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Maximum Likelihood Ratio based small-sample tests for random coefficients in linear regression

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Abstract

Two small-sample tests for random coefficients in linear regression are derived from the Maximum Likelihood Ratio. The first test has previously been proposed for testing equality of fixed effects, but is here shown to be suitable also for random coefficients. The second test is based on the multiple coefficient of determination from regressing the observed subject means on the estimated slopes. The properties and relations of the tests are examined in detail, followed by a simulation study of the power functions. The two tests are found to complement each other depending on the study design: The first test is preferred for a large number of observations from a small number of subjects, and the second test is preferred for the opposite situation. Finally, the robustness of the tests to violations of the distributional assumptions is examined.

MSC: primary 62M10; secondary 62J05

Keywords: Exact test; Hypothesis test; Maximum Likelihood; Pre-test; Random coefficient regression.

1. Introduction and assumptions

Random coefficient regression (RCR) models (Rao [27], Swamy [34]) are generalisations of the classical Gauss-Markov model, where the parameters are allowed to be random quantities. A special case of the RCR models is the random intercept model (Diggle and Heagerty et al. [7]), also known as error components regression (ECR) model, where only the intercept parameter is random. Statistical inference based on RCR models is more demanding since more parameters are introduced in the variance-covariance matrix of the observations. In many cases it is of crucial importance to know whether the simpler ECR model is appropriate, e.g. if one wants to construct tolerance limits by utilising the longitudinal structure of the data (Jonsson [20]).

In this paper tests for random coefficients in linear regression will be considered. Introducing random coefficient variation is to give the dependent variable a different variance at each cross-section. Models with this feature can therefore be transformed into a particular heteroscedastic formulation and tests for heteroscedasticity can hence be used to detect departure from the constant parameter assumption. For detailed reviews of various large-sample tests for heteroscedasticity, see Haggstrom [13], Greene [11], Kmenta [21], Baltagi [4] and Godfrey [9]. However, the aim of this paper will be to utilise knowledge about the model and distribution of the parameters for deriving more specific tests. Instead of using general tests for heteroscedasticity, which are tests for inhomogeneity of variances, we can now test whether the second-order moments of certain parameters are zero or not. Some differences between tests for random coefficients and tests for heteroscedasticity were discussed in Honda [17], where it e.g. was concluded that some proposed large-sample tests for random coefficients were more robust to non-normal disturbances than tests for heteroscedasticity.

Two Maximum Likelihood (ML)-based small-sample tests for random coefficients will be derived and examined. The following linear RCR model will be considered as the alternative hypothesis in the sequel:

$$H_1: \quad Y_{ij} = A_j + \mathbf{x}_i'\mathbf{B}_j + U_{ij},\ i = 1 \ldots T,\ j = 1 \ldots n \qquad (1)$$

where Y_{ij} is the measured response at $\mathbf{x}_i = \left(x_i^{(1)} \ldots x_i^{(r)} \ldots x_i^{(p)}\right)'$ for the j:th subject. The model in (1) is composed of three random components which are, following Swamy [33], assumed to be random drawings from the normal distribution. The random intercept A_j and the random slopes $\mathbf{B}_j = \left(B_j^{(1)} \ldots B_j^{(r)} \ldots B_j^{(p)}\right)'$ reflect factors which are specific for the j:th subject, and U_{ij} is a residual. Let the expected value of the T-dimensional normally distributed vector $\mathbf{Y}_j = (Y_{1j} \ldots Y_{Tj})'$ be $E\left(\mathbf{Y}_j \middle| \tilde{\mathbf{X}}\right)_{T\times 1} = \tilde{\mathbf{X}}(\alpha \mid \boldsymbol{\beta}')'$ where $\tilde{\mathbf{X}} = \left(\mathbf{1} \,\middle|\, (\mathbf{x}_1 \ldots \mathbf{x}_i \ldots \mathbf{x}_T)'\right)_{T\times(p+1)}$ and $\boldsymbol{\beta} = \left(\beta^{(1)} \ldots \beta^{(r)} \ldots \beta^{(p)}\right)'$. Further, under the assumption of independence between the U_{ij}'s and the A_j's and $\mathbf{B}_j$'s respectively, let the variances be $V\left(\mathbf{Y}_j \middle| \tilde{\mathbf{X}}\right)_{T\times T} = \tilde{\mathbf{X}}\boldsymbol{\Sigma}\tilde{\mathbf{X}}' + \sigma_U^2 \mathbf{I}$ where $\boldsymbol{\Sigma} = \left[\begin{array}{c|c} \sigma_A^2 & \boldsymbol{\Sigma}_{AB}' \\ \hline \boldsymbol{\Sigma}_{AB} & \boldsymbol{\Sigma}_{BB} \end{array}\right]$, $\boldsymbol{\Sigma}_{AB} = \left(\sigma_{AB_1} \ldots \sigma_{AB_p}\right)'$ and $\boldsymbol{\Sigma}_{BB} = \begin{bmatrix} \sigma_{B_1}^2 & \cdots & \sigma_{B_1 B_p} \\ & \ddots & \vdots \\ & & \sigma_{B_p}^2 \end{bmatrix}$. Note that the elements of $\boldsymbol{\Sigma}$ are assumed to be equal among the subjects and constant over the study interval. Since $\tilde{\mathbf{X}}$ is equal for all subjects we have a balanced design.

A special case of the general model in (1) will be considered as the null hypothesis

$$H_0: \quad Y_{ij} = A_j + \mathbf{x}_i'\boldsymbol{\beta} + U_{ij},\ i = 1 \ldots T,\ j = 1 \ldots n. \tag{2}$$

This is an ECR model with a random intercept but fixed and equal slopes $\boldsymbol{\beta}$. Under H_0 the variance matrix is reduced to $V\left(\mathbf{Y}_j \middle| \tilde{\mathbf{X}}\right)_{T\times T} = \sigma_A^2 \mathbf{1}\mathbf{1}' + \sigma_U^2 \mathbf{I}$.

There are a number of recent papers on tests for random coefficient covariance structures. For example, Anh and Chelliah [3] extended the analysis-of-covariance test by Swamy [33] to a more general test where the different subjects are allowed to have different covariance structures. Haggstrom [13] showed that the score test by Honda [17] is applicable also for non-linear regression and extended it for possible time effects. In Lundevaller and Laitila [22] another modification of Honda [17]-test was proposed which is robust against heteroscedasticity. Further, in Fujikoshi and von Rosen [8] and Andrews [2] tests of the null hypothesis that some random coefficients have variance equal to zero were proposed. However, only the asymptotic null distributions of these tests are derived, and the properties for finite sample sizes are in general unknown.

In the next section the Maximum Likelihood Ratio (*MLR*) is derived and two potential test statistics based on subparts of the *MLR* are considered. Two small-sample tests based on these test statistics are then proposed in Section 2. The properties of the tests are examined in general, and the power functions are thereafter studied in more detail for the simple case with one explanatory variable in Section 3. A concluding discussion is given in Section 4. Notations not explained in the text are defined in Appendix I, and some stated results are derived in Appendix II.

2. The Maximum Likelihood Ratio and its subparts

Under the given assumptions the Maximum Likelihood (ML) estimators from Rao [27] are minimum variance unbiased. These estimators will be used in the sequel, and further properties are given in Swamy [34] Chap. 1.2, 3.4 and 4.3. In general, the ML estimator of a population parameter φ under H_0 and H_1 will be denoted as $\hat{\hat{\varphi}}$ and $\hat{\varphi}$, respectively, and the corresponding estimators for the j:th subject will be denoted by a subscripted j.

Following Anderson [1] p. 291 the *ML* functions can be written as

$$L_{H_0} = \frac{1}{(2\pi)^{\frac{nT}{2}} \left| \hat{\hat{\sigma}}_A^2 \mathbf{1}\mathbf{1}' + \hat{\hat{\sigma}}_U^2 \mathbf{I} \right|^{\frac{n}{2}}} \exp\left(-\frac{nT}{2} \right) \text{ and } L_{H_1} = \frac{1}{(2\pi)^{\frac{nT}{2}} \left| \tilde{\mathbf{X}} \hat{\boldsymbol{\Sigma}} \tilde{\mathbf{X}}' + \hat{\sigma}_U^2 \mathbf{I} \right|^{\frac{n}{2}}} \exp\left(-\frac{nT}{2} \right),$$

and from Swamy [34] p. 111 it follows that the *MLR* statistic L_{H_0} / L_{H_1} can be written as

$$(MLR)^{\frac{2}{n}} = \frac{\left| \tilde{\mathbf{X}} \hat{\boldsymbol{\Sigma}} \tilde{\mathbf{X}}' + \hat{\sigma}_U^2 \mathbf{I} \right|}{\left| \hat{\hat{\sigma}}_A^2 \mathbf{1}\mathbf{1}' + \hat{\hat{\sigma}}_U^2 \mathbf{I} \right|} = \left(\frac{\hat{\sigma}_U^2}{\hat{\hat{\sigma}}_U^2} \right)^{T-1} \frac{\left| \tilde{\mathbf{X}}' \tilde{\mathbf{X}} \right| \cdot \left| \hat{\sigma}_U^2 \left(\tilde{\mathbf{X}}' \tilde{\mathbf{X}} \right)^{-1} + \hat{\boldsymbol{\Sigma}} \right|}{\left(\hat{\sigma}_U^2 \right)^p S_{\overline{Y}\overline{Y}} \cdot T \big/ (n-1)}. \tag{3}$$

To base a test on the full *MLR* statistic in (3) is appealing since it contains a maximum of information, but there are three potential drawbacks with this approach. First, an important practical problem is that the exact distribution is hard to derive and critical values for tests have to be found by simulation. Second, as noted in Cox and Hinkley [6] p. 172 the strong optimum properties, e.g. the Neyman-Person lemma, associated with the Likelihood Ratio (LR) method for simple hypothesis are not carried over to composite hypothesis problems in general. This means that the test is not guaranteed to be uniformly most powerful. Third, from Figure 1 it can be seen that the test can be biased, i.e. the size of the test under H_0 is correct ($\alpha = 0.05$) but the power under H_1 can be less than the size. Hayakawa [16] and Harris and Peers [14] demonstrated that *MLR* tests are not unbiased in general against local alternatives, which is further discussed by Stuart and Ord et al. [32] p. 259. The criterion of unbiasedness for tests has such strong intuitive appeal that it is natural to restrict oneself to the class of unbiased tests. Altogether, the usefulness of the *MLR* test is limited in practise and it will only be used as a reference in the simulation studies in Section 3.2.

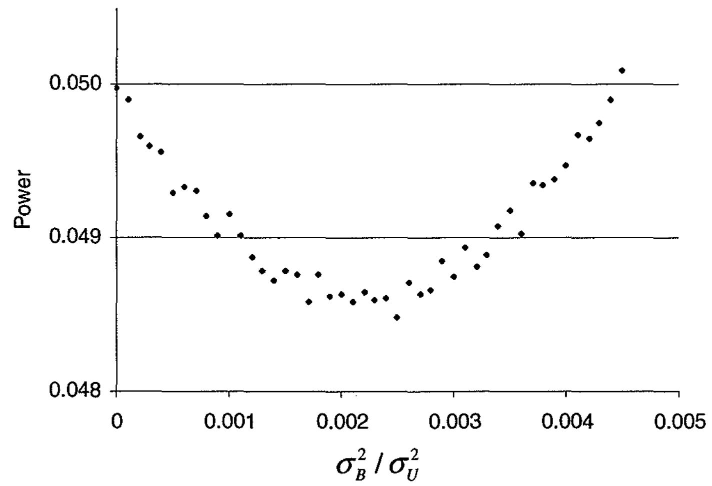

Figure 1. Bias of the *MLR* test with settings from the simulation study in Section 3.2.

2.1 The T_{F_1} -test

Test statistics can also be derived from subparts in (3). An obvious candidate is $\hat{\sigma}_U^2 / \hat{\hat{\sigma}}_U^2$ which expresses the ratio between residual sums of squares where the slopes vary across subjects or not. This quotient can easily be shown to be directly proportional to the analysis-of-covariance test statistic F_1 proposed by Hsiao [18], cf. Appendix IIA. Using the notations in Appendix I the statistic can be written as

$$T_{F_1} = \frac{\sum_{j=1}^{n}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right)' \mathbf{S}_{xx}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right) \Big/ p(n-1)}{SSE / n(T-p-1)}.$$

Under H_0, the distribution of T_{F_1} is well-known to be $F_{p(n-1),n(T-p-1)}$, which for completeness also is shown below, where H_0 is rejected for large values of T_{F_1}. To study the distribution of T_{F_1} in general, notice that the numerator and denominator are independent under $H_0 \cup H_1$ (Rao [27]), and the denominator is distributed as $\sigma_U^2 / n(T-p-1) \cdot \chi^2_{n(T-p-1)}$. The distribution of the numerator becomes clear if we make use of the decomposition

$$D = \sum_{j=1}^{n}\left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}\right)' \mathbf{S}_{xx}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \sum_{j=1}^{n}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right)' \mathbf{S}_{xx}\left(\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}\right) + n\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)' \mathbf{S}_{xx}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = D_1 + D_2,$$

and of the following necessary and sufficient condition for a quadratic form to have a chi-square distribution: Let $\mathbf{z}$ have a multivariate normal distribution with mean vector $\mathbf{0}$ and dispersion matrix $\boldsymbol{\Sigma}$, then any quadratic form $\mathbf{z}'\mathbf{A}\mathbf{z}$ has a chi-square distribution with degrees of freedom $\mathrm{df} = \mathrm{rank}(\mathbf{A})$ if and only if $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{A}$ (Rao [28] Chap 3b.4). Further, under H_0 $\left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}\right)$ and $\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)$ are each normally distributed with mean vector $\mathbf{0}$ and dispersion matrices $\sigma_U^2 \cdot \mathbf{S}_{xx}^{-1}$ and $\sigma_U^2 / n \cdot \mathbf{S}_{xx}^{-1}$, respectively. From the condition above it is now easily verified that D / σ_U^2 and D_2 / σ_U^2 both have chi-square distributions with np and p degrees of freedom (df), respectively. From Cochran [5] it thus follows that D_1 / σ_U^2 is chi-square distributed with $\mathrm{df} = p(n-1)$. This gives the distribution of T_{F_1} under H_0.

Under H_1, $\left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}\right)$ has the dispersion matrix $\boldsymbol{\Sigma}_{BB} + \sigma_U^2 \cdot \mathbf{S}_{xx}^{-1}$ and it is easily concluded from the condition stated above, that neither D nor D_2 can have chi-square distributions in general. Thus for general p the distribution of T_{F_1} under H_1 is complicated. However, the expectation of the statistic can be studied as an indicator of the behaviour of the power function. The expectation can be found by noticing that $E(D_1) = E(D) - E(D_2) = (n-1)\cdot \text{trace}(\mathbf{S}_{xx}\boldsymbol{\Sigma}_{BB} + \sigma_U^2\mathbf{I})$. From this we get

$$E\left(T_{F_1}\right) = \left(1 + \frac{\sum_{r=1}^{p}\sum_{s=1}^{p} s_{rs}\sigma_{B_rB_s}}{p\sigma_U^2}\right)\frac{n(T-p-1)}{\left(n(T-p-1)-2\right)}, \text{ where } s_{rs} = \sum_{i=1}^{T}(x_i^{(r)} - \bar{x}^{(r)})(x_i^{(s)} - \bar{x}^{(s)}).$$

The above expectation will increase with T, through the increasing sums of squares s_{rs}, but will be slowly decreasing with n.

2.2 The T_{R^2}-test

Another interesting subpart of (3) is the determinant $\left|\hat{\sigma}_U^2\left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-1} + \hat{\boldsymbol{\Sigma}}\right|$ which contains the informative variance-covariance estimator $\hat{\boldsymbol{\Sigma}}$. Let $R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}$ be the (sample) multiple coefficient of determination from the unconditional regression of the $\bar{Y}_j$'s on the $\hat{\boldsymbol{\beta}}_j$'s (cf. Appendix ID). From Appendix IIB, $R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}$ can be seen to be a subpart of the latter determinant. Since $R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}$ contains the dispersion matrix $\mathbf{S}_{BB}$ it retains the information about the dispersion pattern of the $\boldsymbol{\beta}_j$'s from $\hat{\boldsymbol{\Sigma}}$. A well known test statistic based on $R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}$ is

$$T_{R^2} = \frac{R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}}{1 - R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}}\cdot\frac{(n-p-1)}{p}.$$

H_0 is then rejected for large values of T_{R^2}, where T_{R^2} under H_0 has the $F_{p,n-p-1}$ distribution which is independent of T, cf. Stuart and Ord et al. [32] p. 528. The distribution of T_{R^2} under H_1 is more complicated, but it can be shown that T_{R^2} then has the same distribution as (cf. Johnson and Kotz et al. [19] p. 618):

$$\frac{\chi^2(p-1) + \left(U + \theta^{1/2}\left(\chi^2_{n-1}\right)^{1/2}\right)^2}{\chi^2(n-p-1)}\cdot\frac{(n-p-1)}{p}. \tag{4}$$

Here $\theta = \rho^2/(1-\rho^2)$, where $\rho^2 = 1 - V\left(\bar{Y}_j \middle| \hat{\boldsymbol{\beta}}_j\right)\cdot V\left(\bar{Y}_j\right)^{-1}$ is the population multiple coefficient of determination, and the three chi-square variables and the standard normal variable U are all independent. By noticing that a non-central t-variable with f df and non-centrality parameter δ can be represented $t_f(\delta) = (U+\delta)/\sqrt{\chi_f^2/f}$ (cf. Johnson and Kotz et al. [19] p. 514), it follows from (4) that T_{R^2} is distributed as

$$\frac{(p-1)}{p} F_{p-1,n-p-1} + \frac{1}{p} t_{n-p-1}^2 \left(\theta^{1/2}\sqrt{\chi_{n-1}^2}\right),$$

where the first term vanishes for p=1 and where all random variables are independent. By utilising that $E\left(t_f^2(\delta)\right) = (1+\delta^2)\cdot n/(n-2)$ for each fixed δ, one obtains the expected value

$$E\left(T_{R^2}\right) = \left(1 + \frac{(n-1)}{p}\theta\right)\frac{(n-p-1)}{(n-p-3)}.$$

In contrast to the expectation of T_{F_1}, the expectation of T_{R^2} increases with n but is quite unaffected by T which appears in the constant θ (cf. Eq. (5) for the $p=1$ case).

3. The simple case with one explanatory variable

The tests based on the T_{F_1} and T_{R^2} statistics utilise information from the data to different extent. To emphasis on inferential issues and to limit the number of parameters the simple case where $p=1$ will be studied in this section:

$$\begin{aligned} H_0: &\quad Y_{ij} = A_j + \beta x_i + U_{ij} \\ H_1: &\quad Y_{ij} = A_j + B_j x_i + U_{ij} \end{aligned}$$

The properties of the T_{F_1} and T_{R^2} tests will be examined in detail, followed by a simulation study of the powers. Finally the robustness to non-normality is studied.

3.1 Some properties of the tests

For $p=1$ the statistic proposed by Hsiao reduces to

$$T_{F_1} = S_{xx}\cdot\frac{S_{BB}/(n-1)}{SSE/(n(T-2))},$$

where SSE is the total residual sum of squares over all subjects, cf. Appendix IC. Since S_{BB} is distributed as $(\sigma_B^2 + \sigma_U^2/S_{xx})\cdot\chi_{n-1}^2$ (cf. Appendix IB), and S_{BB} and SSE

are independent it follows that $T_{F_1} \sim \left(1+S_{xx}(\sigma_B^2/\sigma_U^2)\right)\cdot F_{n-1,n(T-2)}$ under $H_0 \cup H_1$. Notice that the power of the test is an increasing function of the dispersion factor S_{xx} and the quotient σ_B^2/σ_U^2, and does not depend on σ_A^2 and σ_{AB}.

Regarding the T_{R^2} statistic, the χ^2_{p-1}-variable in (4) vanishes for $p=1$ and the coefficient of determination simplifies to $R^2_{\bar{Y}_j\cdot\hat{\beta}_j} = \dfrac{S^2_{\bar{Y}B}}{S_{\bar{Y}\bar{Y}}S_{BB}}$ where $S_{\bar{Y}B}$ and S_{BB} now are scalars. It follows that the test statistic can be written

$$T_{R^2} = \frac{S^2_{\bar{Y}B}}{S_{\bar{Y}\bar{Y}}S_{BB} - S^2_{\bar{Y}B}}\cdot(n-2),$$

which under H_0 has the $F_{1,n-2}$-distribution. Unlike the test based on T_{F_1} the T_{R^2} test has a complicated distribution under H_1 also for $p=1$ and the power cannot be expressed as a known function. However, a maximal power of the T_{R^2} test, i.e. a maximum of (4), is obtained for a maximum of

$$\theta = \frac{\rho^2}{1-\rho^2} = \frac{\left(\rho_{AB} + \bar{x}\sqrt{\dfrac{Q_B}{Q_A}}\right)^2}{S_{xx}^{-1}\left((TQ_AQ_B)^{-1} + Q_B^{-1} + (\bar{x})^2Q_A^{-1}\right) + (TQ_A)^{-1} + 2\bar{x}S_{xx}^{-1}(Q_AQ_B)^{-1/2}\rho_{AB} + 1 - \rho_{AB}^2} \tag{5}$$

where $Q_A = \sigma_A^2/\sigma_U^2$, $Q_B = \sigma_B^2/\sigma_U^2$, $Q_{AB} = \sigma_{AB}/\sigma_U^2$ and ρ_{AB} is the correlation between A_j and B_j. The dependencies in (5) are complicated, but since T_{F_1} does not depend on σ_A^2 and σ_{AB} it is interesting to examine the behaviour of θ regarding these two parameters. First, let $\bar{x}=0$. Considered as a function of ρ_{AB}, θ has one local minimum for $\rho_{AB}=0$ and maximum for $\rho_{AB}=\pm1$. Further, θ is an increasing function of σ_A^2 if $\rho_{AB}=\pm1$ but constant if $\rho_{AB}=0$. Second, let $\bar{x}>0$. Then θ has two local minima for $\rho_{AB}^{(1)} = -\bar{x}\sqrt{Q_B/Q_A}$ and for $\rho_{AB}^{(2)} = \left(1+(TQ_A)^{-1}\right)/\rho_{AB}^{(1)}$, but it is easily seen that only one of these can be larger than -1. Further, for $\rho_{AB} = 0$ it can be seen that θ now is a decreasing function of σ_A^2.

3.2 An illustrative example of the power

The powers of various test statistics may be compared by computing the asymptotic relative efficiency (ARE), cf. Stuart and Ord et al. [32] p. 266. Such a measure, which compares the slopes of the powers at the parameter value specified by H_0, is hard to use in the present situation. One reason for this is that it is difficult to find the distribution function of the T_{R^2} statistic, even in the simple case when $p = 1$. Another reason is that different parameters are involved in the distribution of the statistics. E.g. when $p = 1$, the distribution of T_{F_1} depends only on the variance ratio Q_B, while the distribution of T_{R^2} depends on Q_A, Q_B and Q_{AB}. Due to the complications involved, the comparisons between the powers will be based on simulations.

The T_{F_1} statistic was originally proposed by Hsiao for testing the heterogeneity of a fixed number of subject-specific slope parameters. Since the test only makes use of the observed $\hat{\beta}_j$´s one can suspect that this test will have a relatively larger power when it is possible to estimate the slopes with high precision, i.e. when the number of observations (T) per subject is large. The tests based on the *MLR* and T_{R^2} statistics utilise more information about the stochastic distribution of all the parameters, and it can thus be suspected that the power of the two latter tests would gain relatively more from a large number of subjects (n).

How the power functions depend on n and T was examined in a simulation study for two combinations of n and T, and a nominal test size of 5%. In this section, for simplicity, the x_i´s were chosen as equally spaced on the interval $[-5,5]$ yielding $\bar{x} = 0$ and a maximal power of T_{R^2} for $\rho_{AB} = 1$. From Figure 2a it first seems that the T_{F_1} test has the largest power for $n = 5$ and $T = 20$ throughout the study interval as expected, and from Figure 2b it seems that the opposite is true when $n = 20$ and $T = 5$. However, from Figure 2c we can see that the power of the T_{R^2} is the largest relatively near H_0 using the parameter settings from Figure 2a, and from Figure 2d we can see that the power of T_{F_1} becomes the larger than T_{R^2} for relatively large values of σ_B^2/σ_U^2 using the parameter settings from Figure 2b. This study thus indicates that the power of the T_{R^2} test is larger than the power of the T_{F_1} test for small deviations from H_0 but that this relation will be the opposite for large deviations. The shift where the power of the T_{F_1} becomes larger will appear closer to H_0 if T is relatively large compared to n.

It is notable from Figure 2 that the power of the *MLR* test is dominated in both situations for small deviations from H_0 by the tests based on subparts of the *MLR* statistic. The problem that the optimum properties of the LR method for simple hypothesis are not carried over to the composite case in general was treated in Section 2, which the results here exemplify. From Figure 2 it is obvious that the different subparts of the *MLR* statistic sometimes may work in different directions yielding a smaller power for the *MLR* test than for some of the subpart tests.

Even if there are differences among the three tests, generally the power was found to be relatively large in the studied situations. Also for a very small quotient σ_B^2/σ_U^2 the power is about 80-90%.

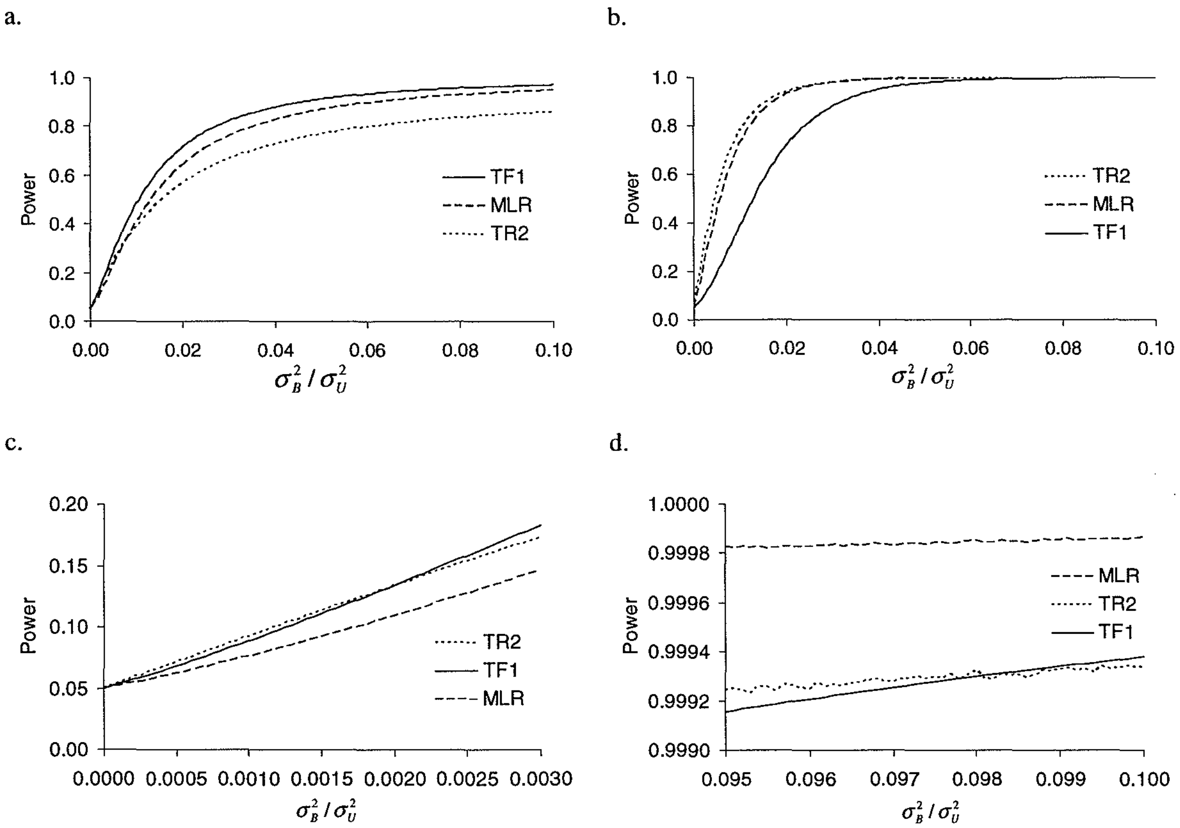

Figure 2. The power of the *MLR*, T_{F_1} and T_{R^2} tests for a.) $n=5$, $T=20$ and b.) $n=20$, $T=5$ where $\sigma_A^2 / \sigma_U^2 = 1/1$ and $\alpha = 0.05$. In c.) and d.) it can be seen that T_{R^2} has a larger power than T_{F_1} for small deviations from H_0 for the settings in a.) and b.), respectively.

3.3 Robustness to non-normality

The two proposed tests are to a different extent based on model assumptions. Here, the effect of deviations from the assumption of normal distributed B_j´s and U_{ij}´s will be examined regarding the nominal test size and power. Two distributional combinations, either only the B_j´s or both the B_j´s and the U_{ij}´s have the exponential distribution, will be treated. Here, since a correlation between the normally distributed A_j´s and the exponentially distributed B_j´s is complicated to construct, the x_i´s were chosen as equally spaced on the interval $[1,10]$ facilitating the use of $\rho_{AB} = 0$.

Starting with the case when both parameters have the exponential distribution we find that the tests do not hold the nominal test size under H_0, cf. Table 1. Since $\sigma_B^2 = 0$ under H_0 this is solely due to the non-normal distribution of the U_{ij}´s. As can be seen, the T_{F_1} test is affected more than the T_{R^2} test in the studied situations. The nominal test size is exceeded by both tests (with up to 60%), and results from a further examination under H_1 will thus be hard to interpret.

	T_{F_1}	T_{R^2}
$n=20,\ T=5$	0.053	0.050
$n=5,\ T=20$	0.080	0.065

Table 1. The observed test size under H_0 for the nominal test size $\alpha = 0.05$.

However, when only the B_j´s have the exponential distribution the properties of the tests can be studied under $H_0 \cup H_1$. In Figure 3a and 3b the quotients $R(T_{F_1}) = Power(T_{F_1} \mid B_j \sim Exp) / Power(T_{F_1})$ and the corresponding $R(T_{R^2})$ are given. A quotient equal to unity means that the power is not affected at all, which e.g. is true under H_0. For small values of $\sigma_B^2 / \sigma_U^2 > 0$ we can see that the power of the tests in the exponential case exceed the powers in the normal case. We also have that $R(T_{F_1}) < R(T_{R^2})$ for small $\sigma_B^2 / \sigma_U^2 > 0$, but this relation shifts to the opposite for larger departures from H_0. The shift appears earlier for $n = 20, T = 5$ than for $n = 5, T = 20$ and the quotients also approach unity earlier in the previous case. However, the main conclusion is that the powers are not heavily affected by the exponential distribution which can be regarded as an extreme deviation from the symmetric normal distribution. In an applied situation less extreme distributions as lognormal and beta may be at hand, which are likely to affect the power even less.

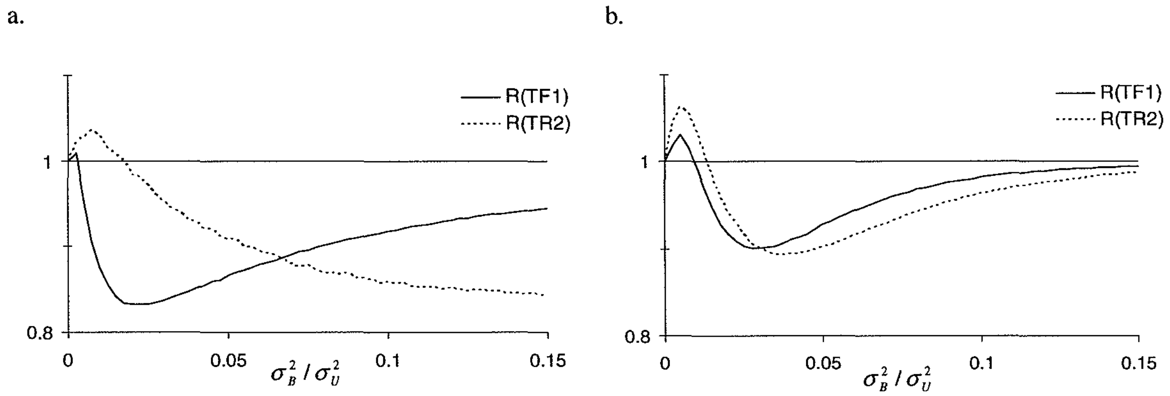

Figure 3. The R-quotients for a.) $n = 5, T = 20$ and b.) $n = 20, T = 5$.

4. Discussion

Two small-sample tests for random coefficients based on subparts of the *MLR* statistic were proposed. One of the tests was equal to the T_{F_1} test proposed by Hsiao [18] for testing the heterogeneity of fixed effects. The explicit connection to the *MLR* statistic found in this paper was not noticed by Hsiao who writes (p. 149): "we can test for random variation indirectly" by using the T_{F_1} test. However, the new result warrants the use of T_{F_1} for testing random coefficients.

To distinguish between the hypothesis where the slopes are assumed to be fixed and different, and the hypothesis where they are assumed to be random variables with a probability distribution, is important. In the former case the inference is conditional on the slopes in the sample while the specific assumptions regarding the distribution of the slopes in the latter case allow an unconditional inference. Because the conditional inference does not make any specific assumptions about the distribution of the slopes, it can be used for a wider range of problems. However, if the restrictive distributional assumption in the unconditional case is correct, this additional information may lead to a more powerful test. The question whether the slopes should be considered as fixed and different or random and different are beyond the scope of this paper but have been discussed by e.g. Mundlak [24] who argues that individual effects should always be treated as random, and by Hausman [15] who proposed a model specification test.

The other proposed test, T_{R^2}, is based on a multiple coefficient of determination derived from the *MLR* statistic. This test utilises more information about the distribution of the parameters. It was found that the T_{R^2} test can be preferable when the number of subjects (n) is relatively large but the number of observations per subject (T) is small. This is a common situation in e.g. routine clinical studies where a large number of patients are measured a few times. Figure 2 indicates that the T_{R^2} test has a larger power for small deviations from H_0 for both combinations of n and T. This is an important property since the power of the tests generally is small near H_0 and all additional contributions to the power are valuable. For larger deviations from H_0 the power of the T_{F_1} then becomes larger, and the shift appears closer to H_0 when T is large.

The tests were for simplicity compared for $p = 1$ in Section 3. Letting $p > 1$ would add relatively more information to the Tr2 test since it also utilise $\sigma_{AB^{(r)}}$, which may increase the power.

The level of the test size has not been discussed in this paper, but it is important to remember that the choice of test size should be guided by the research aim. As discussed by Nelder [26] the tests discussed here can be seen as tests of significant sameness rather than differences. Such tests are relevant in a modeling situation when we are to simplify a complex model by showing that a set of slopes can be replaced by a common slope. We then would like to find a non-significant value of the test statistic for meaningless differences, and hence a small test size is appropriate. This is also the situation when the aim is to predict future observations with small variability where the simpler model under H_0 may be preferable. But if the aim is to describe the data, the more complex model under H_1 may be preferable also for small deviations from H_0. A large test size then helps to ensure that the power of the test is large. The latter is also preferred when testing for poolability of data from different batches of a drug in a drug stability study over time. As discussed in Murphy and Hofer [25] and Ruberg and Stegeman [29] the Type II error is now considered the more serious error. An incorrect pooling of the data may result in unjustifiably long shelf-life, possibly providing the consumer with a drug of reduced potency.

A remark on the use of the tests as pretests followed by a main test has to be done. As noted by Greenland [12], when discussing reanalysis of epidemiologic databases using pretesting in Michalek and Mihalko et al. [23], one has to construct confidence intervals and interpret tests results obtained from a likelihood function chosen by preliminary testing carefully. E.g. it was shown in Sen [30] that pretest estimators potentially have asymptotic non-normality, and in Grambsch and Obrien [10] that the size of the main test can be influenced by the pretest.

The T_{F_1} and T_{R^2} tests were found to complement each other for different situations, and a combined test is thus appealing. Since T_{F_1} and T_{R^2} are subparts of the *MLR* statistic, it can be viewed as the natural combination of the two tests. However, the *MLR* test was examined and some important drawbacks were found. An important extension of this paper would thus be to construct another combined test of the dependent T_{F_1} and T_{R^2} statistics, or some other subparts.

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Appendix I - Definitions of some notations

A. Miscellaneous notations

$$\bar{Y}_j = \frac{1}{T}\sum_{i=1}^{T} Y_{ij}\,,\ \bar{Y} = \frac{1}{n}\sum_{j=1}^{n} \bar{Y}_j\,,\ S_{\bar{Y}\bar{Y}} = \sum_{j=1}^{n}\left(\bar{Y}_j - \bar{Y}\right)^2,\ S_{YY} = \sum_{j=1}^{n}\sum_{i=1}^{T}\left(Y_{ij} - \bar{Y}_j\right)^2,\ \bar{x}^{(r)} = \frac{1}{T}\sum_{i=1}^{T} x_i^{(r)}\,,$$

$$\bar{\mathbf{x}} = \left(\bar{x}^{(1)}\ldots\bar{x}^{(r)}\ldots\bar{x}^{(p)}\right)',\ \mathbf{s}_{xY} = \sum_{j=1}^{n}\sum_{i=1}^{T}(\mathbf{x}_i - \bar{\mathbf{x}})\left(Y_{ij} - \bar{Y}_j\right) = \sum_{j=1}^{n}\mathbf{s}_{xY_j}\,,$$

$$\mathbf{S}_{xx} = \sum_{i=1}^{T}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})',\ \hat{\boldsymbol{\beta}}_j = \left(\hat{\beta}_j^{(1)}\ldots\hat{\beta}_j^{(r)}\ldots\hat{\beta}_j^{(p)}\right)',$$

$$\hat{\boldsymbol{\beta}} = n^{-1}\sum_{j=1}^{n}\hat{\boldsymbol{\beta}}_j = \left(\hat{\beta}^{(1)}\ldots\hat{\beta}^{(r)}\ldots\hat{\beta}^{(p)}\right)'.$$

It can be noted that $\hat{\boldsymbol{\beta}}_j = \mathbf{S}_{xx}^{-1}\mathbf{s}_{xY_j}$ has a p-dimensional normal distribution $N_p(\boldsymbol{\beta}, \boldsymbol{\Sigma}_{BB} + \sigma_U^2\mathbf{S}_{xx}^{-1})$.

B. The dispersion matrix of the regression coefficients

$$\mathbf{S} = \left[\begin{array}{c|c} S_{AA} & \mathbf{s}_{AB}' \\ \hline \mathbf{s}_{AB} & \mathbf{S}_{BB} \end{array}\right] \text{ where } S_{AA} = \sum_{j=1}^{n}\left(\hat{\alpha}_j - \hat{\alpha}\right)^2,\ s\left(\hat{\beta}^{(r)},\hat{\beta}^{(s)}\right) = \sum_{j=1}^{n}\left(\hat{\beta}_j^{(r)} - \hat{\beta}^{(r)}\right)\left(\hat{\beta}_j^{(s)} - \hat{\beta}^{(s)}\right),$$

$$\mathbf{S}_{BB} = \left(s\left(\hat{\beta}^{(r)},\hat{\beta}^{(s)}\right)\right)_{p\times p},\ \mathbf{s}_{AB} = \left(\sum_{j=1}^{n}\left(\hat{\alpha}_j - \hat{\alpha}\right)\left(\hat{\beta}_j^{(1)} - \hat{\beta}^{(1)}\right)\ldots\sum_{j=1}^{n}\left(\hat{\alpha}_j - \hat{\alpha}\right)\left(\hat{\beta}_j^{(p)} - \hat{\beta}^{(p)}\right)\right)'$$

and $\mathbf{s}_{\bar{Y}B} = \left(\sum_{j=1}^{n}\left(\bar{Y}_j - \bar{Y}\right)\left(\hat{\beta}_j^{(1)} - \hat{\beta}^{(1)}\right)\ldots\sum_{j=1}^{n}\left(\bar{Y}_j - \bar{Y}\right)\left(\hat{\beta}_j^{(p)} - \hat{\beta}^{(p)}\right)\right)'$.

From the results in Appendix IA it follows that $\mathbf{S}_{BB} \sim \mathbf{W}_p\left(\boldsymbol{\Sigma}_{BB} + \sigma_U^2\mathbf{S}_{xx}^{-1}, n-1\right)$, i.e. a Wishart distribution with dispersion matrix $\boldsymbol{\Sigma}_{BB} + \sigma_U^2\mathbf{S}_{xx}^{-1}$ and $(n-1)$ df, cf. Srivastava and Khatri [31] p. 78.

C. The total residual sum of squares

$$SSE = \sum_{j=1}^{n}(\mathbf{Y}_j - \tilde{\mathbf{X}}(\alpha|\hat{\boldsymbol{\beta}}_j)')'(\mathbf{Y}_j - \tilde{\mathbf{X}}(\alpha|\hat{\boldsymbol{\beta}}_j)') = S_{YY} - \sum_{j=1}^{n}\mathbf{s}_{xY_j}'\mathbf{S}_{xx}^{-1}\mathbf{s}_{xY_j}\,.$$

It follows from fundamental results in least square theory that SSE is independent of $\mathbf{S}_{BB}$ and that $SSE \sim \sigma_U^2 \cdot \chi^2_{n(T-p-1)}$.

D. The sample multiple coefficient of determination

From the unconditional regression of the $\overline{Y}_j$'s on the $\hat{\boldsymbol{\beta}}_j$'s the sample multiple coefficient of determination is defined as $R^2_{\overline{Y}_j \cdot \hat{\boldsymbol{\beta}}_j} = \dfrac{\mathbf{s}'_{\overline{Y}B}\mathbf{S}^{-1}_{BB}\mathbf{s}_{\overline{Y}B}}{S_{\overline{YY}}}$ (cf. Johnson and Kotz et al. [19] p. 617).

Appendix II – proof of some results

A. Extracting the T_{F_1} test statistic from the MLR

The test statistic proposed by Hsiao [18] p. 18 can be expressed as $T_{F_1} = \dfrac{\left(S_{YY} - \mathbf{S}'_{xY}\hat{\boldsymbol{\beta}} - SSE\right)/p(n-1)}{SSE/n(T-p-1)}$. In Proposition 1 below this statistics is extracted from a subpart of the *MLR* in (3).

Proposition 1: $c_1 T_{F_1} = c_2 \left(\dfrac{\hat{\sigma}^2_U}{\hat{\hat{\sigma}}^2_U}\right)^{-1} - 1$, where c_1 and c_2 are constants.

Proof: Using the estimators in Section 2 we can write the quotient as

$$\left(\frac{\hat{\sigma}^2_U}{\hat{\hat{\sigma}}^2_U}\right)^{-1} = c_2^{-1} \cdot \frac{SSE + \sum_{j=1}^{n}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right)' \mathbf{S}_{xx}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right)}{SSE} = c_2^{-1} \cdot T^*_{F_1}.$$

Considering that $\sum_{j=1}^{n}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right)' \mathbf{S}_{xx}\left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}\right) = \sum_{j=1}^{n}\left(\mathbf{s}'_{xY_j}\mathbf{S}^{-1}_{xx}\mathbf{s}_{xY_j}\right) - \mathbf{s}'_{xY} \cdot n^{-1}\mathbf{S}^{-1}_{xx}\mathbf{s}_{xY}$, $\hat{\boldsymbol{\beta}} = n^{-1}\mathbf{S}^{-1}_{xx}\mathbf{s}_{xY}$ and $SSE = S_{YY} - \sum_{j=1}^{n}\left(\mathbf{s}'_{xY_j}\mathbf{S}^{-1}_{xx}\mathbf{s}_{xY_j}\right)$, it directly follows that $c_1^{-1}\left(T^*_{F_1} - 1\right) = T_{F_1}$. □

B. Extracting the multiple coefficient of determination from the MLR

Since $\mathbf{S}(n-1)^{-1}$ is an unbiased estimator of $\left(\sigma_U^2\left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-1}+\boldsymbol{\Sigma}\right)$ (cf. Rao [27]) we have the equality $\left|\hat{\sigma}_U^2\left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-1}+\hat{\boldsymbol{\Sigma}}\right|=|\mathbf{S}|(n-1)^{-(p+1)}$. It then follows directly from Proposition 2 below that the multiple coefficient of determination $R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}$ can be derived from $\left|\hat{\sigma}_U^2\left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-1}+\hat{\boldsymbol{\Sigma}}\right|$ in (3).

Proposition 2: $|\mathbf{S}|=S_{\bar{Y}\bar{Y}}\left(1-R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}\right)\cdot|\mathbf{S}_{BB}|$

Proof: From Anderson [1] p.40 we have that
$|\mathbf{S}|=S_{AA}\left(1-R^2_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}\right)\cdot|\mathbf{S}_{BB}|=\left(S_{AA}-\hat{\boldsymbol{\beta}}'_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}\mathbf{s}_{AB}\right)\cdot|\mathbf{S}_{BB}|$ where $R^2_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}$ and $\hat{\boldsymbol{\beta}}'_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}=\mathbf{S}_{BB}^{-1}\cdot\mathbf{s}_{AB}$ are the coefficient of determination and the vector of regression coefficients, respectively, from the regression of the $\hat{\alpha}_j$´s on the $\hat{\boldsymbol{\beta}}_j$´s. From regression theory $S_{AA}=S_{\bar{Y}\bar{Y}}-2\bar{\mathbf{x}}'\mathbf{s}_{\bar{Y}B}+\bar{\mathbf{x}}'\mathbf{S}_{BB}\bar{\mathbf{x}}$, $\mathbf{s}_{AB}=\mathbf{s}_{\bar{Y}B}-\mathbf{S}_{BB}\bar{\mathbf{x}}$ and $\hat{\boldsymbol{\beta}}_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}=\mathbf{S}_{BB}^{-1}\left(\mathbf{s}_{\bar{Y}B}-\mathbf{S}_{BB}\bar{x}\right)=\hat{\boldsymbol{\beta}}'_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}-\bar{\mathbf{x}}$ where $\hat{\boldsymbol{\beta}}'_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}=\mathbf{S}_{BB}^{-1}\cdot\mathbf{s}_{\bar{Y}B}$. We may now write
$\hat{\boldsymbol{\beta}}'_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}\cdot\mathbf{s}_{AB}=\hat{\boldsymbol{\beta}}'_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}\cdot\mathbf{s}_{\bar{Y}B}-2\hat{\boldsymbol{\beta}}'_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}\cdot\mathbf{S}_{BB}\cdot\bar{\mathbf{x}}+\bar{\mathbf{x}}'\mathbf{S}_{BB}\bar{\mathbf{x}}$, which finally gives
$\left(S_{AA}-\hat{\boldsymbol{\beta}}'_{\hat{\alpha}_j\cdot\hat{\boldsymbol{\beta}}_j}\mathbf{s}_{AB}\right)|\mathbf{S}_{BB}|=\left(S_{\bar{Y}\bar{Y}}-\hat{\boldsymbol{\beta}}'_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}\cdot\mathbf{s}_{\bar{Y}B}\right)|\mathbf{S}_{BB}|=S_{\bar{Y}\bar{Y}}\left(1-R^2_{\bar{Y}_j\cdot\hat{\boldsymbol{\beta}}_j}\right)|\mathbf{S}_{BB}|$. □

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