IN SEARCH OF A COMMON TERMINOLOGY FOR MATHEMATICS AND THE PHYSICAL WORLD: EXPERIENCE AND REVERSE MATHEMATICS

PETTER JOHANSSON

ABSTRACT. With an *experience*-oriented view on mathematics, we make an argument for its validity as well as the validity of mathematical science, provided that natural science is valid. The central concept here is reverse mathematics, which is presented technically: in particular, we look at mathematical theorems that are equivalent to ACA_0 over RCA_0. Finally, we sketch a proof of the conservation result for Π^0_2-formulas in PRA and WKL as an affirmation for finitistic mathematics.

CONTENTS

Date: May 10, 2007.

1. Presentation of the problem

"The validity of mathematics is under siege", mathematician Stephen G Simpson declares in the article "Partial realization of Hilbert's program" [2]. What he refers to is a more and more widespread doubt among academics about a connection between mathematics and the physical world. For example he mentions an article by physicist and mathematician E P Wigner, suggesting that except for the most elementary parts, mathematics is just a formal game.

The benefit of research in pure mathematics is often far-fetched. The only reliable thing that suggest mathematics is mathematics itself. For the naturalist , this is enough, but mathematical naturalism is thereby mostly appealing for mathematicians.

The purpose of this paper is to show how *reverse mathematics* can help us to validate mathematics philosophically. This relatively new branch of mathematics is looking for correspondences between axioms and theorems: the question is not only which axioms are needed to prove a theorem, but also which axioms can be proved from a given theorem (over a weaker system of axioms). The technical aspects are presented later, but first we will investigate what role a formal theory could play in the study of the world anyway.

In the article mentioned above, Simpson presents an argument for reverse mathematics, its main idea being to settle Hilbert's program partially in finitism. We will use it as one of our arguments, but while Simpson shows that reverse mathematics is a strong candidate for the rehabilitation of mathematics by partial realisation of Hilbert's program, the idea of this paper begins by showing which circumstances implies mathematics, without judging its validity more than through comparing it with the validity of natural science. This will lead to a suggestion that a general plan for mathematical science is motivated, where of course Hilbert's program is inevitable. And then we will see how reverse mathematics appears as attractive for such a plan. We will also see that Hilbert's program and finitistic mathematics is interesting from our point of view in a way directly connected to our terminology.

As will be revealed, the formalism presented below does not lead to any new arguments for the validity of mathematics. But the purpose is also the opposite: the formalism defines mathematics as something out of reach from any common validity arguments and rather tries to clean the discussion from rhetorical dead ends such as "no mathematical objects exist" or the rivalty between platonism and formalism, which will mean nothing in our context. The purpose of this formalism is rather to formulate a question than an answer. The answer is then suggested mainly by technical arguments in section 4-8.

2. How it all have to begin

What is necessary to have in mind when reading this section, is that my opinion is that we understand the world through our language, and that we therefore have to consider the language before we consider who is using it and why. In fact, I would say that we do not define language: language defines us. But of course this does not make sense until we specify what we refer to with the word language.

Sense cannot validate itself. This forces the philosophist to accept some facts that he cannot validate. But in context of our concerns, at least three facts cannot be intellectually rejected:

a The reader exists
b Mathematics exists
c The information here exists

since without a reader, any text is pointless, without mathematics, the validity of mathematics is trivial and without the information represented in this text, the text has failed anyway.

A reader can be a computer with a scanner as well as a human being. Anyhow, it could be said to give meaning to the word subject, with mathematics as well as this information only existing as related to it. What it is to be a reader is thus pointless to describe for the reader: the property of being (being, not working as) a computer would not exist if the only kind of subjects that was, were computers.

When a question is asked, one should start the search for the answer by seing if it is contained in the question. "Is a green apple green?" obviously answers itself. Questions like "what is mathematics?" or "what is everything?" implies an associative ability or a kind of "language" that is tempting to applicate for a reformulation of what we actually know:

1 *Experience* exists
2 *Concepts* exist
3 *Communication* exists

And so, we have defined the foundation of existence by a subjectively associative extension of the axioms a-c. Note that we still have not said anything about the existence of the world, and if this way of arguing needs the world it is only for the validiation of 1-3, which means that to assume the world is to assume 1-3 or more. But before we are making further unvalidable assumptions, we shall decide the relation between 1, 2 and 3.

Here comes the central idea of this section: with the main argument being simplicity (but we will see later that it is not such a bad idea), we represent experience as a nonempty set E, all concepts as a nonempty set B, all communication as a nonempty set T and let

$$B, T \subset E$$

What the elements of E, the experiences, are is still decided subjectively depending on ones view of the existence, even if what you are right now, by the reformulation of a to 1, is built up from them. But one possibility is to think of an element as an "impression", like a certain taste that one can taste. Likewise, the question what the cardinality of E is (e.g. is it finite, continous, singular) do not need an objective answer.

What we will do now though is to specify how models give us access to experience, and how communication gives us access to models. The definitions in the rest of the section are often formulated in a vague way. This is because we are only interested in subjective associations that in many cases would do better without preciseness.

Let R be a binary relation on E and B such that for all b in B there is an e in E such that $R(e, b)$: if this holds, we call b a concept of e and e an object. Even the definition of R is left to the association of the reader, but if e is a taste and $R(e, b)$, b may be thought of as the memory of e or the reflection that you are tasting e: a concept is always an "experience of an experience".

The question what $R(e, b)$ actually means cannot be satisfied by a subjective association, since it resembles to the problem that we started with: the nature of sense cannot be studied through sense. But as we accepted B, we have to accept a nonempty R with no furter discussion. As R is used strictly as implying access of an element e by an element b, there is no loss of generality in letting R be transitive, since we anyway can define a relation T by letting $T(e, b)$ if $R(e, c)$ and $R(c, b)$.

Letting b be as above, a t in T such that $R(t, b)$ can be thought of as the experience of reading or hearing about the taste e. Even if it is subjective, "this" is by our reformulation of c to 3, an element of T. "This" is an experience, not the physical materia that makes the letters perceivable on the printed paper, except from that the latter claim also is an experience and something communicated. Like for every e such that $R(e, b)$, we say that b is a concept of t if $R(t, b)$: if $R("white", m)$, we say that m is a concept of "white".

We say that an element t in T is *successful* to the extent of for how many b in B, $R(t, b)$, and for how big a part of these b:s also $R(e, b)$ for a fixed e in E, $e \neq t$. Communication is not in the first place something that is practiced, but something that relates models for the same objects.

We define *science* as a sequence of sets S of increasingly successful elements in T. Sometimes one says that there are two demands on a scientific model: the intrinsic demand that the model must make sense for us, and the extrinsic demand that all the empirical tests on the object must turn out positive. In this formalism, we can put it like this: $S \subset T$ is intrinsicly satisfying to the extent of how many $b \in B$ satisfies $R(t,b)$ for each $c \in S$. Furthermore S is extrinsicly satisfying in the extent of for how big a part of the b:s in B such that $R(t,b)$ for a fixed $t \in S$, also $R(e,b)$ holds for a fixed $e \in E$. As a suggestion, one could think of b here as someone's perception at a certain time. If $t =$ "the earth is flat" is an element of S, it would make sense for many people, until they see a satellite picture of the earth or the sail on a ship disappearing beneath the horizon. If b is the perception of a satellite picture of the earth, $R(t,b)$ does not hold, while probably $R(e,b)$, for an e that in many senses can be associated with the object earth.

Ultimately, science should converge to a $S \subset T$ such that there is an $e \in E$ such that for all $t \in S$,$R(e,b) \equiv R(t,b)$ for any $b \in B$. In that case, we say that e is *identified* by t.

A subset C of T defines a *communication* , while a subset L of B defines a *language* . While a language is subjective, it can be called objective in a sense if its elements are related to a successful communication. Words and grammar are secondary constructions from C or L if they are motivated. So even if it would lack words and grammar, mathematics is by definition a language: this condition was present already in the reformulation of b to 2.

A concept b is *analytical* if there is a concept $\neg b$ such that for all $a \in B$, $\neg(R(a,b) \wedge R(a,\neg b))$. By the transitivity of R, this means that $\neg R(b,\neg b)$ and hence also $\neg R(c,d)$ for c,d in B such that $R(b,c)$ and $R(d,\neg b)$.

Whatever mathematics means to us, we certainly consider it analytical and furthermore, in a very wide aspect, logics and mathematics can be identified as precisely the analytical concepts. For example, let $b \in B$ be a concept of "warm", $c \in B$ the concept of "prime number" and $\neg c \in B$ the concept of "not a prime number". Given a concept a of a number, both $R(a,c)$ and $R(a,\neg c)$ cannot hold. If a is not a number at all, neither $R(a,c)$ nor $R(a,\neg c)$ holds. Therefore, c is an analytical concept. On the other hand, $R(a,b)$ could possibly hold whatever a is, or b could actually be analytical too. But an analytical concept is in any case not interesting as analytical until we know that it is so, as we arguably know when it comes to the concept of prime numbers.

What separates mathematics from other analytical languages is thus decided rather empirical and from our point of view, this is not important. But as the most extensive analytical language, what characterizes mathematics is the transitivity of R. Note that if b is not analytical, by the transitivity of R there are no concepts of b that are analytical. On the other hand, there is a good chance of finding an analytical concept as a concept of another analytical concept. This fact leads to the fact that mathematics appears to be iterating itself while it is studied: when one makes a concept for a mathematical object, this concept often turns out to be a mathematical concept too. We will now present two examples of this.

We can make a prediction of counting by the concept of addition. In this concept we can add 15 to 24 by first adding 1 to 2 and then 5 to 4, additions that are well-known. Our standard rules then give us that $15+24=39$ and we predict that counting together 15 and 24 will give us 39.

In the next step, we introduce a new concept, namely the concept of multiplication, now with the concept of addition as the object. By experiencing addition and $4+4$, we can also learn to experience multiplication and $2*4$ or $3*4$, both which are applicable as concepts to the method of counting.

Another example is the introduction of the concept of natural numbers and the potential infinity. Here the object is some natural numbers and, again, the method of counting. A person who tries to find out which is the biggest number, probably will reflect upon pairs like 1000000 and 1000001 before accepting the concept of $\mathbb{N}$, and thereby predict for example that 1000000000 cannot be the highest number. In the next step we could introduce concepts of $\mathbb{N}$, for example the theorem that there are infinitely many prime numbers.

3. The use of mathematical science for natural science

The purpose of this paper is to defend the validity of mathematics. In our terms this does not mean that the mathematics itself may not be valid: concepts are of course valid in themselves. Instead, it is the science of mathematics that have to be validated. Since the attack on it primarily stems from the natural sciences, we can assume that these are valid.

If an analytical concept b appears as concept of an element in a science, it is obvious that the science stribes to exclude either b or neg b as concepts for the same communication element: the big challenge for a positivistic science is to avoid contradictions. Sciences that tries to distinguish models for an analytical concept, which is one of the goals of natural science and the science of mathematics, are particulary vulnerable for contradictions. In fact, a condition that these sciences should make sense, is that a supposedly precise language is used: the element

"1+1=2" could be related to the same concepts as "1-1=2", but as a scientist one have no choice but to either invent a new communication or consider a subset of E where such concepts do not appear, so that the interesting element e can be represented perfectly successfully by an element t in T, so that e is identified by t.

Since we have accepted natural science, we have thus accepted that certain communication elements such as ordinary mathematical expressions, are as good as perfectly successful. By now, we can validate basic mathematical objects: numbers, geometrical figures and operators such as sums and differentials are analytical concepts that the natural scientist refers to, and hence, their existence "in the world" cannot be questioned without a simultaneous questioning of the methods of natural science: after all, "the world" is secondary to B from our point of view.

The next question is: could a science applied on mathematics, a mathematical science, be interesting? As natural science tries to identify elements in E, an analogous mathematical science is in many cases trivial, since for example "circle" is supposed to be perfectly successful. On the other hand, a project where all mathematical objects, that is more or less all analytical concepts, are labelled by communication elements, is logically impossible. Rather, one has to choose the analytical concepts that should be identified.

Models that might be useful for natural science can by our assumptions be considered interesting. Hence, the elements of mathematical science that appears in natural science, are immediately approved. These include constructions such as basic geometry and integration theory, the mathematical representation of probability, mechanics, Brownian motion and so forth. Of course, the successor function and the potential infinity can be approved, since they are implicated almost everywhere in the methods of natural science.

The question is to which extent it is useful for natural science to identify analytic concepts of identified concepts of natural science. Should for example the theorem that there are infinitely many prime numbers be identified? What possibly could be gained from such a science, is the finding of new models that are interesting for natural science. According to a certain view, influenced by Darwinism, this would only happen accidentally. But it is through these accidents that a mathematical science could be motivated. Like in every positivistic science, these accidents should be studied so that one can suggest communication elements that identify common properties of them. That is a valid mathematical science, that might produce a communication that can be used by natural science.

We will now mention two basises for mathematical science, both in which reverse mathematics plays a central role: *formalism* and *finitism* . The objective of formalism is to cover mathematics as completely as possible by a formal system, that is a set of axioms or rules by which identifications for mathematical objects are deduced. First order arithmetics, also called Peano's arithmetics or PA, is such an example. Another more general formalism is second order arithmetics that will be discussed more detailed in the next section.

One should have in mind that the axioms of PA or Z_2 do not automatically identify any particular mathematical object. What is meant as an identification is instead the deduction of an "artificial" result or construction, such as we will see that functions and certain theorems will get in the following sections. While this identification seems to be as good or bad as the analytically deduced Bolzano Weierstrass, we can by reverse mathematics manage a further identification from the deduction, namely the axioms that are equivalent to the theorem. As we will see, apparently different theorems will get the same identification in this way, and hence reverse mathematics can identify new analytical concepts of mathematics.

While formalism with reverse mathematics has the advantage that it is a direct method in identifying, by common methods unidentified, properties of commonly used mathematical concepts, finitism has its strengh in that it is closely related to natural science. The idea of finitism is thus to get a representation of mathematics which in a sense could be said to be valid by empirical means, or *constructive* : Due to the finitist, *every mathematical object can be constructed from the natural numbers in a finite number of steps* . In the last section, we will express this fact in our terminology and also see how it is related to traditional mathematics.

4. Second order arithmetics

All formal systems we are going to deal with are subsystems of the theory *second order arithmetics* , denoted Z_2. The differences lies in the comprehension axiom for each system and in some cases in the induction axiom.

Second order arithmetics is of second order in the sense that while first order arithmetics only uses variables ranging over the domain, we also use set variables here. Hence it is convenient to formalise Z_2 in second order logic, allthough it is also possible to do so in first order logic. The language is defined as follows: the constant terms are 0 and 1, there are binary functions $+$ and $\cdot$, and there is one binary relation, $<$. Lowercase letters are used for the *numerical variables* i e the domain variables, and uppercase letters are used for *set variables* .

The axioms of Z_2 are divided into *the basic axioms*:

(1) $n+1 \neq 0$
(2) $n+1=m+1 \rightarrow n=m$
(3) $n+0=n$
(4) $n+(m+1)=(n+m)+1$
(5) $n \cdot 0=0$
(6) $n \cdot (m+1)=(n \cdot m)+n$
(7) $\neg n<0$
(8) $m<n+1 \leftrightarrow (m<n \vee m=n)$

which are included in all relevant subsystems of Z_2, *the induction axiom scheme*

(9) $(\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n \phi(n)$

and *the comprehension axiom scheme*

(10) $\exists X \forall n(n \in X \leftrightarrow \phi(n))$

where ϕ in (9) and (10) is a formula in which X does not occur free. Note that since we have the comprehension axiom scheme, it would have been enough with one single induction axiom:

$$(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)$$

With the natural numbers $\mathbb{N}$ as domain for the numerical values (or *numbers*), the set of all subsets of $\mathbb{N}$, $P(\mathbb{N})$ as domain for the set values and with the constants 0 and 1, the functions $+$ and $\cdot$ and the relation $<$ interpreted in the obvious way, we clearly get a model of Z_2. If we change $P(\mathbb{N})$ to a subset $S \subset P(\mathbb{N})$, we get a model which might satisfy Z_2 or a subsystem of Z_2 as well. We call these kinds of models $\mathbb{N}$-*models* and refer to a specific $\mathbb{N}$-model by its set of sets S.

The subsystems of Z_2 that we will be most interested in are RCA_0 and ACA_0. The central feature that characterizes them is the restrictions of the formula in the comprehension axiom scheme, and these restrictions are the first thing we need to define:

Definition 1. A formula in Z_2 is *arithmetical* if it contains no set quantifiers.

The axioms of ACA_0 are (1)-(8) and (9)-(10) where ϕ is arithmetical. It is obviously a subsystem of Z_2. Note that except from the comprehension axiom its definition looks exactly like that of first order arithmetics. Not so surprising, ACA_0 is first order arithmetic transcribed in the language of Z_2.

Definition 2. Let $k \geq 0$. A formula in Z_2 is Σ_k^0 if it has the form

$$\exists n_1 \forall n_2 \exists n_3 ... n_k \phi$$

where ϕ is arithmetical and only have numerically bounded quantifiers. A formula is Π_k^0 if it has the form

$$\forall n_1 \exists n_2 \forall n_3 ... n_k \phi$$

with ϕ as above. A statement is Δ_k^0 if it is equivalent both to a Σ_k^0-formula and a Π_k^0-formula.

The axioms of RCA_0 are defined as above, with ϕ being Σ_1^0 in the induction scheme and Δ_1^0 in the comprehension scheme. Obviously, RCA_0 is a subsystem of ACA_0.

Another essential definition that puts the definition above in a proper coherence is the following:

Definition 3. Let $k \geq 0$. A formula in Z_2 is Σ_k^1 if it is on the form

$$\exists X_1 \forall X_2 \exists X_3 ... X_k \phi$$

where ϕ is arithmetical. Π_k^1 and Δ_k^1 is defined analogous to the definition above.

The index 1 in the definition above and the index 0 in definition 2 is related to the index 2 in Z_2, telling which order the quantified variables should be of.

Now we will derive some basic definitions and results. Surprisingly many untrivial mathematical results can be deduced in any of these systems, but of course one can do much more in ACA_0 than in RCA_0. Furthermore, mathematical notions that can be defined in ACA_0 in a straightforward way, may not be representable in RCA_0, or if so, rather unintuitively. But as long as it is possible to do it in a straightforward way, we will make our derivations in RCA_0.

First we use the fact that for numbers of the form $(n+m)^2+n$, the numbers n and m are uniquely determined (see [2, p. 66]). Hence, letting $(n, m) = (n+m)^2+n$ and given X and Y, $X \times Y$ exists by Σ_0^0-comprehension.

A finite sequence $< j_1, ..., j_l >$ can be represented as a set X such that

(4.1)
$$n \in X \leftrightarrow \exists i \exists j (n = (i,j) \wedge \forall i \forall j \forall k (((i,j) \in X \wedge (i,k) \in X) \rightarrow j = k)$$

A finite set X in turn, can be represented by a unique number, namely the least $((n, m), k)$ such that

$$\forall i (i \in X \leftrightarrow (i < k \wedge \exists q (m(i+1)+1)q = n))$$

The proof for the existence of such k, m and n is technical but quite direct. It can be found in Simpson [2, p. 67]

Note that (4.1) is Σ_0^0. Thus, the set of the codes for all finite sequences exists. Furthermore, the set of all sequences of length k exists, again by Σ_0^0-comprehension, and is denoted $\mathbb{N}^k$.

Definition 4. A *function* $f : X \to Y$ is a subset of $X \times Y$ such that $\forall i \forall j \forall k(((i,j) \in f \wedge (i,k) \in f) \rightarrow j = k)$. We denote by $f(i)$ the unique j such that $(i,j) \in f$.

If $f : \mathbb{N}^k \rightarrow X$ and $\mathbb{N}^k \ni n =< n_1, ..., n_k >$, we write $f(n)$ as $f(n_1, ..., n_k)$.

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. The uniqueness of $f(n)$ gives that

$$\exists j((i,j) \in f \wedge ((j,k) \in g \text{ iff } \forall j((i,j) \in X \rightarrow ((j,k) \in g$$

Hence by Δ_1^0-comprehension the composition $h : X \rightarrow Z,\ h = f \circ g$, is a function in RCA_0.

Theorem 4.1 says that the universe of k-ary functions are closed under primitive recursion, i. e. defining a sequence of functions recursively is allowed.

Theorem 4.1. *If* $f : \mathbb{N}^k \rightarrow \mathbb{N}$ *and* $g : \mathbb{N}^{k+2} \rightarrow \mathbb{N}$*, there exists a unique* $h : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ *such that*

$$h(0, n_1, ..., n_k) = f(n_1, ..., n_k) \text{ and}$$

$$h(m+1, n_1, ..., n_2) = g(h(m, n_1, ..., n_2), m, n_1, ..., n_k)$$

Proof. We can construct a Σ_0^0-formula $\phi(s, m, < n_1, ..., n_k >)$ saying that $s \in \mathbb{N}^{m+1}$ and that, for all $i < m$, $s(i+1) = g(s(i), i, n_1, ..., n_k)$ where $s(i+1)$ is the $i+1$th element of s. Letting $< n_1, ..., n_k >$ be fixed, $\exists s \phi(s, m)$ is true for each m by Σ_1^0-induction. The uniqueness of the sequence s that satisfies the formula for a given m, also follows by induction so that

$$\exists s(\phi(s,m) \wedge s(m) = j) \leftrightarrow \forall s(\phi(s,m) \rightarrow s(m) = j)$$

Hence by Δ_1^0-comprehension there is a $h : \mathbb{N}^k \rightarrow \mathbb{N}$ such that

$$h(m, n_1, ..., n_k) = i \text{ iff } \exists s(\phi(s, m, < n_1, ..., n_k >) \wedge s(m) = i)$$

Clearly we have the unique h we were looking for.

□

Finally, we will have a look at the numerical spaces $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. The set of integers $\mathbb{Z}$ and the set of rationals $\mathbb{Q}$ can be defined quite directly in RCA_0 by using pairs (n, m) of natural numbers or integers, respectively. The set of real numbers is more tricky in RCA_0. But we will do well with a definition in ACA_0.

Remember that a sequence of rational numbers $< q_n : n \in \mathbb{N} >$ is *cauchy* if

$$\forall \epsilon(\epsilon > 0 \rightarrow \exists n \forall m(n < m \rightarrow |q_n - q_m| < \epsilon)),\ \epsilon \in \mathcal{Q}$$

Definition 5. In ACA_0, a *real number* is a cauchy sequence of rational numbers. If $x =< q_n >$ and $y =< q'_n >$, $x + y =< q_n + q'_n >$, $x \cdot y =< q_n \cdot q'_n >$ and $x = y$ if $\lim |q_n - q'_n| = 0$.

Thus $=$ is an equivalence relation and not an identity for real numbers in ACA_0. But that suffices in most cases. In particular when we continue to the reverse proofs in the next chapter.

5. Real analysis and ACA_0

With the basic concepts and results we deduced in section 4 we can do quite much in reverse mathematics. In this section we will outline a proof for the equivalence between the *Bolzano Weierstrass theorem* and the axiomatic system ACA_0 over RCA_0, i. e. we will show that given the axioms of RCA_0, the claims of Bolzano Weierstrass and ACA_0 are equivalent.

Bolzano Weierstrass is central in real analysis, stating the following: Every bounded sequence of real numbers contains a convergent subsequence. A result which is equivalent to this in real analysis is the *monotone convergence theorem* which says that every bounded increasing sequence of real numbers is convergent. It is not difficult to see a connection between these results and the basic intuition we have for complete spaces such as $\mathbb{R}$.

We will start with some further deductions in RCA_0.

Lemma 5.1. *In* RCA_0 *it is provable that for any infinite* $X \subset \mathbb{N}$*, there is a strictly growing function* $\pi_X : \mathbb{N} \rightarrow \mathbb{N}$ *such that* $X \subset \pi_X(\mathbb{N})$.

Proof. We will construct π_X using primitive recursion. We start with a function $f_X : \mathbb{N} \rightarrow \mathbb{N}$ defined so that $f_X(n)$ is the least $m \in X$ with $m \geq n$. Now we let $\pi_X(0) = f_X(0)$ and $\pi_X(n+1) = f_X(\pi_X(n)+1)$. The result follows by Σ^0_0-induction.

□

Definition 6. Let f be a function. The *range* of f, range(f), is the set $\{y : \exists x(f(x) = y)\}$

The central question in many of our coming proofs is if range(f) exists for a given f. A first simple but important result concerning this is the following:

Lemma 5.2. *In* $\mathsf{RCA_0}$*, let* $\phi(n)$ *be a* Σ^0_1*-formula* $\phi(n)$ *where* X *and* f *do not occur freely. Then either there exists a finite set* X *such that* $\forall n(n \in X \leftrightarrow \phi(n))$ *or there exists an injective function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $n \in \text{range}(f)$ *if and only if* $\phi(n)$ *holds.*

Proof. Write ϕ as $\exists i \theta(i, n)$ for a Σ^0_0-formula θ. Through Σ^0_0-comprehension, we may form the set

$$Y = \{(i, n) : \theta(i, n) \wedge \neg(\exists j < i)\theta(j, n)\}$$

If the first alternative fails so that there is no finite set $X = \{n : \phi(n)\}$, then Y must be infinite. Now use lemma 5.1 to find a function π_Y which enumerates the elements of Y in strictly increasing order. Note also that the projection function $p : \mathbb{N} \to \mathbb{N}$ with $p((i, n) = n$ exists by Σ^0_0-comprehension. Therefore, the second alternative in the theorem is true, with $f = p \circ \pi_Y$.

□

Next, we need a result that allows us to simplify arithmetical formulas:

Lemma 5.3. *In* $\mathsf{RCA_0}$*, comprehension for* Σ^0_1*-formulas is equivalent to comprehension for all arithmetical formulas.*

Proof. Each arithmetical formula is equivalent to a Σ^0_k formula for some k (if it is Π^0_i, it is equivalent to a Σ^0_k formula for $k > i$).Hence it suffice to prove that Σ^0_1 comprehension implies Σ^0_k comprehension. For $k \leq 1$ the assertion is trivial. Assume that the assertion holds for a $k \geq 1$ and let $\phi(n)$ be Σ^0_{k+1} for. Let $\phi(n) = \exists i \theta(n, i)$ for a Π^0_k formula θ. By Σ^1_k comprehension, the set $Y = \{(n, i) : \neg\theta(n, i)\}$ exists. Furthermore, the set $X = \{n : \exists i((n, i) \notin Y)\}$ exists by Σ^0_1 comprehension. But then $n \in X$ if and only if $\phi(n)$. By the arithmetical induction in $\mathsf{RCA_0}$, we are done.

□

With these lemmas at hand, it is only a matter of construction to prove the equivalence between Bolzano Weierstrass theorem and $\mathsf{ACA_0}$ in $\mathsf{RCA_0}$. To prove the former in the latter, let $< x_n >$ be a bounded sequence of real numbers. We may assume that $0 \leq x_n \leq 1$ for all n. By arithmetical comprehension, there is a function $f : \mathbb{N} \to \mathbb{N}$ so that $f(k)$ is the largest $i < k$ such that $2^{-k} i \leq x_n \leq 2^{-k}(i + 1)$ for infinitely many $n \in \mathbb{N}$.

Next, let x be the sequence $< 2^{-k} f(k) >$. The sequence obviously is cauchy and thus a real number. It is also straightforward to verify that $x = \limsup x_n$. Define the subsequence $< x_{n_k} >$ by letting $n_0 = 0$ and letting n_{k+1} be the least $n > n_k$ such that $|x - x_n| \leq 2^{-k}$. Clearly $x = \lim_k x_{n_k}$ and we have thus proved the Bolzano Weierstrass theorem.

Theorem 5.4. *Over* RCA_0*,* ACA_0 *is equivalent to the Bolzano Weierstrass theorem.*

Proof. We have already proved the left-to-right implication, but the "reverse" part remains. Assuming the Bolzano Weierstrass theorem, we now want to prove ACA_0.

As we have hinted, the monotone convergence theorem is equivalent to Bolzano Weierstrass over RCA_0: the proof is just a couple of simple arguments in real analysis. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function and let $c_n = \sum_{i=0}^{n} 2^{-f(i)}$. The sequence $< c_n >$ is then bounded by 2 and increasing, so by the monotone convergence theorem, there is a real number

$$c = \lim c_n = \sum_{i=0}^{\infty} 2^{-f(i)}$$

We observe that

$$\exists i(f(i) = k) \text{ iff } \forall n(|c - c_n| < 2^{-k} \to \exists i \leq n(f(i) = k))$$

By Δ_1^0-comprehension, all k such that $\exists i(f(i) = k)$, thus define a set $X = \text{range}(f)$. But f was arbitrary, so lemma 5.2 gives that for each Σ_1^0-formula ϕ there is a set X such that $n \in X$ if and only if $\phi(n)$, and we have proved Σ_1^0-comprehension. By lemma 5.3, we obtain the desired result.

□

6. Reverse mathematics in topology

An area of mathematics that at first sight might seem to be quite far away from both set theory and analysis, is topology. Hence it would be interesting to see if there are results equivalent to ACA_0 here too. We will start with the mathematical definition of a topology and some adjacent matters.

Definition 7. Let X be a nonempty set. Then $T \subset P(X)$ is a *topology* on X if

- $\emptyset \in T$ and $X \in T$
- If $U_\alpha \in T$, then $\bigcup_\alpha U_\alpha \in T$
- If $U_1, ..., U_n \in T$, then $\bigcup_1^n U_j \in T$

If $U \in T$, U is *open* and U^c is *closed* . The *closure* of a set A is the smallest closed set K such that $A \subset K$.

RCA_0 and ACA_0 are both too weak to obtain much of classical topology. For example, a countable or transfinite union of arbitrary sets U_α

is not allowed since it implies Π^1_1-comprehension. But the following definition, also standard in classical topology, is immediately representable allready in ACA_0:

Definition 8. A *poset* is a nonempty set P with a relation $\preceq$ such that for all $p,\ q,\ r \in P$:

- $p \preceq p$
- If $p \preceq q$ and $q \preceq p$, then $p = q$
- If $p \preceq q$ and $q \preceq r$, then $p \preceq r$

Furthermore, if $A \subset P$, the *upward closure* of A, $\mathrm{ucl}(A) = \{p \in P : \exists q \in A(q \preceq p\}$. We say that A is *upward closed* if $\mathrm{ucl}(A) = A$.

Theorem 6.1. *In* RCA_0*,* ACA_0 *is equivalent to the assertion that every subset of a countable poset has an upward closure.*

Proof. From ACA_0, the assertion follows immediately by arithmetical comprehension. To prove the reverse implication, we will do something similar to the proof of theorem 5.4. Letting $f : \mathbb{N} \to \mathbb{N}$, it suffices to show that the range of f exists, since we thus have Σ^0_1-comprehension by lemma 5.2 which by lemma 5.3 is the same as arithmetical comprehension.

Choose $P = \{2^i : i \in \mathbb{N}\} \cup \{3^j : j \in \mathbb{N}\}$. By Σ^0_0-comprehension, we may define a relation $\prec$ on P by letting

- $2^i \prec 2^j$ if $i > j$
- $3^i \prec 3^j$ for all $i,\ j$
- $2^i \prec 3^j$ if $\exists k \leq i(f(k) = j)$

Obviously $\prec$ is a poset order on P.

Let $A = \{2^k : k \in \mathbb{N}\} \subset P$. We have asserted that $\mathrm{ucl}(A)$ exists. As $3^j \in \mathrm{ucl}(A)$ iff there is a k such that $f(k) = j$, the range of f is given by the Σ^0_0-formula $\phi(k) = 3^{f(k)} \in \mathrm{ucl}(A)$. □

7. Algebra

In this section, we will define some algebraic structures in RCA_0 and see how they corresponds to theories in Z_2. Even here we will find that some classical results are equivalent to ACA_0 over RCA_0.

Definition 9. A *countable ring* R is a subset of $\mathbb{N}$ with operators $+, \cdot : R \times R \to R$ and two elements $0 \neq 1$ satisfying the following axioms:

(1) $(a+b)+c = a+(b+c)$
(3) $a+b = b+a$
(3) $a+0 = a$
(4) For every a there is an element denoted $-a$ such that $a + -a = 0$
(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(6) $1 \cdot a = a$
(7) $a \cdot (b+c) = (a \cdot b + (a \cdot c))$
(8) $(a+b) \cdot c = (a \cdot c + (b \cdot c))$

(To be more specific, we sometimes write e g $+_R$ and 0_R.) The elements of R is denoted by $|R|$. Furthermore, R is *commutative* if

(9) $a \cdot b = b \cdot a$

Examples of countable commutative rings are $\mathbb{N}$ and $\mathbb{Q}$ with the standard definition of $+$ and $\cdot$. Other rings of common interest are the finite *congruence classes* $\mathbb{N}/n$ for $n \geq 1$, where $a +_{\mathbb{N}/n} b = a + b \mod n$ for $a, b \in \mathbb{N}/n$.

Definition 10. Let R be a countable commutative ring. Then $I \subset R$ is an *ideal* of R if

(1) $0 \in I$
(2) $1 \notin I$
(3) $\forall a \forall b((a \in I \wedge b \in I) \to a + b \in I)$
(4) $\forall \lambda \forall a((\lambda \in R \wedge a \in I) \to \lambda \cdot a \in I)$

Furthermore, I is *maximal* if

(5) $\forall \lambda((\lambda \in R \backslash I) \to \exists \mu(\mu \in R \wedge \lambda \cdot \mu - 1 \in I)$

In $\mathbb{N}/n$, an ideal is $\{0\}$, and in the same way $I = \{m : n|m\}$ $(n \geq 2)$ is an ideal of $\mathbb{N}$ (or $\mathbb{Q}$). Furthermore, if n is prime, then I is a maximal ideal on $\mathbb{N}$. An exercise for the reader is to identify the maximal ideals on $\mathbb{N}/n$ for a given $n \geq 1$.

Theorem 7.1. *In* RCA_0, ACA_0 *is equivalent to the assertion that every countable commutative ring has a maximal ideal.*

Proof. In ACA_0, let $R = \{r_n\}$ be a countable commutative ring. By arithmetical comprehension there is an $I \subset R$ such that for each n, $r_n \in I$ iff

$$\sum_{\{k : r_k \in I \text{ and } k<n\}} r_k \cdot a_k \neq 1 \text{ for all } a_k \in R$$

It is now easy to check that (1)-(5) holds for I which thus is a maximal ideal.

To prove the converse, let $f : \mathbb{N} \to \mathbb{N}$. Similarily to our earlier equivalence proofs, we show the implication of ACA_0 by showing that the range of f exists.

Denote the set of polynomials with rational coefficients by R_0. Let $K_0 = \{p/q : p,\ q \in R_0 \wedge q \neq 0\}$. It is easy to check that R_0 and K_0 forms countable commutative rings with $+,\ \cdot,\ 0$ and 1 defined as usual. Let $\phi(a)$ be a Σ^0_1-formula asserting that $a = p/q \in K_0$ where $q = r + \lambda x^{e_1}_{f(n_1)} ... x^{e_k}_{f(n_k)}$ for some $r \in R_0,\ 0 \neq \lambda \in \mathbb{Q},\ k > 0$.

Since countably many elements satisfy ϕ, lemma 5.2 gives that there is an injective function $h : \mathbb{N} \to \mathbb{N}$ such that $\forall a(\phi(a) \leftrightarrow \exists b(h(b) = a))$. By making bijections between $\mathbb{N}$ and $|R_0|$ and $\mathbb{N}$ and $|K_0|$ respectively, we can assume that $h : |R_0| \to |K_0|$ injectively. Note that $\phi(0)$ and $\phi(1)$ holds so that $h^{-1}(0)$ and $h^{-1}(1)$ exists. Hence, define $+,\ \cdot,\ 0$ and 1 on $h^{-1}(|K_0|)$ by letting, for $a,\ b \in h^{-1}(|K_0|)$,

- $a + b = h^{-1}(h(a) +_{K_0} h(b))$
- $a \cdot b = h^{-1}(h(a) \cdot_{K_0} h(b))$
- $0 = h^{-1}(0_{K_0})$
- $1 = h^{-1}(1_{K_0})$.

We check that (1)-(9) holds in definition 9 and construct a ring R with $|R| = |R_0|$ by extending $+$ and $\cdot$ as ring operators to the whole $|R_0|$. By our assumption, let I be a maximum ideal of R.

Assume that $h^{-1}(x_n) \in R$. If there is an m such that $f(m) = n$, then $\phi(1/x_n)$ holds so that $h^{-1}(1/x_n) \in R$. Then $h^{-1}(x_n) \notin I$ since

$$h^{-1}(x_n) \cdot h^{-1}(1/x_n) = h^{-1}(1) = 1 \notin I$$

Now assume that $h^{-1}(x_n) \notin I$ and let $\mu \in R \backslash I$ and $a \in I$ be such that $\mu \cdot h^{-1}(x_n) - 1 = a$. Then $h(a) = p/q$ for some $p,\ q \in R_0$ with q on the form $r + \lambda x^{e_1}_{f(n_1)} ... x^{e_k}_{f(n_k)}$. On the other hand, p is not on that form, since a then would be invertible and thus not an element of I. But

$$p/q = h(a) = h(\lambda) \cdot x_n - 1$$

Hence

$$h(\lambda) \cdot x_n \cdot q = p + q$$

We conclude that $\exists m(n = f(m))$. Together with the result above, we thus have proved that $\exists m(n = f(m))$ if and only if $h^{-1}(x_n) \notin I$. As the latter formula is Σ^0_0, the range of f exists by Σ^0_0-comprehension.

□

8. Combinatorics

Finally, we will prove an equivalence to ACA_0 within discrete mathematics. The equivalence is a combinatorical principle whose general non-discrete form is the Radó selection lemma, an important result for example in differential calculus. For our purposes, the following version (countable Radó lemma) is appropriate: For $k \geq 0$, let $f_k : X_k \to \mathbb{N}$ where $X_k \subset \mathbb{N}$ is finite. Assume that given a finite set $X \subset \mathbb{N}$, there is a k such that $X_k \supset X$. Furthermore, assume that

$$\forall m \exists n \forall k (m \in X_k \to f_k(m) < n)$$

Then there is a $f : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall \textit{ finite } X \subset \mathbb{N}) \exists k (X \subset X_k \wedge f|X = f_k|X)$$

Theorem 8.1. *In* RCA_0, ACA_0 *is equivalent to the countable Radó lemma.*

Proof. Let ACA_0 be given and let $\{f_k\}$ be a sequence of functions as described above. We noted in section 4 that a finite set can be represented by a unique number. Since f_k has a finite domain X_k, it is a finite set so by arithmetical comprehension we might quanitify over $\{f_k\}$ to get the formula

$$\phi(n, m) = \forall i \exists j (j > i \wedge f_j(n) = m) \wedge (k < m \to$$
$$\neg(\forall i \exists j (j > i \wedge f_j(n) = k)))$$

Let $(n, m) \in f$ if and only if ϕ holds. Obviously the criterias we had on f are fullfilled.

Now assume the countable Radó lemma and let $g : \mathbb{N} \to \mathbb{N}$. As usual we prove ACA_0 by showing that the range of g exists.

By Σ^0_0-comprehension there are functions $f_n : \{0, ..., n\} \to \{0, 1\}$ for all $n > 0$ such that

$$f_n = \begin{cases} 1 & \text{if } (\exists i < n)(g(i) = k) \\ 0 & \text{otherwise} \end{cases}$$

Then, by the countable Radó lemma, there exists a function $f : \mathbb{N} \to \{0, 1\}$ such that

$$\forall m \exists n (n > m \wedge f|\{0, ..., m\} = f_n|\{0, ..., m\})$$

Assume now that $k \in \text{range}(g)$. Then $f_n(k) = 1$ for all n big enough and hence $f(k) = 1$. Thus range(f) is given by the set X where $\forall k (k \in X \leftrightarrow f(k) = 1)$.

□

9. Conclusion: The revival of finitistic mathematics

In RCA_0, we let a *binary tree* be a set of finite sequences of 0s and 1s such that any initial segment of a sequence in T also belongs to T. A *path* through T is a function $f : \mathbb{N} \to \mathbb{N}$ such that $(f(0), f(1), ..., f(k)) \in T$ for all $k \in \mathbb{N}$,

A subsystem of second order arithmetics that will turn out to be particulary useful for our argumentation for reverse mathematics is WKL. Defined as RCA_0 with an extra set existence axiom, *weak König's lemma* , stating that every infinite tree has a path, it is weaker than

ACA_0 but still strong enough to actually be equivalent over RCA_0 to the maximum principle, the Heine/Borel covering lemma, Lindenbaum's lemma and several other important results.

In this chapter, we will conclude the thesis by a brief presentation of Simpson's finitistic reductionism of WKL, translated into the terms from the first chapter. We will find that a strong formal mathematical system does not have to be successful only high up in a cumulative hierarchy of sciences of sciences, but through the direct essence of finitism, successful in the science of the nature of natural science. Thereby, nonelementary mathematics shows its usefulness in the understanding of the world.

A concept is essentially related to an experience in the same way as a logical sentence or theory is to a model. Two theories that are satisfied by the same model is a formalisation of how two concepts are related to the same object. If S identifies a part of mathematical science, we can hence formalise a concept A such that $R(S, A)$, by a theory T which is satisfied by a model M of mathematics.

The difference between a formal definition of concepts/objects and of theories/models is that the former can be used also under nonanalytical circumstances while the latter only makes sense as analytical. In the terminology of concepts and objects, it would be possible to discuss the nature of mathematical concepts and concepts of the "physical world" more than in terms of analyticity. Thus the concept of finitism can be said to have things in common not only with traditional mathematical concepts but also with concepts of natural science: in spite of its analytical presumptions, it states a generalisation of the conditions of a natural science. Assuming that the set of natural numbers is the world, one could reformulate the definition of finitism to *everything that we know can be deduced in a finite number of steps from what we observe*
.

So even if finitism concerns mathematics and not the physical world, one could guess that it comprises most of the mathematics that is useful for natural science: the connection between finitistic mathematics and the physical world is not *mystical* .

We will represent finitism by a theory in first order logic: PRA, outspelled primitive recursive arithmetics. In short, it is the same as PA restricted to primitive recursive functions. Its language consists of a constant symbol 0, 1-ary function symbols Z, S and k-ary function symbols P_i^k, $1 \leq i \leq k$. Axioms then are introduced to fix Z as the zero function, S as the successor function and P_i^k as the projection function $P_i^k(x_1, ..., x_k) = x_i$. Finally PRA have the primitive recursive function symbols: if a function symbol f is n-ary and $g_1, ..., g_n$ are k-ary, then $C(f, g_1, ..., g_n)$ is a k-ary function symbol. If a function symbol f is k-ary and g is $k+2$-ary, then $R(f, g)$ is a $k+1$-ary function symbol. By axioms introduced for each of these functions, they are fixed as composition functions $C(f, g_1, ..., g_n) = f(g_1, ..., g_k)$ and primitive recursive functions

$$R(f, g)(0, x_1, ..., x_k) = f(x_1, ..., x_k)$$

$$R(f, g)(S(x), x_1, ..., x_k) = g(x, R(g, f)(x_1, ..., x_k), x_1, ..., x_k)$$

respectively. Additional axioms are given by an axiom scheme which establishes induction on every quantifier-free formula in PRA.

Now the constant 1 and the functions + and · as well as the predicative symbol < can be interpreted in terms of 0 and S, and hence every sentence in PA makes sense also in PRA. The question is now when the sentences that are provable in PA, also are provable in PRA, or more generally, which sentences that are provable in different second order arithmetics, also are provable in PRA. The important result is the following:

Theorem 9.1. *A Π^0_2-sentence is provable in* WKL *if and only if it is provable in* PRA.

We will sketch a model-theoretical proof for this that is fully outlined in Simpson [3]. The most interesting implication is that provability in WKL implies provability in PRA, and hence we need to see that given ψ, not provable in PRA, there is a model for WKL that satisfies $\neg\phi$.

What we know is that there is a model M of PRA in which ϕ is false, and we now have to construct a corresponding model L for WKL. But as M only demands Σ^0_0-induction, it might be a nonstandard model with numbers too big to be reachable by the successor function. On the other hand L have to satisfy Σ^0_1-induction. To handle this matter, we will need a certain terminology:

Definition 11. Let M be a model of PRA. Given the predicative symbol R and $b,\ c_1, ..., c_k \in |M|$ a subset X of $|M|$ is *M-finite* if

$$X = a \in |M| : a <_M b \wedge R_M(a, c_1, ..., c_k)$$

Furthermore the *M-cardinality* of X is then defined as $\mathrm{card_M}(X) = \mathrm{card_M}(X, m)$ where $X \subset \{a : a <_M m\}$ and

$$\mathrm{card_M}(X, 0) = 0$$

$$\mathrm{card_M}(X, a+1) = \begin{cases} \mathrm{card_M}(X, a) + 1 & \text{if } a \in X \\ \mathrm{card_M}(X, a) & \text{otherwise} \end{cases}$$

A subset I of $|M|$ is a *cut* in M if $M(1) \subset I \neq |M|$ and $M(a < b)$, $b \in I$ implies that $c \in I$. We define $\mathrm{coded_M}(I)$ as the set of all $X \subset I$ such that there is an M-finite set X^* with $X^* \cap I = X$. Finally we say that I is *semiregular* if $X \cap I$ is bounded in I for all M-finite sets X such that $\mathrm{card_M}(X) \in I$

The relation between PRA and WKL can now be expressed by the following proposition:

Proposition 9.2. *Given a model M of* PRA*, the restriction of M to any semiregular cut I of M is a model for* WKL*, with* $\mathrm{coded_M}(I)$ *as domain for the set variables*

For the proof we need the following lemma, which is verified straightforwardly:

Lemma 9.3. *Let $\theta(x_1, ..., x_k)$ be a Σ^0_0-formula in the language of* PRA *expanded with the predicative symbol $<$. Then there is a k-ary primitive recursive function symbol f such that*

$$f(x_1, ..., x_k) = 1 \iff \theta(x_1, ..., x_k)\,(and)$$
$$f(x_1, ..., x_k) = 0 \iff \neg\theta(x_1, ..., x_k)$$

in PRA.

Proof of proposition 9.2. Given a semiregular cut I, we denote the restriction of M to I by L. The crucial thing is to see that Σ^0_1-induction holds on L. To this end, let $\phi(x)$ be a Σ^0_1-formula on L and assume that L satisfies 1) $\phi(0)$ and 2) $\forall x(\phi(x) \rightarrow \phi(x+1))$. As a contradiction, assume that there is an e in I such that N satisfies $\neg\phi(e)$. Now it only remains to prove that the set

$$Y = \{a : a <_M e \text{ and } N \text{ satisfies } \phi(a)\}$$

is M-finite, since then there is a least element $b \in I$ such that $b \notin Y$, which contradicts either 1) or 2).

Create a first-order formula ϕ^* from ϕ by replacing each set parameters X by an M-finite set X^* with $X^* \cap I = X$. Let $d \in |M| \backslash I$. Note that letting θ be such that $\phi^* = \exists y\theta(x, y)$, it satisfies the conditions in lemma 9.3. Hence by definition, the set

$$Z = \{(a, b) \in |M| : a <_M e, b <_M d,\ \theta(a, b) \text{ and } \forall c(\theta(a, c) \rightarrow b <_M c)\}$$

is M-finite. Hence, since I is semiregular, $Z \cap I$ is bounded and therefore also M-finite.

Noting that $Y = \{a : \exists b((a, b) \in Z \cap I)\}$, we conclude that Y is M-finite by using lemma 9.3 once again, now on the formula $(a, b) \in Z \cap I$. This completes the proof.

□

It now remains to find a semiregular cut I such that letting L, according to proposition 9.2, be a model of WKL that is the restriction of M to I, L satisfies $\neg\psi$. But $\psi = \forall x\theta(x)$ for some Π^0_1-formula $\theta(x)$. As PRA does not prove ψ, the theory T consisting of PRA, a constant symbol e and the theorem $\neg\theta(e)$ is consistent and satisfied by M with a good interpretation of e. The essential thing is now not to omitt e in the semiregular cut I we choose for constructing L: then obviously L satisfies $\neg\theta(e)$ and thus $\neg\psi$. To show how such an I can be found is outside the range of this paper, but the details are outlined in Simpson p. 379-381.

With theorem 9.1, we have taken one step against comprising the essense of the analytical concepts that accidentally have been useful for natural science, as well as establishing a connection between mathematics and what we say is the world. Furthermore, by the examples lined out in the middle sections we see that reverse mathematical methods produces equivalence results of well-known mathematical theorems. These results proposes a new way to study mathematics and its relevance for the understanding of the physical world.

Acknowledgement

I would like to thank my supervisor, Christian Bennet, for that he encouraged me during all the writing.

References

[1] Carl Mummert: *On the reverse mathematics of general topology*, Penn State (2005).
[2] Stephen G. Simpson: *The journal of symbolic logic, vol. 53, No. 2.*, Partial realization of Hilbert's program. New York (1988)
[3] Stephen G. Simpson: *Subsystems of second order arithmetics*, New York (1999).

Filosofiska institutionen, Göteborgs Universitet, S-412 96 GÖTEBORG, SWEDEN