

THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

**Learning to solve problems
that you have not learned to solve**

Strategies in mathematical problem solving

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Tack!

För tre fina år!
Vi har under
dessa fyra kurser
lärt oss mycket,
framför allt att
gruppera data!

Erik L Johannes
Erik Samuel

Kajsa

Marcus

Gabriella Axel Ek

Erik B

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All
Eriksson

Carl

Eva

Mohamad Patrik

Abstract

This thesis aims to contribute to a deeper understanding of the relationship between problem-solving strategies and success in mathematical problem solving. In its introductory part, it pursues and describes the term strategy in mathematics and discusses its relationship to the method and algorithm concepts. Through these concepts, we identify three decision-making levels in the problem-solving process.

The first two parts of this thesis are two different studies analysing how students' problem-solving ability is affected by learning of problem-solving strategies in mathematics. We investigated the effects of variation theory-based instructional design in teaching problem-solving strategies within a regular classroom. This was done by analysing a pre- and a post-test to compare the development of an experimental group's and a control group's knowledge of mathematics in general and problem-solving ability in particular. The analysis of the test results show that these designed activities improve students' problem-solving ability without compromising their progress in mathematics in general.

The third study in this thesis aims to give a better understanding of the role and use of strategies in the mathematical problem-solving processes. By analysing 79 upper secondary school students' written solutions, we were able to identify decisions made at all three levels and how knowledge in these levels affected students' problem-solving successes. The results show that students who could view the problem as a whole while keeping the sub-problems in mind simultaneously had the best chances of succeeding.

In summary, we have in the appended papers shown that teaching problem-solving strategies could be integrated in the mathematics teaching practice to improve students mathematical problem-solving abilities.

Keywords: Problem-solving strategies, problem-solving ability, variation theory, design principles, classroom teaching, design-based research (DBR)

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The following papers are included in this thesis:

Paper I: Fülöp, E. (2015). Teaching problem-solving strategies in mathematics. *LUMAT*, 3(1), 37-54.

Paper II: Fülöp, E. (under review) Developing problem-solving abilities by learning problem-solving strategies: An exploration of teaching intervention in authentic mathematics classes

Paper III: Fülöp, E. (under review) Connections between chosen problem-solving strategies and success in mathematical problem solving

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1. Introduction

1.1 Area of interest

Improving students' problem-solving skills is a major goal for most mathematic educators. In the preface to the first printing of the book "How to Solve It" George Pólya (1945) wrote:

"Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic deductive science; but mathematics in the making appears as an experimental inductive science. Both aspects are as old as the science of mathematics itself. But the second aspect is new in one respect; mathematics 'in statu nascendi', in the process of being invented, has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public." (Quoted from the 1957 (2nd) edition, p. vii.)

Problem solving has since then emerged as one of major concerns at all levels of school mathematics, becoming a key component in the teaching, learning and mastering of mathematics. Since much of the computational aspects of mathematics now a day can be handled more effectively by computers than humans, there is an increasing need to focus on aspects of problem-solving where the human intellect is most important.

Hence the point of departure for this work is that problem-solving is, and will remain to be, an essential part of the mathematical competence. Therefore, it is relevant to ask the following question: How can we teach students to solve problems in mathematics that they haven't learned to solve? This question has been around as long as problem-solving has been part of the mathematics education, but finding the answer is far from trivial. In problem-solving the general idea is that one should be able to do something that one in beforehand

does not know how to do. This is very different from for example teaching a student how to take the derivative of a function or to solve a standard equation. The general idea in problem-solving is that you don't know how to solve it. If you did it would not be a problem for you. Hence there is, by definition, no list of steps to teach a student that always will give a solution to their mathematical problems.

There have been many different approaches to solve this dilemma. As an example, in the 1980's John Mason wrote about the teaching approach where the teacher acts as a role model in problem-solving. However, he finds that that this does not come natural for all mathematics teachers.

“John naively assumed that all mathematics tutors would ‘be mathematical with and in front of their students’ and so would naturally get students specializing and generalizing, conjecturing and convincing and so on. It took some years before he realized that not all tutors were as self-aware of their own mathematical thinking as he had assumed. The result was a series of training sessions for tutors, designed to get them to experience mathematical thinking for themselves and to reflect on that experience so as to be able to draw student attention to important aspects.”
(Mason, Burton & Stacey, 2010, p. Xiii)

The question above has a number of related questions, such as: What is a mathematical problem? Which are the essential problem-solving competencies (or abilities)? How does one become a competent mathematical problem-solver? The past 40 years were a productive period in research of problem solving in school mathematics (Lester, 1994, 2013; Schoenfeld, 1985, 1992, 2013; Mason, Burton & Stacey, 2010; Cai, 2010; Lester & Cai, 2015; Kilpatrick, Swafford & Findell, 2001; Niss and Højgaard Jensen, 2011). Indeed, much has been learned but much remains to be understood.

In this thesis the focus is on the following related sub-questions: Can mathematical problem-solving strategies be taught? What role does knowledge in mathematical problem-solving strategies play for the

mathematical problem-solving ability and in the problem-solving process? Hence, we want to know how knowledge about problem-solving strategies helps to find new approaches for solving problems and develop students' problem-solving abilities.

However, there is remarkably little agreement on what strategy in mathematical problem-solving is. Therefore, we will discuss what *problem-solving strategy* in mathematics is and what the difference is between the concept *strategy* and the concepts *method* and *algorithm*? Furthermore, we are interested in understanding what is essential when learning about problem-solving strategies and what learning approaches could be used to become successful at using strategies, and what teachers could do in classrooms to reach this goal.

1.2 Purpose and aim of the thesis

The purpose of this thesis is to contribute to a better understanding of how the teaching of problem-solving strategies in mathematics can be organized in a regular classroom setting in upper secondary school without altering the mathematical content. Furthermore, we look at the role of knowledge about problem-solving strategies in the development of the students' problem-solving ability. This is done by (1) identifying what is known about the concept strategy and its relationship to the concepts method and algorithm, (2) developing design principles with the goal to teach problem-solving strategies in mathematics and (3) studying how the knowledge of problem-solving strategies effects the students' problem-solving ability.

The hope is that, knowledge about this can be useful both when specifying the goals and aims of the teaching of mathematical problem-solving, likewise when designing curricula and instruments for formative or summative assessment. One expected takeaway for teachers will be to three design principles exhibited here, to be use in the teaching of problem-solving strategies in mathematics.

1.3 Structure of the thesis

This thesis is organized in five parts. The second chapter introduces the concepts of problem-solving abilities and problem-solving strategies as parts of mathematical knowledge. This includes a background discussing how the strategy-concept has been treated in

different areas and clarifying the difference between strategy, method and algorithm in a problem-solving situation in mathematics. Thereafter follows an explanation how the concept strategy is used in this report. This chapter also includes a presentation of variation theory, the design framework. The Methodology chapter includes descriptions and motivation of the study design and the methods for data analysis. After that follows a chapter where you will find a summary of the appended papers. Their results and their implication are discussed in the last chapter. At the end of the thesis, the three papers are included.

2. Conceptual background

Before we begin to discuss how we teach mathematics, we need first to agree on what we want students to learn. Besides the considerations concerning subject content, this agreement must build on our answer to the following questions: What are the ingredients of mathematical knowledge and how can this knowledge be organized and represented? Thereafter it is relevant to discuss questions like: How do students learn mathematics and how should they be taught? Questions about what knowledge in mathematics is, which type of knowledge is more important or what might be an appropriate balance between them, are important to ask. A detailed description of knowledge in mathematics can give some guidance when deciding how to teach, what to focus on, how to make assessment and how to describe and analyse students' knowledge and abilities in a systematic way. For this purpose, a variety of historical and contemporary views and conceptualizations of what it means to master mathematics are presented in this chapter.

2.1 Historical and contemporary views of knowledge in mathematics and theoretical analyses of the notions

“Formal mathematics is like spelling and grammar – a matter of the correct application of local rules. Meaningful mathematics is like journalism –it tells an interesting story. Unlike some journalism, the story has to be true. The best mathematics is like literature –it brings a story to life before your eyes and involves you in it, intellectually and emotionally.” (Courant & Robbins, 1996, preface to second edition)

What does it mean to master mathematics? Over the past century considerations of mathematical knowledge have taken different forms using different labels. Already in the 1940s mathematicians and mathematics educators pointed to other significant aspects of mastery of mathematics besides factual and procedural knowledge or computational skill. In the early 1960s, the IEA, the International Association for the Evaluation of Educational Achievement (which

later conducted the TIMSS studies), identified five cognitive behaviours: knowledge and information (recall of definitions, notation, concepts); techniques and skills; translation of data into symbols or schema or vice versa; capacity to analyse problems, and reasoning creatively in mathematics (Husén, 1967).

National Council of Teachers of Mathematics 1989 identified five ability or attitude oriented goals for the teaching of mathematics: (1) that students learn to value mathematics, (2) that students become confident in their ability to do mathematics, (3) that students become mathematical problem solvers, (4) that students learn to communicate mathematically, and (5) that students learn to reason mathematically.

Indicating also the mathematical knowledge complexity, the Pentagon Model of the Singapore Mathematics Curriculum Framework (SMCF), published in 1990, emphasizes not only the content to be taught but also the processes and affective aspects of learning mathematics. Aspects such as concepts, processes, metacognition, attitudes, skills and mathematical problem solving link it all together. Finally we also want to mention that the Australian Education Council published in 1994, in the document “Mathematics: a curriculum profile for Australian schools”, in which outcomes of working mathematically were specified, and mathematical ability was subdivided into the areas: investigating, conjecturing, using problem-solving strategies, applying and verifying, using mathematical language, and working in context.

2.2 Competencies and proficiency in the mastery of mathematics

Since then much work has been done to develop notions such as mathematical competencies, capabilities, proficiencies and abilities and some attempts to specify the nature of the competency have been done (Niss et al., 2016). We will now look at three influential models published in the beginning of the millennium, all seeing mathematical knowledge as competence/ proficiency and teaching as creating opportunities to experience and exercise competencies. In the report “*Adding it up*” (American project), sponsored by the National Science Foundation and the U.S. Department of Education

and edited by Kilpatrick, Swafford, & Findell (2001), there is a model consisting of five strands of mathematical proficiency.

“Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we think it means for anyone to learn mathematics successfully.” (Kilpatrick, Swafford & Findell, 2001, p. 5)

Table 1. A summary of the American model’s definitions of the proficiencies

Proficiency	Definition of mastery
conceptual understanding	comprehension of mathematical concepts, operations, and relations
adaptive reasoning	capacity for logical thought, reflection, explanation, and justification
strategic competence	ability to formulate, represent, and solve mathematical problems
procedural fluency	skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
productive disposition	habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy

About the same time, the report *“Matematik och kompetenser”* (Danish KOM project), commissioned by the Danish state and with editors Niss and Højgaard Jensen (2002), suggested a model which consisted of eight competencies in mathematics.

“Mathematical competence means to have knowledge about, to understand, to exercise, to apply, and to relate to and judge mathematics and mathematical activity in a multitude of contexts which actually do involve, or potentially might involve, mathematics.” (Niss and Højgaard Jensen, 2002, p. 43)

Table 2. A summary of the Danish model’s definitions of the competencies

Competency	Definition of mastery
mathematical thinking	pose such questions and be aware of the kinds of answers available
reasoning	the ability to understand, assess and produce arguments to solve mathematical questions
problem tackling	answer questions in and by means of mathematics
modelling	the ability to structure real situations; being able to analyse and build mathematical models, at the same time being able to assess their range and validity
representing	being able to deal with different representations of mathematical entities, phenomena and situations
aids and tools	being able to make use of and relate to the diverse technical aids for mathematical activity
symbol and formalism	being able to deal with the special symbolic and formulaic representations in mathematics

communicating	being able to communicate in, with and about mathematics
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The eight competences in the Danish model can be divided in to two distinct groups, the ability to ask and answer questions in and with mathematics, and to deal with mathematical language and tools.

There are some more conspicuous differences between these models. The American model has some new perception on mathematical knowledge by speaking about *Productive disposition* as a proficiency. It may be unorthodox to consider a positive attitude towards mathematics as a skill in itself, a skill that is developed in interaction with the others, but it highlights the importance of the students' attitude towards both mathematics and their own knowledge. Another difference is that there is no classification of communication or modelling competences in the American model, but it emphasizes procedural fluency, which is not explicitly incorporated into the Danish classification as a competency.

Table 3. Comparing the two models.

American model	Danish model
conceptual understanding	Mathematical thinking competency
adaptive reasoning	reasoning competency
strategic competence	problem tackling competency
	modelling competency
	representing competency
	aids and tools competency
	symbol and formalism competency
procedural fluency	
	communicating competency
productive disposition	

Looking at the similarities, one finds that according to both models different mathematical proficiencies/competencies provide a wider view of mathematics learning, and the teachers' job should be to help students develop this mathematical proficiency/competency. It does not seem as important to distinguish the competencies from each other as it is to integrate them. Both models emphasize that the students' mathematical knowledge is not complete if either kind of competency is deficient or if they remain separate entities.

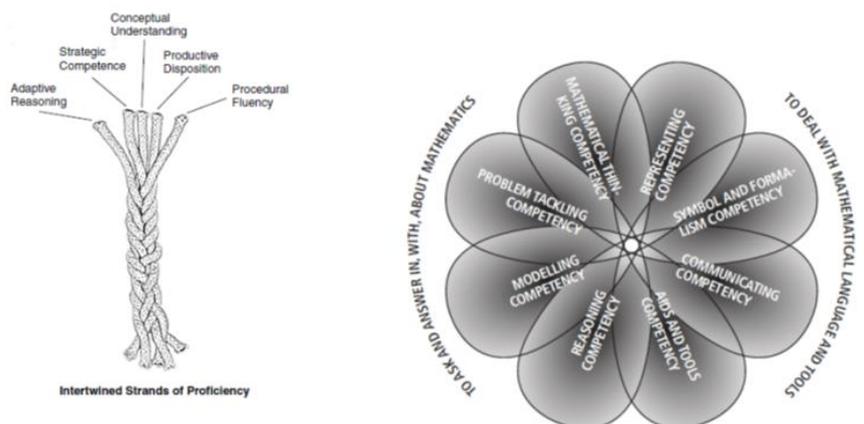


Figure 1. Visual representations of mathematical competencies of the American and Danish models. Figures reprinted with permission from (Kilpatrick, Swafford & Findell, 2001) and (Niss, 2015).

A visual representation of both models shows very clearly that the mathematical competencies and proficiencies are connected to each other within both models:

“Our analyses of the mathematics to be learned, our reading of the research in cognitive psychology and mathematics education, our experience as learners and teachers of mathematics, and our judgment as to the mathematical knowledge, understanding, and skill people need today have led us to adopt a composite, comprehensive view of successful

mathematics learning.”(Kilpatrick, Swafford & Findell, 2001, p.115)

The theoretical framework MCRF (Mathematical Competency Research Framework) inspired by the above mentioned studies, is a framework developed for analysis of empirical data concerning mathematical competencies (Lithner et al., 2010). The framework MCRF is intended to be used as well to develop teaching in mathematics. It can be used to analyse textbooks, tasks and how the competences are made visible in teaching. MCRF defined six competencies: problem solving competency, reasoning competency, procedural competency, representation competency, connection competency, communication competency.

A very important note is that the competencies above can only be held, or discussed, in relation to mathematical content. The point is, however, that each of the competencies can have meaning in relation to any mathematical content. This is actually what gives them their general character.

Most important for this thesis is that all these models contain aspects of problem-solving, called strategic competence, problem tackling competence and, problem-solving competency respectively. All these three models list a number of skills that problem-solving competency consists of, having a common item: the mastery of problem-solving strategies. A good problem-solver's strategic competence includes knowledge to develop strategies for solving non-routine problems, according Kilpatrick, Swafford & Findell (2001), while the problem tackling competency, according to Niss and Højgaard Jensen (2002) focuses on the strategies one can use to answer the questions. The problem-solving competency according to Lithner et al. (2010) includes mastery of applying and adapting various appropriate strategies and methods. All these models highlight the importance of analysis of similarities and differences between strategies and also the ability to represent the problem in different ways when necessary or desirable.

2.3 What is a problem and what is problem solving in mathematics?

In all of the models above a problem is defined as the opposite to a routine task or routine skill. It requires the problem solver to make a special effort to find a solution. In other words, the problem solver does not have easy access to a procedure for solving a problem but does in fact have an adequate background with which to make progress. Furthermore, the person wants to solve the problem and works actively on it (Schoenfeld, 1985; Kilpatrick, 2013; Lester & Cai, 2015).

“In simplest terms for us a mathematics problem is a task presented to students in an instructional setting that poses a question to be answered but for which the students do not have a readily available procedure or strategy for answering it”
(Lester & Cai, 2015, p 8)

Another possible way to define a problem is from the perspective of the teacher. “Rich problems” defined by Taflin (2007) are problems that meet certain conditions. This type of definition focuses on creating discussion and learning possibilities for the students. There are many arguments for why and how students should solve problems. When students are solving problems, it is also essential to distinguish factors that do not have to do with the mathematical solution of the problem, for example to practice mathematical reasoning or creativity.

Much of the research in mathematical problem solving has focused on the thinking processes used by individuals as they solve problems or as they reflect back up on their problem-solving efforts (Pólya, 1973; Lester, 1994; Schoenfeld 1979, 1983, 1985, 1992; Mason, Burton & Stacey, 2010). In some cases, steps required when solving a problem are described. The most well-known of these ideas are the steps identified by Pólya. He identified four basic steps in problem solving: *understand the problem, devise a plan, carry out the plan* and *look back*. The last step is probably the most talked about and the least used. Pólya takes it as given that students’ experience with mathematics must be consistent with the way mathematics is done by mathematicians. It is essential to understand Pólya's conception of mathematics as an activity.

Mason, Burton & Stacey, (2010) separate “Entry”, the thinking phase of the problem-solving process, from the “Attack” phase in which the central activity is conjecturing. “A *conjecture is a statement which appears reasonable, but whose truth has not been established.*” (Mason, Burton & Stacey, 2010, p. 58). During the Attack different approaches are taken, and several plans are formulated and tried out. Those activities depending on whether it provokes “being stuck” or “aha” experiences, which either can lead back to a prior phase or to the next phase, “Review”, the reflecting phase. But what is more apparent, compared with Pólya’s four phases, is the highlighting of the cyclic nature of the problem-solving process.

Schoenfeld (1983, 1992) characterizes some of the defining properties of decision-making in problem-solving situations using the concepts “strategic” and “tactical” decision. He writes about strategic decisions which include selecting goals and deciding on what course of action to pursue, affecting the direction of the problem-solving process. In short, they are decisions about what to do, what direction to take while working on a problem. Once such a strategic decision has been made, a decision about how to implement that choice follows. These “how to do” decisions he calls tactical choices. This characterization highlights the importance of metacognition in the problem-solving processes, giving special attention to the knowledge of the heuristic problem-solving strategies, as one fundamental aspects of thinking mathematically. Schoenfeld argues that domain knowledge interacts with other aspects of problem-solving activities such as strategy use, control and beliefs.

2.4 Development of the concept of problem-solving strategies in mathematics.

The concept of strategy is used in many different areas, such as military theory, business management, game theory, sports, artificial intelligence and in the area of interest for this thesis, mathematical problem-solving.

Playing a game means to select a particular strategy from a set of possible strategies (Zagare, 1984). Strategies are the different options available to players to bring about particular outcomes. In

game theory, strategies can be decomposed into a sequence of decisions called *choices*, made at various decision points called *moves*. Decision theory is often used in the form of decision analysis, which shows how best to acquire information before making a decision. Decision theory is closely connected to game theory, which is formally a branch of mathematics developed to deal with conflict of interest situations in social science (Zagare, 1984).

In military theory (Vego, 2012), *strategy* is a set of ideas implemented by military organizations to pursue desired goals. In contrast, the disposition for and control of military forces and techniques in actual fighting is called *tactics*. Finally, the third level in military theory is the so called *operational level*, which describes how the troops execute operational tasks based on the tactics when the battle has begun. There is a clear hierarchy between these three concepts describing different phases and aspects of war. Essentially, *strategy* is the thinking aspect of organizing war or planning a change by laying out the goals and the ideas for achieving those objectives. Strategy is not a detailed plan or program of instruction. It rather gives coherence and direction to the actions and decisions and can comprise numerous tactics. In contrast to strategy, the *tactics* are the doing aspect that follows the directions, a schema for a specific action. In other words, it is about how people will act on the operational level to fulfil the strategy. According to Vego (2012) wars at sea are won or lost at the strategic and operational levels. With that he emphasizes the importance of the strategy making.

Business can be compared with war. Companies are struggling to survive in a hostile environment, fighting against competitors. In management theory we can see an evolution from corporate planning to strategic management. This was a result of the macroeconomic instability and increased international competition during the 1970's, that made it impossible to forecast and to see far into the future and make corporate planning five years ahead.

So, what is strategy? There is actually remarkably little agreement on what strategy is and generally there is a lack of common definitions of the concept also within any of the above areas. For example, in the world of management there are many diverging views. Andrews (1971), Harvard Business School Professor and

father of Corporate Strategy did not give a detailed description of what strategy is. Instead he argued that “*every business organization, every subunit of an organization, and even every individual should have a clearly defined set of purposes or goals which keeps it moving in a deliberately chosen direction and prevents its drifting in undesired directions.*” Andrews (1971, p.23). Grant (2008) on the question What is strategy? gives the following answer: “*..strategy is the means by which individuals or organizations achieve their objectives. By “means” I am referring not to detailed actions but the plans, policies and principles that guide and unify a number of specific actions*” Grant (2008, p.17).

What seems to be a common aspect is that strategy has to do with high-level decisions. According to Schoenfeld (1983) the core concept behind problem solving is decision-making. He characterized some of the defining properties of decision-making using the concepts strategy and tactics. “*Let us define a heuristic strategy as a general suggestion or technique which helps problem-solvers to understand or to solve a problem... We can think of a heuristic strategy as a "key" to unlock a problem.*” (Schoenfeld, 1980, p.798). For that reason, to become a good problem solver in mathematics one needs to develop a personal collection of problem-solving strategies (Schoenfeld 1985). The second level of decisions, the tactical level, includes the decisions about how to implement the chosen strategy, but in the end, the students need to apply the procedures relevant for the solution of the problem.

From a more practical aspect, Pólya (1945, 1962) and Posamentier & Krulik (1998) present ad hoc examples of strategies, but without giving a general definition or general characteristics of strategies. Posamentier and Krulik (1998) present ten problem-solving strategies in mathematics which seem to be prevalent. They argue for the importance of familiarizing both teachers and students with these strategies until they become a part of their thinking process. The strategies mentioned in the book are *visualization, organizing data, finding a pattern, solving a simpler analogous problem, working backwards, adopting a different point of view, intelligent guessing and testing, logical reasoning, and considering extreme cases*. However, this is not a comprehensive list. Other books include other examples of strategies. In some cases the authors use

the term method, but the meaning behind it seems to be akin to strategy, as we will be define it below. One aspect of strategies is that their applicability is not restricted to a particular topic or subject matter in mathematics.

2.5 Conceptualization of problem-solving strategy and model of problem-solving process in this thesis. Extended framework

In this thesis, based on the above definitions (Section 1.3), we define a problem as a challenge for which the solver does not have direct access to a method or an algorithm which give the solution. We make a distinction in this thesis between three concepts in mathematical problem-solving, namely *strategy*, *method* and *algorithm*.

To begin with, a problem-solving strategy is a general, flexible and overarching manner in which to solve problems. By general we mean that is not domain specific, instead a problem-solving strategy is applicable in all, or at least in many different areas of mathematics, and even outside of mathematics. That a problem-solving strategy is overarching means that it focuses on the goal, the problem as a whole and the overall direction of the problem-solving. Flexible means that it is not a detailed plan but rather allows for several different ways to proceed.

Choosing a strategy imposes some restrictions on how to proceed. Instead of having all possible options available, the strategy introduces high level limitations. This could lift creativity and recognition as similar situations encountered before may come to mind. If the problem solving is fruitless then the problem solver has the option to go back and choose another strategy.

In contrast, we have the concepts of method and algorithm. An algorithm is a predefined set of steps which are followed more or less blindly, involving no uncertain decisions. The relationship to the goal is not considered until the algorithm is completed. A method is a set of ideas and tools that narrow down the possible ways to proceed depending on the specific domain of mathematics. A method involves progressive transition, the initiative of a leading idea through arranging or combining what is otherwise discrete and

independent in accordance to the goals. A method contributes regularity, repeatability and predictability but does not mechanize.

Hence, strategy belongs to the thinking aspect of the problem-solving process, while the algorithm constitutes the doing aspect of the problem solving, describing step by step how to proceed to get an answer. The method is a bridge between the thinking and doing aspects, a set of doing sequences, a description of a systematic way of accomplishing the goal of the problem, which still has a creative aspect with decision possibilities. It is important to note that, in this thesis, problem solving is seen as a series of decisions. These decisions we categorize into three levels: strategy making, choice of method and choice of algorithm. A problem solver can move back and forth between these three levels as the need arises.

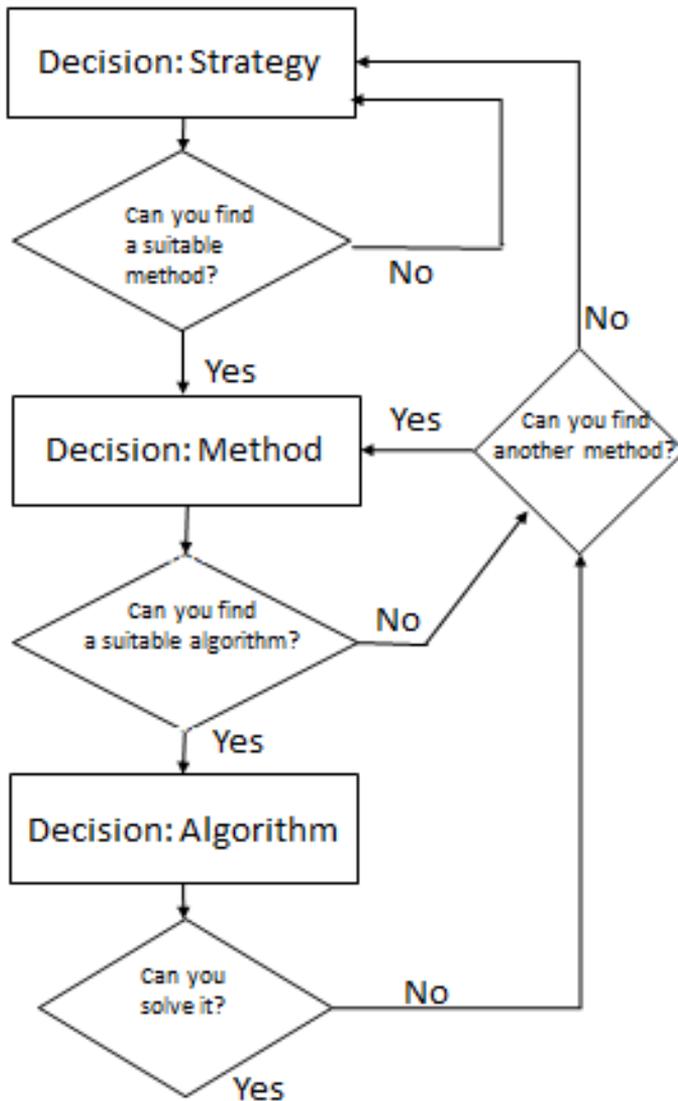


Figure 2. A visualization of the levels of decision making in the problem solving in mathematics described above.

Let us now look at a well known mathematical task that is often used and considered a suitable problem for younger students with the right

background, and use it to exemplify the difference between strategy, method and algorithm. The task is the following:

Calculate the sum of the first 20 odd numbers

$$1+3+5+7+9+11+\dots+37+39=?$$

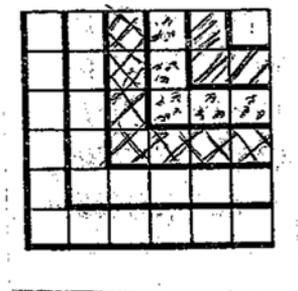
For each strategy chosen below there will follow a choice of method and algorithm.

Strategy 1: *Visualisation*

Each term in the sum will be visualized. Having the goal in mind we want both the terms and the total sum to be visible.

Method 1

We place squares so that they form larger and larger squares together. First, we have one square that corresponds to the number 1, then we add three more squares. In this way we get a 2x2 square followed by a 3x3, 4x4 square and at the end we have got 20x20 squares.



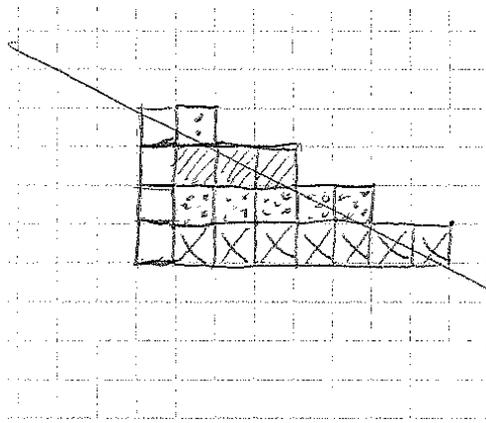
Algorithm

There are not many steps in the algorithm. As the result is a big square with 20x20 small squares this means that the sum consists of 400 squares. This can easily be generalized to some of the first n odd numbers giving that the sum will be

$$n \times n = n^2$$

Method 2

This time, we place squares in a different way, namely under each other, forming a triangle.



Algorithm

Now we need to find an algorithm to count the squares. The height of the triangle includes as many squares as the number of terms added. The base of the triangle contains one square less than twice the number of added terms. We add a column with squares to find an algorithm for calculation of the total number of squares and ultimately the sum of the first 20 odd numbers. In this way the height of the triangle below the line offers still as many squares as the number of terms added but the base of the triangle becomes twice as many as the number of terms. Of course, we should not forget that we have added a certain number of squares and they need to be removed also in the end.

$$\frac{2n \times n}{2} + n - n = n^2$$

Strategy 2: Grouping data

The strategy here is to group the terms so that the sum of the values in the groups can be easily described.

Method 1

We group the first number with the last, then the next number with the second-last, and so on. We finally get half as many pairs as numbers added.

$$1 + 3 + 5 + \dots + 35 + 37 + 39 = (1 + 39) + (3 + 37) + (5 + 35) + \dots + (19 + 21)$$

$$1 + 39 = 40$$

$$3 + 37 = 40$$

$$5 + 35 = 40$$

.....

Algorithm 1

The sum of all pairs giving the same results namely 40. We get the result by multiplying 40 by the number of pairs in this case $40 \times 10 = 400$. In this way calculating the sum of the first 20 odd numbers.

Or generally if we add an even number of odd numbers.

$$2n \times n/2 = n^2.$$

Algorithm 2

If we add an odd number of odd numbers, we need to choose another algorithm giving special treatment to the middle element that does not fit into any pair.

$$2n \times (n - 1)/2 + n = n^2$$

Method 2

This time we group the data in a different way than in *Method 1*. Each number is written as the sum of ones and tens.

$$1 + 3 + 5 + \dots + 35 + 37 + 39 = (1 + 3 + 5 + 7 + 9) + (1 + 3 + 5 + 7 + 9) + 10 \times 5 + (1 + 3 + 5 + 7 + 9) + 20 \times 5 + (1 + 3 + 5 + 7 + 9) + 30 \times 5$$

In the end we add first the ones and then the tens.

$$= (1 + 3 + 5 + 7 + 9) \times 4 + (10 + 20 + 30) \times 5 = 100 + (10 + 20 + 30) \times 5 = 400$$

Strategy 3: Solving a simpler analogous problem and Finding a pattern

The strategy now is to find a similar but simpler problem to derive a hypothesis that we can check or prove.

Method

An obvious simplification is to look at the sum of the first two odd numbers:

$$1 + 3 = 4 = 2^2$$

We continue to look at the sum of the first three odd numbers and compare with the previous case.

$$1 + 3 + 5 = 9 = 3^2$$

We can see a pattern emerging so we check with the next problem which is to add the first four odd numbers if the answer is going to be the quadrat to the number of added odd numbers.

$$1 + 3 + 5 + 7 = 16 = 4^2$$

This strategy gives us an idea about the answer:

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

Algorithm

To prove the hypothesis we choose to use induction over n .

1. The basis (base case): to prove that the statement holds for the natural number $n = 2$ or $n = 3$. We see that already that is true.
2. The inductive step: to prove that, if the statement holds for some natural number n , then the statement holds for $n + 1$.

$$1 + 3 + 5 + \dots + (2n - 1) + (2n + 1) = (n + 1)^2$$

$$n^2 + 2n + 1 = (n + 1)^2$$

Strategy 4: Finding a pattern

Method

Referring to the fact that the difference between two successive terms is constant we note that we have an arithmetic series where $a_1 = 1$ is the first term, $a_n = 39$ is the n^{th} term of the sequence, $d = 2$ is the common difference and $n = 20$ is the number of the term.

Algorithm

This sum can be found quickly by taking the number n of terms being added (here 20), multiplying by the sum of the first and last number in the progression (here $1 + 39 = 40$), and dividing by 2:

$$S_n = \frac{n(a_1 + a_n)}{2} = 400$$

2.6 Teaching problem solving and problem-solving strategies

“If we want students to use them, we must describe them in detail and teach them with the same seriousness that we would teach any other mathematics” (Schoenfeld, 1980, p.795)

Pólya’s book *How to solve it* (Pólya, 1973) and later Schoenfeld’s book *Mathematical Problem Solving* (Schoenfeld, 1985) singled out heuristics and problem-solving strategies. Both argued that, with the right kind of help, students could learn to employ problem-solving strategies and become better problem solvers. Schoenfeld (1992, 1985) defined four categories of problem-solving activities which are necessary and sufficient for the analysis of the success of someone’s problem solving. In his book he paid special attention to understanding how students solve problems as well as how problem solving should be taught. However, this framework has some limitations. Schoenfeld made his analysis of problem solving in a lab environment, not in a regular classroom. Furthermore, the framework did not offer a theory of problem solving, it did not explain how and why the problem solvers made the choices they did.

The understanding and teaching of Pólya’s strategies is then not seen as a theoretical challenge but as an empirical question. Assuming that problem solving is goal-oriented decision making, the new challenge for Schoenfeld (2011) was to build a theory of problem

solving. The role of goals in decision making is a central component in this theory. The basic structure of the general theory is that the individuals, on the basis of their beliefs and available resources, make decisions to pursue their goals. Goal-oriented behaviour is building on available knowledge and on the making of decisions in order to achieve outcomes that you value. The initial questions for his research are not just how issues of learning and development of problem solving can be incorporated into a theory of decision making, but also how students could learn it in complex and knowledge-intensive social environments such as a classroom.

I agree with Schoenfeld, that there is a need for concrete teaching projects that can be used to integrate core concept development with problem solving in mathematics education. It is important to find ways to organise the classroom practice to make problem-solving learning possible for students without losing focus on the mathematical content. We need alternative approaches different from the traditional where concepts, procedures and a repertoire of problem-solving strategies are be taught first, then practiced through problem solving.

During recent decades, there has been an increased interest in teaching methods with the focus on problem solving and whole-class discussions. A reconceptualization of mathematics education as a design science was needed (Lesh and Sriraman, 2005; Schoenfeld, 2010) because much work in mathematics education was, and still is, ideologically driven. Since the classroom “sets the scene” (Niss, 2018) for the mathematical learning experiences, it is important to understand which factors have an impact on students’ learning. Research shows that the didactical contract (Brousseau 1997), the sociomathematical norms established in a classroom (Yackel & Cobb, 1996; Yackel & Rasmussen, 2002; Niss et al., 2016; Niss, 2018) and the dynamic interaction between mathematical concepts and the processes used to solve problems (Lester, 2013; Lester and Cai, 2015) can be important factors.

According to Lester (2013) heuristics and awareness of one’s own thinking develops concurrently with the understanding of mathematical concepts. Problem solving should be an activity which demands the students’ engagement in different cognitive actions in

which metacognition is one of the driving forces. Breaking the isolation of problem solving from other forms of mathematical activity is important. Lester notes that whatever approach the teacher uses, “teaching for problem solving” as an ends approach or “teaching via problem solving” as a means approach, they have to make some decisions anyway. Teachers have to decide which problems to use and how much guidance to give to students. The research to find teaching practices that foster and sustain problem solving activities has been going on for decades.

Rich math problems according to Taflin (2007) create opportunities for learning problem solving. These problems are constructed for mathematics education in a school context. Presenting rich problems in the classroom and holding a joint review at the end of the lesson are ways in which students and teachers together create occasions to utilize known and new mathematical ideas.

Using rich problems allows the teacher to assume other roles than in the traditional approach. An important role involves leading discussions by asking questions, answering questions and looking for interesting solutions. While solving rich problems, the students can show which specific mathematical idea they could apply, but also what they lack to be able to work on the problem. In this way the teacher gets a better understanding of how students start the problem solving and how they find the specific ideas needed to solve the problem. This results in the teacher being able to create more opportunities for mathematical learning and occasions for mathematical thinking.

Creating a “thinking classroom” (Liljedahl, 2015) guarantees not just occasions to think but also to reflect and experience a set of problem-solving strategies. According to Liljedahl (2015) this can be done by initiating problem-solving work in the classroom and teaching the problem-solving process. By giving names to used strategies students can build a resource of these named strategies. They will then become tools for students’ future problem-solving work and for their daily learning of mathematics in general.

Using the guessing technique is another way which stimulates the whole class discussion. It motivates the students to participate in the lessons, making them active learners (Asami-Johansson, 2015). The

guessing technique is used in the Problem-Solving Oriented teaching approach (PSO). PSO is a way to improve the teaching and learning of mathematics developed in Japan. Applying the PSO to Swedish mathematics classrooms Asami-Johansson (2015) found that the discrepancy between the Japanese and Swedish curriculum causes some challenges for the adaptation of the lesson plans. Classroom norms are difficult to bypass (Yackel & Rasmussen, 2002), even when a teacher is motivated to do so. Assami-Johansson (2015) presented some distinct aspects of the PSO approach to explain how this approach encourages students' mathematical learning and the development of their problem-solving ability.

In the PSO approach, all activities are initiated by presenting challenging problems that are carefully chosen to lead to new mathematical understanding. These problems stimulate a whole class discussion motivating students to participate in the lesson. To ensure that the discussion is about the planned subject matter, the teacher must anticipate the students' likely solutions and arguments.

It seems that there is a consensus within the mathematics education community that teaching problem solving and teaching mathematics should be connected. However, there is no consensus about how they should be integrated in the teaching practice (Lester and Cai, 2016; Schoenfeld, 2013; Lester, 2013, Kilpatrick, Swafford & Findell, 2001, Niss, 2018). We know far too little about how problem-solving abilities develop and how students can be helped to become better problem solvers. More research is needed that focuses on the factors that influence student learning in environments such as a classroom (Schoenfeld 2013; Lester 2013).

2.7 Introduction and implementation of ability notions in the curriculum in Sweden

As displayed above (Section 2.2) the research literature has come to include abilities as a fundamental way of describing mathematical knowledge. The Swedish curriculum, Lgr11, does not only use these concepts to describe what should be taught, but also use them to show what to assess. The syllabus for mathematics in Swedish upper secondary school focus on seven abilities that the students should develop and that should be assessed. These are:

(1) To use and describe the meaning of mathematical concepts and the relationship between the concepts. (2) to handle procedures and solve tasks of standard character without tools. (3) to formulate, analyze and solve mathematical problems as well as evaluate selected strategies, methods and results. (4) to interpret a realistic situation and design a mathematical model as well as use and evaluate a model's characteristics and limitations. (5) to follow, bring and assess mathematical reasoning. (6) to communicate mathematical thinking verbally, in writing and in action. (7) to relate mathematics to its significance and use in other subjects, in a professional, social and historical context.

The idea of mathematical abilities is hence very explicit and takes a prominent role in the mathematics syllabus. Problem solving is the only ability that is mentioned as both an ability and as a topic. Teaching of the mathematics course should address some content like arithmetic, algebra and problem solving as well. Furthermore, the teaching in the course should deal with strategies for mathematical problem solving and evaluate selected strategies, methods and results.

However, a clarification of the concept of ability and descriptions of how ability could be achieved are not given. National tests are seen as the main way of communicating what actually should be tested and how this should be done.

3. Methodology

“As an insider I have first-hand knowledge of the designer’s goals, assumptions, and expectations, the teacher’s knowledge of her students and experiences using the materials, and the researcher’s goals, methods, and findings. The voices of these three communities echo in my head as I strive to work within and among them.”
(Magidson, 2005, p.140)

This chapter presents the background and motivation for the study design and the methods for data analysis. Firstly, I describe the chosen research methodology for the intervention study, design-based research. After that, I describe some of the main concepts of variation theory, which help us to understand the design principle used for designing the intervention. Finally, we discuss the methods used to analyse the collected data.

3.1 Design-Based Research (DBR)

There are people from several different areas involved in understanding and improving the teaching and learning of mathematics: classroom teachers, educational researchers and designers (Magidson, 2005). However, historically people from these three communities have seldom collaborated. The result being that educational research for a long time was not connected enough to the problems and issues of everyday practice (DBR, 2003; Wang & Hannafin, 2005; Magidson, 2005).

For that reason, a family of research methods has been developed intended to increase the relevance of research to practice, involving both practitioners and researchers. Among these, one finds *design-based research* (Hoadley, 2002; DBSC 2003, Anderson & Shattuck, 2011, Anderson, 2005), *design experiments* (Bell, 2002a; Brown, 1992, Collins, 1992, 1999; Cobbs et.al, 2003, Zhang et.al., 2009), *design research* (Edelson, 2002), *action research* (Servan et.al., 2009, Rönnerman, K, 2012, Hopkins, D., 2002) and *development research* (van den Akker, 1999, Richey, Klein and Nelson, 2003). They have many similarities, but each research method has a slightly

different focus. All of them include collaboration between practitioners and researchers, designing and exploring innovations and empirical testing of interventions (Wang & Hannafin, 2005).

I have chosen to use Design Based Research, DBR (Wang & Hannafin, 2005; DBSC, 2003) in this study for several reasons. I did not want to make a comparison of multiple innovations like a *design experiment* is meant to do. The goal of my study is rather to conduct a single setting over a long time, in multiple contexts. The aim with the study is to design a learning environment to enhance students' problem-solving abilities. In other words, I did not intend for the design itself to be the main result, as it is if doing *design research*. Nor does the research done in this thesis fall into the category of *development research*, which typically describes and sets a product development process and analyses the final product. The interventions are intended to be designed and progressively refined in collaboration between practitioners and researchers. Finally, while similar to *action research*, DBR is not initiated to answer a local request for improvement. Additionally, the researcher is directly involved in the development process as well as in the refinement in the authentic classroom setting. At the same time, by allowing the selection of a learning theory, DBR contributes to the development of both theory and practice.

In summary, DBR offers a partnership between educational practitioners, designers and researchers, blurring distinctions between them. For this reason, DBR goes beyond merely designing and testing particular interventions. DBR has the potential to generate theories that meet the individual teachers' needs by being useful in designing learning environments, while also generating more collective ideas for educational development.

To define DBR, Wang & Hannafin (2005) use five basic characteristics: pragmatic, grounded, interactive (iterative and flexible), integrative and contextual. It is pragmatic because it refines both theory and practice, grounded because it is theory driven and grounded in relevant research, interactive because the process includes iterative cycles of design, implementation and redesign done by the researchers and teachers together. It is integrative because mixed research methods are used to ensure credibility,

validity and objectivity of research. Finally, it is contextual because research results are connected with the design process and the authentic settings, where research is conducted. The design principles used in the teaching interventions tell us how to implement the design, and support teachers to teach specific skills or concepts for example in my case problem-solving strategies. Design principles work like guidance which is needed to increase the adaptability, the generalisability and external validity of the research. The intention of DBR is to inquire more broadly into the nature of learning and aims at enabling us to create productive learning environment.

“Importantly, design-based research goes beyond merely designing and testing particular interventions. Interventions embody specific theoretical claims about teaching and learning, and reflect a commitment to understanding the relationships among theory, designed artifacts, and practice. At the same time, research on specific interventions can contribute to theories of learning and teaching.” (DBRC, 2003, p.6)

Magdison (2005), Lampert (1990), Roth (2001) and Boaler (2000) advocate the benefits of combining the roles of the teacher, designer and researcher into one person, as I have chosen to do in this study. The fact that the designer and the teacher are the same person can be an advantage in, for example, detecting what the students find difficult and in the improvement of the lesson design for the next cycle. However, there is a risk of teacher-researcher conflicts in the classroom, for example having to choose between helping a student and holding back as a researcher to see what will happen. I have therefore decided to always have the teaching agenda as my main focus during class time and when I am outside the classroom I want to reflect on and scrutinize my teaching with the research goals in mind.

3.2 The design framework. Variation theory

The classroom context is highly dynamic and complex. The design of learning experiences and the analyses of the relationship between teaching and learning in school depends on the theoretical

perspective. I have chosen variation theory as a learning theory to formulate my design principle, because conscious variation can enhance learner's focal awareness and makes it possible for the learner to experience what should be learnt (Marton & Booth, 1997; Marton & Tsui, 2004; Marton & Pang, 2006; Marton, 2015; Pang & Lo, 2012).

“In using variation theory, the role of the teacher is to design learning experiences in such a way that helps students to discern the critical aspects of the object of learning by means of the use of variation and invariance. By consciously varying certain critical aspects, while keeping other aspects invariant a space of variation is created that can bring the learner's focal awareness to bear upon the critical aspects, which makes it possible for the learner to experience the object of learning.” (Pang & Marton, 2005, p.164)

The variation theory has its origins in the phenomenographic research, which investigates and describes qualitatively different ways of understanding the same phenomena. On the other hand, according to variation theory, whatever situation people experience they understand it in a limited number of qualitatively different ways (Marton & Booth, 1997). Furthermore, the theory has an explicit focus on the relationship between teaching and learning, offering a way to discuss potential implications of teaching for student learning. Learning means to see the object of learning in new ways and to be able to discern features of the object of learning that were not discerned earlier.

Choosing variation theory as a learning theory in my design, gives me the possibility to help my students to experience the variation of options to solve a problem, instead of being told. In my case this means to create an environment of learning using the design principles. Several studies have demonstrated that the use of patterns of variation improve student learning outcomes (Runesson, 2005; Marton & Tsui, 2004; Marton & Morris, 2002; Lo, 2012). For that reason, it is important for this study that the design principles enable

the teacher to create a pattern of variation that will direct the students' attention to critical aspects of the object of learning.

3.2.1 Important concepts from variation theory

Object of learning

The object of learning does not necessarily have to be related to the subject matter content, but it always denotes the 'what' aspect of teaching and learning. According to Lo (2012), in the same sense "*it points to the starting point of the learning journey rather than to the end of the learning process*". In this study the object of learning is problem-solving strategy.

We can distinguish two different objects of learning (Lo, 2012). Firstly, the direct object of learning, which refers to content, thus being concerned with specific aspects, for example strategies in mathematical problem solving. It is a short-term educational goal, to know some strategies. The direct object of learning is about the subject knowledge controlled by a centralised curriculum and designated textbooks. Secondly, the indirect object of learning refers to what the learner is supposed to become capable of doing with the content. It is a long-term educational goal, to gain a deeper understanding of the relationship between chosen problem-solving strategies and success in mathematical problem solving.

The object of learning has a dynamic character. For example, it is often very difficult for the teacher as an adult and experienced problem solver to comprehend the difficulties that a novice problem solver experiences. To help students develop the capability to evaluate selected strategies, the teacher must first discover which strategies the students already know. Based on students' reactions and their own understanding of the strategies, teachers can gain better understanding of how students learn. Then the teachers use their own understanding of the object of learning to choose the critical features that they want the students to become able to discern through encountering certain patterns of variation and invariance.

“However, we have to admit that we can never predict exactly what the learning outcome should be, as we must take into account both the dynamic nature of the object of learning and the

unpredictable nature of the classroom, the result being that the enacted object of learning will usually differ from the intended object of learning.” (Lo, 2012, p. 55)

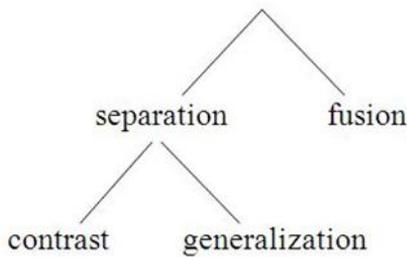
Space of learning

We cannot force students to learn, but we can provide the best opportunities for them to learn through creating a *space of learning*. It is important to note that the space of learning does not describe what students necessarily will learn only what is made possible to learn. Questions structure the learning experience and focus the students’ attention on the object of learning. Ergo, the space of learning should be a description of the enacted object of learning.

In this study, the design principles are encouraging the students to consider a number of possibilities and to formulate answers that make sense not only to themselves but also to the rest of the class. The design principles make room for students’ implementation of meaningful, problem-oriented activities to facilitate learning, aligned with the research context. In addition, variation provides opportunities to study links between how the mathematics is handled in a classroom and what students may possibly learn.

“[...] it is necessary to pay close attention to what varies and what is invariant in a learning situation, in order to understand what it is possible to learn in that situation and what is not.”
(Marton, & Tsui, 2004, p. 16).

Marton and Tsui (2004) identifies four different patterns of variation on a general level: contrast, separation, fusion, and generalisation. Marton (2015) illustrated the relationships between these patterns in the following way:



Patterns of variation in terms of strategies

1. *Separation* When the learner suddenly becomes aware of a strategy (e.g. visualization) by contrasting it with another strategy (e.g. grouping the data), we can say that the strategy is separated from the solution of the problem as an undivided whole. A dimension of variation is opened up. The learner becomes aware of the problem-solving strategy and is capable of focusing on the strategy independently, naming it or even changing it.

2. *Contrast* Experiencing the difference (variation) between two or more problem-solving strategies. In this way, students will experience the variation of the critical feature and will be more likely to be able to discern it and be made aware of different strategies that exist.

3. *Generalization* Keeping the strategy invariant while systematically varying the problem within and different content areas of mathematics one by one, the learner becomes aware of the fact that a strategy is not domain specific, instead a problem-solving strategy is applicable in all, or at least in many different areas of mathematics.

4. *Fusion* An understanding of the strategy depends on the simultaneous awareness of several characteristics (e.g. type, effectiveness) and how these characteristics relate to each other and to the strategy as a whole. Discussing the different characteristics of the strategies may provide opportunities to experience how effective they are in certain problem-solving situation.

The space of learning refers to the pattern of variation which is a necessary condition for learning.

“Students cannot naturally discern the critical features of an object of learning. It is therefore the duty of the teacher to provide them with opportunities to be able to do so.” (Lo, 2012, p. 54).

3.3 Variation in the design principles

The design principles in this study guide how the content is handled during the different lessons, providing students learning experiences, through the opportunity to discern the necessary aspects of the problem-solving strategies. We must not forget that, according to variation theory, learning can take place when students experience variation. These principles are not designed to create decontextualized principles or grand theories that function with equal effect in all contexts. Rather, design principles reflect the conditions in which they operate. These design principles function to help us understand and adjust both the context and the intervention. To develop practical design principles is a key aspect of DBR.

Design principles

Here are the design principles that we have developed on the basis of variation theory.

- (1) Let the problem-solving strategy vary and keep the task invariant.
- (2) Let the task vary and keep the problem-solving strategy invariant.
- (3) Let both the task and the strategy vary and allow students to evaluate the effectiveness of different strategies for different tasks.

In design principle (1) the problem-solving strategy varies while the task is kept invariant. The intention is to offer the students opportunities to discern multiple problem-solving strategies, usually by asking them to solve a task in several different ways. In design principle (2) the task varies while the problem-solving strategy is kept invariant. The intention of design principle (2) is to offer the students opportunities to realize the usefulness of a strategy, that it

can be used to solve different kinds of problems, not only in special domains of mathematics but in tasks from different parts of mathematics. In design principle (3) the intension is to allow students to evaluate the effectiveness of different strategies for different tasks. Effectiveness is an important feature of strategy, saving time in the problem-solving process.

In summary, through the three design principles, the students experience all four patterns of variation mentioned in the previous section. This brings awareness of the existence and the role of strategies in the problem-solving process.

3.4 Mixed research methods

Now we will turn to describing the methods used to analyse the data from the students' written solutions. Mixed methods research is an approach to knowledge that always including the standpoints of qualitative and quantitative research. That attempts to consider multiple viewpoints, perspectives, and standpoints.

3.4.1 Content analysis

We use content analysis, which is a qualitative method, of analysing written and visual communication messages for obtaining access to the words of the text offered by the students' solutions. The method is used to develop an understanding of the meaning of communication (use of strategies) and to identify critical processes (Krippendorff, 1980; Cole, 1988; Lederman, 1991; Cavanagh, 1997 Bryman, 2008). In this study the inductive approach is used. The inductive approach is based on the data and moves from the specific to the general. The particular instances are observed and then combined into a larger whole or general statement. The analysis processes are represented as three main phases: preparation, organising and reporting. Firstly, the aim is to become immersed in the data, which in practice means that the written material is read through several times. The next step is to organize the qualitative data. This process includes coding, creating categories and abstraction. Creating categories is both an empirical and a conceptual challenge. A specific qualitative coding scheme is developed for each problem to examine solution strategies and methods. Observational notes are divided into meaningful units.

Taking into account the context, these meaning units are condensed into a description closely following the text (the manifest content) and into an interpretation of the underlying meaning (the latent content). This model for content analysis of the students' written solutions is employed to qualitatively analyse the decision making, especially the use of problem-solving strategies and methods, which is the criterion of selection. Using this model, three key variables are examined: (1) identified places where the students made decisions, (2) whether the decisions were choices of strategies or methods and which strategies were used, (3) how the choice of strategy and method affected the students' success in problem solving. These selection criteria are rigidly and consistently applied, the post-test is read through several times, in order to ensure the reliability and validity of the findings, and I sought help from my supervisor to carry out a second analysis to establish the validity and reliability of the coding. The results will be presented in a descriptive manner.

3.4.2 Statistical analysis

For the quantitative analysis of the data we use hypothesis testing as it is one of the most powerful ways of making comparisons. To decide whether there exists a connection between the teaching intervention and students' problem-solving ability, the independent samples one-sided t-test is used. We use the independent samples t-test to compare the development of the experimental and control groups in order to determine whether there is statistical evidence that the two groups' development are significantly different. For this reason, we have to be sure that our data set meets a list of requirements, including that the data from the pre- and post-tests has to be comparable. Since the pre- and the post-test scores are measured on different scales, this criterion is not automatically fulfilled.

To aid comparison, we use z-score normalization to convert the students' test scores. We calculate a normalized z-score for each student, for the pre-test scores z_i^{pre} and for the post-test scores z_i^{post} . For the student i , with the result x_i , this is calculated as

$$z_i = \frac{x_i - \bar{x}}{s}$$

where \bar{x} is the mean and s is the standard deviation of the whole sample. The absolute value of the z-score thus represents the distance between the raw score and the sample mean in units of the sample standard deviation. Hence z is negative when the raw score of that student is below the mean, and z is positive when the raw score is above the mean.

Afterwards, we use the difference between the student's z-score on the pre-test and the post-test, $z_i^{\text{diff}} = z_i^{\text{post}} - z_i^{\text{pre}}$ as a measure for the student's relative development. Finally, the procedure is repeated but restricting our attention to only the problem-solving scores from the post-test. The development from the post-test is finally calculated as $z_i^{\text{diff-PLS}} = z_i^{\text{post-PLS}} - z_i^{\text{pre}}$.

The one-sided t-test is used for testing of the difference between experiment group means and control group means. The difference between two groups is statistically significant if it cannot be explained by chance alone, or more specifically if it is less or equal to 5% chance that one and the same distribution function would give the two samples compared in the test, i.e. the experiment and the control group samples.

4. Summary of appended papers

This section contains a summary of the papers appended to the thesis. The emphasis is on presenting the theory and results in a less formal style than in the papers themselves, with special focus on their respective results.

The first two papers investigate the effects that teaching problem-solving strategies have on students' problem-solving abilities and general mathematical knowledge. Finally, paper three strives towards/looks for a deeper understanding of the relationship between chosen problem-solving strategies and success in mathematical problem solving.

4.1 Paper I: Teaching problem-solving strategies in mathematics

By clarifying the distinction and the hierarchical relationship between the three concepts *strategy*, *method* and *algorithm*, the idea

in Paper I is to capture the differences between the three different decision-making levels in a problem-solving situation. In this paper we discuss the nature of the concept strategy and the educational possibilities and effects of teaching problem-solving strategies.

For this reason, three design principles were developed based on variation theory. Educational activities were designed to teach problem-solving strategies and tested in an authentic classroom for four weeks. The design of each lesson, based on the principles, involved goals for what mathematical content within the curriculum that should be learnt, as well as what aspects of problem-solving strategies that should be covered.

To evaluate the effects, we used mixed method. The used method is described in Section 2.4 and methodological consideration is discussed in Section 2.1. We believed that both qualitative and quantitative viewpoints are useful to answer the question. The analysis consisted content analysis of the post-test and descriptive statistic, looks at the results of students' tests from both before (pre-test) and after (post-test) the educational activities and compares with a control group.

The result from the analysis of the post-test of the experimental group show some explicit use of strategies in their problem solutions, already after four weeks. In contrast, the solutions provided by a control group did not display clear strategy choices. Furthermore, compared to the control group, the experimental group had better, or at least comparable, development in their conceptual and procedural knowledge.

We conclude that it is possible to teaching problem-solving strategies, using our three design principles, had positive effect already after four weeks.

4.2 Paper II: Developing problem-solving abilities by learning problem-solving strategies: An exploration of teaching intervention in authentic mathematics classes

The purpose of the work presented in this paper was to extend the results from Paper I to a one-year experiment. The aim of this study was to iterate the designs developed in the previous paper and

analyse their long-term effects on the students' problem-solving abilities and mathematics knowledge in general.

Two tests were used to compare the development of the experimental group with a control group by analysing students' success in solving problems. Pre-test from the study in Paper I and the National test, given by the Swedish National Agency, as a post-test was used in this study. The National test measures all mathematics abilities, including problem-solving ability, and each ability is required in several tasks.

The pre- and the post-test were different tests with different distributions of scores. To compare students' development between the two tests, we used independent samples t-test. The method is described more thoroughly in Section 2.4.2.

The results show that the experimental group had significantly better development in problem-solving abilities compared to the control group. Moreover, our findings suggest that also the general mathematics knowledge of the experimental group was affected in a positive way, however not significant.

In summary, we argue that use of variation theory as a learning theory, was one of the important characteristics of the intervention which is behind the positive development of the students' problem-solving abilities. Making students aware of their decision making on different levels during problem solving and train them to be able to apply something that they learned in one situation in another, are two other important characteristics of the teaching intervention.

In relation to previous research, this study supports the importance of problem-solving strategies in developing students problem-solving ability. We argue that learning problem-solving strategies directly led to improvements in the students' problem-solving skills.

4.3 Paper III: Connections between chosen problem-solving strategies and success in mathematical problem solving

The previous two papers showed that knowledge of problem-solving strategies in general affect students' problem-solving abilities. In this paper the aim was to get a deeper understanding of the relationship between chosen problem-solving strategies and success in

mathematical problem solving. For this reason, a qualitative analysis of students' written responses was conducted to illustrate decision-making at different levels: strategy, method and algorithm. The data was derived from two tasks on the post-test used in Paper II.

The result indicated that the students' success in problem solving was affected by being able to see the problem as a whole. At the same time, the result show that it was necessary that the students being able to operate on all three levels, it was not enough to choose a proper problem-solving strategy. The appropriate choice of strategy also requires corresponding procedural skills. The results suggest, that by increasing students' understanding of the role of strategic decision-making in problem-solving situation, strategies become a part of students' arsenal of problem-solving tools.

5. Discussion and conclusion

The findings are now discussed in relation to the background presented in Section 2. In this thesis, the role of strategies in problem solving are studied from different perspectives. We discuss if and how the teaching of problem-solving strategies affected the students' problem-solving abilities (Paper I and Paper II). We also analyse how the students' possibility to succeed in problem solving depended on their choice of strategies (Paper III). We end by discussing didactical implications and of some limitations of these studies.

5.1 The concept of strategy and its role in mathematical problem solving

In this section the concept of strategy in problem solving is taken as the point of departure for discussion of the following two research questions: What is known about the concept strategy and its relationship to method and algorithm? How are the students' selection of strategies contributing to their success in problem solving?

The theoretical framework of this thesis, described in Section 2.5, make a distinction between three concepts in mathematical problem-solving, namely strategy, method and algorithm. Historically,

military theory divides war into strategic, operational and tactical levels. Similarly, in game theory strategies can be decomposed into a sequence of decisions called choices, made at various decision points, called moves. We argue in Paper I that in mathematical problem solving there are several decisions to deal with as well, there are different decision-making levels with different goals and characteristics. To see how these three levels of decision-making are related to each other, consider Figure 2 in Section 2.5. The findings presented in Paper III showed how choices of strategies, method and algorithm are visible in students' solutions and play a role in the students' progress in problem solving. This confirms that in practice there are differences and a hierarchical relationship between strategy, method and algorithm, which aligns well with the framework as presented in Paper I.

We stress that by distinguishing these three levels, the framework allows the teacher and students to better understand causes and effects of these types of decisions in problem solving. Each of these levels of decision-making involves analysing the situation. Each level is also concerned with choosing or implementing a choice that can be reevaluated at any time. Usually the reevaluation occurs on the basis of incomplete information or lack of understanding or knowledge, adding a dynamic dimension to problem solving.

However, despite their differences, choices concerning strategy, method and algorithm are interdependent. The necessity of being able to operate on all three levels in problem solving is, with respect to the results in Paper III, an important aspect that affects the students' success in problem solving. Findings in Paper III show that the lack of knowledge on algorithm level, for example how to solve a given equation, affects students' selection of strategies in problem solving. Furthermore, the analyses in Paper III showed that when the problem designer removed the students' possibility of making their own strategies, to see the problem as a whole, many students landed in the wrong choice of method.

To understand problem solving in mathematics and to complete it successfully, the students gains from being aware of the three levels in decision making, especially the strategy level, and how they are interrelated (Figure 2 in Section 2.5). The study presented in Paper

III provides evidence that having a vision, in which the problem as a whole, and the parts of the problem are viewed simultaneously, is necessary to succeed in problem solving.

5.2 Teaching problem-solving strategies. What can we learn from the studies?

Contrary to earlier research on teaching strategies, the main goal of this thesis is to develop a teaching intervention, that not only focused on teaching problem solving strategies but also on mathematical content. In other words, to try to infuse strategy thinking in daily teaching of mathematics in an authentic classroom.

A point of departure in this study was that if the teacher increases the students' awareness about different problem-solving strategies, it is then possible and more likely for them to learn to solve problems more successfully. For this reason, three design principles were developed and tested. As shown in Paper I, the design principles aimed at constructing a route by which the mathematical content of the whole course can be redesigned to offer the students opportunities to experience different problem-solving strategies. An important part of the research design was that the proposed sequence of teaching acts during the lessons should achieve both the mathematical curriculum goals and goals related to teaching strategies. Our basic idea was to construct a learning environment that makes it possible for all students to have a good conception of what is to be learned.

Our goal in the study presented in Paper I was not to evaluate the effectiveness of the design. Instead, our goal was to develop and test three design principles based on the conceptual framework described in section 2.5. This study was meant to help us understand whether our design functions in its intended settings.

This study was a demonstration of how the design principles made use of the theory of variation as a pedagogical tool. The design gave opportunity to the students to work with different strategies (variation) in relation to the same content (invariant) and to work with different content (variation) in relation to the same strategy (invariant). If problem-solving strategies, developed by practising in a certain content area, are general enough to be applicable to another

content area, then transfer of learning can occur. The transfer is more likely when the set of skills that are supposed to be generalized (strategy making) are not domain-specific, which we that argue strategy is not.

In addition, the study presented in Paper I explores whether the design made it possible for students to learn about strategies. By examining the post-test written after four weeks, the study showed that the experimental group had been affected in terms of their ability to use problem-solving strategies.

By iterating the design developed in the study presented in Paper I, a one-year-long intervention period led to significant differences between the experimental group and the control group's post-test problem-solving score. This is presented in Paper II. Hence, we argue that the intervention had a notable effect on students' problem-solving ability.

In total, the empirical results presented in Paper I and Paper II suggest that creating the right conditions for learning, using variation theory results in an effective intervention on teaching strategies. The results from Paper II confirm empirically that knowledge of problem-solving strategies is important and is in fact an integral part of the problem-solving ability (Section 2.2).

The results suggest that teaching problem-solving strategies can be an effective tool to promote students' mathematical problem-solving ability. Tool, that can be used to learning to solve problems that students have not learned to solve.

5.3 Ethical considerations and the effects on over all mathematics competence

There are two relevant ethical considerations in this study. Firstly, it is important to ensure that the experiment does not hinder the students from achieving the course goals described in the mathematics curriculum.

The experimental group spent more time in school discussing different ways to solve problems, thereby learning about different problem-solving strategies, than the control group. In this way, they spent less school time solving tasks from the textbooks. Thereby the

experimental group had limited time with activities to practice tasks of a procedural nature, compared with the control groups.

The results in Paper I made it ethically reasonable to continue the intervention study. The results showed that the experimental group had better, or at least comparable, development in their conceptual and procedural knowledge. The results indicated that it is possible to teach with focus on problem-solving strategies without a need to compromise on either the course mathematics content (the same mathematical content was taught in both groups) or the number of available lessons (same number of lessons for both groups). Because of the positive results, the chosen teaching approach was ethically justifiable. Focusing on problem-solving strategies proved not to be an obstacle in the students' development of general mathematics knowledge.

Furthermore, the results from Paper I were further strengthened by the result in Paper II. After the one-year-long intervention period, the analysis of the total score levels on the post-test showed that the experimental group had a higher mean and lower standard deviation than the control group. That means that the results from Paper II confirm that the general mathematics knowledge of the experimental group was at least as good as that of the control groups even at the end of the mathematics course.

Another aspect of this study that needed to be reflected upon from an ethical point of view was the importance of ensuring that the students were aware and give their consent to the analysis of their results. Before conducting the studies, we therefore asked the students for explicit written consent to participate in this research experiment. They were informed of the goals of the experiments and that their contributions would be anonymized (i.e. no personally identifiable information would be included in the analysis or any publications).

5.4 Limitations and strengths

In light of the results, caution must be exercised in attempting to generalize the results of this investigation. The design principles are not instructions that indicate how to teach one or another specific topic, concept, or skill and they are not a collection of effective

lesson plans either. The principles formulate general procedures to apply to teaching any specific content in mathematics and any problem-solving strategy. However, to generalize this conclusion will require further testing; even if the evidence obtained in this thesis was positive and replicable.

As with most empirical studies there are a number of limitations to this current research. The validity and reliability would be higher if this study were conducted during longer time and if different student groups could be included, not just students from math intensive science programs. Students may have different needs or desires when belonging to different groups. The students in this study could certainly have learnt a lot of mathematics before (and after) the studied lessons, as well as outside of school. We only discuss the tasks that were possible to solve in relation to the design from the lessons in this study. Therefore, more research is needed to further substantiate the validity and extend the concept of this study.

A fully objective analysis is not possible since the complexity of a mathematics classroom is considerable. Each mathematics classroom is unique, even if they share common aims. It must be remembered that this complexity is reduced to just a few features in a study like this. The discussion of the outcomes is more or less restricted to these features and can only account for one of many possible ways of seeing and describing the studied activities.

5.5 Didactical implications

The use of computers is becoming an increasingly common supplement in the school classroom. In 2018 the Swedish National Agency for Education introduced programming into the mathematics curriculum. Students are supposed to learn to use computing devices as tools for problem solving. As a general trend, mathematical competency requirements are evolving from knowing how to calculate to improving problem-posing and strategy-making competencies. Mathematics teaching should therefore not focus on educating “the human calculator”. In the development of teaching practices, all students should be given the prerequisites of becoming highly professional and competent thinkers and problem solvers in order to meet the demands in the digital era. A good problem-solver’s ability according to Kilpatrick, Swafford & Findell, (2001),

Niss & Højgaard-Jensen (2002), Lithner et al. (2010) includes knowledge to develop strategies, and mastery of applying and adapting appropriate strategies and methods. Learning problem-solving strategies enhances students problem-solving ability. The results from Paper I and II confirm empirically that knowledge of problem-solving strategies is an integral part of the problem-solving ability.

This thesis has didactical implications related to how teaching problem-solving strategies should be integrated in the teaching practice. In particular, Paper I describes three design principles that teachers can use to help students to become aware of their decision-making in problem solving, especially on the strategy level. At the same time, the students get to know some of the most commonly used problem-solving strategies while also being able to handle the actual content of the course. The three-level decision making model described in section 2.5 can be used in different areas of mathematics.

Paper II offers some examples of practical lessons that can be directly applied in the classroom. The concept can be powerful regardless of how many students there are in a class. How the idea is implemented will of course be dependent on the teacher's knowledge of problem-solving strategies.

This thesis can inform and pave the way for a discussion, among teachers and within teacher education, about the concept strategy and about possible ways to teach problem-solving strategies while also considering the mathematical content.

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