Errata for *Index theory in geometry and physics*

The projection p_T in Chapter $C.2$ does not extend to S^{2n}, and if it did it would be trivializable since $H^2(S^{2n},\mathbb{Z})=0$. In effect the projection p_Y is not well defined. This fact renders Lemma $C.2.1$ and Theorem $C.2.2$ false. By extension, the formulas of Theorem 3 in the introduction, Theorem $C.3.2$, Chapter $C.5$ and Chapter $C.6$ are false in their current form and must be modified as is now described. All references are to Paper C.

The problem is mended by considering the Bott class $\beta \in K^0(\mathbb{R}^{2n})$. The Bott element will be used to define a virtual rank zero bundle on a coordinate neighborhood in Y and extend this to a virtual bundle on Y. The Bott element $\beta \in K^0(\mathbb{R}^{2n})$ is represented by the difference class $(\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n, \wedge^{odd}_{\mathbb{C}}\mathbb{C}^n, c)$ where $\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n$ and $\wedge^{odd}_{\mathbb{C}}\mathbb{C}^n$ are considered as trivial vector bundles on $\mathbb{R}^{2n}$ and $c:\mathbb{R}^{2n}\to \mathrm{Hom}(\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n, \wedge^{odd}_{\mathbb{C}}\mathbb{C}^n)$ is constructed by letting $c(x)\in \mathrm{Hom}(\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n, \wedge^{odd}_{\mathbb{C}}\mathbb{C}^n)$ be the operator defined from the complex spin representation and Clifford multiplication by the vector $x\in\mathbb{R}^{2n}$. Since $c(x)$ is invertible for $x\neq 0$, with inverse $c(x)^*/|x|^2$, this difference class is well defined. See more in Chapter 2.7 of [1]. By Proposition 2.7.2 of [1], the element β generates $K^0(\mathbb{R}^{2n})$. Since $K^0(\mathbb{R}^{2n}) = \ker(K^0(S^{2n})\to K^0(\{\infty\})$, the inclusion $\mathbb{R}^{2n}\subseteq S^{2n}$ induces an injection $K^0(\mathbb{R}^{2n})\to K^0(S^{2n})$, and $K^0(S^{2n})$ is generated by the Bott class and the trivial line bundle. Furthermore, the Bott class, as an element of $K^0(S^{2n})$, does indeed satisfy that

$$\mathrm{ch}_{S^{2n}}\beta = \mathrm{d}V_{S^{2n}}.$$

The problem with this construction of the Bott element is that it does not fit directly into the definition of the Chern character in cyclic cohomology used in Paper C. We will now construct a projection-valued function $p_0:\mathbb{R}^{2n}\to End(\wedge^*_{\mathbb{C}}\mathbb{C}^n) = M_{2^n}(\mathbb{C})$ of rank 2^{n-1} that extends to a projection-valued function p_T on S^{2n} such that $\beta = [p_T]-2^{n-1}[1]$ in $K^0(S^{2n})$. Let us identify the complex Clifford algebra $\mathbb{C}l(\mathbb{R}^{2n})$ with $\mathrm{End}(\wedge^*_{\mathbb{C}}\mathbb{C}^n)$ using the complex spin representation. Define p_0 as:

$$p_0(x) := \frac{1}{1+|x|^2}\begin{pmatrix} |x|^2 & c(x) \\ c(x)^* & 1\end{pmatrix} \in End(\wedge^{odd}_{\mathbb{C}}\mathbb{C}^n \oplus \wedge^{ev}_{\mathbb{C}}\mathbb{C}^n).$$

While

$$p_0(x) - \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} = \frac{1}{1+|x|^2}\begin{pmatrix} 1 & c(x) \\ c(x)^* & 1\end{pmatrix} = \mathcal{O}(|x|^{-1}) \quad \text{as} \quad |x|\to\infty,$$

the function p_0 extends over infinity to a function $p_T\in C^1(S^{2n}, M_{2^n}(\mathbb{C}))$. Let $E_0\to\mathbb{R}^{2n}$ denote the vector bundle associated with p_0 using the Serre-Swan theorem. One has that

$$E_0 = \{(x,v_1,v_2)\in \mathbb{R}^{2n}\times(\wedge^{odd}_{\mathbb{C}}\mathbb{C}^n \oplus \wedge^{ev}_{\mathbb{C}}\mathbb{C}^n) : v_1 = c(x)v_2\}.$$

The vector bundle E_0 is trivializable via the isomorphism

$$\mathrm{id}\oplus c : \mathbb{R}^{2n}\times \wedge^{ev}_{\mathbb{C}}\mathbb{C}^n \to E_0, \quad (x,v)\mapsto (x, c(x)v, v). \tag{1}$$

We define the morphism of vector bundles

$$c_0 : E_0 \to \mathbb{R}^{2n} \times \wedge_{\mathbb{C}}^{odd}\mathbb{C}^n, \quad (x, v_1, v_2) \mapsto (x, v_1). \tag{2}$$

The morphism c_0 is an isomorphism outside the origin, with inverse $(x, v_1) \mapsto (x, v, |x|^{-2}c(x)^*v)$.

Proposition 2.1. *Under the isomorphism $K^0(S^{2n}) \cong K_0(C(S^{2n}))$ the Bott element β is mapped to $[p_T] - 2^{n-1}[1]$, and therefore $\int_{S^{2n}} \mathrm{ch}_{S^{2n}}[p_T] = 1$.*

Proof. The formal difference class $[p_T] - 2^{n-1}[1] \in K_0(C^1(S^{2n}))$ is of virtual rank 0, so it is in the image of the injection $K^0(\mathbb{R}^{2n}) \to K_0(C(S^{2n}))$. The element $[p_T] - 2^{n-1}[1]$ clearly comes from the formal difference $[E_0] - 2^{n-1}[1]$ which in turn is defined as the difference class $(E_0, \wedge_{\mathbb{C}}^{odd}\mathbb{C}^n, c_0) \in K^0(\mathbb{R}^{2n})$, where c_0 is the bundle morphism of equation (2). The latter is isomorphic to the Bott class via the isomorphism $\mathrm{id} \oplus c$ defined in equation (1). It follows that $\mathrm{ch}_{S^{2n}}[p_T] = 2^{n-1} + \mathrm{ch}_{S^{2n}}\beta = 2^{n-1} + \mathrm{d}V_{S^{2n}}$. □

In the general case, let Y be a compact, connected, orientable manifold of dimension $2n$ and U an open subset of Y with a diffeomorphism $U \cong B_{2n}$. This diffeomorphism defines a projection valued Lipschitz function $p_Y : Y \to M_{2^n}(\mathbb{C})$ as is described in Paper C and the following theorem is proved by the same method as in Paper C but instead using Lemma 2.1 as stated above.

Theorem 2.2. *If Y is a compact connected orientable manifold of even dimension and $\mathrm{d}V_Y$ denotes the normalized volume form on Y, then the projection p_Y satisfies*

$$\mathrm{ch}[p_Y] = 2^{n-1} + \mathrm{d}V_Y,$$

in $H^{even}_{dR}(Y)$. Thus, if $f : X \to Y$ is a smooth mapping, then

$$\deg(f) = \int_X f^*\mathrm{ch}[p_Y]$$

We will use the notation $\langle\cdot,\cdot\rangle$ for the scalar product in $\mathbb{R}^{2n}$. For an orthogonal basis $e_1, e_2, \ldots, e_{2n}$ of $\mathbb{R}^{2n}$ the Clifford algebra $\mathbb{C}l(\mathbb{R}^{2n})$ has a basis consisting of multiples $e_{j_1}\cdots e_{j_l}$ for $1 \leq j_1 < \ldots < j_l \leq 2n$. By the universal property of the Clifford algebras, any element u in the complex tensor algebra of $\mathbb{R}^{2n}$ defines an element $\tilde{u} \in \mathbb{C}l(\mathbb{R}^{2n})$. For a tensor u we let $[u]_{2n}$ be the number such that the projection of $\tilde{u}$ onto $e_1e_2\cdots e_{2n}$ is $[u]_{2n}e_1e_2\cdots e_{2n}$. If $u = (u_1, \ldots, u_k) \in (\mathbb{R}^{2n})^{\times k}$ and $1 \leq j_1, \ldots, j_l \leq k$ we will also use the notation $[u|j_1, \ldots j_l]_{2n}$ for $[u_0]_{2n}$ where $u_0 \in (\mathbb{R}^{2n})^{\otimes k-l}$ is defined as the tensor product of all the u_j:s except for $j \in \{j_p\}_{p=1}^l$. For any element $v \in \mathbb{C}l(\mathbb{R}^{2n})$ it holds that

$$\mathrm{tr}_{\wedge_{\mathbb{C}}^{ev}\mathbb{C}^n}(v) - \mathrm{tr}_{\wedge_{\mathbb{C}}^{odd}\mathbb{C}^n}(v) = (-2i)^n[v]_{2n}.$$

For the natural number $l > 0$ we define $\Gamma_m^l \subseteq \{1, 2, \ldots, 2m\}^l$ as the set of all sequences $\mathbb{h} = (h_j)_{j=1}^{2l}$ such that $h_j \neq p$ for any $p \leq j$ and $h_j \neq h_p$ for any $j \neq p$. We define $\varepsilon_l : \Gamma_m^l \to \{\pm 1\}$ by

$$\varepsilon_l(\mathbb{h}) := (-1)^{l + \sum_{j=1}^l h_j}.$$

Lemma 2.3. *For $x = (x_1, x_2, \ldots x_{2m}) \in (\mathbb{R}^{2n})^{\times 2m}$ we have that*

$$\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=1}^{m} c(x_{2l-1})^* c(x_{2l})\right) = (-2)^{n-1} i^n [x_1 \otimes x_2 \otimes \cdots \otimes x_{2m}]_{2n} +$$

$$+ (-2)^{n-1} i^n \sum_{l=1}^{m-1} \sum_{\mathbb{h}\in\Gamma^l_m} \varepsilon_l(\mathbb{h}) \left[x|1, h_1, 2, h_2, \ldots, l, h_l\right]_{2n} \prod_{p=1}^{l} \langle x_p, x_{h_p}\rangle +$$

$$+ 2^{n-1} \sum_{\mathbb{h}\in\Gamma^m_m} \varepsilon_l(\mathbb{h}) \prod_{p=1}^{m} \langle x_p, x_{h_p}\rangle.$$

Proof. Let us calculate these traces using the relations in the Clifford algebra:

$$\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=1}^{m} c(x_{2l-1})^* c(x_{2l})\right) = \frac{1}{2}\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=1}^{m} c(x_{2l-1})^* c(x_{2l})\right) +$$

$$+ \frac{1}{2}\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\left(\prod_{l=1}^{m-1} c(x_{2l})^* c(x_{2l+1})\right) c(x_{2m})^* c(x_1)\right) +$$

$$+ (-2)^{n-1} i^n [x_1 \otimes x_2 \otimes \cdots \otimes x_{2m}]_{2n} =$$

$$= \sum_{j=2}^{2m} (-1)^j \langle x_1, x_j\rangle \mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\widehat{c(x_1)^* c(x_j)}\right) + (-2)^{n-1} i^n [x_1 \otimes x_2 \otimes \cdots \otimes x_{2m}]_{2n},$$

where $\widehat{c(x_1)^* c(x_j)}$ denotes $\prod_{j=1}^{m-1} c(x_{l_{2j-1}})^* c(x_{l_{2j}})$, where $(l_j)_{j=1}^{2m-2}$ is the sequence $1, 2, \ldots, 2m$ with the occurences of 1 and j removed. The sign $(-1)^j$ comes from the number of anti-commutations needed to anti-commute the first operator with the j:th. Continuing in this fashion one arrives at the conclusion of the Lemma. □

Lemma 2.4. *The Chern character of p_Y is given by $\tilde{v}^* \mathrm{ch}[p_T]$ and the Chern character of p_T in cyclic homology can be represented by a cyclic $2k$-cycle that, in the coordinates on $\mathbb{R}^{2n} \subseteq S^{2n}$, is given by the formula*

$$\mathrm{ch}[p_T](x_0, x_1, \ldots, x_{2k}) = \frac{1}{k!}\mathrm{tr}_{\wedge^*_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=0}^{2k} p_0(x_l)\right) =$$

$$= \frac{1}{k!\prod_{l=0}^{2k}(1+|x_l|^2)} \sum_{m=0}^{2k+1} \sum_{0\leq g_1\leq\cdots\leq g_m\leq 2k} \mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=0}^{m} c(x_{g_l})^* c(x_{g_l+1})\right),$$

where we identify $x_{j+2k+2} = x_j$ for $j = 0, 1, \ldots 2k$.

Proof. Define the function $V : \mathbb{R}^{2n} \to \mathrm{Hom}(\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n, \wedge^{odd}_{\mathbb{C}}\mathbb{C}^n \oplus \wedge^{ev}_{\mathbb{C}}\mathbb{C}^n)$ by

$$V(x)v := \frac{c(x)v \oplus v}{\sqrt{|x|^2+1}} \in \wedge^{odd}_{\mathbb{C}}\mathbb{C}^n \oplus \wedge^{ev}_{\mathbb{C}}\mathbb{C}^n, \quad v \in \wedge^{ev}_{\mathbb{C}}\mathbb{C}^n.$$

The vector V is defined so that $p_0(x) = V(x)V(x)^*$. Furthermore, observe that $V(x)^*V(y) = c(x)^*c(y) + 1 \in \mathrm{End}(\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n)$. Therefore

$$\frac{1}{k!}\mathrm{tr}_{\wedge^*_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=0}^{2k} p_0(x_l)\right) = \frac{1}{k!}\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(V(x_{2k})^*V(x_0)\prod_{l=0}^{2k-1} V(x_j)^*V(x_{j+1})\right) =$$
$$= \frac{1}{k!\prod_{l=0}^{2k}(1+|x_l|^2)}\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\left(c(x_{2k})^*c(x_0)+1\right)\prod_{l=0}^{2k-1}\left(c(x_j)^*c(x_{j+1})+1\right)\right) =$$
$$= \frac{1}{k!\prod_{l=0}^{2k}(1+|x_l|^2)}\sum_{m=0}^{2k+1}\sum_{0\le g_1\le\cdots\le g_m\le 2k}\mathrm{tr}_{\wedge^{ev}_{\mathbb{C}}\mathbb{C}^n}\left(\prod_{l=0}^{m} c(x_{g_l})^*c(x_{g_l+1})\right).$$

□

As a consequence, the correct formula for $\tilde{f}_k$ of equation $(C.18)$ is given by

$$\begin{aligned}
&\tilde{f}_k(x_1,\dots,x_{2k}) := \qquad (18)\\
&= 2^{1-n}\sum_{I\in\Gamma_k}\iota(I)Q_I^{p_T}(\tilde{\nu}f(x_1),\dots,\tilde{\nu}f(x_{2k}))H_{\mathrm{d}+\mathrm{d}^*,I}(x_1,\dots,x_{2k}) =\\
&= 2^{1-n}k!\sum_{I\in\Gamma_k}\iota(I)\mathrm{ch}[p_T](\tilde{\nu}f(x_1),\tilde{\nu}f(x_{i_1}),\dots\tilde{\nu}f(x_{i_{2k}}))H_{\mathrm{d}+\mathrm{d}^*,I}(x_1,\dots,x_{2k}),
\end{aligned}$$

where the last expression is calculated as in Lemma 2.4. The correct form of Theorem $C.5.1$ is then given by:

Theorem 5.1. *Suppose that X and Y are smooth, compact, connected manifolds without boundary of dimension $2n$ and $f: X \to Y$ is Hölder continuous of exponent α. When $k > n/\alpha$ the following integral formula holds:*

$$\deg(f) = \frac{1}{2}\left((-1)^k\int_{X^{2k}}\tilde{f}_k(x_1,\dots,x_{2k})\mathrm{d}V_{X^{2k}} - \mathrm{sign}(X)\right)$$

where $\tilde{f}_k$ is as in (18).

References

[1] M.F. Atiyah, *K-theory*, lecture notes by D. W. Anderson, W. A. Benjamin, Inc., New York-Amsterdam 1967 v+166+xlix pp.