. It is clear that

$$T(\Lambda_k, n\mathbf{e}_1) \;\le\; T(\mathbf{0}, n\mathbf{e}_1) \;\le\; T(\Lambda_k, n\mathbf{e}_1) + \sum_{e\in\Lambda_k} \tau_e.$$

In particular, we conclude that for every $k \ge 0$

$$\exists \lim_{n\to\infty} \frac{T(\Lambda_k, n\mathbf{e}_1)}{n} = \lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n}, \quad \text{almost surely and in } L^1. \tag{4.4}$$

However, since $T(\Lambda_k, n\mathbf{e}_1)$ does not depend on τ_e for $e \in \Lambda_k$, the limit in (4.4) cannot either do so. This holds for all $k \geq 0$, which shows that the limit cannot depend on any finite collection of passage times. As a consequence of Kolmogorov's 0-1 law, it has to be constant. $\square$

From Proposition 4.1 we obtain the propagation of the fluid in coordinate directions. Similarly, the Subadditive Ergodic Theorem applies to the sequence $\{T(\mathbf{0}, n\mathbf{z})\}_{n\geq 1}$ for any $\mathbf{z} \in \mathbb{Z}^d$. It is for practical purposes handy to extend the the definition of passage times between vertices to pairs of points in $\mathbb{R}^d$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, define $T(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}^*, \mathbf{y}^*)$ where $\mathbf{x}^*$ and $\mathbf{y}^*$ denotes the points in $\mathbb{Z}^d$ closest to $\mathbf{x}$ and $\mathbf{y}$, respectively (choosing the points closest to the origin in case of a tie, say). Whenever $\mathbb{E}[Y] < \infty$, it is in fact possible (although not immediate from the Subadditive Ergodic Theorem) to show that for any $\mathbf{x} \in \mathbb{R}^2$

$$\exists\, \mu_{\mathbb{Z}^d}(\mathbf{x}) := \lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{x})}{n}, \quad \text{almost surely and in } L^1. \tag{4.5}$$

The limit $\mu_{\mathbb{Z}^d}(\mathbf{x})$ in (4.5) is referred to as the *time constant.*

Hammersley and Welsh (1965) further conjectured that the expected travel time $\mathbb{E}\left[T(\mathbf{0}, n\mathbf{e}_1)\right]$ is monotonic in n. Renewal sequences in the classical sense are, but $\{T(\mathbf{0}, n\mathbf{e}_1)\}_{n\geq 1}$ is not a renewal sequence. Despite the intuitive appeal the conjecture has, van den Berg (1983) constructed an example that essentially shows that the conjecture is false for small n. It remains an open problem to find out whether the expected travel time could be monotonic for sufficiently large n.

4.1 The Shape Theorem

The convergence in (4.5) describes the spatial growth of the process in any fixed direction. To understand the growth of the wet region, the convergence in (4.5) need to be concluded in all directions simultaneously. This can be obtained, and was first realized by Cox and Durrett (1981), inspired by a result of Richardson (1973). In terms of first-passage times, their result can be stated as whenever $\mathbb{E}[Y^d] < \infty$

$$\limsup_{\mathbf{z}\in\mathbb{Z}^d} \left| \frac{T(\mathbf{0}, \mathbf{z}) - \mu_{\mathbb{Z}^d}(\mathbf{z})}{|\mathbf{z}|} \right| = 0, \quad \text{almost surely.} \tag{4.6}$$

(In fact, the convergence also holds in L^1, as seen in Paper II, but I do not want to focus on that here.) Equivalently, (4.6) can be stated in terms of the wet region. Just as first-passage times were extended to pairs of points in $\mathbb{R}^d$, it is convenient to replace $\mathcal{W}_t$, which is a subset of $\mathbb{Z}^d$, with a corresponding subset of $\mathbb{R}^d$. Let

$$\widetilde{\mathcal{W}}_t := \big\{\mathbf{x} \in \mathbb{R}^d : T(\mathbf{0}, \mathbf{x}) \leq t\big\}, \quad \text{for } t \geq 0.$$

Cox and Durrett's result can then be described as how closely $\widetilde{\mathcal{W}}_t$ resembles the set

$$\mathcal{W}^* := \big\{\mathbf{x} \in \mathbb{R}^d : \mu_{\mathbb{Z}^d}(\mathbf{x}) \leq 1\big\}.$$

The geometric properties of the set $\mathcal{W}^*$ can be divided into two regimes. As will be seen shortly,

$$\mathcal{W}^* \text{ is compact, convex and has non-empty interior}, \quad \text{when } \mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0, \tag{4.7}$$

$$\mathcal{W}^* = \mathbb{R}^d, \quad \text{when } \mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0. \tag{4.8}$$

Formulated in terms of the wet region, (4.6) is known as the *Shape Theorem* and this is indeed how it was first described by Cox and Durrett.

Theorem 4.2 (Shape Theorem). *Consider first-passage percolation on the $\mathbb{Z}^d$ lattice with i.i.d. passage times such that $\mathbb{E}\left[Y^d\right] < \infty$, for Y defined as in (4.3). If $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$, then, for all $\epsilon > 0$, almost surely,*

$$(1-\epsilon)\mathcal{W}^* \subset \frac{1}{t}\widetilde{\mathcal{W}}_t \subset (1+\epsilon)\mathcal{W}^*, \quad \textit{for } t \textit{ large enough.} \tag{4.9}$$

If $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0$, then for every compact set K in $\mathbb{R}^d$, almost surely,

$$K \subset \frac{1}{t}\widetilde{\mathcal{W}}_t, \quad \textit{for } t \textit{ large enough.}$$

In the regime $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$ the Shape Theorem, in combination with (4.7), states that the wet region grows with linear speed. It is clear that $\mathcal{W}^*$ is bounded and has non-empty interior is necessary for (4.9) to hold. Except for convexity, it has turned out very hard to prove further characteristics of $\mathcal{W}^*$ in the same regime. My next aim is to prove that (4.7) and (4.8) indeed hold. After that, an argument showing that (4.6) implies the Shape Theorem will be presented.

4.2 The time constant and asymptotic shape

As a start, one would like to characterize the two regimes in the Shape Theorem, that is, when $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$ and not. Kesten (1986) showed that $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0$

if and only if $\mathbb{P}(\tau_e = 0) \geq p_c(d)$, where $p_c(d)$ denotes the percolation threshold for bond percolation on the $\mathbb{Z}^d$ lattice. The precise geometry of the asymptotic shape $\mathcal{W}^*$ is not known. In addition to (4.7) and (4.8), it is easily seen that $\mathcal{W}^*$ has to be symmetric with respect to reflexion in coordinate axis, due to the corresponding symmetry in the lattice. In the following paragraphs, I will show how (4.7) and (4.8) can be obtained as a consequence of a few simple properties of the time constant. The following properties of $\mu_{\mathbb{Z}^d}(\cdot)$ will be used:

$$\mu_{\mathbb{Z}^d}(a\mathbf{x}) = a\mu_{\mathbb{Z}^d}(\mathbf{x}), \qquad \text{for all } a \geq 0 \text{ and } \mathbf{x} \in \mathbb{R}^d, \tag{4.10}$$

$$\mu_{\mathbb{Z}^d}(\mathbf{x}+\mathbf{y}) \leq \mu_{\mathbb{Z}^d}(\mathbf{x}) + \mu_{\mathbb{Z}^d}(\mathbf{y}), \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{4.11}$$

$$\big|\mu_{\mathbb{Z}^d}(\mathbf{x}) - \mu_{\mathbb{Z}^d}(\mathbf{y})\big| \leq d\,\mathbb{E}\left[T(\mathbf{0},\mathbf{e}_1)\right]|\mathbf{x}-\mathbf{y}|, \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \tag{4.12}$$

These properties holds in either regime, and are only subject to the restriction that $\mathbb{E}[Y] < \infty$, in order for $\mu_{\mathbb{Z}^d}(\cdot)$ to be well-defined. How these properties can be derived will be indicated later.

Proof of (4.7). The asymptotic shape $\mathcal{W}^*$ is convex in both regimes. To see this, note that $\mathbf{x}$ is contained in $\mathcal{W}^*$ if and only if $\mu_{\mathbb{Z}^d}(\mathbf{x}) \leq 1$. Thus, if $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathcal{W}^*$, and $\lambda \in (0,1)$, then also $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ belongs to $\mathcal{W}^*$, since according to (4.10) and (4.11)

$$\mu_{\mathbb{Z}^d}\big(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\big) \;\leq\; \lambda\mu_{\mathbb{Z}^d}(\mathbf{x}) + (1-\lambda)\mu_{\mathbb{Z}^d}(\mathbf{y}) \;\leq\; 1.$$

The remaining two properties of $\mathcal{W}^*$ when $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$ can be deduced with help of the convexity. First, note that by (4.10), there are $a > 0$ and $b < \infty$ such that $\mu_{\mathbb{Z}^d}(a\mathbf{e}_1) < 1$ and $\mu_{\mathbb{Z}^d}(b\mathbf{e}_1) > 1$. Together with convexity and reflexion symmetry of $\mathcal{W}^*$, the former implies that $\mathcal{W}^*$ has non-empty interior, whereas the latter that $\mathcal{W}^*$ is bounded. To prove compactness, it remains to conclude that $\mathcal{W}^*$ is closed. However, that is immediate from the continuity of $\mu_{\mathbb{Z}^d}(\cdot)$ in (4.12). □

Proof of (4.8). To conclude that $\mathcal{W}^* = \mathbb{R}^d$ when $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0$, it suffice to prove that either $\mu_{\mathbb{Z}^d}(\cdot) \equiv 0$, or $\mu_{\mathbb{Z}^d}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \neq \mathbf{0}$. Assume that the latter is not the case. First, assume that $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0$, for which it follows that $\mu_{\mathbb{Z}^d}(\mathbf{e}_j) = 0$ for each $j = 1, 2, \ldots, d$ by symmetry. That $\mu_{\mathbb{Z}^d}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^d$ is now immediate from (4.10) and (4.11). In general, if $\mu_{\mathbb{Z}^d}(\mathbf{x}) = 0$ for some $\mathbf{x} \neq \mathbf{0}$, then we can in a similar fashion, via reflexion, obtain d vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ such that $\mu_{\mathbb{Z}^d}(\mathbf{x}_j) = 0$ for each $j = 1, 2, \ldots, d$ and which together span $\mathbb{R}^d$. Again, that $\mu_{\mathbb{Z}^d}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^d$ is immediate from (4.10) and (4.11). □

Hence, (4.7) and (4.8) have been deduced from (4.10), (4.11) and (4.12), which remain to be justified. I will not present all details here, but only indicate why the properties hold. For $a \in \mathbb{N}$, (4.10) follows from (4.5) since

$$\mu_{\mathbb{Z}^d}(a\mathbf{x}) = a \lim_{n\to\infty} \frac{\mathbb{E}\left[T(\mathbf{0}, an\mathbf{x})\right]}{an} = a\mu_{\mathbb{Z}^d}(\mathbf{x}).$$

This extends to all $a \geq 0$ via a comparison of $\mathbb{E}\left[T(\mathbf{0}, an\mathbf{x})\right]$ and $\mathbb{E}\left[T(\mathbf{0}, \lfloor an \rfloor \mathbf{x})\right]$, where $\lfloor \cdot \rfloor$ denotes the integer part. The difference is easily seen to be bounded.

For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}^d$, (4.11) follow directly from (4.1) and (4.5), and a similar comparison can be made to extend (4.11) to arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. For the final property (4.12), note that

$$\begin{aligned}
\left| \frac{\mathbb{E}\left[T(\mathbf{0}, n\mathbf{x})\right]}{n} - \frac{\mathbb{E}\left[T(\mathbf{0}, n\mathbf{y})\right]}{n} \right| &\leq \left| \frac{\mathbb{E}\left[T(n\mathbf{x}, n\mathbf{y})\right]}{n} \right| \\
&\leq \frac{1}{n}\, \mathbb{E}\left[T(\mathbf{0}, \mathbf{e}_1)\right] \left\| (n\mathbf{x})^* - (n\mathbf{y})^* \right\|_1 \\
&\leq \frac{d}{n}\, \mathbb{E}\left[T(\mathbf{0}, \mathbf{e}_1)\right] \left| (n\mathbf{x})^* - (n\mathbf{y})^* \right|.
\end{aligned}$$

Sending n to infinity, we are able to conclude that (4.12) holds.

I will next proceed with the equivalence between the Shape Theorem and (4.6). The proof will for obvious reasons in addition depend on the properties of $\mu_{\mathbb{Z}^d}(\cdot)$ demonstrated above. I will only prove that (4.6) implies the Shape Theorem. This is the more relevant of the two implications, since Paper II repeatedly deals with expressions of the form (4.6). That the Shape Theorem implies (4.6) is easily deduced in a similar fashion.

Proposition 4.3. *(4.6) implies Theorem 4.2.*

Proof. Case $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0$. It suffices to show that for any $M < \infty$, almost surely, all $\mathbf{z} \in \mathbb{Z}^d$ satisfying $|\mathbf{z}| \leq tM$ are included in $\mathcal{W}_t$, for sufficiently large t. According to (4.6) (since $\mu_{\mathbb{Z}^d}(\cdot) \equiv 0$) there is $K = K(M) < \infty$ such that $T(\mathbf{0}, \mathbf{z}) \leq |\mathbf{z}|/M$ for all $|\mathbf{z}| \geq K$, almost surely. Pick M and fix K accordingly. By construction, $T(\mathbf{0}, \mathbf{z}) \leq t$ when $K \leq |\mathbf{z}| \leq tM$. Since only finitely many points in $\mathbb{Z}^d$ have $|\mathbf{z}| < K$, it is possible to choose $t < \infty$ such that $T(\mathbf{0}, \mathbf{z}) \leq t$ for all such points, almost surely.

Case $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$. Pick $\epsilon > 0$ and fix

$$\delta = \min\left(\frac{\epsilon}{\sup_{\mathbf{x}\in\mathcal{W}^*} |\mathbf{x}|}, \frac{\epsilon}{1+\epsilon} \inf_{\mathbf{x}:\,|\mathbf{x}|=1} \mu_{\mathbb{Z}^d}(\mathbf{x}) \right).$$

Importantly, observe that $\delta > 0$, since $\mathcal{W}^*$ is bounded and $\mu_{\mathbb{Z}^d}(\cdot)$ is bounded away from zero on the compact set $|\mathbf{x}| = 1$. According to (4.6), choose $K = K(\delta) < \infty$ such that for all $|\mathbf{x}| \geq K$

$$\mu_{\mathbb{Z}^d}(\mathbf{x}) - \delta|\mathbf{x}| \ \leq \ T(\mathbf{0}, \mathbf{x}) \ \leq \ \mu_{\mathbb{Z}^d}(\mathbf{x}) + \delta|\mathbf{x}|, \quad \text{almost surely.}$$

It suffices to prove that for sufficiently large t

$$(1-\epsilon)t\,\mathcal{W}^* \ \subseteq \ \widetilde{\mathcal{W}}_t \ \subseteq \ (1+\epsilon)t\,\mathcal{W}^*, \quad \text{almost surely.}$$

Let me begin with the lower inclusion, or more precisely, for $\mathbf{x} \in \mathcal{W}^*$, set $\mathbf{y} = (1-\epsilon)t\mathbf{x}$ and prove that $T(\mathbf{0}, \mathbf{y}) \leq t$ for sufficiently large t, almost surely. For $|\mathbf{y}| \geq K$,

$$T(\mathbf{0}, \mathbf{y}) \ \leq \ \mu_{\mathbb{Z}^d}(\mathbf{y}) + \delta|\mathbf{y}| \ \leq \ (1-\epsilon)t + \epsilon t.$$

Since $t_0 := \sup_{\mathbf{y}:\, |\mathbf{y}|<K} T(\mathbf{0}, \mathbf{y})$ is in fact a maximum over finitely many points in $\mathbb{Z}^d$, it is almost surely finite. Therefore, the lower inclusion holds for all $t \geq t_0$, almost surely. For the upper inclusion, it suffices to show that for large t, if $\mathbf{y} \in \widetilde{\mathcal{W}}_t$, then $\mu_{\mathbb{Z}^d}(\mathbf{y}) \leq (1+\epsilon)t$, almost surely. For $|\mathbf{y}| \geq K$,

$$\frac{1}{1+\epsilon}\mu_{\mathbb{Z}^d}(\mathbf{y}) \ = \ \mu_{\mathbb{Z}^d}(\mathbf{y}) - \frac{\epsilon}{1+\epsilon}\mu_{\mathbb{Z}^d}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)|\mathbf{y}| \ \leq \ \mu_{\mathbb{Z}^d}(\mathbf{y}) - \delta|\mathbf{y}| \ \leq \ T(\mathbf{0}, \mathbf{y}) \ \leq \ t.$$

In addition, for $|\mathbf{y}| < K$ we will clearly have $\mu_{\mathbb{Z}^d}(\mathbf{y}) \leq t$ for t large, due to continuity of $\mu_{\mathbb{Z}^d}(\cdot)$. Therefore, also the upper inclusion holds for all large t, almost surely. □

As a final remark, let me say that in addition to (4.12), the time constant is continuous also in other respects. Cox (1980) and Cox and Kesten (1981) have showed that $\mu_{\mathbb{Z}^d}(\mathbf{x})$ varies continuously, for each $\mathbf{x} \in \mathbb{R}^d$, with respect to weak convergence of the passage time distribution. This result needs a greater effort in order to deduce. Essentially, the approach is to compare time constants via a coupling between different distributions. The same approach is used in Paper I, where a similar result is proved in an easier setting.

4.3 Shape fluctuations

Complementing results to the Shape Theorem have so far not been obtained with the same precision. While the expected travel time grows linearly with the distance, the variance of the travel time is believed to have sub-linear growth. The best available bounds on the variance of $T(\mathbf{0}, n\mathbf{e}_1)$ are still not sharp. These bounds generally assume that $\mathbb{E}[\tau_e^2] < \infty$, but are individually subject to additional restrictions. Recall that for a classical renewal sequence, the variance is a linear function in the number of waiting times. In contrast, Hammersley and Welsh (1965) were able, from their early work on subadditive sequences, to conclude that $\mathrm{Var}\big(T(\mathbf{0}, n\mathbf{e}_1)\big)/n^2$ vanishes as n tends to infinity. This was improved upon by Kesten (1993). He showed that for any $d \geq 2$, if in addition $\mathbb{P}(\tau_e = 0) < p_c(d)$, then there are constants $C_1 > 0$ and $C_2 < \infty$ such that

$$C_1 \leq \mathrm{Var}\big(T(\mathbf{0}, n\mathbf{e}_1)\big) \leq C_2 n, \quad \text{for all } n \geq 1. \tag{4.13}$$

Considerable improvements of Kesten's result have so far not been obtained, except for in special cases. In two dimensions, physicists predict that the variance is of order $n^{2/3}$. This is the same order of magnitude as other related planar growth models. Based on this relation, there is even an indication of which limiting distribution to expect when properly scaled. In higher dimensions the picture is even less clear. Newman and Piza (1995) offer a short summary of simulation studies, some of which suggest that the variance might in fact be bounded in sufficiently high dimensions.

The best available bound for $d = 2$ essentially states that for some $C_1 > 0$ and $C_2 < \infty$

$$C_1 \log n \leq \mathrm{Var}\big(T(\mathbf{0}, n\mathbf{e}_1)\big) \leq C_2 \frac{n}{\log n}, \quad \text{for all } n \geq 2.$$

The lower bound is due to Newman and Piza (1995) under the additional assumption that the passage timed distribution does not have a too large point mass at

$$\lambda := \inf\big\{x \geq 0 : \mathbb{P}(\tau_e \in [0, x]) > 0\big\}. \tag{4.14}$$

For the exponential distribution the same bound was obtained simultaneously and independently by Pemantle and Peres (1994). The upper bound is valid for all $d \geq 2$, and was first obtained in a paper by Benjamini, Kalai, and Schramm (2003b) for $\{a, b\}$-valued passage times, where $0 < a < b < \infty$. It was later extended by Benaïm and Rossignol (2008) to a larger class of passage time distributions.

Both Kesten's result and the result due to Newman and Piza are based on a representation of first-passage times in terms of martingale differences. Interestingly, the approach by Benjamini et al. is based on techniques used to study Boolean functions, which have close connection to the study of noise sensitivity (to be discussed in Section 5).

4.4 Minimizing paths

First-passage times between vertices are defined as the infimum of passage times of an infinite number of paths. Given $\mathbf{z} \in \mathbb{Z}^d$, it is therefore a justified question whether there always is a path γ between the origin and $\mathbf{z}$ such that $T(\gamma) = T(\mathbf{0}, \mathbf{z})$. Such path is known to exist as long as the passage times are not assigned according to a distribution with point mass at zero as large as $p_c(d)$. This is easily verified for distributions concentrated to an interval $[a, b]$ for some $0 < a < b < \infty$, since then only finitely many paths can come into consideration. The minimizing path γ is commonly referred to as a *geodesic* or a *route*.

Not much is known about the spatial properties of geodesics. A natural property to study is the length of a geodesic compared to the distance between its endpoints. Let $N(\mathbf{x}, \mathbf{y})$ denote the length of the geodesic between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}^d$. The length is not well-defined if the geodesic is not unique. When several geodesics between $\mathbf{x}$ and $\mathbf{y}$ exist, we let $N(\mathbf{x}, \mathbf{y})$ refer to the shortest such. The sequence $\{N(\mathbf{0}, n\mathbf{e}_1)\}_{n\geq 1}$ is intimately related to the sequence $\{T(\mathbf{0}, n\mathbf{e}_1)\}_{n\geq 1}$, but does not benefit from a subadditive behaviour. Consequently, no precise asymptotic behaviour of $N(\mathbf{0}, n\mathbf{e}_1)/n$ is known, except for in restricted cases. One such case is when edges are given the values 1 or ∞ with probability p and $1 - p$ for some $p > p_c(d)$. This case was previously related to the bond percolation model, and indeed $N(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}, \mathbf{y})$ here. Since $p > p_c(d)$ is assumed, we know that there almost surely exists an infinite component of edges assigned passage time 1. Assume that the origin is located in the unique infinite cluster of 1-valued edges, and that $\{n_k\}_{k\geq 1}$ denotes the subsequence of indices in $\mathbb{N}$ such that $n_k\mathbf{e}_1$ is contained in the infinite cluster. Garet and Marchand (2004) proved that

$$\exists \lim_{k\to\infty} \frac{T(\mathbf{0}, n_k\mathbf{e}_1)}{n_k}, \quad \text{almost surely.}$$

Another case in which even more specific conclusions regarding the asymptotic behaviour of length of geodesics is achievable is given in Paper I. The restriction in Paper I is in the consideration of graphs that are essentially 1-dimensional.

Another related question asks how far away from the straight line segment between the its two endpoints a geodesic wanders. This deviation can be measured as the maximal distance a point in the geodesic is situated from the straight line segment between its endpoints. As before, let $\mathrm{dist}(x, A) = \inf_{a\in A} |x - a|$. The maximal distance of a point in B to the set A is then given by $\overline{\mathrm{dist}}(B, A) := \sup_{b\in B} \mathrm{dist}(b, A)$. Let γ_n denote the geodesic and let $\overline{n\mathbf{e}_1}$ denote the straight line segment between the origin and $n\mathbf{e}_1$. Hence, the deviation of γ_n from $\overline{n\mathbf{e}_1}$ can be measured as

$$\xi := \inf\left\{\alpha \geq 0 : \lim_{n\to\infty} \mathbb{P}\Big(\overline{\mathrm{dist}}(\gamma_n, \overline{n\mathbf{e}_1}) \leq n^\alpha\Big) = 1\right\}.$$

In two dimensions, there is a general belief that $\xi = 2/3$. Although this is not known, upper and lower bounds on ξ that are valid in all dimensions are available, and say that

$$\frac{1}{d+1} \leq \xi \leq \frac{3}{4}. \tag{4.15}$$

The upper bound on ξ is due to Newman and Piza (1995), under the condition that $\mathbb{E}[Y^d] < \infty$, for Y defined as in (4.3), and the additional (and unproven) assumption that the shape $\mathcal{W}^*$ is not flat in the first coordinate direction. Not being flat here refers to that it is possible to find an Euclidean ball B such that $\mathcal{W}^* \subseteq B$ and the point $\mathbf{y} = \mu_{\mathbb{Z}^d}(\mathbf{e}_1)^{-1}\mathbf{e}_1$ that lies on the boundary of $\mathcal{W}^*$ (since $\mu_{\mathbb{Z}^d}(\mathbf{y}) = 1$) also lies on the boundary of B. The lower bound on ξ was deduced by Licea, Newman, and Piza (1996) under the assumption that $\mathbb{E}[\tau_e^2] < \infty$ and the point mass at λ, as defined in (4.14), is not too large.

Determining the variance of first-passage times relates to the problem of describing the magnitude with which the wet region fluctuates around the asymptotic shape. Although I will not do so here, these fluctuations can be described by an exponent χ in a similar way as ξ describes the deviation of a geodesic. The upper bound $\chi \leq 1/2$ has been obtained essentially as a consequence of the upper bound in (4.13). Krug and Spohn (1991) have conjectured the precise relation $\chi = 2\xi - 1$ between the two exponents. In two dimensions this corresponds to the belief that $\xi = 2/3$ and $\chi = 1/3$. Only partial information regarding this relation has been obtained. As a consequence, the bounds on ξ in (4.15) follows from known bounds on χ.

First-passage percolation apparently presents a complicated structure that is still poorly understood. The techniques available have so far not been successful in pinning down other than the almost sure convergence to an asymptotic shape. It is therefore of great interest to develop new techniques that are better suited for this aim. Moreover, it is of interest to obtain partial results that point in the direction of the conjectured behaviour. In this direction, models similar to first-passage percolation have been introduced, at least partly, with the hope to avoid certain difficulties. The survey of Howard (2004) discusses this further.

First-passage percolation is the topic of the first two papers in this thesis. First, Paper I gives a rather precise description of the asymptotic behaviour of first-passage percolation when considered on graphs that are essentially 1-dimensional. Secondly, the convergence of the wet region towards an asymptotic shape is investigated further in Paper II. The almost sure convergence is extended to certain cone-like subgraphs of the $\mathbb{Z}^d$ lattice. Also additional modes of convergence are considered.

Chapter 5

Sensitivity to noise and dynamics

Let me start with a simple example. Consider a Brownian motion $\{B_t\}_{t\geq 1}$ in one dimension, started at the origin. $\mathbb{P}(B_t = 0) = 0$ for each fixed $t > 0$. Moreover, the set of times at which $B_t = 0$ has Lebesgue measure zero. However, B_t will almost surely hit 0 for some $t > 0$. This shows how the occurance of an event may differ substantially when observed at fixed times, and over a time interval. Although this example does not fit into the description I will give next, it does illustrate the phenomena I would like to present.

Quite generally, the phenomena can be described in terms of Markov processes. Let $\{X_t\}_{t\geq 0}$ be a stationary continuous time Markov process. Let A be a subset of its state space for which $\mathbb{P}(X_t \in A) = 1$ for any $t \geq 0$. Via Fubini's theorem, we may change the order of integration, and thus obtain

$$\mathbb{P}\big(X_t \in A \text{ for Lebesgue almost every } t \geq 0\big) = 1.$$

The question is, can this be extended to hold for *all* $t \geq 0$, or are there *exceptional times* at which X_t avoids A. When extendible to all $t \geq 0$, then the event $\{X_t \in A\}$, considered over time, is said to be *dynamically stable.*

Only very specific dynamical processes will be considered in the following. Time dynamics can be introduced also to a static system, in order to observe its behaviour over time. In particular, I will focus on a dynamical version of bond percolation. As will be emphasised, sensitivity in the dynamical percolation model has a close relation to the concept of noise sensitivity. Noise sensitivity is an interesting concept in its own right, and the connection to dynamical percolation adds to the interest further. Existing techniques available when studying noise sensitivity are further developed in Paper III. This motivates a quite detailed introduction to the earlier work on the subject. For a further account on dynamical percolation and noise sensitivity, see Steif (2009) and Garban and Steif (2010).

5.1 Dynamical percolation

Häggström, Peres, and Steif (1997) introduced *dynamical percolation.* Let $\mathcal{G}$ be a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and declare each edge of the graph independently open or closed with probability p and $1-p$. So far this is solely the bond percolation model. Next, assign independent Poisson clocks to the edges of the graph, and when the clock of an edge rings, let the edge refresh its state, i.e., update its state as open or closed with probability p and $1-p$, respectively. Formally, the model can be defined as follows. For each edge $e \in \mathcal{E}$, let $\{\tau_e^{(j)}\}_{j\geq 1}$ be an i.i.d. sequence of rate 1 exponentially distributed random variables, and let $\{\eta_e^{(j)}\}_{j\geq 1}$ be i.i.d. random variables that indicates the state open and closed with probability p or $1-p$. For each edge the stationary state process $\big\{\eta_e(t)\big\}_{t\geq 0}$ is defined as

$$\eta_e(t) = \eta_e^{(j)}, \quad \text{for } \tau_e^{(j-1)} \leq t < \tau_e^{(j)},$$

where $\tau_e^{(0)} = 0$ for all e. Since edges are supposed to act independently, the random element $\{\eta_e(t)\}_{e\in\mathcal{E},t\geq 0}$, that describes the state of the dynamical percolation model over time, is obtained via product measure.

Fundamental to bond percolation is the existence of a critical probability $p_c = p_c(\mathcal{G})$ below which no infinite open component exists, and above which one does exist. The natural question in dynamical percolation is whether there exists exceptional times when the existence of an infinite open component is changed. It is not hard to see that away from criticality, the existence of an infinite open component is dynamically stable. That is, for $p < p_c$ there is almost surely no infinite open component for any $t \geq 0$, and for $p > p_c$ there is almost surely an infinite open component for all $t \geq 0$. This is seen from the following argument given by Häggström et al.. Assume that $p < p_c$. Observe each edge under on the time interval $[0, \epsilon]$. The probability that a given edge will be open at *some* point in $[0, \epsilon]$ is at most p plus the probability of an update in $[0, \epsilon]$. If $\epsilon > 0$ is sufficiently small, then this probability is still strictly smaller than p_c. Hence, on any sufficiently short time interval, no infinite component can possibly exist, with probability one. Covering $[0, \infty)$ with countably many intervals of length ϵ, then countable additivity gives the result. The case when $p > p_c$ is similar. Hence, the only interesting case is at criticality.

At criticality, the situation is much more delicate. Häggström et al. constructed examples of graphs where the existence of an infinite open component exhibit exceptional times, and examples where the existence is dynamically stable. These graphs were quite ad hoc, and it is of particular interest to understand what happens for commonly considered graphs as the usual $\mathbb{Z}^d$ lattice. For $d \geq 19$, Häggström et al. showed that there are no exceptional times of percolation. Their proof was based on properties of the percolation function $\theta_d(p)$ that are known not to hold for $d = 2$. In two dimensions the task turned out to be truly challenging, and was only recently resolved in an extensive work of Garban, Pete, and Schramm (2010). Among other things, they proved that bond percolation on the $\mathbb{Z}^2$ lattice exhibits exceptional times of percolation at criticality. Recall that for $d = 3, 4, \ldots, 18$, an infinite open component at the critical probability is believed not to exsist, but remains as an open question.

The existence of an infinite open component is not the only almost sure property in bond percolation. Another is uniqueness of the infinite component in the supercritical regime. On the $\mathbb{Z}^d$ lattice Peres and Steif (1998) proved that for $p > p_c(d)$, there is a unique infinite component at all times. However, the uniqueness of an infinite open component is not the same as excluding the possibility of coexisting infinite open and closed components. Coexistence is believed, but remains unknown, to occur at exceptional times for the $\mathbb{Z}^2$ lattice (at criticality). However, Garban et al. (2010) proved the existence of exceptional times at which an infinite open and closed component coexist for dynamical (site) percolation on the triangular lattice. The *triangular lattice* is the graph obtained from the $\mathbb{Z}^2$ (square) lattice when adding an edge between $\mathbf{z}$ and $\mathbf{z} + (1, 1)$ for any $\mathbf{z} \in \mathbb{Z}^2$. The reason for the more precise results on the triangular lattice is due to the recent development of SLE and its success in determining critical exponents for percolation, which are known for the triangular lattice but not for the square lattice (see e.g. Werner (2009)). Equivalent results are expected also for the square lattice. In addition, the triangular lattice is known to also exhibit exceptional times of percolation at criticality. This was first proved by Schramm and Steif (2010), and later obtained also by Garban et al. (2010).

When real-valued sequences of i.i.d. random variables are observed in a dynamical perspective, similar phenomena occurs, as observed by Benjamini, Häggström, Peres, and Steif (2003a). Dynamics is in this case introduced in the analogous way by, at each position in the sequence, replacing the variable by independent copies of itself according to a Poisson clock. This is done independently for each position. In particular Benjamini et al. proved that classical results such as the Law of Large Numbers and the Law of the Iterated Logarithm are dynamically stable, whereas a simple random walks in 3 or 4 dimensions exhibit exceptional times at which the random walk is recurrent. Inspired by this work, a dynamically stable version of the Shape Theorem in first-passage percolation is proved in Paper II.

5.2 Noise sensitivity

Noise sensitivity of Boolean functions was introduced by Benjamini, Kalai, and Schramm (1999), motivated by the study of exceptional times in dynamical percolation. Let $f_n : \{0,1\}^n \to \{0,1\}$ be a Boolean function, and let $\omega \in \{0,1\}^n$ be uniformly distributed. The function being *Boolean* simply refers to the domain and range of the function, i.e., that given a Boolean sequence, the function returns a Boolean output. Hence, a Boolean function can be thought of as the indicator of a certain event. When 1 is interpreted as 'heads' and 0 as 'tails', ω can be thought of as the outcome of n independent (fair) coin flips. Given the outcome of the coin flips, the outcome of a given event is known. However, assume that we fail to correctly record the outcome of each coin flip with very low probability. Based on this 'perturbed' sequence of coin flips, can we decide whether the event occurs? That is, is the outcome of the event for the original sequence of coin flips highly or weakly correlated with the outcome for the perturbed sequence?

Example 5.1 (Dictatorship). Let $f_n : \{0,1\}^n \to \{0,1\}$ be the indicator function of the event that 'the first flip turns out heads'. Since the first flip alone decides the outcome of the event, the information about the outcome of f_n is lost only when information about the outcome of the first flip is lost. But this has low probability, so f_n should not be sensitive to small perturbations. □

Example 5.2 (Parity). Let $f_n : \{0,1\}^n \to \{0,1\}$ indicate whether 'the number of heads is even' or not. All information about the outcome of the event is lost as soon as we are unsure about the outcome of a single flip. Since it is very likely to be unsure about the outcome of some flip (when n is large), the event will be sensitive to perturbations. □

Here comes the formal definition. Let $\{f_n\}_{n\geq 1}$ be a sequence of Boolean functions $f_n : \{0,1\}^n \to \{0,1\}$, and let $\omega \in \{0,1\}^n$ be uniformly distributed. Given $\epsilon \in (0,1)$, resample each bit (coordinate) ω_j with probability ϵ independently of each other. Denote the resulting configuration ω^ϵ. Also ω^ϵ is uniformly distributed in $\{0,1\}^n$.

Definition 5.3. *The sequence* $\{f_n\}_{n\geq 1}$ *is said to be* noise sensitive *if for every* $\epsilon > 0$

$$\mathbb{E}\left[f_n(\omega)f_n(\omega^\epsilon)\right] - \mathbb{E}\left[f_n(\omega)\right]^2 \to 0, \quad \text{as } n \to \infty. \tag{5.1}$$

Direct calculations easily show that the sequence $\{f_n\}_{n\geq 1}$ is noise sensitive when f_n is as defined in Example 5.2, but is not when defined as in Example 5.1. Another example of a sequence of functions that are not noise sensitive is given next. To see this is also quite easy, however less direct than than in the examples given above.

Example 5.4 (Majority). Let $\{f_n\}_{n\geq 1}$ be the sequence of functions where $f_n : \{0,1\}^n \to \{0,1\}$ is the indicator function of the event 'more heads than tails'. $\square$

The dictator and majority functions are in fact noise stable. A sequence $\{f_n\}_{n\geq 1}$ is said to be *noise stable* if

$$\lim_{\epsilon\to 0}\sup_{n\geq 1}\mathbb{P}\big(f_n(\omega)\neq f_n(\omega^\epsilon)\big)=0.$$

There exist sequences of Boolean functions that are neither noise stable nor noise sensitive. In addition, there are sequences that are both noise sensitive and noise stable. However, this can only happen for trivial reasons.

Proposition 5.5. *The sequence* $\{f_n\}_{n\geq 1}$ *is noise sensitive and noise stable if and only if* $\mathrm{Var}(f_n)\to 0$ *as* $n\to\infty$*.*

Proof. For the 'only if' part, assume that $\{f_n\}_{n\geq 1}$ is both noise sensitive and noise stable. In

$$\mathrm{Var}(f_n) \;=\; \mathbb{E}\left[f_n(\omega)\big(f_n(\omega)-f_n(\omega^\epsilon)\big)\right]+\mathbb{E}\left[f_n(\omega)f_n(\omega^\epsilon)\right]-\mathbb{E}\left[f_n(\omega)\right]^2,$$

the first term in the right-hand side is at most $\mathbb{P}\big(f_n(\omega)\neq f_n(\omega^\epsilon)\big)$, which due to noise stability can be made arbitrarily small by choosing ϵ small. The remaining expression vanishes as $n\to\infty$, since the sequence is noise sensitive. This shows that $\mathrm{Var}(f_n)\to 0$ as $n\to\infty$.

Now, assume that $\mathrm{Var}(f_n)\to 0$ as $n\to\infty$. Clearly, the sequence is noise sensitive since the covariance between $f_n(\omega)$ and $f_n(\omega^\epsilon)$ is bounded by $\mathrm{Var}(f_n)$, via Cauchy-Schwartz' inequality. Moreover, since $\mathrm{Var}(f_n)=\mathbb{P}(f_n=1)\mathbb{P}(f_n=0)$, then

$$\begin{aligned}\mathbb{P}\big(f_n(\omega)\neq f_n(\omega^\epsilon)\big) \;&=\; \mathbb{P}\big(f_n(\omega)=0,f_n(\omega^\epsilon)=1\big)+\mathbb{P}\big(f_n(\omega)=1,f_n(\omega^\epsilon)=0\big)\\ &\leq\; 2\min\big(\mathbb{P}(f_n=1),\mathbb{P}(f_n=0)\big)\end{aligned}$$

can be made arbitrarily small by choosing n large. Thus, the sequence is noise stable since for each fixed $N\in\mathbb{N}$

$$\lim_{\epsilon\to 0}\sup_{n\leq N}\mathbb{P}\big(f_n(\omega)\neq f_n(\omega^\epsilon)\big) \;\leq\; \lim_{\epsilon\to 0}\max_{n\leq N}\mathbb{P}(\omega\neq\omega^\epsilon) \;=\; 0. \qquad \square$$

As previously mentioned, noise sensitivity is closely related to dynamical percolation. To visualize this, consider dynamical percolation on the $\mathbb{Z}^2$ lattice with $p = 1/2$. Recall that $1/2$ corresponds to the critical probability, and is therefore the interesting case. How does the bond configuration $\eta(t) := \{\eta_e(t)\}_{e\in\mathcal{E}}$ at time t relate to the configuration at time zero? Well, $\eta_e(0) = \eta_e(t)$ if the Poisson clock assigned to e does not ring before time t. This has probability e^{-t}. On the event that the clock rings before time t, then $\eta_e(0)$ gives no information about $\eta_e(t)$. Thus, $\eta(0)^\epsilon$ denotes the configuration obtained when each bit (edge) in $\eta(0)$ is resampled, independently of other bits, with probability ϵ, then

$$\big(\eta(0), \eta(t)\big) \stackrel{d}{=} \big(\eta(0), \eta(0)^\epsilon\big), \quad \text{for } \epsilon = 1 - e^{-t},$$

where the superscript indicates that the equality holds in distribution. This gives a clear relation between the dynamical percolation model and perturbations of binary sequences. In particular, this allows for the correlation between nearby time points in dynamical percolation to be understood through the study of noise sensitivity of certain sequences of Boolean functions. This led to study whether percolation crossings of $(n+1)\times n$-boxes are noise sensitive or not.

Example 5.6. Let $\mathcal{E}_n$ denote the set of edges of the square lattice contained in $[0, n+1] \times [0, n]$. Identify $\omega \in \{0,1\}^{\mathcal{E}_n}$ with a configuration of open and closed edges in $\mathcal{E}_n$ by interpreting an edge e as open if $\omega_e = 1$ and closed otherwise. Recall that $H_{(n+1)\times n}$ denotes the event that there is an open path crossing the rectangle $[0, n+1] \times [0, n]$ horizontally, and let f_n be the indicator function of $H_{(n+1)\times n}$. As already seen, $\mathbb{P}(f_n = 1) = 1/2$, and $\mathrm{Var}(f_n) = 1/4$. Thus, if $\{f_n\}_{n\geq 1}$ is noise sensitive, it will not be for the trivial resons of Proposition 5.5. $\square$

The main part of the work of Benjamini et al. (1999) was carried out to prove the following.

Theorem 5.7. *The sequence $\{f_n\}_{n\geq 1}$, as defined in Example 5.6, is noise sensitive.*

The study of Boolean functions has had large benefit from discrete Fourier analysis. Indeed, a large part of the noise sensitivity literature is based on Fourier techniques. Other important ingredients are concepts like influences and revealments more commonly found in theoretical computer science. In particular, in order to prove noise sensitivity of percolation crossings, Benjamini et al. developed a quite general approach based on discrete Fourier analysis and revealment of algorithms. An essential stepping-stone in this approach is a result that links noise sensitivity to influences of bits. The *influence* of bit j for f_n is defined as

$$\mathrm{Inf}_j(f_n) := \mathbb{E}\left[\left|f_n(\omega) - f_n(\sigma_j\omega)\right|\right],$$

where $\sigma_j\omega$ denotes the element obtained when ω_j is replaced by $1-\omega_j$. The result has come to be referred to as the *BKS Theorem.*

Theorem 5.8 (Benjamini et al. (1999))**.** *Let $\{f_n\}_{n\geq 1}$ be a sequence of Boolean functions $f_n : \{0,1\}^n \to \{0,1\}$. If*

$$\lim_{n\to\infty} \sum_{j=1}^{n} \mathrm{Inf}_j(f_n)^2 = 0,$$

then $\{f_n\}_{n\geq 1}$ is noise sensitive.

The proof of this theorem relies on discrete Fourier analysis. Although these techniques only appears briefly in Paper III of this thesis, basic knowledge thereof is essential when working with noise sensitivity of Boolean functions.

5.3 Fourier-Walsh representation and the spectral measure

There is a natural basis for the space $\mathcal{H}_n$ of real-valued functions defined on the hypercube $\{0,1\}^n$. For each $S \subseteq [n] := \{1,2,\ldots,n\}$ and $\omega \in \{0,1\}^n$, let

$$\chi_S(\omega) := \prod_{j\in S}(-1)^{\omega_j} = \begin{cases} 1, & \text{if the number of } j \in S \text{ with } \omega_j = 1 \text{ is even,} \\ -1, & \text{if the number of } j \in S \text{ with } \omega_j = 1 \text{ is odd,} \end{cases}$$

and $\chi_\emptyset := 1$. When $\{0,1\}^n$ is equipped with uniform probability measure, and inner product between two functions f and g in $\mathcal{H}_n$ is thus given by $\langle f, g\rangle := \mathbb{E}\left[f(\omega)g(\omega)\right]$, then $\{\chi_S\}_{S\subseteq[n]}$ forms an orthonormal basis for the 2^n-dimensional space $\mathcal{H}_n$. To see this, note that $\mathbb{E}\left[\chi_S(\omega)\right] = 0$ for $S \neq \emptyset$, $\mathbb{E}\left[\chi_S(\omega)^2\right] = 1$, and that for disjoint sets S_1 and S_2 we have $\chi_{S_1\cup S_2} = \chi_{S_1}\chi_{S_2}$. Thus, for any two sets $S_1, S_2 \subseteq [n]$

$$\begin{aligned}\mathbb{E}\left[\chi_{S_1}(\omega)\chi_{S_2}(\omega)\right] &= \mathbb{E}\left[\chi_{S_1\cap S_2}(\omega)^2\chi_{S_1\Delta S_2}(\omega)\right] \\ &= \mathbb{E}\left[\chi_{S_1\Delta S_2}(\omega)\right] = \begin{cases} 1, & S_1 = S_2, \\ 0, & S_1 \neq S_2, \end{cases}\end{aligned}$$

where $S_1 \Delta S_2$ denotes the symmetric difference $(S_1 \setminus S_2) \cup (S_2 \setminus S_1)$. We conclude that $\{\chi_S\}_{S \subseteq [n]}$ forms an orthonormal basis, and that each function $f : \{0,1\}^n \to \mathbb{R}$ can be expressed using *Fourier-Walsh representation*

$$f(\omega) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(\omega),$$

where $\hat{f}(S) := \mathbb{E}\left[f(\omega)\chi_S(\omega)\right]$ are the so-called Fourier coefficients. Observe that

$$\mathbb{E}[f] = \hat{f}(\emptyset), \qquad \mathbb{E}[f^2] = \sum_{S \subseteq [n]} \hat{f}(S)^2, \qquad \operatorname{Var}(f) = \sum_{S \neq \emptyset} \hat{f}(S)^2.$$

The concept of noise sensitivity can be characterized in terms of the Fourier coefficients. Given a function $f : \{0,1\}^n \to \mathbb{R}$, define a measure ν_f on the collection of subsets of $[n]$ by giving weight $\hat{f}(S)^2$ to $S \subseteq [n]$. This measure will be referred to as the *spectral measure* or the *Fourier spectrum* of f. The total weight of the spectral measure equals

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = \mathbb{E}[f^2],$$

and is thus a probability measure only when $\mathbb{E}[f^2] = 1$. However, I will let $\mathcal{S}_f$ denote a subset of $[n]$ choosen according to this measure and treat $\mathcal{S}_f$ as a random variable even when the total weight is not equal to 1. In particular, I will write $\mathbb{P}(\mathcal{S}_f = S)$ for $\nu_f(S) = \hat{f}(S)^2$ and $\mathbb{E}\left[g(\mathcal{S}_f)\right]$ also for integral $\nu_f(g) = \sum_{S \subseteq [n]} g(S)\nu_f(S) = \sum_{S \subseteq [n]} g(S)\hat{f}(S)^2$ of a real-valued function g on subsets of $[n]$.

The link that allows noise sensitivity of a sequence of Boolean functions to be characterized via the Fourier spectrum is given next.

Proposition 5.9. *For $f : \{0,1\}^n \to \mathbb{R}$, then*

$$\mathbb{E}\left[f(\omega)f(\omega^\epsilon)\right] - \mathbb{E}\left[f(\omega)\right]^2 = \sum_{S \neq \emptyset} \hat{f}(S)^2 (1-\epsilon)^{|S|} = \mathbb{E}\Big[(1-\epsilon)^{|\mathcal{S}_f|} 1_{\{\mathcal{S}_f \neq \emptyset\}}\Big].$$

Proof. Since bits at different positions are independent, $\mathbb{E}\left[\chi_{S_1}(\omega)\chi_{S_2}(\omega^\epsilon)\right] = 0$ whenever $S_1 \neq S_2$. Thus expanding $f = \sum_S \hat{f}(S)\chi_S$, we obtain

$$\mathbb{E}\left[f(\omega)f(\omega^\epsilon)\right] - \mathbb{E}\left[f(\omega)\right]^2 = \sum_{S \neq \emptyset} \hat{f}(S)^2 \, \mathbb{E}\left[\chi_S(\omega)\chi_S(\omega^\epsilon)\right].$$

Since ω_j and ω_j^ϵ are independent whenever the jth bit is re-randomized, and equal otherwise, $\mathbb{E}\left[\chi_S(\omega)\chi_S(\omega^\epsilon)\right]$ equals the probability that no bit is S is re-randomized. By independence, this probability equals $(1-\epsilon)^{|S|}$. ∎

Several interesting observations can be made from Proposition 5.9. The first one is that the correlation between $f(\omega)$ and $f(\omega^\epsilon)$ is always positive and decreasing in ϵ (unless f is constant). Another interesting observation is that $f(\omega)$ and $f(\omega^\epsilon)$ are more correlated when the Fourier spectrum is concentrated on small sets, and less correlated when concentrated on large sets. This relation can be made more precise.

Proposition 5.10. *For a sequence $\{f_n\}_{n\geq 1}$ of functions $f_n : \{0,1\}^n \to \{0,1\}$ the following are equivalent:*

a) $\{f_n\}_{n\geq 1}$ *is noise sensitive.*

b) $\exists\, \epsilon \in (0,1)$ *such that* $\mathbb{E}\left[f_n(\omega)f_n(\omega^\epsilon)\right] - \mathbb{E}\left[f_n(\omega)\right]^2 \to 0$ *as* $n \to \infty$.

c) $\mathbb{P}\big(0 < |\mathcal{S}_{f_n}| \leq k\big) \to 0$ *as* $n \to \infty$*, for every* $k \in \mathbb{N}$.

Proof. Trivially, *a)* implies *b)*. That *b)* implies *c)* is a consequence of Proposition 5.9, since for the given $\epsilon \in (0,1)$ and each $k \in \mathbb{N}$

$$0 \;=\; \lim_{n\to\infty} \mathbb{E}\left[(1-\epsilon)^{|\mathcal{S}_{f_n}|} 1_{\{\mathcal{S}_{f_n} \neq \emptyset\}}\right] \;\geq\; (1-\epsilon)^k \lim_{n\to\infty} \mathbb{P}\big(0 < |\mathcal{S}_{f_n}| \leq k\big).$$

Similarly, to see that *c)* implies *a)*, take $\epsilon \in (0,1)$ and note that for each $k \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}\left[f_n(\omega)f_n(\omega^\epsilon)\right] - \mathbb{E}\left[f_n(\omega)\right]^2 \;&=\; \mathbb{E}\left[(1-\epsilon)^{|\mathcal{S}_{f_n}|} 1_{\{\mathcal{S}_{f_n} \neq \emptyset\}}\right] \\ &\leq\; (1-\epsilon)^k + \mathbb{P}\big(0 < |\mathcal{S}_{f_n}| \leq k\big). \end{aligned}$$

Sending n to infinity, $\limsup_{n\to\infty} \mathbb{E}\left[f_n(\omega)f_n(\omega^\epsilon)\right] - \mathbb{E}\left[f_n(\omega)\right]^2$ is found to be at most $(1-\epsilon)^k$. However, this holds for every $k \in \mathbb{N}$, so the limit has to be zero. □

Also noise stability can be characterized in terms of the spectral measure.

Proposition 5.11. *A sequence $\{f_n\}_{n\geq 1}$ of functions $f_n : \{0,1\}^n \to \{0,1\}$ is noise stable if and only if the sequence $\big\{|\mathcal{S}_{f_n}|\big\}_{n\geq 1}$ is tight, i.e., for all $\delta > 0$ there is $M < \infty$ such that*

$$\mathbb{P}\big(|\mathcal{S}_{f_n}| > M\big) \leq \delta, \quad \textit{for all } n \in \mathbb{N}.$$

5.4 Noise sensitivity of percolation crossings

The approach developed in Benjamini et al. (1999) can be distinguished into parts. The first part, the BKS Theorem, relates noise sensitivity to influences for any sequence of Boolean functions. It is an interesting fact that the sufficient condition in the theorem is also necessary for sequences of monotone Boolean functions. A function $f : \{0,1\}^n \to \mathbb{R}$ is called *monotone* if for any $\omega, \omega' \in \{0,1\}^n$ such that $\omega_j \leq \omega'_j$ for any $j \in [n]$, then $f(\omega) \leq f(\omega')$.

The remaining part of the approach only applies to monotone functions, and relates influences to the revealment of algorithms. A (deterministic) *algorithm* refers here to a rule that describes which bit of $\omega \in \{0,1\}^n$ to query next, or whether to stop, based on the outcome of the bits already seen. An algorithm can decide to terminate at any point, and does not necessary query all bits. The *revealment* of an algorithm $\mathcal{A}$ with respect to $j \in [n]$ is the probability that bit j is queried by the algorithm. Moreover, the revealment of $\mathcal{A}$ with respect to the set $K \subseteq [n]$ is defined as

$$\delta_{\mathcal{A}}(K) := \max_{j \in K} \mathbb{P}(\mathcal{A} \text{ queries bit } j).$$

Given a function $f : \{0,1\}^n \to \mathbb{R}$, an algorithm is said to determine f if for each $\omega \in \{0,1\}^n$, the outcome of $f(\omega)$ is known at the end of the algorithm.

The approach introduced in Benjamini et al. (1999) could possibly be referred to as the *deterministic algorithm approach*, and, as pointed out to me by Jeff Steif, can be summarized in the following quite general theorem.

Theorem 5.12. *Let $\{f_n\}_{n\geq 1}$, where $f_n : \{0,1\}^n \to \{0,1\}$, be a sequence of monotone functions. Assume that there are $C < \infty$, $\alpha > 0$ and an integer r, such that for each $n \geq 1$ there is a partition of $[n]$ into $K_1, K_2, \dots, K_r$, and (deterministic) algorithms $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_r$ such that for any $j = 1, 2, \dots, r$*

a) $\mathcal{A}_j$ determines f_n,

b) $\delta_{\mathcal{A}_j}(K_j)\big(\log n\big)^6 \to 0$, as $n \to \infty$.

Then $\{f_n\}_{n\geq 1}$ is noise sensitive.

Noise sensitivity of percolation crossings (Theorem 5.7) can be deduced from Theorem 5.12. Let me indicate how this is done. First, we need to define a suitable algorithm. In fact, we shall need a pair of algorithms. Let K_L denote the subset of $\mathcal{E}_n$ of edges of the $\mathbb{Z}^2$ lattice that are contained in left half of the rectangle $[0, n+1] \times [0, n]$, and let K_R denote the remaining edges in $[0, n+1] \times [0, n]$, those on the right half. An algorithm, described in Benjamini et al. (1999), that has low revealment with respect to bits in K_R is described next.

Algorithm. *Let $f_n : \{0,1\}^{\mathcal{E}_n} \to \{0,1\}$ be as in Example 5.6, let $\omega \in \{0,1\}^{\mathcal{E}_n}$, and set $V_0 = \{0\} \times [n]$ to denote the set of vertices on the left side of the rectangle $[0, n+1] \times [0, n]$. For each $k \geq 1$, define the algorithm $\mathcal{A}_R = \mathcal{A}_R(n)$ and the set V_k inductively:*

1. *Query all edges in $\mathcal{E}_n$ which has one endpoint in V_{k-1} and one outside.*

2. *Let U_k denote the set of neighbouring points to V_{k-1} that are endpoints to an edge queried and found open in the previous step. Set $V_k = V_{k-1} \cup U_k$.*

3. *Repeat the above steps until $V_k = V_{k-1}$. At this point $f_n(\omega)$ is known.*

It is easy to see that an edge in K_R will be queried by $\mathcal{A}_R$ if and only if there is a path of open edges from V_0 to one of the endpoints of the edge. Each endpoint of an edge $e \in K_R$ lies in the right half of $[0, n+1] \times [0, n]$. Hence, the event that there is an open path from V_0 to an endpoint of e is contained in the event that there is an open path from that endpoint reaching the boundary of a box of side length $n/2$, centred at the endpoint. This event is recognized as the one-arm event $\mathrm{AE}_{n/2}$, which in (2.2) was seen to have probability at most $(n/2)^{-\alpha}$ to occur, for some $\alpha > 0$ uniformly in n. Hence, for the revealment of an edge in K_R we have $\delta_{\mathcal{A}_R}(K_R) \leq 2^{1+\alpha} n^{-\alpha}$. Likewise, interchanging left and right, one obtains an algorithm $\mathcal{A}_L$ that has low revealment with respect to bits in K_L. This shows that percolation crossings are noise sensitive, and ends the outline of a proof of Theorem 5.7.

5.5 Exceptional times of percolation and noise sensitivity

The main approach to show existence of exceptional times in dynamical percolation is via the second moment method. Above, I gave a short description of how dynamical percolation is linked to perturbations of the edge configuration. I further claimed that correlations between nearby time points in dynamical percolation can be understood via noise sensitivity. Here, I would like to elaborate further on this connection to clearly point to the relevance of the concept of noise sensitivity when deducing exceptional times of percolation. I will present a rough sketch of this for dynamical bond percolation on the $\mathbb{Z}^2$ lattice. I emphasize that the sketch will not argue for the existence of exceptional times, but only indicate the approach.

In order to show that there at criticality exist exceptional times $t \in [0, 1]$ at which an infinite open component exists it suffices, according to Kolmogorov's 0-1 law, to show that

$$\mathbb{P}_{1/2}\big(\exists t \in [0, 1] : |\mathcal{C}| = \infty \text{ at time } t\big) > 0. \tag{5.2}$$

Let $\mathrm{AE}_n(t)$ denote the one-arm event at time t, that is, the event that there in the bond configuration at time t is an open path from the origin reaching the boundary of the box $[-n, n]^2$. Let X_n denote the Lebesgue amount of time during $[0, 1]$ for which $\mathrm{AE}_n(t)$ occurs. That is,

$$X_n = \int_0^1 1_{\mathrm{AE}_n(t)}\, dt.$$

Fubini's theorem gives that $\mathbb{E}[X_n] = \mathbb{P}_{1/2}(\mathrm{AE}_n)$. Employing the second moment method, the key passage is to show that for some $c < \infty$

$$\mathbb{E}[X_n^2] \;\leq\; c\,\mathbb{E}[X_n]^2, \quad \text{uniformly in } n \geq 1. \tag{5.3}$$

When such an estimate is obtained, Cauchy-Schwartz inequality implies that

$$\mathbb{P}(X_n > 0) \;\geq\; \frac{\mathbb{E}[X_n]^2}{\mathbb{E}[X_n^2]} \;\geq\; \frac{1}{c}, \quad \text{uniformly in } n \geq 1.$$

In particular, $X_n > 0$ implies that there exists $t \in [0, 1]$ such that $\mathrm{AE}_n(t)$ occurs. Hence, countable additivity gives that

$$\mathbb{P}\left(\bigcap_{n\geq 1} \{\exists\, t \in [0, 1] : \mathrm{AE}_n(t) \text{ occurs}\}\right) \;\geq\; \frac{1}{c}.$$

This would via a compactness argument easily give (5.2) if the set of times at which $\mathrm{AE}_n(t)$ occurs had been a closed set. It is not, by the fact that the set of times an edge is open is not closed. This is however a minor problem, and resolved by modifying the process such that an edge is open also at the instant it flips. The existence of exceptional times follows for the modified process, and it can easily be seen these times are exceptional times also for the original process. Thus, (5.2) follows from (5.3).

The essential step is thus to show that (5.3) holds for some finite constant c. Note that

$$\begin{aligned}
\mathbb{E}[X_n^2] \;&=\; \int_0^1 \int_0^1 \mathbb{P}_{1/2}\big(\,\mathrm{AE}_n(s) \cap \mathrm{AE}_n(t)\big)\, ds\, dt \\
&\leq\; 2 \int_0^1 \mathbb{P}_{1/2}\big(\,\mathrm{AE}_n(0) \cap \mathrm{AE}_n(t)\big)\, dt.
\end{aligned}$$

If $\mathrm{AE}_n(0)$ and $\mathrm{AE}_n(t)$ would be uncorrelated, then $\mathbb{E}[X_n^2] \leq 2\,\mathbb{P}_{1/2}(\mathrm{AE}_n)^2 = 2\,\mathbb{E}[X_n]^2$. This is not the case, but we have a clear indication that correlation between arm events plays an important rôle. In fact, it seems possible that (5.3) holds if only the correlation between $\mathrm{AE}_n(0)$ and $\mathrm{AE}_n(t)$ is sufficiently weak. More precisely, if f_n denotes the indicator function of the event AE_n, then, via Proposition 5.9 with $\epsilon = 1 - e^{-t}$,

$$\mathbb{P}_{1/2}\big(\,\mathrm{AE}_n(0) \cap \mathrm{AE}_n(t)\big) \;=\; \mathbb{E}\left[f_n(\omega) f_n(\omega^\epsilon)\right] \;=\; \mathbb{P}_{1/2}(\mathrm{AE}_n)^2 + \sum_{S \neq \emptyset} \hat{f}_n(S)^2 e^{-|S|\,t}.$$

This motivates further study of the Fourier spectrum of Boolean functions.

5.6 Quantitative noise sensitivity

The proof of the BKS Theorem is based on Fourier techniques, and an essential ingredient in the proof is a result on hypercontractivity, knows as the Bonami-Beckner inequality. The concept comes from harmonic analysis, but was found useful for the study of Boolean functions when used by Kahn, Kalai, and Linial (1988) to prove a sharp lower bound on the maximal influence over the set of bits. It was previously mentioned that the BKS Theorem gives a necessary and sufficient condition for noise sensitivity for monotone functions. This may give the impression that also this theorem is optimal. However, as will be explained, there is room for improvements.

Noise sensitivity of a sequence $\{f_n\}_{n\geq 1}$ of Boolean functions is equivalent to the corresponding Fourier spectrum to assign asymptotically vanishing weight to fixed levels of the spectrum, i.e., $\mathbb{P}\big(|\mathcal{S}_{f_n}| = k\big) \to 0$ as $n \to \infty$ for each $k \geq 1$. The approach of Benjamini et al. gives a slightly stronger conclusion. Under the conditions of Theorem 5.12, or alternatively, under the assumption that the sum of influences squared decays at inverse polynomial rate, then it is possible to show that there is an $a > 0$ such that

$$\mathbb{P}\big(0 < |\mathcal{S}_{f_n}| \leq a \log n\big) \to 0, \quad \text{as } n \to \infty.$$

This improvement gives additional information regarding the Fourier spectrum, but it is not sufficient in order to conclude the existence of exceptional times of percolation. The part the approach that relates revealment of algorithms with influences of bits was proved in two stages. The first relating influences of bits to correlation with majority functions (of similar form as in Example 5.4), and the second relating correlation with majority functions with revealment of algorithms. These two stages are explained in greater detail in Paper III, where the approach is studied further.

Later, Schramm and Steif (2010) developed a method that directly relates the Fourier coefficients of a function to the revealment of a randomized algorithm that determines the function. A *randomized* algorithm differs from a deterministic one in the way that the next bit queried is chosen at random according to a distribution that is allowed to depend on the previous bits queried, as well as their values. The revealment of a randomized algorithm is defined as $\delta_{\mathcal{A}} := \delta_{\mathcal{A}}([n])$, and is thus measured with respect to all bits. Note that if $\mathcal{A}$ is deterministic, then $\delta_{\mathcal{A}} = 1$. We are interested in finding algorithms with low revealment, from which we obtain the following information.

Theorem 5.13 (Schramm and Steif (2010))**.** *Let $f : \{0,1\}^n \to \mathbb{R}$, and let $\mathcal{A}$ be an algorithm that determines f. Then, for each $k \in [n]$*

$$\mathbb{P}\big(|\mathcal{S}_f| = k\big) \;=\; \sum_{|S|=k} \hat{f}(S)^2 \;\le\; k\,\|f\|_2^2\,\delta_{\mathcal{A}},$$

where $\|f\|_2^2 = \mathbb{E}\left[f^2(\omega)\right]$.

An immediate corollary says that if $\{f_n\}_{n\ge1}$ is a sequence of functions $f_n : \{0,1\}^n \to \{0,1\}$, and for each $n \ge 1$ there is a (randomized) algorithm $\mathcal{A}_n$ that determines f_n and is such that $\delta_{\mathcal{A}_n} \to 0$ as $n \to \infty$, then $\{f_n\}_{n\ge1}$ is noise sensitive. However, assume that the sequence of algorithms $\{\mathcal{A}_n\}_{n\ge1}$ is such that $\delta_{\mathcal{A}_n} \le Cn^{-\alpha}$ uniformly in n, for some $C < \infty$ and $\alpha > 0$. Then Theorem 5.13 gives that for any $\gamma < \alpha$

$$\mathbb{P}\big(0 < |\mathcal{S}_{f_n}| \le n^{\gamma}\big) \to 0, \quad \text{as } n \to \infty. \tag{5.4}$$

With this approach, Schramm and Steif got close to proving the existence of exceptional times of percolation for dynamical percolation on the $\mathbb{Z}^2$ lattice, later obtained by Garban et al. (2010). Instead, they did manage to prove the existence of exceptional times for dynamical (site) percolation on the hexagonal lattice. Again, the reason for this is the very precise information available due to SLE technology.

Given a noise sensitive sequence, which is the largest exponent γ for which (5.4) holds? The same argument used to characterize noise sensitivity in terms of the spectral measure (Proposition 5.10) shows that this corresponds to the largest γ for which we in (5.1) can let ϵ decay as $\epsilon_n = n^{-\gamma}$, while the limit still equals 0. For percolation crossings on the $\mathbb{Z}^2$ lattice, as defined in Example 5.6, the approach in Schramm and Steif (2010) using randomized algorithms is able to show that (5.4) holds for some $\gamma > 0$. For percolation crossings defined similarly on the hexagonal lattice, the approach is able to show that (5.4) holds for all $\gamma < 1/8$. Eventually, Garban et al. (2010) were able to show that on the hexagonal lattice, for any $\epsilon > 0$,

$$\mathbb{P}\big(n^{3/4-\epsilon} < |\mathcal{S}_{f_n}| < n^{3/4+\epsilon}\big) \to 1, \quad \text{as } n \to \infty.$$

This very precise result was obtained from a geometric approach to study the Fourier spectrum. It was with the same approach they were able to deduce the existence of exceptional times for dynamical percolation on the $\mathbb{Z}^2$ lattice.

Chapter 6

Summary of papers

6.1 Paper I: Asymptotics of first-passage percolation on 1-dimensional graphs

The first contribution of this thesis is in the study of first-passage percolation on essentially 1-dimensional periodic graphs. Roughly speaking, the class of graphs considered consists of all graphs that can be constructed from a finite connected graph $\mathcal{G}_0$ as follows. Let $\{\mathcal{G}_n\}_{n\in\mathbb{Z}}$ be a sequence of identical copies of $\mathcal{G}_0$. Construct the infinite graph $\mathcal{G}$ by deciding for, and performing, a fixed way of connecting two consecutive copies in the sequence $\{\mathcal{G}_n\}_{n\in\mathbb{Z}}$. An essentially 1-dimensional periodic graphs of specific interest is the $\mathbb{Z}\times\{0,1,\dots,K-1\}^{d-1}$ nearest neighbour graph, for some $K, d \geq 2$, referred to as the (K,d)-*tube*. The specific interest in this graph lies in its resemblance with the $\mathbb{Z}^d$ lattice, when K is large, although its structure remains 1-dimensional.

As was argued for in Section 4, the asymptotic behaviour of first-passage percolation in two or more dimensions is still poorly understood. It is therefore of interest to investigate how, possibly simpler, but still similar models behave. When studying first-passage percolation on the class of graphs described, I find that the asymptotic behaviour is very much 1-dimensional. Pick a vertex v_0 in $\mathcal{G}_0$, and let v_n denotes the vertex in $\mathcal{G}_n$ corresponding to v_0. In addition to showing that on the graph $\mathcal{G}$, when $\mathbb{E}[\tau_e] < \infty$, then $\exists\, \mu_{\mathcal{G}} := \lim_{n\to\infty} T(v_0, v_n)/n$ almost surely and in L^1, I prove that when $\mathbb{E}[\tau_e^2] < \infty$, then for some $\sigma_{\mathcal{G}} < \infty$, as $n \to \infty$,

$$\frac{T(v_0,v_n)-\mu_{\mathcal{G}}n}{\sigma_{\mathcal{G}}\sqrt{n}} \xrightarrow{d} \chi, \quad \text{in distribution,} \tag{6.1}$$

where χ has a standard normal distribution, and that almost surely

$$\limsup_{n\to\infty} \frac{T(v_0,v_n)-\mu_{\mathcal{G}} n}{\sigma\sqrt{2n\log\log n}} = 1, \quad \text{and} \quad \liminf_{n\to\infty} \frac{T(v_0,v_n)-\mu_{\mathcal{G}} n}{\sigma\sqrt{2n\log\log n}} = -1.$$

I emphasize that this is the characteristics of a 1-dimensional behaviour that is not expected in higher dimensions. In particular, as previously mentioned, the correct normalizing factor in (6.1) when $d = 2$ is believed to be of order $n^{1/3}$. Let μ_K denote the time constant associated with the (K, d)-tube. That this graphs comes to resemble the $\mathbb{Z}^d$ lattice as K increases is reflected in the fact that μ_K is decreasing in K, and

$$\lim_{K\to\infty} \mu_K = \mu_{\mathbb{Z}^d}(\mathbf{e}_1),$$

where $\mu_{\mathbb{Z}^d}$ denotes the time constant associated with the $\mathbb{Z}^d$ lattice.

The key in capturing the 1-dimensional behaviour is by identifying a suitable renewal sequence and using stopped random walk techniques not avaliable in higher dimensions. With the approximation of first-passage times with a renewal sequence I am able to derive certain result whose higher dimensional analogues are also expected, but not known, to hold. In particular, I prove that $\mathbb{E}\big[T(v_0,v_n)\big]$ is increasing in n, for large n, and that

$$\exists\, \alpha_{\mathcal{G}} := \lim_{n\to\infty} \frac{N(v_0,v_n)}{n}, \quad \text{almost surely,}$$

where $N(v_0, v_n)$ denotes the length of the geodesic between v_0 and v_n on the graph $\mathcal{G}$. In addition, I prove that also $\mathrm{Var}\big(T(v_0,v_n)\big)$ is increasing in n, when n is large, a result that is not simply a consequence of the 1-dimensional behaviour, but also of the structure of the model in question. Essentially, the sequence $\{N(v_0,v_n)\}_{n\geq 1}$ is observed to exhibit the same asymptotic behaviour as the sequence $\{T(v_0,v_n)\}_{n\geq 1}$.

Finally, a coupling between two first-passage percolation processes at different initial configurations is constructed, and it is used to derive a 0-1 law. As a complement to the coupling and 0-1 law, an example is given that shows that the corresponding results are not true for first-passage percolation considered on the binary tree. Whether the corresponding results hold for first-passage percolation on the $\mathbb{Z}^d$ lattice for some $d \geq 2$ is unknown.

6.2 Paper II: The asymptotic shape, large deviations and dynamical stability in first-passage percolation on cones

The fact that the time constant of the (K, d)-tube approaches the time constant of the $\mathbb{Z}^d$ lattice as K increases is followed up in the second paper. An immediate consequence thereof is the following. Let $\mathcal{G}$ denote the restriction of the $\mathbb{Z}^d$ lattice to the region obtained when a non-negative increasing function $r : [0, \infty) \to [0, \infty)$ is rotated around the first coordinate axis. If $r(a) \to \infty$ as $a \to \infty$, then consequently on $\mathcal{G}$

$$\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} = \mu_{\mathbb{Z}^d}(\mathbf{e}_1), \quad \text{almost surely and in } L^1.$$

This result states that no matter how slowly the function r increases, the difference between the time it takes a fluid injected at the origin to wet the site $n\mathbf{e}_1$ on the $\mathbb{Z}^d$ lattice, compared with the graph $\mathcal{G}$, grows as $o(n)$. With a little more work this extends to arbitrary directions $\hat{\mathbf{x}} \in \mathcal{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$. For graphs of this form, a Shape Theorem analogue is proven. In the particular case when $\mathcal{G}$ denotes the restriction of the $\mathbb{Z}^d$ lattice to the region obtained when the linear function $r(a) = c \cdot a$, for some $c \in \mathbb{R}_+$, is rotated around the first coordinate axis, a cone in the classical sense is obtained. One of the two main results of the paper states that on $\mathcal{G}$, the rescaled wet region converges almost surely, in L^1 and completely to the restriction of the asymptotic shape $\mathcal{W}^*$ to the obtained region.

The class of graphs can be described as restrictions of the $\mathbb{Z}^d$ lattice to regions obtained when a function is rotated around an axis. Alternatively, which is the form adopted in the paper, such graphs can be described as subgraphs of the $\mathbb{Z}^d$ lattice induced by sets of the form

$$\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, r(a)\big),$$

where $\hat{\mathbf{x}} \in \mathcal{S}^{d-1}$, $r : [0, \infty) \to [0, \infty)$, and $B(\mathbf{x}, r)$ denotes the closed Euclidean ball of radius r, centred at $\mathbf{x}$. Here, a subgraph of the lattice *induced* by a set V refers to the graph with set of vertices given by $V \cap \mathbb{Z}^d$, and where any two vertices are joined by an edge if and only if the were in the lattice, i.e., they are at Euclidean distance 1.

Let $\mathbf{x} \in \mathcal{S}^{d-1}$ and let $r : [0, \infty) \to [0, \infty)$ be any convex or concave function such that $r(a) \to \infty$ as $a \to \infty$. The following result is obtained, in which Y is as defined in (4.3).

Theorem 6.1. *For any $d \geq 2$ there exists a universal constant R_d such that for first-passage percolation on the subgraph $\mathcal{G}$ of the $\mathbb{Z}^d$ lattice induced by $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, r(a) + R_d\big)$, the following holds.*

a) If $\mathbb{E}[\tau_e] < \infty$, then $$\limsup_{\mathbf{z}\in\mathcal{G}\colon |\mathbf{z}|\to\infty} \mathbb{E}\left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right| = 0.$$

b) If $\mathbb{E}[Y^d] < \infty$, then $$\limsup_{\mathbf{z}\in\mathcal{G}\colon |\mathbf{z}|\to\infty} \left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right| = 0,$$ *almost surely.*

c) If $\mathbb{E}[Y^{d+1}] < \infty$, then $$\sum_{\mathbf{z}\in\mathcal{G}} \mathbb{P}\left(\left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right| > \epsilon\right) < \infty,$$ *for all $\epsilon > 0$.*

In view of the inversion argument carried out in Section 4 to show that the Shape Theorem is a consequence of (4.6), then part *b)* of Theorem 6.1 is a generalization of the Shape Theorem. In part *a)* the convergence is deduced also in L^1 sense, and part *c)* shows that the convergence also holds completely. In order to deduce the convergence in part *c)*, it is necessary to deduce some large deviation bounds on first-passage times.

The second main result of the paper is a further generalization of part *b)* of Theorem 6.1, and therefore also of the Shape Theorem. The result concerns the dynamical version of first-passage percolation obtained when edges update their values according to independent Poisson clocks, in analogy with dynamical (bond) percolation. The introduction of dynamics gives rise to a second time dimension, which is not to be confused with the time dimension in which the fluid propagates. This is further emphasized in the paper.

Theorem 6.2. *The almost sure convergence in part* b) *of Theorem 6.1 is dynamically stable with respect to the dynamics described above.*

Another observation made in the paper is the following. Let $\mathcal{G}$ denote the restriction of the $\mathbb{Z}^2$ lattice to the region between the first coordinate axis and the function $f(a) = \alpha \log(1 + a)$, for $a \geq 0$ and some $\alpha \in \mathbb{R}_+$. Grimmett (1983) proved that the critical probability for bond percolation on $\mathcal{G}$ lies strictly between 1/2 and 1. When edges are assigned the values 0 and 1 with equal probability, although we for bond percolation encounter ourselves in the subcritical regime, the time constant along the first coordinate axis equals $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0$. This remains true if the edge distribution is given a slight bias towards the value 0. In particular, I find that the subcritical regimes of first-passage and bond percolation on $\mathcal{G}$ do not coincide, as they do on the $\mathbb{Z}^d$ lattice, for any $d \geq 1$.

6.3 Paper III: Noise sensitivity in continuum percolation

In the final paper, coauthored with Erik Broman, Simon Griffiths and Robert Morris, the concept of noise sensitivity is studied further. Noise sensitivity was introduced as a concept for Boolean functions, and was in particular employed to study the effect of small perturbations of percolation configurations on planar lattices. Subsequently, it is of interest to study similar effects on other planar percolation models. In this paper, noise sensitivity is introduced for continuum percolation, and more specifically so for the Poisson Boolean model. Recall that given a Poisson point process η in $\mathbb{R}^2$ of density λ, then space is partitioned into an occupied and a vacant region by placing a unit disc at each Poisson point. The union of these discs, denoted by $D(\eta)$, is referred to as the occupied region.

In analogy to bond percolation on the $\mathbb{Z}^2$ lattice, we are interested in studying the effect small perturbations of the disc configuration has on the sequence $\{f_n\}_{n\geq 1}$, where $f_n(\eta) = 1$ if there is a horizontal crossing of the square $[0, n]^2$ contained in $D(\eta) \cap [0, n]^2$, and $f_n(\eta) = 0$ otherwise. The function f_n is not Boolean, but a suitable extension of noise sensitivity to the Poisson Boolean model is the following. Given $\epsilon \in (0, 1)$, let η^ϵ be an ϵ-perturbation of η obtained by removing each point in η independently with probability ϵ, and adding an independent Poisson point process of density $\epsilon\lambda$. Observe that also η^ϵ is a Poisson process at density λ.

Definition 6.3. *The Poisson Boolean model is said to be* noise sensitive at density λ, *if for every $\epsilon > 0$, the sequence $\{f_n\}_{n\geq 1}$ satisfies*

$$\lim_{n\to\infty} \mathbb{E}\left[f_n(\eta)f_n(\eta^\epsilon)\right] - \mathbb{E}\left[f_n(\eta)\right]^2 = 0.$$

As for perturbations of a bond percolation configuration, the ϵ-perturbation of η can be seen as the configuration after a short time t (where $\epsilon = 1 - e^{-t}$) of a dynamical version the Poisson Boolean model started in state η. In the dynamical model, unit discs can be thought of as raining down from the sky, and when landing in the plane, the stay, independently of each other, for an exponentially distributed time. The density of discs in the plane is kept constant by regulating the rate at which discs fall from the sky.

For $\lambda \neq \lambda_c$, the Poisson Boolean model is noise sensitive for trivial reasons, as is also the case for percolation crossings on the square lattice. Thus, it is only interesting to consider noise sensitivity for the model at criticality, that is for $\lambda = \lambda_c$. Our main result is the following.

Theorem 6.4. *The Poisson Boolean model is noise sensitive at criticality.*

The means employed to prove this result is by adapting the deterministic algorithm approach due to Benjamini et al. (1999), and described in Section 5. This is arguably the easiest approach to prove noise sensitivity. However, there are several difficulties when carrying out the approach in the continuum. The algorithm as such is a straightforward analogue of the algorithm described for the lattice case. The difficulty lies in the continuum positioning of the Poisson structure. Essentially we approach this problem as follows.

Although we are interested in the Poisson Boolean model at criticality, first pick a Poisson point process η at intensity λ_c/p, where $p \in (0,1)$ is fixed. Next, colour the points in the Poisson process independently either 'green' or 'yellow' with probability p and $1-p$, respectively. Note that the resulting set η_G of green points constitutes a Poisson point process of density λ_c. When 1 is thought of as the colour green, and 0 as yellow, then (given η) the colouring can be represented by an element $\omega_C \in \{0,1\}^\eta$. Thus, the information in η_G is represented by the pair (η, ω_G). Moreover, it is easy to see that an ϵ-perturbation of η_G is given by (η, ω_C^δ), where ω_C^δ is the perturbation obtained from ω_C when re-randomizing each bit independently with probability $\delta = \epsilon/(1-p)$, labeling it as a 1 with probability p. Thus, identifying $f_n(\eta_G) = f_n(\eta, \omega_C)$, to prove Theorem 6.4 it suffices to prove that for every $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{E}\left[f_n(\eta,\omega_C) f_n(\eta,\omega_C^\epsilon)\right] - \mathbb{E}\left[f_n(\eta,\omega_C)\right]^2 = 0.$$

Expressing things in this way has the advantage that, conditioned on the Poisson configuration η, we can identify f_n with the function $f_n^\eta : \{0,1\}^\eta \to \{0,1\}$, via the the identity $f_n^\eta(\omega) = f_n(\eta,\omega)$. Clearly, f_n only depends on the points in η that lies within distance one of the square $[0,n]^2$. For each fixed η, the domain of f_n is therefore really finite dimensional.

Proving noise sensitivity of the Poisson Boolean model proceeds by showing that the sequence $\{f_n^\eta\}_{n\geq 1}$ is noise sensitive, almost surely. For this to prove Theorem 6.4, we also need to show that conditioning on the point configuration η has almost no effect on the probability of having a crossing of green discs after colouring. We prove this in much more general terms, obtaining a result concerning hypergraphs. Since we only manage to prove that the effect of fixing the point configuration is small when p is small, we are led to extending the approach due to Benjamini et al. (1999), in particular the BKS Theorem and Theorem 5.12, to handle the situation $p \neq 1/2$. An even more general version of the BKS Theorem was recently proved by Keller and Kindler (2010). We present an easy deduction of the BKS Theorem for $p \in (0,1)$ from the uniform version. This is done via a simple and non-technical reduction from biased product measure to uniform measure that does not seem to have been used in this context before. We believe that several approaches presented in the paper can find use also in other contexts.

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Paper I

Asymptotics of first-passage percolation on 1-dimensional graphs

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Abstract

In this paper we consider standard first-passage percolation on certain 1-dimensional periodic graphs. One such graph of particular interest is the $\mathbb{Z} \times \{0, 1, \ldots, K-1\}^{d-1}$ nearest neighbour graph for $d, K \geq 1$. Let $T(u, v)$ denote the time it takes for an infection started at u to reach v, and let $N(u, v)$ denote the length of the geodesic (path with minimal passage time) from u to v. We derive asymptotic results that show how the behaviour of first-passage percolation on 1-dimensional graphs differ from what is known or expected in higher dimensions. Let $\mathbf{n} = (n, 0, \ldots, 0)$. By subadditivity $T(0, \mathbf{n})/n \to \mu$ for some $\mu > 0$ as $n \to \infty$, almost surely and in L^1. We show that for some $\sigma > 0$, as $n \to \infty$, $\big(T(0, \mathbf{n}) - \mu n\big)/\sigma\sqrt{n}$ converges in distribution to a standard normal, and moreover, that $\limsup_{n\to\infty} \big(T(0, \mathbf{n}) - \mu n\big)/\sigma\sqrt{2n \log\log n} = 1$, almost surely. We further prove that $\mathrm{E}\big[T(0, \mathbf{n})\big]$ and $\mathrm{Var}\big(T(0, \mathbf{n})\big)$ are monotonic in n, for large enough n. Results for $N(0, \mathbf{n})$ corresponding to the results mentioned for $T(0, \mathbf{n})$ are also derived.

We also allow different sets of initially infected vertices, and construct an exact coupling of two infections with different starting configurations. Using this coupling we prove a 0–1 law.

1 Introduction

First-passage percolation was first considered by Hammersley and Welsh (1965). It can be thought of as a model for the spread of an infection on a connected graph with set of vertices $\mathbb{V}$ and set of edges $\mathbb{E}$. Associate to the edges of the graph non-negative i.i.d. random variables $\{\tau_e\}_{e\in\mathbb{E}}$, referred to as *passage times*. We will denote the passage time distribution by $\mathrm{P}_\tau(\,\cdot\,) := \mathrm{P}(\tau_e \in \cdot\,)$. To avoid trivialities, we assume throughout this paper that P_τ does not concentrate all mass at a single point. With the present interpretation of the model, the passage time of an edge should be thought of as the random time it takes for an infection to spread along the edge. Consider the process where we start with a finite set $I \subset \mathbb{V}$ of infected vertices. As time starts, the infection spreads to adjacent vertices with delays indicated by the passage times.

Let us by a *path* refer to an alternating sequence of vertices and edges; $v_0, e_1, v_1, \ldots, e_m, v_m$, beginning and ending with a vertex, such that v_k is the endpoint of the edges e_k and e_{k+1} that precedes and follows v_k. The vertices v_0 and v_m are referred to as endpoints of the path. A path with one endpoint in U and the other in V, where $U, V \subset \mathbb{V}$, will be referred to as a path from U to V. We will repeatedly abuse notation and identify a path with its set of edges, and occasionally with its set of vertices. For a path Γ, we define the passage time of Γ as $T(\Gamma) := \sum_{e\in\Gamma}\tau_e$, and define the *passage time*, or *first-passage time*, between two sets of vertices $U, V \subset \mathbb{V}$ as

$$T(U,V) := \inf\big\{T(\Gamma) : \Gamma \text{ is a path from } U \text{ to } V\big\}.$$

We are often interested in the case when $U = \{u\}$ or $V = \{v\}$. We will in such case simply write $T(u,v)$ for $T(\{u\},\{v\})$. The main features of first-passage percolation are retained in

$$T(v) := T(I,v)$$

interpreted as the time it takes for the infection started in I to reach the vertex v, and

$$B_t := \{v \in \mathbb{V} : T(v) \le t\},$$

the set of infected vertices at time t.

A typical choice for the underlying graph is the usual $\mathbb{Z}^d$ *lattice*, whose vertices are the elements of $\mathbb{Z}^d$, and where two vertices are connected with an edge if their Euclidean distance is one. In this paper, though, we will consider first-passage percolation on 1-dimensional graphs. However, we begin with a presentation of some of the results for first-passage percolation on the $\mathbb{Z}^d$ lattice. Thereafter, the motivation for considering 1-dimensional graphs, as well as our results themselves, will be better understood. A more detailed survey of first-passage percolation can be found in Howard (2004).

It is customary to consider first-passage percolation with a single initially infected vertex at the origin. However, we have reasons to be interested in different initial configurations of the infection. The results we are about to review regarding the $\mathbb{Z}^d$ lattice hold for any finite initially infected set.

A challenging task, already considered by Hammersley and Welsh (1965), is to describe the behaviour of $T(v)$ when $|v|$ is large. It follows from its definition that $T(u,v)$ is subadditive, i.e.,

$$T(u,v) \le T(u,w) + T(w,v)$$

for any vertices u, v and w in $\mathbb{Z}^d$. Let

$$Y = \min\left(\tau_1, \dots, \tau_{2d}\right), \tag{1.1}$$

where $\tau_1, \dots, \tau_{2d}$ are independent and distributed according to P_τ. Thus, if $\mathrm{E}[Y] < \infty$, Kingman's Subadditive Ergodic Theorem says that there is a constant $\mu(\mathbf{e}_1)$, referred to as the *time constant*, such that

$$\lim_{n\to\infty} \frac{T(\mathbf{n})}{n} = \mu(\mathbf{e}_1), \quad \text{almost surely and in } L^1, \tag{1.2}$$

where $\mathbf{e}_1 = (1,0,\dots,0)$, and $\mathbf{n} = n\mathbf{e}_1$. The same holds in every direction. Let $\bar{x} \in \mathbb{R}^d$ be such that $|\bar{x}| = 1$. If $\lfloor n\bar{x} \rfloor$ denotes the coordinate-wise integer part of $n\bar{x}$, then there is a $\mu(\bar{x})$ such that

$$\lim_{n\to\infty} \frac{T(\lfloor n\bar{x} \rfloor)}{n} = \mu(\bar{x}), \quad \text{almost surely and in } L^1.$$

In fact, one can say more about this asymptotic growth. If we consider B_t, we can state results about the growth in all directions simultaneously. A first such result was due to Richardson (1973). For convenience, we replace B_t by the subset of $\mathbb{R}^d$ defined as

$$\tilde{B}_t := \left\{ x \in \mathbb{R}^d : x \in v + [0,1]^d \text{ for some } v \in B_t \right\}, \tag{1.3}$$

The following version of Richardson's result is due to Cox and Durrett (1981), and states that the set of infected vertices grows linearly with t and has a nonrandom asymptotic shape.

Theorem 1.1 (Shape Theorem)**.** *Consider first-passage percolation on $\mathbb{Z}^d$ with i.i.d. passage times such that*

$$\mathrm{E}[Y^d] < \infty, \tag{1.4}$$

for Y defined as in (1.1). If $\mu(\mathbf{e}_1) > 0$, then there exists a nonrandom, compact, convex subset B^* in $\mathbb{R}^d$ with nonempty interior such that for all $\epsilon > 0$, almost surely,

$$(1-\epsilon)B^* \subset \frac{1}{t}\tilde{B}_t \subset (1+\epsilon)B^*, \quad \textit{for } t \textit{ large enough.}$$

If $\mu(\mathbf{e}_1) = 0$, then for every compact set K in $\mathbb{R}^d$, almost surely,

$$K \subset \frac{1}{t}\tilde{B}_t, \quad \textit{for } t \textit{ large enough.}$$

In addition, it was shown by Kesten (1986) that

$$\mu(\mathbf{e}_1) = 0 \text{ if and only if } \mathrm{P}_\tau(\{0\}) \geq p_c(d),$$

where $p_c(d)$ is the critical value for independent bond percolation on the $\mathbb{Z}^d$ lattice. An elementary argument shows that $\mathrm{E}[\tau_e^2] < \infty$ is sufficient for (1.4) to hold.

As the Shape Theorem establishes a law of large numbers for the sequence $T(\lfloor n\bar{x} \rfloor)$, it is natural to ask about the fluctuations of the same sequence. They have turned out to be harder to understand, and depend on the dimension d. For $d = 1$, $T(n)$ reduces to a sum of i.i.d. random variables, from which it is immediate that

$$\mathrm{Var}\big(T(n)\big) = n\,\mathrm{Var}(\tau_e).$$

Kesten (1993) showed that for any $d \geq 1$, if $\mathrm{P}_\tau(\{0\}) < p_c$ and $\mathrm{E}[\tau_e^2] < \infty$, then there are constants $C_1 > 0$ and $C_2 < \infty$ such that

$$C_1 \leq \mathrm{Var}\big(T(\mathbf{n})\big) \leq C_2 n, \quad \text{for all } n \geq 1.$$

More precise results have been few. Benjamini, Kalai and Schramm (2003) gave an example which showed that for first-passage percolation on $\mathbb{Z}^d$ for $d \geq 2$, with $\{a, b\}$-valued passage times, where $0 < a < b < \infty$, there is a constant C such that

$$\mathrm{Var}\big(T(\mathbf{n})\big) \leq C\frac{n}{\log n}, \quad \text{for all } n \geq 2. \tag{1.5}$$

This result was later extended by Benaïm and Rossignol (2006, 2008) to include a wider class of passage time distributions. This is still far from what is believed to be the precise growth rate of $\mathrm{Var}\big(T(\mathbf{n})\big)$. For $d = 2$ it is believed that $\mathrm{Var}\big(T(\mathbf{n})\big)$ is of the order $n^{2/3}$, and it is not clear which behaviour to expect in higher dimensions (see Newman and Piza (1995); Benjamini et al. (2003) for short resumés). For $d = 2$ Newman and Piza (1995) have shown, additionally assuming that the passage-time distribution does not have a too big point mass at $\inf\{x \geq 0 :\, \mathrm{P}_\tau([0, x]) > 0\}$, that there is a constant $C > 0$ such that

$$\mathrm{Var}\big(T(\mathbf{n})\big) \geq C \log n,$$

for all $n \geq 1$. The same lower bound was found independently by Pemantle and Peres (1994), in the case of exponential passage times.

1.1 Classical limit theorems on 1-dimensional graphs

In this paper we consider first-passage percolation on essentially 1-dimensional periodic graphs defined as follows.

Definition 1.2. *The class of* essentially 1-dimensional periodic graphs *consists of all connected graphs $\mathcal{G}$ that can be constructed in the following manner. Let $\{\mathcal{G}_n\}_{n\in\mathbb{Z}}$ be a sequence of identical copies of some finite connected deterministic graph, each with set of vertices $\mathbb{V}_{\mathcal{G}_n} = \{v_{n,1}, \ldots, v_{n,K}\}$ and set of edges $\mathbb{E}_{\mathcal{G}_n} = \{e_{n,1}, \ldots, e_{n,l}\}$. Fix a nonempty set $J \subseteq \{(i,j) : 1 \leq i,j \leq K\}$, and connect $\mathcal{G}_n$ to $\mathcal{G}_{n+1}$ for each n by adding an edge $e(v_{n,i}, v_{n+1,j})$ between $v_{n,i}$ and $v_{n+1,j}$, for each $(i,j) \in J$. Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ denote the resulting graph, where*

$$\mathbb{V} = \bigcup_{n\in\mathbb{Z}} \mathbb{V}_{\mathcal{G}_n} \quad \text{and} \quad \mathbb{E} = \bigcup_{n\in\mathbb{Z}} \big(\mathbb{E}_{\mathcal{G}_n} \cup \{e(v_{n,i}, v_{n+1,j}) : (i,j) \in J\}\big).$$

We will write $\mathbb{E}^*_{\mathcal{G}_n}$ for $\mathbb{E}_{\mathcal{G}_n} \cup \{e(v_{n,i}, v_{n+1,j}) : (i,j) \in J\}$, and say that a vertex v of $\mathcal{G}$ is at level n *if $v \in \mathbb{V}_{\mathcal{G}_n}$.*

An essentially 1-dimensional periodic graph of particular interest is the $\mathbb{Z} \times \{0, 1, \ldots, K-1\}^{d-1}$ nearest neighbour graph, i.e., the sub-graph of the $\mathbb{Z}^d$ lattice which has set of vertices $\mathbb{Z} \times \{0, 1, \ldots, K-1\}^{d-1}$ for some $d, K \geq 1$, and where any two vertices are connected by an edge if their Euclidean distance is 1. We will refer to this graph as the (K,d)-*tube* (cf. Figure 1). We can think

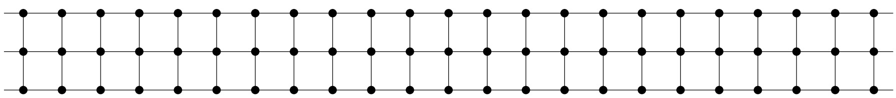

Figure 1: A part of the $(3,2)$-tube.

of the (K,d)-tube as the essentially 1-dimensional periodic graph constructed from a sequence of graphs with vertex set $\mathbb{V}_{\mathcal{G}_n} = \{n\} \times \{0, 1, \ldots, K-1\}^{d-1}$ and where any two vertices at Euclidean distance one are joined by an edge. With this construction, the vertices at level n are exactly the ones with first coordinate n.

Because of the unspecified structure of the underlying graph, it is convenient to consider

$$T_n := T(I, \mathbb{V}_{\mathcal{G}_n}), \tag{1.6}$$

interpreted as the time until a vertex at level n is infected. To consider T_n is natural, but is in no way necessary for the results we obtain. In fact, we shall see that the asymptotic behaviour of the sequence $\{T_n\}_{n\geq 1}$ is the same as that for the sequence $\{\max_{v\in\mathbb{V}_{\mathcal{G}_n}} T(v)\}_{n\geq 1}$, and the sequence $\{T(v_n)\}_{n\geq 1}$, where $\{v_n\}_{n\geq 1}$ is any sequence of vertices such that v_n is at level n. We will for that reason let $\{\hat{T}_n\}_{n\geq 1}$ denote any of the three sequences above, and state several of our results for $\hat{T}_n$. It will then be understood that the result holds for any of the three sequences.

Our main results concerns first-passage percolation on any essentially 1-dimensional periodic graph $\mathcal{G}$, with passage-time distribution that does not concentrate all mass at a single point. We will prove that there are non-negative, finite constants $\mu = \mu(\mathcal{G})$ and $\sigma = \sigma(\mathcal{G})$, such that the following holds.

Theorem 1.3 (Law of Large Numbers)**.** *If* $\mathrm{E}[\tau_e] < \infty$*, then*

$$\lim_{n\to\infty} \frac{\hat{T}_n}{n} = \mu, \quad \textit{almost surely.} \tag{1.7}$$

If $\mathrm{E}[\tau_e^r] < \infty$ *for some* $r \geq 1$*, then*

$$\left\{ \left(\hat{T}_n / n \right)^r \right\}_{n\geq 1} \quad \textit{is uniformly integrable,}$$

and the convergence of (1.7) holds also in L^r*.*

Theorem 1.4 (Central Limit Theorem)**.** *If* $\mathrm{E}[\tau_e^2] < \infty$*, then*

$$\frac{\hat{T}_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \chi, \quad \textit{in distribution,}$$

as $n \to \infty$*, where* χ *has a standard normal distribution.*

Let $\mathcal{L}(\{x_n\}_{n\geq 1})$ denote the set of limit points of a real-valued sequence $\{x_n\}_{n\geq 1}$.

Theorem 1.5 (Law of the Iterated Logarithm)**.** *If* $\mathrm{E}[\tau_e^2] < \infty$*, then*

$$\mathcal{L}\left(\left\{ \frac{\hat{T}_n - \mu n}{\sigma\sqrt{2n\log\log n}} \right\}_{n\geq 3} \right) = [-1, 1], \quad \textit{almost surely.}$$

In particular, almost surely,

$$\limsup_{n\to\infty} \frac{\hat{T}_n - \mu n}{\sigma\sqrt{2n\log\log n}} = 1, \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\hat{T}_n - \mu n}{\sigma\sqrt{2n\log\log n}} = -1.$$

Note that the almost sure and L^1-convergence in Theorem 1.3 actually follows from Kingman's Subadditive Ergodic Theorem. However, it is for the understanding of our approach instructive to state and reprove it, as we do. As a consequence of the regenerative structure explored in Section 2, μ and σ will be given by explicit formulas. For this reason, it will become clear that μ and σ are non-negative, finite, and depend on the underlying graph $\mathcal{G}$ and the passage time distribution, but do not depend on the set of initially infected vertices I, nor on which of the three sequences considered that $\{\hat{T}_n\}_{n\geq 1}$ may represent. We will see (in Proposition 5.7) that μ and σvaries continuously with respect to P_τ. We preferred at this stage to give simple moment conditions in Theorems 1.3, 1.4 and 1.5. But we will later point out that they may in fact be relaxed somewhat (cf. Remark 3.4).

At a comparison with the asymptotic results in higher dimensions, Theorem 1.3 is the 1-dimensional analogue to the Shape Theorem. Theorems 1.4 and 1.5 on the other hand, point out a 1-dimensional behaviour that is not expected in higher dimensions. In particular, $\mathrm{Var}(T_n)$ grows linearly in n, in contrast to the higher dimensional sub-diffusive behaviour in (1.5), pointed out by Benjamini et al. (2003). However, we should also mention a result by Kesten and Zhang (1997) when $d = 2$ and $\mathrm{P}_\tau(\{0\}) = p_c(2) = 1/2$. They have showed that $T(\mathbf{n}) - \mathrm{E}[T(\mathbf{n})]$ converges to a standard normal distribution, when scaled appropriately. This case is considered critical, and the scaling factor is known to grow of order $\log n$.

The classical Central Limit Theorem for i.i.d. sequences extends to a functional central limit theorem, known as *Donsker's theorem*. In contrast to the classical Central Limit Theorem that treats weak convergence of real-valued random variables, Donsker's theorem treats weak convergence of real-valued random functions. Theorem 1.4 also extends to a functional version, with the same limiting distribution as the regular Donsker theorem, i.e., Wiener measure. For the precise statement and a proof, see Theorem 3.6.

We should at this point mention a related, but independent, work by Chatterjee and Dey (2009). They consider first-passage percolation on nearest neighbour graphs of the form $\mathbb{Z} \times \{-K, \dots, K\}^{d-1}$. In our terminology, this is precisely the $(2K + 1, d)$-tube. Introduce the notation $a_n(K)$ for the passage time $T(\mathbf{0}, \mathbf{n})$ between the origin and $\mathbf{n}$ on that graph. Their main result essentially says that if $\mathrm{E}[\tau_e^r] < \infty$ for some $r > 2$, then there exists $\alpha = \alpha(d, r)$ such that if $K_n = o(n^\alpha)$, then

$$\frac{a_n(K_n) - \mathrm{E}[a_n(K_n)]}{\sqrt{\mathrm{Var}\left(a_n(K_n)\right)}} \xrightarrow{d} \chi, \quad \text{in distribution,} \tag{1.8}$$

as $n \to \infty$, where χ has a standard normal distribution. When $\mathrm{E}[\tau_e^r] < \infty$ for all $r \geq 1$, then $\alpha < 1/(d + 1)$ is sufficient for (1.8) to hold. This result is similar to our Theorem 1.4, and applies to cases that Theorem 1.4 does not. The method of proof used in Chatterjee and Dey (2009) is different from ours, and we note that they require a slightly stronger moment condition than we do with our techniques (see also Remark 3.4). In Chatterjee and Dey (2009), (1.8) is also extended to hold for graphs of the form $\mathbb{Z} \times \mathcal{G}$, which is a subclass to the class of essentially 1-dimensional periodic graphs defined in Definition

1.2. Moreover, (1.8) extends to a functional central limit theorem similar to our Theorem 3.6. Again here, Chatterjee and Dey require that $\mathrm{E}[\tau_e^r] < \infty$ for some $r > 2$ in order for the functional limit theorem to hold. We emphasise that the present work was prepared simultaneously and independently of the work by Chatterjee and Dey (2009) by methods distinct from those in their paper. There seem to be advantages with the techniques used in this paper, as well as with the techniques used by Chatterjee and Dey. To further exclude questions of originality, we also mention that this paper is an extended version of the earlier manuscript Ahlberg (2008), in which several of the results presented here were included, among them Theorem 1.4. Theorem 1.4 was also proved by Schlemm (2011) in the particular case of the $(2, 2)$-tube with exponential passage times.

1.2 Monotonicity of mean and variance

It seems natural to believe that the mean and variance of $T(u, v)$ increase with the distance between u and v. We will prove two theorems concerning this.

Theorem 1.6. *Let $v_{n,i}$ denote a specific vertex at level n. For all $i = 1, \ldots, K$, if $\mathrm{E}[\tau_e] < \infty$, then for some $C_i \in \mathbb{R}$, as $n \to \infty$,*

$$\mathrm{E}[T(v_{n,i})] \;=\; \mu n + C_i + o(1).$$

A direct consequence of this result is that

$$\mathrm{E}[T(v_{n+1,i}) - T(v_{n,i})] \to \mu, \quad \text{as } n \to \infty.$$

Since $\mu > 0$, this proves monotonicity of $\mathrm{E}[T(v_{n,i})]$, for large n. This question dates back to Hammersley and Welsh (1965). That we have such monotonicity on $\mathbb{Z}$ is completely trivial, but on $\mathbb{Z}^d$ for $d \geq 2$ it is still an open problem to solve. A counterexample given by van den Berg (1983) shows that such monotonicity result for the expected travel time from $(0, 0)$ to $(n, 0)$ on the $\{0, 1, \ldots, n\} \times \mathbb{Z}$ nearest neighbour graph does not hold for every n. This indicates that it might not be possible to extend Theorem 1.6 to say that the mean travel time is monotonous for *all* n. The same remark should concern also the following result which proves monotonicity of the variance of the travel time for large n.

Theorem 1.7. *Let $v_{n,i}$ denote a specific vertex at level n. For all $i = 1, \ldots, K$, if $\mathrm{E}[\tau_e^2] < \infty$, then for some $C_i \in \mathbb{R}$, as $n \to \infty$,*

$$\mathrm{Var}\big(T(v_{n,i})\big) \;=\; \sigma^2 n + C_i + o(1).$$

1.3 Asymptotic behaviour of geodesics

First-passage percolation offers more than describing the behaviour of the passage time between vertices. One matter which has received a lot of attention is along which edges (path) an infection travels from one vertex to another. Do such paths exist, and if they do, how do they behave? On the $\mathbb{Z}^d$ lattice such minimising paths are known to exist for passage-time distributions with not too big point mass at zero (see Howard (2004) for more precision of the statement). A simple argument that shows that such paths exist on essentially 1-dimensional periodic graphs (for any passage-time distribution) is given in Proposition 5.1. As customary, we will use the term *geodesic* to refer to a path $\gamma(u, v)$ attaining the minimal passage time, i.e., such that $T(\gamma(u, v)) = T(u, v)$. Geodesics are not necessarily unique when the passage-time distribution has atoms (for continuous distributions they are; cf. Proposition 5.1). For this reason, fix a deterministic rule to choose one when several are possible (e.g. the shortest, with some additional rule for breaking ties).

Let N_n and $N(v)$ denote the length of the geodesic realising T_n and $T(v)$, respectively. Let $\{\hat{N}_n\}_{n\geq 1}$ denote either of the sequences $\{\max_{v\in\mathbb{V}_{\mathcal{G}_n}} N(v)\}_{n\geq 1}$, $\{N_n\}_{n\geq 1}$ and $\{N(v_n)\}_{n\geq 1}$, where $\{v_n\}_{n\geq 1}$ is any sequence of vertices such that v_n is at level n. We state here the following result, and refer the reader to Section 5 and Theorem 5.2 and 5.3, for additional result concerning asymptotics of length of geodesics.

Theorem 1.8. *There is a finite constant α such that, for any $r \geq 1$,*

$$\lim_{n\to\infty} \frac{\hat{N}_n}{n} = \alpha, \quad \textit{almost surely and in } L^r.$$

On the $\mathbb{Z}^2$ lattice, Zhang and Zhang (1984) showed that a similar strong law, as the one exhibited in the above theorem, holds for "supercritical" passage-time distributions, i.e., passage-time distributions such that $\mathrm{P}_\tau(\{0\}) > 1/2$. Moreover, Garet and Marchand (2004, 2007) have considered the related case of first-passage percolation on the $\mathbb{Z}^d$ lattice with passage times distributed as $\mathrm{P}_\tau = p\delta_1 + (1-p)\delta_\infty$, for some $p > p_c(d)$. In this situation the length of the geodesic between two vertices equals the passage time between them (given that it is finite). Assume that the origin lies in the unique infinite cluster, and for $z \in \mathbb{Z}^d$ let $\{u_{n,z}\}_{n\geq 1}$ denotes the subsequence of $\{n\}_{n\geq 1}$ such that $u_{n,z}z$ lies in the infinite cluster. They showed that, almost surely,

$$\exists \lim_{n\to\infty} \frac{T(0, u_{n,z}z)}{u_{n,z}}, \quad \text{uniformly in } z \in \mathbb{Z}^d.$$

They further prove exponential decay of deviations away from this limit. If the same limiting behaviour, as in the above theorem, holds for general passage-time distributions on the $\mathbb{Z}^d$ lattice is not known (see Howard (2004) for further reference).

1.4 The (K, d)-tube case

First-passage percolation on (K, d)-tubes is of particular interest, since it can be compared in a natural way to first-passage percolation on the $\mathbb{Z}^d$ lattice. As an example of such a comparison, we can see how Theorem 1.3 is a 1-dimensional analogue to the Shape Theorem. Replace B_t with the set $\tilde{B}_t$ as in (1.3). Let μ_K denote the time constant of Theorem 1.3 associated with the (K, d)-tube, and set

$$B^* = B^*(t) = \left[-\mu_K^{-1}, \mu_K^{-1}\right] \times [0, K/t]^{d-1}.$$

The almost sure convergence in Theorem 1.3 is then equivalent to that for all $\epsilon > 0$, almost surely,

$$(1-\epsilon)B^* \subset \frac{1}{t}\tilde{B}_t \subset (1+\epsilon)B^*, \quad \text{for large } t. \tag{1.9}$$

We can in fact allow ϵ to tend to zero with t. The precise size of the fluctuations in (1.9) follows from Theorem 1.5. We refer the reader to Corollary 3.5 for the precise statement.

Let μ_K denote the time constant associated with the (K, d)-tube (for fixed d). A simple coupling argument shows that $\mu_{K+1} \leq \mu_K$. In fact strict inequality holds for all $K \geq 1$ (cf. Proposition 5.10). Apart from being decreasing, the sequence $\{\mu_K\}_{K \geq 1}$ is bounded below by $\mu(\mathbf{e}_1)$. Thus, the sequence is convergent. In Proposition 5.11 we prove that

$$\lim_{K \to \infty} \mu_K = \mu(\mathbf{e}_1).$$

This shows that the rate of growth of an infection on the (K, d)-tube approaches the rate of growth of an infection on the $\mathbb{Z}^d$ lattice, as K increases. Does the same monotonic behaviour hold for the constants σ_K^2 and α_K, that appear in Theorem 1.4 and 1.8, associated with the (K, d)-tube? There is no argument known to us that implies monotonicity. In view of the belief of the fluctuations in higher dimensions, and the sub-diffusive behaviour shown in (1.5), it seems reasonable to believe that σ_K^2 tends to zero as $K \to \infty$.

To compute the actual values of the constants μ and μ_K (for arbitrary K) does not seem to be practically possible. However, a closed form expression was obtained for the time constant for the $(2, 2)$-tube (that is μ_K with $K = d = 2$) with exponential passage times by both Schlemm (2009) and Renlund (2010).

1.5 Coupling and a 0–1 law

Another main part of this paper consists of the construction of a coupling of two first-passage percolation infections. As an application of the coupling we prove a 0–1 law. Define the σ-algebra $\mathcal{T}_t := \sigma(\{B_s\}_{s\geq t})$ and the tail σ-algebra $\mathcal{T} := \cap_{t\geq 0}\mathcal{T}_t$. We may think of $\mathcal{T}_t$ as the σ-algebra of events that do not depend on the times at which vertices are infected before time t.

Theorem 1.9 (0–1 law). *Consider first-passage percolation on an essentially 1-dimensional periodic graph $\mathcal{G}$, with a finite set of initially infected vertices. Assume that the passage time distribution has an absolutely continuous component (with respect to Lebesgue measure). Then* $\mathrm{P}(A) \in \{0,1\}$, *for any event* $A \in \mathcal{T}$.

The 0–1 law follows from an application of the following coupling.

Proposition 1.10 (Coupling). *Let I and I' be finite subsets of the set of vertices of an essentially 1-dimensional periodic graph $\mathcal{G}$. Assume that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). There exists a coupling of $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ such that $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ form sequences of i.i.d. random variables with distribution P_τ, and if first-passage percolation is performed with $(I, \{\tau_e\}_{e\in\mathbb{E}})$ and $(I', \{\tau'_e\}_{e\in\mathbb{E}})$, respectively, then with probability one there exists $T_c < \infty$, such that*

$$B_t = B'_t, \quad \textit{for all } t \geq T_c.$$

A similar coupling is presented also for discrete passage time distributions, but then on the more restrictive class of (K, d)-tubes (cf. Proposition 6.2). Theorem 1.9 is extended to include this case as well. Motivating examples are given to show why it is not possible to make the coupling as general as Proposition 1.10 also in the discrete case (cf. Remark 6.6 and 6.7).

The mild condition of an absolutely continuous component to be sufficient for the 0–1 law on essentially 1-dimensional periodic graphs, opens up for a discussion. We do not know on which other graphs this condition is sufficient. But, we give an example showing that a 0–1 law analogous to Theorem 1.9 cannot hold on the binary tree $\mathbb{T}^2$. An interesting and challenging case to settle would be on the $\mathbb{Z}^d$ lattice.

The main results of this paper will be based on a "regenerative" nature that arises for first-passage percolation on essentially 1-dimensional periodic graphs. What we mean by a regenerative behaviour will be clarified in the next section, where we also derive the properties of the regenerative structure that will recur throughout this paper. As will become apparent, the idea is to identify a suitable regenerative sequence (cf. Definition 2.1). How the regenerative behaviour arises naturally for exponentially distributed passage times is illustrated in Section 2.1. The general case is thereafter treated in detail in Section 2.2.

Once the regenerative behaviour is understood, some of the results we provide will follow, either from simple arguments or from already known results. Other results that we provide do not follow as easily, and will require an essential amount of additional work. This will be emphasised in connection with their proofs. In Section 3, the regenerative behaviour is used to prove Theorems 1.3, 1.4 and 1.5, among others. Monotonicity of mean and variance of the travel time, i.e., Theorem 1.6 and 1.7, is proved in Section 4. Section 5 is dedicated to study geodesics and properties of μ, σ and α. In the final Section 6 the coupling of Proposition 1.10 is constructed, in its continuous and its discrete version. The 0–1 law Theorem 1.9 is also derived and the counterexample to the 0–1 law on trees is presented at the very end.

2 Regenerative behaviour

Definition 2.1. *We say that a sequence $\{X_k\}_{k\geq 1}$ of random variables is a* regenerative sequence *if there exists an increasing sequence of random variables $\{\lambda_k\}_{k\geq 0}$ such that*

a) $\{\lambda_k - \lambda_{k-1}\}_{k\geq 1}$ forms an i.i.d. sequence, and

b) $\{X_{\lambda_k} - X_{\lambda_{k-1}}\}_{k\geq 1}$ forms a non-negative i.i.d. sequence.

We will refer to $\{\lambda_k\}_{k\geq 0}$ as the sequence of regenerative levels*.*

Some readers may recognise the sub-sequence $\{X_{\lambda_k}\}_{k\geq 0}$ as a renewal sequence, and the sequence $\big\{(X_{\lambda_k}, \lambda_k)\big\}_{k\geq 0}$ as a 2-dimensional renewal sequence.

The idea of how to identify a suitable regenerative sequence arises naturally for first-passage percolation with exponentially distributed passage times. We begin with an illustration of this on the $(2,2)$-tube. In Section 2.2 we will generalise this idea to concern general passage time distributions, and any essentially 1-dimensional periodic graph.

2.1 Exponential passage times

Let the edges of the $(2,2)$-tube be equipped with i.i.d. exponential passage times $\{\tau_e\}_{e\in\mathbb{E}}$, and let both vertices at level zero be initially infected. At any fixed time t, given the infected component B_t, each edge with exactly one endpoint in the infected component is equally likely to be passed by the infection next. Thus, at each level, with probability at least $1/2$, both vertices will become infected before any vertex at the following level. It follows that with probability one, at some level r, both vertices will become infected before any vertex at level $r+1$. Denote by ρ the first level for which this happens, and let τ_ρ denote the time at which this happens. By the lack-of-memory property, the time it takes for the infection from this moment to reach m levels further has the same distribution as the time it would take to reach level m, i.e.,

$$T_{\rho+m} - \tau_\rho \stackrel{d}{=} T_m. \tag{2.1}$$

In fact, at infinitely many levels, both vertices at that level will be infected before any vertex at higher levels. If we repeat the argument, we generate a sequence of (regenerative) levels $\{\rho_k\}_{k\geq 1}$ (see Figure 2), with corresponding sequence of instants $\{\tau_{\rho_k}\}_{k\geq 1}$, such that (2.1) holds. Since the passage times

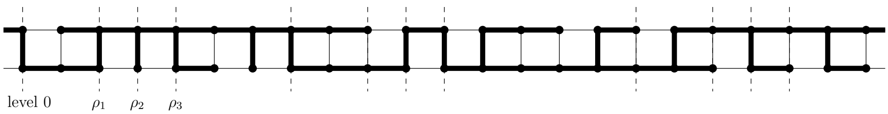

Figure 2: A realisation of the spread of an infection on the $(2,2)$-tube. The broken lines indicate levels at which both vertices will become infected before any vertex ahead.

are i.i.d., the consecutive differences $\rho_{k+1} - \rho_k$ will be i.i.d., as well as the differences $\tau_{\rho_{k+1}} - \tau_{\rho_k}$. It follows that $\left\{\max_{v\in\mathbb{V}_{\mathcal{G}_n}} T(v)\right\}_{n\geq 1}$ is a regenerative sequence.

The point of the regenerative sequence is the following. Note that the nth (regenerative) level and the time at which it occurs may be written as sums of i.i.d. random variables, i.e.,

$$\rho_n = \sum_{k=0}^{n-1} \rho_{k+1} - \rho_k \quad \text{and} \quad \tau_{\rho_n} = \sum_{k=0}^{n-1} \tau_{\rho_{k+1}} - \tau_{\rho_k},$$

where $\rho_0 = 0$ and $\tau_{\rho_0} = 0$. It is easy to see that classical results, such as the Law of Large Numbers, Central Limit Theorem and Law of the Iterated Logarithm, applies to $\{(\tau_{\rho_n}, \rho_n)\}_{n\geq 1}$, the passage time to the nth regeneration, with respect to the level of the same regeneration. Such results can be expanded to include the regenerative sequence in question. This will be further investigated in Section 3.

2.2 The general case

Let us now consider first-passage percolation with general passage time distribution on any essentially 1-dimensional periodic graph. When we refer to an edge *at* some level n, we mean an edge in $\mathbb{E}_{\mathcal{G}_n}$. When we refer to an edge *between* levels n and $n+m$, we mean any edge in $\mathbb{E}^*_{\mathcal{G}_n} \cup \ldots \cup \mathbb{E}^*_{\mathcal{G}_{n+m-1}} \cup \mathbb{E}_{\mathcal{G}_{n+m}}$.

Let M be a positive integer and denote the set of edges between level n and $n+2M$ by E_n. Fix a path γ_n of shortest length between $\mathbb{V}_{\mathcal{G}_n}$ and $\mathbb{V}_{\mathcal{G}_{n+2M}}$, i.e., between two vertices at level n and $n+2M$, respectively. Define the subset $\hat{E}_n$ of E_n as

$$\hat{E}_n := \gamma_n \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M}}. \tag{2.2}$$

Define

$$\begin{aligned} m_\tau &:= \inf\big\{x \geq 0 : \mathrm{P}_\tau\big([0,x]\big) > 0\big\}, \\ M_\tau &:= \sup\big\{x \geq 0 : \mathrm{P}_\tau\big([x,\infty)\big) > 0\big\}. \end{aligned} \tag{2.3}$$

Note that $0 \leq m_\tau < M_\tau \leq \infty$, where the strict inequality holds since we consider only passage-time distributions that do not concentrate all mass at a single point. For constants t' and t'' such that $m_\tau < t' < t'' < M_\tau$, define the regenerative event

$$A_n := \Big\{\tau_e \leq t', \forall e \in \hat{E}_n\Big\} \cap \Big\{\tau_e \geq t'', \forall e \in E_n \setminus \hat{E}_n\Big\}. \tag{2.4}$$

The event A_n is depicted in Figure 3. Trivially $\mathrm{P}(A_n) > 0$. The vertex at

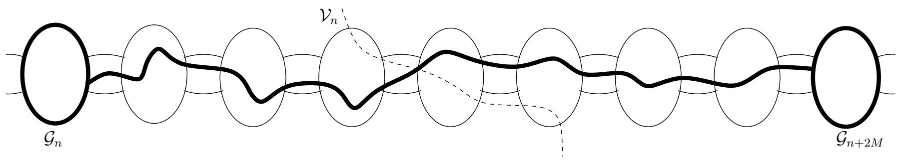

Figure 3: The graph $\mathcal{G}$ between level n and $n+2M$. If A_n occurs, the thick edges at level n, level $n+2M$ and of the path γ_n are "quick".

level $n+M$ first reached via γ_n will be of particular interest, so we introduce the following notation.

Definition 2.2. *Let $\hat{v}_n$ denote the vertex at level n first reached via γ_{n-M}. That is, $\hat{v}_{n+M}$ denotes the vertex at level $n+M$ first reached via γ_n.*

We will require to consider random variables conditioned on the occurrence of events like A_n. We will therefore need a notion of conditional independence. Two random variables X and Y are said to be *conditionally independent* given A, if the random variables X conditioned on A, and Y conditioned on A, are independent.

Lemma 2.3. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$. Then there exists $M \in \mathbb{N}$, such that:*

a) If A_n occurs, then for all $u \in \bigcup_{k\leq n} \mathbb{V}_{\mathcal{G}_k}$ and $v \in \bigcup_{k\geq n+2M} \mathbb{V}_{\mathcal{G}_k}$

$$T(u,v) = T(u,\hat{v}_{n+M}) + T\left(\hat{v}_{n+M}, v\right), \tag{2.5}$$

and $T(\Gamma) > T(u,v)$ for any path Γ between u and v that does not visit $\hat{v}_{n+M}$.

b) $T(u,\hat{v}_{n+M})$ and $T\left(\hat{v}_{n+M}, v\right)$ are conditionally independent given A_n. In addition, given A_n, $T(u,\hat{v}_{n+M})$ is conditionally independent of the passage time of any edge beyond level $n+2M$, and $T(\hat{v}_{n+M}, v)$ is conditionally independent of the passage time of any edge before level n.

Proof. It suffice to prove the lemma for $u \in \mathbb{V}_{\mathcal{G}_n}$ and $v \in \mathbb{V}_{\mathcal{G}_{n+2M}}$. For given t' and t'', choose

$$M > \frac{|\mathbb{E}_{\mathcal{G}_n}|t'}{t''-t'},$$

where $|\cdot|$ denotes the cardinality of the set. Set $\beta := \mathrm{dist}(\hat{v}_{n+M}, \mathbb{V}_{\mathcal{G}_{n+2M}})$, where $\mathrm{dist}(v,V)$ denotes the smallest number of edges one has to pass in order to reach a vertex of V from v, and define (see Figure 3)

$$\mathcal{V}_n := \left\{ v \in \bigcup_{j=n}^{n+2M} \mathbb{V}_{\mathcal{G}_j} : \mathrm{dist}(v, \mathbb{V}_{\mathcal{G}_{n+2M}}) = \beta \right\}.$$

We will prove that, given A_n,

$$T(u,\hat{v}_{n+M}) < T(u,w) \quad \text{and} \quad T(\hat{v}_{n+M}, v) < T(w,v) \tag{2.6}$$

for all $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. This proves that $T(\Gamma) > T(u,v)$ for all paths Γ between u and v that does not visit $\hat{v}_{n+M}$, since each path from u to v has to pass some vertex in $\mathcal{V}_n$. Thus, also (2.5) holds. That $T(u,\hat{v}_{n+M})$ and $T\left(\hat{v}_{n+M}, v\right)$ are conditionally independent given A_n is easily seen from the following observation. When A_n occurs, if follows from (2.6) that $T(u,\hat{v}_{n+M})$ is the infimum of $T(\Gamma)$ over all paths Γ from u to $\hat{v}_{n+M}$ that intersects $\mathcal{V}_n$ only in $\hat{v}_{n+M}$, whereas $T\left(\hat{v}_{n+M}, v\right)$ is the infimum of $T(\Gamma)$ over all paths Γ from $\hat{v}_{n+M}$ to v that intersects $\mathcal{V}_n$ only in $\hat{v}_{n+M}$. Hence, the infima of passage times are taken over paths in disjoint parts of the graph. The remaining statement in *b)* follows similarly.

To deduce (2.6), condition on A_n. By definition of γ_n and $\mathcal{V}_n$,

$$T(w',\hat{v}_{n+M}) < T(w',w)$$

for any vertex w' visited by γ_n, and $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. Let γ_n^- denote the part of the path γ_n between $\mathbb{V}_{\mathcal{G}_n}$ and $\hat{v}_{n+M}$. Let Γ be any path from u to $\mathcal{V}_n$ disjoint from γ_n^-. Note that

$$T(u, \hat{v}_{n+M}) \leq \left(|\mathbb{E}_{\mathcal{G}_n}| + |\gamma_n^-|\right) t' \quad \text{and} \quad T(\Gamma) \geq |\gamma_n^-| t''.$$

(Here γ_n^- is identified with its set of edges.) By the choice of M,

$$T(\Gamma) - T(u, \hat{v}_{n+M}) \geq (t'' - t')|\gamma_n^-| - |\mathbb{E}_{\mathcal{G}_n}|t' \geq (t'' - t')M - |\mathbb{E}_{\mathcal{G}_n}|t' > 0.$$

This proves that $T(u, \hat{v}_{n+M}) < T(u, w)$ for all $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. The proof of the remaining inequality in (2.6) is similar. □

Assume from now on that t', t'' and M are chosen in accordance with Lemma 2.3. We will next introduce an auxiliary random variable Δ. Throughout this paper, Δ will denote any bounded integer-valued random variable independent of $\{\tau_e\}_{e \in \mathbb{E}}$. The auxiliary random variable is not necessary in order to derive the regenerative behaviour we do in this section. In fact, Δ is of no importance to most of our results in this paper. We will in Section 3 set $\Delta \equiv 0$. However, Δ will play a rôle in Section 4, where we need to be more careful to prove monotonicity of mean and variance. At this point we do not specify its distribution further, other than having bounded support.

Let $\rho_I := \max\{n \in \mathbb{Z} : \mathbb{V}_{\mathcal{G}_n} \cap I \neq \emptyset\}$ denote the furthest initially infected level. Define

$$n_k := \rho_I + \Delta + k(2M + 1), \quad \text{for } k \in \mathbb{Z},$$

and note that the sequence of events $\{A_{n_k}\}_{k \in \mathbb{Z}}$ is readily seen to be i.i.d. Let $\kappa = \min\{k \geq 0 : A_{n_k} \text{ occurs}\}$ and set $\rho_0 := n_\kappa + M$. Define further

$$\begin{aligned} \rho_k &:= M + \min\{n_m : n_m > \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\}, && \text{for } k \geq 1, \\ \rho_k &:= M + \max\{n_m : n_m + M < \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\}, && \text{for } k \leq -1. \end{aligned}$$

Since $\{A_{n_k}\}_{k \in \mathbb{Z}}$ is i.i.d. and $\mathrm{P}(A_{n_k}) > 0$, the second Borel-Cantelli lemma gives that

$$\mathrm{P}\left(A_{n_k} \text{ occurs for infinitely many } k \geq 0\right) = 1.$$

The same holds for $k \leq 0$. This generates a sequence $\{\rho_k\}_{k \in \mathbb{Z}}$, where ρ_k is almost surely finite.

Note that $\rho_k \geq \rho_I + M$ for $k \geq 0$. Thus, for $k \geq 0$, Lemma 2.3 says that each path along which any vertex at level $\rho_k + M$ and beyond is infected has to pass the vertex $\hat{v}_{\rho_k}$.

Definition 2.4. *A vertex $\hat{v}_n$ will be referred to as a* regeneration point *if $n = \rho_k$ for some $k \geq 0$.*

For $k \in \mathbb{Z}$, define

$$S_k := \rho_k - \rho_{k-1}, \quad \text{and} \quad \tau_{S_k} := T(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}).$$

For $k \geq 1$, S_k denotes the distance (measured in levels) between two regeneration points, and τ_{S_k} denotes the passage time between two regeneration points. By Lemma 2.3 we see that $\tau_{S_k} = T(\hat{v}_{\rho_k}) - T(\hat{v}_{\rho_{k-1}})$ for $k \geq 1$. With this nota-

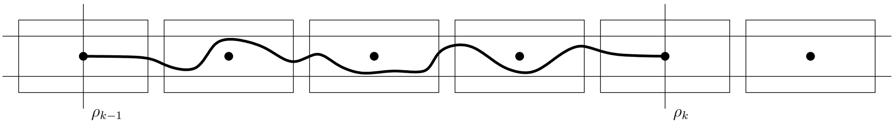

Figure 4: A schematic picture of a graph, in which boxes indicate locations of the sequence $\{A_{n_k}\}_{k\geq 0}$, vertical lines indicate the sequence $\{\rho_k\}_{k\geq 0}$, and dots indicate $\{\hat{v}_{n_k}\}_{k\geq 0}$. The distance between the two vertical lines is S_k, and the thick curve indicates τ_{S_k}.

tion, we may for $n \geq 0$ write the level of the nth regeneration, and the time it takes for the infection to reach the nth regeneration, as

$$\rho_n = \rho_0 + \sum_{k=1}^{n} S_k, \quad \text{and} \quad T(\hat{v}_{\rho_n}) = T(\hat{v}_{\rho_0}) + \sum_{k=1}^{n} \tau_{S_k},$$

respectively.

Lemma 2.5. *Assume that t', t'' and M are chosen in accordance with Lemma 2.3. Then, $\{(\tau_{S_k}, S_k)\}_{k\in\mathbb{Z}}$ forms a sequence of i.i.d. $[0,\infty)\times\mathbb{Z}_+$-valued random variables.*

Proof. That $\{S_k\}_{k\in\mathbb{Z}}$ is an i.i.d. sequence of geometrically distributed random variables, times a factor $2M+1$ can easily be seen, since the events A_{n_k} are pairwise independent with equal success probabilities.

Note that τ_{S_k} is a random variable of the form $T(\hat{v}_{n_i+M}, \hat{v}_{n_j+M})$, for some $i < j$, conditioned (in particular) on the occurrence of the events A_{n_i} and A_{n_j}. Thus, independence of τ_{S_k} and τ_{S_l} for $k \neq l$ follows from Lemma 2.3 part *b)*. That they are identically distributed is due to the events A_{n_k} being pairwise independent with equal success probabilities. □

Proposition 2.6. *The sequence $\{T(\hat{v}_n)\}_{n\geq 1}$ is a regenerative sequence. Moreover, if t', t'' and M are chosen in accordance with Lemma 2.3, then $\{\rho_n\}_{n\geq 0}$ is a sequence of regenerative levels for $\{T(\hat{v}_n)\}_{n\geq 1}$, such that*

$$T(v_{\rho_n+m,i}) - T(\hat{v}_{\rho_n}) \stackrel{d}{=} T(v_{\rho_1+m,i}) - T(\hat{v}_{\rho_1}), \quad \textit{for all } m \geq M, n \geq 1,$$

where superscript d indicates that the equality holds in distribution.

Proof. That $\{T(\hat{v}_n)\}_{n\geq 1}$ is a regenerative sequence with sequence of regenerative levels $\{\rho_n\}_{n\geq 0}$ follows from Lemma 2.5. By Lemma 2.3,

$$T(v_{\rho_n+m,i}) - T(\hat{v}_{\rho_n}) = T(\hat{v}_{\rho_n}, v_{\rho_n+m,i})$$

for $m \geq M$, whose distribution is independent of n, by definition of A_n. $\square$

Let $\mu_\tau := \mathrm{E}[\tau_{S_k}]$ and $\mu_S := \mathrm{E}[S_k]$ denote the expected passage time and distance between two regeneration points, respectively, and define

$$\mu := \frac{\mu_\tau}{\mu_S}, \quad \text{and} \quad \sigma^2 := \frac{\mathrm{Var}(\tau_{S_k} - \mu S_k)}{\mu_S}. \tag{2.7}$$

It is immediate from the construction that the distributions of S_k and τ_{S_k} (and therefore also μ and σ^2) does not depend on the set of initially infected vertices I, nor on Δ. We will in next section see that μ and σ^2 appear as the constants that figure in Theorem 1.3, 1.4 and 1.5. In order to state clear moment conditions, we will also need to know how moments of τ_e relate to moments of S_k and τ_{S_k}. This is given in the following proposition.

Proposition 2.7. *Assume that the passage time distribution P_τ does not concentrate all mass in a single point. Then,*

a) there exists an $\alpha > 0$ such that $\mathrm{E}\left[e^{\alpha S_k}\right] < \infty$.

Assume further that there are $p \geq 1$ (edge) disjoint paths from $\hat{v}_0$ to $\hat{v}_1$. Let $Y = \min(\tau_1, \dots, \tau_p)$, where $\tau_1, \dots, \tau_p$ are independent and distributed as P_τ. Then,

b) if $\mathrm{E}[Y^\alpha] < \infty$, for some $\alpha > 0$, we have $0 < \mathrm{E}\left[\tau_{S_k}^\alpha\right] < \infty$.

In particular, if $\mathrm{E}[\tau_e^\alpha] < \infty$, then $\mathrm{E}[\tau_{S_k}^\alpha] < \infty$, and if $\mathrm{E}[\tau_{S_k}^\alpha] < \infty$ for $\alpha = 1$, and $\alpha = 2$ respectively, then

$$0 < \mu < \infty, \quad \textit{and} \quad 0 < \sigma^2 < \infty.$$

Proof. a) Recall that if θ is geometrically distributed with parameter $p_A = \mathrm{P}(A_n)$, then $S_k \overset{d}{=} (2M+1)\theta$. In particular, $0 < \mathrm{E}[S_k^\alpha] < \infty$ for $\alpha > 0$. Moreover,

$$\begin{aligned}
\mathrm{E}[e^{\alpha S_k}] &= \sum_{n=1}^{\infty} e^{\alpha(2M+1)n}(1-p_A)^{n-1}p_A \\
&= e^{\alpha(2M+1)}p_A \sum_{n=1}^{\infty} \left(e^{\alpha(2M+1)}(1-p_A)\right)^{n-1},
\end{aligned}$$

which is finite if $e^{\alpha(2M+1)}(1-p_A)<1$.

b) Let $\Gamma_j^{(1)},\dots,\Gamma_j^{(p)}$ denote the p disjoint paths from $\hat{v}_{j-1}$ to $\hat{v}_j$. Note that subadditivity gives

$$\tau_{S_k} \;\le\; \sum_{j=\rho_{k-1}+1}^{\rho_k} T(\hat{v}_{j-1},\hat{v}_j). \tag{2.8}$$

For any edge $e\in\mathbb{E}$ we have

$$\begin{aligned} \mathrm{P}(\tau_e>t|\,A_n) \;&\le\; \mathrm{P}(A_n)^{-1}\mathrm{P}(\tau_e>t),\\ \mathrm{P}(\tau_e>t|\,A_n^c) \;&\le\; \mathrm{P}(A_n^c)^{-1}\mathrm{P}(\tau_e>t). \end{aligned} \tag{2.9}$$

Set $\Lambda_n:=\{S_k=(2M+1)n\}$. Note that Λ_n is of the form $\bigcap_{i\in I}A_{n_i}\bigcap_{j\in J}A_{n_j}^c$ for disjoint sets $I,J\subseteq\{l,l+1,\dots,l+n\}$, where l is such that $n_l=\rho_{k-1}-M$. Hence, it follows from (2.9) that when $e\in\Gamma_j^{(i)}$ for some $i=1,\dots,p$ and $j=\rho_{k-1}+1,\dots,\rho_k$, then

$$\mathrm{P}(\tau_e>t|\,\Lambda_n) \;\le\; C_1\mathrm{P}(\tau_e>t), \tag{2.10}$$

where $C_1=\max\big(\mathrm{P}(A_n)^{-1},\mathrm{P}(A_n^c)^{-1}\big)$. We will next prove that

$$\mathrm{E}\big[T(\hat{v}_{j-1},\hat{v}_j)^\alpha\big|\,\Lambda_n\big] \;\le\; C_2\mathrm{E}[Y^\alpha], \tag{2.11}$$

for $j=\rho_{k-1}+1,\dots,\rho_k$ and some $C_2<\infty$. Let λ denote the length of the longest of the paths $\Gamma_j^{(i)}$. Then (2.11) follows immediately from

$$\begin{aligned} \mathrm{P}\big(T(\hat{v}_{j-1},\hat{v}_j)^\alpha>t\big|\,\Lambda_n\big) \;&\le\; \prod_{i=1}^p \mathrm{P}\left(T(\Gamma_j^{(i)})>t^{1/\alpha}\middle|\,\Lambda_n\right)\\ &\le\; \prod_{i=1}^p\left(\sum_{e\in\Gamma_j^{(i)}}\mathrm{P}\big(\tau_e>t^{1/\alpha}/\lambda\big|\,\Lambda_n\big)\right)\\ &\le\; C_1^p\lambda^p\,\mathrm{P}\big(\tau_e>t^{1/\alpha}/\lambda\big)^p \;=\; C_1^p\lambda^p\,\mathrm{P}\big(Y^\alpha>t/\lambda^\alpha\big), \end{aligned}$$

where the second inequality follows since $T\big(\Gamma_j^{(i)}\big)\ge s$ implies that at least one of the edges $e\in\Gamma_j^{(i)}$ has $\tau_e>s/\lambda$, and the third inequality follows from (2.10).

Combining (2.8) and (2.11) we deduce that

$$
\begin{aligned}
\mathrm{E}[\tau_{S_k}^{\alpha}] &\le \sum_{n=1}^{\infty} \mathrm{E}\left[\left(\sum_{j=\rho_{k-1}+1}^{\rho_k} T(\hat{v}_{j-1}, \hat{v}_j) \right)^{\alpha} \middle| \Lambda_n \right] \mathrm{P}(\Lambda_n) \\
&\le \sum_{n=1}^{\infty} n^{\alpha} \sum_{j=\rho_{k-1}+1}^{\rho_{k-1}+n} \mathrm{E}\left[T(\hat{v}_{j-1}, \hat{v}_j)^{\alpha} \middle| \Lambda_n \right] \mathrm{P}(\Lambda_n) \qquad (2.12) \\
&\le C_2 \sum_{n=1}^{\infty} n^{\alpha+1} \mathrm{E}\left[Y^{\alpha}\right] \mathrm{P}(\Lambda_n) \le C_2 \mathrm{E}\left[Y^{\alpha}\right] \mathrm{E}\left[S_k^{\alpha+1}\right],
\end{aligned}
$$

where the second inequality follows since for any non-negative numbers a_j we have

$$
\left(\sum_{j=1}^{n} a_j \right)^{\alpha} \le (n \max_j a_j)^{\alpha} \le n^{\alpha} \sum_{j=1}^{n} a_j^{\alpha}. \qquad (2.13)
$$

Thus, $\mathrm{E}\left[\tau_{S_k}^{\alpha}\right] < \infty$ from part *a)*. We can conclude that $\mathrm{E}[\tau_{S_k}^{\alpha}] > 0$, since the passage times of all edges connecting level $\rho_{k-1} + M$ and $\rho_{k-1} + M + 1$ are independent of Λ_n. □

Remark 2.8. It is worth pointing out that the initially infected component does not need to be finite. But, there needs to be a level m beyond which no vertex is initially infected. Proposition 2.6 holds also in this case. □

3 Asymptotics for first-passage percolation

In this section we will present some variants of classical results for i.i.d. sequences, but here for first-passage percolation considered on essentially 1-dimensional periodic graphs. The fact that $\{T(\hat{v}_n)\}_{n\ge1}$ is a regenerative sequence, and in particular, that $\{\tau_{S_k}\}_{k\ge1}$ and $\{S_k\}_{k\ge1}$ form i.i.d. sequences, will play a central rôle. We will assume throughout this section that t', t'' and M are chosen in accordance with Lemma 2.3, and that the auxiliary variable $\Delta \equiv 0$. In order to approximate $T(\hat{v}_n)$, we will stop the sequence $\{T(\hat{v}_{\rho_k})\}_{k\ge0}$ in a suitable way. The asymptotic behaviour of stopped sums of this form, so called *stopped random walks*, i.e., random walks stopped by some stopping time, has been studied before. Gut (2009) treats this subject. Once the regenerative behaviour is known, results as Theorem 1.3 and 1.5 are easily obtained from the classical Law of Large Numbers and Law of the Iterated Logarithm. So, there is in these cases no need to refer to the theory for stopped sums. However, Theorem 1.4 and 3.6 would require more work, and we will base our proofs of these results on known results for stopped random walks. We should mention that apart from the results presented here, it is possible to deduce other results, such as stable laws, from known results for stopped random walks.

We will without further comment use the fact that if $Y_n \to Y$ and $\eta_n \to \infty$ almost surely as $n \to \infty$, then $Y_{\eta_n} \to Y$ almost surely as $n \to \infty$. We also remind the reader that for any i.i.d. sequence $\{Y_n\}_{n\geq 1}$, a simple application of the Borel-Cantelli lemmas shows that

$$\lim_{n\to\infty} \frac{Y_n^\alpha}{n} = 0, \text{ almost surely} \quad \Leftrightarrow \quad \mathrm{E}[|Y_1|^\alpha] < \infty. \tag{3.1}$$

To see this, note that $\mathrm{E}[|Y_1|^\alpha] < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} \mathrm{P}\big(|Y_n|^\alpha > \epsilon n\big) < \infty, \quad \text{for any } \epsilon > 0.$$

This is by the Borel-Cantelli lemmas equivalent to

$$\lim_{n\to\infty} \frac{|Y_n|^\alpha}{n} \leq \epsilon,$$

and (3.1) follows.

In order to approximate $T(\hat{v}_n)$, we will stop the regenerating sequence when A_{n_k} occurs for the least k such that $n_k \geq n$. In terms of the sequence of regenerative levels, we define

$$\nu(n) := \min\{m \geq 0 : \rho_m \geq n + M\}.$$

Lemma 3.1. $\{\nu(n)\}_{n\geq 0}$ *is a non-decreasing sequence such that*

a) $\displaystyle\lim_{n\to\infty} \frac{n}{\nu(n)} = \mu_S,$ *almost surely.*

b) $\displaystyle\lim_{n\to\infty} \frac{\rho_{\nu(n)}}{n} = 1,$ *almost surely.*

Proof. It is clear that $\nu(n) \uparrow \infty$ as $n \to \infty$. Lemma 2.5 and Proposition 2.7 assure that $\{S_k\}_{k\geq 1}$ forms an i.i.d. sequence with finite mean. Since the definition gives $\rho_{\nu(n)-1} < n + M \leq \rho_{\nu(n)}$, we have

$$\frac{\rho_{\nu(n)}}{\nu(n)} - \frac{S_{\nu(n)}}{\nu(n)} < \frac{n+M}{\nu(n)} \leq \frac{\rho_{\nu(n)}}{\nu(n)}.$$

This, together with the classical Law of Large Numbers and (3.1) proves *a)*. Since

$$\frac{\rho_{\nu(n)}}{n} = \frac{\rho_{\nu(n)}}{\nu(n)} \frac{\nu(n)}{n},$$

part *b)* follows from the Law of Large Numbers and part *a)*. $\square$

Recall that $\{\hat{T}_n\}_{n\geq 1}$ denotes either of $\{T_n\}_{n\geq 1}$, $\{\max_{v\in\mathbb{V}_{\mathcal{G}_n}} T(v)\}_{n\geq 1}$ and $\{T(v_n)\}_{n\geq 1}$, where $\{v_n\}_{n\geq 1}$ is any sequence of vertices such that v_n is at level n. When we prove Theorem 1.3, 1.4 and 1.5, we will first obtain the results for the stopped sequence $\{T(\hat{v}_{\rho_{\nu(n)}})\}_{n\geq 1}$. What we then need to finish the proofs is summarized in the following lemma.

Lemma 3.2. *Assume that there are $p \geq 1$ (edge) disjoint paths from $\hat{v}_0$ to $\hat{v}_1$. Let $Y = \min(\tau_1, \dots, \tau_p)$, where $\tau_1, \dots, \tau_p$ are independent and distributed as P_τ. Then, for any $\alpha > 0$,*

a) $$\lim_{n\to\infty} \frac{\left|\rho_{\nu(n)} - n\right|^\alpha}{n} = 0, \quad \textit{almost surely.}$$

b) if $\mathrm{E}[Y^\alpha] < \infty$, *then* $$\lim_{n\to\infty} \frac{\left|T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}})\right|^\alpha}{n} = 0, \quad \textit{almost surely.}$$

c) if $\mathrm{E}[\tau_e^\alpha] < \infty$, *then* $$\lim_{n\to\infty} \frac{\left|\hat{T}_n - T(\hat{v}_n)\right|^\alpha}{n} = 0, \quad \textit{almost surely.}$$

Proof. Since $\rho_{\nu(n)} - n \leq S_{\nu(n)} + M \overset{d}{=} S_k + M$, then *a)* follows from (3.1) and part *a)* of Proposition 2.7. By subadditivity

$$\left|T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_n)\right| \leq \sum_{j=n+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j) \leq \sum_{j=\rho_{\nu(n)-1}-M+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j),$$

which in the proof of Proposition 2.7 was seen to have finite moment of the same order as Y. Thus, also *b)* follows from (3.1). Finally, also *c)* follows from (3.1). Note that

$$T_n \leq T(v_n) \leq \max_{v\in\mathbb{V}_{\mathcal{G}_n}} T(v) \leq T_n + \sum_{e\in\mathbb{E}_{\mathcal{G}_n}} \tau_e,$$

implies that $\left|\hat{T}_n - T(\hat{v}_n)\right| \leq \sum_{e\in\mathbb{E}_{\mathcal{G}_n}} \tau_e$, which via (2.13) is easily seen to have finite moment of same order as τ_e. □

3.1 Proof of point-wise limit theorems

Proof of Theorem 1.3. *Almost sure convergence.* Lemma 2.5 and Proposition 2.7 gives that $\{\tau_{S_k}\}_{k\geq 1}$ is an i.i.d. sequence with finite mean. Thus, as $n \to \infty$,

$$\frac{T(\hat{v}_{\rho_{\nu(n)}})}{n} = \frac{T(\hat{v}_{\rho_0}) + \sum_{k=1}^{\nu(n)} \tau_{S_k}}{\nu(n)} \frac{\nu(n)}{n} \to \frac{\mu_\tau}{\mu_S}, \quad \text{almost surely,}$$

according to the classical Law of Large Numbers and Lemma 3.1. We conclude that, as $n \to \infty$,

$$\frac{T(\hat{v}_n)}{n} = \frac{T(\hat{v}_{\rho_{\nu(n)}})}{n} + \frac{T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}})}{n} \to \frac{\mu_T}{\mu_S}, \quad \text{almost surely,}$$

by part *b)* of Lemma 3.2. The almost sure convergence of $\hat{T}_n/n$ now follows from part *c)* of the same lemma.

Uniform integrability. According to subadditivity and (2.13)

$$\hat{T}_n^r \leq 3^r \left(T(\hat{v}_0)^r + \left(\sum_{j=1}^{n} T(\hat{v}_{j-1}, \hat{v}_j) \right)^r + \left(\sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e \right)^r \right).$$

By convexity of the function x^r, we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} T(\hat{v}_{j-1}, \hat{v}_j) \right)^r \leq \frac{1}{n} \sum_{k=1}^{n} T(\hat{v}_{j-1}, \hat{v}_j)^r.$$

Note that the distribution of $T(\hat{v}_0)^r$ and $\left(\sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e\right)^r$ does not depend on n, and, via (2.13), have finite mean when $\mathrm{E}[\tau_e^r]$ is finite. Hence, $\{(\hat{T}_n/n)^r\}_{n\geq 1}$ is uniformly integrable if only $\{\sum_{k=1}^n T(\hat{v}_{j-1}, \hat{v}_j)^r/n\}_{n\geq 1}$ is uniformly integrable. However, that holds as soon as $\mathrm{E}\big[T(\hat{v}_{j-1}, \hat{v}_j)^r\big]$ is finite, which follows since $\mathrm{E}[\tau_e^r]$ is finite. (Note that the regenerative behaviour was not used.)

L^r-convergence. The L^r-convergence now follows from the almost sure convergence and uniform integrability. □

Theorem 1.4 will be deduced from the following result sometimes referred to as *Anscombe's theorem*. For a proof, we refer the reader to e.g. Gut (2009, Theorem 1.3.1).

Theorem 3.3 (Anscombe's theorem)**.** *Let $\{\xi_k\}_{k\geq 1}$ be an i.i.d. sequence with mean zero and variance σ_ξ^2. Assume further that*

$$\frac{\eta(n)}{n} \xrightarrow{p} \theta, \quad \textit{in probability,}$$

as $n \to \infty$. Then, as $n \to \infty$,

$$\frac{\sum_{k=1}^{\eta(n)} \xi_k}{\sigma_\xi \sqrt{\theta n}} \xrightarrow{d} \chi, \quad \textit{in distribution,}$$

where χ has a standard normal distribution.

Proof of Theorem 1.4. It follows from Lemma 2.5 and Proposition 2.7 that $\{\tau_{S_k} - \mu S_k\}_{k\geq 1}$ is an i.i.d. sequence with zero mean and finite variance. An application of Anscombe's theorem, together with Lemma 3.1, gives convergence in distribution of the former term in the right-hand side of

$$\frac{T(\hat{v}_n) - \mu n}{\sigma\sqrt{n}} = \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}}{\sigma\sqrt{n}} + \frac{T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}}) - \mu(n - \rho_{\nu(n)})}{\sigma\sqrt{n}},$$

to a standard normal distribution, as $n \to \infty$. The latter term in the above right-hand side vanishes according to part *a)* and *b)* of Lemma 3.2. The convergence of $\hat{T}_n$ now follows from part *c)* of the same lemma. □

Theorem 1.5 will be proved from a version of the Law of the Iterated Logarithm for i.i.d. sequences that is more general than the classical one. The classical version would suffice to prove the second statement in the theorem. A proof of the more general version for i.i.d. sequences can be found in e.g. Gut (2005).

Proof of Theorem 1.5. Recall that $\tau_{S_k} - \mu S_k$ are i.i.d. for $k \geq 1$, with zero mean and finite variance, due to Lemma 2.5 and Proposition 2.7. Trivially

$$\begin{aligned} \frac{T(\hat{v}_n) - \mu n}{\sigma\sqrt{2n\log\log n}} &= \frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}}{\sigma\sqrt{\mu_S 2\nu(n)\log\log\nu(n)}}\sqrt{\mu_S\frac{\nu(n)}{n}}\sqrt{\frac{\log\log\nu(n)}{\log\log n}} \\ &\quad + \frac{T(\hat{v}_n) - T(\hat{v}_{\rho_{\nu(n)}}) - \mu(n - \rho_{\nu(n)})}{\sigma\sqrt{2n\log\log n}}. \end{aligned} \tag{3.2}$$

Since $\nu(n)$ is non-decreasing, and for each $m \in \mathbb{Z}_+$, there is an $n \in \mathbb{Z}_+$ such that $\nu(n) = m$, it follows from the extended version of the Law of the Iterated Logarithm for i.i.d. sequences that

$$\mathcal{L}\left(\left\{\frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}}{\sigma\sqrt{\mu_S 2\nu(n)\log\log\nu(n)}}\right\}_{n:\nu(n)\geq 3}\right) = [-1, 1], \quad \text{almost surely,}$$

and, in particular, that almost surely

$$\limsup_{n\to\infty}\frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}}{\sigma\sqrt{\mu_S 2\nu\log\log\nu}} = 1, \quad \text{and} \quad \liminf_{n\to\infty}\frac{T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}}{\sigma\sqrt{\mu_S 2\nu\log\log\nu}} = -1.$$

Lemma 3.1 gives that $\mu_S\nu(n)/n \to 1$, almost surely, as $n \to \infty$, and we further conclude that

$$\lim_{n\to\infty}\frac{\log\log\nu(n)}{\log\log n} = \lim_{n\to\infty}\frac{\log\left(\log n + \log\frac{\nu(n)}{n}\right)}{\log\log n} = 1, \quad \text{almost surely.}$$

An application of Lemma 3.2 now completes the proof. □

Remark 3.4. Say that we are interested in the sequence $\{T(\hat{v}_0, \hat{v}_n)\}_{n\geq 1}$. A closer look at the proofs of Theorem 1.3, 1.4 and 1.5 reveals that the only moment condition required in order to prove the various modes of convergence for $\{T(\hat{v}_0, \hat{v}_n)\}_{n\geq 1}$ is not $\mathrm{E}[\tau_e^\alpha] < \infty$ for given values of α, but $\mathrm{E}\left[\tau_{S_k}^\alpha\right] < \infty$, for $k \geq 1$ and corresponding α. According to Proposition 2.7, this holds when $\mathrm{E}[Y^\alpha] < \infty$, where Y denotes the minimum of p independent passage times, and p is the number of disjoint paths from $\hat{v}_0$ to $\hat{v}_1$. As an example, on any (K,d)-tube with $K, d \geq 2$, the passage time distribution given by

$$\mathrm{P}(\tau_e > x) = x^{-\alpha}, \quad \text{for } x > 1, \tag{3.3}$$

for some $\alpha > 0$, satisfies $\mathrm{E}[\tau_e^\alpha] = \infty$ but $\mathrm{E}[\min(\tau_1, \tau_2)^\alpha] < \infty$. Hence, Theorem 1.3, 1.4 and 1.5 holds in this example for the sequence $\{T(\hat{v}_0, \hat{v}_n)\}_{n\geq 1}$ and corresponding values of α, even though $\mathrm{E}[\tau_e^\alpha] = \infty$. □

3.2 One dimensional shape theorem

Theorem 1.5 gives the precise rate of convergence towards the asymptotic shape B^* in the case of (K,d)-tubes. We will next rephrase this in terms of the set of infected vertices. The following corollary gives the precise rate of fluctuations of that set. Recall that $B^* = B^*(t) = \left[-\mu_K^{-1}, \mu_K^{-1}\right] \times [0, K/t]^{d-1}$.

Corollary 3.5. *Consider first-passage percolation on a (K,d)-tube, and assume that $\mathrm{E}[\tau_e^2] < \infty$. We have for all $\lambda > \sigma\sqrt{2/\mu_K}$, almost surely, that*

$$\left(1 - \lambda\sqrt{t^{-1}\log\log t}\right) B^* \subset \frac{1}{t}\tilde{B}_t \subset \left(1 + \lambda\sqrt{t^{-1}\log\log t}\right) B^*, \tag{3.4}$$

for all t large enough. Moreover, for all $\lambda < \sigma\sqrt{2/\mu_K}$ and $s \geq 0$, for either of the inclusions in (3.4), there exists, almost surely, $t \geq s$ such that the inclusion does not hold.

Proof. Fix $\epsilon > 0$. By Theorem 1.5, there exists an almost surely finite $N = N(\epsilon)$ such that

$$\mu_K n - (1+\epsilon)\sigma\sqrt{2n\log\log n} \; < \; \min(T_{-n}, T_n),$$

for all $n \geq N$. This implies that

$$\tilde{B}_t \; \subseteq \; [-n, n] \times [0, K]^{d-1},$$

for all

$$t \; \leq \; \mu_K n - (1+\epsilon)\sigma\sqrt{2n\log\log n} \tag{3.5}$$

and $n \geq N$. Write n_t for the least n such that (3.5) holds. By the choice of n_t,

$$\frac{t}{\mu_K} \;\geq\; n_t - 1 - \frac{1+\epsilon}{\mu_K}\sigma\sqrt{2(n_t-1)\log\log(n_t-1)} \;=\; (n_t-1)g(n_t),$$

for some increasing function g such that $g(n) \to 1$ as $n \to \infty$. It follows that

$$n_t - 1 \;\leq\; \frac{1}{\mu_K}\left(t + (1+\epsilon)\sigma\sqrt{\frac{2t}{\mu_K g(n_t)}\log\log\frac{t}{\mu_K g(n_t)}}\right)$$

Since $n_t \to \infty$, also $g(n_t) \to 1$, as $t \to \infty$. Since $\epsilon > 0$ was arbitrary, we have shown that for all $\epsilon > 0$ there exists an almost surely finite $T = T(\epsilon)$ such that

$$\tilde{B}_t \;\subseteq\; \left(t + (1+\epsilon)\sigma\sqrt{\frac{2t}{\mu_K}\log\log t}\right) B^*$$

for all $t \geq T$. The proof of the lower inclusion in (3.4) follows in a similar way from Theorem 1.5 applied to $\max_{v\in\mathbb{V}_{\mathcal{G}_n}} T(v)$.

It remains to prove the second statement of the corollary. Fix $\epsilon > 0$. It follows from Theorem 1.5 that for all $n \geq 1$ there exists, almost surely, $N = N(\epsilon) \geq n$ such that

$$T_N \;\leq\; \mu_K N - (1-\epsilon)\sigma\sqrt{2N\log\log N}$$

In particular,

$$\tilde{B}_{t_N} \;\not\subseteq\; [-N,N]\times[0,K]^{d-1}$$

for $t_N := \mu_K N - (1-\epsilon)\sigma\sqrt{2N\log\log N}$. Since $t_N \leq \mu_K N$, it follows that

$$\tilde{B}_{t_N} \;\not\subseteq\; \left(t_N + (1-\epsilon)\sigma\sqrt{\frac{2t_N}{\mu_K}\log\log\frac{t_N}{\mu_K}}\right) B^*.$$

Since $\epsilon > 0$ was arbitrary, we have shown that for any $\lambda < \sigma\sqrt{2/\mu_K}$, $\epsilon > 0$ and $s > 0$, there exists, almost surely, $t = t(\epsilon) \geq s$ such that the upper inclusion in (3.4) cannot hold. The failure of the lower inclusion follows in a similar way. □

3.3 Functional Donsker theorem

Donsker's theorem can be seen as a functional version of the Central Limit Theorem. In contrast to the Central Limit Theorem that treats weak convergence of real-valued random variables, Donsker's theorem treats weak convergence of sequences of real-valued random functions. Let $D = D[0,\infty)$ denote the set of right-continuous functions with left-hand limits on $[0,\infty)$. Let $\mathcal{D}$ denote the σ-algebra generated by the open sets in D with Skorohod's J_1-topology, defined by the following. Let Λ denote the set of strictly increasing, continuous mappings of $[0, b]$ onto itself. A sequence $\{f_n\}_{n\geq 1}$ of elements in D is said to be J_1-*convergent* to f if, for every $b \geq 0$, there exists a sequence $\{\lambda_n\}_{n\geq 1}$ in Λ such that

$$\sup_{0\leq t\leq b} |\lambda_n(t) - t| \to 0, \quad \text{and} \quad \sup_{0\leq t\leq b} |f_n(\lambda_n(t)) - f(t)| \to 0,$$

as $n \to \infty$. If $\{P_n\}_{n\geq 1}$ is a sequence of probability measures on the measurable space $(D, \mathcal{D})$, then we say that P_n *converge weakly* to P, denoted $P_n \overset{J_1}{\Rightarrow} P$, if

$$\int_D f\, dP_n \to \int_D f\, dP,$$

for all bounded, continuous f from D to $\mathbb{R}$.

Let $\{\xi_k\}_{k\geq 1}$ be an i.i.d. sequence of random variables with zero mean and variance $\sigma_\xi^2 < \infty$, set $S_n = \sum_{k=1}^n \xi_k$, and define

$$X_n(t) := \frac{1}{\sigma_\xi \sqrt{n}} S_{\lfloor nt\rfloor}, \quad \text{for } t \geq 0.$$

Donsker's theorem states that $X_n \overset{J_1}{\Rightarrow} W$, as $n \to \infty$, where W denotes Wiener measure. The following is a result in the same spirit, for our first-passage percolation process.

Theorem 3.6 (Functional Donsker theorem)**.** *If* $\mathrm{E}[\tau_e^2] < \infty$*, then*

$$\frac{\hat{T}_{\lfloor nt\rfloor} - \mu\lfloor nt\rfloor}{\sigma\sqrt{n}} \overset{J_1}{\Rightarrow} W, \quad \textit{as } n \to \infty.$$

As for the point-wise Central Limit Theorem, there is an Anscombe version of Donsker's theorem. We will use it as a lemma to prove Theorem 3.6. Suppose that $\{\eta(n)\}_{n\geq 0}$ is a non-decreasing, right-continuous family of positive, integer valued random variables such that $\eta(n)/n \to \theta$, almost surely, as $n \to \infty$. Define

$$Y_n(t) := \frac{1}{\sigma_\xi \sqrt{n}} S_{\eta(\lfloor nt\rfloor)}, \quad \text{for } t \geq 0.$$

An Anscombe-version of Donsker's theorem states the following.

Lemma 3.7.

$$\theta^{-1/2} Y_n \overset{J_1}{\Rightarrow} W, \quad \textit{as } n \to \infty.$$

We refer the reader to Gut (2009, Theorem 5.2.1) for a proof of Lemma 3.7. The lemma is of great interest in its own right, but we restate it here as a lemma in order to maintain focus on our main aim. We will deduce Theorem 3.6 from Lemma 3.7.

Proof of Theorem 3.6. According to Lemma 2.5 and Proposition 2.7, the sequence $\{\tau_{S_k} - \mu S_k\}_{k\geq 1}$ have i.i.d. elements with zero mean and finite variance. From Lemma 3.7 it follows that

$$\frac{T(\hat{v}_{\rho_{\nu(\lfloor nt\rfloor)}}) - \mu\rho_{\nu(\lfloor nt\rfloor)}}{\sigma\sqrt{n}} \stackrel{J_1}{\Rightarrow} W, \quad \text{as } n \to \infty.$$

It remains to prove that, as $n \to \infty$,

$$\sup_{0\leq t\leq b}\left|\frac{\hat{T}_{\lfloor nt\rfloor} - T(\hat{v}_{\rho_{\nu(\lfloor nt\rfloor)}}) - \mu(\lfloor nt\rfloor - \rho_{\nu(\lfloor nt\rfloor)})}{\sigma\sqrt{n}}\right| \to 0, \quad \text{almost surely.} \tag{3.6}$$

According to Lemma 3.2, as $n \to \infty$,

$$\left|\frac{\hat{T}_n - T(\hat{v}_{\rho_{\nu(n)}}) - \mu(n - \rho_{\nu(n)})}{\sigma\sqrt{n}}\right| \to 0, \quad \text{almost surely.} \tag{3.7}$$

For any sequence of real numbers $\{x_n\}_{n\geq 1}$ it holds that

$$\lim_{n\to\infty}\frac{x_n}{\sqrt{n}} = 0 \quad \Rightarrow \quad \lim_{n\to\infty}\frac{\max_{k\leq bn}|x_k|}{\sqrt{n}} = 0. \tag{3.8}$$

To see this, fix $\epsilon > 0$, and choose N such that $|x_n|/\sqrt{n} \leq \epsilon$ for all $n \geq N$. Then the left-hand side in (3.8) can be made arbitrarily small for large n, since

$$\frac{\max_{k\leq bn}|x_k|}{\sqrt{n}} = \frac{\max_{k<N}|x_k|}{\sqrt{n}} + \frac{\max_{N\leq k\leq bn}|x_k|}{\sqrt{n}} \leq \epsilon + \epsilon\sqrt{b},$$

if n is chosen large enough.

In fact, (3.8) improves (3.7), and we get

$$\frac{\max_{k\leq bn}\left|\hat{T}_k - T(\hat{v}_{\rho_{\nu(k)}}) - \mu(k - \rho_{\nu(k)})\right|}{\sigma\sqrt{n}} \to 0, \quad \text{almost surely,}$$

as $n \to \infty$. But this is equivalent to (3.6). □

Remark 3.8. Remark 3.4 applies also to Theorem 3.6. □

4 Monotonicity of mean and variance

In this section we will prove Theorem 1.6 and 1.7. This shows that mean and variance of $T(v_{n,i})$ are monotonous in n, if n is sufficiently large. The method of proof will use the regenerative behaviour explored in Section 2. In this section, the auxiliary random variable Δ introduced in Section 2 will turn out useful. Throughout this section, Δ will be assumed uniformly distributed on $\{0, 1, \ldots, 2M\}$.

From what is known for stopped random walks (see e.g. Gut (2009, Theorem 4.2.4)), it follows that $\mathrm{E}[T(\hat{v}_{\rho_{\nu(n)}})] = \mu n + C$, for some constant C, and $\mathrm{Var}(T(\hat{v}_{\rho_{\nu(n)}})) = \sigma^2 n + o(n)$, for large n. We will in this section need an essential amount of extra work, in order to improve the latter statement and prove that there is a constant C, such that $\mathrm{Var}(T(\hat{v}_{\rho_{\nu(n)}})) = \sigma^2 n + C$. What then remains in order to prove Theorem 1.6 and 1.7, is to show that the differences between $\mathrm{E}[T(v_{n,i})]$ and $\mathrm{E}[T(\hat{v}_{\rho_{\nu(n)}})]$, and between $\mathrm{Var}(T(v_{n,i}))$ and $\mathrm{Var}(T(\hat{v}_{\rho_{\nu(n)}}))$, converge as $n \to \infty$. We will present full proofs of Theorem 1.6 and 1.7 based on Wald's lemma.

Lemma 4.1 (Wald's lemma)**.** *Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with mean μ_ξ, and set $S_n = \sum_{k=1}^{n} \xi_k$. Let N be a stopping time with* $\mathrm{E}[N] < \infty$.

a) $\mathrm{E}[S_N] = \mu_\xi \mathrm{E}[N]$.

b) If $\sigma_\xi^2 = \mathrm{Var}(\xi_1) < \infty$, *then* $\mathrm{E}\left[(S_N - \mu_\xi N)^2\right] = \sigma_\xi^2 \mathrm{E}[N]$.

c) If X is independent of $\xi_1, \xi_2, \ldots$, then $\mathrm{E}[X S_N] = \mu_\xi \mathrm{E}[X N]$.
In particular, $\mathrm{Cov}(X, S_N) = \mu_\xi \,\mathrm{Cov}(X, N)$.

A proof of Wald's lemma can be found e.g. in Gut (2009, Theorem 1.5.3). The third part of the lemma is a slight extension of the first part, and proved in an analogous way. If $\mathcal{F}_n = \sigma(\{(\rho_0, T(\hat{v}_{\rho_0})), (S_1, \tau_{S_1}), \ldots, (S_n, \tau_{S_n})\})$, then it is immediate from the definition that $\nu(n)$ is a stopping time with respect to the sequence of σ-algebras $\{\mathcal{F}_n\}_{n \geq 1}$.

The importance of the auxiliary variable Δ is the following. A regeneration point may only occur every $2M + 1$ levels. However, introducing a shift uniformly distributed on $\{0, 1, \ldots, 2M\}$ allows every level equal probability to be included in the subset of levels at which regeneration points may occur. This is precisely the rôle of Δ. This allows the following lemma, as well as details in the proof of Theorem 1.6 and 1.7, to become less messy.

Lemma 4.2. *For $n \geq \rho_I$,*

$$\mathrm{E}\big[\nu(n)\big] \;=\; \frac{n - \rho_I}{\mu_S}.$$

Proof. Assume that $n \geq \rho_I$. We may interpret $\nu(n)$ as the number of regeneration points before (but not including) level $n+M$. That is, the number of $k \geq 0$ such that A_{n_k} occurs for $n_k < n$. Since $n_0 = \rho_I + \Delta$, this number is at most $\left\lfloor \frac{n+2M-\rho_I-\Delta}{2M+1} \right\rfloor$. Since the shift Δ is independent of $\{\tau_e\}_{e\in\mathbb{E}}$, we can, conditioned on Δ, think of $\nu(n)$ as the number of successes in $\left\lfloor \frac{n+2M-\rho_I-\Delta}{2M+1} \right\rfloor$ independent Bernoulli trials, each with success probability $p_A := \mathrm{P}(A_{n_k})$. Conditioning on Δ, we see that

$$\mathrm{E}\big[\nu(n)\big] \;=\; p_A\,\mathrm{E}\left[\left\lfloor \frac{n+2M-\rho_I-\Delta}{2M+1} \right\rfloor\right]. \tag{4.1}$$

If $n - \rho_I = (2M+1)k$, for some $k \geq 0$, one realise from (4.1) that

$$\mathrm{E}\big[\nu(n)\big] \;=\; \frac{p_A}{2M+1}(2M+1)k \;=\; \frac{n-\rho_I}{\mu_S},$$

where the latter equality follows from the fact that S_k is geometrically distributed with parameter p_A, times a factor $2M+1$, that is, $\mu_S = (2M+1)/p_A$. Again from (4.1), one realise that as $n - \rho_I$ increase from $(2M+1)k$ to $(2M+1)k + 2M$, then $\mathrm{E}[\nu(n)]$ will have to increase with $p_A/(2M+1)$ for each step. $\square$

We are now ready to prove Theorem 1.6 and 1.7.

Proof of Theorem 1.6. Wald's lemma together with Lemma 4.2 gives that

$$\begin{aligned}\mathrm{E}\left[T(\hat{v}_{\rho_{\nu(n)}})\right] \;&=\; \mathrm{E}\left[T(\hat{v}_{\rho_0}) + \sum_{k=1}^{\nu(n)} \tau_{S_k}\right] \;=\; \mathrm{E}\big[T(\hat{v}_{\rho_0})\big] + \mu_\tau \mathrm{E}[\nu(n)] \\ &=\; \mathrm{E}\big[T(\hat{v}_{\rho_0})\big] + \mu n - \mu\rho_I.\end{aligned}$$

It remains to prove that there is a finite constant C_i such that

$$\mathrm{E}\left[T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})\right] \to C_i, \quad \text{as } n \to \infty. \tag{4.2}$$

Arguments of the type we will use to prove (4.2) will be used repeatedly in the proof of Theorem 1.7. For this reason, we present the argument in detail here. To make the argument clear, we will define a random variable to which $T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})$ converges in distribution. The limit C_i will then be the expectation of this random variable.

Recall that $n_k = \rho_I + \Delta + k(2M+1)$, set $m_{n,k} := n - (2M+1)k$ for $k \geq 1$, and set

$$\begin{aligned} r_+ &:= M + \min\{n_k \geq 0 : A_{n_k} \text{ occurs}\}, \\ r_0 &:= M + \max\{m_{0,k} < 0 : A_{m_{0,k}} \text{ occurs}\}. \end{aligned}$$

Observe that r_+ denotes the first element of the sequence $\{\rho_k\}_{k\geq 0}$ greater than zero, whereas r_0 is not defined along the same subsequence of the integers as $\{\rho_k\}_{k\geq 0}$. Define

$$Y_{k,i} := T(\hat{v}_{r_0}, v_{k,i}), \quad \text{and} \quad Y_+ := T(\hat{v}_{r_0}, \hat{v}_{r_+}),$$

and the events

$$\begin{aligned} D_{T,n} &:= \{A_{m_{n,k}} \text{ occurs for some } k \text{ such that } \rho_I \leq m_{n,k} < n\}, \\ D_{Y,n} &:= \{A_{m_{0,k}} \text{ occurs for some } k \text{ such that } \rho_I \leq m_{0,k} + n < n\}. \end{aligned}$$

Clearly $\mathrm{P}(D_{T,n}) = \mathrm{P}(D_{Y,n}) \to 1$ as $n \to \infty$. Moreover,

$$\left\{T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \leq t\right\} \cap D_{T,n} \stackrel{d}{=} \{Y_{0,i} - Y_+ \leq t\} \cap D_{Y,n}.$$

So, if we let $H_{T,n} = \{T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \leq t\}$ and $H_Y = \{Y_{0,i} - Y_+ \leq t\}$, then as $n \to \infty$,

$$\begin{aligned} \mathrm{P}(H_{T,n}) &= \mathrm{P}(H_{T,n} \cap D_{T,n}) + \mathrm{P}(H_{T,n} \cap D_{T,n}^c) \\ &= \mathrm{P}(H_Y) + \mathrm{P}(H_{T,n} \cap D_{T,n}^c) - \mathrm{P}(H_Y \cap D_{Y,n}^c) \to \mathrm{P}(H_Y). \end{aligned} \tag{4.3}$$

Thus, $T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \xrightarrow{d} Y_0 - Y_+$ as $n \to \infty$. If $\left\{T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})\right\}_{n\geq 1}$, in addition, is uniformly integrable, then

$$\mathrm{E}\left[T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}})\right] \to \mathrm{E}\left[Y_{0,i} - Y_+\right], \quad \text{as } n \to \infty.$$

To deduce uniform integrability, note that subadditivity gives

$$T(v_{n,i}) - T(\hat{v}_{\rho_{\nu(n)}}) \;\leq\; T(v_{n,i}, \hat{v}_n) + \sum_{j=n+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j). \tag{4.4}$$

But, the distribution of the right-hand side of (4.4) does not depend on n. Thus, it suffices to show that it has finite expectation. Conditioning on $\Lambda_k = \{\rho_{\nu(n)} - n = k\}$, one may do so in an analogous way as in (2.12) in the proof of Proposition 2.7, part *b)*. We omit the details. $\square$

The proof of Theorem 1.7 needs a little more work, due to arising covariance terms. Moment convergence arguments similar to the one carried through to prove (4.2) will be used repeatedly.

Proof of Theorem 1.7. To begin with,

$$\begin{aligned}\mathrm{Var}\left(T(v_{n+2M,i})\right) &= \mathrm{Var}\Big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\Big) \\ &+ \mathrm{Var}\Big(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu\rho_{\nu(n)}\Big) \\ &+ 2\,\mathrm{Cov}\Big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\Big) \\ &+ 2\mu\,\mathrm{Cov}\left(T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)}\right) - 2\mu^2\,\mathrm{Var}\left(\rho_{\nu(n)}\right).\end{aligned} \tag{4.5}$$

We will have to treat each of the terms on the right-hand side one by one. Consider the first, and note that

$$\begin{aligned}\mathrm{Var}\Big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\Big) &= \mathrm{Var}\left(T(\hat{v}_{\rho_0}) - \mu\rho_0 + \sum_{k=1}^{\nu(n)}(\tau_{S_k} - \mu S_k)\right) \\ &= \mathrm{E}\left[\left(\sum_{k=1}^{\nu(n)}(\tau_{S_k} - \mu S_k)\right)^2\right] + \mathrm{Var}\left(T(\hat{v}_{\rho_0}) - \mu\rho_0\right) \\ &+ 2\,\mathrm{Cov}\left(\sum_{k=1}^{\nu(n)}(\tau_{S_k} - \mu S_k), T(\hat{v}_{\rho_0}) - \mu\rho_0\right)\end{aligned}$$

So, an application of both the second and third part of Wald's lemma, together with Lemma 4.2, yield

$$\begin{aligned}\mathrm{Var}\Big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\Big) &= \mathrm{Var}\left(\tau_{S_k} - \mu S_k\right)\mathrm{E}[\nu(n)] + \mathrm{Var}\left(T(\hat{v}_{\rho_0}) - \mu\rho_0\right) \\ &+ \mathrm{E}[\tau_{S_k} - \mu S_k]\,\mathrm{Cov}\left(\nu(n), T(\hat{v}_{\rho_0}) - \mu\rho_0\right) \\ &= \sigma^2(n - \rho_I) + \mathrm{Var}\left(T(\hat{v}_{\rho_0}) - \mu\rho_0\right).\end{aligned}$$

To conclude that $\mathrm{Var}(\rho_{\nu(n)})$ is constant, interpret $\rho_{\nu(n)}$ as the level of the first regeneration after level n. Since a regeneration is equally likely to occur at any level, due to the shift variable Δ, it follows that $\mathrm{Var}(\rho_{\nu(n)}) = \mathrm{Var}(\rho_{\nu(n)} - n)$ is independent of n, and therefore constant.

All remaining terms in the right-hand side of (4.5) will in some way or another need an argument similar to the one used to prove (4.2). Recall the notation used for that purpose. We may in an analogous way as in (4.3) divide into cases whether $D_{T,n}$ and $D_{Y,n}$ occurs or not, to show that

$$T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n) \xrightarrow{d} Y_{2M,i} - Y_+ + \mu r_+, \quad \text{as } n \to \infty,$$

Uniform integrability of $\left\{\left(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n)\right)^2\right\}_{n\geq 1}$, can be proved similar to the uniform integrability needed for (4.2). It follows that for

$r = 1, 2$

$$\mathrm{E}\left[\left(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n)\right)^r\right] \to \mathrm{E}\big[(Y_{2M,i} - Y_+ + \mu r_+)^r\big],$$

as $n \to \infty$. From this we conclude that for the second term in the right-hand side of (4.5) we have that, for $C_i = \mathrm{Var}(Y_{2M,i} - Y_+ + \mu r_+)$,

$$\mathrm{Var}\left(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}}) + \mu(\rho_{\nu(n)} - n)\right) = C_i + o(1), \quad \text{as } n \to \infty.$$

Introduce $r_n := M + \max\{m_{n,k} < n : A_{m_{n,k}} \text{ occurs}\}$, and rewrite as follows

$$\begin{aligned}\mathrm{Cov}\left(T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)}\right) &= \mathrm{Cov}\left(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}), \rho_{\nu(n)} - n\right)\\ &\quad + \mathrm{Cov}\left(T(\hat{v}_{r_n}), \rho_{\nu(n)} - n\right).\end{aligned}$$

It is easy to see that $\rho_{\nu(n)} - n \overset{d}{=} r_+$ for $n \geq \rho_I$. Partitioning on whether $D_{T,n}$ and $D_{Y,n}$ occur or not, we see that, as $n \to \infty$,

$$T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}) \xrightarrow{d} Y_+, \quad \text{and} \quad \left(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n})\right)(\rho_{\nu(n)} - n) \xrightarrow{d} Y_+ r_+.$$

Uniform integrability of $\big\{(\rho_{\nu(n)} - n)^2\big\}_{n\geq 1}$ and $\big\{\big(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n})\big)^2\big\}_{n\geq 1}$ is possible to deduce analogously as before; conditioning on the event $\Lambda_k = \{\rho_{\nu(n)} - r_n = k\}$ to deduce that the latter has finite expectation. This implies that also $\big\{\big(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n})\big)(\rho_{\nu(n)} - n)\big\}_{n\geq 1}$ is uniformly integrable. We conclude that, as $n \to \infty$

$$\mathrm{Cov}\left(T(\hat{v}_{\rho_{\nu(n)}}) - T(\hat{v}_{r_n}), \rho_{\nu(n)} - n\right) \to \mathrm{E}\left[Y_+ r_+\right] - \mathrm{E}\left[Y_+\right]\mathrm{E}\left[r_+\right]$$

On the event $D_{T,n}$, $T(\hat{v}_{r_n})$ depends on passage times below level n, but not on Δ, whereas $\rho_{\nu(n)} - n$ is independent of passage times below level n, and hence on $D_{T,n}$. It follows that

$$\mathrm{E}\big[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n)1_{D_{T,n}}\big] = \mathrm{E}\big[T(\hat{v}_{r_n})1_{D_{T,n}}\big]\mathrm{E}[r_+].$$

In particular,

$$\begin{aligned}\mathrm{Cov}\left(T(\hat{v}_{r_n}), \rho_{\nu(n)} - n\right) &= \mathrm{E}\big[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n)1_{D_{T,n}}\big] - \mathrm{E}\big[T(\hat{v}_{r_n})1_{D_{T,n}}\big]\mathrm{E}[r_+]\\ &\quad + \mathrm{E}\big[T(\hat{v}_{r_n})(\rho_{\nu(n)} - n)1_{D^c_{T,n}}\big] - \mathrm{E}\big[T(\hat{v}_{r_n})1_{D^c_{T,n}}\big]\mathrm{E}[r_+]\\ &= \mathrm{E}\big[T(\hat{v}_{r_n})\big(\rho_{\nu(n)} - n - \mathrm{E}[r_+]\big)1_{D^c_{T,n}}\big].\end{aligned}$$

As $n \to \infty$, this expression vanishes, since we can find an upper bound on $E\big[T(\hat{v}_{r_n})\big(\rho_{\nu(n)} - n - \mathrm{E}[r_+]\big)\big]$ in a similar way as before. We conclude that

$$\mathrm{Cov}\left(T(\hat{v}_{\rho_{\nu(n)}}), \rho_{\nu(n)}\right) = \mathrm{Cov}(Y_+, r_+) + o(1), \quad \text{as } n \to \infty.$$

The term $\mathrm{Cov}\big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\big)$ is the only one left in the right-hand side of (4.5) to take care of. An application of the first part of Wald's lemma, we get

$$\mathrm{E}\big[T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\big] = \mathrm{E}\big[T(\hat{v}_{\rho_0}) - \mu\rho_0\big].$$

The sequence $\{\tau_{S_k} - \mu S_k\}_{k=1}^{\nu(n)}$ has until now been considered as a sequence started with at $k = 1$ and stopped at $k = \nu(n)$. But, we can as well see it as a sequence in the opposite direction. That is, as a sequence started at the first point of regeneration after level $n + M$, and that is stopped at the first point of regeneration after level ρ_I. Let $T^* := T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})$. On the event $\{\nu(n) \geq 1\}$, $T^* = T(\hat{v}_{\rho_{\nu(n)-1}}, v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)-1}}, \hat{v}_{\rho_{\nu(n)}})$ and is independent of $\tau_{S_k} - \mu S_k$ for $k < \nu(n)$. The event $\{\nu(n) \geq 1\}$ is itself independent of $\{\tau_{S_k} - \mu S_k\}_{k\geq 1}$. This allows us to apply the third part of Wald's lemma to obtain

$$\begin{aligned}\mathrm{E}\big[\big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\big)T^* 1_{\{\nu(n)\geq 1\}}\big] &= \mathrm{E}\big[\big(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)}\big)T^* 1_{\{\nu(n)\geq 1\}}\big] \\ &\quad + \mathrm{E}\big[\big(T(\hat{v}_{\rho_0}) - \mu\rho_0\big)T^* 1_{\{\nu(n)\geq 1\}}\big].\end{aligned}$$

Since $\mathrm{E}\big[\big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\big)T^* 1_{\{\nu(n)=0\}}\big] = \mathrm{E}\big[\big(T(\hat{v}_{\rho_0}) - \mu\rho_0\big)T^* 1_{\{\nu(n)=0\}}\big]$, we have

$$\begin{aligned}\mathrm{E}\big[\big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}\big)T^*\big] &= \mathrm{E}\big[\big(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)}\big)T^* 1_{\{\nu(n)\geq 1\}}\big] \\ &\quad + \mathrm{E}\big[\big(T(\hat{v}_{\rho_0}) - \mu\rho_0\big)T^*\big],\end{aligned}$$

and, in particular,

$$\begin{aligned}\mathrm{Cov}\big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T^*\big) &= \mathrm{E}\big[\big(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)}\big)T^* 1_{\{\nu(n)\geq 1\}}\big] \\ &\quad + \mathrm{Cov}\big(T(\hat{v}_{\rho_0}) - \mu\rho_0, T^*\big).\end{aligned} \tag{4.6}$$

Let $r_- := M + \max\{n_k < 0 : A_{n_k} \text{ occurs}\}$, $Y_- := T(\hat{v}_{r_-}, \hat{v}_{r_+})$ and $Z_{k,i} := T(\hat{v}_{r_-}, v_{k,i})$. Observe that

$$(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)})T^* 1_{\{\nu(n)\geq 1\}} \overset{d}{=} \big(Y_- - \mu(r_+ - r_-)\big)\big(Z_{2M,i} - Y_-\big)1_H,$$

where $H = \{A_{n_k} \text{ occurs for some } \rho_I + \Delta - n \leq n_k < 0\}$. To conclude that the former term in the right-hand side of (4.6) converges as $n \to \infty$ can now be done via the Monotone Convergence Theorem. That the limit is finite can be seen in a similar way as before, conditioning on $\Lambda_k = \{r_+ - r_- = k\}$. (Note that $Y_- \overset{d}{=} \tau_{S_k}$, and $r_+ - r_- \overset{d}{=} S_k$.) For the latter term in the right-hand side of (4.6), let $Z'_{2M,i}$ and Y'_+ be defined in the same way as $Z_{2M,i}$ and Y_+ above, but now for a set of passage times $\{\tau'_e\}_{e\in\mathbb{E}}$ independent of $\{\tau_e\}_{e\in\mathbb{E}}$ (that defines $T(\hat{v}_{\rho_0}) - \mu\rho_0$), but with the same Δ. By conditioning on the events $\{\nu(n) \geq 1\}$ (with respect to $\{\tau_e\}_{e\in\mathbb{E}}$) and H (with respect to $\{\tau'_e\}_{e\in\mathbb{E}}$), we see that as $n \to \infty$

$$\big(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\big) \xrightarrow{d} (Z'_{2M,i} - Y'_+),$$

and

$$\big(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\big)\big(T(\hat{v}_{\rho_0}) - \mu\rho_0\big) \xrightarrow{d} (Z'_{2M,i} - Y'_+)\big(T(\hat{v}_{\rho_0}) - \mu\rho_0\big).$$

Since $\big\{(T(\hat{v}_{\rho_0}) - \mu\rho_0)^2\big\}_{n\geq 1}$ and $\big\{\big(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\big)^2\big\}_{n\geq 1}$ can be seen to be uniformly integrable, as above, $\big\{\big(T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\big)\big(T(\hat{v}_{\rho_0}) - \mu\rho_0\big)\big\}_{n\geq 1}$ is also uniform integrable, and we have that as $n \to \infty$

$$\operatorname{Cov}\Big(T(\hat{v}_{\rho_0}) - \mu\rho_0, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\Big) \to \operatorname{Cov}\big(T(\hat{v}_{\rho_0}) - \mu\rho_0, Y'_{2M} - Y'_+\big).$$

That is, for some constant C_i,

$$\operatorname{Cov}\big(T(\hat{v}_{\rho_{\nu(n)}}) - \mu\rho_{\nu(n)}, T(v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)}})\big) \;=\; C_i + o(1), \quad \text{as } n \to \infty.$$

This all together amount to that there exists a finite constant C_i such that

$$\operatorname{Var}\big(T(v_{n+2M,i})\big) \;=\; \sigma^2 n + C_i + o(1), \quad \text{as } n \to \infty,$$

which proves the theorem. □

5 Geodesics and time constants

The path along which an infection travels from one vertex to another for first-passage percolation on essentially 1-dimensional periodic graphs is studied in this section. The existence of such minimising paths can be easily derived from Lemma 2.3.

Proposition 5.1. *Let U and V be two finite sets of vertices of an essentially 1-dimensional periodic graph. There is an almost surely finite path γ from U to V, such that*

$$T(\gamma) = T(U, V).$$

Moreover, if the passage-time distribution does not have any point masses, then γ is almost surely unique.

It follows directly from the statement that for any finite I, n and v, there are almost surely finite paths attaining the infima in T_n, $T(v)$ and $\max_{v\in\mathbb{V}_{\mathcal{G}_n}} T(v)$.

Proof. We may assume that $U \cup V \subseteq \bigcup_{k=0}^{m} \mathbb{V}_{\mathcal{G}_k}$. Assume further that t', t'' and M are chosen in accordance with Lemma 2.3. With probability one the event $A_{m+n} \cap A_{-2M-n}$ will occur for infinitely many $n \geq 0$. Let l be the least such n. It follows from Lemma 2.3 that for any path Γ between u and v that reach beyond level $m+l+2M$ in the positive direction, or level $-2M-l$ in the negative direction, there is another path Γ' that only visits vertices in $\bigcup_{k=-2M-l}^{m+l+2M} \mathbb{V}_{\mathcal{G}_k}$, and that satisfies $T(\Gamma) \geq T(\Gamma')$. Thus, since there are only finitely many edges between level $-2M-l$ and $m+l+2M$, it follows that $T(U, V)$ is the minimum of the passage times over an almost surely finite number of paths. This proves the first statement. The second statement also follows from this, together with the fact that the probability of two paths having the same passage time is zero, when the passage-time distribution is free of point masses. □

As in the introduction, we will use the term *geodesic* to refer to a path attaining the minimal passage time between two vertices, or two finite sets of vertices. Since geodesics are not necessarily unique, we assume a fixed deterministic rule to choose one when several are possible, e.g. the shortest (with some additional rule for breaking ties). Let $\gamma(u, v)$ denote the geodesic between u and v. Several properties of geodesics can be investigated. We will in what comes mainly consider the length of geodesics.

Let $N(u, v)$ denote the length of $\gamma(u, v)$. The regenerative behaviour studied in Section 2 will again play an important rôle. It follows from Lemma 2.3 that any geodesic from $u \in \mathbb{V}_{\mathcal{G}_n}$ to $v \in \mathbb{V}_{\mathcal{G}_m}$ (where $n \leq m$) passes $\hat{v}_{\rho_k}$, for all $n + M \leq \rho_k \leq m - M$. Moreover, $\{N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})\}_{k \in \mathbb{Z}}$ forms an i.i.d. sequence, which we may use to write

$$N(\hat{v}_{\rho_n}) = N(\hat{v}_{\rho_0}) + \sum_{k=1}^{n} N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}).$$

It is now easy to see that $\{N(\hat{v}_n)\}_{n \geq 1}$ is a regenerative sequence, with sequence of regenerative levels $\{\rho_n\}_{n \geq 0}$. Since there are only finitely many vertices at each level, say K, it follows that

$$N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}) \leq K S_k,$$

for each $k \in \mathbb{Z}$. In particular, $N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})$ has finite moments of all orders. Set

$$\alpha := \frac{\mathrm{E}[N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})]}{\mu_S}, \quad \text{and} \quad \sigma_N^2 := \frac{\mathrm{Var}(N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}) - \alpha S_k)}{\mu_S}. \tag{5.1}$$

Trivially, $\alpha \geq 1$ for any essentially 1-dimensional periodic graph.

Recall that we let $\{\hat{N}_n\}_{n\geq 1}$ denote either of $\{N_n\}_{n\geq 1}$, $\{\max_{v\in\mathbb{V}_{\mathcal{G}_n}} N(v)\}_{n\geq 1}$ and $\{N(v_n)\}_{n\geq 1}$, where $\{v_n\}_{n\geq 1}$ is any sequence of vertices such that v_n is at level n. By mimicking the proofs of Theorem 1.3, 1.4, 1.5, 1.6, 1.7 and 3.6, one may prove the following two results. Note that no moment conditions are required, since $N(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})$ has finite moments of all orders. The adaptions of the proofs are left to the reader.

Theorem 5.2. *Consider first-passage percolation on any essentially 1-dimensional periodic graph $\mathcal{G}$, with any passage-time distribution that do not concentrate all mass to a single point. Then, the statements of Theorem 1.3, 1.4, 1.5 and 3.6 holds (with constants α and σ_N^2) for the sequence $\{\hat{N}_n\}_{n\geq 1}$.*

Theorem 1.8 is included in Theorem 5.2.

Theorem 5.3. *Consider first-passage percolation on any essentially 1-dimensional periodic graph $\mathcal{G}$, with any passage-time distribution that do not concentrate all mass to a single point. Let $v_{n,i}$ be a specific vertex at level n. Then, for some $C_i, C_i' \in \mathbb{R}$, as $n \to \infty$,*

$$\begin{aligned} \mathrm{E}[N(v_{n,i})] &= \alpha n + C_i + o(1), \\ \mathrm{Var}\big(N(v_{n,i})\big) &= \sigma_N^2 n + C_i' + o(1). \end{aligned}$$

Geodesics are, as seen via Lemma 2.3, locally determined. Thus, it makes sense to talk about an infinite geodesic from $-\infty$ to ∞. Let to this end γ^* denote the unique (subject to the rule for breaking ties) path that between level ρ_{k-1} and ρ_k coincides with $\gamma(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k})$, for each $k \in \mathbb{Z}$. The resulting infinite path is indeed a geodesic, i.e., any finite portion $\tilde{\gamma}^*$ of γ^* with endpoints u and v satisfies $T(\tilde{\gamma}^*) = T(u, v)$. It is possible to characterize time and length constants in terms of the infinite geodesic.

Proposition 5.4.

$$\alpha = \sum_{v\in\mathbb{V}_{\mathcal{G}_0}} \mathrm{P}(v \in \gamma^*) = \sum_{e\in\mathbb{E}^*_{\mathcal{G}_0}} \mathrm{P}(e \in \gamma^*),$$

$$\mu = \sum_{e\in\mathbb{E}^*_{\mathcal{G}_0}} \mathrm{E}[\tau_e 1_{\{e\in\gamma^*\}}] = \sum_{e\in\mathbb{E}^*_{\mathcal{G}_0}} \mathrm{E}[\tau_e | \, e \in \gamma^*]\mathrm{P}(e \in \gamma^*).$$

Proof. We will deduce the characterization of α in terms of vertices, and leave the remaining cases, which are deduced similarly, to the reader. Observe that

$$N(u, w) = \sum_{k\in\mathbb{Z}} \sum_{v\in\mathbb{V}_{\mathcal{G}_k}} 1_{\{v\in\gamma(u,w)\}} - 1.$$

According to Theorem 5.2, we have

$$\frac{\mathrm{E}[N(\hat{v}_{-n},\hat{v}_n)]}{2n} \to \alpha, \quad \text{as } n \to \infty.$$

Define $N^* := \sum_{k=-n+\sqrt{n}}^{n-\sqrt{n}} \sum_{v\in\mathbb{V}_{\mathcal{G}_k}} 1_{\{v\in\gamma^*\}}$. Clearly

$$\frac{\mathrm{E}[N^*]}{2n} = \frac{2(n-\sqrt{n})}{2n} \sum_{v\in\mathbb{V}_{G_0}} \mathrm{P}(v\in\gamma^*) \to \sum_{v\in\mathbb{V}_{\mathcal{G}_0}} \mathrm{P}(v\in\gamma^*), \quad \text{as } n\to\infty,$$

so we are finished if we show that $\mathrm{E}\big[|N(\hat{v}_{-n},\hat{v}_n)-N^*|\big]/n \to 0$, as $n\to\infty$. Let

$$D_n := \big\{A_k \cap A_{-k-2M} \text{ occurs for some } k\in[n-\sqrt{n}, n-2M]\big\},$$

where A_k and M are as defined in Section 2. Let

$$\kappa_n := \min\{k \geq n : A_k \cap A_{-k-2M} \text{ occurs}\}.$$

Trivially, $|N(\hat{v}_{-n},\hat{v}_n)-N^*| \leq 4|\mathbb{V}_{\mathcal{G}_0}|(\kappa_n+2M)$. On the event D_n we have

$$\begin{aligned} N(\hat{v}_{-n},\hat{v}_n) - N^* &= \sum_{\substack{k>n-\sqrt{n}\\ k<-n+\sqrt{n}}} \sum_{v\in\mathbb{V}_{\mathcal{G}_k}} 1_{\{v\in\gamma(u,w)\}} - 1 \\ &\leq 2|\mathbb{V}_{\mathcal{G}_0}|(\kappa_n+2M-n+\sqrt{n}). \end{aligned}$$

Since $\kappa_n - n$ can be dominated by a geometrically distributed random variable, similar to S_k in the proof of the first part of Proposition 2.7, we easily realise that

$$\begin{aligned} \mathrm{E}\big[|N(\hat{v}_{-n},\hat{v}_n)-N^*|\big] &= \mathrm{E}\big[|N(\hat{v}_{-n},\hat{v}_n)-N^*|(1_{D_n}+1_{D_n^c})\big] \\ &\leq 2|\mathbb{V}_{\mathcal{G}_0}|\big(\mathrm{E}[\kappa_n-n]+2M+\sqrt{n}\big) \\ &\quad + 4|\mathbb{V}_{\mathcal{G}_0}|\big(\mathrm{E}[\kappa_n]+2M\big)\mathrm{P}(D_n^c) \\ &\leq 4|\mathbb{V}_{\mathcal{G}_0}|\big(C+\sqrt{n}+n\mathrm{P}(D_n^c)\big) = o(n). \end{aligned}$$

As mentioned, the remaining characterizations are deduced similarly. □

Benjamini et al. (2003) posed the question whether for first-passage percolation on the $\mathbb{Z}^d$ lattice, $\mathrm{P}\big(\mathbf{0}\in\gamma(-\mathbf{n},\mathbf{n})\big)\to 0$ as $n\to\infty$ (given existence of geodesics). One may pose a corresponding question for first-passage percolation on the (K,d)-tube. Let γ_K^* denote the infinite geodesic on the (K,d)-tube. How does $\mathrm{P}(v\in\gamma_K^*)$ behave as $K\to\infty$? In particular, does

$$\max_{v\in\mathbb{V}_{\mathcal{G}_n}} \mathrm{P}(v\in\gamma_K^*) \to 0, \quad \text{as } K\to\infty?$$

If it does, at which rate? Let α_K denote the constant α associated to the (K, d)-tube. By symmetry it is easily realised that for even K,

$$\max_{v \in \mathbb{V}_{\mathcal{G}_n}} \mathrm{P}(v \in \gamma_K^*) \; \leq \; \frac{\alpha_K}{2^{d-1}}.$$

We do not have enough symmetry to conclude a similar upper bound in K. We can increase the symmetry of the $(K, 2)$-tube by connecting the vertices $(n, 0)$ and $(n, K-1)$, for each n, by an edge. On the resulting graph we have, for every vertex v,

$$\mathrm{P}(v \in \gamma^*) = \frac{\alpha}{K}.$$

The same can be done for any (K, d)-tube. Join, for each $j = 2, 3, \dots, d$, the vertices

$$(m_1, m_2, \dots, m_{j-1}, 0, m_{j+1}, \dots, m_{K-1})$$

and

$$(m_1, m_2, \dots, m_{j-1}, K-1, m_{j+1}, \dots, m_{K-1})$$

by an edge, for all $m_1 \in \mathbb{Z}$ and $m_2, \dots, m_{K-1} \in \{0, 1, \dots, K-1\}$. Refer to the resulting graph as a (K, d)-*cylinder*. Let $\tilde{\alpha}_K$ denote the constant α associated to the (K, d)-cylinder. We have for every vertex v of the (K, d)-cylinder

$$\mathrm{P}(v \in \gamma^*) = \frac{\tilde{\alpha}_K}{K^{d-1}}.$$

Thus, in view of Remark 5.5 below, there is a constant $C = C(d)$ such that

$$\frac{1}{K^{d-1}} \; \leq \; \mathrm{P}(v \in \gamma^*) \; \leq \; \frac{C}{K^{d-1}},$$

for every $K \geq 1$, and every vertex v of the (K, d)-cylinder.

Remark 5.5. Provided that $\mathrm{E}[\tau_e] < \infty$ and that $\mathrm{P}_\tau(0)$ is sufficiently small, Kesten (1986) gives an argument that shows that on the $\mathbb{Z}^d$ lattice, there is a constant $C = C(d)$ such that $\mathrm{E}\big[N(u, v)\big] \leq C \operatorname{dist}(u, v)$ for all vertices u and v (cf. Howard (2004, page 146)). It is clear that the argument also applies to (K, d)-tubes and (K, d)-cylinders. That is, on either of these graphs, there exists a $C = C(d)$ such that for all $K \geq 1$

$$\mathrm{E}\big[N(u, v)\big] \; \leq \; C \operatorname{dist}(u, v)$$

for all vertices u and v. A direct consequence of this is that $\alpha_K \leq C$ and $\tilde{\alpha}_K \leq C$, for some finite constant C, for all $K \geq 1$. $\square$

Remark 5.6. An object closely related to geodesics is the *tree of infection* Ψ. Let $v_0 \in \mathbb{V}_{\mathcal{G}_0}$ denote a vertex referred to as the origin. The tree of infection is then defined as the tree $\Psi = \bigcup_{v \in \mathbb{V}} \gamma(v_0, v)$ spanning the underlying graph $\mathcal{G}$ (see Figure 2, page 13 for a realisation on the $(2,2)$-tube). One may ask for the number of infinite self-avoiding paths in Ψ started at the origin, denoted by $\kappa(\Psi)$. On any essentially 1-dimensional periodic graph $\kappa(\Psi) = 2$, almost surely. To see this, for any $M \geq 1$, let κ_M denote the number of self-avoiding paths in Ψ that reach level M. With probability one, for some $n \geq M$ the event A_n will occur. It follows from Lemma 2.3 that the geodesic from u to v, for all u at level n and v at level $n + 2M$, have all to pass a certain vertex at level $n + M$. Thus, only one of the κ_M self-avoiding paths in Ψ will survive beyond level $n + 2M$. This implies that precisely one self-avoiding path will reach infinitely far in positive direction. The same applies in negative direction. From this we conclude that $\kappa(\Psi) = 2$ almost surely.

On the $\mathbb{Z}^d$ lattice for $d \geq 2$, it is believed that $\kappa(\Psi)$ is infinite. So far, it is only known that $\kappa(\Psi) \geq 2d$ almost surely (see Hoffman (2008) and Gouéré (2007)). It would be interesting to prove that $\kappa(\Psi)$ is almost surely constant. That would follow from an higher dimensional version of the Proposition 1.10. It is not known whether such a coupling is possible. □

5.1 Continuity of constants

The following result is inspired by a similar result due to Cox (1980) and Cox and Kesten (1981), who in their case consider first-passage percolation on the $\mathbb{Z}^d$ lattice. The proof of the lattice case is rather lengthy. Due to the regenerative behaviour in the case of essentially 1-dimensional periodic graphs, and in particular the characterization of μ and σ given in (2.7), and of α and σ_N given in (5.1), the proof of our result turns out to be much simpler.

Proposition 5.7. *Let F_m for $m = 1, 2, \ldots, \infty$ be distribution functions such that $F_m \xrightarrow{d} F_\infty$ as $m \to \infty$. Then, as $m \to \infty$,*

$$\alpha(F_m) \to \alpha(F_\infty) \quad \textit{and} \quad \sigma_N(F_m) \to \sigma_N(F_\infty).$$

Assume further that there are $p \geq 1$ (edge) disjoint paths from $\hat{v}_0$ to $\hat{v}_1$, and a distribution function V such that $F_m \geq V$ for all $m \geq 1$. Let Y_V denote the minimum of p independent random variables with distribution V. If, in addition, $\mathrm{E}[Y_V] < \infty$ and $\mathrm{E}[Y_V^2] < \infty$, then as $m \to \infty$, respectively,

$$\mu(F_m) \to \mu(F_\infty) \quad \textit{and} \quad \sigma(F_m) \to \sigma(F_\infty).$$

Remark 5.8. This result will be used in Ahlberg (2011) in order to prove a dynamically stable version of Theorem 1.3. □

In order to compare the different distributions, we will use a coupling of random variables via their inverse distribution functions

$$F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\}.$$

The same approach is used in Cox (1980) and Cox and Kesten (1981). Indeed, if U is uniformly distributed on $[0,1]$, then $F^{-1}(U)$ has distribution F, since

$$\mathrm{P}\big(F^{-1}(U) \leq x\big) = \mathrm{P}\big(U \leq F(x)\big) = F(x), \quad \text{for all } x \in \mathbb{R}.$$

Thus, if we let F run over the class of distribution functions, then $\{F^{-1}(U)\}_F$ generates a coupling of all differently distributed random variables. Note that F^{-1} is nondecreacing since F is, and has at most countably many discontinuity points (since $\mathbb{Q}$ is countable, and for each discontinuity point u, $[F^{-1}(u-), F^{-1}(u)] \cap \mathbb{Q} \neq \emptyset$). It is not hard to prove that (see e.g. Thorisson (2000, Section 1.8.4)), as $m \to \infty$, $F_m \xrightarrow{d} F_\infty$ implies $F_m^{-1}(u) \to F_\infty^{-1}(u)$ for all continuity points $u \in (0,1)$. In particular, $F_m^{-1}(U) \to F_\infty^{-1}(U)$ almost surely, as $m \to \infty$.

Once we have the above coupling, the rest will follow fairly easily. For i.i.d. sequences, if $F_m \xrightarrow{d} F_\infty$, $F_m \geq V$ for all m, and V has finite mean, then $F_m^{-1}(U) \leq V^{-1}(U)$ and $\mathrm{E}\big[F_m^{-1}(U)\big] \to \mathrm{E}\big[F_\infty^{-1}(U)\big]$ as $m \to \infty$, by the Dominated Convergence Theorem. For the proof of the proposition, the idea is similar.

Proof of Proposition 5.7. Let $\{U_e\}_{e\in\mathbb{E}}$ be a collection of independent random variables uniformly distributed on $[0,1]$. Thus, as F ranges over the class of passage-time distributions, then $\big\{\{F^{-1}(U_e)\}_{e\in\mathbb{E}}\big\}_F$ simultaneously couples i.i.d. sets of passage times of the graph. Choose $a \in (0, 1/2)$ such that $F_\infty^{-1}(1-a) > F_\infty^{-1}(a)$, and F_∞^{-1} is continuous in both a and $1-a$. Take $\epsilon > 0$ such that $F_\infty^{-1}(1-a) - F_\infty^{-1}(a) > 2\epsilon$. Choose $L < \infty$ such that

$$\big|F_m^{-1}(a) - F_\infty^{-1}(a)\big| \leq \epsilon \quad \text{and} \quad \big|F_m^{-1}(1-a) - F_\infty^{-1}(1-a)\big| \leq \epsilon,$$

for all $m \geq L$. Recall the definition of $A_n = A_n(M, t', t'')$ in Section 2. Set $t' = F_\infty^{-1}(a) + \epsilon$ and $t'' = F_\infty^{-1}(1-a) - \epsilon$, and let M be chosen in accordance with Lemma 2.3 (recall that M is chosen independently of the passage time distribution F_m). For the same M (and with notation as in Section 2), define

$$\tilde{A}_n = \tilde{A}_n(M) := \Big\{U_e \leq a, \forall e \in \hat{E}_n\Big\} \cap \Big\{U_e \geq 1-a, \forall e \in E_n \setminus \hat{E}_n\Big\}.$$

Since $a > 0$, we have $\mathrm{P}(\tilde{A}_n) > 0$. For all $m \geq L$ we have

$$\begin{cases} F_m^{-1}(u) \leq t', & \text{for } u \leq a, \\ F_m^{-1}(u) \geq t'', & \text{for } u \geq 1-a. \end{cases}$$

With a slight abuse of notation, we let $A_n(F)$ denote the event A_n with respect to $\{F^{-1}(U_e)\}_{e\in\mathbb{E}}$. In particular, this implies that $\tilde{A}_n \subseteq A_n(F_m)$ for all $L \leq m \geq \infty$. Define a sequence $\{\tilde{\rho}_k\}_{k\geq 0}$ with respect to $\tilde{A}_n$ analogously as in Section 2. Note that for $m \geq L$, the sequence $\{\tilde{\rho}_k\}_{k\geq 0}$ is a subsequence of $\{\rho_k\}_{k\geq 0}$ defined with respect to $A_n(F_m)$. The advantage of this is that we get a regenerative sequence valid for all distributions F_m with $L \leq m \leq \infty$.

From here the result follows quickly. Let $T_F(u, v)$ and $N_F(u, v)$ denote the passage time and length of geodesic, respectively, between u and v with respect to $\{F^{-1}(U_e)\}_{e\in\mathbb{E}}$. For $m \geq L$ we have the characterization

$$\alpha(F_m) = \frac{\mathrm{E}\big[N_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big]}{\mathrm{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}, \quad \sigma_N(F_m) = \frac{\mathrm{E}\big[N^2_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big]}{\mathrm{E}[\tilde{\rho}_1 - \tilde{\rho}_0]},$$
$$\mu(F_m) = \frac{\mathrm{E}\big[T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big]}{\mathrm{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}, \qquad \sigma(F_m) = \frac{\mathrm{E}\big[T^2_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big]}{\mathrm{E}[\tilde{\rho}_1 - \tilde{\rho}_0]}.$$

Thus, in order to prove that $\mu(F_m) \to \mu(F_\infty)$ as $m \to \infty$, it suffices to show that

$$\mathrm{E}\big[T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big] \to \mathrm{E}\big[T_{F_\infty}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big], \quad \text{as } m \to \infty. \tag{5.2}$$

But $F_m^{-1}(U) \to F_\infty^{-1}(U)$ almost surely, as $m \to \infty$, and therefore also

$$T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}) \to T_{F_\infty}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}), \quad \text{almost surely.}$$

Since $T_V(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})$ has finite mean when Y_V does (according to Proposition 2.7), and since $T_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}) \leq T_V(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})$, we conclude by the Dominated Convergence Theorem that (5.2) holds when $\mathrm{E}[Y_V] < \infty$. To see that the domination is not necessary in order to prove convergence of $\alpha(F_m)$ to $\alpha(F_\infty)$, it suffices to realize that for $m \geq L$ and some $C < \infty$, we have $N_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1}) \leq C(\tilde{\rho}_1 - \tilde{\rho}_0)$. The remaining conclusions are drawn similarly. □

Remark 5.9. The true condition for the convergence $\mathrm{E}\big[F_m^{-1}(U)\big] \to \mathrm{E}\big[F_\infty^{-1}(U)\big]$ is in fact uniform integrability of $\{F_m\}_{m\geq 1}$. In the same way it is possible to relax the moment condition on Y_V to uniform integrability of $\big\{Y^r_{F_m}\big\}_{m\geq 1}$ (for $r = 1$ or 2), which grants uniform integrability of $\big\{T^r_{F_m}(\hat{v}_{\tilde{\rho}_0}, \hat{v}_{\tilde{\rho}_1})\big\}_{m\geq 1}$. We leave it to the reader to go through the details. □

5.2 Time constant and the (K, d)-tube

Let μ_K denote the time constant associated with the (K, d)-tube. It is easy to realize that $\mu_{K+1} \leq \mu_K$. However, it is hard to prove that without a (trivial) coupling argument. In fact, strict inequality holds, for which we will need the same coupling in order to see. The coupling is as follows. Let $\{\tau_e\}_{e\in\mathbb{E}_{\mathbb{Z}^d}}$ be i.i.d. passage times associated to the $\mathbb{Z}^d$ lattice. The (K,d)-tubes are naturally seen as subgraphs of the $\mathbb{Z}^d$ lattice. Let $T_K(u,v)$ denote the passage time with respect to $\{\tau_e\}_{e\in\mathbb{E}_{\mathbb{Z}^d}}$, between u and v, when only paths in the $\mathbb{Z}\times\{0,\dots,K-1\}^{d-1}$ nearest neighbour graph (the (K,d)-tube) are allowed. This produces a simultaneous coupling of the passage time on (K,d)-tubes for all $K\geq 1$. Trivially, $T_{K+1}(u,v)\leq T_K(u,v)$ for any u and v in $\mathbb{Z}\times\{0,\dots,K-1\}^{d-1}$.

Proposition 5.10. *For all $K\geq 1$,* $\quad \mu_{K+1}<\mu_K$.

Proof. Let A_n^K be the event defined in (2.4) with respect to the (K,d)-tube, for γ_n chosen to be the straight line segment between the points $(n,K,0,\dots,0)$ and $(n+2M,K,0,\dots,0)$. It follows from Lemma 2.3 that if A_n^{K+1} occurs, then

$$\delta := T_K\big(n\mathbf{e}_1,(n+2M)\mathbf{e}_1\big)-T_{K+1}\big(n\mathbf{e}_1,(n+2M)\mathbf{e}_1\big)>0.$$

Thus, if $m_k=(2M+1)k$, then

$$T_{K+1}(0\mathbf{e}_1,m_k\mathbf{e}_1)+\delta\sum_{j=0}^{k-1}1_{A_{m_j}^{K+1}} \;\leq\; T_K(0\mathbf{e}_1,m_k\mathbf{e}_1)$$

for all $k\geq 0$. Dividing by m_k and taking limits as k tends to infinity, gives

$$\mu_{K+1}+\delta\mathrm{P}\big(A_n^{K+1}\big)(2M+1)^{-1} \;\leq\; \mu_K. \qquad \square$$

In order to prove that the limit of the sequence $\{\mu_K\}_{K\geq 1}$ is $\mu(\mathbf{e}_1)$, i.e., equals the time constant for the $\mathbb{Z}^d$ lattice, we will use a coupling similar to the above one. For $K=0,1,\dots,\infty$, let $\tilde{T}_K(u,v)$ denote the passage time with respect to $\{\tau_e\}_{e\in\mathbb{E}_{\mathbb{Z}^d}}$, between u and v, when only paths in the $\mathbb{Z}\times\{-K,\dots,K\}^{d-1}$ nearest neighbour graph (the $(2K+1,d)$-tube) are allowed. This produces a simultaneous coupling of the passage time on (K,d)-tubes for odd K. The case $K=\infty$ corresponds to the $\mathbb{Z}^d$ lattice.

Proposition 5.11. $\lim_{K\to\infty}\mu_K=\mu(\mathbf{e}_1)$.

Proof. Clearly $\tilde{T}_K(0\mathbf{e}_1,n\mathbf{e}_1)\geq\tilde{T}_{K+1}(0\mathbf{e}_1,n\mathbf{e}_1)$. For all n we get

$$\tilde{T}_\infty(0\mathbf{e}_1,n\mathbf{e}_1)=\lim_{K\to\infty}\tilde{T}_K(0\mathbf{e}_1,n\mathbf{e}_1)=\inf_{K\geq 0}\tilde{T}_K(0\mathbf{e}_1,n\mathbf{e}_1),\quad \text{almost surely.}$$

An application of the Monotone Convergence Theorem

$$\mathrm{E}\big[\tilde{T}_\infty(0\mathbf{e}_1,n\mathbf{e}_1)\big] \;=\; \lim_{K\to\infty}\mathrm{E}\big[\tilde{T}_K(0\mathbf{e}_1,n\mathbf{e}_1)\big] \;=\; \inf_{K\geq 0}\mathrm{E}\big[\tilde{T}_K(0\mathbf{e}_1,n\mathbf{e}_1)\big].$$

Since $\exists \lim_{n\to\infty} a_n/n = \inf_{n\geq 1} a_n/n$, for any subadditive real-valued sequence $\{a_n\}_{n\geq 1}$, we have for any $0 \leq K \leq \infty$ that

$$\mu_{2K+1} = \lim_{n\to\infty} \frac{\mathrm{E}\big[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)\big]}{n} = \inf_{n\geq 1} \frac{\mathrm{E}\big[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)\big]}{n}.$$

Thus, since μ_K is non-increasing in K

$$\begin{aligned}\lim_{K\to\infty} \mu_{2K+1} &= \inf_{K\geq 0}\inf_{n\geq 1} \frac{\mathrm{E}\big[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)\big]}{n} = \inf_{n\geq 1}\inf_{K\geq 0} \frac{\mathrm{E}\big[\tilde{T}_K(0\mathbf{e}_1, n\mathbf{e}_1)\big]}{n} \\ &= \inf_{n\geq 1} \frac{\mathrm{E}\big[\tilde{T}_\infty(0\mathbf{e}_1, n\mathbf{e}_1)\big]}{n} = \mu(\mathbf{e}_1). \qquad \square\end{aligned}$$

6 Exact coupling and a 0–1 law

The aim for this section is to couple first-passage percolation infections with different initial configurations, i.e., different initially infected components, in such a way that the infections will eventually coincide. As an application of this, we shall prove a 0–1 law. The method of proof will once again make use of the regenerative behaviour explored in Section 2.

First we must state what we mean by a coupling. A *coupling* of two random variables $X \sim P$ and $Y \sim P'$ on a measurable space $(E, \mathcal{E})$, is a joint distribution $\hat{P}$ of (X, Y), i.e., a measure on $(E^2, \mathcal{E}^2)$, such that its marginal distributions coincide with P and P', respectively. When we couple two time-dependent random elements $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$, we say that the coupling is *exact* if with probability one there exists a $T_c < \infty$ such that $X_t = Y_t$, for all $t \geq T_c$.

We will present an exact coupling of the sets of infected vertices B_t and B'_t of two first-passage percolation processes with different initial configurations. Recall that we let $\mathrm{P}_\tau(\,\cdot\,)$ denote the distribution of τ_e, and let $\mathcal{R}_+$ denote the Borel σ-algebra on $[0, \infty)$. Then $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ are random elements on the product space $\big([0,\infty)^{\mathbb{E}}, \mathcal{R}_+^{\mathbb{E}}\big)$, each with distribution given by the product measure $\mathrm{P}_\tau^{\mathbb{E}}$. Let $\mathbb{E}_n$ denote the set of edges between level $-n$ and n, but not including edges between two vertices at level $-n$ and n. In the same manner $\mathbb{E}_n^c$ denotes the set of edges at and before level $-n$, as well as at level n and beyond.

We shall prove the following result which is slightly stronger than Proposition 1.10.

Proposition 6.1 (Coupling, continuous times)**.** *Let I and I' be finite subsets of the set of vertices $\mathbb{V}$ of an essentially 1-dimensional periodic graph $\mathcal{G}$. Assume* *that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e\in\mathbb{E}_m^c}$ and $\{\tau'_e\}_{e\in\mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e\in\mathbb{E}_m}$ and $\{\tau'_e\}_{e\in\mathbb{E}_m}$ each have distribution $\mathrm{P}_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ are given by the product measure $\mathrm{P}_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $\big(I, \{\tau_e\}_{e\in\mathbb{E}}\big)$ and $\big(I', \{\tau'_e\}_{e\in\mathbb{E}}\big)$, respectively, then with probability one there exists an $N_c < \infty$ and a $T_c < \infty$, such that*

$$T(v_n) = T'(v_n) \quad \text{and} \quad B_t = B'_t, \tag{6.1}$$

for all $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for $n \geq N_c$, and for all $t \geq T_c$.

When the passage time distribution P_τ is discrete, i.e., $\mathrm{P}_\tau(\Lambda) = 1$ for the set of point masses

$$\Lambda := \{t_j \in [0, \infty) : \mathrm{P}_\tau(t_j) > 0\},$$

the statement of Proposition 6.1 is not true in general. More precisely, there are essentially 1-dimensional periodic graphs on which no exact coupling is possible (cf. Remark 6.6). In the discrete case, we will therefore restrict our attention to the case of (K, d)-tubes.

Proposition 6.2 (Coupling, discrete times)**.** *Let I and I' be finite subsets of the set of vertices $\mathbb{V}$ of the (K, d)-tube, for $K, d \geq 2$. Assume that the passage time distribution P_τ is such that $\mathrm{P}_\tau(\Lambda) = 1$ for the set of point masses Λ and that either of the following hold:*

a) there are $t_j \in \Lambda$ and integers n_j for j in some finite set of indices J^*, such that

$$\sum_{j\in J^*} n_j \text{ is odd,} \quad \text{and} \quad \sum_{j\in J^*} n_j t_j = 0.$$

b) $\mathrm{dist}(\mathbf{x}, \mathbf{y})$ is even, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$.

For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e\in\mathbb{E}_m^c}$ and $\{\tau'_e\}_{e\in\mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e\in\mathbb{E}_m}$ and $\{\tau'_e\}_{e\in\mathbb{E}_m}$ each have distribution $\mathrm{P}_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ are given by the product measure $\mathrm{P}_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $\big(I, \{\tau_e\}_{e\in\mathbb{E}}\big)$ and $\big(I', \{\tau'_e\}_{e\in\mathbb{E}}\big)$, respectively, then with probability one there exists an $N_c < \infty$ and a $T_c < \infty$, such that

$$T(v_n) = T'(v_n) \quad \text{and} \quad B_t = B'_t, \tag{6.2}$$

for all $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for $n \geq N_c$, and for all $t \geq T_c$.

Before we construct the couplings, we focus on the promised 0–1 law that follows from Proposition 6.1 and 6.2. For this we will use *Lévy's 0–1 law.* It states that for σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_t \uparrow \mathcal{F}_\infty$ as $t \to \infty$, if $A \in \mathcal{F}_\infty$, then $\mathrm{P}(A|\mathcal{F}_t) \to 1_A$, as $n \to \infty$, almost surely. A proof for the discrete case can be found in e.g. Durrett (2005, Theorem 4.5.8). The continuous case follows via the Martingale convergence theorem.

Recall that we defined the σ-algebra $\mathcal{T}_t = \sigma\big(\{B_s\}_{s\geq t}\big)$, and define $\mathcal{F}_t := \sigma\big(\{B_s\}_{0\leq s\leq t}\big)$, where as before B_s is the set of infected vertices at time s. We may think of $\mathcal{T}_t$ as the σ-algebra of events $A \in \sigma\big(\bigcup_{t\geq 0}\mathcal{F}_t\big)$ that do not depend on the times at which vertices were infected before time t. The 0–1 law we shall prove deals with the tail σ-algebra $\mathcal{T} = \bigcap_{t\geq 0}\mathcal{T}_t$.

Theorem 6.3 (0–1 law)**.** *Consider first-passage percolation performed under the assumptions of either Proposition 6.1 or 6.2. Then* $\mathrm{P}(A) \in \{0,1\}$, *for any event* $A \in \mathcal{T}$.

Note that Theorem 1.9 is a special case of Theorem 6.3.

Proof of Theorem 6.3 from Propositions 6.1 and 6.2. Consider two infections with the respective sets of passage times $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$. For $t \geq 0$, let $\mathcal{F}_t$ and $\mathcal{F}'_t$ be σ-algebras generated by their respective realisations up to time t. Let

$$\nu_t = \max\big\{n \geq 0 : (B_t \cup B'_t) \cap (\mathbb{V}_{\mathcal{G}_n} \cup \mathbb{V}_{\mathcal{G}_{-n}}) \neq \emptyset\big\}$$

denote the furthest level (in positive or negative direction) infected at time t. Since, almost surely, $\rho_k < \infty$ and $T(\hat{v}_{\rho_k})/k \to \mu_\tau > 0$ as $k \to \infty$, then there is a $k = k(t) < \infty$ such that $T(v) > t$ for all $v \in \bigcup_{n\geq\rho_k}\mathbb{V}_{\mathcal{G}_n}$. Thus $\nu_t < \infty$, almost surely, for any $t < \infty$.

For any fixed $t \geq 0$, by Propositions 6.1 and 6.2, there is a coupling of $\{\tau_e\}_{e\in\mathbb{E}^c_{\nu_t+1}}$ and $\{\tau'_e\}_{e\in\mathbb{E}^c_{\nu_t+1}}$, such that there exists an almost surely finite time T_c, such that $B_s = B'_s$ for all $s \geq T_c$. Since $A \in \mathcal{T}_{T_c}$, the outcome of A only depends on B_s for $s \geq T_c$. In particular it has to hold that

$$\mathrm{P}(A|\mathcal{F}_t) = \mathrm{P}(A|\mathcal{F}'_t).$$

Thus, $\mathrm{P}(A|\mathcal{F}_t)$ is nonrandom and equals $\mathrm{P}(A)$, for all $t \geq 0$. But, according to Lévy's 0–1 law, $\mathrm{P}(A|\mathcal{F}_t) \to 1_A$ as $t \to \infty$, almost surely. Hence, $\mathrm{P}(A) = 1_A$ almost surely, and therefore $\mathrm{P}(A)$ equals either 0 or 1. $\square$

It remains only to prove Propositions 6.1 and 6.2.

6.1 Exact coupling of time-delayed infections on $\mathbb{Z}$

Before proving Proposition 6.1 and 6.2, we shall first prove a lemma where we consider two infections on $\mathbb{Z}$. This lemma will figure as a key step in the proof of Proposition 6.1 and 6.2. For first-passage percolation on $\mathbb{Z}$, T_n simply takes the form $T_n = \sum_{k=1}^{n} \tau_k$. If we let the latter infection be delayed for some time T_{delay}, i.e., started at time T_{delay} instead of time zero, then $T'_n = T_{\text{delay}} + \sum_{k=1}^{n} \tau'_k$. We will construct a coupling of the passage times such that $T_n = T'_n$ for large n. The precise statement is as follows.

Lemma 6.4. *Let T_{delay} be any non-negative constant, and assume that either of the following hold:*

a) P_τ *has an absolutely continuous component (with respect to Lebesgue measure).*

b) P_τ *is such that for some finite index set J, there are non-negative integers n_j and n'_j, such that $\sum_{j\in J} n_j = \sum_{j\in J} n'_j$, and for atoms $t_j \in \Lambda$ of* P_τ

$$\sum_{j\in J} n_j t_j \;=\; \sum_{j\in J} n'_j t_j + T_{delay}. \tag{6.3}$$

Then, there exists a coupling of $\{\tau_k\}_{k\geq 1}$ and $\{\tau'_k\}_{k\geq 1}$ such that their marginal distributions are that of i.i.d. random variables with distribution P_τ*, and such that*

$$\sum_{k=1}^{n} \tau_k \;=\; T_{delay} + \sum_{k=1}^{n} \tau'_k, \quad \textit{for large } n. \tag{6.4}$$

The key to prove this lemma is to (in each case separately) identify a suitable random walk. The identification of the random walk in case *a)* heavily exploits ideas similar to those found in Lindvall (2002, Chapter III.5). In case *b)*, a multi dimensional random walk will be based on condition (6.3). This walk is then easily coupled with known techniques found e.g. in Lindvall (2002, Chapter II.12–17).

Proof of case a). Let $[a, b]$ be an interval on which P_τ has density $\geq c$, for some $c > 0$. Define

$$\delta := \max\left\{ d \geq 0 : \; d \leq \frac{b-a}{2}, d = \frac{T_{\text{delay}}}{m} \text{ for some } m \in \mathbb{N} \right\}.$$

Couple $\{\tau_k\}_{k\geq 1}$ and $\{\tau'_k\}_{k\geq 1}$ in the following way. With probability $1 - c2\delta$ we choose $\tau_k = \tau'_k$, drawn from the distribution

$$\tilde{\mathrm{P}_\tau}(\,\cdot\,) := \big(\mathrm{P}_\tau(\,\cdot\,) - c\lambda(\,\cdot \cap [a, a+2\delta])\big)/(1 - c2\delta),$$

where λ denotes Lebesgue measure. With the remaining probability $c2\delta$, draw τ_k uniformly on the interval $[a, a+2\delta]$, and choose τ_k' as

$$\tau_k' = \begin{cases} \tau_k + \delta, & \text{if } \tau_k \le a + \delta \\ \tau_k - \delta, & \text{if } \tau_k > a + \delta \end{cases}.$$

That τ_k' also is uniformly distributed on $[a, a+2\delta]$ is immediate. Thus, it is easy to see that the marginal distribution of both τ_k and τ_k' is P_τ, and this is indeed a coupling of the two infections.

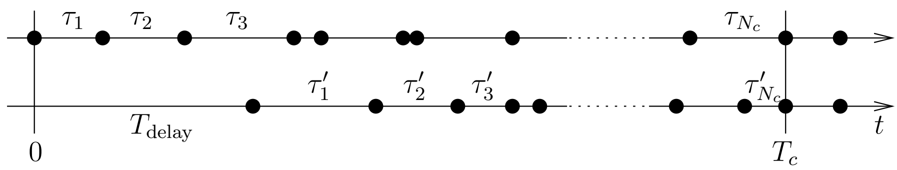

Figure 5: The dots represent the times at which the respective infection spreads. In this realisation $\tau_1 = \tau_1' - \delta$, $\tau_2 = \tau_2'$ and $\tau_3 = \tau_3' + \delta$. The coupling is constructed such that after some time T_c, both infections reach some level N_c simultaneously.

The coupling is such that each time τ_k and τ_k' are chosen differently, the difference $\{D_n\}_{n\ge1}$, where $D_n := T_{\mathrm{delay}} + \sum_{k=1}^{n}(\tau_k' - \tau_k)$ will jump $\pm\delta$. Since $T_{\mathrm{delay}} = m\delta$, for some integer m, D_n constitutes a simple random walk on $\delta\mathbb{Z}$. Let N_c denote the first n for which D_n hits zero. From this moment on, τ_k and τ_k' are chosen identically, and (6.4) holds for $n \ge N_c$. That the coupling is successful is easily seen, since

$$\begin{aligned} \mathrm{P}(N_c < \infty) &= \mathrm{P}(\exists n : D_n = 0) \\ &\ge \mathrm{P}(\exists n : D_n = 0 | \tau_k \ne \tau_k' \text{ i.o.})\mathrm{P}(\tau_k \ne \tau_k' \text{ i.o.}) \;=\; 1, \end{aligned}$$

where 'i.o.' abbreviates 'infinitely often'. The last equality follows from the recurrence of a 1-dimensional simple random walk, and Borel-Cantelli's second lemma.

Proof of case b). By assumption, for some set $\{t_j\}_{j\in J} \subseteq \Lambda$ of atoms for the distribution P_τ, there are non-negative integers n_j and n_j' such that $\sum_{j\in J} n_j = \sum_{j\in J} n_j'$ and (6.3) holds.

It is easily seen that we may assume that J, n_j and n_j' are chosen such that for each $j \in J$, exactly one of the integers n_j and n_j' is positive. We introduce integer valued random variables

$$
\begin{aligned}
X_j^n &= \#\{k \le n : \tau_k = t_j\} - n_j, \\
Y_j^n &= \#\{k \le n : \tau'_k = t_j\} - n'_j.
\end{aligned}
$$

Define $Z_j^n = X_j^n - Y_j^n$. It is clear that we from (6.3) can conclude that (6.4) holds, if $Z_j^n = 0$ for all $j \in J$ and $\tau_k = \tau'_k$ for all $k \le n$ such that $\tau_k \notin \{t_j\}_{j\in J}$ or $\tau'_k \notin \{t_j\}_{j\in J}$.

Let $J_n = \{j \in J : Z_j^n \neq 0\}$, let $p_j = \mathrm{P}_\tau(t_j)$, and $q_n = \sum_{j\in J_n} p_j$. In particular, $J_0 = J$. Couple $\{\tau_k\}_{k\ge1}$ and $\{\tau'_k\}_{k\ge1}$ by choosing τ_k and τ'_k identically from the distribution

$$
\tilde{\mathrm{P}_\tau}(\,\cdot\,) := \frac{1}{1-q_{k-1}}\left(\mathrm{P}_\tau(\,\cdot\,) - \sum_{j\in J_{k-1}} p_j \mathbf{1}_{\{t_j\}}(\,\cdot\,)\right)
$$

with probability $1-q_{k-1}$. With remaining probability q_{k-1} we choose τ_k and τ'_k independently with distribution $\mathrm{P}(\tau = t_j) = \frac{p_j}{q_{k-1}}$, for $j \in J_{k-1}$. The marginal distribution of τ_k and τ'_k is seen to be P_τ, whence this is a coupling of $\{\tau_k\}_{k\ge1}$ and $\{\tau'_k\}_{k\ge1}$.

Note that $\tau_k = \tau'_k$ for all k such that $\tau_k \notin \{t_j\}_{j\in J}$ and $\tau'_k \notin \{t_j\}_{j\in J}$. For each fixed $j \in J$, $\{Z_j^n\}_{n\ge0}$ will, as n increases, jump ±1 with equal probability. Hence, for fixed j, $\{Z_j^n\}_{n\ge0}$ constitutes a simple random walk on $\mathbb{Z}$. Note that if n^* denotes the first n such that $Z_j^n = 0$, then, by definition, $j \in J_n$ for $n < n^*$, but $j \notin J_n$ for $n \ge n^*$.

By assumption we have that

$$
\sum_{j\in J} Z_j^0 \;=\; \sum_{j\in J}\big(n_j - n'_j\big) \;=\; 0.
$$

Moreover, the sum of Z_j^n is constant for all n, i.e.,

$$
\sum_{j\in J} Z_j^n \;=\; \sum_{j\in J} Z_j^0 \;=\; 0.
$$

It follows that it is not possible for $|J_n| = 1$ for some n. There will therefore always be a positive probability to choose $\tau_{n+1} \neq \tau'_{n+1}$ as long as Z_j^n for some j. From this observation, Borel-Cantelli's second lemma and the recurrence of 1-dimensional simple random walks, we may further conclude that $\mathrm{P}\big(\exists n : Z_j^n = 0\big) = 1$ for each $j \in J$. Let $N_c = \min\{n \ge 0 : J_n = \emptyset\}$. For $n \ge N_c$ we have $Z_j^n = 0$ for all $j \in J$, and (6.4) holds for every such n. The coupling is successful since

$$
\mathrm{P}(N_c < \infty) \;=\; \mathrm{P}\left(\bigcap_{j\in J}\{\exists n : Z_j^n = 0\}\right) \;=\; 1. \qquad \square
$$

6.2 Exact coupling of two infections

In order to prove Proposition 6.1 and 6.2, we will arrange matters so that Lemma 6.4 can be applied. First, we need some notation. Recall from Section 2.2 that E_n denotes the set of edges between level n and $n+2M$, including edges at level n and level $n+2M$. In (2.2) we defined $\hat{E}_n = \gamma_n \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M}}$, where γ_n is a path of shortest length between $\mathbb{V}_{\mathcal{G}_n}$ and $\mathbb{V}_{\mathcal{G}_{n+2M}}$. Introduce the notation $\hat{e}_{n+M}$ for the edge in γ_n with endpoints $\hat{v}_{n+M}$ and u, where $\hat{v}_{n+M}$ is the vertex in $\mathbb{V}_{\mathcal{G}_{n+M}}$ first reached by γ_n, and u the vertex first reached after $\hat{v}_{n+M}$ by γ_n. Define the event

$$A_n^* := \left\{\tau_e \le t', \forall e \in \hat{E}_n \setminus \{\hat{e}_{n+M}\}\right\} \cap \left\{\tau_e \ge t'', \forall e \in E_n \setminus \hat{E}_n\right\}.$$

Note that $A_n = A_n^* \cap \{\tau_{\hat{e}_{n+M}} \le t'\}$ for A_n as defined in (2.4).

We will next prove Proposition 6.1, which is a slightly stronger version of Proposition 1.10. We first outline the general idea. It follows from the regenerative behaviour that if $\tau_e = \tau'_e$ for all $e \in \mathbb{E}$, then there is a real number T_d such that

$$B_t \cap \bigcup_{n\ge 0} \mathbb{V}_{\mathcal{G}_n} \;=\; B'_{t+T_d} \cap \bigcup_{n\ge 0} \mathbb{V}_{\mathcal{G}_n} \tag{6.5}$$

for t large enough. The idea for the coupling is to assign identical passage times for both infections, that is $\tau_e = \tau'_e$, except for certain edges which we make sure both infections have to pass. More precisely, for some sequence $\{l_k\}_{k\ge 0}$, for k such that $A^*_{l_k}$ occurs, choose either the passage times for $\hat{e}_{l_k+M}$ independently at most t', or equal. This generates a sequence of edges for which we invoke Lemma 6.4. That is, we make sure that $\{T(\hat{v}_{\rho_n}) - T'(\hat{v}_{\rho_n})\}_{n\ge 1}$ performs a random walk which eventually hits zero. This implies that (6.5) holds, with $T_d = 0$, for t large enough. This will complete the coupling of the infections in the direction of increasing levels. The opposite direction is treated in the same way.

Proof of Proposition 6.1. By assumption, P_τ has an absolutely continuous component, so suppose that $[a,b]$ is an interval on which P_τ has density $\ge c > 0$. Let $a < t' < t'' < b$ and choose M in accordance with Lemma 2.3. We may further assume that $I \cup I'$ contains no vertex beyond level m. Let $l_k := m + k(2M+1)$ for integers $k \ge 0$. Couple $\{\tau_e\}_{e\in\mathbb{E}^c_m}$ and $\{\tau'_e\}_{e\in\mathbb{E}^c_m}$ by choosing $\tau_e = \tau'_e$ with distribution P_τ, independently for all e at level m or beyond such that $e \ne \hat{e}_{l_k+M}$ for some $k \ge 0$. Independently for $k \ge 0$, let

$$(\xi_k, \xi'_k) = \begin{cases} (\theta_k, \theta'_k), & \text{with probability } \mathrm{P}_\tau([0,t']) \\ (\eta_k, \eta_k), & \text{with probability } 1 - \mathrm{P}_\tau([0,t']), \end{cases}$$

where θ_k and θ'_k are to be coupled below, so that they both have marginal distribution $\mathrm{P}_\tau(\,\cdot\,|\tau \leq t')$, and η_k has distribution $\mathrm{P}_\tau(\,\cdot\,|\tau > t')$. For the set of edges $\{\hat{e}_{l_k+M},\ \text{for } k \geq 0\}$, we choose the pair

$$\left(\tau_{\hat{e}_{l_k+M}}, \tau'_{\hat{e}_{l_k+M}}\right) = \begin{cases} (\xi_k, \xi'_k), & \text{if } A^*_{l_k} \text{ occurs} \\ (\tau_k, \tau_k), & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ, independently for all k. One realises from the coupling that the marginal distributions of both τ_e and τ'_e is P_τ, for every edge e.

Note that the only edges for which τ_e and τ'_e may differ, are the edges $\hat{e}_{l_k+M}$ for $k \geq 0$ such that A_{l_k} occurs. Let κ_j denote the index k for which A_{l_k} occurs for the jth time. That

$$\left(\tau_{\hat{e}_{l_k+M}}, \tau'_{\hat{e}_{l_k+M}}\right) = (\theta_k, \theta'_k) \tag{6.6}$$

is equivalent to that A_{l_k} occurs. Since $\mathrm{P}(A_{l_k}) > 0$, we will have an infinite sequence $\{\kappa_j\}_{j\geq 1}$ such that (6.6) holds. We now claim that the proposition will follow if we apply Lemma 6.4 to the sequences $\{\theta_{\kappa_j}\}_{j\geq 1}$ and $\{\theta'_{\kappa_j}\}_{j\geq 1}$, with distribution $\mathrm{P}_\tau(\,\cdot\,|\tau \leq t')$, and

$$T_{\text{delay}} = \left| T(\hat{v}_{l_{\kappa_1}+M}) - T'(\hat{v}_{l_{\kappa_1}+M}) \right|.$$

To see this, we use Lemma 2.3. Given A_{l_k}, the path along which any vertex at level $l_k + 2M$ or beyond is infected has to pass the edge $\hat{e}_{l_k+M}$. By the coupling $\tau_e = \tau'_e$ for all e at level l_{κ_1} or beyond such that $e \neq \hat{e}_{l_{\kappa_j}+M}$ for $j \geq 1$. Moreover, $\tau_e = \theta \leq t'$ and $\tau'_e = \theta' \leq t'$ for $e \in \left\{\hat{e}_{l_{\kappa_j}+M},\ \text{for } j \geq 1\right\}$. It follows that each vertex at level $l_{\kappa_1} + 2M + 1$ and beyond, will be reached in the same order. Since P_τ is absolutely continuous on $[a, b]$ and $t' > a$, $\mathrm{P}_\tau(\,\cdot\,|\tau \leq t')$ is absolutely continuous on $[a, t']$. Condition a) of Lemma 6.4 is therefore fulfilled. Coupling $\{\theta_{\kappa_j}\}_{j\geq 1}$ and $\{\theta'_{\kappa_j}\}_{j\geq 1}$ according to the lemma we will have with probability one that, from some level on, both infections will reach each vertex at the same time, i.e.,

$$T(v_n) = T'(v_n) \tag{6.7}$$

for any $v_n \in \mathbb{V}_{\mathcal{G}_n}$, for n sufficiently large.

The infections may in the same manner be coupled along the negative coordinate axis. Doing this, then there is $N_c \in \mathbb{N}$ such that (6.7) holds for $|n| \geq N_c$. In almost surely finite time, each vertex at level n, for $|n| \leq N_c$, will be infected. Hence, we conclude that for some almost surely finite time T_c,

$$B_t = B'_t, \quad \text{for each } t \geq T_c. \qquad \square$$

In preparation for the proof of Proposition 6.2, we restrict our attention to (K,d)-tubes. Let F_n denote the set of edges between level n and $n+2M+4\beta$, for integers

$$M > \frac{(d-1)(K-1)t'}{t''-t'} \quad \text{and} \quad \beta > \frac{t'}{t''-t'}.$$

Let $\mathbf{e}_1 = (1,0,\ldots,0)$. Denote by $e_{u,n}$ the edge between $(n+M+\beta)\mathbf{e}_1$ and $(n+M+\beta,1,0,\ldots,0)$, and by $e_{d,n}$ the edge between $(n+M+3\beta)\mathbf{e}_1$ and $(n+M+3\beta,1,0,\ldots,0)$. Let γ_n^* denote the path of shortest length from $\mathbf{n} = n\mathbf{e}_1$ to $(n+2M+4\beta)\mathbf{e}_1$. Let γ_n^{**} denote the path of shortest length from $\mathbf{n}$ to $(n+2M+4\beta)\mathbf{e}_1$ that visits the four endpoints of $e_{u,n}$ and $e_{d,n}$. Let $\hat{F}_n$ and $\hat{H}_n$ be defined as (see Figure 6)

$$\begin{aligned}
\hat{F}_n &:= \gamma_n^* \cup \{e_{u,n}, e_{d,n}\} \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M+4\beta}} \\
\hat{H}_n &:= \gamma_n^{**} \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M+4\beta}}.
\end{aligned}$$

For constants t' and t'' such that $m_\tau < t' < t'' < M_\tau$, define the events

$$\begin{aligned}
C_n &:= \left\{\tau_e \le t', \forall e \in \hat{F}_n\right\} \cap \left\{\tau_e \ge t'', \forall e \in F_n \setminus \hat{F}_n\right\}, \\
D_n &:= \left\{\tau_e \le t', \forall e \in \hat{H}_n\right\} \cap \left\{\tau_e \ge t'', \forall e \in F_n \setminus \hat{H}_n\right\}.
\end{aligned}$$

Trivially $\mathrm{P}(C_n) = \mathrm{P}(D_n) > 0$, since $\hat{F}_n$ and $\hat{H}_n$ contain equally many edges.

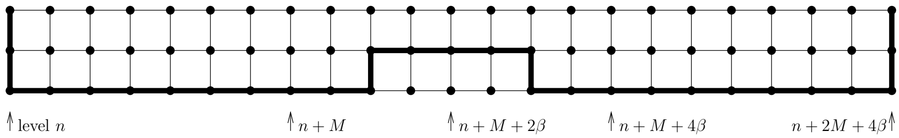

Figure 6: The $(3,2)$-tube between level n and $n+2M+4\beta$. If D_n occurs, the infection will advance along the thick edges.

Recall that $\rho_I = \max\{n \in \mathbb{Z} : \mathbb{V}_{\mathcal{G}_n} \cap I \neq \emptyset\}$. The following lemma says that given that the event C_n (or D_n) occurs, the infection will in order to reach level $n+2M+4\beta$ from level n do so via γ_n^* (or γ_n^{**}).

Lemma 6.5. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$, and assume that $n \ge \rho_I$. Given C_n (respectively D_n), then*

$$T(v) = T(\mathbf{n}) + T(\Gamma) + T\big((n+2M+4\beta)\mathbf{e}_1, v\big),$$

for each v at level $n+2M+4\beta$ or beyond, where $\Gamma = \gamma_n^*$ (respectively $\Gamma = \gamma_n^{**}$).

It is easy to see that the infection, from level n to level $n + 2M + 4\beta$, inevitably has to follow the paths γ_n^* and γ_n^{**}, in their respective cases, reasoning in a similar way as in the proof of Lemma 2.3. We leave the details to the reader.

The coupling of Proposition 6.2 will be constructed in two steps. The second part is similar to the coupling in the proof of Proposition 6.1. The first part is needed to make sure that condition b) of Lemma 6.4 will be satisfied. Before we give the somewhat technical proof, we present the idea behind the first step.

The events C_n and D_n were defined with respect to passage times from the sequence $\{\tau_e\}_{e\in\mathbb{E}}$. Let C'_n and D'_n denote the analogous events with respect to the sequence $\{\tau'_e\}_{e\in\mathbb{E}}$. Assign identical passage times for both infections, except for some edges in F_{l_k}, for some sequence $\{l_k\}_{k\geq 0}$. The remaining edges we couple in order to make the event C_{l_k} occur simultaneously as D'_{l_k}, and D_{l_k} occur simultaneously as C'_{l_k}. When they happen, the difference in length of the minimising paths in $T_{n+2M+4\beta}$ and $T'_{n+2M+4\beta}$ will either increase or decrease by 2. Thus, the difference in length constitutes a random walk. End the first step when it hits either 0 or the odd number $\omega = \sum_{j\in J^*} n_j$, for $\{n_j\}_{j\in J^*}$ as in assumption a) of Proposition 6.2. We will see that condition b) of Lemma 6.4 is then satisfied for $T_{\text{delay}} = |T'(v) - T(v)|$, for some vertex v.

Proof of Proposition 6.2. We may assume that $I \cup I'$ contains no vertex beyond level m. Set $l_k := m + k(2M + 4\beta + 1)$ for $k \geq 0$. For $j = 1, 2, \ldots, 2\beta$, let $f_{k,j}$ (and $h_{k,j}$) denote the edge in $\hat{F}_{l_k}$ (and $\hat{H}_{l_k}$) between level $l_k + M + \beta + j - 1$ and $l_k + M + \beta + j$ (respectively).

Couple $\{\tau_e\}_{e\in\mathbb{E}^c_m}$ and $\{\tau'_e\}_{e\in\mathbb{E}^c_m}$ in the following way. For one k at the time, choose $\tau_e = \tau'_e$ with distribution P_τ, independently for every edge e between level l_k and l_{k+1}, not at level l_{k+1} nor among $\{f_{k,j}, h_{k,j} : j = 1, 2, \ldots, 2\beta\}$. For $j = 1, 2, \ldots, 2\beta$, choose $\tau_{f_{k,j}}$ and $\tau_{h_{k,j}}$ independently with distribution P_τ, and set

$$\left(\tau'_{f_{k,j}}, \tau'_{h_{k,j}}\right) = \begin{cases} \left(\tau_{h_{k,j}}, \tau_{f_{k,j}}\right), & \text{if } C_{l_k} \cup D_{l_k} \\ \left(\tau_{f_{k,j}}, \tau_{h_{k,j}}\right), & \text{otherwise.} \end{cases}$$

Trivially τ_e has distribution P_τ, and it is easy to see that the marginal distribution of τ'_e for each e also is P_τ. Note that the coupling is such that C'_{l_k} occurs if and only if D_{l_k} occurs. In addition, D'_{l_k} occurs if and only if C_{l_k} does.

Let z_k, for $k \geq 1$, denote the length of the path of shortest passage time from I to level $l_k + 2M + 4\beta$, with respect to $\{\tau_e\}_{e\in\mathbb{E}}$. When several paths are possible, choose one. Similarly, let z'_k denote the length of the path of shortest passage time from I' to level $l_k + 2M + 4\beta$, with respect to $\{\tau'_e\}_{e\in\mathbb{E}}$. When several paths are possible, choose one that minimises $|z_k - z'_k|$. Set $\zeta_k := z_k - z'_k$.

With help from Lemma 6.5, we draw the following conclusions. For each k such that C_{l_k} (and therefore also D'_{l_k}) occurs, $\zeta_k - \zeta_{k-1} = -2$. When D_{l_k} (and therefore also C'_{l_k}) occurs, $\zeta_k - \zeta_{k-1} = 2$. Otherwise $\zeta_k = \zeta_{k-1}$. Thus, $\{\zeta_k\}_{k\geq 1}$ constitutes a simple random walk on either $2\mathbb{Z}$ or $2\mathbb{Z}+1$, depending on the value of ζ_1. Such walk is recurrent and will with probability one, reach either zero or the odd number $\omega := \sum_{j\in J^*} n_j$, respectively. Let κ denote the first k for which this happens. Couple the infections along the negative coordinate axis in the same way.

The first part of the coupling is done, and before we continue with the second part, we shall verify that assumption b) of Lemma 6.4 is satisfied. We may assume that $T_{l_\kappa+2M+4\beta} \leq T'_{l_\kappa+2M+4\beta}$. Set

$$T_{\text{delay}} = T'_{l_\kappa+2M+4\beta} - T_{l_\kappa+2M+4\beta}.$$

Given $\left\{(\tau_e, \tau'_e)\right\}_{e\in\mathbb{E}_{l_\kappa+2M+4\beta}}$, we may represent the passage time for each infection as

$$T_{l_\kappa+2M+4\beta} = \sum_{j\in J} m_j t_j \quad \text{and} \quad T'_{l_\kappa+2M+4\beta} = \sum_{j\in J'} m'_j t_j,$$

for index sets J and J', $t_j \in \Lambda$, and positive integers m_j and m'_j that indicate the number of edges e in the minimising path to level $l_\kappa + 2M + 4\beta$ such that $\tau_e = t_j$ and $\tau'_e = t_j$, respectively.

If $\zeta_\kappa = 0$, then $\sum_{j\in J} m_j = \sum_{j\in J'} m'_j$, and assumption b) of Lemma 6.4 is directly satisfied, since

$$T_{\text{delay}} + T_{l_\kappa+2M+4\beta} = T'_{l_\kappa+2M+4\beta}.$$

Note that this will be the case if $\text{dist}(\mathbf{x}, \mathbf{y})$ is even, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$, since then $\zeta_k \in 2\mathbb{Z}$. If rather $\zeta_\kappa = \omega$ is odd, we need the additional assumption that $\sum_{j\in J^*} n_j t_j = 0$ for some index set J^*, point masses t_j, and integers n_j such that $\sum_{j\in J^*} n_j = \omega$. Then, assumption b) of Lemma 6.4 is again satisfied, since

$$T_{\text{delay}} + T_{l_\kappa+2M+4\beta} = T'_{l_\kappa+2M+4\beta} + \sum_{j\in J^*} n_j t_j.$$

We will now go on with the second part of the coupling. Let

$$t^* := \max\{t_j \in \Lambda : \ j \in J \cup J' \cup J^*\}.$$

It may be the case that $t^* = M_\tau$ as defined in (2.3). This makes it necessary to introduce some extra notation. Write E'_n for the set of edges between level

n and $n+2M+1$, and let γ_n' denote the path of shortest length from $\mathbf{n}$ to $(n+2M+1)\mathbf{e}_1$. Let

$$\hat{E}_n' = \gamma_n' \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M+1}}.$$

Denote by $\hat{e}_n$ the edge between $\mathbf{n}$ and $(n+1)\mathbf{e}_1$. Let X_n denote the set of edges connecting a vertex at level n with one at level $n+1$, excluding the edge $\hat{e}_n$. Define the event

$$\begin{aligned} A_n^{**} \;=\; & \left\{\tau_e \le t', \forall e \in \hat{E}_n' \setminus \{\hat{e}_{n+M}\}\right\} \cap \{\tau_e \ge t^*, \forall e \in X_{n+M}\} \\ & \cap \left\{\tau_e \ge t'', \forall e \in E_n' \setminus \left(\hat{E}_n' \cup X_n\right)\right\}. \end{aligned}$$

Let $\lambda_k := l_{\kappa+1} + k(2M+2)$ for $k \ge 0$. Continue the coupling of $\{\tau_e\}_{e\in\mathbb{E}_m^c}$ and $\{\tau_e'\}_{e\in\mathbb{E}_m^c}$ by choosing $\tau_e = \tau_e'$ with distribution P_τ, independently for all e at level λ_0 or beyond such that $e \ne \hat{e}_{\lambda_k+M}$ for some $k \ge 0$. Independently for $k \ge 0$, let

$$(\xi_k, \xi_k') = \begin{cases} (\theta_k, \theta_k'), & \text{with probability } \mathrm{P}_\tau([0,t^*]) \\ (\eta_k, \eta_k), & \text{with probability } 1 - \mathrm{P}_\tau([0,t^*]), \end{cases}$$

where θ_k and θ_k' have marginal distribution $\mathrm{P}_\tau(\,\cdot\,|\tau \le t^*)$, and η_k has distribution $\mathrm{P}_\tau(\,\cdot\,|\tau > t^*)$ (η_k is not needed when $t^* = M_\tau$). For the set of edges $\{\hat{e}_{\lambda_k+M},$ for $k \ge 0\}$ we couple their passage times as

$$\left(\tau_{\hat{e}_{\lambda_k+M}}, \tau'_{\hat{e}_{\lambda_k+M}}\right) = \begin{cases} (\xi_k, \xi_k'), & \text{if } A_{\lambda_k}^{**} \text{ occurs} \\ (\tau_k, \tau_k), & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ, independently for all k. One realises from the coupling that the marginal distributions of both τ_e and τ_e' is P_τ.

Note that the only edges for which τ_e and τ_e' may differ, are the edges $\hat{e}_{\lambda_k+M}$ for $k \ge 0$ such that $A_{\lambda_k}^{**} \cap \left\{\tau_{\hat{e}_{\lambda_k+M}} \le t^*\right\}$ occurs. Let κ_j denote the index k for which $A_{\lambda_k}^{**} \cap \left\{\tau_{\hat{e}_{\lambda_k+M}} \le t^*\right\}$ occurs for the jth time. That

$$\left(\tau_{\hat{e}_{\lambda_k+M}}, \tau'_{\hat{e}_{\lambda_k+M}}\right) = (\theta_k, \theta_k') \tag{6.8}$$

is equivalent to that $A_{\lambda_k}^{**} \cap \left\{\tau_{\hat{e}_{\lambda_k+M}} \le t^*\right\}$ occurs. Since

$$\mathrm{P}\left(A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \le t^*\}\right) > 0,$$

we will have an infinite sequence $\{\kappa_j\}_{j\ge1}$ such that (6.8) holds. We now claim that the proposition will follow if we apply Lemma 6.4 to the sequences $\{\theta_{\kappa_j}\}_{j\ge1}$ and $\{\theta'_{\kappa_j}\}_{j\ge1}$, with distribution $\mathrm{P}_\tau(\,\cdot\,|\tau \le t^*)$ and T_{delay} as defined above.

To see this, argue as in the proof of Lemma 2.3. Given $A^{**}_{\lambda_k} \cap \{\tau_{\hat{e}_{\lambda_k+M}} \leq t^*\}$, the path along which any vertex at level $\lambda_k + 2M + 1$ or beyond is infected inevitably has to pass the edge $\hat{e}_{\lambda_k+M}$. By the coupling, $\tau_e = \tau'_e$ for all e at level λ_{κ_1} or beyond such that $e \neq \hat{e}_{\lambda_{\kappa_j}+M}$ for some $j \geq 1$. Moreover, $\tau_e = \theta \leq t'$ and $\tau'_e = \theta' \leq t'$ for $e \in \{\hat{e}_{l_{\kappa_j}+M} \text{ for } j \geq 1\}$. Therefore, each vertex at level $\lambda_{\kappa_1} + 2M + 1$ and beyond, will be reached in the same order for both infections. Coupling $\{\theta_{\kappa_j}\}_{j\geq 1}$ and $\{\theta'_{\kappa_j}\}_{j\geq 1}$ according to Lemma 6.4 we will have with probability one that, from some level on, both infections will reach each vertex at the same time, i.e.,

$$T(v_n) = T'(v_n) \tag{6.9}$$

for any $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for n sufficiently large. Since we chosen t^* as large as we did, we made sure that $\mathrm{P}_\tau(\,\cdot\,|\tau \leq t^*)$ meets assumption b) of Lemma 6.4.

The infections may in the same manner be coupled along the negative coordinate axis. Doing this, then there is $N_c \in \mathbb{N}$ such that (6.7) holds for $|n| \geq N_c$. In almost surely finite time, each vertex at level n, for $|n| \leq N_c$, will be infected. Hence, we conclude that for some almost surely finite time T_c,

$$B_t = B'_t, \quad \text{for each } t \geq T_c. \qquad \square$$

Remark 6.6. There exists in general no exact coupling of two infections with discrete passage time distribution on arbitrary 1-dimensional periodic graphs. Consider the distribution $\mathrm{P}_\tau(1) = \mathrm{P}_\tau(1 + 3/5) = 1/2$. P_τ satisfies the assumption of Proposition 6.2, whence there is an exact coupling of two infections on the (K, d)-tube, for $K, d \geq 2$.

Consider instead the graph with set of vertices $\mathbb{Z} \times \{0, 1\}$ and where two vertices are connected by an edge if their Euclidean distance is $\leq \sqrt{2}$. Note that with the above passage time distribution, in order to reach any vertex at level n, an infection will always pass exactly n edges. This is easily seen by realising that no vertical edge will ever be used in order to reach an uninfected vertex. Thus, for two infections started with $I = \{(0, 0)\}$ and $I' = \{(m, 0)\}$, we will have

$$\begin{aligned} \big|T(\mathbf{n}) - T'(\mathbf{n})\big| \;&\geq \inf_{\substack{a+b=n \\ a'+b'=n-m}} \left| a - a' + (b - b')\left(1 + \frac{3}{5}\right)\right| \\ &= \inf_{b-b' \in \mathbb{Z}} \left| m - \frac{3(b - b')}{5}\right| \;\geq\; \frac{1}{5}, \end{aligned}$$

for any m that is not a multiple of 3. As we can see, an exact coupling is not possible. $\square$

Remark 6.7. Condition a) of Proposition 6.2 is due to the fact that the (K, d)-tube is bipartite, i.e., that every circuit has even length. As seen in Remark 6.6, not every non-bipartite graph has an exact coupling without condition a). But, condition a) and b) of Proposition 6.2 could be dropped for e.g. the class of triangular graphs with vertex set $\mathbb{Z} \times \{0, 1, \ldots, K-1\}$ and where two vertices at Euclidean distance is 1 and every two vertices (n, m) and $(n+1, m+1)$ for any $n \in \mathbb{Z}$ and $m = 0, 1, \ldots, K-2$, are connected by an edge. The necessary modifications of the first part of the proof, and of the event D_n in particular, are easily made. $\square$

Remark 6.8. If $\mathrm{dist}(\mathbf{x}, \mathbf{y})$ is odd, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$, then condition a) of Proposition 6.2 is necessary. To see this, assume that an exact coupling is possible. In particular, $T(v) = T'(v)$ for some vertex v. But, if one infection has an even number of edges to pass in order to reach v, the other has an odd number of edges to pass. Thus,

$$0 \;=\; T(v) - T'(v) \;=\; \sum_{j \in J} n_j t_j - \sum_{j \in J} n'_j t_j,$$

for integers n_j and n'_j such that $\sum_{j \in J}(n_j - n'_j)$ is odd. Hence, condition a) holds. $\square$

Remark 6.9. Condition a) of Lemma 6.4 can be weakened to distributions P_τ whose convolution with itself has an absolutely continuous component. In fact, it is sufficient if P_τ convoluted with itself n times, for some $n \geq 0$, has an absolutely continuous component. Since the distribution of a sum of independent random variables is the convolution of the individual distributions, we may instead of specifying how to choose (τ_j, τ'_j) for $j \geq 1$, choose $\big(\sum_{k=(j-1)n+1}^{jn} \tau_k, \sum_{k=(j-1)n+1}^{jn} \tau'_k\big)$ according to the same specification. Consequently, the assumption on P_τ of Proposition 6.1 can be weakened to involve distributions whose convolution with itself n times has an absolutely continuous component. The modifications are left to the reader.

An example of a distribution that does not have an absolutely continuous component, but whose convolution does, is given by the following. Let $\xi_0, \xi_1, \ldots$ be i.i.d. Bernoulli(1/2)-distributed random variables. Define τ to have binary expansion

$$\tau := \begin{cases} (0, \xi_1, 0, \xi_3, 0, \ldots), & \text{with probability } \dfrac{1}{2} \\ (\xi_0, 0, \xi_2, 0, \xi_4, \ldots), & \text{otherwise.} \end{cases}$$

Let τ_1 and τ_2 be two independent random variables distributed as τ, and let A denote the event that one of τ_1 and τ_2 has all even coordinates equal to zero and the other has all odd coordinates equal to zero. Neither τ_1 nor τ_2

is absolutely continuous, but the conditional distribution of $\tau_1 + \tau_2$ given A is uniformly distributed on $[0,1]$. Hence the distribution of $\tau_1 + \tau_2$ has an absolutely continuous component. $\square$

6.3 No exact coupling possible on trees

We have seen that there is an exact coupling of two first-passage percolation infections on any essentially 1-dimensional periodic graph when the passage time distribution has an absolutely continuous component. We also saw how this sort of coupling gave rise to a 0–1 law. One may ask whether a continuous component is sufficient for an analogous coupling, and corresponding 0–1 law, on any graph? We will answer this question no, by showing that the binary tree $\mathbb{T}^2$ constitutes a counterexample. $\mathbb{T}^2$ is the infinite graph that does not contain any circuit, and where each vertex has three neighbours. The graph is completely homogeneous and one vertex, called the *root*, is chosen for reference. Let $\{\tau_e\}_{e\in\mathbb{E}}$ be a set of independent and exponentially distributed passage times associated with the edge set $\mathbb{E}$ of $\mathbb{T}^2$, and analogous to before, let

$$B_t = \big\{v \in \mathbb{V} : T\,(\mathrm{root}, v) \leq t\big\}.$$

The following argument is based on the theory of continuous branching processes. Define the front line of the infection at time t as

$$F_t := \#\{v \notin B_t : v \text{ shares an edge with some } u \in B_t\}.$$

Note that $F_0 = 3$ and that F_t increases by one, when B_t does. Hence, F_t can be seen as a continuous time branching process with F_t individuals at time t. Each individual gives with probability one birth to two children (and dies) after an exponentially distributed time, independent of one another. It is well-known (see e.g. Athreya and Ney (1972, Theorems III.7.1–2)) that, for some Malthusian parameter $\lambda > 0$,

$$\exists W := \lim_{t\to\infty} F_t e^{-\lambda t}, \quad \text{almost surely,} \tag{6.10}$$

and that $\mathrm{E}[W] = 3$. Let τ_{e_1}, τ_{e_2} and τ_{e_3} denote the passage time of the edges connected to the root, and let $\tilde{F}_t$ denote F_t conditioned on $\{\tau_{e_1}, \tau_{e_2}, \tau_{e_3} \geq 1\}$. Then, by the lack-of-memory property of the exponential distribution, we have that $\tilde{F}_{t+1} \overset{d}{=} F_t$ for any $t \geq 0$. Thus, by (6.10) we have almost surely

$$\lim_{t\to\infty} \tilde{F}_t e^{-\lambda t} \;\overset{d}{=}\; e^{-\lambda} \lim_{t\to\infty} F_t e^{-\lambda t} \;=\; e^{-\lambda} W,$$

and we conclude that W is almost surely non-constant. Note that the event

$$\big\{W = \lim_{t\to\infty} F_t e^{-\lambda t} \leq x\big\} \in \mathcal{T}, \quad \text{for every } x.$$

Then, a 0–1 law analogous to Theorem 6.3 cannot hold for first-passage percolation on $\mathbb{T}^2$, since this would imply that $\mathrm{P}(W \leq x) \in \{0, 1\}$, i.e., that W is almost surely constant.

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Paper II

The asymptotic shape, large deviations and dynamical stability in first-passage percolation on cones

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Abstract

In this paper we consider first-passage percolation on certain subgraphs of the $\mathbb{Z}^d$ nearest neighbour graph. We present a three-fold extension of the Shape Theorem. Firstly, we show that the convergence holds not only almost surely and in L^1, but also completely. For this, we deduce certain large deviation estimates for first-passage times under the assumption of finite power moment on the passage time distribution. Secondly, we prove that there are no exceptional times at which the almost sure convergence fails, when edges update their values according to independent Poisson clocks. With respect to the mentioned dynamics, this provides a dynamically stable version of the Shape Theorem. Finally, we prove that all of the above extends to cone-like subgraphs of the lattice, for which their associated asymptotic shapes can be expressed in terms of the asymptotic shape of the lattice.

1 Introduction

First-passage percolation was first considered by Hammersley and Welsh (1965). It can be thought of as a model for the spread of an infection on an underlying graph $\mathcal{G}$ with set of vertices $\mathcal{V}_\mathcal{G}$ and set of edges $\mathcal{E}_\mathcal{G}$. Associate to the edges of the graph non-negative i.i.d. random variables $\{\tau_e\}_{e\in\mathcal{E}_\mathcal{G}}$, referred to as *passage times*. To avoid trivialities, we assume throughout this paper that the passage-time distribution does not concentrate all mass at a single point. With the present interpretation of the model, the passage time of an edge should be thought of as the random time it takes for an infection to spread along that edge.

One of the main achievement in first-passage percolation is known as the *Shape Theorem* (cf. Theorem 1.1), and describes the almost sure rate of the growth of an infection started at the origin on the $\mathbb{Z}^d$ nearest neighbour graph, in all directions simultaneously. The $\mathbb{Z}^d$ nearest neighbour graph is also commonly referred to as the $\mathbb{Z}^d$ *lattice*, and is the graph whose vertices are the points in $\mathbb{Z}^d$ and where every two vertices at Euclidean distance one are joined by an edge. We assume throughout that $d \geq 2$. The two main theorems of this paper together give a three-fold extension of the Shape Theorem. The first main result, Theorem 1.2, says that a statement analogous to the Shape Theorem holds for certain cone-like subgraphs of the $\mathbb{Z}^d$ lattice. Moreover, it states that the convergence holds almost surely, in L^1 and completely. The second main result, Theorem 1.6, is a dynamically stable version of the Shape Theorem. For this we will introduce a dynamical version of first-passage percolation, in which edges update their values according to i.i.d. Poisson clocks. As will be emphasized later on, the time dimension in which this takes place should not be confused with the interpretation given to the random values assigned to the edges as 'times'.

Let us by a *path* refer to an alternating sequence of vertices and edges; $v_0, e_1, v_1, \ldots, e_n, v_n$, beginning and ending with a vertex, such that v_k is the endpoint of the edges e_k and e_{k+1} that precedes and follows v_k. The vertices v_0 and v_n are referred to as endpoints of the path. A path with endpoints u and v will be referred to as a path from u to v. We will repeatedly abuse notation and identify a path with its set of edges. For a path Γ, we define the passage time of Γ as $T(\Gamma) := \sum_{e\in\Gamma} \tau_e$, and define the *passage time*, or *first-passage time*, between two vertices $u, v \in \mathcal{V}_\mathcal{G}$ as

$$T(u,v) := \inf\big\{T(\Gamma) : \Gamma \text{ is a path from } u \text{ to } v\big\}.$$

Given a vertex $\mathbf{0} \in \mathcal{V}_\mathcal{G}$, referred to as the origin, we define the set of vertices reachable within time t as

$$\mathcal{W}_t := \big\{v \in \mathcal{V}_\mathcal{G} : T(\mathbf{0}, v) \leq t\big\}.$$

Interpreting passage times as times it takes an infection to traverse the corresponding edges, then the first-passage time $T(u,v)$ should be thought of as the time it takes an infection started at u to reach v. Starting with a single infected vertex at the origin, $\mathcal{W}_t$ is interpreted as the spatial propagation, i.e., the set of infected vertices, at time t.

When the underlying discrete structure is given by the $\mathbb{Z}^d$ lattice, we will for practical reasons extend the definition of $T(\mathbf{x},\mathbf{y})$ from $\mathbb{Z}^d$ to $\mathbb{R}^d$. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$ we assigning $T(\mathbf{x},\mathbf{y})$ the passage time $T(\mathbf{x}^*,\mathbf{y}^*)$, where $\mathbf{x}^*$ and $\mathbf{y}^*$ denotes the two points in $\mathbb{Z}^d$ closest to $\mathbf{x}$ and $\mathbf{y}$, respectively (choosing the point closest to the origin in case of a tie, say). The definition of $\mathcal{W}_t$ extends consequently to let $\mathcal{W}_t$ contain all points $\mathbf{x}\in\mathbb{R}^d$ such that $T(\mathbf{0},\mathbf{x})\leq t$.

1.1 The Shape Theorem

An important breakthrough was achieved by Kingman (1968) and his Subadditive Ergodic Theorem. The theorem implies that for any $\hat{\mathbf{z}}$ of the form $\hat{\mathbf{z}}=\mathbf{z}/|\mathbf{z}|$, for some $\mathbf{z}\in\mathbb{Z}^d$,

$$\exists\,\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}):=\lim_{n\to\infty}\frac{T(\mathbf{0},n\hat{\mathbf{z}})}{n},\quad\text{almost surely and in } L^1, \tag{1.1}$$

under the assumption that $\mathbb{E}[Y]<\infty$, where

$$Y=\min(\tau_1,\tau_2,\ldots,\tau_{2d}), \tag{1.2}$$

and $\tau_1,\tau_2,\ldots,\tau_{2d}$ are independent and distributed as τ_e. The limit $\mu_{\mathbb{Z}^d}(\cdot)$ that figures in (1.1) is referred to as the *time constant*. Given the radial convergence in (1.1), a fair amount of additional work provides the asymptotic growth in all directions simultaneously. In particular, if $\mathbb{E}\left[Y^d\right]<\infty$, then

$$\limsup_{\mathbf{z}\in\mathbb{Z}^d:\,|\mathbf{z}|\to\infty}\left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|}-\mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right|=0,\quad\text{almost surely.} \tag{1.3}$$

Alternatively, we can present (1.3) in terms of how closely $t^{-1}\mathcal{W}_t$ resembles the set

$$\mathcal{W}^*:=\left\{\mathbf{x}\in\mathbb{R}^d:|\mathbf{x}|\leq\mu_{\mathbb{Z}^d}(\mathbf{x}/|\mathbf{x}|)^{-1}\right\}.$$

(As will be seen in Section 2, $\mu_{\mathbb{Z}^d}(\cdot)$ extends continuously to all unit vectors in $\mathbb{S}^{d-1}$.) Stated in terms of spatial propagation, the result is known as the Shape Theorem and due to Cox and Durrett (1981), inspired by a result of Richardson (1973).

Theorem 1.1 (Shape Theorem)**.** *Consider first-passage percolation on $\mathbb{Z}^d$ with i.i.d. passage times such that $\mathbb{E}\left[Y^d\right]<\infty$, for Y defined as in (1.2). If $\mu_{\mathbb{Z}^d}(\mathbf{e}_1)>0$, then, for all $\epsilon>0$, almost surely,*

$$(1-\epsilon)\mathcal{W}^*\subset\frac{1}{t}\mathcal{W}_t\subset(1+\epsilon)\mathcal{W}^*,\quad\textit{for } t \textit{ large enough.}$$

If $\mu_{\mathbb{Z}^d}(\mathbf{e}_1)=0$, then for every compact set K in $\mathbb{R}^d$, almost surely,

$$K\subset\frac{1}{t}\mathcal{W}_t,\quad\textit{for } t \textit{ large enough.}$$

Theorem 1.1 can be seen to be equivalent to (1.3) via an inversion argument. For the purpose of this paper, it will be more convenient to consider limits of the form in (1.3). It was shown by Kesten (1986) that

$$\mu_{\mathbb{Z}^d}(\mathbf{e}_1) = 0 \text{ if and only if } \mathbb{P}(\tau_e = 0) \geq p_c(d),$$

where $p_c(d)$ denotes the critical probability for independent bond percolation on the $\mathbb{Z}^d$ lattice. Moreover, it is known that if $\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) = 0$ for some $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$, then it does for all. In the regime $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$, the shape $\mathcal{W}^*$ can be seen to be compact, convex and to have non-empty interior.

1.2 A shape theorem for subgraphs

The Shape Theorem gives a Law of Large Numbers for first-passage percolation on the $\mathbb{Z}^d$ lattice. In this paper we will show that similar limit result can be achieved for certain subgraphs of the $\mathbb{Z}^d$ lattice. A subgraphs $\mathcal{G}$ of the $\mathbb{Z}^d$ lattice will be called *induced* if any two vertices in $\mathcal{G}$ are connected by an edge if and only if the same thing holds in the $\mathbb{Z}^d$ lattice. An induced subgraph is uniquely determined by its set of vertices. Subsets of $\mathbb{Z}^d$ is in turn uniquely determined by a subsets of $\mathbb{R}^d$. Thus, we say that $\mathcal{G}$ is the subgraph of the $\mathbb{Z}^d$ lattice *induced by* $V \in \mathbb{R}^d$, if $\mathcal{G}$ is an induced subgraph of the $\mathbb{Z}^d$ lattice and $\mathcal{V}_\mathcal{G} = V \cap \mathbb{Z}^d$.

Let $B(\mathbf{y}, r) = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{y}| \leq r\}$ denote the closed Euclidean ball. We will in this paper focus on subgraphs $\mathcal{G}$ of the $\mathbb{Z}^d$ lattice induced by sets of the form $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a)\big)$, for some $\hat{\mathbf{x}} \in \mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ and $\omega : [0,\infty) \to [0,\infty)$. Note that when $\omega(a) = r \cdot a$ for some $r \in (0,1)$, then $\mathcal{G}$ is simply a d-dimensional cone. One of the main results of this paper is the following extension of (1.3) (and hence the Shape Theorem). The constant R_d that figures in the statement of the theorem will be explained and quantified in Lemma 2.4.

Theorem 1.2. *For any $d \geq 2$ there exists a universal constant R_d such that the following holds. Let $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and let $\omega : [0,\infty) \to [0,\infty)$ be a convex or concave function such that $\omega(a) \to \infty$, as $a \to \infty$. Consider first-passage percolation on the subgraph $\mathcal{G}$ of the $\mathbb{Z}^d$ lattice induced by $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a) + R_d\big)$.*

a) If $\mathbb{E}[\tau_e] < \infty$, then
$$\limsup_{\mathbf{z}\in\mathcal{V}_\mathcal{G}:\,|\mathbf{z}|\to\infty} \mathbb{E}\left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right| = 0.$$

b) If $\mathbb{E}[Y^d] < \infty$, then
$$\limsup_{\mathbf{z}\in\mathcal{V}_\mathcal{G}:\,|\mathbf{z}|\to\infty} \left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right| = 0,$$

almost surely.

c) If $\mathbb{E}[Y^{d+1}] < \infty$, *then* $\quad \sum_{\mathbf{z}\in\mathcal{V}_\mathcal{G}} \mathbb{P}\left(\left|\frac{T(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)\right| > \epsilon\right) < \infty,$

for all $\epsilon > 0$.

We continue with a series of remarks on the statement of Theorem 1.2. As mentioned above, $\omega(a) = r \cdot a$ gives rise to a cone in the classical sense when $r \in (0,1)$. When $r = 1$, $\mathcal{G}$ is the subgraph induced by the half-space $\{\mathbf{z} \in \mathbb{Z}^d : \langle \mathbf{z}, \hat{\mathbf{x}}\rangle \geq -R_d\}$, where $\langle\cdot,\cdot\rangle$ denotes inner product, and for $r > 1$, $\mathcal{G}$ equals the $\mathbb{Z}^d$ lattice. Hence, part *b)* of Theorem 1.2 extends (1.3).

We will prove part *b)* and *c)* of Theorem 1.2 under the stronger condition that $\mathbb{E}[\tau_e^2] < \infty$. This will save us from a few additional technicalities, which can be found in the paper of Cox and Durrett (1981). The necessary steps for the proof to go through without the stronger condition are indicated in Remark 6.4.

The condition $\mathbb{E}[\tau_e] < \infty$ for the L^1-convergence in part *a)* to hold can not be relaxed in general. However, it is not hard to see that if the limsup is taken over $\mathbf{z} \in \mathcal{V}_\mathcal{G} \cap \bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a)\big)$ instead of over $\mathbf{z} \in \mathcal{V}_\mathcal{G}$, then $\mathbb{E}[Y] < \infty$ is sufficient. In particular, $\mathbb{E}[Y] < \infty$ is sufficient when $\mathcal{G}$ is the $\mathbb{Z}^d$ lattice.

The case when $\hat{\mathbf{x}}$ is of the form $\mathbf{z}/|\mathbf{z}|$, for some $\mathbf{z} \in \mathbb{Z}^d$, and $\omega(a) \equiv K - R_d$ was treated in Ahlberg (2010). Also in this case the limit exists and satisfies $\mu_K(\hat{\mathbf{x}}) > \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})$ for all $K \geq R_d$. Indeed $\mu_K(\hat{\mathbf{x}}) \to \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})$ as $K \to \infty$ (cf. Proposition 2.9), which explains the presence of $\mu_{\mathbb{Z}^d}(\cdot)$ in Theorem 1.2.

For $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and $r \in (0,1)$, let $\mathcal{C}(\hat{\mathbf{x}}, r) := \bigcup_{a\geq 0} B(a\hat{\mathbf{x}}, r\cdot a)$. The cone $\mathcal{C}(\hat{\mathbf{x}}, r)$ is with elementary trigonometry found to have radius equal to $\frac{r}{\sqrt{1-r^2}}\cdot a$ at distance a from its tip. Consequently, an alternative way to generate $\mathcal{C}(\hat{\mathbf{x}}, r)$ is as the volume obtained when the function $g(a) = \frac{r}{\sqrt{1-r^2}}\cdot a$, for $a \geq 0$, is rotated around the axis $\{a\hat{\mathbf{x}}\}_{a\in\mathbb{R}}$. The other way around, the volume obtained when the function $g(a) = r\cdot a$, for $a \geq 0$ and some $r > 0$, is rotated around the axis $\{a\hat{\mathbf{x}}\}_{a\in\mathbb{R}}$ equals $\mathcal{C}\big(\hat{\mathbf{x}}, \frac{r}{\sqrt{1+r^2}}\big)$. In fact, for any non-decreasing convex function $g : [0,\infty) \to [0,\infty)$, there is a convex function $\omega : [0,\infty) \to [0,\infty)$ such that the rotation volume obtained when g is rotated around the axis $\{a\hat{\mathbf{x}}\}_{a\in\mathbb{R}}$ equals $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a)\big)$.

The convergence in part *c)* of Theorem 1.2 is also known as *complete convergence*, a concept first introduced by Hsu and Robbins (1947). More generally, a random sequence $\{X_n\}_{n\geq 1}$ is said to *converge completely* to X as $n \to \infty$ if

$$\sum_{n=1}^{\infty} \mathbb{P}\big(|X_n - X| > \epsilon\big) < \infty, \quad \text{for all } \epsilon > 0.$$

Observe that complete convergence of a random sequence implies almost sure convergence, via Borel-Cantelli's lemma. For Theorem 1.2 this implies that part *b)* is included in part *c)*.

Finally, the following observation is made in connection to Theorem 1.2. Recall that the sub-critical regime of bond percolation on the $\mathbb{Z}^d$ lattice coincides with $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$, that is $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$ if and only if $\mathbb{P}(\tau_e = 0) < p_c(d)$. In addition, $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$ coincides with linear growth of the spatial propagation of the infection, via the Shape Theorem. This is not true for all graphs. Consider the subgraph of the $\mathbb{Z}^2$ lattice induced by $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq a \log(1 + x)\}$, for some $a \in \mathbb{R}_+$. Grimmett (1983) proved that for each $a \in \mathbb{R}_+$ the critical probability for bond percolation on this graph lies strictly between 1/2 and 1. Although this graph is not treated by Theorem 1.2, it is by similar means easy to see that $\lim_{n\to\infty} T(\mathbf{0}, n\mathbf{e}_1)/n = \mu_{\mathbb{Z}^d}(\mathbf{e}_1)$, almost surely, also for this graphs. In particular, we conclude that the sub-critical regime of bond percolation does not coincide with the regime in which $\lim_{n\to\infty} T(\mathbf{0}, n\mathbf{e}_1)/n$ is almost surely positive.

1.3 Large deviations

Hsu and Robbins (1947) proved that the sequence of arithmetic averages of i.i.d. random variables converges completely to its common mean, given that their variance is finite. That this is also necessary was proved by Erdős. We will from an extension of their results (cf. Theorem 3.1) deduce the following result which we will need to prove the large deviations estimate in part *c)* of Theorem 1.2.

Proposition 1.3. *Let $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and $\alpha \geq 1$. Consider first-passage percolation on the $\mathbb{Z}^d$ lattice with $\mathbb{E}[Y^\alpha] < \infty$. Then,*

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{P}\Big(T(\mathbf{0}, n\hat{\mathbf{x}}) > \big(\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) + \epsilon\big)n\Big) < \infty, \quad \text{for any } \epsilon > 0.$$

This result is a kind of large deviations estimate for first-passage times above the time constant, and as we will see (cf. Proposition 4.1), it also holds for subgraphs of the $\mathbb{Z}^d$ lattice. We will also prove a large deviations estimate below the time constant, in which case we have exponential decay.

Proposition 1.4. *Consider first-passage percolation on the $\mathbb{Z}^d$ lattice. For any $\epsilon > 0$, there are $\alpha = \alpha(\epsilon) < \infty$ and $\beta = \beta(\epsilon) > 0$ such that*

$$\mathbb{P}\Big(T(\mathbf{0}, n\hat{\mathbf{x}}) < (\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) - \epsilon)n\Big) \leq \alpha e^{-\beta n}, \quad \text{for all } n \geq 0 \text{ and } \hat{\mathbf{x}} \in \mathbb{S}^{d-1}.$$

Proposition 1.4 extends a result by Grimmett and Kesten (1984) ($d = 2$) and Kesten (1986) ($d \geq 2$). They treat the case when $\hat{\mathbf{x}} = \mathbf{e}_1$. The convergence obtained in Proposition 1.3 does not seem to be previously known under the given hypothesis. However, Grimmett and Kesten also show that $\mathbb{P}\big(T(\mathbf{0}, n\mathbf{e}_1) > (\mu_{\mathbb{Z}^d}(\mathbf{e}_1) + \epsilon)n\big)$ decays at least exponentially, but under the stronger assumption that $\mathbb{E}[e^{\gamma\tau_e}] < \infty$ for some $\gamma > 0$. First-passage times as defined above are sometimes referred to as point-to-point passage times. Related results have also been obtained by Chow and Zhang (2003) for so-called face-to-face passage times, and by Garet and Marchand (2007) for the chemical distance in bond percolation clusters. We should mention that it does not follow from the proof we present whether $\mathbb{E}[Y^\alpha] < \infty$ is a necessary condition for the conclusion of Proposition 1.3 or not.

In fact, it is possible to prove something stronger that Proposition 1.4. We will introduce what could be referred to as *point-to-shape* passage times, which we define as

$$T_{0,n}^{\mathcal{W}} := \inf\left\{T(\mathbf{0}, \mathbf{z}) : \ \mathbf{z} \in \mathbb{Z}^d \text{ with } |\mathbf{z}| \geq \frac{n}{\mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)}\right\}.$$

This definition only makes sense when $\mu_{\mathbb{Z}^d}(\cdot) > 0$, which is known to hold when $\mathbb{P}(\tau_e = 0) < p_c(d)$, where $p_c(d)$ denotes the critical probability in bond percolation on the $\mathbb{Z}^d$ lattice. For these cases, $T_{0,n}^{\mathcal{W}}$ equals the time it takes to reach a vertex at the boundary of, or outside the blow-up $n\mathcal{W}^*$ of the shape $\mathcal{W}^*$. In the light of the Shape Theorem, it natural that (see Section 3)

$$\lim_{n\to\infty} \frac{T_{0,n}^{\mathcal{W}}}{n} = 1, \quad \text{almost surely.} \tag{1.4}$$

In fact, the following is true, from which Proposition 1.4 comes as an easy corollary. Our proof of this result is heavily influenced by the proof presented in Kesten (1986) for the case $\hat{\mathbf{x}} = \mathbf{e}_1$ of Proposition 1.4.

Proposition 1.5. *For any $\epsilon > 0$ there exists $\alpha = \alpha(\epsilon) < \infty$ and $\beta = \beta(\epsilon) > 0$ such that*

$$\mathbb{P}\Big(T_{0,n}^{\mathcal{W}} < (1-\epsilon)n\Big) \leq \alpha e^{-\beta n}, \quad \textit{for all } n \geq 0.$$

1.4 Dynamical first-passage percolation

We will also consider a dynamical version of first-passage percolation. This is inspired by so-called dynamical percolation introduced by Häggström, Peres, and Steif (1997). From bond percolation, dynamical (bond) percolation is obtained by allowing the edges of a graph to flip between 'open' and 'closed' according to i.i.d. Poisson clocks. At each fixed time, an infinite open component exists with probability either 0 or 1 (depending on the probability of an edge being open). Is this property dynamically stable in the sense that for almost every realization we will see an infinite open component either present or absent at all times? This question was first studied in Häggström et al. (1997), and continued by Benjamini, Kalai, and Schramm (1999), Schramm and Steif (2010) and Garban, Pete, and Schramm (2010). It is not hard to conclude that away from criticality, the answer is 'yes'. However, at criticality, the $\mathbb{Z}^2$ lattice exhibits exceptional times at which an infinite open component exists, as proved by Garban et al. (2010). In a related work, Benjamini, Häggström, Peres, and Steif (2003) consider similar questions in the context of i.i.d. sequences.

Analogously, we obtain a dynamical version of first-passage percolation by assigning i.i.d. passage times, together with independent Poisson clocks, to the edges of a graph $\mathcal{G}$. When a clock rings, the passage time of the corresponding edge is re-sampled from the same distribution. More formally, associate independently to each edge e of the underlying graph a random process $\{\tau_e(s)\}_{s\geq 0}$ defined as follows. Given non-negative i.i.d. random variables $\{\tau_e^{(j)}\}_{e\in\mathcal{E}_\mathcal{G},j\geq 1}$ and i.i.d. rate 1 exponentially distributed random variables $\{\xi_e^{(j)}\}_{e\in\mathcal{E}_\mathcal{G},j\geq 1}$, let $\xi_e^{(0)}=0$ for each e, and define for all $e\in\mathcal{E}_\mathcal{G}$ and $s\geq 0$

$$\tau_e(s) := \tau_e^{(j)}, \quad \text{for } \sum_{k=0}^{j-1}\xi_e^{(k)} \leq s < \sum_{k=0}^{j}\xi_e^{(k)}.$$

By construction, the processes $\{\tau_e(s)\}_{s\geq 0}$ are independent for different edges, and $\{\tau_e(s)\}_{s\geq 0}$ will sometimes be referred to as the *dynamical passage time* of the edge e. If $\tau_e^{(j)}$ has probability measure ν, then for any $s\geq 0$, the distribution of $\{\tau_e(s)\}_{e\in\mathcal{E}_\mathcal{G}}$ is given by the product measure $\nu^{\mathcal{E}_\mathcal{G}}$. Let for each $s\geq 0$, $T^{(s)}(u,v)$ denote the passage time between the points u and v with respect to $\{\tau_e(s)\}_{e\in\mathcal{E}_\mathcal{G}}$. It follows that also

$$T^{(0)}(u,v) \overset{d}{=} T^{(s)}(u,v), \quad \text{for all } s\geq 0.$$

It is natural to think of dynamical passage times as evolving with time. This gives us two time dimensions. To better picture the dynamical setting, it may help not to think of $\tau_e(s)$ as the *time*, but as the *cost* related to crossing e at time s. Then $T^{(s)}(u,v)$ is interpreted as the minimal cost to travel between u and v at time s, and $\{T^{(s)}(u,v)\}_{s\geq 0}$ as the evolution over time of the cost for the passage between u and v.

On the $\mathbb{Z}^d$ lattice (1.1) states that if $\mathbb{E}[Y]<\infty$, then

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{T^{(s)}(\mathbf{0},n\hat{\mathbf{z}})}{n} = \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\right) = 1, \quad \text{for each } s\geq 0,$$

and, (1.3) that if $\mathbb{E}[Y^d] < \infty$, then we have the stronger

$$\mathbb{P}\left(\limsup_{\mathbf{z}\in\mathbb{Z}^d\colon |\mathbf{z}|\to\infty} \left| \frac{T^{(s)}(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|) \right| = 0 \right) = 1, \quad \text{for each } s \geq 0.$$

An application of Fubini's theorem strengthens this to

$$\mathbb{P}\left(\limsup_{\mathbf{z}\in\mathbb{Z}^d\colon |\mathbf{z}|\to\infty} \left| \frac{T^{(s)}(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|) \right| = 0 \text{ for Lebesgue-a.e. } s \geq 0 \right) = 1.$$

Our next result strengthened this to hold for *every* $s \geq 0$. The Shape Theorem is so to say *dynamically stable* with respect to the dynamics introduced. The same is true for the almost sure convergence in part *b)* of Theorem 1.2.

Theorem 1.6. *For any $d \geq 2$ there exists a universal constant R_d such that the following holds. Let $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and let $\omega : [0,\infty) \to [0,\infty)$ be a convex or concave function such that $\omega(a) \to \infty$, as $a \to \infty$. Consider first-passage percolation on the subgraph $\mathcal{G}$ of the $\mathbb{Z}^d$ lattice induced by $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a) + R_d\big)$. If $\mathbb{E}[Y^d] < \infty$, then*

$$\mathbb{P}\left(\limsup_{\mathbf{z}\in\mathcal{V}_{\mathcal{G}}\colon |\mathbf{z}|\to\infty} \left| \frac{T^{(s)}(\mathbf{0},\mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|) \right| = 0 \textit{ for all } s \geq 0 \right) = 1.$$

Like part *b)* and *c)* of Theorem 1.2, we will prove Theorem 1.6 under the somewhat stronger assumption that $\mathbb{E}[Y^d] < \infty$.

1.5 Outline of proof and paper

The two main results of this paper, Theorem 1.2 and 1.6, together extends the Shape Theorem in three different directions. The three extensions are consequences of the following tasks:

(i) Prove that the convergence in (1.3) holds almost surely, in L^1 and completely.

(ii) Prove that the almost sure convergence in (1.3) is dynamically stable.

(iii) Prove that the convergence in (i) and (ii) also holds on the cone-like subgraphs we consider.

We will prove task (i)-(iii) step by step. We recall the reader that the almost sure convergence in (1.3) is equivalent to the Shape Theorem. The L^1-convergence is normally not emphasized, but can be deduced in the same way. The novelty in task (i) is the complete convergence.

The accomplishment of the three tasks will proceed along the following lines.

Section 3 Obtain the large deviation estimate in radial directions given by Proposition 1.3 and 1.4. This is sufficient, together with existing proofs of the Shape Theorem, to obtain complete convergence in (1.3) under the condition that $\mathbb{E}[Y^{d+1}] < \infty$. This would complete the first of the three tasks above.

Section 4 Show that the radial almost sure and L^1-convergence in (1.1) extends to cone-like subgraphs of the lattice. In the same way we extend the large deviations estimate obtained in Section 3. The results are easily obtained for rational directions, i.e., $\hat{\mathbf{x}} \in \mathbb{U}^{d-1}$, whereas the general case, $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$, is harder. With this we take the first step towards an analogue of (1.3) for the cone-like graphs in consideration.

Section 5 Here we prove that the almost sure convergence in radial directions, which is given in (1.1) and extended in Section 4, is dynamically stable. A dynamically stable version of (1.1) is sufficient, again together with existing proofs of the Shape Theorem, to obtain a dynamically stable version of (1.3) under the condition that $\mathbb{E}[Y^d] < \infty$. This completes the second task.

Section 6 Finally, we complete the proof of Theorem 1.2 and 1.6. In addition to the results obtained in Section 4 and 5, the missing piece is a geometric argument needed to prove Lemma 6.2. The assumption $\mathbb{E}[\tau_e^2] < \infty$ is used in the proof of Lemma 6.2, and only there. That assumption is made to avoid additional technicalities. Under a even stronger moment assumption, also the geometric argument is redundant. Together with the work carried out in Section 4, the third task is accomplished.

First of all, we dedicate Section 2 to recall some additional facts about first-passage percolation, as well as introducing some notation, that will recur in the rest of the paper.

2 Preliminaries

Consider first-passage percolation on the $\mathbb{Z}^d$ lattice. For the purpose of this paper it will suffice to consider the case when $\mathbb{E}[Y] < \infty$, where Y is as defined in (1.2). That $\mathbb{E}[Y] < \infty$ is sufficient to obtain $\mathbb{E}\big[T(\mathbf{0},\mathbf{z})\big] < \infty$ for any $\mathbf{z} \in \mathbb{Z}^d$ is given by Proposition 2.5 below. An very central concept in first-passage percolation is the *subadditive property*, that is, that

$$T(\mathbf{x},\mathbf{y}) \leq T(\mathbf{x},\mathbf{z}) + T(\mathbf{z},\mathbf{y}), \quad \text{for any } \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^d.$$

The property is immediate from the definition of first-passage times, and holds for first-passage percolation on any graph. Let $\mathbb{U}^{d-1} \subseteq \mathbb{S}^{d-1}$ denote the set of $\hat{\mathbf{z}} \in \mathbb{S}^{d-1}$ such that $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$ for some $\mathbf{z} \in \mathbb{Z}^d$. In (1.1) the time constant $\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$ was defined for $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$ as the almost sure limit of $T(\mathbf{0}, n\hat{\mathbf{z}})/n$ as $n \to \infty$. Existence of the limit is a consequence of the Subadditive Ergodic Theorem, and crucially based on the subadditive property. Alternatively, $\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$ equals $\lim_{n\to\infty} \mathbb{E}\big[T(\mathbf{0}, n\hat{\mathbf{z}})\big]/n = \inf_{n\geq 1} \mathbb{E}\big[T(\mathbf{0}, n\mathbf{z})\big]/n$. This is a more elementary consequence of the convergence of real-valued subadditive sequences, realized already by Fekete (1923). Existence of the limit is easily extended to $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ as follows. By subadditivity and lattice symmetry,

$$
\begin{aligned}
\left| \frac{\mathbb{E}\big[T(\mathbf{0}, n\hat{\mathbf{x}})\big]}{n} - \frac{\mathbb{E}\big[T(\mathbf{0}, n\hat{\mathbf{y}})\big]}{n} \right| &\leq \left| \frac{\mathbb{E}\big[T(n\hat{\mathbf{x}}, n\hat{\mathbf{y}})\big]}{n} \right| \\
&\leq \frac{\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big] \|n\hat{\mathbf{x}} - n\hat{\mathbf{y}}\|_1}{n} \\
&\leq d\, \mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big] |\hat{\mathbf{x}} - \hat{\mathbf{y}}|.
\end{aligned} \tag{2.1}
$$

Now, assume that $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and $\hat{\mathbf{y}} \in \mathbb{U}^{d-1}$, let $n \to \infty$, and conclude that

$$
\exists\, \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) := \lim_{n\to\infty} \frac{\mathbb{E}\big[T(\mathbf{0}, n\hat{\mathbf{x}})\big]}{n} = \lim_{\substack{\hat{\mathbf{y}} \in \mathbb{U}^{d-1}: \\ \hat{\mathbf{y}} \to \hat{\mathbf{x}}}} \mu_{\mathbb{Z}^d}(\hat{\mathbf{y}}), \quad \text{for any } \hat{\mathbf{x}} \in \mathbb{S}^{d-1}. \tag{2.2}
$$

From (2.1) we also conclude, letting $M = d\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]$ and sending n to infinity, that

$$
\big|\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{y}})\big| \leq M|\hat{\mathbf{x}} - \hat{\mathbf{y}}|, \quad \text{for all } \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{S}^{d-1}. \tag{2.3}
$$

Hence, the time constant is Lipschitz continuous on $\mathbb{S}^{d-1}$. The time constant is also known to be continuous with respect to weak convergence of passage-time distributions. This fact will be used in order to prove dynamical stability in Theorem 1.6. Let $V, F_1, F_2, \dots, F_\infty$ denote passage time distribution functions, and let $\mu^F_{\mathbb{Z}^d}$ denote the time constant on the $\mathbb{Z}^d$ lattice associated with the distribution F.

Proposition 2.1 (Cox and Kesten (1981))**.** *If $F_n \to F_\infty$ weakly, then for each $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$*

$$
\mu^{F_n}_{\mathbb{Z}^d}(\hat{\mathbf{x}}) \to \mu^{F_\infty}_{\mathbb{Z}^d}(\hat{\mathbf{x}}), \quad \textit{as } n \to \infty.
$$

Remark 2.2. The proposition was proved, for $d = 2$ and $\hat{\mathbf{x}} = \mathbf{e}_1$, in Cox (1980) under the assumption that there exists a distribution $V \leq F_n$, for all n, with $\int_0^\infty \big(1 - V(u)\big)du < \infty$. That condition was removed in Cox and Kesten (1981). The proof extends to all $d \geq 2$ and directions (cf. Kesten (1986, Theorem 6.9 and Remark 6.18)). □

2.1 The trivial coupling

There is a very natural coupling between passages times on a graph and its subgraphs. We will throughout this paper assume passage times on a graph are coupled with passage times on its subgraphs in this way. We are interested in the $\mathbb{Z}^d$ lattice and subgraphs thereof. We will next present the coupling, and some notation, which will be in force for the rest of this paper.

Let $\mathcal{E}_{\mathbb{Z}^d}$ denote the edge set of the $\mathbb{Z}^d$ lattice, and let $\{\tau_e\}_{e\in\mathcal{E}_{\mathbb{Z}^d}}$ be a family of i.i.d. non-negative random variables associated with the edges of the $\mathbb{Z}^d$ lattice. Let $\mathcal{G}$ be a subgraph of the $\mathbb{Z}^d$ lattice, with vertex set $\mathcal{V}_\mathcal{G}$ and edge set $\mathcal{E}_\mathcal{G}$. In particular, $\mathcal{V}_\mathcal{G} \subseteq \mathbb{Z}^d$ and $\mathcal{E}_\mathcal{G} \subseteq \mathcal{E}_{\mathbb{Z}^d}$. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{V}_\mathcal{G}$, let $T_\mathcal{G}(\mathbf{x},\mathbf{y})$ denote the first-passage time on $\mathcal{G}$ between $\mathbf{x}$ and $\mathbf{y}$ with respect to the set of passage times $\{\tau_e\}_{e\in\mathcal{E}_{\mathbb{Z}^d}}$. This generates a simultaneous coupling of first-passage times among all subgraphs of the $\mathbb{Z}^d$ lattice. In particular, if $\mathcal{G}_1$ is a subgraph of the $\mathbb{Z}^d$ lattice, and $\mathcal{G}_2$ is a further subgraph of $\mathcal{G}_1$, then

$$T_{\mathcal{G}_1}(\mathbf{x},\mathbf{y}) \le T_{\mathcal{G}_2}(\mathbf{x},\mathbf{y}), \quad \text{for all } \mathbf{x},\mathbf{y}\in\mathcal{V}_{\mathcal{G}_2}.$$

If $\mathcal{G}$ is a subgraph of the $\mathbb{Z}^d$ lattice induced by a set $B\subseteq\mathbb{R}^d$, then we define $T_\mathcal{G}(\mathbf{x},\mathbf{y}) = T_\mathcal{G}(\mathbf{x}^*,\mathbf{y}^*)$ for any $\mathbf{x},\mathbf{y}\in B$, where $\mathbf{x}^*$ and $\mathbf{y}^*$ denotes the points in $B\cap\mathbb{Z}^d$ closest to $\mathbf{x}$ and $\mathbf{y}$.

We will also use some additional notation. We will let $T(\cdot,\cdot)$ denote $T_{\mathbb{Z}^d}(\cdot,\cdot)$ for short. For $K\ge 1$ and $\hat{\mathbf{z}}\in\mathbb{U}^{d-1}$, we will let $T_{K,\hat{\mathbf{z}}}(\cdot,\cdot)$ denote passage times on the graph induced by $\bigcup_{a\in\mathbb{R}} B(a\hat{\mathbf{z}},K)$, and referred to as the $(K,d,\hat{\mathbf{z}})$-*tube* (more on this graph in Section 2.3). For $K\ge 1$ and $\mathbf{x}$, $\mathbf{y}\in\mathbb{R}^d$, we will let $\tilde{T}_K(\mathbf{x},\mathbf{y})$ denote the passage time between $\mathbf{x}^*$ and $\mathbf{y}^*$ on the subgraph of the $\mathbb{Z}^d$ lattice induced by the set $\bigcup_{a\in[0,1]} B\big(\mathbf{x}^*+a(\mathbf{y}^*-\mathbf{x}^*),K\big)$. In all these cases we assume the above coupling present. In particular, for any $K\ge 1$, $\mathbf{z}\in\mathbb{Z}^d$, and with $\hat{\mathbf{z}}=\mathbf{z}/|\mathbf{z}|$,

$$T(m\mathbf{z},n\mathbf{z}) \;\le\; T_{K,\hat{\mathbf{z}}}(m\mathbf{z},n\mathbf{z}) \;\le\; \tilde{T}_K(m\mathbf{z},n\mathbf{z}), \quad \text{for any } m,n\in\mathbb{Z}.$$

When we consider dynamical passage times $\{\tau_e(s)\}_{e\in\mathcal{E}_{\mathbb{Z}^d}}$ in Section 5 and 6, the analogous coupling to the above one will be assumed to be in force. The notation above will be adopted also in this case, with an additional superscript (s), as in $T_\mathcal{G}^{(s)}(\cdot,\cdot)$, to indicate that dynamical passage times are considered.

2.2 Geometry of the lattice

We will in this section deduce some basic properties of the graphs we consider, such as connectivity. Furthermore, we will also describe what requirements are necessary for the entities we are interested in to have finite moments.

Lemma 2.3. *For $\mathbf{z} \in \mathbb{Z}^d$ and $r \geq \sqrt{d}$, the subgraph of the $\mathbb{Z}^d$ lattice induced by $\bigcup_{a\in[0,1]} B(a\mathbf{z}, r)$ is connected.*

Proof. Let $r \geq \sqrt{d}$. Observe that for any $\mathbf{x} \in \mathbb{R}^d$, the graph induced by $B(\mathbf{x}, r)$ is connected and non-empty. Moreover, $\mathbb{Z}^d \cap \bigcup_{a\in[0,c]} B(a\mathbf{z}, r)$ can be written as a union of $B(a_j\mathbf{z}, r)$ for finitely many j's. Since the induced graph of each $B(a_j\mathbf{z}, r)$ is connected, it suffice to choose an increasing sequence of a_j's such that $B(a_j\mathbf{z}, r) \cap B(a_{j+1}\mathbf{z}, r) \cap \mathbb{Z}^d \neq \emptyset$, for each j. □

Given $\mathbf{x} \in \mathbb{R}^d$, recall that $\mathbf{x}^*$ denotes the point in $\mathbb{Z}^d$ closest to $\mathbf{x}$. Clearly $|\mathbf{x} - \mathbf{x}^*| \leq \sqrt{d}/2$.

Lemma 2.4. *There exists $R_d < \infty$ such that for each $\mathbf{z} \in \mathbb{Z}^d$ and $b, c > 0$, there are $2d$ disjoint paths between $(b\mathbf{z})^*$ and $(c\mathbf{z})^*$ contained in the subgraph of the $\mathbb{Z}^d$ lattice induced by $\bigcup_{a\in[b,c]} B(a\mathbf{z}, R_d)$.*

From now on and throughout the paper, R_d will be considered as a fixed constant chosen as in Lemma 2.4.

Proof. Set $\mathbf{y} = (c\mathbf{z})^* - (b\mathbf{z})^*$. Pick $2d-1$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{2d-1} \in \mathbb{Z}^d$ such that the $2d$ tubes $\bigcup_{a\in[0,1]} B\big((b\mathbf{z})^* + a\mathbf{y}, \sqrt{d}\big)$, and $\bigcup_{a\in[0,1]} B\big(\mathbf{x}_j + a\mathbf{y}, \sqrt{d}\big)$ for $j = 1, 2, \ldots, 2d-1$ are pairwise disjoint. Since each tube is connected, by Lemma 2.3, we obtain $2d$ paths from $(b\mathbf{z})^*$ to $(c\mathbf{z})^*$ by connecting $(b\mathbf{z})^*$ and $(c\mathbf{z})^*$ by disjoint paths to $\mathbf{x}_j$ and $\mathbf{y} + \mathbf{x}_j$, respectively. If the points $\mathbf{x}_1, \ldots, \mathbf{x}_{2d-1}$ are chosen at distance at most M from $(b\mathbf{z})^*$, then the $2d$ paths will be included in $\bigcup_{a\in[0,1]} B\big((b\mathbf{z})^* + a\mathbf{y}, M + \sqrt{d}\big)$. Thus, it suffices to choose $R_d \geq M + 3\sqrt{d}/2$. □

We now turn to questions regarding conditions for existence of finite moments. For any $\mathbf{z} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, we have $\mathbb{P}\big(T(\mathbf{0}, \mathbf{z}) > s\big) \geq \mathbb{P}(Y > s)$. It is therefore immediate that for any $\alpha > 0$

$$\mathbb{E}[Y^\alpha] = \infty \quad \Rightarrow \quad \mathbb{E}\big[T(\mathbf{0}, \mathbf{z})^\alpha\big] = \infty.$$

The converse is also true.

Proposition 2.5. *Let $\mathbf{z} \in \mathbb{Z}^d$, $m, n \in \mathbb{Z}$, $K \geq R_d$, $\alpha > 0$, and set $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$. When $\mathbb{E}[Y^\alpha] < \infty$, then*

$$\mathbb{E}\big[T(\mathbf{0}, \mathbf{z})^\alpha\big] < \infty, \quad \mathbb{E}\big[T_{K,\hat{\mathbf{z}}}(m\hat{\mathbf{z}}, n\hat{\mathbf{z}})^\alpha\big] < \infty, \quad \text{and} \quad \mathbb{E}\big[\tilde{T}_K(\mathbf{0}, \mathbf{z})^\alpha\big] < \infty.$$

Proof. One may easily realize, e.g. via induction, that there are $2d$ disjoint paths between $\mathbf{0}$ and $\mathbf{z}$ of length at most $\|\mathbf{z}\|_1 + 8$. Let Γ denote the longest of the $2d$ disjoint paths between $\mathbf{0}$ and $\mathbf{z}$, and let λ denote its length. Then

$$\begin{aligned} \mathbb{P}\big(T(\mathbf{0}, \mathbf{z}) > s\big) \;&\leq\; \mathbb{P}\big(T(\Gamma) > s\big)^{2d} \\ &\leq\; \lambda^{2d}\mathbb{P}(\tau_e > s/\lambda)^{2d} \;=\; \lambda^{2d}\mathbb{P}(Y > s/\lambda), \end{aligned} \tag{2.4}$$

where the second inequality holds because if $T(\Gamma) > s$, then for at least one of the λ edges $\tau_e > s/\lambda$. Moreover, for any non-negative random variable X, then $\mathbb{E}[X^\alpha] = \alpha \int_0^\infty x^{\alpha-1}\mathbb{P}(X > x)\,dx$ for $\alpha > 0$. In particular,

$$\mathbb{E}[X^\alpha] < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} n^{\alpha-1}\mathbb{P}(X \geq n) < \infty, \tag{2.5}$$

We conclude that $\mathbb{E}[Y^\alpha] < \infty$ implies $\mathbb{E}\big[T(\mathbf{0},\mathbf{z})^\alpha\big] < \infty$. In combination with Lemma 2.4, the other conclusions follow in a similar manner. □

Theorem 1.2 and 1.6 will be proved assuming that $\mathbb{E}[\tau_e^2] < \infty$. That assumption simplifies the presentation somewhat, since Chebyshev's inequality applies. The same inequality can be applied to find that indeed

$$\mathbb{E}[\tau_e^2] < \infty \quad \Rightarrow \quad \mathbb{E}[Y^{4d}] < \infty. \tag{2.6}$$

2.3 Tube convergence

In this section we shall review some results from Ahlberg (2010) that will be needed in the present paper. In that paper, the key to understand the behaviour of first-passage times on essentially 1-dimensional periodic graphs was to identify a regenerative structure. In this paper, we will have a particular interest in the graph induced by the set $\bigcup_{a\in\mathbb{R}} B(a\hat{\mathbf{z}}, K)$, above referred to as the $(K, d, \hat{\mathbf{z}})$-tube. Its regenerative structure is specified in the following result.

Proposition 2.6. *Assume that $K \geq R_d$, $\mathbf{z} \in \mathbb{Z}^d$ and $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$. There is a sequence of non-negative integer-valued random variables $\{\rho_j\}_{j\geq 0}$ such that*

a) $T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_j\mathbf{z}) - T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_{j-1}\mathbf{z}) = T_{K,\hat{\mathbf{z}}}(\rho_{j-1}\mathbf{z}, \rho_j\mathbf{z})$.

b) $\Big\{\big(T_{K,\hat{\mathbf{z}}}(\rho_{j-1}\mathbf{z}, \rho_j\mathbf{z}), \rho_j - \rho_{j-1}\big)\Big\}_{j\geq 1}$ forms an i.i.d. sequence.

The behaviour specified in the above proposition is referred to as a *regenerative behaviour*. In particular, $T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_n\mathbf{z})$ can be written as a sum of independent variables

$$T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_n\mathbf{z}) = T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_0\mathbf{z}) + \sum_{j=1}^{n} T_{K,\hat{\mathbf{z}}}(\rho_{j-1}\mathbf{z}, \rho_j\mathbf{z}).$$

This can be exploited in order to approximate the value of $T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\mathbf{z})$. Indeed, $T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\mathbf{z})$ can be approximated by $T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_{\nu(n)}\mathbf{z})$, where

$$\nu(n) := \min\{j \in \mathbb{N} : \rho_j \geq n\}.$$

Since $\{\rho_j\}_{j\geq 0}$ is an increasing sequence, we must have that $\nu(n) \leq n$. Define further

$$\mu_K(\hat{\mathbf{z}}) := \frac{\mathbb{E}\big[T_{K,\hat{\mathbf{z}}}(\rho_{j-1}\mathbf{z}, \rho_j\mathbf{z})\big]}{|\mathbf{z}|\mathbb{E}[\rho_j - \rho_{j-1}]},$$
$$\Delta_j := T_{K,\hat{\mathbf{z}}}(\rho_{j-1}\mathbf{z}, \rho_j\mathbf{z}) - |\mathbf{z}|\mu_K(\hat{\mathbf{z}})(\rho_j - \rho_{j-1}).$$

Proposition 2.7. *The sequence $\{\rho_j\}_{j\geq 0}$ can be chosen such that the distributions of $\rho_{\nu(n)} - n$ and $T_{K,\hat{\mathbf{z}}}(n\mathbf{z}, \rho_{\nu(n)}\mathbf{z})$ does not depend on n, and such that for any $\alpha \geq 1$ and $j \geq 1$,*

a) $\mathbb{E}\big[(\rho_j - \rho_{j-1})^\alpha\big] < \infty$ *and* $\mathbb{E}\big[(\rho_{\nu(n)} - n)^\alpha\big] < \infty$.

b) if $\mathbb{E}[Y^\alpha] < \infty$, *then*

$$\mathbb{E}\big[T_{K,\hat{\mathbf{z}}}(\rho_{j-1}\mathbf{z}, \rho_j\mathbf{z})^\alpha\big] < \infty, \quad \mathbb{E}\big[T_{K,\hat{\mathbf{z}}}(n\mathbf{z}, \rho_{\nu(n)}\mathbf{z})^\alpha\big] < \infty,$$
$$\mathbb{E}\big[|\Delta_j|^\alpha\big] < \infty, \qquad \mathbb{E}\Big[\big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_0\mathbf{z}) - |\mathbf{z}|\rho_0\mu_K(\hat{\mathbf{z}})\big|^\alpha\Big] < \infty.$$

Moreover, if $\mathbb{E}[Y] < \infty$, *then* $\mathbb{E}[\Delta_j] = 0$ *for all* $j \geq 1$.

The regenerative behaviour described in Proposition 2.6 and 2.7 was deduced in Ahlberg (2010) and used to prove the following limiting behaviour.

Proposition 2.8. *Let $K \geq R_d$ and $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$. If $\mathbb{E}[Y] < \infty$, then*

$$\lim_{n\to\infty} \frac{T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\hat{\mathbf{z}})}{n} = \mu_K(\hat{\mathbf{z}}), \quad \textit{almost surely and in } L^1.$$

The same conclusion holds for $T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\hat{\mathbf{z}})$ exchanged for $\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})$.

The above theorem was proved for so called essentially 1-dimensional periodic graphs. The $(K, d, \hat{\mathbf{z}})$-tube is an example of such a graph. The statement regarding $\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})$, was in Ahlberg (2010) not directly stated. However, it is easily seen to follow analogously. Proposition 2.8 could in fact be derived from the Subadditive Ergodic Theorem. However, the approach using regenerative sequences allows for much more detailed picture to be derived. A central limit theorem and a law of the iterated logarithm are two examples of additional results obtained in Ahlberg (2010).

There it was also seen that $\mu_K(\hat{\mathbf{z}}) > \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$ for all K. However, as K increases the constants approach each other. This fairly simple consequence the subadditive behaviour will be essential in this paper. It was indicated already in Chayes and Chayes (1984), and proofs appeared in Ahlberg (2008, 2010); Chatterjee and Dey (2009). Due to its central rôle to this paper, we recall the proof also here.

Proposition 2.9. $\lim_{K\to\infty} \mu_K(\hat{\mathbf{z}}) = \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}), \quad$ *for all* $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$.

Proof. Clearly $\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})$ is decreasing in K. For all n we get

$$T(\mathbf{0}, n\hat{\mathbf{z}}) = \lim_{K\to\infty} \tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}}) = \inf_{K\geq 0} \tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}}), \quad \text{almost surely.}$$

An application of the Monotone Convergence Theorem gives

$$\mathbb{E}\big[T(\mathbf{0}, n\hat{\mathbf{z}})\big] \;=\; \lim_{K\to\infty} \mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big] \;=\; \inf_{K\geq 0} \mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big].$$

By Fekete's lemma $\exists \lim_{n\to\infty} a_n/n = \inf_{n\geq 1} a_n/n$, for any subadditive real-valued sequence $\{a_n\}_{n\geq 1}$. Hence, for any $0 \leq K \leq \infty$

$$\mu_K(\hat{\mathbf{z}}) \;=\; \lim_{n\to\infty} \frac{\mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big]}{n} \;=\; \inf_{n\geq 1} \frac{\mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big]}{n}.$$

(Here $\mu_\infty(\hat{\mathbf{z}})$ refers to $\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$, and $\tilde{T}_\infty(\mathbf{0}, n\hat{\mathbf{z}})$ to $T(\mathbf{0}, n\hat{\mathbf{z}})$.) Thus, since μ_K is non-increasing in K we conclude that

$$\begin{aligned}
\lim_{K\to\infty} \mu_K(\hat{\mathbf{z}}) \;&=\; \inf_{K\geq 0} \inf_{n\geq 1} \frac{\mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big]}{n} \;=\; \inf_{n\geq 1} \inf_{K\geq 0} \frac{\mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big]}{n} \\
&=\; \inf_{n\geq 1} \frac{\mathbb{E}\big[T(\mathbf{0}, n\hat{\mathbf{z}})\big]}{n} \;=\; \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}).
\end{aligned}$$

□

3 Large deviation estimates for the lattice

The aim of this section is to prove Proposition 1.3, 1.4 and 1.5. The proof of the first of the three will be based on a characterization of the rate of convergence of large deviations of i.i.d. sums. The characterization is a generalized version of a result due to Hsu, Robbins and Erdős. We refer the reader to Gut (2005, Theorem 12.1) for a more complete statement as well as a proof.

Theorem 3.1. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with mean* μ, *and let* $S_n = \sum_{k=1}^{n} X_k$. *For* $\alpha \geq 1$, *the following are equivalent.*

a) $\mathbb{E}\big[|X_k|^\alpha\big] < \infty$.

b) $\sum_{n=1}^{\infty} n^{\alpha-2}\, \mathbb{P}\big(|S_n - \mu n| > n\epsilon\big) < \infty$, *for all* $\epsilon > 0$.

c) $\sum_{n=1}^{\infty} n^{\alpha-2}\, \mathbb{P}\big(\max_{1\leq k\leq n} |S_k - \mu k| > n\epsilon\big) < \infty$, *for all* $\epsilon > 0$.

The proof of Proposition 1.5 will be an adaptation of a proof given in Kesten (1986) of a less general statement. Proposition 1.5 will be derived under the additional assumption that $\mathbb{E}[Y^d] < \infty$, and the reader is referred to Kesten's paper to see how this assumption can be avoided. The details of how Proposition 1.4 follows from Proposition 1.5 are easy to sort out, and therefore left to the reader. A proof of (1.4) will also be presented.

3.1 Above the time constant

In order to derive the large deviation estimate for the $\mathbb{Z}^d$ lattice, we will first do so for tubes. The following result was not included in Ahlberg (2010), but will be proved by similar means, based on Proposition 2.6 and 2.7.

Proposition 3.2. *Let $K \geq R_d$, $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$ and $\alpha \geq 1$. If $\mathbb{E}[Y^\alpha] < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{P}\Big(\big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\hat{\mathbf{z}}) - n\mu_K(\hat{\mathbf{z}})\big| > n\epsilon\Big) < \infty, \quad \textit{for any } \epsilon > 0.$$

Proof. Fix $\mathbf{z} \in \mathbb{Z}^d$ such that $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$. Let $m_n := \min\{m \in \mathbb{N} : |\mathbf{z}|m \geq n\}$. Clearly $0 \leq |\mathbf{z}|m_n - n < |\mathbf{z}|$. Due to subadditivity,

$$\begin{aligned}\Big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\hat{\mathbf{z}}) - n\mu_K(\hat{\mathbf{z}})\Big| \;\leq\; & T_{K,\hat{\mathbf{z}}}(n\hat{\mathbf{z}}, m_n\mathbf{z}) + T_{K,\hat{\mathbf{z}}}(m_n\mathbf{z}, \rho_{\nu(m_n)}\mathbf{z}) \\ & + \Big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_{\nu(m_n)}\mathbf{z}) - |\mathbf{z}|\rho_{\nu(m_n)}\mu_K(\hat{\mathbf{z}})\Big| \\ & + |\mathbf{z}|\mu_K(\hat{\mathbf{z}})\Big|\rho_{\nu(m_n)} - m_n\Big| \\ & + \mu_K(\hat{\mathbf{z}})\Big||\mathbf{z}|m_n - n\Big|.\end{aligned} \tag{3.1}$$

If we denote the terms in the right-hand side of (3.1) by $X_1, X_2, \ldots, X_5$, then it suffices to show that for each $j = 1, 2, \ldots, 5$

$$\sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}(X_j > n\epsilon/5) < \infty. \tag{3.2}$$

The last term in the right-hand side of (3.1) is non-random and bounded. Thus $\mathbb{P}(X_5 > \epsilon n/5) = 0$ for large n. Thus, (3.2) does hold for $j = 5$. According to Proposition 2.7, $\rho_{\nu(m_n)} - m_n$ has finite moment of any order, and its distribution does not depend on n. Thus, (2.5) implies that (3.2) holds also for $j = 4$.

The only term that is essentially contributing is that for $j = 3$. Since $\nu(m) \leq m$, we have

$$\begin{aligned}\mathbb{P}(X_3 > \epsilon n/5) \;\leq\; & \mathbb{P}\left(\max_{i \leq m_n}\Big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_i\mathbf{z}) - |\mathbf{z}|\rho_i\mu_K(\hat{\mathbf{z}})\Big| > n\epsilon/5\right) \\ \leq\; & \mathbb{P}\left(\Big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_0\mathbf{z}) - |\mathbf{z}|\rho_0\mu_K(\hat{\mathbf{z}})\Big| + \max_{i \leq m_n}\left|\sum_{j=1}^{i}\Delta_j\right| > n\epsilon/5\right) \\ \leq\; & \mathbb{P}\Big(\Big|T_{K,\hat{\mathbf{z}}}(\mathbf{0}, \rho_0\mathbf{z}) - |\mathbf{z}|\rho_0\mu_K(\hat{\mathbf{z}})\Big| > n\epsilon/10\Big) \\ & + \mathbb{P}\left(\max_{i \leq m_n}\left|\sum_{j=1}^{i}\Delta_j\right| > n\epsilon/10\right).\end{aligned}$$

According to Proposition 2.6 and 2.7 the sequence $\{\Delta_j\}_{j\geq 1}$ is i.i.d., and both $T_{K,\hat{\mathbf{z}}}(\mathbf{0},\rho_0\mathbf{z}) - |\mathbf{z}|\rho_0\mu_K(\hat{\mathbf{z}})$ and Δ_j (for $j \geq 1$) have finite moments of order α. Thus, via (2.5) and Theorem 3.1 (respectively) we conclude that (3.2) holds for $j = 3$.

That (3.2) holds for $j = 2$ and $j = 1$ will follow in a similar way from (2.5) as for $j = 4$. Again, Proposition 2.7 gives that the distribution of $X_2 = T_{K,\hat{\mathbf{z}}}(m\mathbf{z}, \rho_{\nu(m)}\mathbf{z})$ does not depend on m, and has finite moment of order α. Finally, the distribution of $X_1 = T_{K,\hat{\mathbf{z}}}(n\hat{\mathbf{z}}, m_n\mathbf{z})$ may depend on n. However, it can only vary among a finite number of different ones, each with finite moment of order α (there are $2d$ disjoint paths between $n\hat{\mathbf{z}}$ and $m_n\mathbf{z}$, due to Lemma 2.4). $\square$

We will derive Proposition 1.3 from the estimate on the deviations from the time constant on tubes, that we have just proved.

Proof of Proposition 1.3. Let $\epsilon > 0$, $\alpha \geq 1$ and assume that $\mathbb{E}[Y^\alpha] < \infty$. We will first prove that for any $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{P}\Big(T(\mathbf{0}, n\hat{\mathbf{z}}) > \big(\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) + \epsilon\big)n\Big) < \infty.$$

Take K large enough for $\mu_K(\hat{\mathbf{z}}) \leq \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) + \frac{\epsilon}{2}$ (which is possible according to Proposition 2.9). Then,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\Big(T(\mathbf{0}, n\hat{\mathbf{z}}) > \big(\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) + \epsilon\big)n\Big) \\
&\qquad \leq \sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\Big(T(\mathbf{0}, n\hat{\mathbf{z}}) > \big(\mu_K(\hat{\mathbf{z}}) + \epsilon/2\big)n\Big) \\
&\qquad \leq \sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\Big(T_{K,\hat{\mathbf{z}}}(\mathbf{0}, n\hat{\mathbf{z}}) > \big(\mu_K(\hat{\mathbf{z}}) + \epsilon/2\big)n\Big),
\end{aligned}$$

which Proposition 3.2 says is finite when $\mathbb{E}[Y^\alpha] < \infty$.

We proceed with the general case. For $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$, take $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$ such that $|\hat{\mathbf{x}} - \hat{\mathbf{z}}| \leq \epsilon$. Now

$$T(\mathbf{0}, n\hat{\mathbf{x}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) \;\leq\; \Big(T(\mathbf{0}, n\hat{\mathbf{z}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\Big) + T(n\hat{\mathbf{z}}, n\hat{\mathbf{x}}) + n\Big(\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\Big)$$

According to 2.3 we have $n\big|\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\big| \leq d\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]n\epsilon$. We will show that for some $M < \infty$

$$\sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\Big(T(n\hat{\mathbf{z}}, n\hat{\mathbf{x}}) > Mn\epsilon\Big) < \infty. \tag{3.3}$$

All of the above then easily gives that

$$\sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\Big(T(\mathbf{0},n\hat{\mathbf{x}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) > \big(1+M+d\mathbb{E}\big[T(\mathbf{0},\mathbf{e}_1)\big]\big)n\epsilon\Big) \;<\; \infty,$$

which is sufficient since $\epsilon > 0$ was arbitrary. It remains to prove (3.3).

Choose a path between $(n\hat{\mathbf{z}})^*$ and $(n\hat{\mathbf{x}})^*$ of length $\lambda_n := \big\|(n\hat{\mathbf{x}})^* - (n\hat{\mathbf{z}})^*\big\|_1$. Let $v_0, v_1, \ldots, v_{\lambda_n}$ denote the sequence of vertices in the path. Subadditivity gives that

$$T(n\hat{\mathbf{z}}, n\hat{\mathbf{x}}) \;\le\; \sum_{j=1}^{\lambda_n} T(v_{j-1}, v_j) \;\le\; \sum_{j=1}^{\lambda_n} \hat{T}(v_{j-1}, v_j),$$

where $\hat{T}(v_{j-1}, v_j) = \min\big(T(\Gamma_1), T(\Gamma_2), \ldots, T(\Gamma_{2d})\big)$, and $\Gamma_1, \Gamma_2, \ldots, \Gamma_{2d}$ denotes the $2d$ disjoint paths between v_{j-1} and v_j of length at most 9. The variables $\hat{T}(v_{j-1}, v_j)$ and $\hat{T}(v_{i-1}, v_i)$ are not necessarily independent for $i \neq j$. However, they are if $|v_j - v_i| > 4\sqrt{2}$. Since $|j-i| = \|v_j - v_i\|_1 \le d|v_j - v_i|$, they will be independent when $|j-i| > 4d\sqrt{2}$. We can therefore partition $\{0, 1, \ldots, \lambda_n\}$ into at most $8d\sqrt{2}+1 \le 13d$ sets $J_1, J_2, \ldots, J_{13d}$, such that for each $i = 1, 2, \ldots, 13d$, the elements in $\big\{\hat{T}(v_{j-1}, v_j)\big\}_{j \in J_i}$ are independent. Each J_i contains at most $\lambda_n \le d(\sqrt{d}+n\epsilon)$, which for large n is at most $2dn\epsilon$, indices. Thus, since when $\mathbb{E}[Y^\alpha] < \infty$ also $\mathbb{E}\big[\hat{T}(v_{j-1}, v_j)^\alpha\big] < \infty$, Theorem 3.1 assures that

$$\sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\left(\sum_{j \in J_i} \hat{T}(v_{j-1}, v_j) > \Big(\mathbb{E}\big[\hat{T}(\mathbf{0},\mathbf{e}_1)\big]+1\Big)2dn\epsilon\right) < \infty.$$

We conclude that

$$\sum_{n=1}^{\infty} n^{\alpha-2}\,\mathbb{P}\left(T(n\hat{\mathbf{z}}, n\hat{\mathbf{x}}) > 26d\Big(\mathbb{E}\big[\hat{T}(\mathbf{0},\mathbf{e}_1)\big]+1\Big)dn\epsilon\right) < \infty.$$

This proves (3.3). □

In preparation for Section 4, we also prove that radial L^1-convergence holds in any direction on the $\mathbb{Z}^d$ lattice. The proof of which is similar to the proof of Proposition 1.3.

Proposition 3.3. *Let $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and $d \ge 2$. If $\mathbb{E}[Y] < \infty$, then*

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{T(\mathbf{0}, n\hat{\mathbf{x}})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| = 0.$$

Proof. Let $\epsilon > 0$ and take $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$ such that $|\hat{\mathbf{x}} - \hat{\mathbf{z}}| \leq \epsilon$. As in the proof of Proposition 1.3 we have

$$\begin{aligned}
\big|T(\mathbf{0}, n\hat{\mathbf{x}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\big| \;&\leq\; \big|T(\mathbf{0}, n\hat{\mathbf{z}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\big| + n\big|\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\big| \\
&\quad + T(n\hat{\mathbf{z}}, n\hat{\mathbf{x}}) \\
&\leq\; \big|T(\mathbf{0}, n\hat{\mathbf{z}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\big| + d\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]n\epsilon \\
&\quad + \sum_{j=1}^{\lambda_n} \hat{T}(v_{j-1}, v_j),
\end{aligned}$$

where $\lambda_n = \big\|(n\hat{\mathbf{x}})^* - (n\hat{\mathbf{z}})^*\big\|_1 \leq d\big(\sqrt{d} + n|\hat{\mathbf{x}} - \hat{\mathbf{z}}|\big)$. Together with (1.1), we obtain that

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{T(\mathbf{0}, n\hat{\mathbf{x}})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| \;\leq\; d\Big(\mathbb{E}\big[\hat{T}(\mathbf{0}, \mathbf{e}_1)\big] \;+\; \mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]\Big)\epsilon.$$

Since, $\mathbb{E}\big[\hat{T}(\mathbf{0}, \mathbf{e}_1)\big] < \infty$ when $\mathbb{E}[Y] < \infty$, and $\epsilon > 0$ arbitrary, the L^1-convergence follows. □

3.2 Below the time constant

We first present a proof of (1.4). We will prove (1.4) and Proposition 1.5 under the additional assumption that $\mathbb{E}[Y^d] < \infty$. With this assumption, the derivation of (1.4) and Lemma 3.5 below can be simplified somewhat, since it enables us to appeal to the Shape Theorem directly. An alternative version of the Shape Theorem (see Kesten (1986)) can be used to obtain our results without moment assumption. The reader can in Kesten's paper find the additional details that are needed in that case.

Proof of (1.4). Assume that $\mathbb{E}[Y^d] < \infty$. By definition, we easily obtain that, almost surely,

$$\frac{T^{\mathcal{W}}_{0,n}}{n} \leq \frac{T\left(\mathbf{0}, \left(\frac{n}{\mu_{\mathbb{Z}^d}(\mathbf{e}_1)} + 1\right)\mathbf{e}_1\right)}{n} = \frac{\frac{n}{\mu_{\mathbb{Z}^d}(\mathbf{e}_1)} + 1}{n} \cdot \frac{T\left(\mathbf{0}, \left(\frac{n}{\mu_{\mathbb{Z}^d}(\mathbf{e}_1)} + 1\right)\mathbf{e}_1\right)}{\frac{n}{\mu_{\mathbb{Z}^d}(\mathbf{e}_1)} + 1} \;\to\; 1,$$

as $n \to \infty$. So, it suffice to show that

$$\liminf_{n\to\infty} \frac{T^{\mathcal{W}}_{0,n}}{n} \;\geq\; 1, \quad \text{almost surely.} \tag{3.4}$$

Assume that there is $\delta > 0$ such that

$$\liminf_{n\to\infty} \frac{T^{\mathcal{W}}_{0,n}}{n} \;\leq\; 1 - \delta \tag{3.5}$$

has positive probability. For any realization satisfying (3.5) there are $n_1 < n_2 < \ldots$ such that $T^{\mathcal{W}}_{0,n_k}/n_k \le 1-\delta/2$. In particular, we can pick $\mathbf{z}_1, \mathbf{z}_2, \ldots$ such that $\mathbf{z}_k \in \mathcal{W}_{n_k(1-\delta/2)}$ and

$$|\mathbf{z}_k| \;\ge\; \frac{n_k}{\mu_{\mathbb{Z}^d}(\mathbf{z}_k/|\mathbf{z}_k|)},$$

for each $k = 1, 2, \ldots$. The Shape Theorem says that with probability one

$$\mathcal{W}_{n(1-\delta/2)} \subseteq n(1-\delta/4)\mathcal{W}^*, \tag{3.6}$$

for any n sufficiently large. Thus, there are realizations satisfying both (3.5) and (3.6). But, (3.6) implies that for all n sufficiently large, if $\mathbf{z} \in \mathcal{W}_{n(1-\delta/2)}$, then

$$|\mathbf{z}| \le \frac{n(1-\delta/4)}{\mu_{\mathbb{Z}^d}(\mathbf{z}/|\mathbf{z}|)},$$

which is a contradiction. Hence, (3.4) must hold. $\square$

We now prepare for the proof of Proposition 1.5. That will require to extend the definition of point-to-shape passage times somewhat. Assume that $\mu_{\mathbb{Z}^d}(\mathbf{e}_1) > 0$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, let $u(\mathbf{y}-\mathbf{x}) = (\mathbf{y}-\mathbf{x})/|\mathbf{y}-\mathbf{x}|$. Define for any $\mathbf{z} \in \mathbb{Z}^d$ and integers $0 \le n \le m$

$$T^{\mathcal{W}}_{n,m}(\mathbf{z}) := \inf\left\{ T(\mathbf{x},\mathbf{y}) :\; |\mathbf{x}-\mathbf{z}| \le \frac{n}{\mu_{\mathbb{Z}^d}\big(u(\mathbf{x}-\mathbf{z})\big)},\; |\mathbf{y}-\mathbf{z}| \ge \frac{m}{\mu_{\mathbb{Z}^d}\big(u(\mathbf{y}-\mathbf{z})\big)} \right\}.$$

For short, we write $T^{\mathcal{W}}_{n,m} = T^{\mathcal{W}}_{n,m}(\mathbf{0})$. One should think of $T^{\mathcal{W}}_{n,m}$ as the minimal travel time from some vertex within $n\mathcal{W}^*$ to some vertex at the boundary, or in the complement, of $m\mathcal{W}^*$. Following Kesten's approach, the first step in proving Proposition 1.5 is this next lemma.

Lemma 3.4. *Let $X^{(q)}_{N,N+M}$ for $q = 1, 2, \ldots$ denote independent random variables distributed as $T^{\mathcal{W}}_{N,N+M}$. There exists $C < \infty$ such that for any $n \ge M \ge N \ge 0$ and $x > 0$ we have*

$$\mathbb{P}\big(T^{\mathcal{W}}_{0,n} < x\big) \;\le\; \sum_{Q+1 \ge n/(M+CN)} n^{d-1}\left(C\frac{M}{N}\right)^{d(Q-1)} \mathbb{P}\left(\sum_{q=1}^{Q} X^{(q)}_{N,N+M} < x\right).$$

The proof of this lemma does not require any moment assumption.

Proof. Pick $\mathbf{z} \in \mathbb{Z}^d$ such that $|\mathbf{z}| \ge n/\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$, where $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$. Let $\gamma = \gamma(\mathbf{z})$ be a self-avoiding path from $\mathbf{0}$ to $\mathbf{z}$. Choose a subsequence $v_0, v_1, \ldots, v_Q$ of the vertices in γ as follows. Set $v_0 = \mathbf{0}$. Given v_q, choose v_{q+1} to be the first vertex in γ succeeding v_q such that

$$|v_{q+1} - v_q| \geq \frac{M + 2N}{\mu_{\mathbb{Z}^d}\big(u(v_{q+1} - v_q)\big)}.$$

When no such vertex exists, stop and set $Q = q$. To find a lower bound on Q, pick a plane tangent to $a\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\mathcal{W}^*$ at the point $a\hat{\mathbf{z}}$ and denote it by $H(a, \hat{\mathbf{z}})$. That at least one such plane exists follows from convexity of $\mathcal{W}^*$. Take $a_q \in \mathbb{R}$ such that $v_q \in H(a_q, \hat{\mathbf{z}})$ for each $q = 0, 1, \dots, Q$. It is easily seen that $a_q \leq q\left(\frac{M+2N}{\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})} + 1\right)$. Moreover, $\mathbf{z} \in H(a, \hat{\mathbf{z}})$ for some a satisfying

$$n/\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \;\leq\; a \;\leq\; a_Q + \frac{M + 2N}{\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})} \;\leq\; (Q+1)\left(\frac{M + 2N}{\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})} + 1\right).$$

In particular, since $\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \leq \sqrt{d}\mu_{\mathbb{Z}^d}(\mathbf{e}_1)$, we see that Q must satisfy

$$n \;\leq\; (Q+1)\Big(M + \big(2 + \sqrt{d}\mu_{\mathbb{Z}^d}(\mathbf{e}_1)\big)N\Big). \tag{3.7}$$

Next, pick $r > 0$ such that $[-r, r]^d \subseteq \mathcal{W}^*$ and tile $\mathbb{Z}^d$ with copies of $(-rN, rN]^d$ such that each box is centred at a point in $\mathbb{Z}^d$, and each point in $\mathbb{Z}^d$ is contained in precisely one box. Let Λ_q denote the box that contains v_q, and let w_q denote the centre of Λ_q. Of course, the tiling can be chosen such that $w_0 = v_0 = \mathbf{0}$. Denote by γ_q the part of the path γ that connects v_q and v_{q+1}. Note that for $q_1 \neq q_2$ the two pieces γ_{q_1} and γ_{q_2} are edge disjoint. By construction v_q is included in the copy of $N\mathcal{W}^*$ centred at w_q. Moreover, v_{q+1} is not included in the interior of the copy of $(M + 2N)\mathcal{W}^*$ centred at v_q. By convexity of $\mathcal{W}^*$, v_{q+1} cannot either lie in the interior of the shape $(M+N)\mathcal{W}^*$ centred at w_q. That is,

$$|v_q - w_q| \;\leq\; \frac{N}{\mu_{\mathbb{Z}^d}\big(u(v_q - w_q)\big)} \quad \text{and} \quad |v_{q+1} - w_q| \;\geq\; \frac{M + N}{\mu_{\mathbb{Z}^d}\big(u(v_{q+1} - w_q)\big)}. \tag{3.8}$$

Given $x > 0$, $Q \in \mathbb{Z}_+$ and $w_1, \dots, w_{Q-1}$, let $A(x, w_1, w_2, \dots, w_{Q-1})$ denote the event that there exists a path γ from $\mathbf{0}$ to $\mathbf{z}$ which contains edge disjoint pieces $\gamma_0, \gamma_1, \dots, \gamma_{Q-1}$ such that

a) $\sum_{q=0}^{Q-1} T(\gamma_q) < x$,

b) for each $q = 0, 1, \dots, Q-1$, the endpoints v_q and v_{q+1} of γ_q satisfy (3.8) (where $w_0 = \mathbf{0}$).

Since $T(\gamma) \geq \sum_{q=0}^{Q-1} T(\gamma_q)$, together with (3.7), we obtain that

$$\{T(\mathbf{0},\mathbf{z}) < x\} \subseteq \bigcup_{Q+1\geq n/(M+bN)} \bigcup_{w_1,w_2,\ldots,w_{Q-1}} A(x, w_1, w_2, \ldots, w_{Q-1}) \quad (3.9)$$

where $b = 2 + \sqrt{d}\,\mu_{\mathbb{Z}^d}(\mathbf{e}_1)$. Note that given w_q, the passage time of any path between any two vertices $\mathbf{x}$ and $\mathbf{y}$ that satisfy $|\mathbf{x} - w_q| \leq \frac{N}{\mu_{\mathbb{Z}^d}(u(\mathbf{x}-w_q))}$ and $|\mathbf{y} - w_q| \geq \frac{M+N}{\mu_{\mathbb{Z}^d}(u(\mathbf{y}-w_q))}$ is stochastically larger than $T^{\mathcal{W}}_{N,N+M}$. Hence, for fixed $w_0, w_1, \ldots, w_{Q-1}$, the event $A(x, w_1, w_2, \ldots, w_{Q-1})$ has probability at most

$$\mathbb{P}\Big(X^{(1)}_{N,N+M} + X^{(2)}_{N,N+M} + \ldots + X^{(Q)}_{N,N+M} < x\Big). \quad (3.10)$$

It remains to count the number of possible choices for $w_1, w_2, \ldots, w_{Q-1}$. Assume that w_q has already been chosen. The distance which the vertex v_{q+1} can have to w_q is bounded by

$$|v_{q+1}-w_q| \;\leq\; |v_{q+1}-v_q|+|v_q-w_q| \;\leq\; \frac{M+2N}{\mu_{\mathbb{Z}^d}\big(u(v_{q+1}-v_q)\big)}+1+\frac{N}{\mu_{\mathbb{Z}^d}\big(u(v_q-w_q)\big)}.$$

In particular, v_{q+1} is contained in the cube centred at w_q of side length

$$2\sqrt{d}\frac{M + \big(3 + \mu_{\mathbb{Z}^d}(\mathbf{e}_1)\big)N}{\mu_{\mathbb{Z}^d}(\mathbf{e}_1)}.$$

This cube is intersected by at most $(CM/N)^d$ boxes of the form $(-rN, rN]^d$ that tiles $\mathbb{Z}^d$, for some $C < \infty$. This bounds the number of choices for w_{q+1}, and since for each $q = 1, 2, \ldots, Q-1$ we cannot have more choices than this, the total number of choices for $w_1, w_2, \ldots, w_{Q-1}$ is at most $(CM/N)^{d(Q-1)}$. From (3.9) and (3.10) we conclude that

$$\mathbb{P}\big(T(\mathbf{0},\mathbf{z}) < x\big) \;\leq\; \sum_{Q+1\geq n/(M+CN)} \left(C\frac{M}{N}\right)^{d(Q-1)} \mathbb{P}\left(\sum_{q=1}^{Q} X^{(q)}_{N,N+M} < x\right),$$

for some $C < \infty$. The lemma now follows observing that, by convexity of $\mathcal{W}^*$, the number of $\mathbf{z} \in \mathbb{Z}^d$ that satisfies $|\mathbf{z}| \geq n/\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$ and has a neighbour within $n\mathcal{W}^*$ are of order n^{d-1}. □

Lemma 3.5. *For any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that*

$$\lim_{M\to\infty} \max_{N\leq\eta M} \mathbb{P}\Big(T^{\mathcal{W}}_{N,N+M} < M(1-\epsilon)\Big) = 0.$$

Proof. We will prove this lemma under the additional, but not necessary, assumption that $\mathbb{E}[Y^d] < \infty$. For $\mathbf{z} \in \mathbb{Z}^d$ with $|\mathbf{z}| \le N/\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})$, let γ be a path from $\mathbf{z}$ to some point $\mathbf{y}$ with $|\mathbf{y}| \ge (N+M)/\mu_{\mathbb{Z}^d}(\mathbf{y}/|\mathbf{y}|)$. Clearly

$$T^{\mathcal{W}}_{0,N+M} \;\le\; T(\mathbf{0},\mathbf{z}) + T(\gamma).$$

In particular, we may pick $\mathbf{z} \in N\mathcal{W}^*$ and $\gamma = \gamma(\mathbf{z})$ such that $T(\gamma) = T^{\mathcal{W}}_{N,N+M}$. It follows that

$$T^{\mathcal{W}}_{0,M} \;\le\; T^{\mathcal{W}}_{0,N+M} \;\le\; \max\left\{T(\mathbf{0},\mathbf{z}) : \mathbf{z} \text{ with } |\mathbf{z}| \le \frac{N}{\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})}\right\} + T^{\mathcal{W}}_{N,M+N}.$$

By (1.4), $\mathbb{P}\Big(T^{\mathcal{W}}_{0,M} < M(1-\epsilon/2)\Big) \to 0$ as $M \to \infty$. Thus, it suffices to prove that

$$\lim_{M\to\infty} \mathbb{P}\left(\max\left\{T(\mathbf{0},\mathbf{z}) : \mathbf{z} \text{ with } |\mathbf{z}| \le \frac{N}{\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})}\right\} > \epsilon M/4\right) \;=\; 0. \tag{3.11}$$

However, the event in (3.11) is contained in the event $\{N\mathcal{W}^* \not\subseteq \mathcal{W}_{\epsilon M/4}\}$. According to the Shape Theorem, the probability of this event tends to zero as $M \to \infty$ for any $N \le (1-\delta)\epsilon M/4$, and $\delta > 0$. Hence, the result follows for any $\eta < \epsilon/4$. $\square$

Proof of Proposition 1.5. Fix $\epsilon > 0$. Let $X^{(1)}_{N,N+M}, X^{(2)}_{N,N+M}, \ldots, X^{(Q)}_{N,N+M}$ and $C < \infty$ be as in Lemma 3.4, and choose $\eta = \eta(\epsilon)$ according to Lemma 3.5. Let

$$N = \min\left(\eta M, \left\lfloor \frac{M\epsilon}{4C} \right\rfloor\right).$$

Markov's inequality and independence give that for any $\gamma > 0$

$$\mathbb{P}\left(\sum_{q=1}^{Q} X^{(q)}_{N,N+M} < n(1-\epsilon)\right) \;\le\; e^{\gamma n(1-\epsilon)}\mathbb{E}\Big[e^{-\gamma X^{(1)}_{N,N+M}}\Big]^{Q},$$

which in turn is at most

$$e^{\gamma(M+CN)(1-\epsilon)}\left[e^{\gamma(M+CN)(1-\epsilon)}\left(e^{-\gamma M(1-\epsilon/2)} + \mathbb{P}\Big(X^{(1)}_{N,N+M} < M(1-\epsilon/2)\Big)\right)\right]^{Q}.$$

Since $CN - M\epsilon/2 \le -M\epsilon/4$, the expression within square brackets is at most

$$e^{-\gamma M\epsilon/4} + e^{(1+\eta C)\gamma M}\mathbb{P}\Big(X^{(1)}_{N,N+M} < M(1-\epsilon/2)\Big). \tag{3.12}$$

According to Lemma 3.5 we can make this expression arbitrarily small by choosing γ and M such that γM is large and M is as large as necessary. Fix γ and M such that $N \geq 1$ and (3.12) is at most

$$(2C)^{-d} \max\left(\frac{2}{\eta}, \frac{8C}{\epsilon}\right)^{-d} \leq \left(2C\frac{M}{N}\right)^{-d}.$$

Appealing to Lemma 3.4 with these γ, M and N, we find that

$$\begin{aligned}
&\mathbb{P}\Big(T_{0,n}^{\mathcal{W}} < n(1-\epsilon)\Big) \\
&\quad\leq\ e^{\gamma(M+CN)} \sum_{(Q+1)\geq n/(M+CN)} n^{d-1}\left(C\frac{M}{N}\right)^{d(Q-1)}\left(2C\frac{M}{N}\right)^{-dQ} \\
&\quad\leq\ e^{\gamma(M+CN)} \cdot n^{d-1} \cdot 2^{-d(n/(M+CN)-1)+1},
\end{aligned}$$

which is of the required form. □

4 Radial convergence on cones

In this section we prove that we have almost sure, L^1 and complete convergence in any radial direction on cone-like subgraphs of the $\mathbb{Z}^d$ lattice. The result will be stated next, and extends several of the previous results stated for the $\mathbb{Z}^d$ lattice. It is the first step in the proof of Theorem 1.2.

Proposition 4.1. *Let $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$, and $\omega : [0,\infty) \to [0,\infty)$ be any function such that $\omega(a) \to \infty$ as $a \to \infty$. Let $\mathcal{G}$ denote the subgraph of the $\mathbb{Z}^d$ lattice induced by the set $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a) + R_d\big)$.*

a) If $\mathbb{E}[Y] < \infty$, then $\lim_{n\to\infty} \dfrac{T_{\mathcal{G}}(\mathbf{0}, n\hat{\mathbf{x}})}{n} = \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})$, almost surely and in L^1.

b) If $\alpha \geq 1$ and $\mathbb{E}[Y^\alpha] < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{P}\Big(\big|T_{\mathcal{G}}(\mathbf{0}, n\hat{\mathbf{x}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\big| > n\epsilon\Big) < \infty, \quad \textit{for any } \epsilon > 0.$$

Two comments. We will not explicitly prove that $\mathbb{E}[Y] < \infty$ is sufficient for the almost sure convergence to hold, but rather remark at the end of the proof on how this can be obtained. That $\mathbb{E}[Y^2] < \infty$ is sufficient follows from part *b)* and Borel-Cantelli's lemma.

In case $\hat{\mathbf{x}} \in \mathbb{U}^{d-1}$, we can easily bound the passage time on the cone between the passage time on the full lattice and the passage time on a tube. However, in case $\hat{\mathbf{x}} \notin \mathbb{U}^{d-1}$, we do not know that we have convergence on the tube. Therefore we will have to make use of a shifting trick.

Proof. Let $\hat{\mathbf{x}} \in \mathbb{S}^{d-1}$ and $\epsilon > 0$. Next, choose $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$ such that both $|\hat{\mathbf{z}} - \hat{\mathbf{x}}| \leq \epsilon$ and $\big|\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\big| \leq \epsilon/3$, choose $K \geq R_d + 1$ such that $\mu_K(\hat{\mathbf{z}}) \leq \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) + \epsilon/3$, and choose n such that $n|\hat{\mathbf{z}} - \hat{\mathbf{x}}| > 2K + 3\sqrt{d}/2$ and $\big|\mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big] - n\mu_K(\hat{\mathbf{z}})\big| \leq n\epsilon/3$. This is possible according to (2.3), and Proposition 2.9 and 2.8, in that order. In particular, for the $\hat{\mathbf{z}}$, K and n chosen,

$$\Big|\mathbb{E}\big[\tilde{T}_K(\mathbf{0}, n\hat{\mathbf{z}})\big] - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\Big| \leq n\epsilon. \tag{4.1}$$

We will need to bound $T_{\mathcal{G}}(\mathbf{0}, m\hat{\mathbf{x}})$ from below and above by known entities. In order to obtain an upper bound, choose an integer $M = M(K, n)$ such that $\omega(x) \geq K + n + \sqrt{d}/2$ for all $x \geq M$. Let $\mathbf{x}_n = (n\hat{\mathbf{x}})^*$, i.e., the point in $\mathbb{Z}^d$ closest to $n\hat{\mathbf{x}}$. By the choice of M, it is clear that for $m \geq M$, the subgraph induced by $\bigcup_{a\in[0,n]} B(a\hat{\mathbf{z}}+\mathbf{x}_m, K)$ is a subgraph of $\mathcal{G}$. Therefore, together with subadditivity, it is clear that

$$\begin{aligned} T_{\mathcal{G}}\big(\mathbf{0}, (M+kn)\hat{\mathbf{x}}\big) \;\leq\; & T_{\mathcal{G}}(\mathbf{0}, \mathbf{x}_M) + \sum_{j=0}^{k-1} \tilde{T}_K(\mathbf{x}_{M+jn}, \mathbf{x}_{M+jn} + n\hat{\mathbf{z}}) \\ & + \sum_{j=0}^{k-1} \tilde{T}_K(\mathbf{x}_{M+jn} + n\hat{\mathbf{z}}, \mathbf{x}_{M+(j+1)n}). \end{aligned} \tag{4.2}$$

Recall that $|\mathbf{x} - \mathbf{x}^*| < \sqrt{d}$. Since

$$\big|(\mathbf{x}_{M+jn} + n\hat{\mathbf{z}})^* - \mathbf{x}_{M+(j+1)n}\big| \;\geq\; n|\hat{\mathbf{z}} - \hat{\mathbf{x}}| - 3\sqrt{d}/2 \;>\; 2K,$$

the summands in each of the two sums in the above expression are independent. A lower bound on $T_{\mathcal{G}}\big(\mathbf{0}, (M+kn)\hat{\mathbf{x}}\big)$ is obtained from $T\big(\mathbf{0}, (M+kn)\hat{\mathbf{x}}\big)$. Now, given $m \geq M$, set $m = M + k_m n + a_m$ for integers $k_m \geq 0$ and $a_m \in [0, n)$. The triangle inequality and subadditivity gives

$$\begin{aligned} \left|\frac{T_{\mathcal{G}}(\mathbf{0}, m\hat{\mathbf{x}})}{m} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| \;\leq\; & \left|\frac{T(\mathbf{0}, m\hat{\mathbf{x}})}{m} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| + \frac{T_{\mathcal{G}}(\mathbf{0}, m\hat{\mathbf{x}}) - T(\mathbf{0}, m\hat{\mathbf{x}})}{m} \\ \leq\; & 2\left|\frac{T(\mathbf{0}, m\hat{\mathbf{x}})}{m} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| + \frac{\tilde{T}_K\big((M+k_m n)\hat{\mathbf{x}}, m\hat{\mathbf{x}}\big)}{m} \\ & + \left(\frac{T_{\mathcal{G}}\big(\mathbf{0}, (M+k_m n)\hat{\mathbf{x}}\big)}{m} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right). \end{aligned} \tag{4.3}$$

When (4.2) is substituted into (4.3), we obtain

$$\left|\frac{T_{\mathcal{G}}(\mathbf{0},m\hat{\mathbf{x}})}{m}-\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| \;\le\; 2\left|\frac{T(\mathbf{0},m\hat{\mathbf{x}})}{m}-\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right|+\frac{\tilde{T}_K\big((M+k_mn)\hat{\mathbf{x}},m\hat{\mathbf{x}}\big)}{m}$$
$$+\;\frac{T_{\mathcal{G}}(\mathbf{0},\mathbf{x}_M)}{m}+\left(\frac{1}{k_m}\sum_{j=0}^{k_m-1}\frac{\tilde{T}_K(\mathbf{x}_{M+jn},\mathbf{x}_{M+jn}+n\hat{\mathbf{z}})}{n}-\frac{\mathbb{E}\big[\tilde{T}_K(\mathbf{0},n\hat{\mathbf{z}})\big]}{n}\right)$$
$$+\;\left(\frac{\mathbb{E}\big[\tilde{T}_K(\mathbf{0},n\hat{\mathbf{z}})\big]}{n}-\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right)+\frac{1}{k_m}\sum_{j=0}^{k_m-1}\frac{\tilde{T}_K(\mathbf{x}_{M+jn}+n\hat{\mathbf{z}},\mathbf{x}_{M+(j+1)n})}{n}. \tag{4.4}$$

Denote the terms in the right-hand side of (4.4) by $X_1, X_2, \ldots, X_6$.

Part *a)*. Assume that $\mathbb{E}[Y]<\infty$, and let m tend to infinity. Proposition 3.3 tells us that $\mathbb{E}[X_1]\to 0$. Since

$$|\mathbf{x}_m-\mathbf{x}_{M+k_mn}| \;\le\; \sqrt{d}+\big|m-(M+k_mn)\big| \;<\; \sqrt{d}+n$$

for $m\ge M$, and $\mathbb{E}\big[\tilde{T}_K(\mathbf{0},\mathbf{z})\big]<\infty$ when $|\mathbf{z}|<\sqrt{d}+n$, it is clear that $\mathbb{E}[X_2]$ vanishes as $m\to\infty$. Since $\mathbb{E}\big[T_{\mathcal{G}}(\mathbf{0},\mathbf{x}_M)\big]<\infty$, also $\mathbb{E}[X_3]$ vanishes. The fourth term is the average of k_m i.i.d. random variables, minus their mean. Its mean is therefore zero. The only terms that do not vanish as $m\to 0$, are the last two. The fifth term is constant, and at most ϵ, due to (4.1). The final term X_6 is an average of independent random variables, each of which is distributed according to one of a finite number of possible distributions. Since

$$\big\|(\mathbf{x}_{M+jn}+n\hat{\mathbf{z}})^*-\mathbf{x}_{M+(j+1)n}\big\|_1 \;\le\; d\big(3\sqrt{d}/2+n|\hat{\mathbf{z}}-\hat{\mathbf{x}}|\big) \;\le\; 2dn\epsilon,$$

due to the choice of n, we obtain for each $j\ge 0$ that

$$\frac{1}{n}\mathbb{E}\Big[\tilde{T}_K\big(\mathbf{x}_{M+jn}+n\hat{\mathbf{z}},\mathbf{x}_{M+(j+1)n}\big)\Big] \;\le\; 2d\mathbb{E}\big[\tilde{T}_{K-1}(\mathbf{0},\mathbf{e}_1)\big]\epsilon. \tag{4.5}$$

Hence, we conclude that

$$\lim_{k\to\infty}\mathbb{E}\left|\frac{T_{\mathcal{G}}(\mathbf{0},m\hat{\mathbf{x}})}{m}-\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| \;\le\; \Big(2d\mathbb{E}\big[\tilde{T}_{K-1}(\mathbf{0},\mathbf{e}_1)\big]+1\Big)\epsilon.$$

Since $\epsilon>0$ was arbitrary, this proves the L^1-convergence in part *a)*.

Part *b)*. Assume that $\mathbb{E}[Y^\alpha]<\infty$. We aim to prove that for each $j=1,2,\ldots,6$,

$$\sum_{n=1}^{\infty}n^{\alpha-2}\mathbb{P}(X_j>\epsilon)<\infty. \tag{4.6}$$

For $j=1$, this follows from Proposition 1.3 and 1.4. Since $\mathbb{E}\big[\tilde{T}_K(0,\mathbf{z})^\alpha\big] < \infty$ for any $\mathbf{z}\in\mathbb{Z}^d$, and $\tilde{T}_K\big((M+k_m n)\hat{\mathbf{x}}, m\hat{\mathbf{x}}\big)$ is distributed according to one of finitely many possible distributions (as m varies), it follows from (2.5) that (4.6) holds for $j=2$. For $j=3$, (4.6) follows also from (2.5). For $j=4$ the same follows from Theorem 3.1, since the terms in X_4 are i.i.d. The term X_5 is again constant, so $j=5$ is fine. For $j=6$ we do not quite have (4.6). However, since the terms in X_6 are i.i.d. with finite moment of order α, and can be dominated by a random variable with mean at most $2d\mathbb{E}\big[\tilde{T}_{K-1}(\mathbf{0},\mathbf{e}_1)\big]\epsilon$, as in (4.5), we have

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{P}\Big(X_6 > \Big(2d\mathbb{E}\big[\tilde{T}_{K-1}(\mathbf{0},\mathbf{e}_1)\big]+1\Big)\epsilon\Big) < \infty.$$

We conclude that

$$\sum_{m=1}^{\infty} n^{\alpha-2}\mathbb{P}\left(\left|\frac{T_{\mathcal{G}}(\mathbf{0},m\hat{\mathbf{x}})}{m} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| > \Big(2d\mathbb{E}\big[\tilde{T}_{K-1}(\mathbf{0},\mathbf{e}_1)\big]+6\Big)\epsilon\right) < \infty.$$

Since $\epsilon>0$ was arbitrary, the proof is complete. □

Remark 4.2. Here comes an explanation of how to see that $\mathbb{E}[Y]<\infty$ is sufficient for the almost sure convergence in Proposition 4.1. It is easily deduced from (1.4) (or Proposition 1.4) that $\liminf_{n\to\infty} T_{\mathcal{G}}(\mathbf{0},n\hat{\mathbf{x}})/n \geq \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})$ almost surely. The reason we cannot draw the conclusion we wish directly from (4.4) is that we have not yet proved that $\mathbb{E}[Y]<\infty$ is sufficient for convergence of $T(\mathbf{0},m\hat{\mathbf{x}})/m$. However, we do not need to know that in order to show, when $\mathbb{E}[Y]<\infty$, that $\limsup_{n\to\infty} T_{\mathcal{G}}(\mathbf{0},n\hat{\mathbf{x}})/n \leq \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})$. To see this, bound $T_{\mathcal{G}}(\mathbf{0},m\hat{\mathbf{x}})$ from above as in (4.2), and obtain an upper bound similar to (4.4), that does not include $T(\mathbf{0},m\hat{\mathbf{x}})$. Of course, from this it follows that also $\lim_{m\to\infty} T(\mathbf{0},m\hat{\mathbf{x}})/m$ exists when $\mathbb{E}[Y]<\infty$. □

5 Dynamical stability of radial convergence

In this section we consider dynamical first-passage percolation. The key in order to understand how dynamical first-passage times behaves, will be to compare them to non-dynamical first-passage times. We will start to introduce some notation that will be in force for the rest of this section. Recall that $\{\tau_e(s)\}_{e\in\mathcal{E}_{\mathbb{Z}^d}}$ denotes the i.i.d. family of dynamical passage times associated with the $\mathbb{Z}^d$ lattice. Let $N_e=N_e(\delta)$ denote the number or updates of $\tau_e(s)$ on the interval $[0,\delta]$. Clearly $N_e\sim\text{Poisson}(\delta)$ and $\mathbb{E}[N_e]=\delta$. Define

$$\bar{\tau}_e := \tau_e(0)\cdot 1_{\{N_e(\delta)=0\}}, \quad \text{and} \quad \hat{\tau}_e := \sup_{s\in[0,\delta]} \tau_e(s).$$

Let $\bar{T}_{\mathcal{G}}(\mathbf{x},\mathbf{y})$ denote the passage time between $\mathbf{x}$ and $\mathbf{y}$ on $\mathcal{G}$ with respect to $\{\bar{\tau}_e\}_{e\in\mathcal{E}_{\mathbb{Z}^d}}$, and let $\hat{T}_{\mathcal{G}}(\mathbf{x},\mathbf{y})$ denote the passage time between $\mathbf{x}$ and $\mathbf{y}$ on $\mathcal{G}$ with respect to $\{\hat{\tau}_e\}_{e\in\mathcal{E}_{\mathbb{Z}^d}}$. Clearly, for any $\delta \geq 0$, and $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$,

$$\bar{T}_{\mathcal{G}}(\mathbf{x},\mathbf{y}) \;\leq\; T_{\mathcal{G}}^{(s)}(\mathbf{x},\mathbf{y}) \;\leq\; \hat{T}_{\mathcal{G}}(\mathbf{x},\mathbf{y}), \quad \text{for all } s\in[0,\delta].$$

The first thing we do will be to compare moments.

Lemma 5.1. *For $\delta \geq 0$, $\alpha \geq 0$ and $q \geq 1$*

$$\mathbb{E}\left[\min\left(\sup_{s\in[0,\delta]}\tau_1(s),\ldots,\sup_{s\in[0,\delta]}\tau_q(s)\right)^{\alpha}\right] \;\leq\; (1+\delta)^q\mathbb{E}\Big[\min\big(\tau_1(0),\ldots,\tau_q(0)\big)^{\alpha}\Big],$$

where $\tau_1(s),\tau_2(s),\ldots,\tau_q(s)$ are i.i.d. and distributed as $\tau_e(s)$.

Proof. Since $\mathbb{E}[X^\alpha] = \int_0^\infty n^{\alpha-1}\mathbb{P}(X > x)\,dx$ and $\mathbb{P}\big(\min(X_1,\ldots,X_m) > x\big) = \mathbb{P}(X_1 > x)^m$ for i.i.d. non-negative random variables $X, X_1,\ldots,X_m$, the result follows from the following observation.

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\in[0,\delta]}\tau_e(s) > x\right) &= \sum_{k=0}^{\infty}\mathbb{P}\left(\max_{j=1,\ldots,k+1}\tau_e^{(j)} > x \,\middle|\, N_e = k\right)\mathbb{P}(N_e = k)\\
&\leq \sum_{k=0}^{\infty}(k+1)\mathbb{P}(\tau_e > x)\mathbb{P}(N_e = k)\\
&= \mathbb{P}(\tau_e > x)\mathbb{E}[1+N_e]. \qquad \square
\end{aligned}$$

The first step in order to prove a dynamically stable version of the Shape Theorem is to show that the almost sure convergence in radial directions is dynamically stable.

Proposition 5.2. *Let $\hat{\mathbf{x}}\in\mathbb{S}^{d-1}$, and $\omega : [0,\infty)\to[0,\infty)$ be any function such that $\omega(a)\to\infty$ as $a\to\infty$. Let $\mathcal{G}$ denote the subgraph of the $\mathbb{Z}^d$ lattice induced by the set $\bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}},\omega(a)+R_d\big)$. If $\mathbb{E}[Y]<\infty$, then, almost surely, for every $\epsilon > 0$, there exists an $M = M(\epsilon) < \infty$ such that*

$$\left|\frac{T_{\mathcal{G}}^{(s)}(\mathbf{0},n\hat{\mathbf{x}})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| < \epsilon, \quad \textit{for all } s\in[0,1] \textit{ and } n\geq M.$$

The proof of this proposition is heavily inspired by the proof of the dynamical version of the Law of Large Numbers due to Benjamini et al. (2003).

Proof. Fix $\epsilon > 0$. Let $\hat{Y}$ denote the minimum of $2d$ independent variables distributed as $\hat{\tau}_e$. According to Lemma 5.1, $\mathbb{E}[Y] < \infty$ implies $\mathbb{E}[\hat{Y}] < \infty$, so the time constants $\bar{\mu}_{\mathbb{Z}^d}(\hat{\mathbf{x}})$ and $\hat{\mu}_{\mathbb{Z}^d}(\hat{\mathbf{x}})$ defined as limits of first-passage times with respect to $\{\bar{\tau}_e\}_{e\in\mathbb{E}_{\mathbb{Z}^d}}$ and $\{\hat{\tau}_e\}_{e\in\mathbb{E}_{\mathbb{Z}^d}}$, respectively, exist and are finite. As δ tends to zero, the distributions of $\bar{\tau}_e = \bar{\tau}_e(\delta)$ and $\hat{\tau}_e = \hat{\tau}_e(\delta)$ converges weakly to the distribution of τ_e. Hence, by Proposition 2.1 we can choose $\delta > 0$ such that

$$\left|\hat{\mu}_{\mathbb{Z}^d}(\hat{\mathbf{x}}) - \bar{\mu}_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| < \epsilon/2.$$

According to Proposition 4.1 we can, almost surely, find $M = M(\epsilon, \delta) < \infty$ such that for all $n \geq M$

$$\left|\frac{\bar{T}_{\mathcal{G}}(\mathbf{0}, n\hat{\mathbf{x}})}{n} - \bar{\mu}_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| < \frac{\epsilon}{2}, \quad \text{and} \quad \left|\frac{\hat{T}_{\mathcal{G}}(\mathbf{0}, n\hat{\mathbf{x}})}{n} - \hat{\mu}_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| < \frac{\epsilon}{2}.$$

We conclude that

$$\left|\frac{T_{\mathcal{G}}^{(s)}(\mathbf{0}, n\hat{\mathbf{x}})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\right| < \epsilon, \quad \text{for all } s \in [0, \delta] \text{ and } n \geq M.$$

The result is now obtained by covering $[0, 1]$ with finitely many intervals of length δ. □

Similarly, one may derive a dynamically stable version of Proposition 2.8.

Proposition 5.3. *For $K \geq R_d$, $\hat{\mathbf{z}} \in \mathbb{U}^{d-1}$ and $\mathbb{E}[Y] < \infty$, almost surely, for every $\epsilon > 0$, there exists an $M = M(\epsilon) < \infty$ such that*

$$\left|\frac{T_{K,\hat{\mathbf{z}}}^{(s)}(\mathbf{0}, n\hat{\mathbf{z}})}{n} - \mu_K(\hat{\mathbf{z}})\right| < \epsilon, \quad \textit{for all } s \in [0, 1] \textit{ and } n \geq M.$$

Proof. The proof is analogous to the proof of Proposition 5.2. The reader may see that it goes through smoothly replacing the reference to Proposition 4.1 with Proposition 2.8, and Proposition 2.1 with its analogue for $(K, d, \hat{\mathbf{z}})$-tubes, which can be found in Ahlberg (2010). □

6 Extending the Shape Theorem to cones

The proofs of Theorem 1.2 and 1.6 will naturally follow the proof of the Shape Theorem closely. We follow the approach given in Howard (2004). An essential step in the proof of the Shape Theorem, and subsequently in order to prove Theorem 1.2 and 1.6, is to show that

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}\big(T(\mathbf{0},\mathbf{z}) \geq M|\mathbf{z}|\big) < \infty, \quad \text{for some } M > 0. \tag{6.1}$$

Under the assumption of finite exponential moment, that is $\mathbb{E}[e^{\theta\tau_e}] < \infty$ for some $\theta > 0$, this follows by standard large deviation estimates for i.i.d. sequences. Cox and Durrett (1981) showed that

$$\mathbb{E}[Y^d] < \infty \quad \Leftrightarrow \quad \sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}\big(T(\mathbf{0},\mathbf{z}) \geq M|\mathbf{z}|\big) < \infty, \quad \text{for some } M > 0. \tag{6.2}$$

Let $Y(\mathbf{z})$ denote the minimum over the passage times of the $2d$ edges connected to $\mathbf{z}$. When $\mathbb{E}[Y^d] = \infty$, in fact $T(\mathbf{0},\mathbf{z}) \geq Y(\mathbf{z})$ implies

$$\sum_{\mathbf{z}\in(2\mathbb{Z})^d} \mathbb{P}\big(T(\mathbf{0},\mathbf{z}) \geq M|\mathbf{z}|\big) \;\geq\; \sum_{\mathbf{z}\in(2\mathbb{Z})^d} \mathbb{P}\big(Y(\mathbf{z}) \geq M|\mathbf{z}|\big) \;=\; \infty$$

for any $M < \infty$. This proves one implication. Moreover, the Borel-Cantelli lemma gives that $T(\mathbf{0},\mathbf{z})/|\mathbf{z}| > M$ for infinitely many $\mathbf{z} \in \mathbb{Z}^d$. This shows that $\mathbb{E}[Y^d] < \infty$ is necessary also for (1.3) to hold.

The remaining direction of (6.2) requires more work. The proof can be simplified if we make the somewhat stronger assumption $\mathbb{E}[\tau_e^2] < \infty$ (as in Kesten (1986)).

Lemma 6.1. *Let $\mathcal{G}$ be a graph, and let $\mathbf{x}$ and $\mathbf{y}$ be two vertices connected with q disjoint paths, each of length at most λ. If $\mathbb{E}[\tau_e^2] < \infty$, then*

$$\mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{x},\mathbf{y}) \geq \big(\mathbb{E}[\tau_e] + 1\big)\lambda\Big) \;\leq\; \left(\frac{\mathbb{E}[\tau_e^2]}{\lambda}\right)^q.$$

(Since there on the $\mathbb{Z}^d$ lattice are $2d$ disjoint paths from $\mathbf{0}$ to $\mathbf{z}$ of length of order $\|\mathbf{z}\|_1$, Lemma 6.1 implies that $\mathbb{E}[\tau_e^2] < \infty$ is sufficient for (6.1) to hold.)

Proof. Denote the disjoint paths by $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$. Note that

$$\mathbb{E}\big[T(\Gamma_j)\big] \;\leq\; \mathbb{E}[\tau_e]\lambda, \quad \text{and} \quad \mathrm{Var}\,\big(T(\Gamma_j)\big) \;\leq\; \mathrm{Var}(\tau_e)\lambda \;\leq\; \mathbb{E}[\tau_e^2]\lambda$$

for all $j = 1, \ldots, q$. Since $T_{\mathcal{G}}(\mathbf{x},\mathbf{y}) \leq T(\Gamma_j)$ for each j, we have

$$\begin{aligned}\mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{x},\mathbf{y}) \geq \big(\mathbb{E}[\tau_e] + 1\big)\lambda\Big) \;&\leq\; \prod_{j=1}^{q} \mathbb{P}\Big(T(\Gamma_j) \geq \big(\mathbb{E}[\tau_e] + 1\big)\lambda\Big)\\ &\leq \left(\frac{\mathbb{E}[\tau_e^2]\lambda}{\lambda^2}\right)^q,\end{aligned}$$

where we in the last step have applied Chebyshev's inequality. $\square$

We will prove part *b)* and *c)* of Theorem 1.2, and Theorem 1.6, under the additional assumption that $\mathbb{E}[\tau_e^2] < \infty$. The stronger assumption will be used in order to derive Lemma 6.2 from Lemma 6.1. The main ideas are still present, and the remaining piece needed to obtain Lemma 6.2 with the relaxed condition $\mathbb{E}[Y^d] < \infty$ is indicated in Remark 6.4.

6.1 Proof of Theorem 1.2

Let $S_{\mathcal{G}} := \{\hat{\mathbf{u}} \in \mathbb{S}^{d-1} : (a\hat{\mathbf{u}})^* \in \mathcal{V}_{\mathcal{G}} \text{ for all } a \text{ large enough}\}$. Clearly $\hat{\mathbf{x}} \in S_{\mathcal{G}}$, and if $\lim_{a\to\infty} \omega(a)/a = 0$, then $S_{\mathcal{G}} = \{\hat{\mathbf{x}}\}$. Fix $\epsilon \in \big(0, \frac{1}{3\sqrt{d}}\big)$, and choose $\hat{\mathbf{u}}^{(1)}, \ldots, \hat{\mathbf{u}}^{(m)} \in S_{\mathcal{G}}$ such that for some $M_1 < \infty$

$$\bigcup_{j=1}^{m} \bigcup_{a\geq 0} B\big(a\hat{\mathbf{u}}^{(j)}, \epsilon a\big) \supseteq \bigcup_{a\geq M_1} B\big(a\hat{\mathbf{x}}, \omega(a) + R_d\big).$$

If $S_{\mathcal{G}} = \{\hat{\mathbf{x}}\}$, then $m = 1$, $\hat{\mathbf{u}}^{(1)} = \hat{\mathbf{x}}$, and we may directly apply Proposition 4.1 to obtain desired convergence of $\big|T_{\mathcal{G}}(\mathbf{0}, n\hat{\mathbf{x}}) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{x}})\big|$. If instead $S_{\mathcal{G}} \neq \{\hat{\mathbf{x}}\}$, then we can choose $\hat{\mathbf{u}}^{(1)}, \ldots, \hat{\mathbf{u}}^{(m)} \in S_{\mathcal{G}} \cap \mathbb{U}^{d-1}$ such that they are interior points in $S_{\mathcal{G}}$ (seen as a subset of $\mathbb{S}^{d-1}$). Then there are $\delta > 0$ and $M_2 = M_2(\delta) < \infty$ such that

$$\bigcup_{a\geq M_2} B\big(a\hat{\mathbf{u}}^{(j)}, \delta a + R_d\big) \subseteq \bigcup_{a\geq 0} B\big(a\hat{\mathbf{x}}, \omega(a) + R_d\big), \quad \text{for each } j = 1, \ldots, m.$$

Choose $a_j \geq M_2$ such that $a_j\hat{\mathbf{u}}^{(j)} \in \mathbb{Z}^d$. In particular,

$$\Big|T_{\mathcal{G}}\big(\mathbf{0}, n\hat{\mathbf{u}}^{(j)}\big) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)})\Big| \leq T_{\mathcal{G}}\big(\mathbf{0}, a_j\hat{\mathbf{u}}^{(j)}\big) + \Big|T_{\mathcal{G}}\big(a_j\hat{\mathbf{u}}^{(j)}, n\hat{\mathbf{u}}^{(j)}\big) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)})\Big|.$$

We conclude from Proposition 4.1 that when $\mathbb{E}[Y] < \infty$, for each $j = 1, 2, \ldots, m$,

$$\lim_{n\to\infty} \frac{T_{\mathcal{G}}\big(\mathbf{0}, n\hat{\mathbf{u}}^{(j)}\big)}{n} = \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)}), \quad \text{almost surely and in } L^1. \tag{6.3}$$

Moreover, when $\mathbb{E}[Y^\alpha] < \infty$, for some $\alpha \geq 1$, we obtain for each $j = 1, 2, \ldots, m$

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{P}\Big(\big|T_{\mathcal{G}}\big(\mathbf{0}, n\hat{\mathbf{u}}^{(j)}\big) - n\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)})\big| > n\epsilon\Big) < \infty. \tag{6.4}$$

Set $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$. For $\hat{\mathbf{z}}$ close to $\hat{\mathbf{u}}^{(j)}$, we will compare $T_{\mathcal{G}}(\mathbf{0}, \mathbf{z})$ with $T_{\mathcal{G}}\big(\mathbf{0}, a\hat{\mathbf{u}}^{(j)}\big)$, for a suitably chosen. Define

$$H_n := \left\{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{y}\|_\infty := \max_{j=1,2,\ldots,d} |y_j| = n\right\}.$$

Given $\mathbf{z} \in H_n$, let $j(\mathbf{z})$ denote that of the indices $j = 1, 2, \ldots, m$ that minimizes $\left|\hat{\mathbf{z}} - \hat{\mathbf{u}}^{(j)}\right|$. With a slight abuse of notation we write $\hat{\mathbf{u}}^{(\mathbf{z})}$ for $\hat{\mathbf{u}}^{(j(\mathbf{z}))}$. Denote by $\mathbf{u}_{\mathbf{z}}$ the point in $H_n \cap \mathbb{Z}^d$ closest to the unique point in $H_n \cap \{a\hat{\mathbf{u}}^{(\mathbf{z})}\}_{a \geq 0}$.

Claim 1. *If* $\epsilon \in \left(0, \frac{1}{3\sqrt{d}}\right)$ *and* $\mathbf{z} \in H_n$, *then*

$$|\mathbf{u}_{\mathbf{z}} - \mathbf{z}| \;\leq\; 8dn\epsilon + \sqrt{d} \;\leq\; 8d|\mathbf{z}|\epsilon + \sqrt{d}.$$

Proof of Claim 1. This claim is easily proved with trigonometry, observing that the angle α at which the line $\{a\hat{\mathbf{u}}^{(\mathbf{z})}\}_{a \geq 0}$ intersects H_n satisfies $\tan(\alpha) \geq \sqrt{d}$, the extremal case being $\hat{\mathbf{u}}^{(\mathbf{z})} = \frac{1}{\sqrt{d}}(1, \ldots, 1)$. The details are left to the reader. □

By subadditivity

$$\begin{aligned}
\Big|T_{\mathcal{G}}(\mathbf{0}, \mathbf{z}) - |\mathbf{z}|\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\Big| \;\leq\; & \Big|T_{\mathcal{G}}(\mathbf{0}, \mathbf{z}) - T_{\mathcal{G}}(\mathbf{0}, \mathbf{u}_{\mathbf{z}})\Big| \\
&+ \Big|T_{\mathcal{G}}(\mathbf{0}, \mathbf{u}_{\mathbf{z}}) - |\mathbf{u}_{\mathbf{z}}|\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})\Big| \\
&+ \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})\Big||\mathbf{u}_{\mathbf{z}}| - |\mathbf{z}|\Big| \\
&+ |\mathbf{z}|\Big|\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\Big|.
\end{aligned} \tag{6.5}$$

The latter two terms in the right-hand side of (6.5) are non-random. Claim 1 gives

$$\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})\Big||\mathbf{u}_{\mathbf{z}}| - |\mathbf{z}|\Big| \;\leq\; \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})|\mathbf{u}_{\mathbf{z}} - \mathbf{z}| \;\leq\; \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})\big(8d|\mathbf{z}|\epsilon + \sqrt{d}\big), \tag{6.6}$$

and, by (2.3), $|\mathbf{z}|\left|\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\right|$ is bounded from above by

$$\begin{aligned}
|\mathbf{z}|d\,\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]|\hat{\mathbf{u}}^{(\mathbf{z})} - \hat{\mathbf{z}}| \;&\leq\; |\mathbf{z}|d\,\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]\epsilon/\sqrt{1 - \epsilon^2} \\
&\leq\; 2d\,\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big]\,|\mathbf{z}|\epsilon.
\end{aligned} \tag{6.7}$$

The first term in the right-hand side of (6.5) is, again by subadditivity, at most

$$\Big|T_{\mathcal{G}}(\mathbf{0}, \mathbf{z}) - T_{\mathcal{G}}(\mathbf{0}, \mathbf{u}_{\mathbf{z}})\Big| \;\leq\; T_{\mathcal{G}}(\mathbf{u}_{\mathbf{z}}, \mathbf{z}).$$

Proof of part *a)*. Assume that $\mathbb{E}[\tau_e] < \infty$. Since $\|\mathbf{z}\|_1 \leq d|\mathbf{z}|$, we conclude via Claim 1 that

$$\mathbb{E}\big[T_{\mathcal{G}}(\mathbf{u}_{\mathbf{z}}, \mathbf{z})\big] \;\leq\; \mathbb{E}[\tau_e]\,\|\mathbf{u}_{\mathbf{z}} - \mathbf{z}\|_1 \;\leq\; d\,\mathbb{E}[\tau_e]\,\big(8d|\mathbf{z}|\epsilon + \sqrt{d}\big).$$

When $\mathbb{E}[\tau_e] < \infty$, also $\mathbb{E}[Y] < \infty$. So, (6.3) assures that when $|\mathbf{z}|$, and then also $|\mathbf{u}_{\mathbf{z}}|$, is sufficiently large, then $\mathbb{E}\Big|T_{\mathcal{G}}(\mathbf{0}, \mathbf{u}_{\mathbf{z}}) - |\mathbf{u}_{\mathbf{z}}|\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})\Big| \leq |\mathbf{u}_{\mathbf{z}}|\epsilon$, which in turn is at most $(1+8d\epsilon)|\mathbf{z}|\epsilon + \sqrt{d}\epsilon$. Finally, together with (6.6) and (6.7), we have obtained that

$$\begin{aligned}
&\mathbb{E}\Big|T_{\mathcal{G}}(\mathbf{0},\mathbf{z}) - |\mathbf{z}|\mu_{\mathbb{Z}^d}(\hat{\mathbf{z}})\Big| \\
&\quad\leq \Big(8d^2\,\mathbb{E}[\tau_e] + (1+8d\epsilon) + 8d\,\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) + 2d\,\mathbb{E}\big[T(\mathbf{0},\mathbf{e}_1)\big]\Big)|\mathbf{z}|\epsilon \\
&\qquad + \big(d\,\mathbb{E}[\tau_e] + \epsilon + \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})})\big)\sqrt{d},
\end{aligned}$$

for $|\mathbf{z}|$ large enough. Since $\epsilon > 0$ was arbitrary, this proves *a)*. $\square$

In order to prove either of the remaining two parts, *b)* and *c)*, the pending step is to prove a variant of (6.1). With the approach we have chosen, we will prove the following.

Lemma 6.2. *Let the notation above be in force and assume that $\mathbb{E}[\tau_e^2] < \infty$. There exists $M < \infty$, that does not depend on ϵ, such that*

$$\sum_{\mathbf{z}\in\mathcal{V}_{\mathcal{G}}} \mathbb{P}\big(T_{\mathcal{G}}(\mathbf{u}_{\mathbf{z}},\mathbf{z}) \geq M|\mathbf{z}|\epsilon\big) < \infty.$$

Since not every vertex of $\mathcal{G}$ has the same degree, we will need the following classification.

Claim 2. *There is a partition $D_0, D_1, \ldots, D_{d-1}$ of the vertices in $\mathcal{V}_{\mathcal{G}}$, such that for some $C < \infty$,*

$$\big|D_q \cap H_n\big| \leq \begin{cases} Cn^{q-1}, & \textit{for } q = 1,2,\ldots,d-1, \\ Cn^{d-1}, & \textit{for } q = 0. \end{cases}$$

and if $\mathbf{z} \in D_q \cap H_n$ for some $n, q \geq 1$, then there is a $\mathbf{v}_{\mathbf{z}} \in D_0 \cap H_n$ such that $\|\mathbf{v}_{\mathbf{z}} - \mathbf{z}\|_1 \leq C$, and there exist (at least) q disjoint paths between $\mathbf{z}$ and $\mathbf{v}_{\mathbf{z}}$ of length $\|\mathbf{v}_{\mathbf{z}} - \mathbf{z}\|_1$. For $\mathbf{z} \in D_0$ there are $2d$ disjoint paths from $\mathbf{z}$ to $\mathbf{u}_{\mathbf{z}}$ of length at most $C\|\mathbf{u}_{\mathbf{z}} - \mathbf{z}\|_1$.

Remark 6.3. In most cases it is possible to choose D_q so $\big|D_q \cap H_n\big| \leq Cn^{q-2}$ for $q \geq 2$, and $D_1 = \emptyset$. (When $\hat{\mathbf{x}} = \mathbf{e}_1$ this is always possible.) However, there are cases when this is not possible. For example, when $d = 3$ and $\hat{\mathbf{x}} = \frac{1}{\sqrt{3}}(1,1,1)$, it is possible to choose ω such that the intersection of $\mathcal{V}_{\mathcal{G}}$ with $\{-R_d\} \times \mathbb{Z}^2$ equals the points along the line $\big\{a(0,1,1) + (-R_d,0,0)\big\}_{a\geq 0}$. These vertices will only have one neighbour. $\square$

Proof of Claim 2. We will prove the case when $\omega(a) = c \cdot a$ and $\hat{\mathbf{x}} = \mathbf{e}_1$ only, and leave it to the reader to verify that the proof extends to the remaining cases. There are three cases: $c > 1$, $c = 1$ and $c < 1$. In the first case set $D_0 := \mathcal{V}_\mathcal{G} = \mathbb{Z}^d$. In the second case set $D_0 := \mathcal{V}_\mathcal{G} = \{0, 1, \ldots\} \times \mathbb{Z}^{d-1}$ and $D_{d-1} := \{-R_d, -R_d + 1, \ldots, -1\} \times \mathbb{Z}^{d-1}$. Both these cases are easy, but the final case is more tricky. In that, let

$$D_0 := \big\{\mathbf{z} \in \mathcal{V}_\mathcal{G} : B(\mathbf{z}, R_d) \subseteq \mathcal{V}_\mathcal{G}\big\} = \bigcup_{a \geq 0} B(a\mathbf{e}_1, c \cdot a). \tag{6.8}$$

(Here and below we identify subsets of $\mathbb{R}^d$ with its restriction to $\mathbb{Z}^d$.) Since H_n is a $(d-1)$-dimensional subset of $\mathbb{R}^d$, then there is an $C_1 < \infty$ such that

$$|D_0 \cap H_n| \;\leq\; |H_n| \;\leq\; C_1 n^{d-1}, \quad \text{and} \quad \big|(\mathcal{V}_\mathcal{G} \setminus D_0) \cap H_n\big| \;\leq\; C_1 n^{d-2}.$$

For $q = 1, 2, \ldots, d-1$ we define

$$D_q := \big\{\mathbf{z} \in \mathcal{V}_\mathcal{G} \setminus D_0 :\; z_j \neq 0 \text{ for } q \text{ indices } j \geq 2\big\}.$$

Since $\mathcal{V}_\mathcal{G} \setminus D_0 = \big(\bigcup_{a\geq 0} B(a\mathbf{e}_1, c\cdot a + R_d)\big) \setminus \big(\bigcup_{a\geq 0} B(a\mathbf{e}_1, c\cdot a)\big)$, it is clear that fixation of $z_j = 0$ for some $j \geq 2$ in $\mathcal{V}_\mathcal{G} \setminus D_0$ reduces the degree of freedom (in the choice of $\mathbf{z}$) by one, in the sense that going from D_{q+1} to D_q we loose one dimension. Hence, there is a $C_2 < \infty$

$$|D_q \cap H_n| \leq C_2 n^{q-1}, \quad \text{for each } q = 1, 2, \ldots, d-1.$$

Move on to the second part of the statement. Take $\mathbf{z} \in D_q \cap H_n$. Due to lattice symmetry, we may assume that $z_j \geq 0$ for all $j = 1, 2, \ldots, d$. We will choose $\mathbf{v_z}$ suitably in the rectangle

$$\mathcal{R}_\mathbf{z} = \Big\{\mathbf{v} \in \mathcal{V}_\mathcal{G} \cap H_n :\; z_1 \leq v_1 \leq n, \text{ and } 0 \leq v_j \leq z_j \text{ for } j \geq 2\Big\}.$$

Due to (6.8) we can find $C_3 < \infty$ such that, for $\mathbf{v} \in \mathcal{R}_\mathbf{z}$, if $z_1 + C_3 \leq v_1 \leq n$, or if $v_1 = n$ and $|\mathbf{v} - n\mathbf{e}_1| \leq |\mathbf{z} - n\mathbf{e}_1| - C_3$, then $\mathbf{v} \in D_0$. Choose $\mathbf{v_z}$ accordingly. Since $\mathbf{v_z}$ and $\mathbf{z}$ differ in q coordinates, it is easy to find q disjoint paths from $\mathbf{v_z}$ to $\mathbf{z}$ of length $\|\mathbf{v_z} - \mathbf{z}\|_1$.

It remains to conclude that there are $2d$ disjoint paths between $\mathbf{z} \in D_0$ and $\mathbf{u_z}$. This follows from Lemma 2.4, since $\bigcup_{a\in[0,1]} B\big(\mathbf{u_z} + a(\mathbf{z} - \mathbf{u_z}), R_d\big) \subseteq \mathcal{V}_\mathcal{G}$. ☐

Proof of Lemma 6.2. Let $1 \leq C < \infty$ and $D_0, D_1, \ldots, D_{d-1}$ be as in Claim 2. We will treat each D_q separately, starting with $q = 0$. Claim 1 says that $\|\mathbf{u_z} - \mathbf{z}\|_1 \leq 8dn\epsilon + \sqrt{d}$ for $\mathbf{z} \in H_n$. Via Lemma 6.1 we deduce that for $\mathbf{z} \in D_0 \cap H_n$

$$\mathbb{P}\Big(T_\mathcal{G}(\mathbf{u_z}, \mathbf{z}) \geq \big(\mathbb{E}[\tau_e] + 1\big)C(8dn\epsilon + \sqrt{d})\Big) \;\leq\; |D_0 \cap H_n| \left(\frac{\mathbb{E}[\tau_e^2]}{n\epsilon}\right)^{2d},$$

and, since $|D_0 \cap H_n| \leq Cn^{d-1}$,

$$\sum_{\mathbf{z}\in D_0} \mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{u_z},\mathbf{z}) \geq \big(\mathbb{E}[\tau_e]{+}1\big)C(8d|\mathbf{z}|\epsilon{+}\sqrt{d})\Big) \;\leq\; C\sum_{n=1}^{\infty} n^{d-1}\left(\frac{\mathbb{E}[\tau_e^2]}{n\epsilon}\right)^{2d} \;<\; \infty.$$

For each $\mathbf{z} \in D_q \cap H_n$, where $q \geq 1$, we obtain that

$$\mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{u_z},\mathbf{z}) \geq \big(\mathbb{E}[\tau_e]+2\big)C(8dn\epsilon+\sqrt{d}+C)\Big) \leq \mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{v_z},\mathbf{z}) \geq n\epsilon\Big) \\ + \mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{u_z},\mathbf{v_z}) \geq \big(\mathbb{E}[\tau_e]+1\big)C(8dn\epsilon+\sqrt{d}+C)\Big).$$

Let $\mathcal{T}_q$ denote the minimum passage times of q disjoint paths of length C. Since $\|\mathbf{v_z}-\mathbf{z}\|_1 \leq C$, then $\mathbb{P}\big(T_{\mathcal{G}}(\mathbf{v_z},\mathbf{z}) \geq n\epsilon\big)$ is at most as large as $\mathbb{P}(\mathcal{T}_q \geq n\epsilon)$. Thus,

$$\sum_{\mathbf{z}\in D_q\cap H_n} \mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{v_z},\mathbf{z}) \geq n\epsilon\Big) \;\leq\; |D_q \cap H_n|\mathbb{P}(\mathcal{T}_q \geq n\epsilon) \;\leq\; Cn^{q-1}\mathbb{P}(\mathcal{T}_q \geq n\epsilon),$$

which, summing over n, is finite, since $\mathbb{E}[\mathcal{T}_q^q] < \infty$. Moreover, $\|\mathbf{u_z}-\mathbf{v_z}\|_1 \leq 8dn\epsilon+\sqrt{d}+C$, and an application of Lemma 6.1 gives that

$$\sum_{\mathbf{z}\in D_q\cap H_n} \mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{u_z},\mathbf{v_z}) \geq \big(\mathbb{E}[\tau_e]{+}1\big)C(8dn\epsilon{+}\sqrt{d}{+}C)\Big) \;\leq\; |D_q{\cap}H_n|\left(\frac{\mathbb{E}[\tau_e^2]}{n\epsilon}\right)^{2d}.$$

Again $|D_q \cap H_n| \leq Cn^{q-1}$, and we conclude that for each $q \geq 1$

$$\sum_{\mathbf{z}\in D_q} \mathbb{P}\Big(T_{\mathcal{G}}(\mathbf{u_z},\mathbf{z}) \geq \big(\mathbb{E}[\tau_e]+2\big)C(8d|\mathbf{z}|\epsilon+\sqrt{d}+C)\Big) < \infty.$$

Hence, it suffices to choose $M = \big(\mathbb{E}[\tau_e]+3\big)8Cd$. □

Remark 6.4. The condition in Lemma 6.2 can be relaxed to $\mathbb{E}[Y^d] < \infty$. That is essentially obtained by showing that the set D_0 can be chosen in a way that under the relaxed condition

$$\sum_{\mathbf{z}\in D_0} \mathbb{P}\big(T_{\mathcal{G}}(\mathbf{u_z},\mathbf{z}) \geq M|\mathbf{z}|\epsilon\big) < \infty,$$

for some $M < \infty$ not depending on ϵ. To see that this is true, consult Lemma 2.4 and the paper of Cox and Durrett (1981, Lemma 3.3). □

Recall that complete convergence implies almost sure convergence.

Proof of part *b)* and *c)*. Assume that $\mathbb{E}[\tau_e^2] < \infty$. We will prove that

$$\sum_{\mathbf{z} \in \mathcal{V}_{\mathcal{G}}} \mathbb{P}\Big(\big| T_{\mathcal{G}}(\mathbf{0}, \mathbf{u}_{\mathbf{z}}) - |\mathbf{u}_{\mathbf{z}}| \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) \big| > |\mathbf{z}| \epsilon \Big) < \infty. \tag{6.9}$$

Once this is done, it will follow from (6.5) and Lemma 6.2 that sum of

$$\mathbb{P}\bigg(\Big| T_{\mathcal{G}}(\mathbf{0}, \mathbf{z}) - |\mathbf{z}| \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \Big| > \Big(M + 1 + (8d + \sqrt{d}) \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) + 2d\, \mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1) \big] \Big) |\mathbf{z}| \epsilon \bigg)$$

over all $\mathbf{z} \in \mathcal{V}_{\mathcal{G}}$ is finite. Since $\epsilon > 0$ has been arbitrarily chosen, this would prove *c)*, and therefore also *b)*.

In order to prove (6.9), note that $\mathbf{u}_{\mathbf{z}}$ may obtain the same value for at most $|H_n|$ vertices $\mathbf{z} \in \mathcal{V}_{\mathcal{G}}$. Since $|H_n| = 2d(2n)^{d-1}$, then (6.9) follows easily from (6.4), given that $E[Y^{d+1}] < \infty$. However, that indeed holds since $\mathbb{E}[\tau_e^2]$ is assumed finite, according to (2.6). $\square$

6.2 Proof of Theorem 1.6

Assume that $\mathbb{E}[\tau_e^2] < \infty$. Only minor adjustments are necessary in order to obtain Theorem 1.6 from the proof of Theorem 1.2. We proof will proceed along the same lines. For $\mathbf{z} \in \mathcal{V}_{\mathcal{G}}$, set $\hat{\mathbf{z}} = \mathbf{z}/|\mathbf{z}|$. The statement we will prove is that, almost surely, for every $\epsilon > 0$, there exists $M = M(\epsilon) < \infty$ such that

$$\left| \frac{T_{\mathcal{G}}^{(s)}(\mathbf{0}, \mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \right| < \epsilon, \quad \text{for all } s \in [0,1] \text{ and } |\mathbf{z}| \geq M. \tag{6.10}$$

This is sufficient for Theorem 1.6 to follow.

Fix $\epsilon \in \big(0, \frac{1}{3\sqrt{d}}\big)$. Let $\hat{\mathbf{u}}^{(1)}, \ldots, \hat{\mathbf{u}}^{(m)} \in S_{\mathcal{G}}$ be chosen as in the proof of Theorem 1.2. It follows directly from Proposition 5.2 that there exists an almost surely finite $M = M(\epsilon)$ such that

$$\left| \frac{T_{\mathcal{G}}^{(s)}(\mathbf{0}, n\hat{\mathbf{x}})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{x}}) \right| < \epsilon, \quad \text{for all } s \in [0,1] \text{ and } n \geq M.$$

For $\hat{\mathbf{u}}^{(j)} \in S_{\mathcal{G}} \cap \mathbb{U}^{d-1}$ that are interior points in $S_{\mathcal{G}}$, we may again choose a_j large, such that

$$\begin{aligned} \left| \frac{T_{\mathcal{G}}^{(s)}(\mathbf{0}, n\hat{\mathbf{u}}^{(j)})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)}) \right| \leq\ & \frac{T_{\mathcal{G}}^{(s)}(\mathbf{0}, a_j\hat{\mathbf{u}}^{(j)})}{n} \\ & + \left| \frac{T_{\mathcal{G}}^{(s)}(a_j\hat{\mathbf{u}}^{(j)}, n\hat{\mathbf{u}}^{(j)})}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)}) \right|, \end{aligned}$$

and where the right-hand side is, almost surely, at most ϵ uniformly in $s \in [0,1]$, for sufficiently large n, again according to Proposition 5.2. That is, there is an almost surely finite $M = M(\epsilon)$ such that for each $j = 1, 2, \dots, m$

$$\left| \frac{T_{\mathcal{G}}^{(s)}\big(\mathbf{0}, n\hat{\mathbf{u}}^{(j)}\big)}{n} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(j)}) \right| < \epsilon, \quad \text{for all } s \in [0,1] \text{ and } n \geq M. \tag{6.11}$$

For any given $\mathbf{z} \in \mathcal{V}_{\mathcal{G}}$, let $\mathbf{u}_{\mathbf{z}}$ be specified as before. Analogously to (6.5) we have that

$$\begin{aligned} \left| T_{\mathcal{G}}^{(s)}(\mathbf{0}, \mathbf{z}) - |\mathbf{z}| \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \right| \;\leq\; & T_{\mathcal{G}}^{(s)}(\mathbf{u}_{\mathbf{z}}, \mathbf{z}) \\ & + \left| T_{\mathcal{G}}^{(s)}(\mathbf{0}, \mathbf{u}_{\mathbf{z}}) - |\mathbf{u}_{\mathbf{z}}| \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) \right| \\ & + \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) \big| |\mathbf{u}_{\mathbf{z}}| - |\mathbf{z}| \big| \\ & + |\mathbf{z}| \big| \mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) - \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \big|. \end{aligned} \tag{6.12}$$

According to (6.11), for $s \in [0,1]$, the second term in the right-hand side of (6.12) is not greater than $|\mathbf{u}_{\mathbf{z}}|\epsilon \leq (1 + 8d\epsilon)|\mathbf{z}|\epsilon + \sqrt{d}\epsilon$ whenever $|\mathbf{u}_{\mathbf{z}}| \geq \big(|\mathbf{z}| - \sqrt{d}\epsilon\big)/(1 + 8d\epsilon) \geq M$. Recall the notation introduced in Section 5. Since

$$T_{\mathcal{G}}^{(s)}(\mathbf{u}_{\mathbf{z}}, \mathbf{z}) \;\leq\; \hat{T}_{\mathcal{G}}(\mathbf{u}_{\mathbf{z}}, \mathbf{z}),$$

and $\mathbb{E}[\hat{\tau}_e^2] < \infty$ whenever $\mathbb{E}[\tau_e^2] < \infty$, it follows from Lemma 6.2 and Borel-Cantelli's lemma that there is an $M' < \infty$ such that

$$\sup_{s \in [0,1]} T_{\mathcal{G}}^{(s)}(\mathbf{u}_{\mathbf{z}}, \mathbf{z}) \;\leq\; \hat{T}_{\mathcal{G}}^{(s)}(\mathbf{u}_{\mathbf{z}}, \mathbf{z}) \;\leq\; M'|\mathbf{z}|\epsilon$$

for all but finitely many $\mathbf{z} \in \mathcal{V}_{\mathcal{G}}$. From all of the above, we conclude that there is an almost surely finite $M'' = M''(\epsilon)$ such that when $|\mathbf{z}| \geq M''$

$$\left| \frac{T_{\mathcal{G}}^{(s)}(\mathbf{0}, \mathbf{z})}{|\mathbf{z}|} - \mu_{\mathbb{Z}^d}(\hat{\mathbf{z}}) \right| \leq \Big(M' + (1 + 8d\epsilon) + 8d\mu_{\mathbb{Z}^d}(\hat{\mathbf{u}}^{(\mathbf{z})}) + 2d\,\mathbb{E}\big[T(\mathbf{0}, \mathbf{e}_1)\big] + 1 \Big)\epsilon,$$

for all $s \in [0,1]$. Since $\epsilon > 0$ was arbitrary, this proves (6.10). □

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Paper III

Noise sensitivity in continuum percolation

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Abstract

We prove that the Poisson Boolean model, also known as the Gilbert disc model, is noise sensitive at criticality. This is the first such result for a continuum percolation model, and the first for which the critical probability $p_c \neq 1/2$. Our proof uses a version of the Benjamini-Kalai-Schramm Theorem for biased product measure. A quantitative version of this result was recently proved by Keller and Kindler. We give a simple deduction of the non-quantitative result from the unbiased version. We also develop a quite general method of approximating continuum percolation models by discrete models with p_c bounded away from zero; this method is based on an extremal result on non-uniform hypergraphs.

1 Introduction

The concept of noise sensitivity of a sequence of Boolean functions was introduced by Benjamini, Kalai, and Schramm (1999), and has since developed into one of the most exciting areas in probability theory, linking percolation with discrete Fourier analysis and combinatorics. So far, most attention has been focused on percolation crossings in two dimensions, either for bond percolation on the square lattice $\mathbb{Z}^2$, or for site percolation on the triangular lattice $\mathbb{T}$. In this paper we study the corresponding question in the setting of continuum

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percolation. In particular, we shall prove that the Poisson Boolean model, also known as the Gilbert disc model, is noise sensitive at criticality.

Roughly speaking, a sequence $(f_n)_{n\geqslant 1}$ of Boolean functions $f_n : \{0,1\}^n \to \{0,1\}$ is said to be *noise sensitive* if a slight perturbation of the state ω asymptotically causes all information about $f_n(\omega)$ to be lost. More precisely, let $\varepsilon > 0$ and suppose that $\omega \in \{0,1\}^n$ is chosen uniformly at random. Define $\omega^\varepsilon \in \{0,1\}^n$ to be the (random) state obtained by re-sampling each coordinate (independently and uniformly) with probability ε, and note that ω^ε is also a uniform element of $\{0,1\}^n$. Then the sequence $(f_n)_{n\geqslant 1}$ is said to be *noise sensitive* (NS) if

$$\lim_{n\to\infty} \mathbb{E}\big[f_n(\omega)f_n(\omega^\varepsilon)\big] - \mathbb{E}\big[f_n(\omega)\big]^2 \;=\; 0, \quad \text{for every } \varepsilon > 0. \tag{1.1}$$

One can easily see, using the Fourier representation of Section 4, that if (1.1) holds for some $\varepsilon > 0$, then it holds for all $\varepsilon > 0$. For example, the Majority function ($f_n(\omega) = 1$ iff $\sum \omega_j > n/2$) and the Dictator function ($f_n(\omega) = 1$ iff $\omega_1 = 1$) are not noise sensitive, but the Parity function ($f_n(\omega) = 1$ iff $\sum \omega_j$ is even) is noise sensitive.

Noise sensitivity was first defined by Benjamini et al. (1999), who were partly motivated by the problem of exceptional times in dynamical percolation (see e.g. Steif (2009)). In this model, which was introduced independently by Benjamini (unpublished) and by Häggström, Peres, and Steif (1997), each bond in $\mathbb{Z}^2$ (or site in $\mathbb{T}$) has a Poisson clock, and updates its state every time the clock rings. At any given time, the probability that there is an infinite component of open edges is zero at p_c (see Bollobás and Riordan (2006a) or Grimmett (1999), for example). However, there might still exist certain exceptional times at which such a component appears. Building on the work of Benjamini et al. (1999), Schramm and Steif (2010) were able to prove that, for the triangular lattice $\mathbb{T}$, such exceptional times do exist, and moreover the Hausdorff dimension of the set of such times lies in $[1/6, 31/36]$. Even stronger results were later obtained by Garban, Pete, and Schramm (2010), who were able to prove, via an extremely precise result on the Fourier spectrum of the 'percolation crossing event', that the dimension of the exceptional set for $\mathbb{T}$ is $31/36$, and that exceptional times also exist for bond percolation on $\mathbb{Z}^2$.

Following Benjamini et al. (1999), we shall study Boolean functions which encode 'crossings' in percolation models. For example, consider bond percolation on $\mathbb{Z}^2$ at criticality (i.e., with $p = p_c = 1/2$), and let f_N encode the event that there is a horizontal crossing of R_N, the $N \times N$ square centred at the origin, using only the open edges of the configuration. In other words, let $f_N : \{0,1\}^E \to \{0,1\}$, where E is the set of edges of $\mathbb{Z}^2$ with an endpoint in R_N, be defined by $f_N(\omega) = 1$ if and only if there is such a crossing using only edges $e \in E$ with $\omega_e = 1$. Benjamini et al. proved that the sequence $(f_N)_{N \geqslant 1}$ is noise sensitive.

Continuum percolation describes the following family of random geometric graphs: define

$$\Omega := \big\{ \eta \subset \mathbb{R}^2 : \; |\eta \cap F| < \infty \text{ for every bounded } F \subset \mathbb{R}^2 \big\},$$

and pick $\eta \in \Omega$ according to some distribution. We then join two points of η with an edge in a deterministic way, based on their relative position. Two especially well-studied examples are Voronoi percolation (see e.g. Bollobás and Riordan (2006b)), and the Poisson Boolean model, which was introduced by Gilbert (1960), and is further accounted for in Meester and Roy (1996); Alexander (1996). In the latter model, $\eta \in \Omega$ is chosen according to a Poisson point process with intensity λ, and for each point $x \in \eta$, a disc of radius 1 is placed with its centre on x; let $D(\eta)$ denote the union of these discs. The model is said to *percolate* if there exists an infinite connected component in $D(\eta)$. It is well known that there exists a critical intensity $0 < \lambda_c < \infty$ such that if $\lambda < \lambda_c$ then the model almost surely does not percolate, while if $\lambda > \lambda_c$ it almost surely percolates. See the books by Meester and Roy (1996) and Bollobás and Riordan (2006a) for a detailed introduction to continuum percolation.

We shall be interested in the problem of noise sensitivity of the Poisson Boolean model at criticality, that is, with $\lambda = \lambda_c$. Let $f_N^G : \Omega \to \{0,1\}$ be the function which encodes whether or not there is a horizontal crossing of R_N using only points of $D(\eta) \cap R_N$ for $\eta \in \Omega$. That is, for every $\eta \in \Omega$,

$$f_N^G(\eta) = 1 \quad \Leftrightarrow \quad H\big(\eta, R_N, \bullet\big) \text{ occurs},$$

where $H(\eta, R_N, \bullet)$ denotes the event that such a crossing exists in the 'occupied space' $D(\eta)$.

Since f_N^G is defined on Ω, we shall need to modify the definition of noise sensitivity. Let $\varepsilon > 0$ and $\lambda > 0$, and pick $\eta \in \Omega$ according to a Poisson point process of intensity λ. We shall denote the measure associated to this Poisson process by $\mathbf{P}_\lambda$, expectation with respect to this measure by $\mathbf{E}_\lambda$ and variance by $\mathbf{Var}_\lambda$. We define $\eta^\varepsilon \in \Omega$ to be the set obtained by deleting each element of η independently with probability ε, and then adding a new Poisson point process of intensity $\varepsilon\lambda$. It is clear that η^ε has the same distribution as η. With a minor abuse of notation, we will let $\mathbf{P}_\lambda$ denote also the measure by which the pair (η, η^ε) is chosen.

Definition 1.1. *We say that the Poisson Boolean model is* noise sensitive *at λ if the sequence of functions $(f_N^G)_{N \geqslant 1}$ satisfies*

$$\lim_{N \to \infty} \mathbf{E}_\lambda\big[f_N^G(\eta) f_N^G(\eta^\varepsilon)\big] - \mathbf{E}_\lambda\big[f_N^G(\eta)\big]^2 \; = \; 0, \quad \textit{for all } \varepsilon > 0.$$

We remark that the Poisson Boolean model is trivially noise sensitive at every $\lambda \neq \lambda_c$. The reason is simply that when $\lambda > \lambda_c$ (or $\lambda < \lambda_c$), then $\lim_N f_N^G = 1$ almost surely (or $\lim_N f_N^G = 0$ almost surely), as is well known. We shall say that the model is *noise sensitive at criticality* if it is noise sensitive at λ_c.

The following theorem is our main result. It is the analogue for the Poisson Boolean model of the result of Benjamini et al. (1999) mentioned above concerning bond percolation on $\mathbb{Z}^2$.

Theorem 1.2. *The Poisson Boolean model is noise sensitive at criticality.*

The proof of Theorem 1.2 is based on two very general theorems, neither of which uses any properties of the specific model which we are studying. The first is a version of one of the main theorems of Benjamini, Kalai, and Schramm (1999), a result referred to as the BKS Theorem. It gives a sufficient condition (based on the concept of influence) for an arbitrary sequence of functions to be noise sensitive at density p (see Theorem 1.4). A quantitative version of the BKS Theorem for biased product measure was recently proved by Keller and Kindler (2010). Their result is therefore a strengthening of the qualitative result of Benjamini et al.. We shall give a short deduction of the BKS Theorem for general $p \in (0,1)$ from the uniform case.

The second main tool is an extremal result on arbitrary non-uniform hypergraphs (i.e., arbitrary events on $\{0,1\}^n$), which allows us to bound the variance that arises when two stages of randomness are used to choose a random subset. We shall use this bound (see Theorem 1.6) to prove noise sensitivity for the Poisson Boolean model via a corresponding result when we condition on a 'much larger' Poisson configuration (see Theorem 1.5). These tools are quite general, and we expect both to have other applications; we shall therefore state them here, and in some detail, for easy reference.

In order to state the BKS Theorem for product measure, we first need to define noise sensitivity in this setting. Let $\mathbb{P}_p$ denote product measure with density $p \in (0,1)$ with which we pick $\omega \in \{0,1\}^n$, i.e. $\mathbb{P}_p(\omega_i = 1) = p$ independently for every $i \in [n] := \{\mathbf{1}, 2, \dots, n\}$. We let $\mathbb{E}_p$ denote expectation with respect to this measure. When $p = 1/2$ this corresponds to picking an element of $\{0,1\}^n$ uniformly at random, and so we refer to it as the uniform case. Define ω^ε as above, by re-randomizing each bit (coordinate) independently with probability ε.

Definition 1.3. *A sequence $(f_n)_{n\geqslant 1}$ of functions $f_n : \{0,1\}^n \to [0,1]$ is said to be* noise sensitive at density p *(NS_p) if*

$$\lim_{n\to\infty} \mathbb{E}_p\big[f_n(\omega)f_n(\omega^\varepsilon)\big] - \mathbb{E}_p\big[f_n(\omega)\big]^2 = 0, \quad \textit{for every } \varepsilon > 0. \tag{1.2}$$

When $p = 1/2$, this is equivalent to (1.1), the definition of noise sensitivity of Benjamini et al. (1999).

The *influence at density* p, denoted $\mathrm{Inf}_{p,i}(f)$, of a coordinate $i \in [n]$ in a function $f : \{0,1\}^n \to [0,1]$, is defined by

$$\mathrm{Inf}_{p,i}(f) := \mathbb{E}_p\big[\big|f(\omega) - f(\sigma_i\omega)\big|\big],$$

where σ_i is the function that flips the value of ω at position i. We denote the sum of the squares of the influences of f by

$$I\!I_p(f) := \sum_{i=1}^{n} \mathrm{Inf}_{p,i}(f)^2.$$

The theorem we are about to present was first proved by Benjamini et al. (1999) in the case $p = 1/2$, and further remarked to hold for general $p \in (0,1)$. A quantitative version of the result was obtained by Keller and Kindler (2010). (In Benjamini et al. (1999) it was stated only for functions into $\{0,1\}$, but the more general result follows by the same method as one can check, see Keller and Kindler (2010, Page 3), for example.)

When proving statements concerning functions on the hypercube $\{0,1\}^n$ endowed with biased product measure, it can be favourable to strive for a simple deduction of the case $p \neq 1/2$ from its uniform counterpart. One technique used to reduce the biased case to uniform was considered by Friedgut (2004) and also Keller (2010). We will present a different reduction, with which we shall give a simple deduction of the following theorem from the known uniform analogue.

Theorem 1.4 (BKS Theorem for product measure)**.** *Let $(f_n)_{n\geqslant 1}$ be a sequence of functions $f_n : \{0,1\}^n \to [0,1]$. For every $p \in (0,1)$,*

$$\lim_{n\to\infty} I\!I_p(f_n) = 0 \qquad \Rightarrow \qquad (f_n)_{n\geqslant 1} \text{ is } \mathrm{NS}_p.$$

We remark that the approach we use to prove Theorem 1.4 is quite general, and may be used to extend various other results from uniform to biased product measure, as can be seen in Section 2.2. Before introducing our second main tool, Theorem 1.6, let us give some more context, by describing our general approach to the proof of Theorem 1.2.

Given $\eta \in \Omega$, $D(\eta)$ corresponds to the geometric graph obtained by connecting any two points in η at Euclidean distance at most 2. For $p \in (0,1)$ and a countable set S, define the (random) *p-subset* S_p of S as the random set obtained by including each element of S independently with probability p. For a p-subset η_p of $\eta \in \Omega$, $D(\eta_p)$ can be thought of as the resulting structure when site percolation at density p is performed on (the graph induced by)

$D(\eta)$. Possibly, this should rather be referred to as 'disc' percolation, but we will stick to calling it site percolation on $D(\eta)$. In particular, when $\eta \in \Omega$ is chosen according to the measure $\mathbb{P}_{\lambda_c/p}$, then $D(\eta_p)$ corresponds to a critical configuration of the Poisson Boolean model. An intermediate step in proving Theorem 1.2 will be a statement regarding noise sensitivity of site percolation on $D(\eta)$.

Given $\eta \in \Omega$, let f_N^η denote the restriction of f_N^G onto η. Formally, let $\Omega_\eta := \{\xi \in \Omega : \xi \subseteq \eta\}$, and define $f_N^\eta : \Omega_\eta \to \{0,1\}$ by letting $f_N^\eta(\xi) = f_N^G(\xi)$ for every $\xi \in \Omega_\eta$. We can identify Ω_η and $\{0,1\}^\eta$ and so, equivalently, we can view f_N^η as a function from $\{0,1\}^\eta$ to $\{0,1\}$, defined by letting, for every $\xi \in \{0,1\}^\eta$,

$$f_N^\eta(\xi) = 1 \quad \Leftrightarrow \quad H(\xi, R_N, \bullet) \text{ occurs.}$$

We will say that site percolation on $D(\eta)$ is NS_p if the sequence $(f_N^\eta)_{N\geqslant 1}$ is NS_p. Given $\varepsilon \in (0,1)$, let $(\eta_p)^\varepsilon$ denote an ε-perturbation of the site percolation configuration η_p, that is, each point in η is independently with probability ε re-evaluated to be included in η_p. NS_p of $(f_N^\eta)_{N\geqslant 1}$ then corresponds to

$$\lim_{N\to\infty} \mathbb{E}\big[f_N^G(\eta_p) f_N^G\big((\eta_p)^\varepsilon\big) \,\big|\, \eta\big] - \mathbb{E}\big[f_N^G(\eta_p) \,\big|\, \eta\big]^2 \;=\; 0, \quad \text{for each } \varepsilon > 0.$$

Naturally, we will be interested in site percolation on $D(\eta)$ when $\eta \in \Omega$ is chosen according to $\mathbb{P}_{\lambda_c/p}$. The proof of Theorem 1.2 proceeds via the following.

Theorem 1.5. *Site percolation on $D(\eta)$ is NS_p for $\mathbb{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$, for each sufficiently small $p > 0$.*

Thus, the proof of Theorem 1.2 divides naturally into two parts. In the first we adapt the methods of Benjamini et al. (1999) to prove noise sensitivity in Theorem 1.5; in the second we use our bound on the variance (Theorem 1.6, below) to prove that this noise sensitivity transfers to the continuous Poisson Boolean model. Interestingly, Theorem 1.6 will also be a key tool in the proof of Theorem 1.5.

Recall that $[n] = \{1, 2, \ldots, n\}$. A hypergraph $\mathcal{H}$ is simply a collection of subsets of $[n]$; or, equivalently, it is a subset of $\{0,1\}^n$. We call these sets 'edges', and remark that if every edge has exactly two elements then $\mathcal{H}$ is a graph. Given a hypergraph $\mathcal{H}$, a set $B \subseteq [n]$ and $p \in (0,1)$, define

$$r_{\mathcal{H}}(B, p) \;:=\; \mathbb{P}(B_p \in \mathcal{H}).$$

We sometimes write $\mathbb{P}(B_p \in \mathcal{H} |\, B)$ if we explicitly want to stress the choice of B. In our applications, $[n]$ will correspond to a discrete approximation of a rectangle $R \subseteq \mathbb{R}^2$ while $\mathcal{H}$ will be the hypergraph which encodes crossings of R in $D(B)$. Further, we will let $q = q(n)$ be chosen so that $[n]_q$ has (expected) density λ_c/p in R. Note that a p-subset of $[n]_q$ has the same distribution as $[n]_q \cap [n]_p$, and so we have that $r_{\mathcal{H}}\big([n]_q, p\big) = \mathbb{P}\big([n]_q \cap [n]_p \in \mathcal{H} \,\big|\, [n]_q\big)$.

Theorem 1.6. *There is a universal constant $C < \infty$ such that if $p \in \big(0, \frac{1}{2}\big]$, $q \in (0,1)$ and $n \in \mathbb{N}$ satisfy $n \geqslant 128(pqn)^3$, $n \geqslant 4p(qn)^2$ and $pqn \geqslant 32 \log \frac{1}{p}$, then, for every hypergraph $\mathcal{H}$ on vertex set $[n]$,*

$$\mathrm{Var}\left(r_{\mathcal{H}}\big([n]_q, p\big)\right) \;=\; Cp\left(\log \frac{1}{p}\right)^2.$$

We emphasize the crucial point, which is that our bound on Var $\big(r_{\mathcal{H}}\big([n]_q, p\big)\big)$ goes to zero as $p \to 0$ *uniformly* in $\mathcal{H}$. At occasions, we shall use the notation $f(x) = O\big(g(x)\big)$ to denote the existence of a universal constant $C < \infty$, independent of all other variables, such that $\big|f(x)\big| \leqslant C\big|g(x)\big|$ for all x in some given range.

As noted above, we shall use Theorem 1.6 in order to prove Theorem 1.5 as well as to deduce Theorem 1.2 from Theorem 1.5. Indeed, we shall use Theorem 1.6 together with the 'deterministic algorithm method' (see Sections 2.3 and 5) to obtain bounds on the influences of variables; Theorem 1.5 then follows from the BKS Theorem for product measure.

The rest of the paper is organized as follows. In Section 2 we give a full overview of the proof, and state several other results which may be of independent interest. In Section 3 we recall some facts about the Poisson Boolean model, and in the Sections 4 and 5 we prove Theorem 1.4 and extend the 'deterministic algorithm method' of Benjamini et al. (1999) to general $p \in (0,1)$. In Section 6 we prove Theorem 1.6, and deduce some simple consequences, and in Section 7 we prove Theorem 1.5 and deduce Theorem 1.2. Finally, in Section 8 we state some open questions.

Throughout the article we treat elements of $\{0,1\}^n$ as subsets of $[n]$, and vice versa, without comment, by identifying sets with their indicator functions.

2 Further results, and an overview of the proof

In this section we introduce a number of auxiliary methods and results that we shall use in the proof of Theorem 1.2, and which may also be of independent interest. In particular, we introduce a new way of deducing results for biased product measure from results in the uniform case. We shall use this method in Sections 4 and 5 to generalize the BKS Theorem and the 'deterministic algorithm method' of Benjamini, Kalai, and Schramm (1999).

Let us begin by outlining how Theorem 1.2 follows from Theorems 1.5

and 1.6. Fix $p \in (0,1)$, and observe that

$$\begin{aligned}\mathbf{E}_{\lambda_c/p}\Big[\mathbb{E}\big[f_N^G(\eta_p) \,|\, \eta\big]\Big] &= \mathbf{E}_{\lambda_c/p}\Big[\mathbb{P}\big(H(\eta_p, R_N, \bullet) \,|\, \eta\big)\Big] \\ &= \mathbf{P}_{\lambda_c}\big(H(\eta, R_N, \bullet)\big) = \mathbf{E}_{\lambda_c}\big[f_N^G(\eta)\big],\end{aligned} \tag{2.1}$$

where the second equality follows since if $\eta \in \Omega$ is chosen according to a Poisson point process of intensity λ_c/p, then η_p is distributed as a Poisson point process of intensity λ_c.

Using Theorem 1.6, and a straightforward discretization of the square R_N (see Section 7), we shall prove the following proposition.

Proposition 2.1. $\displaystyle \lim_{p\to 0} \limsup_{N\to\infty} \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p, R_N, \bullet) \,|\, \eta\big)\Big) = 0.$

In Section 7 we shall show that the Poisson Boolean model is noise sensitive at criticality if and only if for every $\varepsilon > 0$, as $N \to \infty$,

$$\mathbf{E}_{\lambda_c/p}\Big[\mathbb{E}\big[f_N^G(\eta_p) f_N^G\big((\eta_p)^\varepsilon\big) \,|\, \eta\big] - \mathbb{E}\big[f_N^G(\eta_p) \,|\, \eta\big]^2\Big] + \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{E}\big[f_N^G(\eta_p) \,|\, \eta\big]\Big) \to 0.$$

For $p > 0$ small enough, Theorem 1.5 says exactly that the first term is $o(1)$ as $N \to \infty$, and Proposition 2.1 shows that the second can be made arbitrarily small by choosing p appropriately.

In the rest of the section we shall outline the proofs of Theorems 1.5 and 1.6; we begin by stating the key property of the Poisson Boolean model that we shall need.

2.1 Non-triviality of crossing probabilities

If the probability (in $\mathbf{P}_{\lambda_c}$) of the crossing event $H(\eta, R_N, \bullet)$ were trivial, in the sense that it converged to 0 or 1 as $N \to \infty$, then Theorem 1.2 would itself be trivial. However, this is not the case. Further, one may deduce, using Theorem 1.6, that for N large enough, with high probability (in $\mathbf{P}_{\lambda_c/p}$), $\eta \in \Omega$ will be such that $\mathbb{P}\big(H(\eta_p, R_{N\times tN}, \bullet) \,|\, \eta\big)$ is also non-trivial (see Proposition 2.3). The following fact will be a vital tool in our proof of the noise sensitivity of the sequence $(f_N^\eta)_{N\geqslant 1}$, as it will allow us to bound the probability of the 'one-arm event' (see Section 2.3). Throughout $R_{a\times b}$ denotes the rectangle with side lengths a and b, centred at the origin.

Theorem 2.2 (Alexander (1996)). *For every $t > 0$ there exists $c = c(t) > 0$ such that,*

$$c \leqslant \mathbf{P}_{\lambda_c}\Big(H\big(\eta, R_{N\times tN}, \bullet\big)\Big) \leqslant 1 - c, \quad \textit{for every } N \in \mathbb{N}.$$

Theorem 2.2 is in fact a slight extension of Alexander (1996, Theorem 3.4), but it follows similarly. For completeness, we shall sketch the proof in Section 3. From this bound, together with Theorem 1.6, we shall deduce the following bound (see Section 7).

Proposition 2.3. *For every $t, \gamma > 0$ there exist constants $c = c(t) > 0$ and $p^* = p^*(t, \gamma) > 0$ such that if $p \in (0, p^*)$, then*

$$\mathbf{P}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p, R_{N\times tN}, \bullet)\,\big|\,\eta\big) \notin (c, 1-c)\Big) < \gamma$$

for every sufficiently large $N \in \mathbb{N}$.

This bound will allow us to show that, with high $\mathbf{P}_{\lambda_c/p}$-probability, the 'one-arm event' of *every* point of $\eta \cap R_N$ is bounded above by $N^{-\delta}$, for some $\delta > 0$ (see Section 2.3).

2.2 A new method for proving results for biased product measure

We outline here a new method for deducing results in the setting of a density p product measure from the uniform case (i.e. p=1/2). The idea is the following. Rather than considering directly the function $f : \{0,1\}^n \to \{0,1\}$ where $\{0,1\}^n$ is endowed with density p product measure, we consider a related function $h_f : \{0,1\}^n \to [0,1]$ where $\{0,1\}^n$ is endowed with uniform measure. This function h_f is obtained from f by an averaging operation. By relating various parameters of f and h_f, results about one may be deduced from results concerning the other. In particular, we use this method both in our proof of Theorem 1.4 and in our generalization of the 'deterministic algorithm method' (see Section 2.3).

The following coupling of the uniform measure on $\{0,1\}^n$ and the p-density product measure $\mathbb{P}_p$ is key to the definition of h_f. In what follows let $p \in (0,1)$ be fixed and let $\bar{p} = \min\{p, 1-p\}$. (We are generally interested in the case $p \leqslant 1/2$ but there is little extra difficulty in presenting the argument in the general case.)

Let $X \in \{0,1\}^n$ be uniformly distributed. Let $Y \in \{0,1\}^n$ be a $2\bar{p}$-subset of $[n]$, i.e., $\mathbb{P}(Y_i = 1) = 2\bar{p}$ for each $i \in [n]$, all independently. We define the random variable $Z \in \{0,1\}^n$ as follows

$$Z_i := \begin{cases} X_i Y_i & \text{if } p \leqslant 1/2 \\ 1 - (1 - X_i) Y_i & \text{if } p > 1/2 \end{cases} \tag{2.2}$$

for each $i \in [n]$. Note that Z is a p-subset of $[n]$, i.e. distributed according to $\mathbb{P}_p$, and, by construction, $Z \subseteq X$ if $p \leqslant 1/2$ and $Z \supseteq X$ if $p \geqslant 1/2$.

With the random variables X, Y, Z as defined above, and given any function $f : \{0,1\}^n \to [0,1]$ we define

$$h_f(X) := \mathbb{E}\big[f(Z) \,\big|\, X\big], \tag{2.3}$$

and observe that $h_f : \{0,1\}^n \to [0,1]$. We shall use h_f to reduce statements about $f(Z)$, where Z has a p-biased distribution, to statements about $h_f(X)$, where X is chosen uniformly. The connection between f and h_f is given by the following proposition. Here and below, by f being *monotone* we mean that $f(\omega) \le f(\omega')$ for every $\omega, \omega' \in \{0,1\}^n$ such that $\omega_j \le \omega'_j$ for each $j \in [n]$.

Proposition 2.4. *Let* $f : \{0,1\}^n \to [0,1]$, $p \in (0,1)$, *and set* $\bar{p} = \min\{p, 1-p\}$.

a) If f is monotone then h_f is monotone.

b) $\mathrm{Inf}_{1/2,i}(h_f) \leqslant 2\bar{p}\cdot\mathrm{Inf}_{p,i}(f)$, *and moreover equality holds if f is monotone.*

c) $(f_n)_{n\geqslant 1}$ is NS_p $\quad\Leftrightarrow\quad$ $\big(h_{f_n}\big)_{n\geqslant 1}$ is NS.
Moreover, if $p \neq 1/2$ *then this is also equivalent to* $\lim_{n\to\infty} \mathrm{Var}\,\big(h_{f_n}\big) = 0$.

Proposition 2.4 is a key step in order to give short proofs of Theorem 1.4 and 2.6 from their uniform counterparts. A related reduction method has previously been employed to give short deductions (from the uniform case) of results for biased measure; see e.g. Friedgut (2004); Keller (2010); Keller, Mossel, and Schlank (2011).

2.3 The deterministic algorithm method

In order to prove Theorem 1.5, we shall use the 'algorithm approach', which was also introduced by Benjamini, Kalai, and Schramm (1999) in the case $p = 1/2$. (We would like to thank Jeff Steif for pointing out to us that the approach of Benjamini et al. can be synthesized in the way it is presented here.) Given a function $f : \{0,1\}^n \to \mathbb{R}$, let $\mathcal{A}^*(f)$ denote the collection of deterministic algorithms which determine f.[1]

Definition 2.5 (Revealment of an algorithm)**.** *Let* $f : \{0,1\}^n \to [0,1]$ *and let* $\mathcal{A} \in \mathcal{A}^*(f)$. *For each* $p \in (0,1)$, $K \subseteq [n]$ *and* $j \in K$, *define*

$$\delta_j(\mathcal{A}) = \delta_j(\mathcal{A}, p) := \mathbb{P}_p\big(\mathcal{A} \text{ queries bit } j \text{ when determining } f(\omega)\big).$$

[1] An algorithm is simply a rule which, given the information about ω received so far, tells you which bit of ω to query next. It determines f if it determines $f(\omega)$ for any input $\omega \in \{0,1\}^n$.

The revealment $\delta_K(\mathcal{A})$ *of* $\mathcal{A}$ *with respect to* K *is defined to be* $\max_{j\in K}\delta_j(\mathcal{A})$.

Using Theorem 1.4, we shall prove the following theorem, which generalizes the method of Benjamini et al. (1999) to the non-uniform set-up. We emphasize that $\delta_K(\mathcal{A})$ depends on p.

Theorem 2.6. *Let* $r \in \mathbb{N}$ *be fixed, and let* $(f_n)_{n\geqslant 1}$ *be a sequence of monotone functions* $f_n : \{0,1\}^n \to [0,1]$. *For each* $n \in \mathbb{N}$, *let* $\mathcal{A}_1, \ldots, \mathcal{A}_r \in \mathcal{A}^*(f_n)$, *and let* $[n] = K_1 \cup \ldots \cup K_r$. *If for* $p \in (0,1)$

$$\delta_{K_i}(\mathcal{A}_i)\big(\log n\big)^6 \;\to\; 0$$

as $n \to \infty$ *for each* $i \in [r]$, *then* $(f_n)_{n\geqslant 1}$ *is* NS_p.

In order to apply Theorem 2.6, we shall need to define a deterministic algorithm which determines f_N^η, and show that it has low revealment with high probability. The algorithm which we shall use is analogous to that used by Benjamini et al. Roughly speaking, we 'pour water' into the left-hand side of the square R_N, and query every element of η that we reach via a path in $D(\eta_p)$ (see Section 7 for a precise definition). For elements in the left half of R_N we pour water into the right-hand side.

It is easy to see that the probability that an element $x \in \eta \cap R_N$ is queried by $\mathcal{A}$ is at most the probability of the corresponding 'one-arm event', i.e., the probability that there is a path from x to the side of R_N in $D(\eta_p)$ (for background on arm-events, see e.g. Bollobás and Riordan (2006a)). In the original Poisson Boolean model a bound can be deduced from the RSW Theorem for the vacant space due to Roy (1990). However, in order to apply Theorem 2.6 we need a bound for the one-arm-events of f_N^η simultaneously for each $x \in \eta \cap R_N$; we obtain such a bound using Proposition 2.3.

In order to apply Proposition 2.3, we simply surround each point $x \in \eta \cap R_N$ by $c \log N$ disjoint annuli, and show that, with *very* high probability (in $\mathbf{P}_{\lambda_c/p}$), at least half of them are 'good', in the sense that the probability that there is a vacant loop around x is at least γ, for some small constant $\gamma > 0$. It will then follow that with high probability (in $\mathbf{P}_{\lambda_c/p}$), every $x \in \eta \cap R_N$ has probability at most $N^{-\delta}$ of being queried by $\mathcal{A}$ (see Section 7).

2.4 Hypergraphs

Theorem 1.6 provides a very general bound on the variance that arises in settings where two stages of randomness are used to select a random subset. We remark that alternative versions in which one or both of the sets $A \subseteq B \subseteq [n]$ have a fixed size (see Proposition 6.1 and 6.4) are also proved in Section 6. We prove these results on the way to proving Theorem 1.6.

The main step in the proof of Theorem 1.6 is to prove a variance bound (Proposition 6.1) for the case where the random sets $A \subseteq B \subseteq [n]$ have fixed sizes m and $k \geqslant m$. It is then relatively straightforward to deduce a corresponding bound on $\mathrm{Var}\big(r_{\mathcal{H}}([n]_q, p)\big)$, and thus prove Theorem 1.6, by bounding other factors that might contribute towards the variance. These bounds are obtained using Chernoff's inequality (see Section 6).

We shall control $\mathrm{Var}\big(X_m(B_k)\big)$, where B_k is a uniformly chosen k-element subset of $[n]$ and $X_m(B) = X_m(B, \mathcal{H})$ counts the number of hypergraph edges of size m contained in $B \subseteq [n]$, using the following theorem of Bey (2003) concerning the sum of squares of degrees in hypergraphs. It generalizes results of Ahlswede and Katona (1978) and de Caen (1998), and answered a question of Aharoni (1980).

Let $e(\mathcal{H})$ denote the number of edges in a hypergraph $\mathcal{H}$, and, given a set $T \subseteq [n]$, let $d_{\mathcal{H}}(T)$ denote the *degree* of T in $\mathcal{H}$, i.e., the number of edges of $\mathcal{H}$ which contain T. The following result bounds the sum of the squares of the degrees over sets of size t in an m-uniform hypergraph, i.e., one in which all edges have size m. By convention, we let $\binom{n}{k} := 0$ for $k < 0$ and $k > n$.

Bey's inequality (Bey (2003)). *Let $\mathcal{H}$ be an m-uniform hypergraph on n vertices, and let $t \in [m]$. Then*

$$d_2\big(\mathcal{H}, t\big) := \sum_{T \subseteq [n] : |T| = t} d_{\mathcal{H}}(T)^2 \;\leqslant\; \frac{\binom{m}{t}\binom{m-1}{t}}{\binom{n-1}{t}} e(\mathcal{H})^2 + \binom{m-1}{t-1}\binom{n-t-1}{m-t} e(\mathcal{H}).$$

To see how Bey's inequality is related to the variance of $X_m(B_k)$, observe that $d_{\mathcal{H}}(T)^2$ counts the number of (ordered) pairs of edges of size m in $\mathcal{H}$ which both contain T. Thus, summing over t (with appropriate weights), we obtain an upper bound on $\mathbb{E}\big[X_m(B_k)^2\big]$.

2.5 Summary of the proof

Let us finish this section by summarizing the proof of Theorem 1.2:

(i) We use Bey's inequality to prove Theorem 1.6.

(ii) We use Proposition 2.4 to deduce the BKS Theorem, as well as Theorem 2.6, for product measure, from the uniform case.

(iii) We deduce Proposition 2.1 and 2.3 from Theorem 1.6 and 2.2.

(iv) Using Proposition 2.3 we prove that, with high probability, there exists an algorithm (in fact, a pair of algorithms) for f_N^η with low revealment.

(v) Using our algorithm, Theorem 2.6 implies that $(f_N^\eta)_{N\geqslant 1}$ is NS_p for $\mathbf{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$, completing the proof of Theorem 1.5.

(vi) Finally, we shall show that Theorem 1.5 and Proposition 2.1 together imply that $(f_N^G)_{N\geqslant 1}$ is noise sensitive, completing the proof of Theorem 1.2.

Theorem 2.2 is proved in Section 3, Step 2 is performed in Sections 4 and 5, Step 1 is performed in Section 6, and the missing parts are completed in Section 7.

3 Non-triviality of the crossing probability at criticality

In this section we shall state the RSW Theorem for vacant space in the Poisson Boolean model, which was proved by Roy (1990). We shall sketch the proof of Theorem 2.2, which says that at criticality, the probability of crossing a square is bounded away from zero and one. The proof is based on the RSW Theorem. It is with help from Theorem 2.2 we will will be able to deduce that the algorithm we shall use has low revealment.

Recall that the probability measure $\mathbf{P}_\lambda$ indicates that the set $\eta \in \Omega$ is chosen according to a Poisson process on $\mathbb{R}^2$ with intensity λ. Here and later, $H\big(\eta, R, \circ\big)$ (or $V\big(\eta, R, \circ\big)$) will denote the event that the rectangle $R \subseteq \mathbb{R}^2$ contains a horizontal (or vertical) crossing of $\mathbb{R}^2 \setminus D(\eta)$.

Vacant RSW Theorem (Roy (1990), see Meester and Roy (1996, Theorem 4.2))**.** *For every $\delta, t, \lambda > 0$, there exists an $\varepsilon = \varepsilon(\delta, t, \lambda) > 0$ such that the following holds for every $a, b, c > 0$ with $c \leqslant 3a/2$. If*

$$\mathbf{P}_\lambda\Big(H\big(\eta, R_{a\times b}, \circ\big)\Big) \geqslant \delta, \quad \text{and} \quad \mathbf{P}_\lambda\Big(V\big(\eta, R_{c\times b}, \circ\big)\Big) \geqslant \delta,$$

then $\mathbf{P}_\lambda\Big(H\big(\eta, R_{ta\times b}, \circ\big)\Big) \geqslant \varepsilon$.

We remark that this result was in fact proved in substantially greater generality: it holds for random radii, with arbitrary distribution on $(0, r)$ (where $r \in \mathbb{R}$ is arbitrary). Alexander (1996) proved the corresponding statement for the occupied space (for fixed radii), and used this result to prove the following characterization.

Theorem 3.1 (Alexander (1996, Theorem 3.4))**.** *In the Poisson Boolean model, there exists $\theta > 0$ such that the following are equivalent:*

a) There is almost surely an infinite occupied component.

b) $\lim_{N\to\infty} \mathbf{P}_\lambda\Big(H\big(\eta, R_N, \bullet\big)\Big) = 1.$

c) $\lim_{N\to\infty} \mathbf{P}_\lambda\Big(H\big(\eta, R_{3N\times N}, \bullet\big)\Big) = 1.$

d) There exists $N \in \mathbb{N}$ *such that* $\mathbf{P}_\lambda\Big(H\big(\eta, R_{3N\times N}, \bullet\big)\Big) > 1 - \theta.$

The same holds true if 'occupied' is changed for 'vacant' throughout.

It follows immediately that there is no percolation at criticality for either the occupied or vacant space.

Corollary 3.2 (Alexander (1996, Corollary 3.5))**.** *At* $\lambda = \lambda_c$*, there is almost surely no infinite component in the occupied space* $D(\eta)$*, and no infinite component in the vacant space* $\mathbb{R}^2 \setminus D(\eta)$*.*

Proof. A standard argument shows that $\mathbf{P}_\lambda\big(H(\eta, R_{3N\times N}, \bullet)\big)$ is a continuous function of λ, and so the set of $\lambda \in \mathbb{R}$ for which property *d)* of Theorem 3.1 holds is an open set. □

Theorem 2.2 follows immediately from Corollary 3.2, together with the following slight extension of Theorem 3.1.

Theorem 3.3. *For every* $t > 0$*,*

a) $\sup_{N\geqslant 1} \mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times tN}, \bullet\big)\Big) = 1 \quad \Rightarrow \quad D(\eta)$ *percolates almost surely.*

b) $\sup_{N\geqslant 1} \mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times tN}, \circ\big)\Big) = 1 \quad \Rightarrow \quad \mathbb{R}^2 \setminus D(\eta)$ *percolates almost surely.*

The proof of Theorem 3.3 is almost identical to that of Theorem 3.1; for completeness, we shall sketch the argument.

Sketch proof of Theorem 3.3. We shall prove only *a)*; part *b)* follows by the same proof, except using the Occupied RSW Theorem of Alexander (1996, Theorem 2.1) in place of the Vacant RSW Theorem. We claim that our assumption implies property *d)* in Theorem 3.1, and hence (by property *a)* of the theorem) that $D(\eta)$ percolates.

To show that property *d)* of Theorem 3.1 holds, we note that a straightforward argument shows that

$$\sup_{N\geqslant 1} \mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times tN}, \bullet\big)\Big) = 1 \quad \Rightarrow \quad \sup_{N\geqslant 1} \mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times N}, \bullet\big)\Big) = 1,$$

by the Vacant RSW Theorem, applied with $a = b = c = N$. The latter implies that for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \geqslant 1$ such that

$$\mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times N}, \bullet\big)\Big) \;=\; 1 \,-\, \mathbf{P}_\lambda\Big(V\big(\eta, R_{N\times N}, \circ\big)\Big) \;>\; 1 - \varepsilon. \tag{3.1}$$

Next, observe that for every $N > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}\mathbf{P}_\lambda\Big(V\big(\eta, R_{3N\times N}, \circ\big)\Big) \;\leqslant\; & (2k+1)\mathbf{P}_\lambda\Big(V\big(\eta, R_{N\times N}, \circ\big)\Big)\\ & +\, 2k \cdot \mathbf{P}_\lambda\Big(H\big(\eta, R_{\frac{k-1}{k}N\times N}, \circ\big)\Big),\end{aligned}$$

and moreover that

$$\begin{aligned}\mathbf{P}_\lambda\Big(V\big(\eta, R_{3N\times N}, \circ\big)\Big) \;\leqslant\; & 2k \cdot \mathbf{P}_\lambda\Big(V\big(\eta, R_{\frac{k+1}{k}N\times N}, \circ\big)\Big)\\ & +\, (2k-1)\mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times N}, \circ\big)\Big).\end{aligned}$$

To see these, partition the rectangle $R_{3N\times N}$ into $\frac{N}{k} \times N$ rectangles $B_1, \ldots, B_{3k}$, and consider the leftmost and rightmost pieces B_j touched by a vertical path across $R_{3N\times N}$. The first follows because either an $N \times N$ square (made up of B_j's) is crossed vertically, or a $\big(\frac{k-1}{k}\big)\, N \times N$ rectangle is crossed horizontally. The second follows because either a $\frac{k+1}{k}N \times N$ rectangle is crossed vertically, or an $N \times N$ square is crossed horizontally.

Thus, either $\mathbf{P}_\lambda\big(V(\eta, R_{3N\times N}, \circ)\big)$ can be made arbitrarily small, as required, or there exists $\delta > 0$ such that

$$\mathbf{P}_\lambda\Big(H\big(\eta, R_{\frac{k-1}{k}N\times N}, \circ\big)\Big) \;\geqslant\; \delta \quad \text{and} \quad \mathbf{P}_\lambda\Big(V\big(\eta, R_{\frac{k+1}{k}N\times N}, \circ\big)\Big) \;\geqslant\; \delta \tag{3.2}$$

for every $N = N(\varepsilon)$ and every $\varepsilon > 0$.

Now, apply the Vacant RSW Theorem with $a = \frac{k-1}{k}N$, $b = N$, $c = \frac{k+1}{k}N$, for some $N > 0$. Note that $c \leqslant 3a/2$ if $2(k+1) \leqslant 3(k-1)$, which holds if $k \geqslant 5$. Setting $t = \frac{k}{k-1}$, it follows that if (3.2) holds for N, then

$$\mathbf{P}_\lambda\Big(H\big(\eta, R_{N\times N}, \circ\big)\Big) \;\geqslant\; \varepsilon'. \tag{3.3}$$

where $\varepsilon' = \varepsilon(\delta, t, \lambda) > 0$ is given by the Vacant RSW Theorem.

Hence if (3.2) holds for $N = N(\varepsilon')$ then (3.3) also holds, and (3.3) contradicts (3.1). Thus (3.2) must fail to hold for $N = N(\varepsilon')$, and so, by the observations above, $\mathbf{P}_\lambda\big(V(\eta, R_{3N\times N}, \circ)\big)$ can be made arbitrarily small, as required. $\square$

4 BKS Theorem for biased product measure

A tool that has turned out to be very useful in connection with the study of Boolean functions is discrete Fourier analysis. For $\omega \in \{0,1\}^n$ and $i \in [n]$, we define

$$\chi_i^p(\omega) = \begin{cases} -\sqrt{\frac{1-p}{p}} & \text{if } \omega_i = 1 \\ \sqrt{\frac{p}{1-p}} & \text{otherwise.} \end{cases}$$

Furthermore, for $S \subseteq [n]$, let $\chi_S^p(\omega) := \prod_{i \in S} \chi_i^p(\omega)$. (In particular, $\chi_\emptyset^p$ is the constant function 1.) We observe that for $i \neq j$

$$\mathbb{E}_p\big[\chi_i^p(\omega)\chi_j^p(\omega)\big] \;=\; \left(\frac{1-p}{p}\right) p^2 + \left(\frac{p}{1-p}\right)(1-p)^2 \;-\; 2p(1-p) \;=\; 0.$$

In fact, from this it is easily seen that the set $\{\chi_S^p\}_{S \subseteq [n]}$ forms an orthonormal basis for the set of functions $f : \{0,1\}^n \mapsto \mathbb{R}$. We can therefore express such functions using the so-called *Fourier-Walsh representation* (see Paley (1932); Walsh (1923)):

$$f(\omega) = \sum_{S \subseteq [n]} \hat{f}^p(S)\chi_S^p(\omega), \tag{4.1}$$

where $\hat{f}^p(S) := \mathbb{E}_p[f\chi_S^p]$.

Although our results throughout Section 4 and 5 hold for arbitrary $p \in (0,1)$, we will often prove them only for $p \leqslant 1/2$, since this is the case we shall need in our applications. The proofs for $p > 1/2$ all follow in exactly the same way. From now on, we will not stress that $S \subseteq [n]$ in the notation. Furthermore, when $p = 1/2$ we shall write χ_S for χ_S^p and $\hat{f}(S)$ for $\hat{f}^p(S)$.

The following lemma was proved by Benjamini et al. (1999) in the uniform case; its generalization to arbitrary (fixed) p is similarly straightforward.

Lemma 4.1. *Let $p \in (0,1)$, and let $(f_n)_{n\geqslant 1}$ be a sequence of functions $f_n : \{0,1\}^n \mapsto [0,1]$. The following two conditions are equivalent.*

a) The sequence $(f_n)_{n\geqslant 1}$ is NS_p.

b) For every $k \in \mathbb{N}$, $\displaystyle\lim_{n\to\infty} \sum_{0<|S|\leqslant k} \hat{f_n}^p(S)^2 = 0.$

Proof. Note that $\mathbb{E}_p\big[\chi_S^p(\omega)\chi_{S'}^p(\omega^\varepsilon)\big] = 0$ if $S \neq S'$, that $\mathbb{E}_p\big[f_n(\omega)\big] = \hat{f_n}^p(\emptyset)$, and that

$$\mathbb{E}_p\big[\chi_S^p(\omega)\chi_S^p(\omega^\varepsilon)\big] = (1-\varepsilon)^{|S|},$$

since this is zero whenever at least one of the coordinates $\{\omega_i : i \in S\}$ is re-randomized, and one otherwise. By (4.1), it follows that

$$\mathbb{E}_p\big[f_n(\omega)f_n(\omega^\varepsilon)\big] - \mathbb{E}_p\big[f_n(\omega)\big]^2 \;=\; \mathbb{E}_p\left[\sum_{S\neq\emptyset} \hat{f_n}^p(S)\chi_S^p(\omega) \sum_{S'\neq\emptyset} \hat{f_n}^p(S')\chi_{S'}^p(\omega^\varepsilon)\right]$$

$$= \sum_{S\neq\emptyset} \hat{f_n}^p(S)^2\mathbb{E}_p\big[\chi_S^p(\omega)\chi_S^p(\omega^\varepsilon)\big] \;=\; \sum_{S\neq\emptyset} \hat{f_n}^p(S)^2(1-\varepsilon)^{|S|},$$

from which both implications follow easily. $\square$

Recall from (2.3) that we define $h_f(X) := \mathbb{E}\big[f(Z)\big|X\big]$, where (for $p \leqslant 1/2$) X and Y in $\{0,1\}^n$ are independent random variables, X uniformly distributed and Y with density $2p$, and $Z_i = X_iY_i$ for every $i \in [n]$. The key fact - that the sequence $(f_n)_{n\geqslant 1}$ is NS_p if and only if $(h_{f_n})_{n\geqslant 1}$ is NS - will follow directly from Lemma 4.1, together with the following result.

Proposition 4.2. *Let $f : \{0,1\}^n \to [0,1]$, $p \in (0,1)$, and set $\bar{p} = \min\{p, 1-p\}$. Then, for every $S \subseteq [n]$,*

$$\hat{h}_f(S) \;=\; \left(\frac{\bar{p}}{1-\bar{p}}\right)^{|S|/2} \hat{f}^p(S).$$

Proof. We shall prove the proposition in the case $p \leqslant 1/2$; the other case follows similarly. Let $f : \{0,1\}^n \to [0,1]$ and $S \subseteq [n]$. By the definitions, we have

$$\begin{aligned} \hat{h}_f(S) \;&=\; \mathbb{E}\big[h_f(X)\chi_S(X)\big] \;=\; \mathbb{E}\Big[\mathbb{E}\big[f(Z)\,\big|\,X\big]\chi_S(X)\Big] \\ &=\; \mathbb{E}\Big[\mathbb{E}\big[f(Z)\chi_S(X)\,\big|\,X\big]\Big] \;=\; \mathbb{E}\big[f(Z)\chi_S(X)\big] \\ &=\; \mathbb{E}\Big[f(Z)\mathbb{E}\big[\chi_S(X)\,\big|\,Z\big]\Big]. \end{aligned} \tag{4.2}$$

Furthermore, $Z_i = 1$ implies $X_i = 1$, which implies $\chi_i(X) = -1$, so

$$\mathbb{E}\big[\chi_i(X)\,\big|\,Z_i = 1\big] \;=\; -1,$$

while $Z_i = 0$ and $X_i = 1$ implies that $Y_i = 0$, so

$$\begin{aligned} \mathbb{E}\big[\chi_i(X)\big|Z_i = 0\big] \;&=\; 1 \;-\; 2\cdot\mathbb{P}\big(X_i = 1\,\big|\,Z_i = 0\big) \\ &=\; 1 \;-\; 2\cdot\frac{\mathbb{P}(Z_i = 0\,|\,X_i = 1)\mathbb{P}(X_i = 1)}{\mathbb{P}(Z_i = 0)} \\ &=\; 1 - \frac{1-2p}{1-p} \;=\; \frac{p}{1-p}. \end{aligned}$$

We conclude that $\mathbb{E}\big[\chi_i(X)\big|Z_i\big] = \sqrt{\frac{p}{1-p}}\chi_i^p(Z)$. Therefore, since the X_i and Y_i are all independent,

$$\begin{aligned}\mathbb{E}\big[\chi_S(X)\,\big|\,Z\big] &= \prod_{i\in S}\mathbb{E}\big[\chi_i(X)\,\big|\,Z_i\big]\\ &= \prod_{i\in S}\sqrt{\frac{p}{1-p}}\chi_i^p(Z) \;=\; \left(\frac{p}{1-p}\right)^{|S|/2}\chi_S^p(Z).\end{aligned}$$

Inserting this into (4.2) gives the result. □

It is now straightforward to deduce Proposition 2.4 from Proposition 4.2.

Proof of Proposition 2.4. We shall assume that $p \leqslant 1/2$; once again, the other case follows similarly. Let $f : \{0,1\}^n \to [0,1]$.

a) Suppose that f is monotone; we claim that h_f is also monotone. Indeed, observe that

$$h_f(X) \;=\; \mathbb{E}\big[f(Z)\big|X\big] \;=\; \mathbb{E}\big[f(X\cdot Y)\big|X\big] \;=\; \sum_{\xi\in\{0,1\}^n} f(X\cdot\xi)\mathbb{P}(Y=\xi). \tag{4.3}$$

But if f is monotone, then $f(X\cdot\xi)$ is also monotone in X for every $\xi \in \{0,1\}^n$. Thus (4.3) implies that h_f is monotone, as required.

b) We next claim that $\mathrm{Inf}_{1/2,i}(h_f) \leqslant 2p\cdot\mathrm{Inf}_{p,i}(f)$ for every $i \in [n]$. For $i \in [n]$ and $k \in \{0,1\}$, let $X^{i\to k} \in \{0,1\}^n$ be defined by $X_j^{i\to k} = X_j$ if $j \neq i$, and $X_i^{i\to k} = k$.

By the definition, we have

$$\begin{aligned}\mathrm{Inf}_{1/2,i}(h_f) &= \mathbb{E}\big[\big|h_f(X) - h_f(\sigma_i X)\big|\big]\\ &= \mathbb{E}\Big[\Big|\mathbb{E}\big[f(Z)\big|X^{i\to1}\big] - \mathbb{E}\big[f(Z)\big|X^{i\to0}\big]\Big|\Big].\end{aligned} \tag{4.4}$$

Now, if $X_i = 1$, then $Y_i = 1$ if and only if $Z_i = 1$, and if $X_i = 0$ then $Z_i = 0$, so the right-hand side of (4.4) is equal to

$$\mathbb{E}\Big[\Big|2p\cdot\mathbb{E}\big[f(Z)\,\big|\,X^{i\to1}, Z_i = 1\big] \;+\; (1-2p)\mathbb{E}\big[f(Z)\,\big|\,X^{i\to1}, Z_i = 0\big] - \mathbb{E}\big[f(Z)\,\big|\,X^{i\to0}, Z_i = 0\big]\Big|\Big].$$

But given Z_i, the value of X_i is irrelevant to $f(Z)$, so we have (with obvious notation $X_{\{i\}^c}$)

$$\begin{aligned}\mathrm{Inf}_{1/2,i}(h_f) &= 2p\cdot\mathbb{E}\Big[\big|\mathbb{E}\big[f(Z^{i\to1}) \;-\; f(Z^{i\to0})\,\big|\,X_{\{i\}^c}\big]\big|\Big]\\ &\leqslant 2p\cdot\mathbb{E}\Big[\big|f(Z^{i\to1}) \;-\; f(Z^{i\to0})\big|\Big] \;=\; 2p\cdot\mathrm{Inf}_{p,i}(f),\end{aligned} \tag{4.5}$$

as required. Finally, note that the inequality in (4.5) be replaced by an equality when f is monotone.

c) We are required to show that $(f_n)_{n\geqslant 1}$ is NS_p if and only if $(h_{f_n})_{n\geqslant 1}$ is NS. Indeed, by Lemma 4.1, $(f_n)_{n\geqslant 1}$ is NS_p if and only if $\sum_{0<|S|\leqslant k} \hat{f_n}^p(S)^2 \to 0$ as $n \to \infty$ for every fixed k, and by Proposition 4.2,

$$\lim_{n\to\infty} \sum_{0<|S|\leqslant k} \hat{h}_{f_n}(S)^2 = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \sum_{0<|S|\leqslant k} \hat{f_n}^p(S)^2 = 0$$

for every such k. But by Lemma 4.1 (applied with $p = 1/2$), we have that $(h_{f_n})_{n\geqslant 1}$ is NS if and only if $\sum_{0<|S|\leqslant k} \hat{h}_{f_n}(S)^2 \to 0$ as $n \to \infty$ for every fixed k, so the result follows.

Finally, note that $\text{Var}(h_{f_n}) = \sum_{S\neq\emptyset} \hat{h}_{f_n}(S)^2$. Thus, by Proposition 4.2, if $p \neq 1/2$ then

$$\text{Var}(h_{f_n}) \;=\; \sum_{S\neq\emptyset} \left(\frac{p}{1-p}\right)^{|S|} \hat{f_n}^p(S)^2 \;\to\; 0$$

as $n \to \infty$ if and only if $\sum_{0<|S|\leqslant k} \hat{f_n}^p(S)^2 \to 0$ as $n \to \infty$ for every fixed k, as claimed. ∎

The BKS Theorem for biased product measure follows almost immediately from the uniform case, together with Proposition 2.4.

Proof of Theorem 1.4. Let $(f_n)_{n\geqslant 1}$ be a sequence of functions $f_n : \{0,1\}^n \to [0,1]$, let $p \in (0,1)$, and assume that $I\!I_p(f_n) \to 0$ as $n \to \infty$. We are required to show that $(f_n)_{n\geqslant 1}$ is NS_p.

By part *b)* of Proposition 2.4, we have

$$I\!I(h_{f_n}) \;\leqslant\; 4\bar{p}^2 \cdot I\!I(f_n),$$

and so $I\!I(h_{f_n}) \to 0$ as $n \to \infty$. By the BKS Theorem (i.e., Theorem 1.4 in the case $p = 1/2$), which was proved by Benjamini et al. (1999), it follows that $(h_{f_n})_{n\geqslant 1}$ is NS. But, by part *c)* of Proposition 2.4, we have $(h_{f_n})_{n\geqslant 1}$ is NS if and only if $(f_n)_{n\geqslant 1}$ is NS_p. Hence $(f_n)_{n\geqslant 1}$ is NS_p, as required. ∎

5 The deterministic algorithm approach

In this section we shall prove Theorem 2.6. Throughout this section let X and Z be the random variables defined in (2.2), so $X \in \{0,1\}^n$ is uniformly distributed, and $Z \in \{0,1\}^n$ is given by $Z_i = X_i Y_i$, where Y is chosen according to product measure with density $2p$. (We assume again for simplicity that $p \leqslant 1/2$.)

We need the following definition.

Definition 5.1 (The Majority function)**.** *For every $K \subseteq [n]$ define the function $M_K : \{0,1\}^n \to \{-1,0,1\}$ by*

$$M_K(X) := \begin{cases} 1 & \text{if } \sum_{i \in K} (2X_i - 1) > 0 \\ 0 & \text{if } \sum_{i \in K} (2X_i - 1) = 0 \\ -1 & \text{if } \sum_{i \in K} (2X_i - 1) < 0. \end{cases}$$

Theorem 2.6 will follow by combining the BKS Theorem with the following two propositions, which were both obtained by Benjamini et al. (1999) in the case $p = 1/2$. We shall generalize them to the biased setting.

Proposition 5.2. *There exists a universal constant $C < \infty$ such that, if $f : \{0,1\}^n \to [0,1]$ is monotone, $p \in (0,1)$ and $K \subseteq [n]$, then*

$$\sum_{j \in K} \mathrm{Inf}_{p,j}(f) \leqslant \frac{C}{\bar{p}} \sqrt{|K|}\, \mathbb{E}\big[f(Z) M_K(X)\big] \left(1 + \sqrt{-\log \mathbb{E}\big[f(Z) M_K(X)\big]}\right),$$

where $\bar{p} = \min(p, 1-p)$.

Recall that the revealment of an algorithm with respect to a set $K \subseteq [n]$ is defined as $\delta_K(\mathcal{A}) := \max_{j \in K} \mathbb{P}_p\big(\mathcal{A}$ queries coordinate $j\big)$.

Proposition 5.3. *There exists a universal constant $C > 0$ such that, if $f : \{0,1\}^n \to [0,1]$, $p \in (0,1)$, $K \subseteq [n]$ and $\mathcal{A} \in \mathcal{A}^*(f)$, then*

$$\mathbb{E}\big[f(Z) M_K(X)\big] \leqslant C\, \delta_K(\mathcal{A})^{1/3} \log n.$$

We begin by proving Proposition 5.2, which follows almost immediately from the uniform case, together with Proposition 2.4.

Proof of Proposition 5.2. The proposition was proved by Benjamini et al. (1999, Corollary 3.2) in the case $p = 1/2$; we apply this result to the function h_f. It follows that

$$\sum_{j \in K} \mathrm{Inf}_{1/2,j}(h_f) \leqslant C \sqrt{|K|} \mathbb{E}\big[h_f(X) M_K(X)\big] \left(1 + \sqrt{-\log \mathbb{E}\big[h_f(X) M_K(X)\big]}\right).$$

for some $C > 0$. Next, observe that

$$\mathbb{E}\big[h_f(X) M_K(X)\big] = \mathbb{E}\Big[\mathbb{E}\big[f(Z) M_K(X) \,\big|\, X\big]\Big] = \mathbb{E}\big[f(Z) M_K(X)\big].$$

Since f is monotone, we have $\mathrm{Inf}_{1/2,j}(h_f) = 2\bar{p} \cdot \mathrm{Inf}_{p,j}(f)$, by Proposition 2.4, and so the result follows. $\square$

Also the proof of Proposition 5.3 will be based on the argument of Benjamini et al. (1999, Section 4), but modified to fit in the current setting. The strategy is roughly as follows: let V denote the set of coordinates queried by the algorithm. Then with high probability, $V \cap K$ is small enough so that $M_K(X)$ will (probably) be determined by the values of bits of X in $K \setminus V$. By a careful coupling, we can make these independent of the value of f, and thus $\mathbb{E}\big[f(Z)M_K(X)\big]$ is small.

We shall use Chernoff's inequality; see e.g. Alon and Spencer (2008, Appendix A). Throughout the rest of the paper, $\xi_{n,p}$ will denote a binomially distributed random variable with parameters $n \in \mathbb{N}$ and $p \in (0,1)$.

Chernoff's inequality. *Let $n \in \mathbb{N}$ and $p \in (0,1)$, and let $a > 0$. Then*

$$\mathbb{P}\Big(\big|\xi_{n,p} - pn\big| > a\Big) \; < \; 2\exp\left(-\frac{a^2}{4pn}\right) \tag{5.1}$$

if $a \leqslant pn/2$, and $\mathbb{P}\big(|\xi_{n,p} - pn| > a\big) < 2\exp\big(-pn/16\big)$ otherwise. If $p = 1/2$, then (5.1) holds for every $a \geq 0$.

We shall also use the following simple property of the binomial distribution, which follows by Stirling's formula.

Observation 5.4. *There exists $C > 0$ such that for every $n \in \mathbb{N}$, $p \in (0,1)$ and $a \in \mathbb{N}$,*

$$\mathbb{P}\big(\xi_{n,p} = a\big) \; \leqslant \; \frac{C}{\sqrt{np(1-p)}}.$$

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. We assume as before that $p \leqslant 1/2$; the proof for $p > 1/2$ is similar. Let $f : \{0,1\}^n \to [0,1]$ and $\emptyset \neq K \subseteq [n]$ (if $|K| = 0$ then both sides are zero). We begin by defining our coupling; the purpose is to make the values of X_i outside V independent of those inside.

We shall obtain the random variables X and Z, defined in (2.2), as follows. Let $Z^1 \in \{0,1\}^K$, $Z^2 \in \{0,1\}^{[n]\setminus K}$ and $Z^3, Z^4 \in \{0,1\}^n$ be such that

$$\mathbb{P}\big(Z_i^j = 1\big) = p,$$

independently for each i and j. Similarly, let $W^1 \in \{0,1\}^K$, $W^2 \in \{0,1\}^{[n]\setminus K}$ and $W^3 \in \{0,1\}^n$ be independent of the Z_i^j, and such that

$$\mathbb{P}\big(W_i^j = 1\big) \; = \; \frac{1-2p}{2(1-p)},$$

independently for every i and j. Set $X_i^j = \max\{Z_i^j, W_i^j\}$, and observe that

$$\mathbb{P}\big(X_i^j = 1\big) \;=\; \mathbb{P}\big(Z_i^j = 1\big) + \mathbb{P}\big(Z_i^j = 0\big)\mathbb{P}\big(W_i^j = 1\big) \;=\; p + \frac{(1-p)(1-2p)}{2(1-p)} \;=\; \frac{1}{2}$$

for every i and j.

Next, we describe how to use the X_i^j and Z_i^j to assign values to coordinates, depending on the order in which they are queried by $\mathcal{A}$. Indeed, run the algorithm, and do the following:

(i) If $j \in K$ is queried, and is the k^{th} element of K to have been queried by $\mathcal{A}$, then set $Z_j := Z_k^1$ and $X_j := X_k^1$.

(ii) If $j \notin K$ is queried, and is the k^{th} element of $[n] \setminus K$ to have been queried by $\mathcal{A}$, then set $Z_j := Z_k^2$ and $X_j := X_k^2$.

(iii) When the algorithm stops, let $\pi : K \setminus V \to [|K \setminus V|]$ be an arbitrary bijection, and for each $j \in K \setminus V$ set $Z_j := Z_k^3$ and $X_j := X_k^3$, where $k = \pi(j)$.

(iv) Finally, let $Z_j := Z_j^4$ and $X_j := X_j^4$ for each $j \in [n] \setminus (V \cup K)$.

Note that X is chosen uniformly and Z according to the product measure with density p. Moreover, note that if $Z_i = 1$ then $X_i = 1$, so the coupling is as in (2.2), as claimed.

Let $V \subseteq [n]$ be the (random) set of coordinates which are queried by the algorithm, and note that V is independent of Z^3 and W^3. We first show that the set $V \cap K$ is likely to be small. Indeed, we have

$$\mathbb{E}\big[|V \cap K|\big] \;=\; \sum_{j \in K} \delta_{\mathcal{A}}(j) \;\leqslant\; |K|\delta_K(\mathcal{A}),$$

and so, if we define

$$B_1 \;:=\; \big\{|V \cap K| \geqslant |K|\delta_K(\mathcal{A})^{2/3}\big\},$$

then $\mathbb{P}(B_1) \leqslant \delta_K(\mathcal{A})^{1/3}$, by Markov's inequality.

Next we shall deduce that, with high probability, the difference between the number of 0s and 1s on $V \cap K$ is less than that on $K \setminus V$. Indeed, let $S_k := \sum_{j=1}^k (2X_j^1 - 1)$ denote this difference on the first k coordinates of X^1, and let $T_k := \sum_{j=1}^k (2X_j^3 - 1)$ denote the same thing for X^3. Let

$$B_2 \;:=\; \Big\{\exists\, k \leqslant |K|\delta_K(\mathcal{A})^{2/3} \;:\; |S_k| \geqslant \sqrt{|K|}\delta_K(\mathcal{A})^{1/3}\log n\Big\},$$

and let

$$B_3 := \left\{ |T_{|K \setminus V|}| \leqslant \sqrt{|K|}\delta_K(\mathcal{A})^{1/3} \log n \right\}.$$

Claim. $\mathbb{P}\big(B_1 \cup B_2 \cup B_3\big) = O\left(\delta_K(\mathcal{A})^{1/3} \log n\right)$.

Before proving the claim, let's see how it implies the proposition. Set $Q = \big(B_1 \cup B_2 \cup B_3\big)^c$, and let $\mathcal{F}$ be the sigma-algebra generated by Z^1, Z^2 and W^1. Then

$$\mathbb{E}\big[M_K(X)\,\mathbf{1}_Q \,\big|\, \mathcal{F}\big] = \mathbb{P}_p\big(Q \,\big|\, \mathcal{F}\big)\mathbb{E}\big[M_K(X) \,\big|\, \mathcal{F}, Q\big] = 0,$$

by symmetry, since $T_{|K\setminus V|}$ is equally likely to be positive or negative, and Q implies $|T_{|K\setminus V|}| > |S_{|V\cap K|}|$. Thus by the claim, and since $\mathcal{F}$ determines $f(Z)$, we have

$$\begin{aligned}
\big|\mathbb{E}\big[f(Z)M_K(X)\big]\big| &\leqslant \big|\mathbb{E}\big[f(Z)M_K(X)\,\mathbf{1}_Q\big]\big| + \mathbb{P}(Q^c) \\
&= \Big|\mathbb{E}\Big[f(Z)\mathbb{E}\big[M_K(X)\,\mathbf{1}_Q \,\big|\, \mathcal{F}\big]\Big]\Big| + \mathbb{P}(Q^c) \\
&= O\left(\delta_K(\mathcal{A})^{1/3}\log n\right),
\end{aligned}$$

as required.

Thus, it only remains to prove the claim, which follows easily using Chernoff's inequality. We have already shown that $\mathbb{P}(B_1) \leqslant \delta_K(\mathcal{A})^{1/3}$, and so it will suffice to prove corresponding bounds for B_2 and $B_3 \cap B_1^c$. The bound for B_2 follows using Chernoff and the union bound. Indeed, let $t = |K|\delta_K(\mathcal{A})^{2/3}$, and recall that X^1 was chosen uniformly. Thus, by Chernoff's inequality,

$$\begin{aligned}
\mathbb{P}(B_2) &\leqslant \sum_{k=1}^{t} \mathbb{P}\Big(\big|2\cdot\xi_{k,1/2} - k\big| > \sqrt{|K|}\delta_K(\mathcal{A})^{1/3}\log n\Big) \\
&\leqslant 2\sum_{k=1}^{t}\exp\left(-\frac{|K|\delta_K(\mathcal{A})^{2/3}\log^2 n}{8k}\right) \leqslant t\cdot e^{-\log^2 n/8} \leqslant \delta_K(\mathcal{A})^{1/3},
\end{aligned}$$

as required.

Finally, we shall bound the probability of $B_3 \cap B_1^c$; that is, the probability that

$$|V \cap K| \leqslant t = |K|\delta_K(\mathcal{A})^{2/3} \quad \text{and} \quad |T_{|K\setminus V|}| \leqslant \sqrt{|K|}\delta_K(\mathcal{A})^{1/3}\log n.$$

By Observation 5.4 and the union bound, we have

$$\mathbb{P}\Big(\big|2\cdot\xi_{m,1/2} - m\big| \leqslant \sqrt{|K|}\delta_K(\mathcal{A})^{1/3}\log n\Big) \leqslant \sqrt{|K|}\delta_K(\mathcal{A})^{1/3}\log n\cdot\frac{C_1}{\sqrt{m}},$$

for some constant $C_1 > 0$ and every $m \geqslant 1$. Since V is determined by the information in $\mathcal{F}$, and since X^3 is uniformly distributed, we have

$$\begin{aligned}
\mathbb{P}(B_3 \cap B_1^c) \;&=\; \mathbb{E}\big[\mathbb{P}(B_1^c \cap B_3|\mathcal{F})\big] \;=\; \mathbb{E}\big[\mathbf{1}_{B_1^c} \cdot \mathbb{P}(B_3|\mathcal{F})\big] \\
&\leqslant\; \mathbb{E}\left[\mathbf{1}_{B_1^c} \cdot \sqrt{|K|}\delta_K(\mathcal{A})^{1/3} \log n \frac{C_1}{\sqrt{|K \setminus V|}}\right] \\
&\leqslant\; \sqrt{|K|}\delta_K(\mathcal{A})^{1/3} \log n \frac{2C_1}{\sqrt{3|K|}} \;=\; O\left(\delta_K(\mathcal{A})^{1/3} \log n\right),
\end{aligned}$$

where we in the second inequality used that on B_1^c, we have

$$|K \setminus V| \;\geqslant\; |K| - t \;\geqslant\; 3|K|/4,$$

assuming that $t \leqslant |K|/4$ (since otherwise $\delta_K(\mathcal{A}) \geqslant 1/8$, and the proposition is trivial). This completes the proof of the claim, and hence of the proposition as well. $\square$

It is now easy to deduce Theorem 2.6. We shall use the following straightforward optimization lemma.

Lemma 5.5. *If $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$, then*

$$\max\left\{\sum_{i=1}^n c_i^2 \,:\, c_1 \geqslant \ldots \geqslant c_n \geqslant 0, \text{ and } \sum_{i=1}^k c_i \leqslant \sum_{i=1}^k a_i \text{ for all } k \in [n]\right\} = \sum_{i=1}^n a_i^2.$$

We are ready to prove Theorem 2.6.

Proof of Theorem 2.6. Let $r \in \mathbb{N}$ be fixed, and let $(f_n)_{n\geqslant 1}$ be a sequence of monotone functions $f_n : \{0,1\}^n \to [0,1]$. For each $n \in \mathbb{N}$, let $\mathcal{A}_1, \ldots, \mathcal{A}_r \in \mathcal{A}^*(f)$ and let $K_1, \ldots, K_r$ be a partition of $[n]$. Let $p \in (0,1)$, and suppose that

$$\delta_{K_i}(\mathcal{A}_i)\big(\log n\big)^6 \;\to\; 0$$

as $n \to \infty$ for each $i \in [r]$. We shall show that $I\!I_p(f_n) \to 0$ as $n \to \infty$, and hence deduce, by Theorem 1.4, that $(f_n)_{n\geqslant 1}$ is NS_p.

Choose $C > 0$ so that Propositions 5.2 and 5.3 both hold for C, and assume that $n \in \mathbb{N}$ is sufficiently large so that $\delta_{K_i}(\mathcal{A}_i)^{1/3} \log n \leqslant 1/(2C)$ for each $i \in [r]$. To bound $I\!I_p(f_n) = \sum_{j=1}^n \mathrm{Inf}_{p,j}(f_n)^2$ from above, we shall first bound $\sum_{j \in K} \mathrm{Inf}_{p,j}(f_n)$ for every $K \subseteq [n]$, and then apply Lemma 5.5. Let us assume for simplicity that $\delta_{K_i}(\mathcal{A}_i) \geqslant 1/n$ for some $i \in [r]$; the other case follows by an almost identical calculation.

Claim. For every $K \subseteq [n]$, we have

$$\sum_{j \in K} \mathrm{Inf}_{p,j}(f_n) \;\leqslant\; \frac{C^2 r}{\bar{p}} \sqrt{|K|} \max_{i \in [r]} \Big\{ \delta_{K_i}(\mathcal{A}_i)^{1/3} \Big\} \big(\log n \big)^{3/2}.$$

Proof of claim. By Proposition 5.2, for every $K \subseteq [n]$ we have

$$\sum_{j \in K} \mathrm{Inf}_{p,j}(f_n) \;\leqslant\; \frac{C}{\bar{p}} \sqrt{|K|} \mathbb{E}\big[f_n(Z) M_K(X) \big] \left(1 + \sqrt{ - \log \mathbb{E}\big[f_n(Z) M_K(X) \big] } \right).$$

Moreover, by Proposition 5.3, for every $i \in [r]$ and every $K \subseteq K_i$,

$$\mathbb{E}\big[f_n(Z) M_K(X) \big] \;\leqslant\; C \, \delta_{K_i}(\mathcal{A}_i)^{1/3} \log n.$$

Recall that $C \, \delta_{K_i}(\mathcal{A}_i)^{1/3} \log n \leqslant 1/2$, and note that $x\big(1 + \sqrt{-\log x}\big)$ is increasing on $(0, 1/2)$. Thus, if $K \subseteq K_i$ for some $i \in [r]$, then

$$\sum_{j \in K} \mathrm{Inf}_{p,j}(f_n) \;\leqslant\; \frac{C^2}{\bar{p}} \sqrt{|K|} \max_{i \in [r]} \Big\{ \delta_{K_i}(\mathcal{A}_i)^{1/3} \Big\} \big(\log n \big)^{3/2},$$

since $\max_{i \in [r]} \delta_{K_i}(\mathcal{A}_i) \geqslant 1/n$. Summing over $i \in [r]$, the claim follows. $\square$

Without loss of generality, assume that

$$\mathrm{Inf}_{p,1}(f_n) \;\geqslant\; \ldots \;\geqslant\; \mathrm{Inf}_{p,n}(f_n),$$

and apply Lemma 5.5 with $c_j = \mathrm{Inf}_{p,j}(f_n)$, and

$$a_j \;=\; \frac{C^2 r}{\bar{p}} \max_{i \in [r]} \Big\{ \delta_{K_i}(\mathcal{A}_i)^{1/3} \Big\} \big(\log n \big)^{3/2} \big(\sqrt{j} - \sqrt{j-1} \big).$$

By the claim applied to $K = [k]$, we have, for each $k \in [n]$,

$$\sum_{j=1}^{k} c_j \;=\; \sum_{j=1}^{k} \mathrm{Inf}_{p,j}(f_n) \;\leqslant\; \frac{C^2 r}{\bar{p}} \sqrt{k} \max_{i \in [r]} \Big\{ \delta_{K_i}(\mathcal{A}_i)^{1/3} \Big\} \big(\log n \big)^{3/2} \;=\; \sum_{j=1}^{k} a_j,$$

and hence, since p is fixed and $\sum_j \big(\sqrt{j} - \sqrt{j-1} \big)^2 = O\big(\log n \big)$, by Lemma 5.5 there is $C' < \infty$ such that

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{Inf}_{p,j}^2(f_n) \leqslant \sum_{j=1}^{n} a_j^2 \;&=\; \frac{C^2 r}{\bar{p}} \max_{i \in [r]} \Big\{ \delta_{K_i}(\mathcal{A}_i)^{2/3} \Big\} \big(\log n \big)^3 \sum_{j=1}^{n} \big(\sqrt{j} - \sqrt{j-1} \big)^2 \\ &\leqslant\; \frac{C'}{\bar{p}} \max_{i \in [r]} \Big\{ \delta_{K_i}(\mathcal{A}_i)^{2/3} \Big\} \big(\log n \big)^4 \;\to\; 0 \end{aligned}$$

as $n \to \infty$, as claimed. Thus, by Theorem 1.4, $(f_n)_{n \geqslant 1}$ is NS_p, as required. $\square$

We finish this section by proving the following closely related result, which was also proved by Benjamini et al. (1999, Theorem 1.5) in the case $p = 1/2$. In fact we shall not need it, but since it follows immediately from the uniform case and Proposition 2.4, and may be of independent interest, we include it for completeness.

Given a function $h : \{0,1\}^n \to [0,1]$, define

$$\Lambda(h) \;:=\; \max_{K \subseteq [n]} \mathbb{E}\big[h(X) M_K(X)\big].$$

In particular, $\Lambda(h_f) = \max_{K \subseteq [n]} \mathbb{E}\big[f(Z) M_K(X)\big]$.

Theorem 5.6. *There exists a constant $C > 0$ such that, if $f : \{0,1\}^n \to [0,1]$ is monotone and $p \in (0,1)$, then*

$$I\!I_p(f) \;\leqslant\; \frac{C}{\min(p^2,(1-p)^2)} \Lambda^2(h_f)\big(1 - \log \Lambda(h_f)\big) \log n.$$

Proof. We apply the uniform case to the function h_f. By Proposition 2.4, it follows that

$$4\bar{p}^2 \cdot I\!I_p(f) \;=\; I\!I_{1/2}(h_f) \;\leqslant\; C\Lambda^2(h_f)\big(1 - \log \Lambda(h_f)\big) \log n,$$

as required. $\square$

6 Hypergraphs

In this section we shall prove Theorem 1.6, which will allow us to bound the variance (in $\mathbf{P}_{\lambda_c/p}$) of the probability $\mathbb{P}\big(H(\eta_p, R_N, \bullet) \,|\, \eta\big)$. Although one can think of all the results in this section in terms of events on the cube $\{0,1\}^n$, it will be convenient for us to use the language of hypergraphs. For background on graph theory, see Bollobás (1998).

Recall that a hypergraph $\mathcal{H}$ is just a collection of subsets of $[n]$, which we refer to as edges of $\mathcal{H}$. We shall write $\mathcal{H}_m$ for the m-uniform hypergraph contained in $\mathcal{H}$, that is, the collection of edges with m elements, and recall that for $B \subseteq [n]$,

$$r_{\mathcal{H}}(B,p) \;:=\; \mathbb{P}\big(B_p \in \mathcal{H}\big).$$

The proof of Theorem 1.6 is in two parts: first we shall prove the corresponding result for sets of fixed size. That is, instead of considering $[n]_q$ we consider a uniformly chosen set $B_k \subseteq [n]$ of size k and instead of $[n]_q \cap [n]_p$, we consider a uniformly chosen subset of B_k of size $m \leqslant k$.

6.1 The proof for sets of fixed size

Let us begin by informally illustrating the central idea with a simple example. Let G be a (large) graph with vertex set $[n]$, and consider the restriction of G to a random subset $S \subseteq [n]$ selected uniformly at random from the sets of size k. If $k = 2$ then the resulting graph $G[S]$ will have density either 0 or 1, which will typically be quite far from the density of the original graph. However, once k is a large constant the density of $G[S]$ is already unlikely to be far from the density of G.

The following proposition extends this result to hypergraphs. Given a hypergraph $\mathcal{H}$ on vertex set $[n]$, a subset $S \subseteq [n]$ and an integer $0 \leqslant m \leqslant n$, define

$$X_m(S) \; := \; \big|\{e \in \mathcal{H}_m : e \subseteq S\}\big|,$$

and $\tilde{X}_m(S) := X_m(S)/\binom{|S|}{m}$.

Proposition 6.1. *Let $n, m, k \in \mathbb{N}$, and suppose that $n \geqslant k \geqslant m$, and that $n \geqslant 2m^3$ and $n \geqslant km/2$. Let $\mathcal{H}$ be a hypergraph on vertex set $[n]$, and let $B_k \subseteq [n]$ be a uniformly chosen subset of size k. Then*

$$\operatorname{Var}\big(X_m(B_k)\big) \; \leqslant \; \frac{48m}{k}\binom{k}{m}^2$$

and

$$\operatorname{Var}\big(\tilde{X}_m(B_k)\big) \; \leqslant \; \frac{48m}{k}.$$

We remark that with a little extra effort, one could improve the upper bounds in Proposition 6.1 by a factor of $\beta_m := e(\mathcal{H}_m)/\binom{n}{m}$. Since we shall not need such a strengthening, however, we leave it to the reader to verify that that follows from the proof. In order to keep the presentation simple, we also make no attempt to optimize the constant.

We shall use some straightforward relations between binomial coefficients in the proof of Proposition 6.1; we state them here for convenience.

Observation 6.2. *Let n, k, m, t be integers such that $k \geqslant m \geqslant t \geqslant 1$ and $n \geqslant 2m^3$. Then*

a) $$\binom{k}{m}^2 \; = \; \sum_{t=0}^{m} \binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t}.$$

b) $$\binom{m-1}{t}\binom{n}{m}^2 \; \leqslant \; \binom{n}{2m-t}\binom{2m-t}{m}\binom{n-1}{t}.$$

c) $\binom{m-1}{t-1}\binom{n-t-1}{m-t}\binom{n}{m} \leqslant \frac{2t}{m}\binom{n}{2m-t}\binom{2m-t}{m}\binom{m}{t}$.

Proof. For *a)*, note that both sides count the number of (ordered) pairs of m-subsets of set of size k; on the right-hand side we have partitioned according to the size of their intersection. By convention, $\binom{n}{k} := 0$ for $k > n$.

For *b)* and *c)*, note that the case $m = t$ is trivial. So, assume $n \geqslant 2m^3$ and $m > t \geqslant 1$, simply cancel common terms and note that it suffices to prove that

$$x \cdot \frac{(n-m)!(n-m)!}{(n-2m+t)!(n-t)!} \geqslant 1, \tag{6.1}$$

where in *b)* $x = \frac{m(n-t)}{n(m-t)}$ and in *c)* $x = \frac{2(n-t)}{n-m}$. Since $(a+c)/(b+c) \geqslant a/b$ for $b \geqslant a > 0$ and $c \geqslant 0$, we obtain

$$\begin{aligned} \frac{(n-m)!(n-m)!}{(n-2m+t)!(n-t)!} &= \frac{(n-2m+t+1)\dots(n-m)}{(n-m+1)\dots(n-t)} \\ &\geqslant \left(\frac{n-2m}{n-m}\right)^{m-t} \geqslant 1 - \frac{m^2}{n-m}, \end{aligned}$$

where the last inequality follows since $(1-a)^m \geqslant 1 - m \cdot a$ for all $a \leqslant 2$. Moreover,

$$\frac{m(n-t)}{n(m-t)} \geqslant \frac{m(n-1)}{n(m-1)} \quad \text{and} \quad \frac{2(n-t)}{n-m} \geqslant 2.$$

To prove *b)* via (6.1), it therefore suffices to show that $\frac{m(n-1)}{n(m-1)} \cdot \left(1 - \frac{m^2}{n-m}\right) \geqslant 1$, or equivalently that $n(n-m^3-2m)+m^3+m^2 \geqslant 0$. For *c)* it similarly suffices to show that $1 - \frac{m^2}{n-m} \geqslant \frac{1}{2}$, which is equivalent to $n \geqslant 2m^2 + m$. Certainly, $n \geqslant 2m^3$ is sufficient in both cases. □

We shall use Bey's inequality in order to prove the following lemma, from which Proposition 6.1 follows easily. Let

$$Y_t(k,m) := \binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t},$$

which counts the number of pairs of m-subsets of a fixed k-set which have t common elements.

Lemma 6.3. *Let $k, m, n \in \mathbb{N}$, with $n \geqslant k \geqslant 2m$ and $n \geqslant 2m^3$. Let $\mathcal{H}$ be a hypergraph on $[n]$, and let $B_k \subseteq [n]$ be a uniformly chosen subset of size k. Then*

$$\mathrm{Var}\left(X_m(B_k)\right) \leqslant 2\beta_m \sum_{t=1}^{m} \left(\frac{t}{m} + \frac{k}{2n}\right) Y_t(k,m).$$

Proof. Let $\alpha(\mathcal{H},t) := \big|\big\{(e,f) : e,f \in \mathcal{H} \text{ and } |e \cap f| = t\big\}\big|$ denote the number of pairs (e,f) of edges of $\mathcal{H}$ with t common vertices. We first claim that

$$\mathbb{E}\big[X_m(B_k)^2\big] \;=\; \sum_{t=0}^{m} \alpha(\mathcal{H}_m,t)\frac{\binom{k}{2m-t}}{\binom{n}{2m-t}}. \tag{6.2}$$

Indeed, writing $\mathbf{1}_A$ for the indicator function of the event A, and $\binom{[n]}{k}$ for the collection of subsets of $[n]$ of size k, we obtain

$$\begin{aligned}\mathbb{E}\big[X_m(B_k)^2\big] \;&=\; \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} \sum_{e,f \in \mathcal{H}_m} \mathbf{1}_{\{e \cup f \subseteq S\}} \\ &=\; \frac{1}{\binom{n}{k}} \sum_{t=0}^{m} \sum_{e,f\in\mathcal{H}_m} \mathbf{1}_{\{|e\cap f| = t\}} \sum_{S\in\binom{[n]}{k}} \mathbf{1}_{\{e\cup f\subseteq S\}}\,.\end{aligned}$$

But if $|e \cap f| = t$ then $|e \cup f| = 2m - t$, and so there are exactly $\binom{n-2m+t}{k-2m+t}$ sets S of size k such that $e \cup f \subseteq S$. Moreover, $\binom{n-2m+t}{k-2m+t}\binom{n}{2m-t} = \binom{n}{k}\binom{k}{2m-t}$, and hence

$$\mathbb{E}\big[X_m(B_k)^2\big] \;=\; \sum_{t=0}^{m} \sum_{e,f\in\mathcal{H}_m} \mathbf{1}_{\{|e\cap f|=t\}} \,\frac{\binom{k}{2m-t}}{\binom{n}{2m-t}},$$

as claimed.

Next, observe that $\alpha(\mathcal{H}_m,t) \leqslant d_2(\mathcal{H}_m,t)$, where $d_2(\mathcal{H}_m,t)$ denotes the sum of $d_{\mathcal{H}}(T)^2$ over all t-sets in $[n]$, and recall that $e(\mathcal{H}_m) = \beta_m\binom{n}{m}$. Hence, by Bey's inequality and Observation 6.2 part *b)* and *c)*,

$$\begin{aligned}&\alpha(\mathcal{H}_m,t)\frac{\binom{k}{2m-t}}{\binom{n}{2m-t}} \\ &\quad\leqslant \left(\frac{\binom{m}{t}\binom{m-1}{t}}{\binom{n-1}{t}} e(\mathcal{H}_m)^2 + \binom{m-1}{t-1}\binom{n-t-1}{m-t} e(\mathcal{H}_m)\right)\frac{\binom{k}{2m-t}}{\binom{n}{2m-t}} \qquad (6.3)\\ &\quad\leqslant \left(\beta_m^2 + \frac{2t}{m}\cdot\beta_m\right)\binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t}\end{aligned}$$

for every $1 \leqslant t \leqslant m$. Moreover, $\alpha(\mathcal{H}_m,0) \leqslant e(\mathcal{H}_m)^2 = \beta_m^2\binom{n}{m}^2$, so by part *a)* of Observation 6.2

$$\alpha(\mathcal{H}_m,0)\frac{\binom{k}{2m}}{\binom{n}{2m}} \;\leqslant\; \beta_m^2\binom{n}{m}^2\frac{\binom{k}{2m}}{\binom{n}{2m}} \;=\; \beta_m^2\frac{\binom{k}{2m}}{\binom{n}{2m}}\sum_{t=0}^{m}\binom{n}{2m-t}\binom{2m-t}{m}\binom{m}{t}.$$

Cancelling common terms, we easily see that for each $1 \leqslant t \leqslant m$

$$\frac{\binom{k}{2m}\binom{n}{2m-t}}{\binom{n}{2m}\binom{k}{2m-t}} \;\leqslant\; \left(\frac{k}{n}\right)^t \;\leqslant\; \frac{k}{n}.$$

Hence,

$$\alpha(\mathcal{H}_m, 0)\frac{\binom{k}{2m}}{\binom{n}{2m}} \;\leqslant\; \beta_m^2 \binom{k}{2m}\binom{2m}{m} + \frac{k}{n} \cdot \beta_m^2 \sum_{t=1}^{m} \binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t}. \tag{6.4}$$

Finally,

$$\mathbb{E}\big[X_m(B_k)\big]^2 \;=\; \beta_m^2 \binom{k}{m}^2 \;=\; \beta_m^2 \sum_{t=0}^{m} \binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t}, \tag{6.5}$$

by part *a)* of Observation 6.2. Combining (6.2), (6.3), (6.4) and (6.5), we obtain

$$\mathrm{Var}\,\big(X_m(B_k)\big) \;\;\leqslant\;\; \sum_{t=1}^{m} \left(\frac{2t}{m} \cdot \beta_m + \frac{k}{n} \cdot \beta_m^2\right) \binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t},$$

as required. $\square$

It is easy to deduce Proposition 6.1 from Lemma 6.3.

Proof of Proposition 6.1. We deduce the claimed bound on $\mathrm{Var}\,\big(X_m(B_k)\big)$; the second statement follows immediately from the first, since $X_m(B_k) = \binom{k}{m}\tilde{X}_m(B_k)$. The result is trivial for $m \leqslant k \leqslant 48m$, so we can assume that $k \geqslant 48m$. (In fact we shall only use that $k \geqslant 4m$.)

First, note that by Lemma 6.3, and since $n \geqslant km/2$, we have

$$\mathrm{Var}\,\big(X_m(B_k)\big) \;\leqslant\; 2\sum_{t=1}^{m} \frac{t+1}{m} Y_t(k,m), \tag{6.6}$$

where $Y_t(k,m) = \binom{k}{2m-t}\binom{2m-t}{m}\binom{m}{t}$. We shall see that most of the weight of the $Y_t(k,m)$ is concentrated on terms with small t. We split into two cases, depending on the size of m.

Case 1. $k \geqslant 3m^2$.

We shall prove that

$$\sum_{t=1}^{m} \frac{t+1}{m} Y_t(k,m) \;\leqslant\; \frac{4}{m} Y_1(k,m) \;\leqslant\; \frac{4m}{k}\binom{k}{m}^2. \tag{6.7}$$

Indeed, first note that

$$\begin{aligned}\frac{(t+2)Y_{t+1}(k,m)}{(t+1)Y_t(k,m)} \;&=\; \frac{(t+2)(m-t)^2}{(t+1)^2(k-2m+t+1)}\\ &\leqslant\; \frac{3m^2}{2(t+1)(k-2m)} \;\leqslant\; \frac{1}{2},\end{aligned} \tag{6.8}$$

since $k - 2m \geqslant k/2$ and $(t+1)k \geqslant 2k \geqslant 6m^2$. This proves the first inequality in (6.7); for the second, observe that

$$Y_1(k,m) \;=\; \frac{m^2}{k} \cdot \frac{(k-m)!(k-m)!}{(k-2m+1)!(k-1)!} \cdot \binom{k}{m}^2 \;\leqslant\; \frac{m^2}{k}\binom{k}{m}^2,$$

as claimed. By (6.6), we obtain $\mathrm{Var}\big(X_m(B_k)\big) \leqslant \dfrac{8m}{k}\dbinom{k}{m}^2$.

Case 2. $k \leqslant 3m^2$.

Let $a := \lfloor 6m^2/k \rfloor$, and observe that (6.8) holds whenever $t \geqslant a$. Thus

$$\sum_{t=a}^{m} \frac{t+1}{m} Y_t(k,m) \;\leqslant\; \frac{2(a+1)}{m} Y_a(k,m) \;\leqslant\; \frac{18m}{k}\binom{k}{m}^2,$$

since $Y_a(k,m) \leqslant \binom{k}{m}^2$ and $a + 1 \leqslant 9m^2/k$. Moreover, it is immediate that

$$\sum_{t=1}^{a-1} \frac{t+1}{m} Y_t(k,m) \;\leqslant\; \frac{a}{m}\sum_{t=1}^{a-1} Y_t(k,m) \;\leqslant\; \frac{6m}{k}\binom{k}{m}^2.$$

By (6.6), we obtain $\mathrm{Var}\big(X_m(B_k)\big) \leqslant \dfrac{48m}{k}\dbinom{k}{m}^2$, as required. □

6.2 The proof for random-sized sets

We shall now deduce Theorem 1.6 from Proposition 6.1. The task can be divided into two part, via the conditional variance formula,

$$\mathrm{Var}(X) \;=\; \mathrm{Var}\big(\mathbb{E}[X \mid Y]\big) + \mathbb{E}\big[\,\mathrm{Var}(X \mid Y)\big], \tag{6.9}$$

applied with $X = r_{\mathcal{H}}\big([n]_q, p\big)$ and $Y = \big|[n]_q\big|$. To illustrate the idea behind the proof, let $K \sim \mathrm{Bin}(n,q)$ and given K, let $M \sim \mathrm{Bin}(K,p)$. Roughly, we obtain upper bounds on the right-hand side of (6.9) as follows.

Proposition 6.1 will be applied to bound the latter of the two terms. By Chernoff's inequality $M \leqslant 2pK$ occurs with high probability when pK is large, which in turn is likely to occur when pqn is large. Thus, if pqn is large, then $M/K \leqslant 2p$ with high probability, which together with Proposition 6.1, shows that $\mathbb{E}\big[\operatorname{Var}\big(X_M(B_K)\big|K\big)\big]$ is small.

The size of $[n]_q$, here represented by K, will roughly fluctuate by $\sqrt{qn}$ around its mean. This will influence the mean of M, representing the size of $[n]_q \cap [n]_p$, roughly by $p\sqrt{qn}$. However, M will naturally vary by $\sqrt{pqn}$ which is much larger than $p\sqrt{qn}$ when p is small. Hence, conditioning on the size of $[n]_q$ will not affect the size of $[n]_q \cap [n]_p$ much.

As a first step towards Theorem 1.6, we obtain a result for fixed k and a randomly chosen $m \sim \operatorname{Bin}(k,p)$. Indeed, given a hypergraph $\mathcal{H}$ on vertex set $[n]$, a subset $S \subseteq [n]$ of size k, and $p \in (0,1)$, observe that

$$r_{\mathcal{H}}(S,p) \;=\; \sum_{m=0}^{k} \mathbb{P}\big(\xi_{k,p} = m\big)\tilde{X}_m(S),$$

where $\xi_{k,p} \sim \operatorname{Bin}(k,p)$, as in the previous section. The following proposition is an easy consequence of Proposition 6.1.

Proposition 6.4. *Let $p \in (0,1)$ and let $n,k \in \mathbb{N}$, with $n \geqslant 16(pk)^3$ and $n \geqslant pk^2$. Let $\mathcal{H}$ be a hypergraph on vertex set $[n]$, and let $B_k \subseteq [n]$ be a uniformly chosen subset of size k. Then*

$$\operatorname{Var}\big(r_{\mathcal{H}}(B_k,p)\big) \;\leqslant\; 96p \,+\, 4\exp\big(-pk/16\big).$$

Proof. The result follows from Proposition 6.1 and Chernoff's inequality, since if $m \leqslant 2pk$ then the variance of $\tilde{X}_m(B_k)$ is at most $96p$, and the probability that $\xi_{k,p} > 2pk$ is at most $2e^{-pk/16}$.

To spell it out, note that if $p \leqslant 1/2$ and $m \leqslant 2pk$, then $m \leqslant k$, $n \geqslant 2m^3$ and $n \geqslant km/2$, and so, by Proposition 6.1,

$$\operatorname{Var}\big(\tilde{X}_m(B_k)\big) \;\leqslant\; \frac{48m}{k} \;\leqslant\; 96p.$$

Since $\operatorname{Var}\big(\tilde{X}_m(B_k)\big) \leqslant 1$, the same bound trivially holds for $p > 1/2$.

Now since $r_{\mathcal{H}}(B_k,p) = \sum_{m=0}^{k} \mathbb{P}\big(\xi_{k,p} = m\big)\tilde{X}_m(B_k)$,

$$\operatorname{Var}\big(r_{\mathcal{H}}(B_k,p)\big) \;=\; \sum_{m_1,m_2} \mathbb{P}\big(\xi_{k,p} = m_1\big)\mathbb{P}\big(\xi_{k,p} = m_2\big)\operatorname{Cov}\big(\tilde{X}_{m_1}(B_k),\tilde{X}_{m_2}(B_k)\big).$$

By Cauchy-Schwarz's inequality, we have $\operatorname{Cov}(X,Y) \leqslant \sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}$, and thus

$$\begin{aligned}
&\operatorname{Var}\big(r_{\mathcal{H}}(B_k,p)\big)\\
&\qquad\leqslant\; \sum_{m_1,m_2} \mathbb{P}\big(\xi_{k,p} = m_1\big)\mathbb{P}\big(\xi_{k,p} = m_2\big)\sqrt{\operatorname{Var}\big(\tilde{X}_{m_1}(B_k)\big)\operatorname{Var}\big(\tilde{X}_{m_2}(B_k)\big)}\\
&\qquad\leqslant\; 96p \,+\, 2\cdot\mathbb{P}\big(\xi_{k,p} > 2pk\big) \;\leqslant\; 96p \,+\, 4\exp\big(-pk/16\big),
\end{aligned}$$

where the last step is by Chernoff's inequality, as required. □

In order to deduce Theorem 1.6 from Proposition 6.4, we shall use the following simple bounds on binomial random variables, which follow immediately from Chernoff's inequality.

Observation 6.5. *Let $p \in (0,1/2]$, $q \in (0,1)$ and $n \in \mathbb{N}$, be such that $pqn \geqslant 32\log(1/p)$. Then*

a) $\mathbb{P}\Big(\big|\xi_{n,q} - qn\big| \geqslant 2\sqrt{qn\log(1/p)}\Big) \leqslant 2p.$

b) $\mathbb{P}\left(\xi_{n,q} \leqslant \dfrac{16}{p}\log\dfrac{1}{p}\right) \leqslant \mathbb{P}\Big(\xi_{n,q} \leqslant qn/2\Big) \leqslant 2e^{-qn/16} \leqslant 2p.$

We shall also need the following bound, relating nearby binomial coefficients.

Lemma 6.6. *Let $p \in (0,1/4]$, $q \in (0,1)$ and $n \in \mathbb{N}$ satisfy $pqn \geqslant 16\log(1/p)$. If*

$$qn - 2\sqrt{qn\log(1/p)} \leqslant k \leqslant k' \leqslant qn + 2\sqrt{qn\log(1/p)},$$

and $\big|m - pqn\big| \leqslant 4\sqrt{pqn\log(1/p)}$, then

$$\frac{\mathbb{P}\big(\xi_{k',p} = m\big)}{\mathbb{P}\big(\xi_{k,p} = m\big)} = 1 + O\bigg(\sqrt{p}\log\frac{1}{p}\bigg).$$

Proof. First observe that $\sqrt{qn\log(1/p)} \leqslant \sqrt{p}qn/4$, by assumption, so that

$$k \geqslant qn - \frac{\sqrt{p}qn}{2} \geqslant \frac{3qn}{4}, \quad \text{and} \quad 0 \leqslant m \leqslant 2pqn \leqslant \frac{qn}{2}.$$

In particular, $k > m$. For each $k < \ell \leqslant k'$,

$$\frac{\mathbb{P}\big(\xi_{\ell,p} = m\big)}{\mathbb{P}\big(\xi_{\ell-1,p} = m\big)} = \frac{\ell(1-p)}{\ell - m} = 1 + \frac{m - p\ell}{\ell - m} = 1 + \frac{\frac{m-p\ell}{\ell}}{1 - \frac{m}{\ell}}.$$

Next observe that $0 \leqslant \frac{m}{\ell} \leqslant \frac{2}{3}$, and

$$\left|\frac{m - p\ell}{\ell}\right| \leqslant \frac{4\sqrt{pqn\log\frac{1}{p}} + 2p\sqrt{qn\log\frac{1}{p}}}{\frac{3}{4}qn} \leqslant \frac{20\sqrt{p\log\frac{1}{p}}}{3\sqrt{qn}}.$$

In particular,

$$\frac{\mathbb{P}\big(\xi_{\ell,p} = m\big)}{\mathbb{P}\big(\xi_{\ell-1,p} = m\big)} \leqslant 1 + 10\sqrt{\frac{p}{qn}\log\frac{1}{p}}.$$

Expressing the quantity of interest as a telescoping product, we obtain

$$\frac{\mathbb{P}\big(\xi_{k',p}=m\big)}{\mathbb{P}\big(\xi_{k,p}=m\big)} = \prod_{\ell=k+1}^{k'} \frac{\mathbb{P}\big(\xi_{\ell,p}=m\big)}{\mathbb{P}\big(\xi_{\ell-1,p}=m\big)} \leqslant \left(1+10\sqrt{\frac{p}{qn}\log\frac{1}{p}}\right)^{k'-k} = 1+O\left(\sqrt{p}\log\frac{1}{p}\right),$$

where we in the final step used that $(1+x)^m \leqslant e^{mx} \leqslant 1+mx\,e^{mx}$, that $k'-k \leqslant 4\sqrt{qn\log(1/p)}$ and the fact that $\sqrt{p}\log\frac{1}{p}$ is bounded. This provides the upper bound, and a lower bound is obtained similarly. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. The result is trivial for any $p \in \big(\frac{1}{4},\frac{1}{2}\big]$, since the variance is at most 1. Let $0 < p \leqslant 1/4$, $0 < q < 1$ and $n \in \mathbb{N}$, and suppose that $n \geqslant 128(pqn)^3$, $n \geqslant 4p(qn)^2$ and $pqn \geqslant 32\log(1/p)$. Observe that therefore Observation 6.5 and Lemma 6.6 applies, and that the assumptions of Proposition 6.4 are met for every $k \leqslant 2qn$.

Apply the conditional variance formula (6.9) with $X = r_{\mathcal{H}}\big([n]_q,p\big)$ and $Y = \big|[n]_q\big|$ to obtain

$$\begin{aligned}\operatorname{Var}\Big(r_{\mathcal{H}}\big([n]_q,p\big)\Big) &= \operatorname{Var}\Big(\mathbb{E}\Big[r_{\mathcal{H}}\big([n]_q,p\big)\,\Big|\,\big|[n]_q\big|\Big]\Big) \\ &\quad + \mathbb{E}\Big[\operatorname{Var}\Big(r_{\mathcal{H}}\big([n]_q,p\big)\,\Big|\,\big|[n]_q\big|\Big)\Big].\end{aligned} \tag{6.10}$$

Note that in the former term on the right-hand side, the variance is over the size k of $[n]_q$, and the expectation over the (uniform) choice of a subset in $[n]$ of size k. In the latter, the variance is over the uniform choice of the set $[n]_q$ of prescribed size. First, we use Proposition 6.4 and Observation 6.5 to show that the latter part of (6.10) is small.

Claim 1. $\mathbb{E}\Big[\operatorname{Var}\Big(r_{\mathcal{H}}\big([n]_q,p\big)\,\Big|\,\big|[n]_q\big|\Big)\Big] \leqslant 104p.$

Proof of Claim 1. First note that $\big|[n]_q\big| \sim \operatorname{Bin}(n,q)$, so if $B_k \subseteq [n]$ is a uniformly chosen set of size k, we have the expression

$$\mathbb{E}\Big[\operatorname{Var}\Big(r_{\mathcal{H}}\big([n]_q,p\big)\,\Big|\,\big|[n]_q\big|\Big)\Big] = \sum_{k=0}^{n}\operatorname{Var}\Big(r_{\mathcal{H}}(B_k,p)\Big)\mathbb{P}\big(\xi_{n,q}=k\big).$$

Now, $\mathbb{P}\big(\xi_{n,q} \leqslant \frac{16}{p}\log\frac{1}{p}\big) \leqslant 2p$, by Observation 6.5. On the other hand, by Chernoff's inequality $\mathbb{P}(\xi_{n,q} > 2qn) \leqslant 2\exp(-qn/16) \leqslant 2p$. For $k \leqslant 2qn$

Proposition 6.4 applies, as observed above. So if $\frac{16}{p}\log\frac{1}{p} \leqslant k \leqslant 2qn$, then we have

$$\mathrm{Var}\Big(r_{\mathcal{H}}(B_k,p)\Big) \leqslant 100p,$$

and the claim follows. $\square$

For each $k \in [n]$, set $\alpha_k := \mathbb{E}\big[r_{\mathcal{H}}(B_k,p)\big]$, where $B_k \subseteq [n]$ is a uniformly chosen set of size k. Let K and K' be independent random variables with distribution $\mathrm{Bin}(n,q)$. Note that $\mathbb{E}\big[r_{\mathcal{H}}\big([n]_q,p\big) \,\big|\, \big|[n]_q\big| = k\big] = \alpha_k$, so the remaining term in (6.10) may be re-written as

$$\mathrm{Var}\left(\mathbb{E}\Big[r_{\mathcal{H}}\big([n]_q,p\big) \,\Big|\, \big|[n]_q\big|\Big]\right) = \mathrm{Var}\left(\alpha_K\right) = \frac{1}{2}\,\mathbb{E}\Big[\big(\alpha_K - \alpha_{K'}\big)^2\Big].$$

Again by Observation 6.5, the probability that

$$qn - 2\sqrt{qn\log\frac{1}{p}} \leqslant K, K' \leqslant qn + 2\sqrt{qn\log\frac{1}{p}} \tag{6.11}$$

is at least $1-4p$. Hence, it will suffice to prove the following claim.

Claim 2. For $qn - 2\sqrt{qn\log\frac{1}{p}} \leqslant k \leqslant k' \leqslant qn + 2\sqrt{qn\log\frac{1}{p}}$

$$\big|\alpha_{k'} - \alpha_k\big| = O\left(\sqrt{p}\log\frac{1}{p}\right).$$

Proof of Claim 2. Note that α_k can be expressed as

$$\alpha_k = \mathbb{E}\big[r_{\mathcal{H}}(B_k,p)\big] = \sum_{m=0}^{k} \mathbb{P}\big(\xi_{k,p} = m\big)\mathbb{E}\big[\tilde{X}_m(B_k)\big] = \sum_{m=0}^{k} \mathbb{P}\big(\xi_{k,p} = m\big)\beta_m.$$

Set $S := \Big\{m : |m - pqn| \leqslant 4\sqrt{pqn\log\frac{1}{p}}\Big\}$, and re-write the difference as

$$\begin{aligned}\big|\alpha_{k'} - \alpha_k\big| &= \left|\sum_{m=0}^{k'} \mathbb{P}\big(\xi_{k',p} = m\big)\beta_m - \sum_{m=0}^{k} \mathbb{P}\big(\xi_{k,p} = m\big)\beta_m\right| \\ &\leqslant \sum_{m\in S} \Big|\mathbb{P}\big(\xi_{k',p} = m\big) - \mathbb{P}\big(\xi_{k,p} = m\big)\Big| \\ &\qquad + \mathbb{P}\big(\xi_{k',p} \notin S\big) + \mathbb{P}\big(\xi_{k,p} \notin S\big).\end{aligned} \tag{6.12}$$

Next, we show that $\mathbb{P}\big(\xi_{k,p} \notin S\big) \leqslant 2p$; the same is obtained for k replaced by k' analogously. By the triangle inequality, we find

$$\big|\xi_{k,p} - pqn\big| \leqslant \big|\xi_{k,p} - pk\big| + p|k - qn| \leqslant \big|\xi_{k,p} - pk\big| + 2p\sqrt{qn\log\frac{1}{p}}.$$

Since $pqn \geqslant \log \frac{1}{p}$ and $p \leqslant \frac{1}{4}$, we obtain $k \leqslant 2qn$. As a consequence

$$\begin{aligned}\mathbb{P}\Big(\big|\xi_{k,p} - pqn\big| > 4\sqrt{pqn\log(1/p)}\Big) \;&\leqslant\; \mathbb{P}\Big(\big|\xi_{k,p} - pk\big| > 3\sqrt{pqn\log(1/p)}\Big) \\ &\leqslant\; \mathbb{P}\Big(\big|\xi_{k,p} - pk\big| > 2\sqrt{pk\log(1/p)}\Big) \;\leqslant\; 2p,\end{aligned}$$

via Chernoff's inequality. Hence, $\mathbb{P}\big(\xi_{k,p} \notin S\big) \leqslant 2p$ and also $\mathbb{P}\big(\xi_{k',p} \notin S\big) \leqslant 2p$.

The claim now follows from (6.12) and Lemma 6.6, which states that

$$\Big|\mathbb{P}\big(\xi_{k',p} = m\big) - \mathbb{P}\big(\xi_{k,p} = m\big)\Big| \;=\; O\left(\sqrt{p}\log\frac{1}{p}\right)\cdot\mathbb{P}\big(\xi_{k,p} = m\big)$$

for every $m \in S$. □

Theorem 1.6 is easily deduced from Claim 1 and 2, and (6.11). Indeed,

$$\begin{aligned}\operatorname{Var}\Big(r_{\mathcal{H}}\big([n]_q, p\big)\Big) \;&=\; \frac{1}{2}\,\mathbb{E}\Big[\big(\alpha_K - \alpha_{K'}\big)^2\Big] \;+\; 104p \\ &\leqslant\; \mathbb{P}\Big(\big|\xi_{n,q} - qn\big| > 2\sqrt{qn\log(1/p)}\Big) \;+\; O\left(p\Big(\log\frac{1}{p}\Big)^2\right) \\ &=\; O\bigg(p\left(\log\frac{1}{p}\right)^2\bigg),\end{aligned}$$

as required. □

We end the section by presenting an easy consequence of Theorem 1.6. It provides a general framework from which Proposition 2.3 can be deduced from Theorem 2.2. It follows almost immediately from Theorem 1.6, together with Chebyshev's inequality.

Corollary 6.7. *Given $\varepsilon > 0$, there exists $p^* = p^*(\varepsilon) > 0$ such that, if $p \in (0, p^*)$, $q \in (0,1)$ and $n \in \mathbb{N}$ satisfy $n \geqslant 128(pqn)^3$, $n \geqslant 4p(qn)^2$ and $pqn \geqslant 32\log\frac{1}{p}$, then*

$$\mathbb{P}\bigg(\Big|\mathbb{P}\big([n]_p \cap [n]_q \in \mathcal{H} \,\big|\, [n]_q\big) - \mathbb{P}\big([n]_{pq} \in \mathcal{H}\big)\Big| > \varepsilon\bigg) \;<\; \varepsilon,$$

for every hypergraph (event) $\mathcal{H} \subseteq \{0,1\}^n$.

Proof. Fix $\varepsilon > 0$. According to Theorem 1.6 there exists a universal constant $C < \infty$ such that for all $p \in \big(0, \frac{1}{2}\big)$, $q \in (0,1)$ and $n \in \mathbb{N}$ that satisfy $n \geqslant 128(pqn)^3$, $n \geqslant 4p(qn)^2$ and $pqn \geqslant 32\log\frac{1}{p}$

$$\operatorname{Var}\Big(r_{\mathcal{H}}\big([n]_q, p\big)\Big) \;\leqslant\; Cp\left(\log\frac{1}{p}\right)^2.$$

Pick $p^* = p^*(\varepsilon)$ such that $Cp\big(\log \frac{1}{p}\big)^2 \leqslant \varepsilon^3$ for all $p \in (0, p^*)$. Hence, for all $p \in (0, p^*)$, $q \in (0,1)$ and $n \in \mathbb{N}$ that satisfy the given conditions, we obtain via Chebyshev's inequality that

$$\mathbb{P}\bigg(\Big|\mathbb{P}\big([n]_p \cap [n]_q \in \mathcal{H} \,\big|\, [n]_q\big) - \mathbb{P}\big([n]_{pq} \in \mathcal{H}\big)\Big| > \varepsilon\bigg) \;\leqslant\; \frac{\mathrm{Var}\,\big(r_{\mathcal{H}}\big([n]_q, p\big)\big)}{\varepsilon^2} \;\leqslant\; \varepsilon.$$

□

It is immediately seen from Corollary 6.7 that its statement could alternatively be formulated as follows.

Corollary 6.8. *Given $c \in (0,1)$ and $\varepsilon \in \big(0, \frac{c}{2}\big)$, there exists $p^* = p^*(\varepsilon) > 0$ such that, if $r \in (0, p^*)$ and $n \in \mathbb{N}$ satisfy $n \geqslant 128(rn)^3$, $n \geqslant 4(rn)^2/p^*$ and $rn \geqslant 32 \log \frac{1}{p^*}$, then*

$$\mathbb{P}\big([n]_r \in \mathcal{H}\big) \;\geqslant\; c \quad \textit{implies} \quad \mathbb{P}\Big(\mathbb{P}\big([n]_{p^*} \cap [n]_q \in \mathcal{H} \,\big|\, [n]_q\big) < \varepsilon\Big) \;<\; \varepsilon$$

for each event (hypergraph) $\mathcal{H} \subseteq \{0,1\}^n$, where $q = r/p^*$.

7 Proof of Theorem 1.2

In this section we shall put together the pieces, and prove Theorem 1.2. We shall first deduce Propositions 2.1 and 2.3 from Theorem 1.6; then we shall use the deterministic algorithm method to prove Theorem 1.5; finally we shall deduce Theorem 1.2.

7.1 Variance bound – Proof of Proposition 2.1 and 2.3

We shall prove the following slight generalization of Proposition 2.1, which follows easily from Theorem 1.6, together with an easy discretization argument.

Proposition 7.1. $$\lim_{p \to 0} \limsup_{a,\, b \to \infty} \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p, R_{a\times b}, \bullet) \,\big|\, \eta\big)\Big) \;=\; 0.$$

In order to apply Theorem 1.6 we need to construct a discretization of the rectangle $R_{a\times b}$. In order to do so, for each $\delta > 0$ consider the lattice

$$\Lambda \;=\; \Lambda_{a,b}^{\delta} \;:=\; R_{(a+2)\times(b+2)} \cap \delta\mathbb{Z}^2,$$

and set $n = |\Lambda_{a,b}^{\delta}|$, the number of vertices of $\delta\mathbb{Z}^2$ in the rectangle $R_{(a+2)\times(b+2)}$. (Note that we consider the rectangle $R_{(a+2)\times(b+2)}$ because it contains all the points which can affect the event $H(\eta, R_{a\times b}, \bullet)$.) Let $p > 0$, and set

$$q \;=\; q(n) \;:=\; 1 - e^{-\lambda_c \delta^2/p}. \tag{7.1}$$

Let Λ_q denote a random q-subset of Λ, and $\Lambda_q \cap \Lambda_p$ denote a random p-subset of Λ_q. Of course, the distribution of $\Lambda_q \cap \Lambda_p$ equals that of Λ_{pq}.

The following lemma is an immediate consequence of Theorem 1.6.

Lemma 7.2. *There exists a universal constant $C < \infty$ such that for each $p \in \left(0, \frac{1}{2}\right]$, if $a = a(p)$, $b = b(p) \geqslant 1$ are sufficiently large and $\delta = \delta(a,b) > 0$ is sufficiently small, then the following holds. Let $\Lambda = \Lambda^\delta_{a,b}$, $n = |\Lambda|$ and $q > 0$ be as described above. Then*

$$\operatorname{Var}\Big(\mathbb{P}\big(H(\Lambda_q \cap \Lambda_p, R_{a\times b}, \bullet) \,\big|\, \Lambda_q\big)\Big) \leqslant Cp \left(\log \frac{1}{p}\right)^2.$$

Proof. We apply Theorem 1.6 to the hypergraph $\mathcal{H}$ which encodes crossings of the rectangle $R_{a\times b}$. That is, we identify the vertices of $\Lambda = \Lambda^\delta_{a,b}$ with the elements of $[n]$, and define $\mathcal{H}$ by the relation that for any $B \subseteq [n]$

$$B \in \mathcal{H} \quad \Leftrightarrow \quad H\big(B, R_{a\times b}, \bullet\big) \text{ occurs.}$$

It only remains to check that for every fixed p the conditions of Theorem 1.6 are satisfied if a and b are chosen sufficiently large, and thereafter δ is chosen sufficiently small. Clearly, there are $0 < c_1 \leqslant c_2 < \infty$ such that $c_1 \frac{ab}{\delta^2} \leqslant n \leqslant c_2 \frac{ab}{\delta^2}$. One can further show that $\delta^2 \leqslant \frac{p}{\lambda_c}$ implies $\frac{\lambda_c \delta^2}{2p} \leqslant q \leqslant \frac{\lambda_c \delta^2}{p}$, and consequently that

$$c_1 \frac{\lambda_c ab}{2p} \leqslant pqn \leqslant c_2 \frac{\lambda_c ab}{p}.$$

Thus, choose ab large so that $pqn \geqslant 32 \log \frac{1}{p}$ holds, and thereafter $\delta^2 \leqslant \frac{p}{\lambda_c}$ small enough for $n \geqslant 128(pqn)^3$ and $n \geqslant 4(pqn)^2/p$ to hold. $\square$

In order to deduce Proposition 7.1, we need to provide a coupling between our two probability spaces – one discrete, the other continuous – which approximately maps the crossing event $H(\eta, R_{a\times b}, \bullet)$ onto itself (in the sense of (7.2)). More precisely, we will construct a mapping ψ from Ω to subsets of $\Lambda = \Lambda^\delta_{a,b}$ such that when $\eta \in \Omega$ is chosen according to $\mathbf{P}_{\lambda_c/p}$, then $\big(\eta, \psi(\eta)\big)$ represents a coupling between a Poisson point process of intensity λ_c/p and a q-subset of Λ, where q is as in (7.1).

To construct the mapping, cover $R_{(a+2)\times(b+2)}$ with disjoint $\delta \times \delta$ squares, centred on elements of $\Lambda^\delta_{a,b}$, and let ψ map points of η to the centre of the square in which they lie. That is, given $\eta \in \Omega$, let

$$\psi(\eta) := \Big\{ y \in \Lambda^\delta_{a,b} \,:\, x \in y + (-\delta/2, \delta/2]^2 \text{ for some } x \in \eta \Big\}.$$

Observe that if η is picked according to $\mathbf{P}_{\lambda_c/p}$, then $\psi(\eta)$ is distributed as Λ_q.

We define a bad event $E^\delta_{a,b} \subseteq \Omega$, by saying that $E^\delta_{a,b}$ occurs if either of the following holds:

a) Two points of η lie in the same $\delta \times \delta$ square, i.e., $\left|\eta \cap \left(y + \left(-\frac{\delta}{2}, \frac{\delta}{2}\right]^2\right)\right| > 1$ for some $y \in \Lambda^{\delta}_{a,b}$.

b) There exist x, $y \in \eta \cap R_{(a+2)\times(b+2)}$ with $2 - 2\delta \leqslant \|x - y\|_2 \leqslant 2 + 2\delta$.

c) There exist $x \in \eta$ such that $1 - \delta \leqslant \|x - \partial R_{a\times b}\|_2 \leqslant 1 + \delta$,

where $\partial R_{a\times b}$ denotes the boundary of $R_{a\times b}$ and the distance between a point and a set is defined in the canonical way. Observe that *a)* and *b)* implies that, if $E^{\delta}_{a,b}$ does not occur, then the graphs naturally induced by the points of η and $\psi(\eta)$ (connecting any two point at distance at most 2) are identical, and the vertices are in 1-1 correspondence. Hence, together with *c)*, we have that for every $\eta \in \Omega \setminus E^{\delta}_{a,b}$,

$$\big\{\xi \subseteq \eta : H(\xi, R_{a\times b}, \bullet) \text{ occurs}\big\} \;=\; \big\{\xi \subseteq \eta : H(\psi(\xi), R_{a\times b}, \bullet) \text{ occurs}\big\}. \qquad (7.2)$$

In particular, we find that if $\mathbb{P}\big(H(\eta_p, R_{a\times b}, \bullet) \,|\, \eta\big) \neq \mathbb{P}\big(H(\psi(\eta)_p, R_{a\times b}, \bullet) \,|\, \eta\big)$, then $\eta \in E^{\delta}_{a,b}$. For the coupling $\big(\eta, \psi(\eta)\big)$ to be favourable, we need to the occurrence of $E^{\delta}_{a,b}$ to be unlikely.

Lemma 7.3. *For every $\varepsilon > 0$, $\lambda > 0$ and $a, b \geqslant 1$, there is $\delta = \delta(\varepsilon, \lambda, a, b) > 0$ such that $\mathbf{P}_\lambda\big(E^{\delta}_{a,b}\big) \leqslant \varepsilon$.*

Proof. To bound $\mathbf{P}_\lambda\big(E^{\delta}_{a,b}\big)$, we estimate the probabilities of *a)*, *b)* and *c)* separately.

For property *a)*, the probability is $O(\delta^2\lambda^2 ab)$, since each $\delta \times \delta$-square has probability at most $(\delta^2\lambda)^2$ of containing at least two points of η, and there are about ab/δ^2 such squares.

Property *b)* has probability at most $O(\delta\lambda + \delta\lambda^2 ab)$. Informally, the reason is that conditioned on the number of points in $\eta \cap R_{(a+2)\times(b+2)}$, the points are uniformly distributed. Each pair of uniformly distributed points in $R_{(a+2)\times(b+2)}$ has probability $O\big(\delta/(ab)\big)$ of falling within the right distance of each other. Furthermore, since the expected number of pairs of points are $O\big(\lambda ab + (\lambda ab)^2\big)$, we arrive at the claimed probability. It is not hard to make this argument precise.

For *c)*, it is immediate that the probability is $O(\delta ab)$. It is clear that $\mathbf{P}_\lambda\big(E^{\delta}_{a,b}\big)$ can be made arbitrarily small if only δ is chosen sufficiently small. $\square$

We can now easily deduce Proposition 7.1 from Lemmas 7.2 and 7.3.

Proof of Proposition 7.1. First, note that for any random variable X taking values in $[0, 1]$, and any event E,

$$\operatorname{Var}(X \cdot \mathbf{1}_E) - 2\,\mathbb{P}(E^c) \;\leqslant\; \operatorname{Var}(X) \;\leqslant\; \operatorname{Var}(X \cdot \mathbf{1}_E) + \mathbb{P}(E^c).$$

To see this, note that

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}(X\cdot\mathbf{1}_E)+\mathrm{Var}(X\cdot\mathbf{1}_{E^c})-2\,\mathbb{E}[X\cdot\mathbf{1}_E]\mathbb{E}[X\cdot\mathbf{1}_{E^c}]\\ &= \mathrm{Var}(X\cdot\mathbf{1}_E)+\mathbb{E}[X^2\,\mathbf{1}_{E^c}]-\big(\mathbb{E}[X]+\mathbb{E}[X\cdot\mathbf{1}_E]\big)\mathbb{E}[X\cdot\mathbf{1}_{E^c}].\end{aligned}$$

When applied with $X=\mathbb{P}\big(H(\eta_p,R_{a\times b},\bullet)|\,\eta\big)$ and $E=\Omega\setminus E^\delta_{a,b}$, we obtain via (7.2) that

$$\begin{aligned}&\mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p,R_{a\times b},\bullet)|\,\eta\big)\Big)\\ &\quad\leqslant\ \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\psi(\eta)_p,R_{a\times b},\bullet)\,|\,\eta\big)\cdot\mathbf{1}_{\Omega\setminus E^\delta_{a,b}}\Big)\ +\ \mathbf{P}_{\lambda_c/p}\big(E^\delta_{a,b}\big)\\ &\quad\leqslant\ \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\psi(\eta)_p,R_{a\times b},\bullet)\,|\,\eta\big)\Big)\ +\ 3\,\mathbf{P}_{\lambda_c/p}\big(E^\delta_{a,b}\big)\\ &\quad=\ \mathrm{Var}\,\Big(\mathbb{P}\big(H(\Lambda_q\cap\Lambda_p,R_{a\times b},\bullet)\,\big|\,\Lambda_q\big)\Big)\ +\ 3\,\mathbf{P}_{\lambda_c/p}\big(E^\delta_{a,b}\big),\end{aligned}\tag{7.3}$$

since $\psi(\eta)$, by construction, has the same distribution as Λ_q, when η is chosen according to $\mathbf{P}_{\lambda_c/p}$, and $\Lambda=\Lambda^\delta_{a,b}$ and $q>0$ are as above. Now, for each $p\in\big(0,\frac12\big]$, let $\varepsilon=p\big(\log\frac1p\big)^2$ and choose $a,b\geqslant 1$ sufficiently large and $\delta>0$ sufficiently small for Lemma 7.2 and 7.3 to hold (the latter with $\lambda=\lambda_c/p$). Together with (7.3) we obtain that

$$\mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p,R_{a\times b},\bullet)\,|\,\eta\big)\Big)\ \leqslant\ (C+3)\,p\left(\log\frac1p\right)^2,$$

where $C<\infty$ is as given in Lemma 7.2. □

Our second consequence of Theorem 1.6 is Proposition 2.3. To save us from repeating the discretization, we shall deduce it from Proposition 7.1.

Proof of Proposition 2.3. Fix $t,\gamma>0$, and recall that, by Theorem 2.2, we have

$$c\ \leqslant\ \mathbf{E}_{\lambda_c/p}\Big[\mathbb{P}\big(H(\eta_p,R_{N\times tN},\bullet)\,|\,\eta\big)\Big]\ \leqslant\ 1-c$$

for some $c=c(t)>0$. Moreover, by Proposition 7.1, there exists a constant $p^*=p^*(t,\gamma)>0$ such that

$$\limsup_{N\to\infty}\ \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p,R_{N\times tN},\bullet)\,\big|\,\eta\big)\Big)\ <\ \frac{c^2\gamma}{4},$$

for every $0<p<p^*$. Now, setting $c'=c/2$, apply Chebyshev's inequality to obtain

$$\mathbf{P}_{\lambda_c/p}\Big(\mathbb{P}\Big(H\big(\eta_p,R_{N\times tN},\bullet\big)\,\big|\,\eta\Big)\notin(c',1-c')\Big)\ <\ \frac{c^2\gamma/4}{(c/2)^2}\ =\ \gamma,$$

for every sufficiently large $N\in\mathbb{N}$, as required. □

7.2 Site percolation on $D(\eta)$ – Proof of Theorem 1.5

Our next aim is to prove that for all sufficiently small $p > 0$, the sequence $(f_N^\eta)_{N \geqslant 1}$ is NS_p for $\mathbf{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$. The proof is based on Theorem 2.6, the deterministic algorithm method. We begin by defining the algorithm which we shall use; it is a straightforward adaptation of that used by Benjamini et al. (1999) to prove noise sensitivity of bond percolation crossings.

Recall that, given any $\eta \in \Omega$, the function $f_N^\eta : \{0,1\}^\eta \to \{0,1\}$ is defined by

$$f_N^\eta(\xi) = 1 \quad \Leftrightarrow \quad H\big(\xi, R_N, \bullet\big) \text{ occurs}, \quad \text{for each } \xi \in \Omega_\eta.$$

The following algorithm determines $f_N^\eta(\xi)$ for any $\eta \in \Omega$, and any $\xi \in \{0,1\}^\eta$. The name given to the algorithm is inspired by the following way of visualizing it: imagine pouring water into the left-hand side of R_N, and allowing it to enter only balls $D(x)$ such that $\xi(x) = 1$, i.e., such that $x \in \xi$. If water can flow to the other side of R_N, there is a connection. Recall that points of η that lie outside R_{N+2} cannot affect the outcome of f_N^η. Hence, the domain of f_N^η is really finite dimensional, and we can (and will) in the following identify η and its restriction $\eta \cap R_{N+2}$.

Algorithm (The Water Algorithm)**.** *Define the algorithm $\mathcal{A}_W$ as follows. Let $\eta \in \Omega$ and $\xi \in \{0,1\}^\eta$. Further, let*

$$A_0 := \big\{(x,y) \in \mathbb{R}^2 : x = -N/2\big\} \quad \textit{and} \quad Q_0 := \emptyset$$

denote the 'active' and 'queried' points at time zero. For each $k \in \mathbb{N}$, if Q_{k-1} and A_{k-1} have already been chosen, then define Q_k and A_k as follows:

1. *Set $Q_k := D\big(D(A_{k-1})\big) \cap \eta$, and query the elements of Q_k.*
2. *Let A_k denote the set $x \in Q_k$ such that $\xi(x) = 1$.*
3. *If $A_k = A_{k-1}$, then stop, and set $A_\infty = A_k$, otherwise go to step 1.*
4. *If $H(A_\infty, R_N, \bullet)$ occurs, then output 1, otherwise output 0.*

Define the algorithm $\mathcal{A}_W^*$ similarly, except with $A_0 := \big\{(x,y) \in \mathbb{R}^2 : x = N/2\big\}$.

To see that the Water Algorithm determines $f_N^\eta(\xi)$, note that at each step $k \in \mathbb{N}$ the algorithm queries each $x \in \eta$ such that $D(x) \cap D(A_{k-1}) \neq \emptyset$. Hence, an element $x \in \eta$ is queried if and only if there is a path from the left edge of R_N to $D(x)$, using only points of $D(\xi) \cap R_{N+2}$. Thus, if $f_N^\eta(\xi) = 1$ then the algorithm will find a horizontal path across R_N contained in $D(\xi) \cap R_N$; conversely, if $f_N^\eta(\xi) = 0$ then the algorithm will output zero, since $A_\infty \subseteq \xi$.

As we shall see, the algorithm $\mathcal{A}_W$ is unlikely to query points in the right-hand half of R_{N+2}, and $\mathcal{A}_W^*$ unlikely to query those in the left-hand half. Define

$$K_L := R_{N+2} \cap \Big(\big(-\infty, 0 \big) \times \mathbb{R} \Big), \quad \text{and} \quad K_R := R_{N+2} \cap \Big(\big[0, \infty \big) \times \mathbb{R} \Big).$$

Recall Definition 2.5. Given $\eta \in \Omega$, it will be convenient to introduce a notation for the revealment of the algorithm $\mathcal{A}_W$ in a bounded region $K \subseteq \mathbb{R}^2$. We denote this by $\delta_K(\mathcal{A}_W) = \delta_K(\mathcal{A}_W, N, p, \eta)$, and is defined as

$$\delta_K(\mathcal{A}_W) := \max_{x \in \eta \cap K} \mathbb{P}\big(\mathcal{A}_W \text{ queries } x \text{ when determining } f_N^G(\eta_p) \,\big|\, \eta \big).$$

The following lemma will allow us to deduce Theorem 1.5 from Theorem 2.6.

Lemma 7.4. *For every $C > 0$, there exists $\delta > 0$ and $p^* = p^*(C)$ such that for $p \in (0, p^*)$,*

$$\mathbf{P}_{\lambda_c/p}\Big(\delta_{K_R}(\mathcal{A}_W) \geqslant N^{-\delta} \Big) \;\leqslant\; N^{-C}$$

for every sufficiently large $N \in \mathbb{N}$. (The same holds for K_R and $\mathcal{A}_W$ substituted for K_L and $\mathcal{A}_W^*$.)

Informally, what the statement says is that with very low $\mathbf{P}_{\lambda_c/p}$-probability ($\leqslant N^{-C}$), the configuration $\eta \in \Omega$ will be 'bad' in the sense that the algorithm has high probability ($\geqslant N^{-\delta}$) to find its way to some $x \in \eta \cap K_R$.

Partition R_{N+2} into $(N+2)^2$ squares of side-length 1 in the canonical way, and denote these squares by $S_1, \ldots, S_{(N+2)^2}$. Define $\mathbb{A}_\ell$ to be the annulus centred at the origin, and consisting of all point with ℓ_∞-norm between ℓ and 2ℓ, and for $1 \leqslant i \leqslant (N+2)^2$, let $\mathbb{A}_\ell(S_i)$ denote $\mathbb{A}_\ell$ shifted to be concentric to S_i. For fixed $\eta \in \Omega$, let $\mathcal{C}\big(\mathbb{A}_\ell(S_i), \eta_p\big)$ denote the (monotone decreasing) event that there is a loop of vacant space in $\mathbb{A}_\ell(S_i)$; equivalently, it is the event that there is no path between the two faces of $\mathbb{A}_\ell(S_i)$ which is contained in $D(\eta_p)$.

Now, consider the $t = \log_4(N/4)$ annuli $\mathbb{A}_{\ell(1)}(S_i), \ldots, \mathbb{A}_{\ell(t)}(S_i)$, where $\ell(j) = 4^j$. Of course, t might not be an integer, but this is easily adjusted for by the reader. Note that the distance between $\mathbb{A}_{\ell(j)}(S_i)$ and $\mathbb{A}_{\ell(j+1)}(S_i)$ is at least 2 for each j, so the events $\mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big)$ are independent.

Proof of Lemma 7.4. It will suffice to prove that for every $C > 0$, there exists $\delta > 0$ and $p^* = p^*(C)$ such that if $p \in (0, p^*)$, then for every S_i that intersects K_R,

$$\mathbf{P}_{\lambda_c/p}\Big(\delta_{S_i}(\mathcal{A}_W) \geqslant N^{-\delta} \Big) \;\leqslant\; N^{-C} \tag{7.4}$$

for every sufficiently large $N \in \mathbb{N}$. To realize that this suffices, note that there are less than $(N+2)^2$ squares S_i that intersect K_R, so

$$\mathbf{P}_{\lambda_c/p}\Big(\delta_{K_R}(\mathcal{A}_W) \geqslant N^{-\delta}\Big) \;\leqslant\; N^{-C}(N+2)^2,$$

as required.

For $\eta \in \Omega$ and $S_i \subseteq K_R$, note that if $\bigcap_{j\in[t]} \mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big)^c$ does not occur, i.e. that $\mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big)$ occurs for some $j \in [t]$, then no point $\eta \cap S_i$ will be queried by the algorithm $\mathcal{A}_W$ (or $\mathcal{A}_W^*$ if instead $S_i \subseteq K_L$). This follows because $\mathcal{A}_W$ queries $x \in \eta \cap K_R$ if and only if there exists a path from the left edge of R_N to $D(x)$, using only points of $D(\eta_p)$, and since $x \in K_R$, so x is ℓ_∞-distance at least $N/2$ from the left edge of R_N. However, no such path can exist if $\mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big)$ occurs for some $j \in [t]$. Therefore, to obtain (7.4), we will show that

$$\mathbf{P}_{\lambda_c/p}\left(\mathbb{P}\left(\bigcap_{j\in[t]} \mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big)^c \,\middle|\, \eta\right) \geqslant N^{-\delta}\right) \leqslant N^{-C}. \tag{7.5}$$

Fix $C > 0$ and choose $\gamma = \gamma(C) > 0$ such that $2^t \gamma^{t/4} \leqslant N^{-C}$ (when N is large). By Proposition 2.3 and the FKG inequality, we conclude that there is $p^* = p^*(C) > 0$ such that if $p \in (0, p^*)$, then

$$\mathbf{P}_{\lambda_c/p}\Big(\mathbb{P}\big(\mathcal{C}\big(\mathbb{A}_\ell(S_i), \eta_p\big) \,\big|\, \eta\big) \geqslant c^4\Big) \;\geqslant\; 1 - \gamma \tag{7.6}$$

for all sufficiently large $\ell \in \mathbb{N}$, where $c > 0$ is a fix constant given by Proposition 2.3. Define the good event $G \subseteq \Omega$ as

$$G := \left\{\eta \in \Omega : \left|\left\{j \in [t] : \mathbb{P}\big(\mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big) \,\big|\, \eta\big) \geqslant c^4\right\}\right| \geqslant \frac{t}{2}\right\}.$$

Note that if $\eta \in G^c$, then $\mathbb{P}\big(\mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big) \,\big|\, \eta\big) \geqslant c^4$ must fail for at least half of the j's in $[t]$, of which at least $\frac{t}{4}$ fulfill $j \geqslant \frac{t}{4}$. Thus, if N is sufficiently large, then (7.6) applies and gives that

$$\begin{aligned} \mathbf{P}_{\lambda_c/p}(G^c) \;&\leqslant\; \mathbf{P}_{\lambda_c/p}\left(\left|\left\{j \geqslant \frac{t}{4} : \mathbb{P}\big(\mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big) \,\big|\, \eta\big) < c^4\right\}\right| \geqslant \frac{t}{4}\right) \\ &\leqslant\; 2^t \gamma^{t/4} \;\leqslant\; N^{-C}, \end{aligned} \tag{7.7}$$

by the choice of γ. Finally, for $\eta \in G$ at least $t/2$ of the annuli $\mathbb{A}_{\ell(j)}(S_i)$ have a reasonable probability ($\geqslant c^4$) of containing a loop that prevents the algorithm to reach S_i. This implies that for $\eta \in G$

$$\mathbb{P}\left(\bigcap_{j\in[t]} \mathcal{C}\big(\mathbb{A}_{\ell(j)}(S_i), \eta_p\big)^c \,\middle|\, \eta\right) \;\leqslant\; \big(1 - c^4\big)^{t/2} \;\leqslant\; N^{-\delta}, \tag{7.8}$$

for some $\delta > 0$.

Combining (7.7) and (7.8) we obtain (7.5), and therefore also (7.4). $\square$

We can now deduce Theorem 1.5.

Proof of Theorem 1.5. We prove that if $p > 0$ is sufficiently small, then $(f_N^\eta)_{N\geqslant 1}$ is NS_p for $\mathbf{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$. Indeed, according to Lemma 7.4 and Borel-Cantelli's lemma, we can find $\delta > 0$ and p^* such that for every $p \in (0, p^*)$

$$\mathbf{P}_{\lambda_c/p}\Big(\delta_{K_R}\big(\mathcal{A}_W\big) \geqslant N^{-\delta} \text{ for infinitely may } N\Big) = 0.$$

By symmetry, the same holds for $\delta_{K_L}\big(\mathcal{A}_W^*\big)$, and hence

$$\Big(\delta_{K_R}\big(\mathcal{A}_W\big) + \delta_{K_L}\big(\mathcal{A}_W^*\big)\Big)\big(\log N\big)^6 \;\to\; 0, \quad \text{as } N \to \infty,$$

for $\mathbf{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$. Since f_N^η is monotone for each $N \geqslant 1$ and $\eta \in \Omega$, we may apply Theorem 2.6 to conclude that for $\mathbf{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$, the sequence $(f_N^\eta)_{N\geqslant 1}$ is NS_p. $\square$

7.3 The Poisson Boolean model is noise sensitive at criticality

We are finally ready to deduce Theorem 1.2; as we remarked in Section 2, it follows easily from Theorem 1.5 and Proposition 2.1. Recall that f_N^G is the function that for each $\eta \in \Omega$ encodes whether or not there is a horizontal crossing of R_N in the occupied space $D(\eta) \cap R_N$. To prove the noise sensitivity of the Poisson Boolean model we must prove that for some $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbf{E}_{\lambda_c}\big[f_N^G(\eta) f_N^G(\eta^\varepsilon)\big] - \mathbf{E}_{\lambda_c}\big[f_N^G(\eta)\big]^2 = 0, \quad \text{for every } \varepsilon \in (0,1). \tag{7.9}$$

For $p \in (0,1)$ and $\varepsilon \in (0,1)$ such that $\varepsilon < 1 - p$, set $\delta = \varepsilon/(1-p)$. We observe the equivalence of the following two constructions of a pair in Ω.

(i) Recall that the pair (η, η^ε) is chosen as follows. Pick $\eta \in \Omega$ according to the measure $\mathbf{P}_{\lambda_c}$. Obtain η^ε by deleting each element of η independently with probability ε, and proceed by adding a new configuration picked independently according to $\mathbf{P}_{\varepsilon\lambda_c}$.

(ii) Recall that the pair $\big(\eta_p, (\eta_p)^\delta\big)$ is chosen as follows. Pick $\eta \in \Omega$ according to the measure $\mathbf{P}_{\lambda_c/p}$. Obtain η_p by deleting each element of η independently with probability p. Construct $(\eta_p)^\delta$ by independently, with probability δ for every $x \in \eta$, re-randomizing the decision to delete or keep it.

Obviously, picking (η, η^ε) according to $\mathbf{P}_{\lambda_c}$ is equivalent to $\big(\eta_p, (\eta_p)^\varepsilon\big)$ when η is picked according to $\mathbf{P}_{\lambda_c/p}$. In particular,

$$\begin{aligned}\mathbf{E}_{\lambda_c}\big[f_N^G(\eta) f_N^G(\eta^\varepsilon)\big] - \mathbf{E}_{\lambda_c}\big[f_N^G(\eta)\big]^2 \\ = \mathbf{E}_{\lambda_c/p}\big[f_N^G(\eta_p) f_N^G\big((\eta_p)^\delta\big)\big] - \mathbf{E}_{\lambda_c/p}\big[f_N^G(\eta_p)\big]^2.\end{aligned} \tag{7.10}$$

Proof of Theorem 1.2. Fix $\varepsilon \in (0,1)$ and $\gamma \in (0,1)$. Pick $p \in (0, 1-\varepsilon)$ sufficiently small in order for $(f_N^\eta)_{N\geqslant 1}$ to be NS_p for $\mathbf{P}_{\lambda_c/p}$-almost every $\eta \in \Omega$, and for

$$\limsup_{N\to\infty} \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{E}\big[f_N^G(\eta_p)\big|\, \eta\big]\Big) < \gamma$$

to hold. The former is possible according to Proposition 1.5, and the latter according to Proposition 2.1. Set $\delta = \varepsilon/(1-p)$. We have, as a consequence of (7.10), that

$$\begin{aligned}&\mathbf{E}_{\lambda_c}\big[f_N^G(\eta) f_N^G(\eta^\varepsilon)\big] - \mathbf{E}_{\lambda_c}\big[f_N^G(\eta)\big]^2 \\ &\quad = \mathbf{E}_{\lambda_c/p}\Big[\mathbb{E}\big[f_N^G(\eta_p) f_N^G\big((\eta_p)^\delta\big) \,\big|\, \eta\big]\Big] - \mathbf{E}_{\lambda_c/p}\Big[\mathbb{E}\big[f_N^G(\eta_p) \,\big|\, \eta\big]\Big]^2 \\ &\quad = \mathbf{E}_{\lambda_c/p}\Big[\mathbb{E}\big[f_N^G(\eta_p) f_N^G\big((\eta_p)^\delta\big) \,\big|\, \eta\big] - \mathbb{E}\big[f_N^G(\eta_p) \,\big|\, \eta\big]^2\Big] \\ &\qquad + \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{E}\big[f_N^G(\eta_p) \,\big|\, \eta\big]\Big).\end{aligned} \tag{7.11}$$

Sending N to infinity, we obtain by the choice of p that (7.11) is at most γ. Since both γ and ε were arbitrary, (7.9) has been established, so the Poisson Boolean model is noise sensitive at criticality. $\square$

We remark that, using (7.11), we obtain as an immediate consequence of Theorems 1.2 and 1.5 the following strengthening of Proposition 2.1: For every sufficiently small $p > 0$

$$\lim_{N\to\infty} \mathbf{Var}_{\lambda_c/p}\Big(\mathbb{P}\big(H(\eta_p, R_N, \bullet) \,\big|\, \eta\big)\Big) = 0.$$

8 Open problems

In this paper we have laid out a fairly general approach to the problem of proving noise sensitivity in models of continuum percolation, and we expect that our method could be applied to prove similar results in more general settings. In this section we shall state a few of these open problems.

8.1 More general Poisson Boolean models

The simplest extension of the Poisson Boolean model considered in this paper would be to allow discs to be assigned with random (but bounded) radii. Given $R > 0$ and some distribution μ_R on $(0, R)$, we obtain a configuration of occupied space in this model by picking $\eta \in \Omega$, according to the measure $\mathbf{P}_\lambda$, and placing a disc of radius $r(x)$ at $x \in \eta$, where $r(x)$ is chosen according to μ_R, independently for each vertex.

In this paper we consider sensitivity to small perturbations of the Poisson point configuration with respect to the positions of the points. In the Poisson Boolean model with random radii, perturbations can be achieved in several different ways, as is informally described below.

(i) We can add and remove a small proportion of the balls, much like in this paper.

(ii) We can leave the Poisson configuration unaffected, but re-randomize some of the radii.

(iii) We can do a mix of both.

For sensitivity to noise as described in (i), the missing ingredient is an RSW Theorem for the occupied space which allows for random radii.

Conjecture 8.1. *For every $R > 0$ and μ_R, the Poisson Boolean model with random radii chosen according to μ_R is noise sensitive at criticality with respect to perturbations as described in (i).*

An alternative generalization would allow us to use an arbitrary shape S instead of a disc. Given such an $S \subseteq \mathbb{R}^2$, and a Poisson point process η, place a copy of S on every point $x \in \eta$; that is, set $D(\eta) = \bigcup_{x\in\eta}(x + S)$. It seems likely that if S is bounded and has positive Lebesgue measure, then results similar to those presented in this paper could hold.

8.2 Voronoi percolation

Given a configuration $\eta \in \Omega$, the Voronoi tiling of η (see Bollobás and Riordan (2006a), for example) is constructed by associating each point of $\mathbb{R}^2$ with the point of η closest to it. We call the set of points associated to $x \in \eta$ in this way the Voronoi cell of x. In Voronoi percolation we choose a random subset of the cells, by colouring each blue with probability p, and say that the model percolates if there exists an infinite component of blue space. Bollobás and Riordan (2006b) proved that if η is picked according to $\mathbf{P}_\lambda$, the critical probability is $1/2$.

Given a Voronoi tiling $V = V(\eta)$ of $\mathbb{R}^2$, let $f_N^V : \{0,1\}^V \to \{0,1\}$ be the function which, given a colouring of V, replies whether there is a blue crossing of R_N. Picking η according to a Poisson process, we say that Voronoi percolation is noise sensitive at criticality if $(f_N^V)_{N\geqslant 1}$ is almost surely NS, that is for $\mathbf{P}_\lambda$-almost every η.

Benjamini, Kalai, and Schramm (1999, Section 5) conjectured that knowing the Voronoi tiling, but not the colouring, gives almost no information as to whether or not there exists a blue crossing of R_N. We make the following conjecture.

Conjecture 8.2. *Voronoi percolation is noise sensitive at criticality.*

8.3 Stronger results for the Gilbert model

There is a concept of *quantitative* noise sensitivity where, in (1.1), one lets $\varepsilon = \varepsilon(n)$ depend on n. Recently, *very* strong results have been proved by Schramm and Steif (2010) and Garban et al. (2010) in the case of bond percolation on the square lattice and site percolation of the triangular lattice. An interesting problem, inspired by their work, would be to prove a quantitative version of Theorem 1.2.

Problem 1. *Determine the exact dependence on N such that with $\varepsilon = \varepsilon(N)$,*

$$\lim_{N\to\infty} \mathbf{E}_{\lambda_c}\big[f_N^G(\eta) f_N^G(\eta^\varepsilon)\big] - \mathbf{E}_{\lambda_c}\big[f_N^G(\eta)\big]^2 = 0.$$

One possible application of a solution to Problem 1 would be to dynamical (continuum) percolation. To define this model, consider a Poisson point process η of density λ_c in the plane, and suppose points in the plane disappear at rate one. Next, let new points 'rain down' at a rate that keeps the intensity of points in the plane constant. Once landed, also these points disappear at rate one. This yields a stationary process for which we let η_t denote the set of points in the plane at time t. By Corollary 3.2, at any given time there is (almost surely) no infinite component in $D(\eta_t)$; we therefore say that t is an *exceptional time* if there is an infinite component in $D(\eta_t)$ at time t.

The analogue of the following conjecture was proved for site percolation on the triangular lattice by Schramm and Steif (2010), and for bond percolation on the square lattice by Garban, Pete, and Schramm (2010).

Conjecture 8.3. *There exist exceptional times in dynamical continuum percolation at criticality, almost surely.*

A related problem was studied by Benjamini and Schramm (1998).

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