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**Monotonicity restrictions
used in a system of early
warnings applied to monthly
economic data**

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When making political decisions, the ability to make correct predictions about the behaviour of the business cycle in the future is important. The ability to forecast business cycles determines the success of, for example, governmental programmes. By business cycles we generally refer to the fluctuations over time in the total economy. A variety of techniques for predicting the business cycles have been proposed. Some of the proposed techniques aim to predict the values of the business cycle. These techniques often prove to give very bad fits near turning points. Other forecasting methods are designed to predict the time of the turning points in the cycle. Many of the turning point prediction methods are based on discovery of turning points in a leading indicator. A leading indicator represents activity, which lead business cycle turning points, thus having its turning points before the general business cycle. It is often the case that the times of the turning points of the business cycle and a lagged leading index coincide remarkably well, even though the actual values of the two time series do not.

This licentiate thesis consists of two parts:

- I** On monotonicity and early warnings with applications in economics,
Research report 1999:1, Department of Statistics, Göteborg University, Sweden
- II** Monotonicity aspects on seasonal adjustment
Research report 1999:3, Department of Statistics, Göteborg University, Sweden

In report **I** a method is proposed that uses the theory of statistical surveillance and monotonicity conditions to decide if a leading indicator, X , has reached a turning point. The proposed alarm function is based on the maximum likelihood ratio. The time series X is modelled as consisting of two components, namely trend cycle and error, where the cycles in the first component are not periodic. Non-parametric regression is used to estimate the trend cycle. The only information used is the fact that the trend cycle is monotonically increasing up to a peak and thereafter monotonically decreasing. A simulation study is made in order to illustrate the proposed method. The median run length to the first false alarm and the expected delay time were considered to be relevant measures of the performance of the proposed method. The results from the simulation study indicate that when there is no turning point, the average time to the first false alarm is five years, whereas when there is a turning point the average time to the alarm is 3 months.

A possible complication in the monitoring of a leading indicator X is the fact that the observations on X will be made monthly and thus they are likely to exhibit seasonal variation. The proposed method is not designed to deal with seasonality, and therefore the observations must be adjusted for seasonality, prior to surveillance. This possible complication is considered to be solved in report **I**, but it is treated in report **II**. The seasonal variation should be eliminated only in order that the turning points should be identified. Since the proposed method of surveillance uses robust regression under monotonicity restrictions the seasonal

adjustment method must not change the monotonicity. Moving average techniques, used for seasonal adjustment, are investigated as to their monotonicity preserving properties. The results from this study indicate that when there is no turning point the moving average estimator does preserve the monotonicity. If there, however, is a turning point the moving average estimator does not preserve the time of the turning point, except for some special cases.

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On monotonicity and early warnings with applications in economics

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Abstract

In this report a method for monitoring time series with cycles is presented. It is a non-parametric approach for detecting the turning point of the cycles. Time series of business indicators often exhibit cycles that can not easily be modelled with a parametric function. Forecasting the turning points is important to economic and political decisions. One approach to forecasting the business cycles is to use a leading indicator. The method presented in this report uses statistical surveillance to detect the turning points of a leading indicator. Statistical surveillance is a methodology for detecting a change in the underlying process as soon as possible. Observations on the leading indicator are gathered once a month and the change in the process is a turning point. Only a part of a series that contains one turning point at most will be investigated. The time series is assumed to consist of two additive components: a trend cycle part and a stochastic error part. No parametric model is assumed for the trend cycle, estimation is instead made by robust regression under different monotonicity restrictions. The aim is to detect a turning point as soon as possible, not to predict the value of the time series at the turning point. Evaluation of this surveillance method is done by means of simulation. The number of false alarms and the delay time are analysed. The evaluation shows that if there is no turning point then the median time to the first false alarm is five years, whereas if there is a turning point after three years, the median time to an alarm is 3 months.

Keywords: Turning point detection, monitoring, leading indicator, non-parametric, robust regression

1. Introduction

By business cycles we generally refer to the major fluctuations over time in the total economy. Forecasting the business cycle is important, both for the players in the economic process and for economic policy. Several methodologies have been suggested that use a leading indicator to predict the turning points. A leading indicator represents activity that leads business cycle turning points for sound economic reasons.

The report is an attempt to develop a system for early warnings of turning points in a leading indicator. This can be used to predict the turning points of the business cycle. Frisé (1994) showed that the times of the turning points of the Swedish business cycle and a lagged leading index have coincided remarkably well for a period of 30 years. In economic literature a system for early warnings can also be called monitoring, while in the statistical literature it is sometimes called statistical surveillance.

It is important to distinguish between surveillance and, for example, a test of structural change. The latter case has a fixed null hypothesis and a fixed number of observations. The surveillance case, however, has no fixed null hypothesis and the number of observations is increasing. The surveillance continues, even if there has been no signal of change for a long time and the evidence for no change in the beginning is strong. Thus the null hypothesis is never accepted.

The performance of a method of surveillance is not evaluated using Type I and Type II error probabilities. Instead evaluations can be made by investigating the delay time for an alarm, the predictive value of an alarm or different utility functions (Frisé and Wessman, 1998). For research regarding the theory of statistical surveillance and different kinds of optimality, see Frisé and de Maré (1991).

Much research has been done in the area of statistical surveillance. Industrial quality control using control charts is one area where the theory of statistical surveillance is applied. Some control charts are designed to detect a sudden change in the mean or the standard deviation of the process, whereas others are designed to detect a slower change (Wetherill and Brown, 1991).

The aim of the method of surveillance presented here is to detect a turning point in the economic time series as soon as possible after occurrence. Observations are gathered and at each new additional observation there has to be a decision of whether there is enough evidence to conclude that a turning point has occurred. In a surveillance situation the number of observations are not fixed but increases at every new time point, thus improving the information about the time series. The timeliness of the detection is important. The observation of the time series at time t is denoted $X(t)$. Based on the data available at time s a discrimination is made between two states: 1) that the turning point has occurred and 2) that the turning point has not yet occurred. This report presents an alarm statistic that can be used to discriminate between these two states.

Business cycles, leading indicators and earlier suggestions of methods are discussed in Section 2. In Section 3 the suggested new method is presented. Section 4 contains the results of a simulation study regarding the properties of the method. The results and plans for continuation of research in this area are discussed in Section 5.

2. Business cycles and leading indicators

2.1 Business cycles - a review

By business cycles we generally refer to the phenomenon that years of rapid expansion are followed by a period of slower growth or even contraction. These expansions and recessions affect both the performance of business firms and more general aspects of a nation's economy. Forecasting the business cycle, for example the time of the end of an ongoing recession is important as a basis for decisions regarding economic policy. Much research has been devoted to finding a method for predicting the business cycle.

The definition of business cycles offered by Mitchell and Burns (1946) is:

Business cycles are a type of fluctuation found in the aggregate economic activity of nations that organise their work mainly in business enterprise: a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle; this sequence of changes is recurrent but not periodic; in duration business cycles vary from more than one year to ten or twelve years; they are not divisible into shorter cycles of similar character with amplitudes approximating their own.

For a general review of research on business cycles, see Oppenländer (1997a, eds).

Theories claim that business cycles can be caused by mechanisms within the economic system as well as external factors (Samuelson and Nordhaus, 1992). Most commonly accepted theories of the business cycle processes and mechanisms entail dynamic models. The so-called internal theories often focus on the internal dynamics of the economic system. Various exogenous shocks are also expected to propagate business cycles, such as war, population growth, migration and scientific discoveries (Westlund, 1993). Questions have also been raised as to whether business fluctuations in a country can be divided into two unobservable components, world business cycle and country specific business cycle (Bergman, 1992).

Sometimes it is claimed that the standard practice is to adjust the original values for seasonal influences, irregular components and trend-related processes, see Oppenländer (1997b). However, Westlund (1993) says that separating trend and cycle components is always more or less judgmental.

The most frequently used business cycle reference series are the Gross National Product and Industrial Production series (Westlund, 1993). Apart from Gross National Product the production index for the manufacturing sector gives good information about the monthly developments of economic activity (Lindlbauer, 1997). Yet another possibility is to use the seasonally adjusted capacity utilisation in manufacturing industry (Köhler, 1997).

When describing the business cycle the following characteristics are often used, namely the shape and magnitude of the turning points and the length and pattern of the upturns and downturns. As a rule a boom phase lasts longer than a recession phase. The shape of the period of economic boom could be called a crest or plateau whereas the recession phase is

characterised by a funnel shape (Oppenländer, 1997b). Neftci (1984) showed that the underlying process of economic time series in expansion periods is different from that in recession periods.

A recession is among other things characterised by decreasing investments in machinery and equipment and decreasing labour demand (Samuelson and Nordhaus, 1992).

2.2 Leading indicators - a review

A business cycle indicator is a time series that can be used as an indicator for the business cycle. For a general review, see Lahiri and Moore (1991, eds).

Moore (1961) classified business cycle indicators into three groups of leading, roughly coincident and lagging indicators. In Oppenländer (1997c) an even finer classification is made of the indicators into leading indicators (for example business expectations), tension indicators (for example change in order stocks), coincident indicators (for example change in production) and lagging indicators (for example change in number of employed).

In Köhler (1997) it is said that the objective of leading indicator research is described as isolating those macroeconomic variables that have an especially long and stable lead relative to the business cycle. The reason that some time series are leading can be explained by the fact that for example opinions are formed in business before orders are placed and sales are achieved (Oppenländer, 1997c). The usefulness of leading indicators comes from the fact that they are often released more frequently and some indicators are available with a relatively short delay (Parigi and Schlitzer, 1997).

The economic indicators are selected with respect to their performance according to a number of criteria, among others economic significance, length and consistency of the lead and freedom from excessive revisions (Westlund, 1993).

A leading indicator can be quantitative or qualitative: The leading indicators published regularly by OECD (Organisation for Economic Co-operation and Development) are for instance to 40% based on judgmental and expectation variables. The lower turning points have been shown to be more difficult to predict. Also the lead times of the indicators may differ at upper and lower turning points. Good leading indicators for the upper turning points are often judgement regarding order stocks, finished goods inventories and the present business situation. The lower turning points are often indicated by export expectations, assessment of finished goods inventories and business expectations for the next six months (Nerb, 1997). The lead is often shorter for the upper turning point than for the lower (Oppenländer, 1997c). Indicators of business conditions and business expectations could be carried out in the form of a survey to companies. It should be considered that different companies do not evaluate the same constellations of economic data in the same way. Judgements and expectations are included in their assessments (Lindlbauer, 1997).

The employment trends are considered to be lagged indicators because of social legislation and training costs. It is often more profitable to keep the staff even during periods of slacker demand even if they are under-employed. Money supply is considered a good early indicator.

Share index could be an early indicator, but due to the quickly response needed the investors often fall for rumours (Lindlbauer, 1997).

One example of an index of leading indicators is the one consisting of 11 variables that is compiled by the U.S. Department of Commerce (Niemira, 1991). For Swedish data, a method to calculate an index for short time predictions was proposed by Lyckeberg, Pramsten and Ruist (1979). Another example of an index for the activity of the Swedish economy is the one that is calculated and published in SCB Indikatorer. This activity index consists of the four series of industrial production index, the development of the production volume for the energy sector, index of sales for the retail trade and number of hours worked in the public sector. These four time series are deseasonalised and weighted together to form the index.

2.3 Some earlier suggestions of methods for the prediction of turning points in economic time series

Much research has been done in the area of forecasting the turning points of business cycles and several methodologies use a leading composite index or individual leading indicators to predict the turning points. Some of the methods suggested are mentioned below.

A sequential signal system was proposed by Zarnowitz and Moore (1982). The motivation is that a decline in the leading index rate is an early sign of an ongoing expansion that is starting to decelerate. A sustained decline of the growth rate in the leading index puts it below the average 3.3% line. If the leading index rate then falls below zero and the coincident index rate falls below 3.3% the probability of recession is heightened. Finally if the coincident index rate follows the leading index rate by turning negative chances are high that the slowdown is being succeeded by an actual decline in overall economic activity, that is, a recession. This method was applied on Swedish data by Frisén (1993) for the period between January 1960 to April 1987. For this set of data the proposed method did not give clear-cut results.

Chaffin and Talley (1989) proposes a test of diffusion indexes that can be used to predict the business cycles. A diffusion index may be defined as the number of series in a group that are rising, expressed as a fraction of the total number of series in this group. The equality of the diffusion index at time $t-L$ and the diffusion index at time $t-L-r$ is tested using the McNemar test. As an evaluation the method was applied to monthly data for thirty leading indicators for a period of fourteen years.

Keen, as cited in Silver (1991), observed that the last eleven recessions were preceded by two months of negative and decelerating growth in the composite leading index. He suggested a signalling system based “the rule of two months of negative and decelerating growth in the composite leading index”.

One sequential analysis method was suggested by Neftci (1982), where the probability distribution of a leading indicator, X , can be either F^0 or F^1 , depending on the regime (normal or downturn). The turning points are characterised by sudden switches in the distribution of X . A prior probability, concerning when downturns are expected to occur, has been developed from observing past cyclical downturns. The posterior distribution is used as an alarm statistic. This method was evaluated by an example.

Jun and Joo (1993) proposed a method for predicting turning points by detecting slope changes in a leading composite index, Z . The Z consists of a random level, L , and a white noise process, a . The level, L , is modelled $L(t+1) = L(t) + T(t) + b(t+1)$, where b is a white noise process and the slope, T , follows a random walk process except at the turning point. At an unknown time a random slope change is represented by a random shock which will cause either a peak or a trough. A Bayesian approach is used so that if the posterior probability of the alternative hypothesis exceeds a limit, it is concluded that the leading index has passed through its turning point. Data on US leading and coincident composite index was used to evaluate the method.

Lahiri (1997) refers to Hamilton's (1989, 1993) non-linear filter. This model postulates a data generating process with two different regimes - expansion and recession, respectively. The process is subject to discrete shifts by a two-state Markov process. The posterior distribution is used as an alarm statistic. In his article Lahiri (1997) applies Hamilton's filter to a time-series of interest rate spreads, that are the differences on a given date between interest rates on alternative financial assets. These interest rate spreads has been shown to be good predictors for the future economic activity.

Koskinen and Öller (1998) proposed using a hidden Markov regime-switching model as a Markov-Bayesian classifier. The data generating process has two hidden classes and the regime posterior probabilities are used to signal a turning point. Parameter estimation and model selection are based on a probability score to minimise the turning point forecast error. The method was applied to Swedish and US data.

2.4 Model discussion

Much research has been done in the area of modelling and forecasting the business cycles. In Makridakis and Wheelwright (1979) econometric forecasting models are divided into single equation regression models, multi-equation models and time series models. The single equation regression model is one where the variable of interest is related to a single function of explanatory variables and an implicit additive error term. In the multi-equation models, the interrelationships among a set of variables are simultaneously accounted for. This is a more involved process than simply constructing and combining a set of individual regression equations. Among the models defined as time series models in this book, the most basic linear model is the ARMA model.

In Christ (1996) econometric models are defined as systems of equations intended to determine a vector of endogenous variables in terms of a vector of exogenous variables, vectors of lagged endogenous variables, a matrix of parameters and a vector of stochastic disturbances.

For the methods reviewed in Section 2.3 the use of models differ. The methods proposed by Zarnowitz and Moore (1982), Chaffin and Talley (1989) and Keen (s. s. Silver, 1991) do not demand any model for the data. The others use different parametric models for the behaviour before and after the turning point.

One aspect of the time series, for which the method of surveillance presented in this report is intended, is that they are observed once a month and hence likely to contain seasonal

variation. For the methods reviewed in Section 2.3 the aspect of seasonality is only mentioned by Jun and Joo (1993). They assume the series to be free of seasonal variation. The method of surveillance presented in this report is based on the assumption that the turning points of the process under surveillance are unobservable because of random fluctuations, not because of both random and seasonal fluctuations. Therefore, in order to use the method proposed here, data must be seasonally adjusted prior to the surveillance. The topic of seasonal adjustment using moving average techniques has been treated by Andersson (1998). Here however the problem is considered solved and the time series is assumed to be free of seasonal variation.

3. A non-parametric method for surveillance of cycles

Many methods have been suggested for detecting changes in a time series process. Some of the methods used, for example in the results by Garbade (1977), concern a single decision and a fixed set of observations. This approach will not be considered in this report. When monitoring a time series in order to detect a change it is important to consider that the inferential situation is one of repeated decisions: we need to make a new decision with the entrance of each new observation. The methods in 2.3 are all designed for sequential decisions, but they are based on different methodology. The decision rules used by Zarnowitz and Moore (1982), Chaffin and Talley (1989) and Keen (s. s. Silver 1991) are based on rather ad-hoc assumptions, whereas Neftci (1982), Jun and Joo (1993) and Hamilton (1989) proposes to use the posterior distribution as an alarm function. This report proposes an alarm function that is developed using the theory of statistical surveillance. It was shown by Frisé and de Maré (1991) that the use of the posterior distribution is equivalent to the alarm statistic of the likelihood ratio method when there are only two states. This is the likelihood ratio method that will be applied below. Many sequential methods are based on assumptions of a parametric model where sometimes the estimation problem is not taken seriously. This report proposes a non-parametric estimation procedure that is based only on assumptions of monotonicity. Nearly all the method mentioned in Section 2.3 are evaluated by one or two examples. The method presented in this report will be evaluated by Monte Carlo techniques, using 16 000 replicates. An additional aspect that is important in the evaluation of sequential methods is using relevant measures of performance, for example the time to a false alarm and the delay time. The method proposed here will be evaluated by both these measures.

3.1 Model

This is a first investigation of how to use the theory of statistical surveillance and monotonicity conditions for turning point detection of a leading indicator. The ambition of the method presented here is not to predict the actual value of the time series at the turning point. Instead the method will be used to detect a turning point or, to be more specific, to decide whether the turning point has occurred or not after each new observation. Thus a very simple model for a leading indicator X is used. Only a part of X that contains one turning point at most will be investigated. The set of all data available at time s is denoted $X_s = \{X(t): t = 1, 2, \dots, s\}$. The time series X consists of two components: the trend cycle and the error term. Thus no decomposition of trend and cycle will be made. The trend cycle is not modelled as a stochastic process. If no other prior information is given this approach is often associated with modelling the trend cycle component as a polynomial function. This is not the case here. The

trend cycle function is unknown, apart from the important aspect of monotonicity and unimodality, which is not an assumption but follows from the definition of a turning point.

The model used in this report for an observation of the time series at time t is

$$X(t) = \mu(t) + \varepsilon(t) \quad (3.1)$$

where $\mu(t) \in \mathcal{F}$, \mathcal{F} is the family of all unimodal functions and $\varepsilon(t)$ are iid $N(0; \sigma^2)$. Without loss of generality $\sigma^2 = 1$ is used.

As mentioned in Section 1, at time s a discrimination is made between the two states $C(s) = \{\tau \leq s\}$ and $D(s) = \{\tau > s\}$, where τ is the unknown time of the turning point. The case treated in this report is when the aim is to detect the next peak. The opposite case is solved correspondingly. The first possible time for decision is set to $t = 1$ (see Section 3.1.1). Thus

$$\mu(t-1) \leq \mu(t) \text{ for } t < \tau \text{ and} \quad (3.2)$$

$$\mu(t-1) \geq \mu(t) \text{ for } t \geq \tau$$

where at least one inequality is strict in the second part.

The frame work is similar to that of Neftci (1982) and others. The distribution of $X(t)$ is assumed to differ, depending on the state (the turning point having occurred or not). Once a month a new observation is made on $X(t)$, where

$$X(t) \sim \begin{cases} f\{x(t)|C\} & \text{if } 1 \leq \tau \leq t \\ f\{x(t)|D\} & \text{if } \tau > t. \end{cases} \quad (3.3)$$

3.1.1 The relation between the start of surveillance and the turning point

Surveillance is based on repeated decisions. At each new additional observation, a decision has to be made of whether the change has yet occurred. The system of surveillance presented here starts at the latest confirmed turning point (here a trough). In a realistic situation the confirmation is not likely to come directly after the occurrence of the turning point, thus producing a delay. However, this delay is not considered in this report. The following notation of the first time that a decision can be made is used.

Denote the times after the latest confirmed turning point as $t = 0, t = 1$, etc, and denote the observations $x(0), x(1)$, etc. The first decision of whether a turning point has occurred, can be made at time $t = 1$.

An illustration of the trend cycle near a confirmed trough is found below.

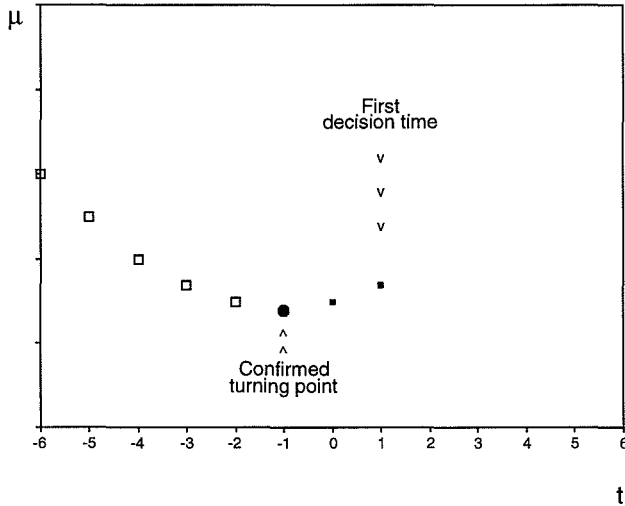


Fig. 1. The figure shows the values of the (unobservable) trend cycle, μ , near a confirmed trough. The first decision of whether a peak has occurred, can be made at time $t = 1$.

Another important definition that is somewhat special to this application is the definition of the time of change, τ . The change treated in this report is the event when the trend cycle starts to decline after an expansion.

If the latest confirmed turning point is a trough and observations are made in order to detect the next peak, then τ is defined as the first time the trend cycle is decreasing. By the notation above, $\tau \geq 1$.

Examples of turning points at time 1 and 2 are shown in Figure 2.

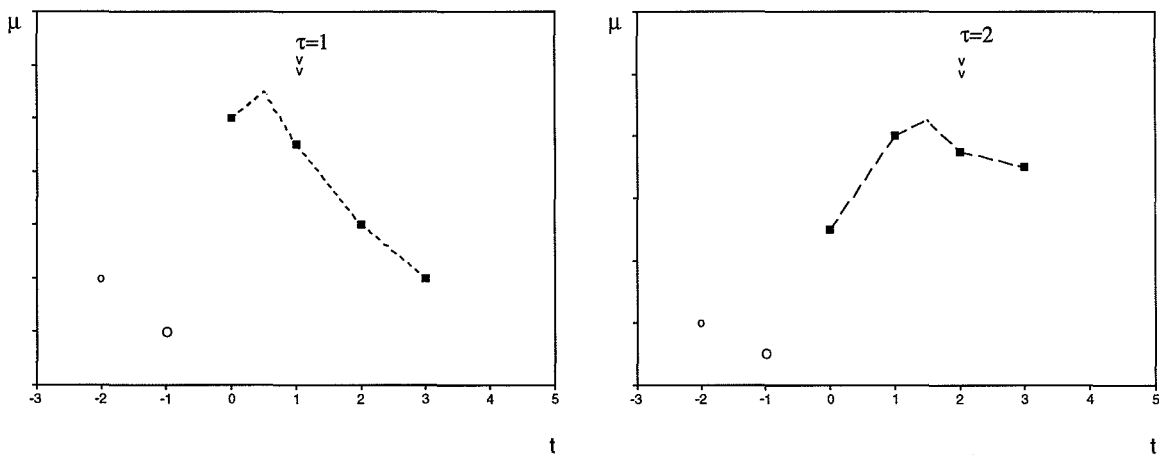


Fig. 2. (a) Example where $\tau = 1$; (b) Example where $\tau = 2$.

3.1.2 Estimation of the trend cycle

In this report a non-parametric approach for turning point detection is suggested. No parametric model is needed to estimate the trend cycle component in (3.1). This component is estimated using only the knowledge that the monotonicity of the trend cycle is changed at a

turning point. There are several advantages with this approach. One advantage is that since the cycles are likely to be asymmetric and irregular and may contain plateaux, the task of finding a suitable function to fit such data is difficult. In this approach therefore the trend cycle is estimated using robust regression. Only the restriction implicit in the definition of τ in Section 3.1.1 is used.

Thus the estimate of the trend cycle vector between a trough and a successive peak is

$$\hat{\mu}^D : \max_{\mu \in \mathfrak{S}} f(x_s | \mu), \quad (3.4)$$

where \mathfrak{S} is the family of μ such that $\mu(t) \leq \mu(t+1)$, $t \geq 1$

The estimate of the trend cycle vector, if there is a peak at time τ , is

$$\hat{\mu}^{C\tau} : \max_{\mu \in \mathfrak{K}\tau} f(x_s | \mu), \quad (3.5)$$

where $\mathfrak{K}\tau$ is the family of μ such that $\mu(\tau-k) \leq \mu(\tau-k+1) \leq \dots \leq \mu(\tau-1)$
and

$$\mu(\tau-1) \geq \mu(\tau) \geq \dots \geq \mu(\tau+m),$$

where at least one inequality is strict in the second part and $k, m > 1$

The trend cycle vector is estimated using a least square criterion under these monotonicity restrictions (Frisén, 1986 and Robertson, Wright, Dykstra, 1988). Under the conditions used in this report these estimates are also the maximum likelihood estimates.

3.2 Surveillance

As mentioned in Section 3.1 the observations available for the decision at time s are the vector x_s . The goal of the surveillance is to detect a turning point in the business cycle as soon as possible after occurrence. An alarm set, $A(s)$, is constructed with the property that when X_s belongs to $A(s)$ it is an indication that $C(s)$ occurs. Optimal methods to discriminate between C and D are based on the likelihood ratios under different conditions (Frisén and de Maré, 1991). A possible way to adapt the likelihood ratio method to the case of turning points was indicated by Frisén (1994).

3.2.1 The alarm statistic

The vectors μ^D and $\mu^{C\tau}$ are unknown and thus the optimal likelihood ratio can not be used as an alarm function. Instead the maximum likelihood ratio statistic will be used. The properties of the maximum likelihood ratio statistic when used as an alarm function are not known. In this report some of its properties will be investigated. Thus, the alarm set consists of those X_s for which the ratio of the maximum likelihood functions exceeds a limit k_s , i.e.

$$\text{MLR}(s) = \frac{\max f\{x_s | C\}}{\max f\{x_s | D\}} > k_s \quad (3.6)$$

The estimates of the trend cycle vector are the maximum likelihood estimates shown in 3.4 and 3.5. Thus, for the case of independently normally distributed variables with standard deviation one and $\pi_j = P(\tau = j)$, the alarm statistic in (3.6) can be written as

$$\begin{aligned}
\text{MLR}(s) &= \frac{\max f\{x_s|C\}}{\max f\{x_s|D\}} = & (3.7) \\
&= \sum_{j=1}^s \frac{\pi_j}{P(\tau \leq s)} \frac{f\{x_s; \hat{\mu}^{Cj}\}}{f\{x_s; \hat{\mu}^D\}} = \\
&= \sum_{j=1}^s \frac{\pi_j}{P(\tau \leq s)} \frac{\left[\prod_{u=1}^s f\{x(u); \hat{\mu}^{Cj}\} \right]}{\left[\prod_{u=1}^s f\{x(u); \hat{\mu}^D\} \right]} = \\
&= \sum_{j=1}^s \frac{\pi_j}{P(\tau \leq s)} \frac{\left[\prod_{u=1}^s \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x(u) - \hat{\mu}^{Cj})^2}{2}\right\} \right]}{\left[\prod_{u=1}^s \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x(u) - \hat{\mu}^D)^2}{2}\right\} \right]} = \\
&= \sum_{j=1}^s \frac{\pi_j}{P(\tau \leq s)} \frac{\left[\exp\left\{-\sum_{u=1}^s \frac{(x(u) - \hat{\mu}^{Cj})^2}{2}\right\} \right]}{\left[\exp\left\{-\sum_{u=1}^s \frac{(x(u) - \hat{\mu}^D)^2}{2}\right\} \right]} = \\
&= \sum_{j=1}^s \frac{\pi_j}{P(\tau \leq s)} \exp\left\{\left(-\frac{1}{2}\right)\left(\sum_{u=1}^s (x(u) - \hat{\mu}^{Cj})^2 - \sum_{u=1}^s (x(u) - \hat{\mu}^D)^2\right)\right\} = \\
&= \frac{1}{P(\tau \leq s)} \sum_{j=1}^s \pi_j \exp\left\{\frac{1}{2}Q^D - \frac{1}{2}Q^{Cj}\right\}
\end{aligned}$$

where

Q^{Cj} = quadratic deviation from the best model with turning point at $t = j$

Q^D = quadratic deviation from the best model with no turning point.

In the alarm function specified in (3.7) the distribution of the change point, τ , is included. If the distribution of τ is known it should be used. However, for the application in this report the distribution of τ is not known and thus a robust approximation of the alarm function in

3.7 is used. It was shown by Frisé and Wessman (1998) that the method of surveillance called Shiryaev-Roberts method can be used as a good approximation of the likelihood ratio method. The Shiryaev-Roberts method uses equal weights for all components and the limit k_s is constant. In this report the Shiryaev-Roberts approach will be used as an approximation to the alarm function (3.7). Thus the alarm function is given by

$$\text{MSR}(s) = \sum_{j=1}^s \exp\left(\frac{1}{2}Q^D - \frac{1}{2}Q^{Cj}\right) \quad (3.8)$$

The time of the alarm, t_A , is defined as

$$t_A = \min[t: \text{MSR}(t) > k] \quad (3.9)$$

where the alarm limit, k , is a constant.

3.2.2 The alarm limit

The alarm limit, k , is determined so that the median run length to the first false alarm is known and fixed. Which value of the median run length to the first false alarm that is suitable depends on the application. The MSR method of surveillance presented in this report is developed to decide if a leading indicator has reached a turning point. The time from one turning point to the next among Swedish business cycles is rarely not longer than 4 years. In this illustration the median time to the first false alarm, named MedRL^0 , is set to a little over 5 years. The reason for not using the median run length of exactly 5 years is that the simulations are very time consuming and calibrating the method to give a MedRL^0 equal to 60 months exactly would take a disproportionately long time. This is merely an illustration of the MSR method and the important thing has been to use a value for the MedRL^0 that is not considered too unrealistic. For a specific application great care must of course be taken in setting the correct MedRL^0 . The implication of setting MedRL^0 equal to 62 months is that if there is no turning point the median time to the first false alarm is a little over 5 years.

$$\begin{aligned} &\text{Let RL represent the run length.} \\ &\text{The alarm limit } k \text{ is the value} \\ &\text{for which } P[(RL > 62 \mid \tau = \infty)] = 1/2, \text{ i.e. } \text{MedRL}^0 = 62 \end{aligned} \quad (3.10)$$

Using this criterion the alarm limit, k , can be determined.

$$RL = t_A = \min[t: \text{MSR}(t) > k]. \quad (3.11)$$

In order to evaluate the MSR method a simulation study is made.

4. Evaluation

4.1 Specifications

Observations on the process $X(t)$, described in Section 3.1, are simulated using a fixed MedRL^0 . The results from the simulation will be used to evaluate some of the properties of the MSR method for the situation described below. In order for the representation of the trend cycle to be realistic, the trend cycle structure used in the simulation is based on a part of a real Swedish time series (Fig 4). The false alarm probabilities are examined for the case of constant growth (Section 4.2.1.) and for the case of the observed growth (Section 4.2.2.). The expected delay of a justified alarm is investigated for the case of a peak after three years (Section 4.3).

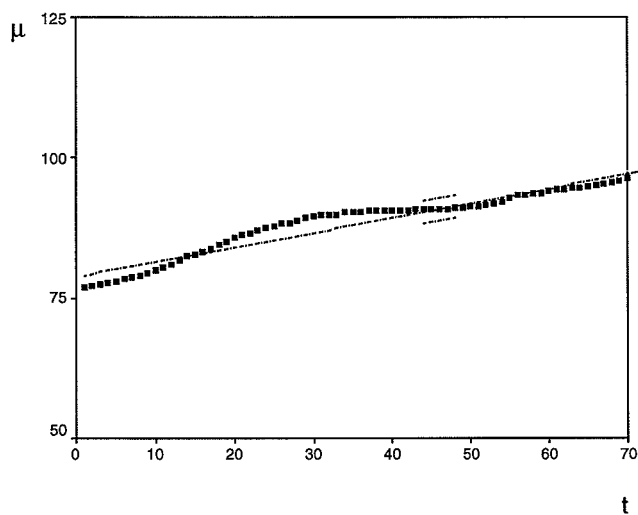


Fig. 4. Part of a real Swedish time series (\dots) is used as the basis for the trend cycle structure. The case of constant growth is indicated by --- . The vertical axis has been cut and the 2.5 and 97.5 percentiles are indicated at $t = 45$.

Simulations are made only for the case when the surveillance starts at a trough and the aim is to detect the next peak. However, the result applies for the opposite situation too.

4.2 Run length distribution for first false alarm

4.2.1. A monotonically increasing trend cycle function with constant absolute growth

The alarm limit, k , is the limit for which the median run length, MedRL^0 , is set to 62, i.e. the limit for which the probability $P[(RL > 62 | \tau = \infty)]$ equals 0.50. The alarm limit is determined by simulations. In order that a 95% confidence interval for the estimated probability $P[(RL > 62 | \tau = \infty)]$ should be of maximum length 0.016, the number of replicates used is 16 000. The distribution of the run length is presented below for the case when $\mu(t)$ is monotonically increasing and has a constant absolute growth (Fig 4).

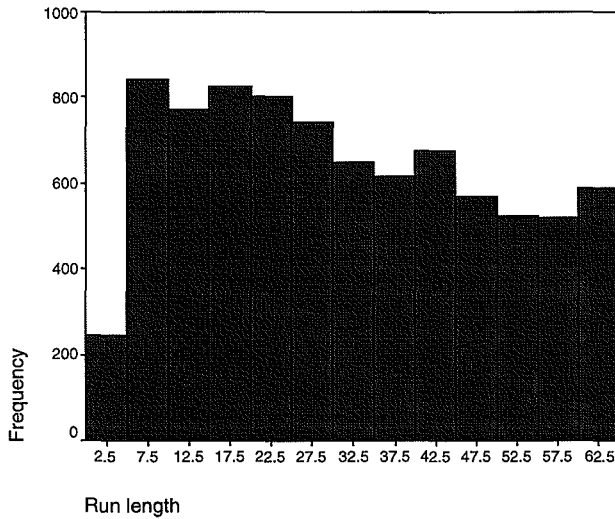


Fig. 5. The run length distribution for the case when $\tau = \infty$ and the trend cycle has a constant growth. The horizontal axis has been cut.

The probability of a false alarm no later than at time t from the start is denoted by α_t^{MSR} for the MSR method. The maximum standard error of $\hat{\alpha}_t^{\text{MSR}}$, retrieved when $\hat{\alpha}_t^{\text{MSR}} = 0.50$, is 0.004. Thus, the random error of the estimation is negligible. For the Shewhart method of surveillance (Frisén, 1992) the corresponding $\alpha_t^{\text{S}} = 1 - (2\phi(g) - 1)^t$, where ϕ is the normal probability distribution function. The α_t functions from the Shewhart method and the MSR method are compared in Figure 6 (see also Appendix A).

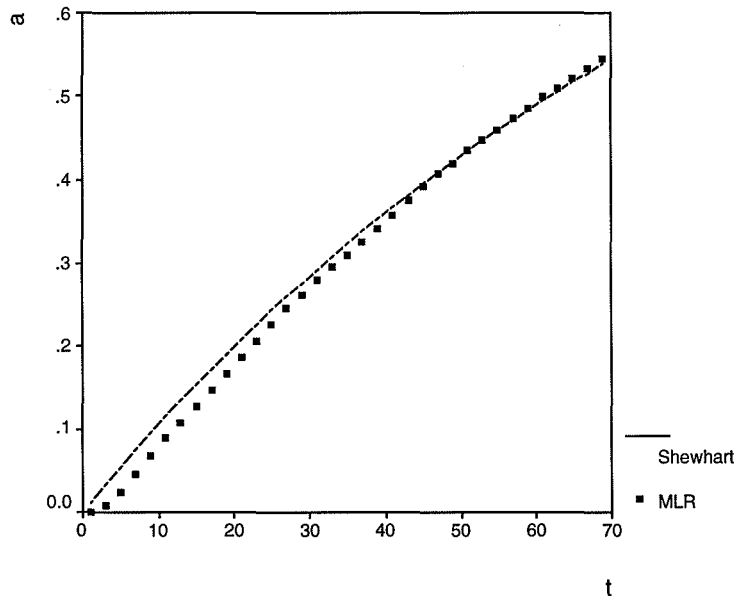


Fig. 6. The probability of a false alarm no later than at time t is shown, for the Shewhart method and for the MSR method.

The probability of a false alarm can also be presented for each time t , $P(t_A = t)$. Some methods of surveillance have a relatively high probability of a false alarm at the start of the surveillance. For the Shewhart method with $\text{MedRL}^0 = 62$ the probability of a false alarm at $t = 1$ equals $P^S(t_A = 1) = 0.011$. For the situation described in Section 4.1 this probability is considerably lower for the MSR method. The corresponding false alarm probability is $P^{\text{MSR}}(t_A = 1) = 0.00005$ (see Appendix B).

4.2.2. A special case: A monotonically increasing trend cycle function with plateaux

The results presented in Section 4.2.1 are based on the trend cycle being represented by a constant absolute growth (Fig 4). An interesting topic for future research is to investigate how robust the MSR method is with regard to a deviation from an absolute constant growth. As an illustration, preliminary results regarding the robustness in question are presented in this Section. These preliminary results are based on 5000 replicates. A vector that is monotonic, but does not have constant absolute growth, see Fig 4, represents the trend cycle. The alarm limit used is the same as in Section 4.2.1. One interesting result from this preliminary investigation is that the run length distribution will show peaks. This is due to the fact that the trend cycle function in Fig 4 does not have a constant absolute growth. The alarm statistic assumes smaller values for the case when $x(t) < x(t+1)$ than for the case when $x(t) = x(t+1)$ (see Appendix C). The situation when $x(t) = x(t+1)$ is a special case of a monotonically increasing function.

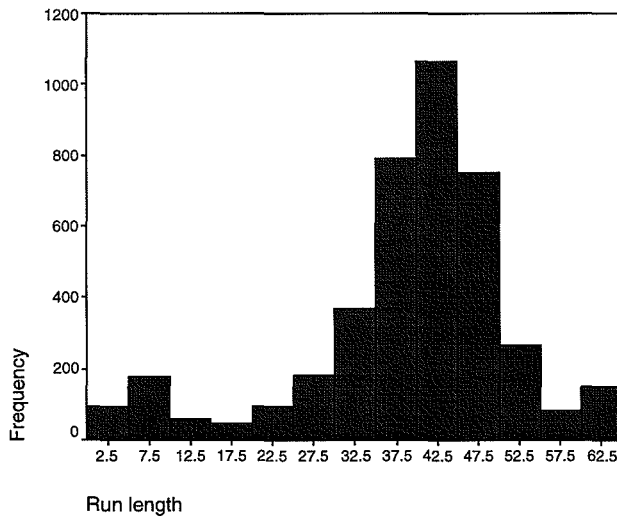


Fig. 7. The run length distribution for the case when $\tau = \infty$ and the trend cycle is a monotonically increasing function, but not with constant growth. The horizontal axis has been cut.

The estimated probability of a false alarm no later than at time t from the start is shown in Figure 8. The maximum standard error of $\hat{\alpha}_t^{\text{MSR}}$, retrieved when $\hat{\alpha}_t^{\text{MSR}} = 0.50$, is 0.007.

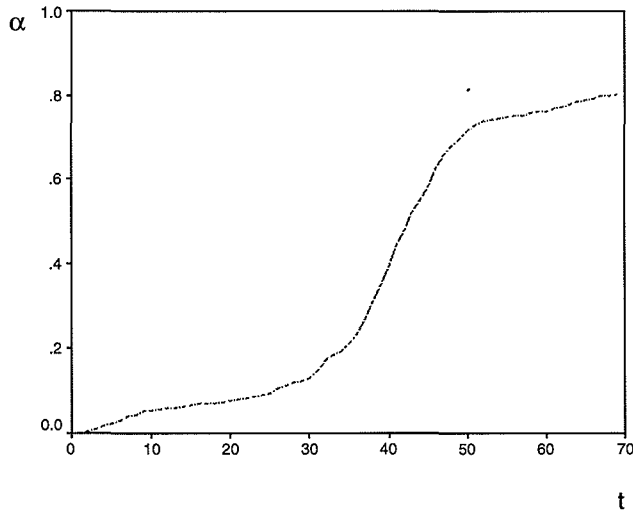


Fig. 8. The probability of a false alarm no later than at time t is shown for the case when the trend cycle contains plateaux.

For the illustration in Section, where the trend cycle is represented by a constant absolute growth, the median time to the first false alarm is 62 months. For a case when the trend cycle is represented by a vector that is monotonic but not constantly growing, illustrated by the observations in Fig 4, the estimated median run length is considerably shorter, 43 months.

4.3. Expected delay

An important aspect of a surveillance system used to detect the turning points of an economic time series is that the delay, i.e. $t_A - \tau$, is fairly short. The median delay time to a justified alarm is simulated for the MSR method, for the case of a peak 36 months after the start of the surveillance. The specifications are the same as in Section 4.1, with the exception of the trend cycle structure. The structure used for these simulations is shown in Figure 9.

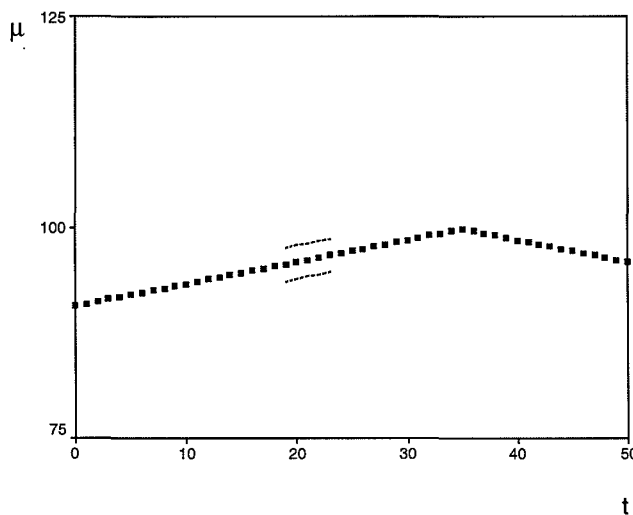


Fig. 9. The structure of the trend cycle function used for simulating the case of $\tau = 36$. The vertical axis has been cut and the 2.5 and 97.5 percentiles are indicated at $t = 20$.

4.3.1 Run length distribution for the first alarm in the case of a peak after three years

The probability of an alarm no later than at time t from the start is determined for the case when $\tau = 36$. For the MSR method, denote this probability by ${}_{36}\gamma_t^{\text{MSR}} = P^{\text{MSR}}(t_A \leq t \mid \tau = 36)$. In order that the length of a 95% confidence interval for the estimated probability ${}_{36}\gamma_t^{\text{MSR}}$ should be at most 0.016, the number of replicates used are 16 000. Since 0.016 is the maximum length of interval, retrieved when ${}_{36}\gamma_t^{\text{MSR}} = 0.50$, the random error is negligible.

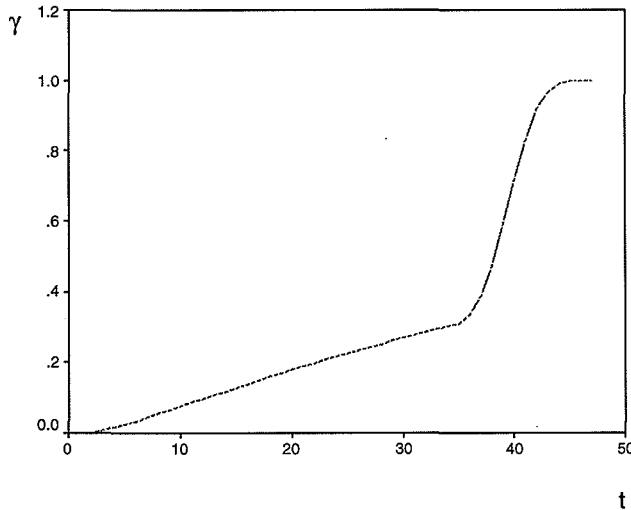


Fig. 10. The probability of an alarm no later than at time t is shown for the case when the trend cycle has a peak at time $t = 36$.

From the results in Section 4.2.1 and 4.3 it can be concluded that if there is no turning point in the trend cycle then the median time to an alarm is 62 months. If there however is a turning point after three years the median time to an alarm is 3 months.

5. Discussion

A method for detecting turning points in a time series exhibiting cycles has been proposed. This method combines the field of surveillance and unimodal regression. In this study the proposed method was illustrated by one example. The results indicate that the maximum likelihood ratio can be used for surveillance of cycles to detect the turning points.

In the sections of the time series where the trend cycle is monotonic, the median time to a (false) alarm is slightly longer than 5 years. If however there is a turning point in the trend cycle after three years, the median time to an alarm is 3 months.

An approximation of the maximum likelihood ratio, using constant weights, is used as alarm function. In a more sophisticated model, a probability distribution for τ can be used.

In this study a simple model is used, in which it is assumed that the observations are independent of each other. This can be questioned, but this study is made as a first illustration of how to use surveillance to detect changes in monotonicity. The next step is to evaluate and modify this proposed MSR method when the time series under surveillance is an auto-

regressive process. Surveillance of an autoregressive process for the purpose of detecting a change in the mean has been investigated by Pettersson (1998).

The application described in this study is that one leading indicator or one function of several leading indicators is monitored. The ability to predict the general business cycle is likely to increase if separate information on several leading indicators is available. In this situation a system for multivariate surveillance can be used. Results from the investigations made by Wessman (1998, 1999) concerning surveillance of multivariate processes might be applied.

Acknowledgement

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Appendix A. The cumulative probability of a false alarm for the Shewhart method

For the Shewhart method $ARL = 1/p$, where $p = P(|x_t| > g)$ (Frisén, 1992).

Also $\alpha_t = 1 - (2\phi(g) - 1)^t$, where ϕ is the normal probability distribution function.

Setting $MedRL^0 = 62 \Rightarrow \alpha_{62} = 1 - (2\phi(g) - 1)^{62} = 0.50 \Rightarrow \alpha_t = 1 - (0.98888)^t$.

Appendix B. The probability of a false alarm at time 1, $P(t_A = 1 \mid D)$

The probability of a false alarm at time $t=1$ equals $P(t_A = 1 \mid D) = P(\text{MSR}(1) > 68)$

At $t = 1$ we have observed $x(0)$ and $x(1)$. They can be related in three different ways:

i) $x(0) < x(1)$

Result: $\hat{\mu}^D = \hat{\mu}^{\text{CI}} = \{x(0), x(1)\} \Rightarrow \text{MSR}(1) = 1$

ii) $x(0) = x(1)$

Result: $\hat{\mu}^D = \hat{\mu}^{\text{CI}} = \{x(0), x(1)\} \Rightarrow \text{MSR}(1) = 1$

iii) $x(0) > x(1)$

Result: $\hat{\mu}^{\text{CI}} = \{x(0), x(1)\}$, $\hat{\mu}^D = \left\{ \frac{x(0) + x(1)}{2}, \frac{x(0) + x(1)}{2} \right\}$

$$\text{MSR}(1) = \exp\left(\frac{(x(1) - \hat{\mu}^D(1))^2}{2}\right)$$

There will be an alarm if $\exp\left(\frac{(x(1) - \hat{\mu}^D)^2}{2}\right) > 68$.

That is, there will be an alarm if $\Rightarrow |x(1) - \hat{\mu}^D| > 2.905$

The distribution of X , conditional on D , is $X \mid D \in N(\mu^D; 1)$

$$P(|x(1) - \hat{\mu}^D| > 2.905) = P\left(\left|\frac{x(1) - x(0)}{2}\right| > 2.905\right)$$

$$X(t) \mid D \in N(79.052 + 0.260129t; 1)$$

$$P\left(\left|\frac{x(1) - x(0)}{2}\right| > 2.905\right) = P(Z > 3.924) + P(Z < -4.292) \approx 0.0000524$$

Appendix C. The alarm function at monotonically increasing sections versus at strictly monotonically increasing sections.

The alarm statistic assumes smaller values for the case when $x(t) < x(t+1)$ than for the case when $x(t) = x(t+1)$. The alarm statistic for these two cases is investigated for different numbers of observations.

$$s = 1 \ (n = 2)$$

$$x_1 = \{x(0), x(1)\}$$

Case 1; $x(0) < x(1)$
 $\hat{\mu}^D = \{x(0), x(1)\}$
 $\hat{\mu}^{C1} = \{x(0), x(1)\}$
 $MSR(1) = 1$

Case 2; $x(0) = x(1)$
 $\hat{\mu}^D = \{x(0), x(1)\}$
 $\hat{\mu}^{C1} = \{x(0), x(1)\}$
 $MSR(1) = 1$

$$s = 2 \ (n = 3)$$

$$x_2 = \{x(0), x(1), x(2)\}$$

Case 1; $x(0) < x(1) < x(2)$
 $\hat{\mu}^D = \{x(0), x(1), x(2)\}$
 $\hat{\mu}^{C1} = \{x(0), ave_2^1, ave_2^1\}$
 $\hat{\mu}^{C2} = \{x(0), x(1), x(2)\}$
 $MSR(2) = \frac{\exp(-Q^{C1} / 2)}{1} + 1 = a + 1, \text{ where } a < 1$

Case 2; $x(0) = x(1)$
 $\hat{\mu}^D = \{x(0), x(1), x(2)\}$
 $\hat{\mu}^{C1} = \{x(0), x(1), x(2)\}$
 $\hat{\mu}^{C2} = \{x(0), x(1), x(2)\}$
 $MSR(2) = 1 + 1 = 2$

$$s = j \ (n = j+1)$$

$$x_j = \{x(0), x(1), \dots, x(j)\}$$

Case 1;

$$x(0) < x(1) < x(2) < \dots < x(j)$$

$$\hat{\mu}^D = \{x(0), x(1), x(2), \dots, x(j)\}$$

$$\hat{\mu}^{C1} = \{x(0), ave_j^1, ave_j^1, \dots, ave_j^1\}$$

$$\hat{\mu}^{C2} = \{x(0), x(1), ave_j^2, \dots, ave_j^2\}$$

...

$$\hat{\mu}^{Cj} = \{x(0), x(1), x(2), \dots, x(j)\}$$

$$MSR(j) = \frac{\exp(-Q^{C1}/2)}{1} + \frac{\exp(-Q^{C2}/2)}{1} + \dots + 1 = \sum_{i=1}^{j-1} a_i + 1, \text{ where } a_i < 1, \forall i$$

Case 2;

$$x(0) = x(1) = \dots = x(j)$$

$$\hat{\mu}^D = \{x(0), x(1), x(2), \dots, x(j)\}$$

$$\hat{\mu}^{C1} = \{x(0), x(1), x(2), \dots, x(j)\}$$

$$\hat{\mu}^{C2} = \{x(0), x(1), x(2), \dots, x(j)\}$$

...

$$\hat{\mu}^{Cj} = \{x(0), x(1), x(2), \dots, x(j)\}$$

$$MSR(j) = \sum_{i=1}^j 1$$

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Monotonicity aspects on seasonal adjustment

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Abstract

Monotonicity is an important property in time series analysis. It is often of interest to know if the seasonal adjustment method used has altered the monotonicity or changed the time of turning points in a time series that exhibits cycles. The issue of whether the monotonicity of the trend cycle component of the original non-stationary time series is preserved after the series has been adjusted is treated in this report. The time of a turning point is defined as the time when the cycle changes from recession to expansion (or vice versa). In this report seasonal adjustment with moving average methods is analysed from monotonicity aspects. The time series is assumed to consist of three additive components: a trend cycle part, a seasonal part and a stochastic error part. No parametric model is assumed for the trend cycle. The behaviour of the adjusted series is analysed for two cases: a monotonically increasing trend cycle and a trend cycle with a peak. If the trend cycle is monotonic within the entire observed section the monotonicity is preserved. Unimodality is preserved but not always the time of the turning point.

Keywords: Seasonal adjustment, monotonicity, turning point, moving average

1. Introduction

When non-stationary time series are analysed, one important aspect is the monotonicity of the series. The time series examined in this report are non-stationary economic time series exhibiting cycles. The observation on the time series at time t is denoted $Y(t)$, where Y consists of three additive components, namely the component of long term change (hereafter trend cycle), the seasonal component and the error term. The cycles in the trend cycle component are not periodic. Monthly observations are made on Y and some of the variation is due to seasonality. The series is adjusted for seasonality in order to make the turning points of the trend cycle more easily identified. Moving average methods, used for seasonal adjustment, are analysed with regard to their monotonicity preserving properties. The question of whether the seasonally adjusted series preserves the time of the turning points in the cycles is treated. The timeliness of the estimates is also discussed. This is of importance in many cases, especially in order to obtain warnings.

In Section 2 the model for the time series is specified. Different suggestions for seasonal adjustment are discussed in Section 3. Two moving average estimators are analysed with regard to their ability to preserve the monotonicity in Section 4. The conclusions from this study are presented in Section 5.

2. Specifications

Seasonal variation in a time series is not often defined rigorously, but Wallis (1974) gives examples of some more explicit statements. A common description is that seasonal variation is fluctuations that are periodical with a period of one year. The series under observation, Y , contains a trend cycle and thus Y is not a stationary series. The seasonal adjustment is made in order to distinguish the trend cycle. It is sometimes proposed that a polynomial function should be used to model a deterministic trend. In this report the trend cycle is not restricted to any parametric function. The trend cycle function is unknown, apart from the important aspect of monotonicity and unimodality, which is not an assumption, but follows from the definition of a turning point. The errors are modelled as white noise. The case when the error term is modelled as an ARMA process will not be studied in this report. In this report the case when the accessible data is for a short or moderate time period. Only a part of a series that contains one turning point at most will be investigated.

The model used in this report for an observation of the time series at time t is

$$Y(t) = \mu(t) + S(t) + \varepsilon(t) \quad (2.1)$$

$\mu(t)$ is the trend cycle component,
 $\mu(t) \in \mathcal{F}$, \mathcal{F} is the family of all unimodal functions,
 $S(t)$ is the seasonal component
and $\varepsilon(t)$ are iid $N(0; \sigma^2)$.

The seasonal component in (2.1) is not modelled as a stochastic process. The case of a seasonal cycle of 12 time periods is studied. The seasonal component and $\mu(t)$ are defined by

$$\sum_{k=1}^{12} S(k+h) = 0, \quad h = 0, 1, 2, \dots \quad (2.2)$$

and by the restriction that there exists no decomposition of $\mu(t)$ such that $\mu(t) = X(t) + Z(t)$,

$$\text{with } Z(t) \neq 0 \text{ for some } t, \text{ where } \sum_{k=1}^{12} Z(k+h) = 0, \quad h = 0, 1, 2, \dots$$

3. Different suggestions for adjustment of seasonality

3.1 The seasonal component is known

If the seasonal component, $S(t)$, is known, the estimation of the trend cycle component is reduced to a simple subtraction.

$$\hat{\mu}(t) = Y(t) - S(t) = \mu(t) + \varepsilon(t) \quad (3.1)$$

$$E[\hat{\mu}(t)] = \mu(t) + E[\varepsilon(t)] = \mu(t)$$

$$\text{Var}[\hat{\mu}(t)] = \text{Var}[\varepsilon(t)] = \sigma^2$$

3.2 The seasonal component is unknown

In Harvey (1993) some models for seasonal time series are discussed. One suggestion of how to model a deterministic seasonality is

$$Y(t) = \alpha + \beta * t + \sum_{j=1}^{s-1} \gamma_j z_j(t) + u(t),$$

where α is the mean, β is the slope, z_j are dummy variables, γ_j are seasonal coefficients that sum to zero and u is a stationary stochastic process. An alternative way of modelling a seasonal pattern is by a set of trigonometric terms at the seasonal frequencies, that is

$$Y(t) = \alpha + \beta * t + \sum_{j=1}^{s/2} (\gamma_j \cos \lambda_j t + \gamma_j^* \sin \lambda_j t) + u(t).$$

Regression methods can be used for estimating the components of the models above.

The observed variable Y can be assumed to comprise three unobserved component, namely trend cycle, seasonal and irregular components, that is

$$Y(t) = C(t) + S(t) + I(t)$$

If it can be assumed that these components each follows an ARMA process, an estimate of S can be obtained by applying a linear filter to the observations (Burrige and Wallis, 1990).

The time series can be modelled as consisting of a seasonal component and a non-seasonal component, i.e.

$$Y(t) = S(t) + N(t)$$

Several authors have suggested seasonal adjustment methods that involve fitting an ARIMA model to Y above and using this along with some assumptions to determine models for S and N . Some authors suggest using ARIMA models and deterministic terms to allow for both stochastic and deterministic components. These methods involve determining ARIMA models for the components and then using signal extraction theory to estimate them (Bell and Hillmer, 1984).

One method for seasonal adjustment is the X-11 method. The multiplicative model used for the X-11 method is assumed to be the following

$$Y(t) = TC(t) * S(t) * TD(t) * H(t) * I(t)$$

where $TC(t)$, $S(t)$, $TD(t)$, $H(t)$ and $I(t)$ are the unobserved trend-cycle component, the seasonal component, the trading-day component, the holiday component and the irregular component, respectively. The program for the X-11 method is divided into seven steps. The first version of this method, X-1, was a refinement of the ratio to moving average method (Hylleberg, 1992).

The aim of the seasonal adjustment in this report is to produce an estimate, $\hat{\mu}(t)$, that pertains the monotonicity of $\mu(t)$. Two moving average techniques are used, denoted M_1 and M_2 respectively. Data consists of monthly observations and a natural estimator is a 12-month moving average. A known property of the twelve point moving average is that it tends to cut corners at turning points (Leong, 1962). This report does not, however, deal with preservation of the level. An implication of this property is that a turning point does become less pronounced and thus more difficult to detect. Another aspect of using the moving average as a trend cycle estimator is the Slutsky-Yule effect, cited in Jorgenson (1964). This effect refers to the fact that if the random component of the original series is independently distributed over time, then the random component of a moving average of this series is not.

For the model in Section 2 other adjustment methods than a moving average might be more efficient. However, it is important to use a technique that is robust against slow changes in the seasonal component over years. It should also be considered that the number of available observations is not very large. Another reason for examining the technique of moving average is that this technique is frequently used.

Both a centred (M_1) and a non-centred (M_2) moving average are shown below. For the time series under study, Y , it is assumed that the seasonal component is constant over time or at least that any possible change is very slow.

The estimate of the trend cycle component by a centred moving average (M_1) is

$$\hat{\mu}^{M_1}(t) = \sum_{j=-6}^6 w_j Y(t+j) \quad (3.2)$$

where $w_{-6}=w_6=1/24$ and $w_{-5}=w_{-4}=\dots=w_5 = 1/12$.

$$E[\hat{\mu}^{M_1}(t)] = \sum_{j=-6}^6 w_j \mu(t+j)$$

$$\text{Var}[\hat{\mu}^{M_1}(t)] = \sigma^2 \left(\frac{23}{288} \right).$$

The estimate of the trend cycle component by a non-centred moving average (M_2) is

$$\hat{\mu}^{M_2}(t) = \sum_{j=-11}^0 w_j Y(t+j) \quad (3.3)$$

where $w_{-11}=w_{-10}=\dots=w_0=1/12$.

$$E[\hat{\mu}^{M_2}(t)] = \frac{1}{12} \sum_{j=-11}^0 \mu(t+j)$$

$$\text{Var}[\hat{\mu}^{M_2}(t)] = \sigma^2 \left(\frac{24}{288} \right).$$

The centred estimator is based on the latest thirteen observations and the non-centred estimator is based on the latest twelve observations. A difference between these two estimators is that the centred moving average at time t , per definition, can not be calculated until six months later, thus producing an systematic delay.

4. Results

The investigation concerns whether the monotonicity of the trend cycle is preserved by the expected value of a moving average estimator. Two methods of estimation, M_1 and M_2 , are investigated for two different cases, namely the case when the trend cycle is monotonic and the case of a peak in the trend cycle. The observations under study are denoted $y(1), y(2), \dots, y(n)$.

4.1 Moving average estimators for a monotonically non-decreasing trend cycle

The case when the μ -vector is monotonically non-decreasing within the entire observed section is studied in this section, that is

$$\mu(t-1) \leq \mu(t), \quad \text{where } 2 \leq t \leq n.$$

The question of whether the monotonicity is preserved when two moving average techniques (centred and non-centred respectively) are used to estimate the monotonically non-decreasing μ -vector, is analysed in this section. That is, the correctness in the relation

$$E[\hat{\mu}^{M_i}(t-1)] \leq E[\hat{\mu}^{M_i}(t)], \quad \text{where } i = \{1, 2\} \quad (4.1)$$

is investigated. The results are valid also for the opposite case (a monotonically non-increasing μ -vector).

4.1.1 Centred 12-month moving average

The trend cycle estimate is the centred 12-month moving average,

$$\begin{aligned}
 \hat{\mu}^{M_1}(t) &= \frac{1}{24}(Y(t-6) + Y(t+6)) + \frac{1}{12} \sum_{j=-5}^5 Y(t+j) = \\
 &= \frac{1}{24}(\mu(t-6) + \mu(t+6) + S(t-6) + S(t+6) + \varepsilon(t-6) + \varepsilon(t+6)) + \\
 &+ \frac{1}{12} \sum_{j=-5}^5 \mu(t+j) + S(t+j) + \varepsilon(t+j).
 \end{aligned} \tag{4.2}$$

The expected value, denoted $\kappa(t)$, is defined as

$$\kappa(t) = E[\hat{\mu}^{M_1}(t)] = \frac{1}{24}(\mu(t-6) + \mu(t+6)) + \frac{1}{12} \sum_{j=-5}^5 \mu(t+j). \tag{4.3}$$

Statement 1: *If the trend cycle is a monotonic function, the expected value of the centred 12-month moving average is also a monotonic function.*

The proof is given in the appendix.

4.1.2 Non-centred 12-month moving average

The trend cycle estimate is the non-centred 12-month moving average,

$$\begin{aligned}
 \hat{\mu}^{M_2}(t) &= \frac{1}{12} \sum_{j=-11}^0 Y(t+j) = \\
 &= \frac{1}{12} \sum_{j=-11}^0 \mu(t+j) + \frac{1}{12} \sum_{j=-11}^0 S(t+j) + \frac{1}{12} \sum_{j=-11}^0 \varepsilon(t+j)
 \end{aligned} \tag{4.4}$$

The expected value, denoted $\eta(t)$, is

$$\eta(t) = E[\hat{\mu}^{M_2}(t)] = \frac{1}{12} \sum_{j=-11}^0 \mu(t+j). \tag{4.5}$$

Statement 2: *If the trend cycle is a monotonic function, the expected value of the non-centred 12-month moving average is also a monotonic function.*

The proof is given in the appendix.

4.2 Moving average estimators at a turning point in the trend cycle vector

The case when the μ -vector is inversely U-shaped with a peak at time $t = p$ is studied in this section, that is

$$\begin{aligned} \mu(1) < \mu(2) < \dots < \mu(p) \quad \text{and} \\ \mu(p) > \mu(p+1) > \dots > \mu(n) \end{aligned}$$

The question of whether the monotonicity and the time of the turning point is preserved when the two moving average techniques are used to estimate the inversely U-shaped μ -vector, is investigated in this section. That is, the correctness in the relation

$$\begin{aligned} E[\hat{\mu}^{M_i}(1)] < E[\hat{\mu}^{M_i}(2)] < \dots < E[\hat{\mu}^{M_i}(p)] \quad \text{and} \\ E[\hat{\mu}^{M_i}(p)] > E[\hat{\mu}^{M_i}(p+1)] > \dots > E[\hat{\mu}^{M_i}(n)] \end{aligned} \quad (4.6)$$

where $i = \{1, 2\}$

is investigated. The results are valid also for the opposite case (a trough).

4.2.1 Centred 12-month moving average

The observations under study are denoted $y(1), y(2), \dots, y(p), \dots, y(n)$, where p is the time of the peak. The expected value of $\hat{\mu}^{M_i}(t)$ is $\kappa(t)$. The values of κ for different time intervals are presented in Table 1. Note that the analysis at time t assumes that the observations $y(t+1), y(t+2), \dots, y(t+6)$ are available.

Table 1

The expected value of the estimated trend cycle, $\kappa(t)$, at a peak at time p in the μ -vector

Decision time, t		$\mu(t)$		$E[\hat{\mu}^1(t)] = \kappa(t)$
$2 \leq t \leq (p-6)$		$\mu(t-1) < \mu(t) \nearrow$		$\kappa(t-1) < \kappa(t) \nearrow$
$(p-5) \leq t \leq p$		$\mu(t-1) < \mu(t) \nearrow$		Several possibilities
$(p+1) \leq t \leq (p+6)$		$\mu(t-1) > \mu(t) \searrow$		Several possibilities
$(p+7) \leq t \leq n$		$\mu(t-1) > \mu(t) \searrow$		$\kappa(t-1) > \kappa(t) \searrow$

In Table 1 the function $\kappa(t)$ is shown for different time intervals. The results assume that the observations $\{y(t+1), y(t+2), \dots, y(t+6)\}$ are available at time t . Nothing definite

can be concluded about $\kappa(t)$ in the interval $\{p-5: p+6\}$. However, the time of the turning point of $\kappa(t)$ must occur in this interval. Thus the indication of a turning point can come before, at or after time $t = p$. The μ -vector is monotonically decreasing from time $t = p$ and forward. From Table 1 we have that from time $t = p+7$ the value of κ is monotonically decreasing. This indicates that the delay of the expected value of this estimator is, at maximum, six time units. However the usual situation would be that the observations $\{y(t+1), y(t+2), \dots, y(t+6)\}$ are not available at time t . This situation does cause a systematic delay of six time units for the centred moving average estimator. This systematic delay must be added to the delay times presented in Table 1. Therefore the total delay of κ is, at maximum, twelve months. Because of this systematic delay, the centred moving average estimator will not be considered further.

4.2.2 Non-centred 12-month moving average

The observations under study are denoted $y(1), y(2), \dots, y(p), \dots, y(n)$. The expected value of $\hat{\mu}^{M_2}(t)$ is $\eta(t)$. The values of η for different time intervals are presented in Table 2.

Table 2

The expected value of the estimated trend cycle, $\eta(t)$, at a peak at time p in the μ -vector

Decision time, t	$\mu(t)$	$E[\hat{\mu}^2(t)] = \eta(t)$
$2 \leq t \leq p$	$\mu(t-1) < \mu(t) \nearrow$	$\eta(t-1) < \eta(t) \nearrow$
$(p+1) \leq t \leq (p+12)$	$\mu(t-1) > \mu(t) \searrow$	Several possibilities
$(p+13) \leq t \leq n$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) > \eta(t) \searrow$

In Table 2 the function $\eta(t)$ is shown for different time intervals. Nothing definite can be concluded about $\eta(t)$ in the interval $\{p+1: p+12\}$. However, the time of the turning point of $\eta(t)$ must occur in this interval. The maximum delay of $\eta(t)$ is twelve months. Without further specifications of the kind of turning point nothing definite can be concluded.

The non-centred moving average estimator will now be further studied in three special cases of a peak at time $t = p$. The three cases considered are a symmetric turning point, an almost flat curve after the turning point and a steep slope after the turning point. For all cases we have that

$$\begin{aligned} \mu(1) < \mu(2) < \dots < \mu(p) \quad \text{and} \\ \mu(p) > \mu(p+1) > \dots > \mu(n) \end{aligned} \quad (4.7)$$

The observations under study for all three cases are denoted $y(1), y(2), \dots, y(p), \dots, y(n)$.

In the first case investigated, the μ -vector is symmetric around the peak at time $t = p$, i.e.

$$\mu(p - q) = \mu(p + q), \quad q \geq 1. \quad (4.8)$$

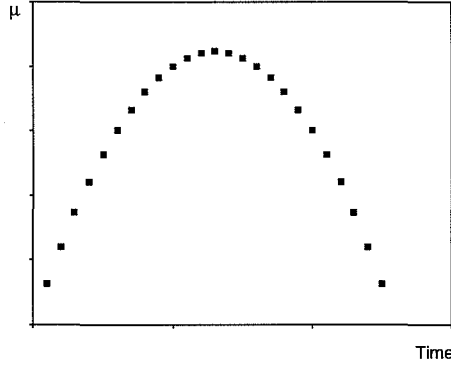


Fig. 1. Case 1, the μ -vector is symmetric around the peak

The table below shows the non-centred moving average estimator at a symmetric peak.

Table 3

The expected estimated trend cycle, $\eta(t)$, at a symmetric peak at time p in the μ -vector (case 1)

Decision time, t	$\mu(t)$	$E[\hat{\mu}^2(t)] = \eta(t)$
$2 \leq t \leq (p)$	$\mu(t-1) < \mu(t) \nearrow$	$\eta(t-1) < \eta(t) \nearrow$
$(p+1) \leq t \leq (p+5)$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) < \eta(t) \nearrow$
$t = (p+6)$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) = \eta(t) \rightarrow$
$(p+7) \leq t \leq n$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) > \eta(t) \searrow$

Table 3 shows the expected estimated trend cycle, $\eta(t)$, at a symmetric peak at time p . The function $\eta(t)$ is shown for different time intervals. The delay for the expected value of this estimator at a symmetric peak is six time units.

The second case considered is where the μ -vector forms an almost flat curve after the turning point at time $t = p$, that is

$$\mu(p + q) > \mu(p - 1), \quad q \in \{1, 2, \dots, 11\}. \quad (4.9)$$

Since $\mu(1) < \dots < \mu(p-2) < \mu(p-1)$, it follows that $\mu(p+m) > \mu(p-m)$, $m \in \{1, 2, \dots, 11\}$.

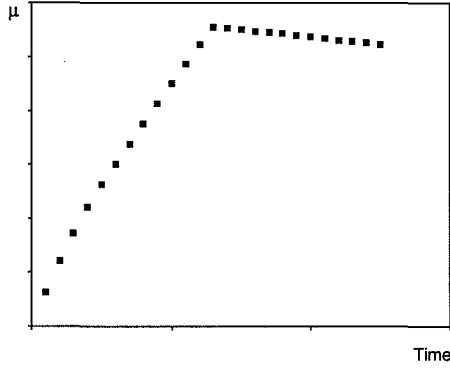


Fig. 2. Case 2, the μ -vector forms an almost flat curve after the peak.

The table below shows the non-centred moving average estimator at an unsymmetrical peak (case 2).

Table 4

The expected estimated trend cycle, $\eta(t)$, at an unsymmetrical peak at time p in the μ -vector (case 2)

Decision time, t	$\mu(t)$	$E[\hat{\mu}^2(t)] = \eta(t)$
$2 \leq t \leq (p)$	$\mu(t-1) < \mu(t) \nearrow$	$\eta(t-1) < \eta(t) \nearrow$
$(p+1) \leq t \leq (p+11)$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) < \eta(t) \nearrow$
$(p+12) \leq t \leq n$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) > \eta(t) \searrow$

Table 4 shows the expected estimated trend cycle, $\eta(t)$, at an unsymmetrical peak at time p . The function $\eta(t)$ is shown for different time intervals. The delay for the expected value of this estimator at this kind of peak is eleven time units.

The third case considered is where the μ -vector forms a steep slope after the turning point at time $t = p$. It is assumed that

$$\mu(p+1) < \mu(p-11). \quad (4.10)$$

Since $\mu(p+1) > \mu(p+2)$ and $\mu(p-11) < \mu(p-10)$ the inequality $\mu(p+2) < \mu(p-10)$ must hold. From this result it is implicit that $\mu(p+m) < \mu(p-12+m)$, $m \in \{3, 4, \dots, 11\}$.

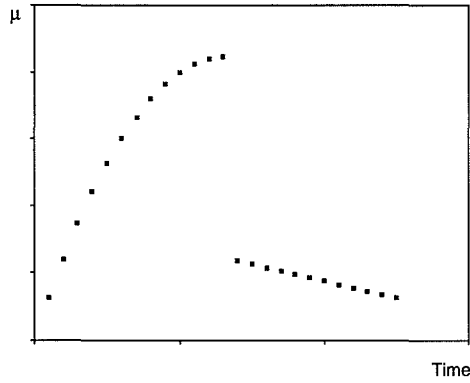


Fig. 3. Case 3, the μ -vector forms a steep slope after the peak.

The table below shows the non-centred moving average estimator at an unsymmetrical peak (case 3).

Table 5

The expected estimated trend cycle, $\eta(t)$, at a peak at time p in the μ -vector (case 3)

Decision time, t	$\mu(t)$	$E[\hat{\mu}^2(t)] = \eta(t)$
$2 \leq t \leq (p)$	$\mu(t-1) < \mu(t) \nearrow$	$\eta(t-1) < \eta(t) \nearrow$
$(p+1) \leq t \leq n$	$\mu(t-1) > \mu(t) \searrow$	$\eta(t-1) > \eta(t) \searrow$

Table 5 shows the expected estimated trend cycle, $\eta(t)$, at an unsymmetrical peak at time p . The function $\eta(t)$ is shown for different time intervals. There is no delay for the expected value of this estimator at this kind of peak.

The ability of $\eta(t)$ to preserve the time of the turning point depends on the shape of the μ -vector at the peak. For a symmetric peak, a delay of seven time units is to be expected. For a μ -vector that is almost flat after the turning point, the delay in $\eta(t)$ is twelve time units. For a μ -vector with a very steep slope after the turning point, there is no delay in $\eta(t)$.

Statement 3: *If the trend cycle is unimodal, the expected value of the non-centred 12-month moving average will preserve the unimodality (Frisén, 1986) but not always the time of the turning point in the trend cycle.*

5. Discussion

The centred and the non-centred moving average techniques have been investigated as possible methods of adjusting a time series for seasonality. The properties of these moving averages have been evaluated for a monotonic trend cycle and for a turning point in the trend cycle. The assumptions for the model used in this report might be too strong for many applications. However, some problems with the monotonicity evaluation of the seasonal adjustment were demonstrated even for a model with these assumptions.

Never the less, the following can be concluded from this monotonicity study: Using the simple moving average technique for seasonal adjustment does preserve the monotonicity of a monotonically non-decreasing trend cycle. It has been shown that if all observations of the moving average are within a monotonic section of the time series, both the centred and the non-centred moving average will preserve the monotonicity. For the case of an unspecified peak in the trend cycle, no definite conclusions can be made regarding the time of the turning point for either of the moving averages.

Because of a systematic delay in the timeliness, the centred moving average was not investigated further.

The non-centred moving average was investigated for three different kinds of peaks. For a unimodal section of the trend cycle, the non-centred moving average will preserve the unimodality. However, it has been shown in this investigation that the non-centred moving average does not always preserve the monotonicity of all parts in the unimodal case. Thus, the time of the turning point is not always preserved. In some cases the use of the moving average technique results in a delayed indication of a turning point.

The non-centred moving average as a method for seasonal adjustment is conservative in the sense that it does not give any false indications of a turning point. This moving average performs well at monotonic sections, but because of the possible delay at a turning point, it is important to try to use other methods. One possibility, if it agrees with the structure in the data on hand, is to use a large historical data set to estimate the seasonal components.

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Appendix. Proof of Statement 1 - 3

For **Statement 1** we have that the difference between two consecutive expected estimates of the trend cycle is

$$\begin{aligned} \kappa(t) - \kappa(t-1) &= \\ &= \frac{1}{24} [\mu(t+6) - \mu(t-1+6)] + \frac{1}{12} [\mu(t+5) - \mu(t-1-5)] + \frac{1}{24} [\mu(t-6) - \mu(t-1-6)] \end{aligned}$$

The differences inside the three brackets are each greater than or equal to zero for all $t \geq 7$, according to the assumption of a non-decreasing function. Therefore $\kappa(t) - \kappa(t-1) \geq 0$ and κ is a non-decreasing function for all $t \geq 7$.

For **Statement 2** we have that the difference between two consecutive expected estimates of the trend cycle is

$$\eta(t) - \eta(t-1) = \frac{1}{12} [\mu(t) - \mu(t-1-11)]$$

The difference inside the brackets is greater than or equal to zero, for all $t \geq 12$, according to the assumption of a non-decreasing function. Therefore $\eta(t) - \eta(t-1) \geq 0$ and η is a non-decreasing function for all $t \geq 12$.

For **Statement 3** the proof is divided into the three cases that have been investigated. For each case investigated in Section 4.2.2 the proof for the different time intervals, denoted i), ii), iii) and iv), are given separately. The proof for the interval i) is the same as for Statement 2. For the rest of the intervals the proof for the first time point is showed, the rest follow easily.

Case 1

- i) $\eta(p-a) - \eta(p-1-a) > 0$, for $0 \leq a \leq (p-2)$. See proof of Statement 2.
- ii) $\eta(p+1) - \eta(p) = (1/12) * (\mu(p+1) - \mu(p-11)) = (1/12) * (\mu(p-1) - \mu(p-11)) > 0$
 $\eta(p+b) - \eta(p-1+b) > 0$, for $2 \leq b \leq 5$
- iii) $\eta(p+6) - \eta(p+5) = (1/12) * (\mu(p+6) - \mu(p-6)) = (1/12) * (\mu(p-6) - \mu(p-6)) = 0$
- iv) $\eta(p+7) - \eta(p+6) = (1/12) * (\mu(p+7) - \mu(p-5)) = (1/12) * (\mu(p-7) - \mu(p-5)) < 0$
 $\eta(p+b) - \eta(p-1+b) < 0$, for $b \geq 8$

Case 2

i) $\eta(p-a) - \eta(p-1-a) > 0$, for $0 \leq a \leq p-2$. See proof of statement 2.

ii) $\eta(p+1) - \eta(p) = (1/12)*(\mu(p+1) - \mu(p-1)) > 0$
 $\eta(p+b) - \eta(p-1+b) > 0$, for $2 \leq b \leq 11$

iii) $\eta(p+12) - \eta(p+11) = (1/12)*(\mu(p+12) - \mu(p)) < 0$
 $\eta(p+b) - \eta(p-1+b) < 0$, for $b \geq 13$

Case 3

i) $\eta(p-a) - \eta(p-1-a) > 0$, for $0 \leq a \leq p-2$. See proof of statement 2.

ii) $\eta(p+1) - \eta(p) = (1/12)*(\mu(p+1) - \mu(p-1)) < 0$
 $\eta(p+b) - \eta(p-1+b) < 0$, for $b \geq 2$

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