ABSTRACT

We present four papers. The common theme is real-rootedness and unimodality of polynomials occurring in combinatorics.

In the first paper we introduce a new class of labeled posets, *sign-graded posets*, which contains the class of graded naturally labeled posets. We prove that the Eulerian polynomial (W-polynomial) of a sign-graded poset has symmetric and unimodal coefficients. This verifies the motivating consequence of the Neggers-Stanley conjecture on real zeros for this class of posets. It also extends a recent result of Reiner and Welker who proved unimodality of the Eulerian polynomials of naturally labeled graded posets by associating to each graded poset a simplicial polytopal sphere. By proving that the Eulerian polynomial of a sign-graded poset has the right sign at -1 we are able to prove the Charney-Davis conjecture for these spheres (whenever they are flag).

In the second paper we refine a technique used in a paper by Schur, from 1914, on polynomials with real zeros. This amounts to an extension of a theorem of Wagner on Hadamard products of Pólya frequency sequences. We also apply our results to polynomials for which the Neggers-Stanley conjecture is known to hold. More precisely, we settle interlacing properties for E-polynomials of series-parallel posets and column-strict labeled Ferrers posets.

The third paper is a continuation of the second. It is concerned with linear operators that preserve the Pólya frequency property and real-rootedness. We apply our results to settle some conjectures and open problems in combinatorics proposed by Bóna, Brenti and Reiner-Welker.

In the last paper we provide the first counterexamples to the Neggers-Stanley conjecture on real zeros. This conjecture asserts that the polynomial whose coefficients count the linear extensions of a labeled partially ordered set by their number of descents has real zeros only. We provide a family of labeled posets such that for any integer $M > 0$ there is a labeled poset whose corresponding polynomial has more than M non-real zeros.

Keywords and phrases: Neggers-Stanley conjecture, partially ordered set, linear extension, Eulerian polynomial, real zeros, unimodality, Pólya frequency sequence

2000 Mathematics Subject Classification: 06A07, 26C10, 05E99, 13F55, 05A15, 05A19, 05A05, 20F55