Abstract

Title: Making Sense of Negative Numbers

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Numbers are abstract objects that we conceptualize and make sense of through metaphors. When negative numbers appear in school mathematics, some properties of number sense related to natural numbers become contradictory. The metaphors seem to break down, making a transition from intuitive to formal mathematics necessary. The general aim of this research project is to investigate how students make sense of negative numbers, and more specifically what role models and metaphorical reasoning play in that process. The study is based on assumptions about mathematics as both a social and an abstract science and of metaphor as an important link between the social and the cognitive. It is an explorative study, illuminating the complexity of mathematical thinking and the richness of the concept of negative numbers. The empirical data were collected over a period of three years, following one Swedish school class being taught by the same teacher, using recurrent interviews, participant observations and video recordings. Conceptual metaphor theory was used to analyse teaching and learning about negative numbers. In addition to the four grounding metaphors for arithmetic described in the theory, a metaphor of Number as Relation is suggested as essential for the extension of the number domain. Different metaphors give different meanings to statements such as finding the difference between two numbers, and result in incoherent mappings onto mathematical symbols. The analyses show affordances but also many constraints of the metaphors in their role as tools for sense making. Stretching metaphors, from the domain of natural numbers to fit the domain of signed numbers, changes the metaphor, with unfamiliarity, inconsistency and limited applicability as a result. This study highlights the importance of understanding limitations and conditions of use for different metaphors, something that is not explicitly brought up during the lessons or in the textbook in the study. Findings also indicate that students are less apt to make explicit use of metaphorical reasoning than the teacher. Although metaphors initially help students to make sense of negative numbers, extended and inconsistent metaphors can create confusion. This suggests that the goal to give metaphorical meaning to specific tasks with negative numbers can be counteractive to the transition from intuitive to formal mathematics. Comparing and contrasting different metaphors could give more insight to the meaning embodied in mathematical structures than trying to fit the mathematical structure into any particular embodied metaphor. Participants in the study showed quite different learning trajectories concerning their development of number sense. Problems that students had were often related to similar problems in the historical evolution of negative numbers, suggesting that teachers and students could benefit from deeper knowledge of the history of mathematics. Students with a highly developed number sense for positive numbers seemed to incorporate negatives more easily than students with a poorly developed numbers sense, implying that more time should be spent on number sense issues in the earlier years, particularly with respect to subtraction and to the number zero.

CONTENTS

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A Fish called wonder *by Genevieve Ryan*[1]

I fish through my stream of consciousness
I hook onto a thought
It isn't light.
I wind it up to get a closer look
It struggles and pulses,
It flaps itself around the deck of my mind
It tries to swim away
So I hold it down
I see the fear of capture in its eye,
as though it wishes not to be interpreted.

At a closer look
I see it is more complicated than I first thought
It shines
It's coloured
It has lines to read
It has feelings
a life of its own.

It begins to tear at the mouth
and the fear in its eye accelerates to terror
Then – it ceases flailing
It holds quite still…

It is hardly there now…
And I begin to wonder – I begin to
It's going…
I'm loosing it…
I can't think where it could have –

Although I am hungry
I pull the hook out as quickly as possible
and proceed to toss the thought back into the stream
It swims away,
Faster
as though the water is a source of strength

I watch it think away
And smile
Because it did not sink, or die, or float –
It was more alive now that it flowed with the stream

And I was not hungry
I was filled with a stream of consciousness and many fish of wonder.

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[1] From the book "…regards, some girl with words. Genevieve's Journey" by Elisabeth Ryan, SidHarta Publishers, 2006.

Introduction

Personal background

As a child my favourite subject at school was mathematics. I can't remember ever not understanding, not until I reached university level. I loved the feeling when things worked out nice and neatly, like a puzzle. Algebra was particularly fun. Things could look so complicated and then you worked on them a bit and they cleared up and ended in a single number! I spent a lot of my time in maths class explaining things to my classmates, who often found my explanations easier to understand than those of the teacher. Choosing a profession as a teacher was not far fetched. During my years as a primary school teacher (grade 4-6) I loved teaching mathematics. I was, however, puzzled by the fact that some students did not understand things that seemed so evident for me. Why did they not understand? Why couldn't they *see* how it was all connected; the structures, the patterns? It seemed as if some children just didn't make friends with numbers. After 12 years as a primary school teacher I went back to university and studied mathematics. A master of science in mathematics degree later I started teaching in the teachers' education program at the University of Gothenburg.

When I first encountered the problem of understanding negative numbers among my pre-service teacher students I was teaching an off-campus course using on-line learning. My students produced essays every week on different topics related to mathematics education. Usually they prepared the essays by reading literature, doing exercises and discussing in the chat-room. If the essay didn't pass I would return it with comments for rewriting. Normally, one or two out of 19 essays would need rewriting, but when it came to negative numbers 9 students failed. After sending in a second version many of them still didn't pass and some of them had to rewrite four times. It was obviously the most difficult assessment during the year. I asked myself why. When I asked the students they all replied that it was because negative numbers were so abstract and lacked connections to the real world. I looked for more literature and tried to find a good way of teaching negative numbers but realized that there was not much to be found on the topic, and so I entered into a field of research.

When I embarked upon this journey I had no insight into the stormy waters I would have to travel on or where the final destination was to be. I have gained a lot of experience along the way and felt lost and bewildered many times. Of vital importance for the journey were the shores along the way where I could find inspiration and a sense of direction. Theoretically the most important inspiration came from ideas about conceptual metaphors which I encountered in the book *Where Mathematics Comes From* (Lakoff & Núñez, 2000), as well as other books about philosophy of mathematics and philosophy of mathematics education.

Methodologically, I was inspired by longitudinal studies (e.g. Helldén, 2006) and attracted by the possibility of following students' over a long period of time listening to them without having the responsibility of teaching them. I had come to realize how much there is to learn from one's students and what a pity it is that as a teacher one has so little time to really listen.

The PhD project was made possible by the Centre for Educational Sciences and Teacher Research at Gothenburg University. In 2005 a Graduate School in Educational Science was started and I was granted one of the first doctoral research stipends in this graduate school. Switching identity from teacher to researcher is not always easy and has its pros and cons. With a teacher background I know my questions are relevant for practice. I also have easy access to practice. Teachers, headmasters and students all treat me as an insider rather than an outsider. Being so closely connected to everyday questions of school life it was natural for me to do my research within the field of subject matter didactics (mathematics education). From an academic point of view this close relation to practice is sometimes seen as a complication. Certainly the idea of being an 'objective observer' is ruled out. Dislocating myself enough from practice to look at it with the eyes of a researcher is something I have worked on all along. The question of what good my research will do and to whom the outcome will be communicated is another intricate question. As a PhD project the first and foremost aim of the project is educational; it is a learning process for me. The results that come out of the project should be of interest to the international community of researchers in mathematics education and contribute to their field of knowledge. On top of this, or underlying it perhaps, is a desire to improve mathematics education in schools. I feel confident that I am not alone in this endeavour, and my hope is that this thesis will be a small contribution to the field of knowledge that lies in between that of teachers and researchers, and in between that of subject matter and pedagogical matters. The work presented here could be seen as part of Pedagogical Content Knowledge (Schulman, 1987) and closely related to the type of research conducted for example within the Mathematical Knowledge for Teaching and Concept Study (Ball et al., 2009; Ball, Thames & Phelps, 2008) and the Mathematics for Teaching project (Davis & Renert, 2009)

Negative numbers and their role in mathematics education

At the heart of the interest of this research project is the claim that the extension of the numerical domain from natural numbers to integers is an essential element in mathematical competence expected from students in schools in Europe. This extension is often made at about the same time as students are in the process of acquiring an algebraic language. Vlassis (2002) found that many errors made when solving equations are caused by the presence of negative numbers and concludes that it is the degree of abstraction created by the negatives that creates these difficulties rather than the presence of variables or the structure of the equation. This conclusion supports earlier findings suggesting that success in algebra may depend in part on a structural understanding of the relationship of addition and subtraction of directed numbers (Shiu, 1978).

Prather & Alibali (2008) pose the question of how people acquire knowledge of principles of arithmetic with negative numbers. Is it a process of detecting and extracting regularities through repeated exposure to operations on negative numbers or do they transfer known principles from operations on positive numbers? Exposure to operations on negative numbers is fairly scarce. For many problems in a school or every-day context it is often possible to find a solution without including negative numbers. Prather & Alibali (2008) used the following task in a study concerning knowledge of principles of arithmetic:

> Jane's checking account is overdrawn by $378. This week she deposits her pay check of $263 and writes a check for her heating account. If her checking account is now overdrawn by $178, how much was her heating bill?

This problem was represented by one student as -378 + 263 – x = -178 and by another as 378 – 263 + x = 178. Both representations are mathematically correct but only the first one involves negative numbers. As the problem is posed there is no mention of negative numbers. In many situations people avoid negatives if they can. For example when Celsius constructed the thermometer he originally placed zero at boiling point and 100 at freezing point so that for most everyday situations we would not have to deal with negatives, and on the Fahrenheit scale normal temperatures are all positive. Another example is the sewage workers in the municipality of Gothenburg, who have decided to place altitude 0 at 10.2 meters below sea level in order to avoid working with negative altitudes when digging in the ground.

An exercise taken from the section about negative numbers in a school textbook introduces negative numbers as a measure of time (Carlsson, Hake, & Öberg, 2002, p 19, original in Swedish):

> Emperor Augustus was born the year 63 BC. That could be written as year -63. He died year 14 AC. How old was he when he died?

This problem is easily solved by relating to zero and just adding 63 years before and 14 years after zero: 63 + 14 = 77. Writing 63 BC as -63 can give the calculation 14 – (-63) = 77. Writing it this way does not make it easier to solve the problem, so the motivation for doing so must be found elsewhere. The task could be used to illustrate the fact that 14 – (-63) = 14 + 63, but that is not self-evident and would need a lot of explicit reasoning. The important thing to ask about the exercise is whether the goal is to solve the problem or to develop reasoning with negative numbers. If students work by themselves with these kinds of exercises they will probably focus on solving the problem, and in doing so they might choose not to involve negative numbers. It cannot be easy to motivate students to solve problems using negative numbers which are new to them when they can easily solve such problems using numbers with which are comfortable. The main interest of this research is to explore how students make sense of negative numbers when they appear as part of school mathematics.

Many textbooks use visual representations such as the number line, a scale, a time line, and everyday life representations such as temperatures or money to explain subtraction with negative numbers. Most commonly such representations are referred to as models. Thomaidis (1993, p 81) claims that "…the various concrete models employed … are not convincing enough for the necessity of these numbers. Students know quite well that they can work out the difference between two temperatures or determine the position of a moving point on an axis without having to resort to the operations between negative numbers". In this thesis such models and their metaphorical underpinnings are assumed to play an important role in the sense making process. To discover more about that role and what mathematical development these models afford or constrain is an important research interest.

In recent years there has been a large amount of research about the importance of metaphors in mathematics education (cf. Danesi, 2003; English, 1997b; Frant et al., 2005; Parzysz, Pesci, & Bergsten, 2005). Some scholars, in particular Lakoff and Núñez (2000), claim that mathematics would not exist without its metaphors. Making use of a theory of conceptual metaphors (Lakoff & Johnson, 1980), Lakoff and Núñez assert that basic arithmetic is understood through four grounding metaphors. In these metaphors experiences from the physical world are source domains that give meaning to mathematical objects. Mathematical objects are created through these metaphors. In that sense abstract ideas such as mathematical concepts, inherit the structure of physical, bodily and perceptual experiences (Lakoff & Johnson, 1980; Sfard, 1994). The aim of this thesis is to use the theory of conceptual metaphors to better understand why the topic of negative numbers is so difficult to teach and to learn. Freudenthal dismisses what he calls "old models" with the words: "they are unworthy of belief" (1983, p 437). He points out what he calls a didactical asymmetry between positive and negative numbers in the models. His didactical analysis is sharp and clear. It is now 25 years ago and the lack of influence this has had on the teaching tradition in Sweden is remarkable. By placing a metaphorical perspective on the models they will be analysed here as source domains for conceptual metaphors.

Chapter 1 of the thesis will explore the historical and mathematical evolution of negative numbers and review the current state of educational research concerning the topic. After the theoretical framework discussed in chapter 2, a metaphorical analysis of contemporary negative number models is carried out and reported in chapter 3. In the second part of the thesis issues of sense making and use of metaphors when teaching and learning negative numbers are empirically explored.

Description of the project

The research project includes two studies described in detail in chapter 4. As an introduction to the research project a survey of prospective pre-school and primary school teachers was conducted, using a "testing and assessment" research style (Cohen*,* Manion & Morrison, 2000, p 80) with the purpose of measuring achievement and diagnosing strengths and weaknesses. This pilot study is reported in chapter 1.7. Following the pilot study, a design was made consisting of two interconnected case studies. The whole project was designed and conducted as a one-person PhD project. A decision was made to focus on one single class, which placed the project in the category of case studies where the purpose is "to portray, analyse and interpret the uniqueness of real individuals and situations through accessible accounts" (Cohen et al., 2000, p 79). All the characteristics of case studies listed by Cohen et al, such as in-depth detailed data, participant observation, non-intervention, an empathic approach and a holistic treatment of phenomena are characteristics of these two studies.

The research project as a whole can be characterized as mainly qualitative. It encompasses an epistemological position described as interpretive and a constructivist ontological position (Bryman, 2004, p 266). Whenever mathematical tasks are involved there are usually only a limited number of different answers plausible and a quantification of such answers gives a general idea of the level of mathematical achievement of a whole group. Such numbers, along with grades in the school report, are used in order to relate the researchers interpretations of a particular student's activities to his/her level of achievement and to national standards.

The aim of the study is to observe learning rather than to assess instruction. There was no presupposed 'best way' to teach negative numbers. Therefore the teacher of the class was given the responsibility of deciding when and how to teach the topic. However, it must be acknowledged that the presence of the researcher as a participant observer, recurrent interviews with the students, video recordings of some lessons and the awareness from the teacher's point of view of the researcher's interest influenced the observed activities.

Research aim

The general aim of this research project is to investigate how students make sense of negative numbers, and more specifically what role models and metaphors play in that process. This can be expressed as an interest in how students think mathematically about numbers and how their thinking changes when negative numbers are introduced in mathematics classrooms, and as a consequence: why do some students not learn to handle negative numbers the way we want them to? Since it is not possible to study thinking directly it is the external results of thinking, i.e. what students express in actions, words and writing, which make up the empirical data.

CHAPTER 1

Negative Numbers

This thesis is about negative numbers: what they are, where they come from, how they are taught, and most of all about how students understand them and make sense of them. The ambition of this first chapter is to make the reader acquainted with negative numbers both historically and mathematically. From section 1.3 and onwards the focus is shifted towards educational research about how negative numbers are taught and learned. Even if the picture drawn is not complete, it covers a wide variety of different studies explicitly about negative numbers as well as about related topics such as the number line and subtraction.

At one stage a systematic search for articles concerning negative numbers was made in all the volumes between the years 2000 and 2007 of nine international journals for mathematics education research. Only five relevant articles were found and of these only two explicitly dealt with negative numbers. When a wider search was made using the data bases ERIC and MATHDI around 30 relevant articles, including conference proceedings, appeared after 1990[2]. Based on these articles, the current state of research concerning teaching and learning about negative numbers can be described as a research field dominated by fairly small, isolated design based or experimental studies. Most of the research is done in a tradition of cognitive psychology with influences from sociocultural theory, although in many of the empirical articles the theoretical framework was missing or tacit. Questions that have been asked greatly concern the use of models. It was seen in the systematic review that the amount of research articles concerning teaching and learning negative numbers was fairly limited. The challenge was to find it. A few older references appeared in many of the articles, particularly Glaeser (1981), Freudenthal (1983), Janvier (1985) and several articles by Gérard Vergnaud. In section 1.3 to 1.6 this research about teaching and learning negative numbers is reported and discussed, thematically organised around four themes; *integers*, *subtraction*, *the minus sign*, and *the role of models and metaphors*. The last section of the chapter reports on a small study made at the start of the research project about making sense of negative numbers. This pilot study gave direction to the rest of the project and initiated the research questions.

[2] Only articles in English were included.

1.1 Historical evolution of the negative number concept

uråldrig är den väg vår tanke följer,
i urtidsdunkel sig dess början döljer
Pär Lagerkvist[3]

A mathematical concept is not a definite and durable thing; on the contrary it changes and evolves over time. It can be both an object and a process; processes can turn into objects and vice versa. Hersch (1997, p 81) gives *2* as an example of an evolving mathematical concept and writes:

> When Euclid went to market, he knew that two oboli plus two oboli was four oboli. If we're talking about '2' and '4' as *adjectives* modifying 'oboli', and + and = as a commercial operation, we and Euclid agree, just as we would agree that the sun rises in the East. But if we're talking about the *nouns* '2' and '4' – meaning some sort of autonomous objects – and operations and relations on *them*, there are differences between us and Euclid.

For Euclid, 2 was a counting number; a natural number. For us it is an integer with an additive inverse: -2. For Euclid it was discrete, isolated. For us it is a rational number and a point on a continuous number line. Numbers that were once adjectives modifying and describing other objects eventually evolved into nouns that acquired adjectives of their own, such as positive, negative, rational or real. In time, these adjectives too would turn into nouns (Kaplan, 1999).

This section will describe and discuss some of the changes the mathematical concept we today name 'negative number' has gone through. Scholars of mathematical history are well attuned to historical facts and decisive changes in the historical evolution of negative numbers as a mathematical concept (cf. Beery, Cochell, Dolazal, Sauk, & Shuey, 2004; Glaeser, 1981; Heeffer, 2008; Ifrah, 2002; B. G. Johansson, 2004; Mumford, 2010; Schubring, 2005; Thompson, 1996). There are traces of an early notion of what was to become the concept of negative numbers in old cultures such as the Han dynasty in China around 200 BC – 200 AD, in ancient Greece around 500BC – 250 AD, and in ancient India and Persia during the 1st century AD.

Eastern mathematics, originating in Babylon and flourishing in ancient India, was mainly concerned with counting and tallying; numbers represented discrete quantities, but also order. It was in India that the place value system and base 10 became the dominant numerical system, opening up the way for efficient algorithms. In that sense the Eastern culture can be said to be the birthplace of algebra. For Eastern mathematicians numbers did not need to make geometrical sense; they only needed to make sense in relation to each other. Zero and negative numbers made sense as ordered numbers when subtractions were carried out. In modern notation this could be shown as: 2-1=1; 2-2=0; 2-3=-1.

[3] Ancient is the path our thought does follow, in prehistoric obscurity its origin hides.

Western mathematics was more concerned with geometry. It originated in ancient Egypt and spread to Greece where it flourished and became a philosophy as well as a technical science. Numbers were created as a means of measuring space. Numbers came to represent space and had to make geometrical sense to be accepted. There was no meaning of negative distances or areas. A line with no length was not a line so zero only made sense as a representation of nothing. Eastern and Western mathematics met in the Islamic world, developed and was preserved for the future in the works of al-Khwarizmi during the 1st century (cf. Seife, 2000). When Eastern and Western mathematics met, the Indian place value system eventually took over, but not much seemed to have happened to the concept of negative numbers for a long while.

> The problem of elaborating a coherent mathematical status for negative numbers developed over long periods of time … It challenged the traditional first understanding of mathematics, its first 'paradigm' in Kuhn's terms, its understanding of being a science of quantities: of quantities that, while being abstracted to attain some autonomy from objects of the real world, continued at the same time to be epistemologically legitimized by the latter. The various cultures succeeded over a long time in finding various auxiliary constructions that permitted them to remain within the existing paradigm. (Schubring, 2005, p 149)

The real development of negative numbers as mathematical objects came with the introduction of algebra. Although algebra was born in Indian mathematics at the time of Brahmagupta it was not until the 16th century AD that algebra flourished, and at that time very much as a result of an increasing symbolisation of mathematics and the acceptance of zero as a number. An overview of the concept of negative numbers in Western mathematics from the 16th century on is given in stages of development suggested by Arcavi and Bruckheimer (1983):

- *16th century*: Non recognition of negatives, e.g. Viète
- *17th century*: Recognition of negatives as roots of equations, e.g. Descartes
- *End of 17th and beginning of 18th century*: Use of negatives with reservations because of the contradictions arising from their use, e.g. Arnauld, Wallis
- *18th century*: Free use of negatives and their entry into textbooks, but without mathematical definitions, e.g. Saunders, Euler, *and* opposition to negatives, e.g. Frend, Masères
- *19th century*: Attempts to give a mathematical foundation to negatives, e.g. Peacock, de Morgan, Hamilton
- *End of 19th century*: Formal mathematical definition of negative numbers

In the study of historical texts, Gleaser (1981) identified about 20 different 'obstacles' for understanding negative numbers. He highlights particularly six authors and the obstacles identified in their texts. These obstacles can be described as follows:

1. Inability to manipulate isolated negative quantities.
2. Difficulty to make sense of isolated negative quantities.

3. Difficulty to unify the number line, that is to see it as one line, one axis, instead of two semi-lines opposite one another with different symbols, or understanding positive and negative quantities as having different quality.
4. Difficulty to accept two different conceptions of zero: *zero as absolute,* where zero is understood as the bottom, below which there is nothing; and *zero as origin*, where zero is an arbitrary point on an axis of orientation from which there is two directions.
5. Stagnation in the phase of concrete operations and not entering the phase of formal operations, i.e. seeing numbers as representing something substantial, concrete.
6. The wish for a unified model that will cover addition and multiplication.

Schubring (2005) is critical to the theory of epistemological obstacles described by Glaeser, pointing out that the historical development of a concept is not necessarily linear. He also criticises the anachronistic view of concept development visible when an obstacle is presented as having been always self-evident. It is only in the hindsight of the development of the concept of negative numbers that for example unifying the number line could be seen as an obstacle. According to Schubring the development of a concept follows a winding path, and many different conceptions can exist simultaneously in a culture, but also in the writings of a single mathematician. Schubring claims that a study of the concept development must not look exclusively to the development of the rule of signs, but to the more general one of the *existence* of negative numbers. "For clarifying the concept of negative numbers, the separation between the concept of numbers from that of magnitudes or *quantités* will prove to have been decisive" (Schubring, 2005, p 16).

Mumford (2010) describes the evolution of negative numbers as culturally different, claiming that China and India both seemed ready to extend the number domain to include zero and negative numbers, whereas European mathematics resisted the extension. He blames Euclidian mathematics for this, stating that in Euclidian mathematics numbers only appear in three forms, of which none can be negative or zero. The three forms are: i) number as magnitude, ii) number as a multitude composed of units, and iii) number as ratio between two magnitudes of the same kind.

Whether the different changes in the conception of negative numbers be labelled 'obstacles' to be overcome or clarifying 'insights' once they appear, different historical accounts tend to point at the same aspects as thresholds in the evolution of the concept. Once these different aspects came to influence the concept, it changed and adjusted, opening new possibilities of interpretation and meaning. The following account of this evolution will focus different aspects of the concept of negative numbers and follow these aspects through the years. These aspects are: *the notion of opposite quantities*, *the difference between quantity and number*, *the sign rules*, *the different meanings of the minus sign*, *zero as a number*, *the number line* and *the genesis of symbolic algebra*.

Opposite quantities

Originally all signs and symbols used to represent numbers and operations were first order representations, i.e. direct representations of physical experiences. At different times and places through history various cultures have had similar conceptions of numbers closely tied to quantities. Negative numbers as quantities of an opposite quality have been used to keep track of economic transactions. Red and black counting rods were used for negative and positive quantities in China 2000 years ago. They thought of the negative quantity as a *quantity to be subtracted* from another quantity or as an *amount yet to be paid*. Even if they used the rule *same signs take away, different signs add together* they would never deal with subtractions that did not originate in problems concerning concrete objects. The 'negative' numbers that arose in the manipulation of counting rods were always numbers that could be represented as negative quantities; as quantities to be subtracted.

In India, Brahmagupta wrote about negative numbers in his work *Brahmasputa-siddhanta* from the year 628. He called positive quantities *property* or *fortune*, and negative quantities *debt* or *loss* (Schubring, 2005). This idea of symbolising economic transactions, and particularly the notions of owning or gaining as opposed to losing or having debts, is found in many historical texts, for example in *Liber Abbaci*, written by Leonardo of Pisa**,** better known as Fibonacci, in 1202. Fibonacci mentioned negative numbers as *debitum* (loss). Viéte mentioned opposite magnitudes: *nomed adfirmatum* and *nomen negatum* in his work *In Artem Analyticum Isagoge*. Johann Widman was in the 15th century the first to use the symbols + for addition and - for subtraction in print in his book *Mercantile Arithmetic* in 1489. (Beery et al., 2004). A surplus in measure was to be denoted with the sign + and a deficiency with the sign -. At this time the idea of a negative quantity became accepted, even if many mathematicians still did not accept negative solutions.

In Europe during the 18th century negative numbers were usually regarded as being 'less than nothing', metaphorically linked to debt, as opposed to possession. Fontanelle deepened the concept of opposites by differentiating between quantitative aspects and qualitative aspects: "every positive or negative magnitude does not have just its *numerical* being, by which it is a certain number, a certain quantity, but has in addition its *specific* being, by which it is a certain *Thing* opposite to another" Fontanelle (1727), (quoted in Schubring, 2005, p 100).

Since the idea of something less than nothing was difficult to accept, many mathematicians proclaimed that if a problem had a negative solution it was wrongly stated. Stating the problem in the opposite way would produce a positive answer. A solution with a negative value can be interpreted as a debt. 3-7 = -4 so having 3 and paying 7 renders a dept of 4. Note that you do not get a debt of -4. By interpreting it as a debt, the negativity is removed from the number (Heeffer, 2008). If the problem is to find out how much money there is left, the answer will be negative, but if the problem is stated in terms of how large the debt will be, the answer will be positive. In his work with the *encyclopaedia*, d'Alembert (1717-1783) described negative quantities as opposite of positive quantities by the idea that where the positives finish the negatives start. Furthermore, he defines negative quantities as "less than nothing" and "the absence of"; -3 is the absence of 3 (Glaeser, 1981). According to Gleaser, Kant (1724-1804) tried 1762 to define opposites in two different ways*: l'opposition logique*; that is opposite and contradictory, and *l'opposition réelle*; opposite but not contradictory.

Mumford (2010, p 116) distinguishes between quantities that are *naturally positive* and *signed* quantities. Naturally positive quantities are all measures of weight, length, area and volume as well as numbers of objects and proportions. Examples of signed quantities are money transactions (profit/loss) and measures in relation to a centre point of reference (north/south, above/below, backwards/forwards). When mathematics is a science for modelling quantitative problems many of the modelled problems do not make any sense at all of negative quantities. Modern notation, writes Mumford, obscures this subtle differences of meaning.

It seems as if the idea of opposite quantities, although a very old notion and at the root of what was to become a negative number, has constraints as well as affordances[4]. Symbolising a problem about quantities is one thing, but thinking about numbers as quantities limits the possible interpretations. "What facilitates thought impoverishes imagination" (Kaplan, 1999).

From quantities to numbers

Although negative numbers were by and by tolerated and accepted in procedures, it took humankind many centuries to acknowledge them as numbers in their own right. Many attempts were made by European mathematicians during the 16th, 17th and 18th century to free the concept of number from the concept of quantity, but operations with numbers were in fact practical manipulations of concrete magnitudes. Although negative numbers were used quite frequently during the 18th century, they were mainly justified by analogy with various physical interpretations such as debt. 19th century mathematicians sought a more rigorous mathematical justification. Instead of focusing on a meaning of algebraic symbols they shifted their attention to the laws of operations on these symbols (Katz, 1993, p 611). They shifted attention from

[4] The terms *affordance* and *constrain* are here used as defined by Donald Norman: an affordance is a perceived actionable property, what a person perceives as possible to do. A constrain is what is perceived as not possible. [http://www.jnd.org/dn.mss/affordances_and.html, 100118]

what numbers and operations *were* to what they *did* and how they *acted*, more in line with Indian mathematics (see also Kaplan, 1999, p 141).

The appearance of negative numbers in the process of solving equations caused many mathematicians to ponder on the very existence of these numbers. In the Islamic world al-Khwarizmi, in his work *Arithmetica*, wrote texts on arithmetic and algebra avoiding negative numbers as much as possible. Influenced by Greek geometry al-Khwarizmi would verify all his procedures geometrically. Negative solutions could not exist since there was no logical geometrical justification for negative lengths. He had no notion of negative numbers standing by themselves and rejected negative solutions calling them *absurd* (Schubring, 2005). In India, Bhaskara II used negative numbers in his calculations but rejected negative solutions to quantitative problems since "people did not approve of negative absolute values". However, he reinterpreted negative geometrical line segments as having the opposite direction (Schubring, 2005, p 38).

Viète **(**1540-1603), the 'father of modern algebra', completely avoided negative numbers. Although the term negative and the law of signs appear in his writings, negative numbers as such are not there. He writes that in order to subtract A minus B, A must be greater than B. "Magnitudes A and B, the former is required to be greater than the latter. Subtraction is a disjoining or removal of the lesser from the greater" (quoted in; Arcavi & Bruckheimer, 1983). In 17th century Europe negative numbers were partially accepted thanks to their efficiency in calculations, but the question of a meaning of negative quantities was still a major problem. A large number of algebra textbooks dealt with negative numbers although they were still not mathematically defined. Descartes **(**1596-1650) accepted negatives as *roots of equations* but did still not assign to them the status of numbers. He called them *racines fausses* (false roots). They were also considered as additive inverses; *le default d'une quantité* (the absence of, or defect of, a quantity). "It often happens that some of the roots are false, or less than nothing. Thus if we suppose x to stand also for the defect of a quantity, 5 say, we have $x + 5 = 0$", Descartes in *Œuvres VI Géométrie* (quoted in Schubring, 2005, p 47).

Heeffer (2008) states that "…isolated negative quantities formed a conceptual barrier for the Renaissance habit of mind". A solution to the problem of negative quantities was to redefine number. Around 400 BC Aristotle made the distinction between *number* and *magnitude*, or more precisely between discrete and continuous (Katz, 1993, p 52). The implications of this distinction would much later prove to be utterly important for the acceptance of negative numbers. Stevin (1548-1620) developed ideas about negative numbers as *computational artefacts*. He suggested that instead of saying take away 3 one should say add -3 (Glaeser, 1981). The idea that positive and negative numbers had equally legitimate mathematical status appeared in books at the beginning of the 18th century. Rivard (1697-1778) wrote: "negatives are not the negation or absence of positives; but they are certain magnitudes opposite to those which are regarded as positive" (quoted in; Schubring, 2005, p 84). The idea that numbers do not in all respects need to be the same as quantities was pointed out by Saunderson (1682-1739).

One step along the way was the separation between arithmetic and algebra. Arnauld (1612-1694) handled letters as representations of quantities and investigated their *identities* and *relations* in respect to mathematical principals instead of their meanings. Euler (1707-1783) made a clear distinction between arithmetic and algebra:

> Arithmetic treats of numbers in particular, and is the science of numbers properly so called; but this science extends only to certain methods of calculation, which occur in common practice. Algebra, on the contrary, comprehends in general all the cases that can exist in the doctrine and calculation of numbers. Euler (1770) in *Elements of Algebra*, (quoted in Beery et al., 2004).

With this distinction made, it was no longer a problem for Euler to handle negative and imaginary numbers algebraically. Numbers, in an algebraic context, did not need to have a physical meaning.

MacLaurin (1698-1746) thought of algebra as generalised arithmetic. He saw negative quantities as just as real as positive quantities, as in excess and deficit, money owed to and by a person, the right-hand and left-hand direction along a horizontal line, the elevation above and depression below the horizon. (Beery et al*., 2004)*. He allowed the subtraction of a greater quantity from a lesser of the same kind if it made physical sense. One could subtract a greater height from a smaller height to get a negative height, but not a greater quantity of matter from a smaller one (Katz, 1993, p 552). A quantity can in itself not be negative, it is negative only in comparison to its opposite. MacLaurin, like Saunderson, stressed the difference between concrete quantities and abstract quantities (numbers). "While abstract quantities can be both negative and positive, concrete quantities are not always capable of being the opposite of each other", MacLaurin (1748) (quoted in; Schubring, 2005, p 94). MacLaurin stated that science is more apt to examine *relations* between things than these things inner essence. It does not really matter if mathematical objects exist independently of us or not, or if we can describe them and their characteristics perfectly, the important thing is that their connections are coherent and clearly deduced. According to Glaeser (1981), MacLaurin reasoned axiomatically, starting with non questionable principles. He maintained that physical evidence of mathematical objects is not essential, although it can be helpful to confirm our conclusions.

By the18th century there was a clear separation between aspects of number that could be physically justified (quantity and magnitude), and aspects of number that could only be formally or axiomatically justified. However, the latter's expansion into the modern science of mathematics was not done without opposition. As a comment to MacLaurins work on negatives, Frend (1758-1841) wrote in *The principals of algebra* (1796), (quoted in Arcavi & Bruckheimer, 1983):

> Now, when a person cannot explain the principles of a science without referring to metaphor, the probability is that he has never thought accurately upon the subject. A number may be greater or less than another; it may be added to, taken from, multiplied into, and divided by another number; but in other respects it is very untraceable: though the whole world should be destroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it may obey: it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous … This is all jargon. (Frend, 1796)

Frend's conception of numbers was that of a physical magnitude. He was unwilling to extend the number concept and could therefore not accept a subtraction of a greater from a smaller. Another opponent was Masères (1731-1824) who wrote his *Dissertation on the Use of the Negative Sign in Algebra* in 1759, where he discarded the use of negative numbers except to indicate the subtraction of a larger quantity from a lesser. He argued that negative roots should never have been admitted into algebra.

At the beginning of the 19th century negative numbers did not exist in daily life. Shopkeepers and bankers kept a double bookkeeping combining debts and assets at the very end, and not until 100 years later did people start speaking of temperatures below zero. The gap between everyday use of mathematics and the science of mathematics became apparent as the ideas of algebra advanced.

Sign rules

Rules for calculations with positives and negatives have been formulated in many cultures. For example by the Chinese in 250, the Arabs at the time of al-Kwarizmi, in India around year 600 and in Greece in 250 (Beery et al., 2004). All these mainly express rules without producing mathematical proofs for them. Treating numbers as quantities, as in India, produced sign rules relating to the magnitude of the numbers involved, making it necessary to separate the sign rules into many different cases. In Brahmagupta's treatise *Bramasphuta-siddhanta* from 628 we find these rule, (quoted in Mumford, 2010, p 123-124):

> [The sum] of two positives [is] positive, of two negatives, negative; of positive and negative [the sum] is their difference; if they are equal, it is zero. … [If] a smaller [positive] is to be subtracted from a larger positive, [the result] is positive; [if] a smaller negative from a larger negative, [the result] is negative; [if] a larger from a smaller their difference is reversed – negative becomes positive and positive negative. [chapter 30 verses 30 - 31]

If a justification existed it was generally a geometrical one, as show in figure 1.1. Multiplication with two negatives appear in problems of the type $(a\text{-}b) \cdot (c\text{-}d)$, where *a*>*b*>0 and *c*>*d*>0, the numbers *b* and *c* being not proper negative numbers in the modern meaning but simply numbers to be subtracted. Diaphantos stated that "a number to be subtracted, multiplied by a number to be subtracted, gives a number to be added" (Cajori, 1991). Al-Khwarizmi (780-850) explained multiplication of two negatives using the law of distribution similar to that in figure 1.1, but in Western mathematics a proof based on the distributive law of arithmetic first appeared around 1380 in the works of Maestro Dardi titled *Aliabraa argibra* (Heeffer, 2008). Knowing that 3 ¾ times itself must be the same as (4 -¼) · (4 -¼), the distributive law gives -¼ times -¼ equal to $+^{1}/_{16}$.

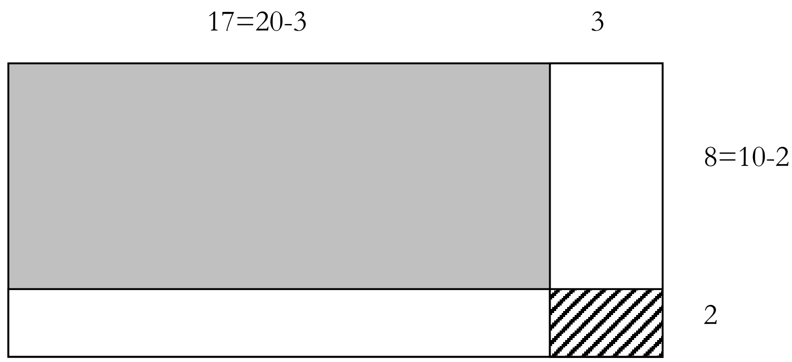

FIGURE 1.1: Geometrical illustration of the law of distribution: 8 · 17 = (10-2) · (20-3) = 200 – 40 – 30 + 6 = 136

When negative numbers started to appear as isolated objects it was noted that the rule of signs justified geometrically as in figure 1.1 was in fact not dealing with negative numbers, which initiated alternative ideas. Cardano (1501-1576) argued that the +6 in the calculation was not the result of a multiplication of -2 by -3 but an area we must add since we subtracted it twice (Heeffer, 2008). Cardano introduced an alternative rule of signs stating that multiplying minus by minus gave minus. He thought of positives and negatives as constituting two different areas that were to be held separate. Multiplying positives resulted in a positive, multiplying negatives resulted in a negative (Schubring, 2005). Arnauld (1612-1694) claimed that the basic principle of multiplication is that the ratio of unity to one factor is equal to the ratio of the second factor, i.e. in the product *ab* $1/a = b/ab$ or $1/b = a/ab$. Accepting that multiplication of two minuses be a plus would violate the principle, implying that 1/-4 = -5/20, which means that a bigger number is to a smaller number (1 to -4) as a smaller number is to a bigger number (-5 to 20), which contradicts the proportion concept (Arcavi & Bruckheimer, 1983; Heeffer, 2008; Schubring, 2005). Leibniz answered Arnauld that these are not truly ratios, but since they are useful they are tolerable, in spite of their lack of rigor[5]. The two concepts of ratio and size (i.e. order) do not a priori carry over to the extended number domain without creating

[5] "Veras illas ratione non esse. Habent tamen usum magnum in calculando toleranter verae, rigorem quidem non", in Leibniz, G.W. (1712) *Math. Schriften* Vol. 5, Ch. 29, p. 387-389 (quoted in Arcavi & Bruckheimer, 1983).

contradictions. The relationship between order and proportion is a property of positive numbers only.

Leibniz' willingness to use what was useful was followed by several attempts to find justification. Prestet (1648-1691); interpreted (-1) · (-4) = 4 as a *negative addition* of -4, much like Euler who in his work for beginners tried to justify the sign rules by distinguishing between the multiplier and the multiplicand. For the multiplication $(-a) \cdot (-b)$ it is clear, wrote Euler, that the absolute value is ab, and since $(-a) \cdot b$ already is $-ab$ then $(-a) \cdot (-b)$ must be ab. Saunderson (1682-1739) referred to grammatical issues; "that two negatives make an affirmative; which is undoubtedly true in Grammar" in his work *The Elements of Algebra* (1741), (quoted in; Arcavi & Bruckheimer, 1983). Along the same line, d'Alembert (1717-1783) suggested that $(-a) \cdot (-b)$ be interpreted as $-(a) \cdot (-b) = -(-ab) = +ab$.

Saunderson also used arithmetic progressions to show the rule of signs for multiplication. If the progression, 3; 2; 1; … is multiplied by the common multiplier 4, the products will also be in arithmetical progression, 12; 8; 4; …. He used this assumption in his proof, shown in figure 1.2, an assumption not necessarily proven to hold for the extended number domain.

progression	multiplier	new progression	this shows
4 ; 0 ; -4	3	12; 0; -12	3 · (-4) = -12
3; 0; -3	-4	-12; 0 12	(-4) · (-3) = 12

FIGURE 1.2: Saunders proof of the signs rules for multiplication.

During the 18th century negative numbers caused frustration among mathematicians and mathematics educators in particular. Boyer writes in his *History of Mathematics* (Boyer, 1968, quoted in Arcavi & Bruckheimer, 1983):

> Algebra textbooks of the 18th century illustrate a tendency toward increasingly algorithmic emphasis, while at the same time there remained considerable uncertainty about the logical bases for the subject. Most authors felt it necessary to dwell at length on the rules governing multiplication of negative numbers, and some rejected categorically the possibility of multiplication of two negative numbers.

A fully algebraic verification of the sign rules was given by Laplace (1749-1827), and thus, Gleaser (1981) notes, finally the six obstacles were overcome and a formal approach was attempted.

$(-a) \cdot (b + (-b)) = (-a) \cdot 0 = 0$ gives us that $-ab + (-a) \cdot (-b) = 0$
since $-ab + ab = 0$ we have that $(-a) \cdot (-b) = ab$.

Different meanings of the minus sign

Today the minus sign is said to have two different functions; one is a binary function symbolizing the operation subtraction, the other is a unary function symbolizing a negative number or an additive inverse to a number. These two meanings have not always been acknowledged.

The Chinese thought of a negative number as a number to be subtracted from another quantity or an amount yet to be paid (Beery et al., 2004) and thus did not differentiate between the two meanings of the minus sign. Subtracting 4 from 2 was put down in the counting boards as two red rods and four black rods, eliminating two of each, giving a result of 2 red rods representing 2 yet to be subtracted. In modern notation: 2 - 4 = -2. Although the first minus sign has a binary function and the second one a unary function, they have the same meaning in the Chinese interpretation. In contrast, the algebraic view on multiplication as iterated addition differentiates between the multiplier and the multiplicand. This view was extended by Arnauld so that if the multiplier was negative it represented an iterative subtraction; -5 times -3 meant to *take away* 5 times *negative* 3, which was the same as *setting down positive* 15 (Schubring, 2005). In this example the two minus signs, although both are unary signs, are given different meanings.

As long as numbers were on a par with quantities, there was no differentiation between the two meanings of the minus sign. In a textbook widely spread during the 17th century the German mathematician Wolff treated negatives as quantities to be subtracted. "The quantities marked with a sign of - have to be regarded as nothing else but debts, and by contrast the others bearing the sign of + as ready money. And therefore the former are called less than nothing, because one must first give away enough to settle one's debt before having nothing. Wolff (1750), (quoted in Schubring, 2005, p 96). Since 0+3 = +3 it follows that 0-3 = -3. Writing -3 was simply seen as a shorter version of writing 0-3.

A clear distinction between the two meanings came with Whitehead (1861-1947) who argued that: "mathematicians have a habit, which is puzzling to those engaged in tracing out meanings, but very convenient in practice, of using the same symbol in different though allied senses. The one essential requisite for a symbol in their eyes is that, whatever its possible varieties of meaning, the formal laws for its use shall always be the same.", in Whitehead *An Introduction to Mathematics* (quoted in; Arcavi & Bruckheimer, 1983). Thus, + can be used to symbolise the *addition of a number* or the *addition of an operation*. Consequently, (+3) + (+1) = +4 shows the addition of the two operations of adding 3 and adding 1 as the operation of adding 4. Whitehead emphasizes the concept of extension. The extension principle implies attaching new or extended meanings to familiar symbols

Zero: from nothing to number

> In the history of culture the discovery of zero will always stand out as one of the greatest single achievements of the human race.
>
> *Tobias Danzig*

> The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought.
>
> *Alfred Whitehead*

When zero first appeared it was not as a number, but as a 'placeholder', a symbol for a blank place in the abacus. 20 means 2 tens and *no* units. The use of a position system including zero is found in many documents from India and Southeast Asia in the 7th and 8th century (Ifrah, 2002). Zero was a digit, not a number. It had no value since a number's value comes from its position in the order relation with other numbers. Zero had no position since we always start counting with one, and it could therefore be placed anywhere in the numbers sequence (Seife, 2000). Today we still see zero being placed after 9 in some places, for example on a telephone or a computer. Zero thus became a symbol for 'nothing'. Although Greek mathematicians developed great mathematics they lacked a specific word for zero, which might explain why zero for so long failed to gain the status of a proper number but stayed only as a representation of the empty set. "Zero – balanced on the edge between an action and a thing (and what are numbers when it comes to that: adjectives or nouns?)", writes Kaplan (1999). When numbers were related to shape zero did not make sense. What shape could zero have? What length and what area? Even though the Greeks saw the usefulness of zero in their calculations they still rejected it as a number (Seife, 2000).

In India, nothingness and the infinite were considered the beginning and end of everything. These concepts were actively explored and often used interchangeably, and gave rise to a great number of different names for zero such as: *kha* (space, the universe), *nabha* (heaven, sky), *randhra* (hole), *sunya* (emptiness), *vintu* (point) and *ananta* (the vast heaven). The signs used for this concept were either a circle or a point (Ifrah, 2002).

Gradually zero changed from being an idea of an absence of any number to the idea of a number *for* such absence (Kaplan, 1999). In *Brahmasputa-siddhanta*, from the year 628, zero was considered a number as "real" as the rest, yet different from all the others (Schubring, 2005). Also rules for adding, subtracting, multiplying and dividing with zero were formulated. Zero was defined as the result of a number subtracted from itself: $a - a = 0$. Instead of dwelling on what zero *is* mathematicians became concerned with how it *behaved.*

The counting numbers have a neat relationship between their cardinality and their ordinality. One is the first number, two the second and so on. When zero is included we get the sequence 0 1 2 3… and suddenly that neat relationship is ruined. One becomes the second number, two the third. This complication became evident when measuring time. When the first year of a person's life ends we say that she is one year old. Was she zero years before that day? She lives for another day, week, month, and we will still refer to her as one year old. Which is the first year, is it the first year she lives or the year when she is one? In Korean age a person is one year (sal) from the day she is born and during the first calendar year of her life. According to Kaplan (1999), the Roman calendar had no year zero between 1 BC and 1 AD. Considering a number line as in figure 1.3, it is easy to see that between the year -1 and the year 1 is a span of two years. But which of those two years is the year zero? And when asked how many numbers there are between 2 and 6 is the correct answer 3 or 4? The answer depends on if we are talking about the numbers as points on the line or as measurements of segments of the line.

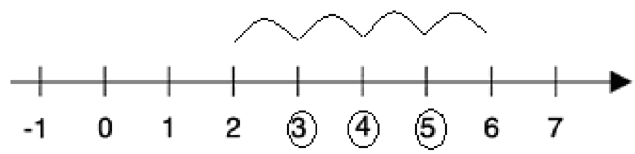

FIGURE 1.3: A number line with the numbers between 2 and 6 marked as four segments and as three points.

The West kept the notion of zero at a distance until Fibonacci, in his *Liber Abaci*, reintroduced it in the 13th century. As a consequence of a growing monetary accounting in the 14th century, zero took on a role as a balance-point between negative and positive quantities. In double-entry book keeping a loss was entered twice in different columns illustrating debit from one point of view and credit from another point of view. This made negative numbers as real as positive numbers (Kaplan, 1999). However, a very strong conception of zero as a nil zero, an absolute starting point, made any number less than zero impossible to comprehend. Stifel (1487-1567) clearly states that negative numbers are less than zero (infra 0, id est infra nihil) and terms them *numera absurdi* or *ficti*, in contrast to the positive numbers termed *numera veri* (Schubring, 2005).

Subtracting a larger quantity from a smaller quantity is absurd when zero is only interpreted as nil zero, or zero absolute, but would be perfectly intelligible with zero as origin (see section 1.3 for more details). In connection with the two interpretations of zero, Gleaser (1981) brings up another obstacle, the number 1. For Euclide numbers denoted measures, and 1 was considered the unity. The unity could always be changed, it was not a number! This difficulty is still present today when it comes to defining powers in mathematics. If $x \cdot x = x^2$, what is x^1 and x^0 ? The index number symbolises the number of factors. What is the product of one factor, or no factors? Many contradictions in terms also appear when real problems are posed using these numbers. What is for example speed calculated to zero? That is no speed! One single model or context cannot at the same time incorporate the two different interpretations of zero as absolute and as origin.

The change from zero as absolute to zero as origin was initiated by Descartes (1596-1650) who placed zero as the number all other numbers would relate to in the coordinate system. Descartes unified numbers and shapes; the Western art of geometry with the Eastern art of algebra became different perspectives of the same domain. Every shape could be expressed by an algebraic expression in the form of a function $f(x,y)$. Zero was at the centre of the coordinate system and as such implicit in all geometric shapes (Seife, 2000). The new notion of zero caught on, and in 1714 Reynaud (quoted in Schubring, 2005, p 78) published a textbook where zero was treated as a separating term. He wrote:

> It is evident that zero, or nothing, is the term between the positive and negative magnitudes that separates them from each other. The positives are magnitudes added to zero; the negatives are, as it were, below zero or nothing; or to put it in a better way, zero or nothing lies between the positive and negative magnitudes; and it is the term between the positive and negative magnitudes where they both begin. … We call this term the origin of the positive and negative magnitudes; and at this term there are neither positive nor negative magnitudes; thus there is zero or nothing. (Reynauld, 1714)

The number line

Illustrating the extended number domain on a number line seems to be a fairly recent invention. Apart from a very early notation of a negative as a line segment in a contrary direction found in the works of the 12th century Bhaskara II (Mumford, 2010, p 126) there is no historical evidence of this conception until it appears in Europe in the 17th century. In *A treatise of Algebra both Historical and Practical* from 1673, Wallis was perhaps the first to introduce the number line showing negative numbers (Thomaidis, 1993). Wallis used the number line to illustrate addition and subtraction of negative numbers by imagining a moving man, starting from a point A, advancing 5 yards and then retreating 8 yards, and thus ending up 3 yards backward from his starting point. According to Mumford (2010, p 138) Wallis wrote: "That is to say, he advanced 3 Yards less then nothing … (which) is but what we should say (in ordinary form of Speech), he is Retarded 3 Yards; or he wants 3 Yards of being so Forward as he was at A". The subtlety here lies in the fact that in ordinary speech a negative forward is always spoken of as a positive backward. At about the same time as the works of Wallis appeared, Newton also illustrated subtraction of a greater number from a smaller one by drawing a number line.

Visualizing numbers as points on a line slowly spread, and towards the end of the 17th century all sorts of measuring scales had developed. However, negative numbers were often avoided in different ways. Even the first temperature scales avoided negative numbers; Fahrenheit by setting 0 very low and Celsius by setting 0 as the boiling point for water and 100 as the freezing point, later reversed (Johansson, 2004, p 411). Only at the end of the 19th century did people start speaking of temperatures below zero.

Descartes, to whom we attribute the system of coordinates, did not in fact, ever use an axis ranging from -∞ to +∞ (Glaeser, 1981). In all his work he utilized two opposite, separate semi-lines such that a negative line had the opposite direction of a positive line. He also showed most of the functions he worked with only in the first quadrant, thus only dealing with positive values. Instead of handling sign rules he tried changing the position of the origin in order to obtain equations where all the roots were positive. Carnot (1803) used a complete number line as an axis rather than the two semi lines that Descartes used. The revolutionary input from this was a change of perspective about the number line and distances on that line. Instead of speaking about numbers as distances from zero, they were related to an arbitrary point of departure. The number line had become unified and all numbers on it were equal since it was possible to shift the origin.

The negatives on the number line are ordered opposite to the positives. Girard (1595-1632) interpreted negative solutions in a geometric way: "the minus solution is explicated in geometry by retrograding; the minus goes backward where the plus advances" (Katz, 1993, p 407). Hamilton (1805-1865) believed that our intuitive notion of order in time is more deep-seated than that of order in space and, as geometry is founded on the latter, so can algebra be founded on the former" (Arcavi & Bruckheimer, 1983). He developed his theory and wrote: "the opposition of the Negative and the Positive being referred … *not* to the opposition of increasing and diminishing a *magnitude*, but to the simple and more extensive contrast between the relation of *Before* and *After*, or between the direction of *Forward* and *Backward*" (Hamilton, quoted in Beery et al., 2004). He justified the product of two negatives as positive in the following way: assuming a forward starting position, a product of two negatives represents exactly two reversals of direction.

Symbolical algebra

With the increasing symbolisation and the growth of algebra as a mathematical science new questions were posed about the negative numbers. Peacock (1791-1858) described the important changes in the use of algebra in his two books: *Arithmetical Algebra* (1842) and *Symbolical Algebra* (1845). The old principle of permanence of equivalent forms implies that an extended number domain should obey the same laws for operations as the natural numbers. Peacock realised that this does not necessarily hold. "From a philosophical point of view there is no reason why the principle should hold unless we want it to. But from a didactical point of view … [it] is often extremely useful". Symbolical algebra adopts all the rules from arithmetical algebra but removes all restrictions, "thus, symbolic subtraction differs from the same operation in arithmetic algebra in being possible for all relations of value of the symbols or expressions…" (Arcavi & Bruckheimer, 1983). Furthermore, Peacock did not acknowledge negative numbers in arithmetical algebra but assigned them strictly to the province of symbolical algebra. This is a more narrow meaning of arithmetic dealing solely with natural numbers compared to today's meaning of arithmetic which deals with all real numbers. When the principle of permanence is used to prove arithmetical laws in an extended number domain, it does not lead to real "proofs", it merely accepts the principal as an axiom in the new domain. It can be used to suggest the extension of concepts, but it is not a universal principal.

Peacock's symbolic algebra provided the necessary logical justifications for negative numbers that mathematicians were looking for, and the acceptance came with the work of mathematicians such as Hamilton and Weierstrass. Weierstrass introduced the mathematical symbol for absolute value, enabling us to distinguish between magnitude (absolute value) and value (order relation) of a number. Hence, rules for working with signed numbers became easier to write and negative numbers became more acceptable. Negative numbers were at last freed from the "less than nothing" definition attached by Stifel in the 16th century (Beery et al., 2004).

In his Theory of Complex Numbers from 1867, Hankel finally showed a complete change of perspective by accepting the negative numbers as formal constructs in an algebraically consistent structure unnecessary to justify through a concrete model. He thus freed the concept of number from the concept of quantities and started dealing with numbers as second order representations, that is, material representations not of real objects but of "mental models" (Damerow, 2007, p 25). The construction of the logical foundations for the real number system as we know it today is a result of the works of Weierstrass, Dedekind and Cantor, and was published in 1872 (Beery et al., 2004).

A short summary

> Since mathematics indeed was deemed to be 'the science of quantity', those who criticized notions of negatives and imaginaries were justified in their objections. The way in which mathematicians on the whole circumvented such ambiguities was by *redefining the nature of mathematics*. They made arithmetic deal with something far more abstract than quantity, namely, numbers.
>
> *Albert A. Martínez*

The above short history of negative numbers describes how these numbers took a long time to evolve into what they are today. Epistemological changes were necessary concerning the idea of what numbers represented. The idea that numbers represented quantities where every quantity, however small it may be, is greater than zero needed to give way to ideas of numbers representing a value in relation to a midpoint, an origin or a point of balance. The minus sign was first used to represent a quantity yet to be subtracted, implying a very subtle difference between subtraction and negativity. But in order to operate more generally with negative numbers they needed to become reified as mathematical objects in their own right and thus the meaning of negativity became detached from subtraction. It will become clear in the following section that in modern mathematics subtraction and negative numbers can *either* be defined as two separate things, *or* as the same thing (the axiomatic approach where subtraction is defined as the addition of an additive inverse).

In early history negative numbers were accepted in practical situations, to represent for example debts, without being accepted as mathematical objects. With the introduction of algebra as an independent field and the increasing use of algebraic methods instead of geometric, the limitation of subtraction to an operation on *quantities* became at last an insurmountable conceptual conflict. Accepting and legitimizing *zero* was analogously problematic. Eventually a radical solution outside the paradigm was necessary, and algebraization opened up the way. Zero and negative numbers were not given meaning as quantities but through their *relations* and *connections* with other mathematical entities in an algebraic structure.

1.2 A modern definition of negative numbers.

As far as the laws of mathematics refer to reality, they are not certain;
and as far as they are certain, they do not refer to reality.
Albert Einstein

During the 20th century mathematics has become a much more formalised science. The formal entry of the negative numbers into mathematics is based on a set of axioms for natural numbers. Natural numbers (**N**) are defined on the basis of set theory, and integers (**Z**) are defined as equivalence classes of ordered pairs of natural numbers (cf. Arcavi & Bruckheimer, 1983). The following section will give a brief account of how the number domain is formally extended from **N** to **Z**.

Extension of the set of natural numbers to the set of integers

Take the set of natural numbers: **N**={1,2,3…}
On the set of natural numbers we have defined the operations addition, subtraction, multiplication and division with the restriction that they must produce a number in **N**. The set **N** is closed under addition and multiplication, meaning that all additions and multiplications with numbers in **N** will produce a new number in **N**. The set **N** is not closed under subtraction and division. In order to create closure, the number domain is extended. When the set **N** is extended to the set of integers, **Z**, it becomes closed under subtraction, and

when the set **Z** is extended to the set of rational numbers, **Q**, we have also closure under division.

Defining integers on the basis of natural numbers

Take $a + n = b$.

The solution $n = b - a$ is true for $n \in$ **N** under the restriction $b > a$.

Since a solution to $a + n = b$ is determined uniquely by the two natural numbers a and b, an *integer* n is defined as the ordered pair (a, b) for $a, b \in$ **N** *without* the restriction of $b > a$. The *equivalence* classes of all such ordered pairs of natural numbers is defined as the set of integers; **Z**.

An equivalence relation (~) is defined in the following way:

$(a, b) \sim (c, d)$ **if and only if** $a + d = b + c$.

We thus have (10, 8) ~ (4, 2) **if and only if** 10 + 2 = 8 + 4

(10, 8) ~ (4, 2), corresponding to the integer -2, and so 10+(-2)=8 and 4+(-2)=2.

An ordered pair (a, b), where $b > a$ is denoted by n

An ordered pair (a, b), where $a > b$ is denoted by $-n$

Defining addition, multiplication and subtraction with integers

Since addition and multiplication are well defined in **N** we can extend addition and multiplication to these pairs as follows: for any pairs (a, b) and $(c, d) \in$ **Z**

- ~ *Addition* is defined as: $(a, b) + (c, d) = (a+c, b+d) \in$ **Z**
- ~ *Multiplication* is defined as: $(a, b) \cdot (c, d) = (ad + bc, bd + ac) \in$ **Z**

The final task is to define subtraction on **Z**. Take $a, b, c, d \in$ **N**

Subtraction of $(a, b) - (c, d)$ = [in natural number arithmetic] = $(a\text{-}c, b\text{-}d)$

To show that $(a\text{-}c, b\text{-}d) \in$ **Z** it is necessary to make sure that $a\text{-}c$ and $b\text{-}d \in$ **N**

Instead of the pair (a, b) choose the equivalent ordered pair $(a+c+d, b+c+d)$.

Now $(a+c+d, b+c+d) - (c, d) = (a+c+d-c, b+c+d-d) = (a+d, b+c)$ and since $a, b, c, d \in$ **N** it follows that $a+d$ and $b+c \in$ **N** which proves that $(a+d, b+c) \in$ **Z**. So **Z** is closed under subtraction.

- ~ *Subtraction* is defined as: $(a, b) - (c, d) = (a+d, b+c)$

Numerical examples:

- ~ Addition: 5 + (-2):

Let the integers 5 and -2 be represented by the ordered pairs (1, 6) and (3, 1)

(1, 6) + (3, 1) = [by def.] = (1+3, 6+1) = (4, 7) which corresponds to -3 ∈ **Z**

- ~ Multiplication: $2 \cdot 3$:

Let the integers 2 and 3 be represented by the ordered pairs (3, 5) and (4, 7)

$(3, 5) \cdot (4, 7)$ = [by def.] = $(3 \cdot 4 + 3 \cdot 7, 3 \cdot 4 + 5 \cdot 7)$ = (41, 47)

(41, 47) corresponds to 6 ∈ **Z**

~ Multiplication: (-2) · (-3):
Let the integers -2 and -3 be represented by the ordered pairs (5, 3) and (7, 4)
(5, 3) · (7, 4) = [by def.] = (5 · 4 + 3 · 7 , 3 · 4 + 5 · 7) = (41, 47)
(41, 47) corresponds to 6 ∈ **Z**

~ Subtraction: 3 - (-5):
Let the integers 3 and -5 be represented by the ordered pairs (10, 13) and (20, 15)
(10, 13) - (20, 15) = [by def.] = (10+15, 13+20) = (25, 33)
(25, 33) corresponds to 8 ∈ **Z**

An axiomatic approach

Defining negative numbers in terms of ordered pairs of natural numbers, and all operations in terms of additions, made the negative numbers fit into a mathematical structure already well established. A slightly different way of achieving the same thing is to start with the axioms that define the basic structure of arithmetic, as in the following section (based on Rudin, 1976).

Let S be an *ordered set*. An order on S is a relation, denoted <, with the following properties: If $x \in S$ and $y \in S$, then one and only one of the statements: $x < y$, $x = y$, or $y < x$ is true. If $x, y, z \in$ S and if $x < y$ and $y < z$, then $x < z$

Let an *ordered field* F be an *ordered* set with two operations called *addition* and *multiplication* which satisfy the axioms for addition, multiplication and distributivity.

~ Axioms for addition:
Let ⊕ be an operation on F called addition.
If $x \in F$ and $y \in F$, then their sum $x \oplus y \in F$.
Addition is commutative: $x \oplus y = y \oplus x$ for all $x, y \in$ F
Addition is associative: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
F contains an element 0 such that $0 \oplus x = x$ for every $x \in$ F
To every $x \in$ F corresponds an element $-x \in$ F such that $x \oplus -x = 0$.
$-x$ is the additive inverse to x

~ Axioms for multiplication:
Let multiplication, denoted by ⊗, be an operation on F.
If $x \in F$ and $y \in F$, then their product $x \otimes y \in$ F.
Multiplication is commutative: $x \otimes y = y \otimes x$ for all $x, y \in$ F
Addition is associative: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
F contains an element $1 \neq 0$ such that $1 \otimes x = x$ for every $x \in$ F
To every $x \in$ F, $x \neq 0$, corresponds an element $1/x \in$ F such that $x \otimes 1/x = 1$
$1/x$ is the multiplicative inverse to x.

~ Axiom for the distributive law:
For all $x, y, z \in$ F holds that: $x \otimes (y + z) = (x \otimes y) \oplus (x \otimes z)$
The axioms imply that:
If $x \oplus y = 0$, then $y = -x$ and also that $-(-x) = x$
For $x \in$ F: if $0 < x$ we call x positive and if $x < 0$ we call x negative.

In this second version subtraction is not defined as an operation. Negative numbers are defined as additive inverses to positive numbers, and follow the same axioms for addition and multiplication as the positive numbers. Subtraction is thereby defined as adding the additive inverse, i.e. adding the negative number. With this approach the meaning of the signs rules is implicit in the axioms, and the difference between a binary and a unary function of the minus sign becomes a non issue; it is always unary, and when it acts binary there is implicitly an addition of an additive inverse.

Properties of the new number domain

When the number domain is extended some properties of number do carry over into the new domain. These properties are for example: Property of order, commutativity and associativity for addition and multiplication; and distributivity for the relation between addition and multiplication. This was visible in the axiomatic system described in the previous section. However, one important property does *not* carry over: *the principles of ratio*; i.e. the relation between order and proportion.

For natural numbers, if $a > b$ and $a/b = c/d$ then $c > d$ and there exists a number $n > 0$ such that $a = nc$ and $b = nd$. Hence for any $a, b, c \in \mathbf{N}$ if $a > b$ then $ac > bc$. Including zero in the domain and taking $c = 0$ will lead to a contradiction: if $a > b$ then $0 > 0$. Including negative numbers in the domain and taking $a \neq 0$ and $b = -a$ implies that if $a > -a$ then $ac < -ac$. Taking $c = -1$ will lead to a contradiction: if $a > -a$ then $a(-1) < -a(-1)$, in short: if $a > -a$ then $-a < a$. This contradicts the basic property of order.

To overcome this flaw in the system the concept of absolute value was defined so that $|a|=|-a|= a$ for any $a \in \mathbf{N}$, and within the boundaries of absolute values the principles of ratio was restored. Thus, negative numbers came to have two contradictory features of value; one connected to the order and another to the magnitude. If $-a < -b$, then $|-a|>|-b|$.

With the historical evolution and modern definition of negative numbers in mind the rest of this chapter will review research concerning teaching and learning about these numbers. It will be shown that many of the aspects of negative numbers brought up in the historical review, and many of the various conceptions of negative numbers that have flourished in past times, will appear in modern educational practice.

1.3 Conceptualizing negative numbers and zero

> Interpretation of the magnitude and direction of negative numbers in the minds of pupils is the most important stage in learning the concept of negative numbers.
> *Altiparmak & Özdoan*

Do negative numbers exist? That is a good question. When children in first and third and fifth grade were interviewed, Peled, Mukhopadhyay and Resnick (1989) found that they often totally lacked a conception of negative numbers. They had two different strategies for subtractions like 5 - 7; either they inverted the subtraction and treated it as 7 - 5, or they stated that 5 - 7 yielded zero. When asked about something like -5 + 8 they simply ignored the minus sign and treated it as 5 + 8. In third and fifth grade they were more likely to generate negative numbers as answers showing that they believed in their existence, even if they did not treat them within the conventional rules, i.e. answering that -5 + 8 = 13. Since these students did not receive any formal instruction about negative numbers before 6th grade the researchers conclude that they construct mental models that include negative numbers before school instruction drawing on their models for positive numbers to do this.

When the existence of negative numbers has been accepted, many questions arise about the properties of these numbers. Stacey and her colleagues have done a lot of work on students understanding of fractions and decimals. One of the common misinterpretations they found was that decimals sometimes were associated with negative numbers and conceived of as smaller than zero. A Decimal Comparison Test developed and used in a previous study (Stacey & Steinle, 1998) was given to 553 teacher education students (Stacey et al., 2001b) and interviews were teacher students who had made errors in zero comparisons (Stacey, Helme, & Steinle, 2001a). About 1% of the students completely identified decimal numbers with negative numbers and about 7% could order non zero decimals, but thought that decimals such as 0.6 and 0.22 were less than zero. When reasoning about the items in the interview the students revealed conceptions of zero as a whole number, and since decimals are parts, they must be smaller than a whole. Stacey and her colleagues sought for an explanation of the confusion between decimals, fractions and negative numbers that proved to be so common and proposed as an explanation that:

> i) The natural numbers are the primary elements from which concepts of other numbers are constructed
>
> ii) A metaphor of the mirror is involved in the psychological construction of fractions, negative numbers and place value in three different ways.
>
> iii) The observed confusion is a result of students' merging (confusing or not distinguishing between) the different targets of the same feature of the mirror metaphor under different analogical mappings.
>
> (Stacey et al., 2001a, p 224)

Fractions are (multiplicative) reciprocals and mirror the whole number with 1 as the point of reflection. Negative numbers are additive inverses and mirror the positive numbers with 0 as the reflection point. Both these metaphors show an increase to the right on the number line. Place value, as a contrast, has the ones column as the reflection point with values increasing by multiples of ten to the left of the ones column and decreases to the right.

A mental number line is an important feature in the conceptualization of negative numbers. The SNARC effect (Spatial-Numerical Association of Response Codes) indicates that people in general associate large magnitudes with right side of space and small magnitudes with left side of space (Fischer, 2003; Fischer & Rottmann, 2005). This effect has been taken as evidence for an existing mental number line where numbers are ordered from left to right, and increasing to the right. Lately these researchers have investigated if the mental number line extends to the left of zero. In studies of the existence of a SNARC effect on the extended number line results have been somewhat contradictory and two possible representations are discussed: i) the holistic representation where absolute magnitude is integrated with polarity, and ii) the components representation where absolute value is stored separately from polarity (Fischer & Rottmann, 2005; Ganor-Stern & Tzelgov, 2008). According to Fisher and Rotmann's results negative numbers differ from positive numbers in that they are not *automatically* associated with space. While positive numbers are perceived as larger, negative numbers are not necessarily perceived as smaller. In a number such as -34, the minus sign signals small and the magnitude of 34 signals large. The most plausible indication is that negative numbers are not represented as such but generated when required from positive numbers. If the sign or the absolute value is processed first depends on the task (Ganor-Stern & Tzelgov, 2008).

There were two forms of number line representation in the historical development described by Glaeser (1981). He claimed that one of the big obstacles that needed to be overcome was the unification of the number line; i.e. to see it as one line, one axis, instead of two semi-lines opposite one another with different symbols, or understanding positive and negative quantities as having different quality. These two versions of the number line were also found by Peled et al. (1989) among children in grades 1, 3, 5, 7 and 9.

The Continuous Number Line model (CNL) is more advanced, representing positive and negative numbers as ordered and increasing from left to right as shown in figure 1.4. In contrast to the one line of the CNL, the Divided Number Line (DNL) joins two symmetrical strings of numbers at zero and stresses movement away from zero in both directions. This model requires special rules for crossing zero, usually in the form of partitioning the number to be added or subtracted into the amount needed to reach zero and then the rest. Children who display the DNL often speak of operating 'on the negative side' (Peled et al., 1989, p 108).

FIGURE 1.4: Illustrations of the continuous number line and the divided number line.

Ball (1993) introduced negative numbers as a subject matter in a teaching experiment in 3rd grade, in an endeavour to engage children in "intellectual and practical forays and help to extend their ways of thinking mathematically" (p 374) by letting the students explore different representations and models. She noted that in spite of the fact that they had experiences with negative numbers (such as temperatures below zero and owing things or scoring negative points in games) they would still assert that "you can't take 9 away from 0" and that "zero is the lowest number" (Ball, 1993, p 378). She also found that the absolute value aspect (the magnitude) of negative numbers was very powerful. When negative numbers enter the scene the natural numbers become positive, and both types of numbers have *magnitude* and *direction*. For positive numbers these size properties coincide, but for negative numbers they diverge. Ball (1993) writes:

> Simultaneously understanding that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers.

With the presence of both negative and positive numbers, zero emerges as a special number without either magnitude or direction. Gleaser (1981) identified two historical conceptions of zero; i) *zero as absolute*; understood as the bottom, below which there is nothing, and ii) *zero as origin*; an arbitrary point on an axis of orientation from which there are two directions. Although different, these are both conceptions of zero as a position on the number line. A study of 40 students age 13-15 solving additions and subtractions using a graded number line (Gallardo & Hernández, 2007) revealed that some students interpreted zero as origin on the number line, but others avoided zero. Either zero was not symbolized at all, or symbolized but ignored during operations. For some students the numbers one and minus one were considered origins on the number line. Even among students who accepted negative numbers there were some who did not accept zero as a number. The researchers concluded that recognizing negativity does not necessarily entail identifying zero as a number. Gallardo and Hernández (2005, 2006) also investigated 12 and 13 year olds conceptions of zero and found five meanings of zero:

1. nil zero: that which has no value
2. implicit zero: that which is used during operations but does not appear in writing
3. total zero: that which is made up of opposite numbers
4. arithmetic zero: that which arises as a result of arithmetic operations
5. algorithmic zero: that which emerges as a solution to an equation

Consequently, conceptualizing zero includes seeing the duality of zero on the one hand as a null element ($a + 0 = a$), and on the other hand as contained by opposites ($a + (-a) = 0$), as well as seeing zero as a point of origin. Steinbring (1998) brings up an additional meaning of zero as a *relation* between other mathematical objects. Zero is the *whole set* of all possible pairs of opposite numbers. This meaning of zero can be extended to all numbers so that they are interpreted as *differences* or as *relations* in the general form [$a – b$]. 5 - 7 can thus be solved by a generalisation of the specific: {… 5 - 7, 4 - 6, … 0 - 2, (-1) - 1, (-2) - 0, (-3) - (-1), …}all represent the same relation (Steinbring, 1998, p 523).

Another aspect of zero is its role as a reference point (benchmark number) and an indicator of symmetry. In a study where adults were asked to quickly give the midpoint of two displayed numbers (Tsang & Schwartz, 2009) it was found that they were faster if the number pair was symmetric around zero or anchored (one number being zero) and that problems furthest from symmetry were solved the slowest. The authors concluded that the adults' representation of integers relied on zero as a structural pillar to aid operations due to its indication of symmetry.

Numbers are a precise way of expressing quantities, but quantities and numbers are not the same. Vergnaud (1982) distinguished three types of problems that can be represented by natural numbers: *quantities*, *transformations* and *relations*. Relations can be perceived without quantification (the same, more or less) but are often quantified. An example of a problem (from Nunes & Bryant, 2009) involving relations would be: *In Ali's class there are 8 boys and 6 girls. How many more boys than girls are there?* The number 2 that is asked for is not a quantity; it is the *relation* (the difference) between the two quantities. Often problems that involve relations are rephrased so that all numbers refer to quantities. In the above problem we could for example say that all boys go and find a girl partner, how many girls will not have a partner? (ibid, p 5). Vergnaud (1982) showed that children found problems involving relations much more difficult than problems about quantities or transformations of quantities. He hypothesised that when working only with transformations and relations children have to go beyond natural numbers and operate in the domain of whole numbers. Vergnaud also described six main categories of problems involving quantities (measures), transformations and relations. Categories IV and V are problems that can be represented by the equation $a + x = b$ where a, b and x are directed numbers.

Category IV: *Composition of two transformations*: Peter won 6 marbles in the morning. He lost 9 marbles in the afternoon. Altogether he lost 3 marbles.

Category V: *A transformation links two static relationships*: Problem: Peter owes Henry 6 marbles. He gives him 4. He still owes him 2 marbles.

Vergnaud stresses that time transformations and static relations are not adequately represented by natural numbers" because they involve elements that should be represented by directed numbers. However, students meet these categories long before that learn about directed numbers. Thus, there is a discrepancy between the structure of problems that children meet and the mathematical concepts they are taught" (Vergnaud, 1982, p 46). This result raises the question of whether directed numbers could be introduced earlier as a means of dealing with these types of problems.

To summarize: accepting the existence of zero and numbers smaller than zero, distinguishing between decimals and negative numbers, conceptualizing the duality of zero, seeing numbers as relations, distinguishing between the magnitude and the value of numbers, locating negative numbers to the left space and unifying the number line are some of the features emphasized by researchers as important in the process of making sense of negative numbers.

1.4 Understanding subtraction

> In the case of negative numbers, [interiorization] is the stage when a person becomes skilful in performing subtractions.
> *Anna Sfard*

Addition and subtraction are inverse operations: 7 + 9 = 16 implies that 16 – 7 = 9. Nunes and Bryant (2009) report on a series of studies concerning what they call 'the complement question'. A child is told that $a + b = c$, and thereafter asked what $c - a$ is. These studies show that the step from the first to the second operation is extremely difficult for children in their first years at school. Gilmore (in; Nunes & Bryant, 2009, p 24) showed that children need to understand the inverse relation before they can learn to add and subtract efficiently.

Understanding the inverse relation between addition and subtraction is a first step, and when negative numbers are concerned addition and subtraction also need to be identified as interchangeable. Additive problems can often be solved as either additions or subtractions and identifying these is an important feature according to Bruno and Martinón (1999). In a classroom intervention study teaching negative numbers in 7th grade in Spain, Bruno and Martinón studied students' understanding of the identification of addition with subtraction and found three levels of identification (ibid, p 798). One of the questions asked was if the equality 4 - (-7) = 4 + 7 was true. The levels of identification were:

1. the two operations were not identified since one is a subtraction and the other an addition
2. the two operations were identified on the operational level: students could state that 'any subtraction can be transformed into an addition of the opposite', or 'the minus changes the sign inside and that makes plus'
3. the two operations were identified by their meaning through the use of double language: 'owing -7 is the same as having 7'

In ordinary use of language we have different words to distinguish direction. We say for example that the temperature increased or decreased. We do not normally say that the temperature increased by -5 degrees, or that we have -5 kronor when we mean that we are 5 kronor short. Using a double language is by Bruno and Martinón taken as an important step in developing knowledge. They conclude that: "The meaning assigned to these two operations is controlled by students' prior knowledge of positive numbers. Our results show that such identification requires long, thoroughgoing classroom work." (Bruno & Martinón, 1999, p 808).

The double language was also dealt with in an animation experiment with 75 6th grade students and a control group of the same size (Altiparmak & Özdoan, 2010). The students were given an animation that was meant to illustrate that taking away -3 was the same as giving 3, as shown in figure 1.2. It is not self-evident that this double language becomes meaningful simply because it is repeated and visualized.

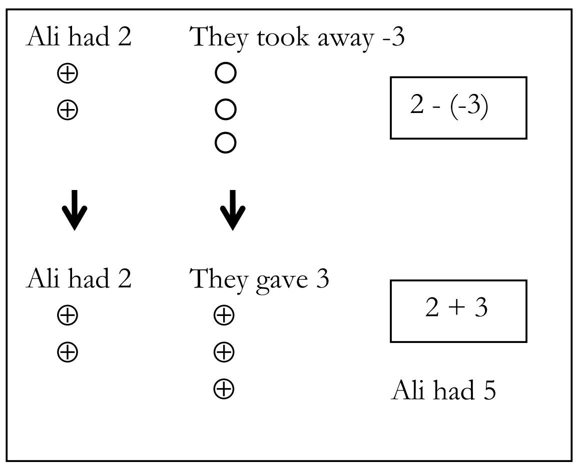

FIGURE 1.5: Animation of 2 - (-3) = 2 + 3 = 5 (Altiparmak & Özdoan, 2010).

In another teaching experiment with students in 6th grade, Linchevsky and Williams (1999) used a double abacus with markers of two colours to keep track of opposite objects. In one case the different colours marked people coming in and going out of a room, and in the other case the colours marked points given to different teams in a dice game. The major difference between this model and the animation shown in figure 1.5 is that the focus was on the *difference* between number of positive and negative markers shown on the double abacus. Every now and again a control was reported about the state of people in the room. If the abacus had 3 negative markers and two positive markers the state was at that moment -1. After that four more people came in and in the next report the difference was +3. (3 negative markers and 6 positive markers). This could be written mathematically as (-1) + (+4) = (+3), or more conventionally: -1 + 4 = 3. Now, as the work progressed a situation arouse where the students ran out of one kind of markers and realized that instead of adding another positive marker representing a person coming into the room they could just as well take away a negative marker representing a person who had left the room. If the difference is +4 and another person comes in the new difference is +5, and if the difference is +4 and a person who has left the room comes back in the new difference is also +5, so 4 + 1 = 4 - (-1) = 5. In the second context adding a point to the score of one team would give the same score difference as taking away one point from the other team. Experimentally they arrived at the identification of addition as subtraction and the double language was used in a situation where it had meaning. Both the above experiments treat additive situations of the type state-variation-state (Bruno & Martinón, 1999), but in the latter the states represent *relations* and in the former *quantities*.

Subtraction situations with natural numbers are characterised in terms of take away, combine and compare (Fuson, 1992). Often 'take away' situations dominate in early algebra education, and Fuson writes: "consideration of the full range of addition and subtraction situations requires an extension to the integers, which necessitates an avoidance of terminology or educational practices in the lower grades that interfere with later comprehension of these integers" (ibid p 247). Vergnaud's (1982) analysis indicates that integers need to enter into the classroom mathematical discourse at an earlier stage to facilitate working with compare and combine situations. Consequently, when integers have been introduced, it becomes necessary to work less with take away situations and more with combine and compare situations. In a learning study where teachers collaborated in the planning and revising of a lesson on negative numbers they found that switching from the take-away situations to compare situations was a critical aspect for understanding negative numbers (Kullberg, 2010).

1.5 Understanding the minus sign

> A book is made up of signs that speak of other signs, which in their turn speak of things. Without an eye to read them, a book contains signs that produce no concepts
> *Umberto Eco*

The minus sign is used both as a sign of operation (subtraction) and as a sign indicating a negative number (polarity). Many researchers have studied procedural errors that are caused by students' lack of conceptual understanding of the minus sign (Gallardo, 1995; Kullberg, in press; Küchemann, 1981; Vlassis, 2004). In some ways it is unfortunate that the sign is the same, and there has been experimental research and teaching strategies where different signs are used for the two different purposes. For example Ball (1993) used a small circumflex (^) above the numeral instead of a minus sign for the purpose of focusing the children's attention on the idea of a negative number as a *number*, not as an *operation*. Küchemann (1981) wrote a smaller elevated sign ($^{-}4$, $^{+}8$) and Shiu (1978) wrote a horizontal line on top of the numeral. The convention in Swedish mathematics textbooks is to write negative numbers in brackets. However, this is not systematically done. In the equality: -4 - (-3) = -1 all the included numbers are negative but only one of them is marked as such by the brackets. In a commonly used Swedish textbook for 8th grade negative numbers are introduced as follows (Carlsson et al., 2002, p 17)[6]:

> Positive numbers are numbers that are larger than zero.
> Negative numbers are numbers that are smaller than zero.
> Negative numbers are written with a minus sign in front.
> Often brackets are put around a negative number, e.g. (-4), to show that it is the negative number 4 and not the subtraction minus 4.

Subtracting a positive number gives the same result as adding a negative number and once you have fully understood this, using the same sign opens the possibility to choose whether to treat for example -4 as a subtraction by 4 or as a negative number 4. In some cases these are simply two perspectives of the same thing, just as ¾ can be viewed as the division of 3 by 4 or as the fraction three fourths. These different meanings of the minus sign could be described as the operational and the structural aspects of the sign. In school education the operational meaning (minus as the operation sign for subtraction) is introduced long before the structural meaning.

Vlassis (2004) coined the term 'negativity' to show the multidimensionality of the minus sign. Negativity is referred to as a map of the different uses of the minus sign in elementary algebra. These are:

1. Unary function; relates to the sign as attached to the number to represent a negative number as a relative number or as a solution or a result. It represents the formal concept of negative number.
2. Binary function; relates to the minus sign as an operational sign. It represents activities of taking away, completing, finding differences and moving on the number line.
3. Symmetry function; also operational but relates to the activity of taking the opposite, of inverting.

Vlassis (2002) noted that many of the difficulties students had when solving equations did not depend on the structure of the equation or the appearance of variables but was a result of the degree of abstraction caused by the presence of negative numbers. One of the major types of errors found was detachment of the number from the minus sign preceding it (previously described by Herscovics & Linchevski, 1994). In a second study 8th grade students were interviewed about solutions to tasks with polynomials that had previously been given in a test. The research questions concerned their procedures as well as what meanings they gave to the minus sign (Vlassis, 2004, 2008, 2009). Vlassis found that for most students the minus sign had its meaning only in relation to the procedure, generally as subtraction (binary function) when it was placed

[6] Original in Swedish.

between two like terms, but for some students also as a splitting sign separating for example *20 + 8 - 7n - 5n* into the two calculations *20 + 8* and *7n - 5n*. One student said: "*4 - 6n - 4n*, is only to split. I first make that operation, then the other one. I made 4, there was no other without a letter. I wrote *4-* , kept the -, then I made *6n - 4n*, I found *2n*, then I have *4 - 2n"* (Vlassis, 2004, p 479).

A minus sign at the beginning of a polynomial was always considered as a unary sign. Vlassis particularly noted that no student explicitly considered that the minus sign could have a double status. Learning about the minus sign is a two-step process. First it is necessary to discern the different meanings, and then to see that the meanings are interchangeable depending on the context. Interpreting -3 + 8 as the same as 8 - 3 can be explained in colloquial language by saying that "if I first subtract 3 and then add 8 or if I first add 8 and then subtract 3 I will end up with the same amount." To reason like that entails an imaginary 'starting number', so what is really treated is the expression $x - 3 + 8 = x + 8 - 3$. However, since subtraction is not commutative as a rule, a mathematically correct justification could be to treat the expression as an addition of signed numbers: $^{-}3 + {}^{+}8 = {}^{+}8 + {}^{-}3$. Once this has been done it is possible to again see it as a subtraction and view $^{+}8 + {}^{-}3$ as 8 - 3. The flexible use of the different meanings of the number sign as described above is what is named being flexible in negativity. Vlassis (2009) concluded that the 8^{th} grade students had not entered into an algebraic discourse characterized by flexibility in the use of the minus sign and did not move in an appropriate manner between the unary and binary point of view. To reconcile ones initial conceptions about operating with natural numbers and the algebraic rules required for negative numbers, and to become flexible in negativity are described by Vlassis (2004) as two major conceptual changes.

The symmetry function of the minus sign appears strangely absent from empirical data in all the reported research. Another missing feature is the conceptualization of the plus sign. In the domain of integers all numbers are signed (except zero[7]), but the positive numbers are very rarely symbolized with a plus sign. In the domain of integers 6 - 4 means $^{+}6 - {}^{+}4$. Should perhaps 'becoming flexible in negativity' also include 'becoming flexible in positivity'?

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7 In more advanced mathematics zero sometimes does have a sign, but in such cases it can have either of the two signs.: +0 and -0.

1.6 The role of models and metaphors

> The various approaches to teaching integers can be classified into two main types, one of which is essentially abstract while the other relies on the use of concrete models or embodiments to give meaning to the integers and the operations to be performed upon them.
>
> *Dietmar Küchemann*

In the findings reported from a large test and interview study investigating 818 students age 13-15 concerning their understanding of positive and negative numbers (Küchemann, 1981), the most difficult items turned out to be two types of subtractions: ${}^{-}2 - {}^{-}5 = \_\_$ (44% correct solutions), and ${}^{-}2 - {}^{+}3 = \_\_$ (36% correct solutions). As a comparison, the multiplication item ${}^{-}4 \cdot {}^{-}2$ was correctly solved by around 75%, and the subtraction item ${}^{+}8 - {}^{-}6$ by 77% of the children. It appeared that most children tried to solve the items by making use of, or inventing, sign rules. Both of the latter items can be unambiguously solved making use of the rule 'two minus make plus', writes Küchemann, which is not the case for the two most difficult items. These findings indicate that rules will be misleadingly applied and difficult to check for consistency if they lack meaningful support. The question is therefore, how can these rules be given meaning?

Intra-mathematical explanations, principles and justifications

During the 19th century the algebraic permanence principle became important for the development of formal mathematics and the formation of new mathematical concepts, and hence influenced mathematics education (Semadeni, 1984). The principle states that if we want to extend the definition of a basic algebraic operation beyond its original domain (e.g. from the set **N** of natural numbers to the set **Z** of integers), then among all logically possible (noncontradictory) extensions the one to be chosen is that which best preserves the rules of calculation.

Mathematics education researchers have argued about whether to teach negative numbers through models or to wait until students are ready to cope with intra-mathematical justifications (Galbraith, 1974; Linchevski & Williams, 1999). When is a student ready for that? If we do not see development of mathematical thinking as qualitative changes in biological modes of functioning but rather as increasingly sophisticated ways of reasoning about mathematics, the question should perhaps be: How do we make students ready for intra-mathematical justifications?

According to Sfard the necessary change is a change in discourse (see Sfard, 2008 for an extensive description of the theory of commognition). In short, she proposes that a discourse is governed by a set of discursive rules. Object level rules can be changed through a process of constructing new routines, extending the vocabulary and producing new narratives. In contrast, changes in metalevel rules mean that familiar tasks will be done in unfamiliar ways. An example of a necessary change of meta-rules is given by Sfard (2007) concerning the learning of negative numbers. The mathematical statement that the product of two negative numbers is a positive number can not be justified by following the old routines of using concrete models. The new meta-rule justifies the statement by referring to the inner coherence of the discourse, as shown in table 1.1.

Table 1.1: Old and new mathematical meta-rules for endorsements of definitions (adapted from Sfard, 2007, p 582).

Old meta-rule	New meta-rule
The set of object-level rules to be fulfilled by the defined object *must be satisfied by a concrete model.*	The set of object-level rules to be fulfilled by the defined object *must be consistent with a predetermined set of other object-level rules called axioms.*
2 · 3 = 6 because it can be interpreted as 2 times jumping 3 steps on the number line. 2 · -3 = -6 because it can be interpreted as 2 times jumping 3 steps to the left of zero on the number line.	-2 · -3 = 6 because 0 · -3 = 0 and 2 + -2 = 0 , so (2 + -2) · -3 = 0 and we know that (2 + -2) · -3 = (2 · -3) + (-2 · -3) so -6 + (-2 · -3) = 0 This is true if and only if (-2 · -3) = 6

The difficulty is two-fold: first it is necessary to understand the need of a change of metarules, and secondly, when the algebraic permanence principle is accepted there is a question of which rules of calculation and which properties of numbers will be preserved. Sfard (2007) reports on a participant observation study in a class of 12 to 13 year olds who were taught the topic of negative numbers. Focus was mainly on the classroom discourse and how it changed during the 30 hours of observation. Symbols and images were chosen so as to ensure that they would not be treated in terms of natural numbers more easily than in terms of signed numbers. According to the commognitive analysis, learning about negative numbers involves a transition to a new, incommensurable discourse, particularly the change of rules for endorsement. The observed teaching was not very successful because, reports Sfard; "the new metarule for endorsement was enacted by the teacher but not made explicit. As a result, the students were unaware of the metalevel change and looked for sources of their bewilderment elsewhere" (Sfard, 2007, p 595). Furthermore, she proposed as remedy for this to engage students in an ongoing conversation about the sources of mathematics, about the human agency in mathematics, about the fact that mathematics is a matter of human decisions rather than of externally imposed necessity. Sfard clearly does not advocate 'waiting' for the students to become ready, but gives the teacher the responsibility of introducing and cultivating a discourse that involves intra-mathematical justifications.

Semadeni (1984) has a somewhat different approach to reach the goal of understanding formal reasoning. He proposes the use of a concretization permanence principle (c.p.p.) put to work in four steps: 1) select a suitable concretization schema, 2) let the students explore this with numbers within the familiar domain, 3) extend the examples to include numbers of the broader domain, and 4) let the students use the new case and answer questions expressed in colloquial language (ibid, pp 380). For negative numbers he claims that no real life motivation is satisfactory and that it requires formal reasoning to conclude that 'subtracting debt' is equivalent to income. Instead he suggests a semi-concrete motivation where integers are represented by counters denoted as positive and negative. To manipulate these counters implies accepting that a pair consisting of one of each counter can be taken away or added without affecting the value of the collection. (There is a distinction made here between the *number* of counters displayed and the *value* of those counters meaning the relation between them, similar to the one described in the Linchevsky and Williams (1999) study with a double abacus described in section 1.4.) Addition in the semi concrete model is visualized in figure 1.6 and subtraction in figure 1.7.

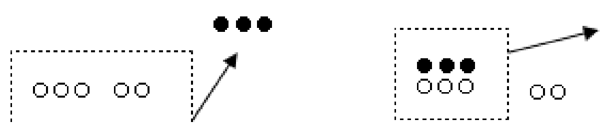


FIGURE 1.6: Illustration of the addition of three positive with five negative: 3 + (-5) = (-2).

Semadeni claims that the great advantage, of the c.p.p. is that subtraction as an inverse to addition is present already at the level of operations. A formal justification says that (-2) - (-5) must be 3 because 3 is the only number satisfying (-5) + x = (-2). "Whereas a formal justification extends the property that the number a - b is the solution of the equation $a = x + b$; the latter, conformable with c.p.p, extends the schema: subtraction means taking away what was joined during addition" (Semadeni, 1984, p 391-392).

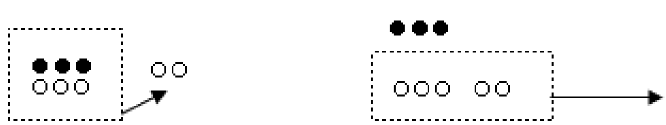


FIGURE 1.7: Illustration of the subtraction: (-2) - (-5) = (+3).

The c.p.p. illustrates a stance somewhere in between those that propose real life situations and models and those that propose formal reasoning. This model is concrete, but not realistic in the sense that it illustrates some out of school experience. Semadeni's article is theoretical and he calls for empirical studies about the role of c.p.p. in teaching children as well as teacher education.

When analyzing discussions during a workshop in a teacher development study Schorr and Alston (1999) report a basic misconception about using concrete materials and generating a real situation. One teacher gave the following example (Schorr & Alston, 1999, p 173):

> Sandy got squares for positive and negative numbers.
> -1 = red. 1 = blue. -(-1) = blue.
> She took 3 red squares and then subtracted 4 blue.
> How many squares in what colour did she have?
> -3 - (-4) = -3 + 4 = 1

The other teachers did not understand and asked how this could be considered a 'real' situation. When she attempted to show this with concrete tiles she could not produce a situation where she started with 3 red tiles, removed 4 blue ones and ended up with 1 blue. In the following discussion all the teachers agreed that they had been writing their stories to match an answer they already knew.

A quite different kind of intra-mathematical justification is what Freudenthal (1973) named the *induction extrapolatory method,* where the algebraic rules for negative numbers are discovered through structure and patterns like:

$3 + 2 = 5$	$3 - 2 = 1$	$3 \cdot 2 = 6$	$(-3) \cdot 2 = -6$
$3 + 1 = 4$	$3 - 1 = 2$	$3 \cdot 1 = 3$	$(-3) \cdot 1 = -3$
$3 + 0 = 3$	$3 - 0 = 3$	$3 \cdot 0 = 0$	$(-3) \cdot 0 = 0$
$3 + (-1) = \_$	$3 - (-1) = \_$	$3 \cdot (-1) = \_$	$(-3) \cdot (-1) = \_$
$3 + (-2) = \_$	$3 - (-2) = \_$	$3 \cdot (-2) = \_$	$(-3) \cdot (-2) = \_$

The above examples show that the *reason* for using a certain concrete model and the *connection* one expects to make with formal mathematical reasoning is more crucial than the model as such. If the teacher knows what the expected learning is to be and makes it explicit, the choice of method is secondary. Freudenthal writes: "however one proceeds in extending the number concept, it is necessary that the fact and the mental process of extending are made conscious" (Freudenthal, 1983, p 460).

Font, Bolite and Acevedo (2010) showed that mathematics teachers are not always aware of the metaphors they use. In their study of teaching and learning graph functions teachers were amazed by the metaphors they used when these were pointed out by the researchers.

Making a simple problem complicated

Symbolic representations should help students solve
problems they would otherwise fail to solve.
Gérard Vegnaud

People tend to avoid negative numbers in their daily lives if they can. When students are presented with a problem to solve, it would therefore not come as a surprise if they avoid negative numbers if they can. Altiparmak & Özdoan (2010) gave the following problems to 150 pupils in 6th grade:

> A building has 20 floors: 5 below ground level and 15 above ground level. The lift comes from the 7th floor down to the 3rd floor below ground. How many floors did the lift move? Show the operation on the number line.

The easiest way to solve this problem is to imagine travelling in the lift, first you go down 7 floors to reach ground level and then you go another 3 floors down. The calculation to write is 7 + 3 = 10. No negatives are involved. A more algebraic way of writing it is: 7 - x = -3. Here the last state is negative but the unknown x it not and can easily be illustrated on the number line as in fig 1.8.

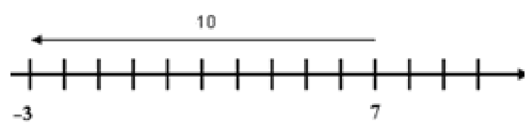


FIGURE 1.8: Illustration of the subtraction: 7 - x = (-3).

In their study an example of an incorrect student answer is 7 - 3 = 4 (Altiparmak & Özdoan, 2010, p 11). A correct student reply given by a student in the research group was 7 - (-3) = 10, with the same illustration as in fig 1.8. If the student first drew the number line (visualized the problem) and then wrote the mathematical expression, it would have been more natural to write 7 - 10 = (-3). The other way of writing which gives you a subtraction of a negative number is a way of making an easy problem difficult. The exercise could be used to illustrate that 7 - (-3) = 7 + 3. Bruno & Martinon (1999) found that students often first solve a problem on the number line and then look for a calculation where the results matches the result previously obtained on the number line.

A task given in a Swedish textbook in the introduction chapter about negative numbers uses a real life context of a time line stretching before and after a year zero (Carlsson et al., 2002, p 19)[8].

> Emperor Augustus was born in the year 63 BC. That can be written as year -63. He died year 14 AC. How old was he when he died?

The expected solution to this task is to write 14 - (-63) = 77. A much more straightforward way of solving the task is to add 14 + 63 (number of years

[8] Original in Swedish.

before and after zero). Introducing negative numbers here is again making an easy task more complicated. It cannot be easy to motivate students to solve tasks using negative numbers when the tasks are easier to solve using numbers they are more comfortable with. Gallardo (2003) uses a similar problem in a case study with 41 8th grade students. The problem posed was:

> Socrates, the Greek philosopher was born in 469 BC and died in 399 BC. How old was he when he died?

Using negative number notation here would generate the expression -399 - (-469) = 70 which means subtracting Socrates birth year from his death year, whereas the easier way of solving the problem would be to write 469 - 399 = 70. Gallardo reports that many students chose the simple way of solving the problem which she refers to as 'meaningless' (Gallardo, 2003, p 406). However, in the domain of natural numbers it is not meaningless, because to find the difference between two numbers you always subtract the smaller from the larger. If a person has lived between the years 469 and 399 (BC or AC) they must have lived 70 years. If you mark these years on a timeline the distance will be 70. The rule *'subtracting the smaller from the larger'* will always give the absolute value of the difference if the magnitude is used. In this problem if the numbers are treated as natural numbers the difference is 70. If the numbers are treated as negative numbers the difference will be the same: -399 - (-469) = 70. However, in this case it is not the magnitude but the value of the numbers that is referred to. $-469 < -399$, but $|-469| > |-369|$.

To expect students to choose the more complicated version involving negative numbers when there is an easy way using natural numbers is to underestimate their skill of mathematical reasoning. "Many real-life situations supposedly supporting the use of negative numbers; e.g. questions about changes in temperatures do not, in fact, necessitate manipulating negative numbers" (Sfard, 2007). This argument is supported by Thomaidis (1993, p 81) who made a historico-didactical study of negative numbers:

> [T]he properties of the operations between negative numbers do not express quantitative relationships of the real world, but are the result of certain conventions that help us to solve problems. It is common knowledge that the understanding of this fact and the transgression of the quantitative conception of number has required a large amount of time and considerable conceptual change. … the various concrete models employed … are not convincing enough for the necessity of these numbers. Students know quite well that they can work out the difference between two temperatures or determine the position of a moving point on an axis without having to resort to the operations between negative numbers.

One model or many? Number line or discrete models?

Galbraith (1974) separates conceptualizing negative numbers (including addition) from operating with them, and refers to the formal operational stage in Piaget's levels of intellectual development when she proposes to wait with the latter. Whether or not one believes that the ability to use formal reasoning is biologically driven or not, the noteworthy point is that this type of reasoning is necessary for understanding the operations of subtraction, multiplication and division of negative numbers. Some way of bridging the gap from the use of models to the use of formal or intra-mathematical reasoning is needed.

> Most models only partially represent the phenomenon to be illustrated, so a *collection* of such models must be used to illustrate different aspects of the problem. I see no reason why signed numbers cannot be introduced at an early stage …. However, the operations of subtraction, multiplication and division of integers are best approached not trying to extend our models to embody the operations. (Galbraith, 1974)

Several researchers recommend the use of many models and embodiments; whereas others entertain apprehensions that many models will create confusion and bewilderment. Lincheviski and Williams (1999, p 143) claim that at least subtraction with negative numbers can be understood through models, not one single model but a multiplicity of models, even if they do *"*acknowledge that multiplication and division may require a purely algebraic approach*"*. Contrary to that, in a teaching experiment where different teaching strategies were compared it was found that *systematic explorations* of a single embodiment combined with intensive practice of skills produced both greater immediate learning and sustained achievement than a *guided discovery* teaching strategy where several embodiment were explored (Shiu, 1978).

Concerns about clarity versus confusion are brought forward, for instance Kilborn (1979) points out that teachers used several different models and metaphors simultaneously during observed lessons on addition and subtraction with negative numbers and that these models seemed to confuse the students. Ball (1993, p 384) states that no representation captures all aspects of an idea and "teachers need alternative models to compensate for imperfections and distortions in any given model*"*. However, she articulates the "dilemma of content and representation" when she asks whether she confuses the children by letting them explore multiple dimensions of negative numbers through different metaphors.

In a study of novices' and experts' use of metaphors to understand and solve arithmetic problems with negative numbers Chiu (2001) concludes that novices and experts used the same metaphors but used them differently. For experts the metaphors served as a resource when difficulties were faced or to connect different ideas, whereas novices more often used metaphors to get started on a problem and to justify their answers. Metaphors could be said to serve as scaffolding for the experts but were more a point of reference for the novices.

Some researchers advocate the use of number line models whereas others prefer discrete models. For example Küchemann (1981) and Gallardo (1995) recommend that the number line should be abandoned in favour of discrete models where whole numbers represent objects of an opposing nature. As a contrast, Bruno and Martinón (1999) found that the number line became an indispensible tool when solving problems for the students in their experiment working with additive problems where the identification of addition with subtraction was a primary focus.

Freudenthal (1983, ch 15) has made a didactic phenomenological analysis of the concept of negative numbers that has become a classic in the field. In this work he distinguishes "old models", such as gains and losses, temperatures going up and down and walking the stairs. These old models, he claims, are useful when restricted to adding and subtracting with a positive number, but are unable to justify the identification of adding as the same as subtracting the opposite. Instead he proposes a model where numbers are described as inverses of each other that "undo" each other. One such model is the one described above in relation to the c.p.p. (Semadeni, 1984), and the other one is a number line model where numbers are represented by arrows instead of points. Arrows have both magnitude and direction, which is not the case with points or segments. Consequently, according to Freudenthal it is not the case of number line versus discrete models, it is more about the properties put into the model and the metaphorical reasoning connected to it. Altiparmak and Özdoan (2010) made an experiment using both kinds of models with positive results but relate their positive results more to the fact that the students were actively working with the models than to the models themselves.

Judging from the presented review, there does not seem to be any consensus around which models to use or how many. Some critique is legitimate concerning some of the results presented. For example in the Altiparmak and Özdoan study one group was taught to use models whereas the control group was exposed to "traditional teaching" which was not described. Both Küchemann and Gallardo base their recommendations on analysis of tests and interviews without any knowledge of what the participants had met in their school instruction. Discovering the need for one kind of model does not imply that other models be abandoned. A common concern that emerges from this literature review is the importance of being explicit about the purpose of the instruction given and conscious of the relation between a chosen model and formal mathematics. Questions that are still seeking answers concern how models are in fact interpreted by the students: what metaphorical reasoning they support and to what extent students are apt to find them helpful or confining.

1.7 Negative numbers and metaphorical reasoning among pre-service teachers

Concerning why pupils might find it difficult to solve (-3)- (-8):
"Perhaps they, like me, have heard something about this 'magic' of
minus becoming plus. I would like to do it with some concrete material
in some way, but I still have difficulties knowing how it is with negatives"
Quote from a pre-service teacher

This section will report some result from a study about pre-service teachers knowledge of negative number (Kilhamn, 2009a, 2009b). Negative numbers are taught as a topic in grade 8 in Sweden. At higher levels negative numbers appear in algebra and calculus and students are expected to handle them fluently. A group of students in the teacher training program for preschool and primary school teachers was selected as participants in a study with the aim to explore the nature of knowledge about negative numbers among students who had passed through Swedish secondary school. Concerning their level of secondary school mathematics the study population is representative of a population of Swedish students leaving secondary school. A positive correlation between level of secondary school mathematics (number of courses passed) and correct solutions on negative number tasks was hypothesized.

A written test was given to students (n=99) in the teacher training program prior to the topic being dealt with in their mathematics course. The test included calculation tasks on negative numbers with follow-up free text questions and self-estimate ratings. Particular attention was given to the way students explained how they arrived at an answer when calculating with negative numbers; whether they used models and metaphors (referred to as metaphorical reasoning) or sign rules, laws of arithmetic and symbolic manipulations (referred to as formal, arithmetical or symbolic reasoning[9]). Two of the items were tasks that had previously been identified to be among the most difficult types of tasks involving negative integers (Küchemann, 1981). Item 1 was a subtraction of a negative number: (-3) - (-8) =__. Of those students who completed this task (n=94), 30% gave an incorrect answer. The distribution of different solutions was as follows:

[5] 70% ;   [-11] 25% ;   [-5] 4% ;   [5 and -11] 1%.

Table 1.2 shows how the different solutions correspond to different solution strategies using either metaphorical or formal reasoning. It was found that students who solved the task directly by means of metaphorical reasoning (n=14) *all gave the incorrect* answer: [-11], whereas students who *first transformed the expression formally* from (-3)-(-8) to (-3)+8 and *then* used metaphorical reasoning

[9] In one report this category was referred to as formal reasoning, in another report as arithmetical reasoning. The category could of course be more differentiated, for example separating those who only apply rules in a procedural and imitative way from those who use rules they know how to justify.

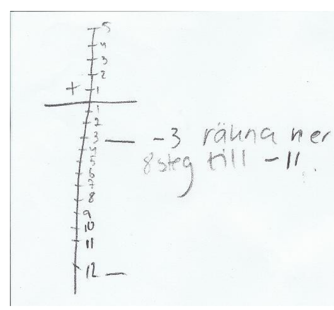
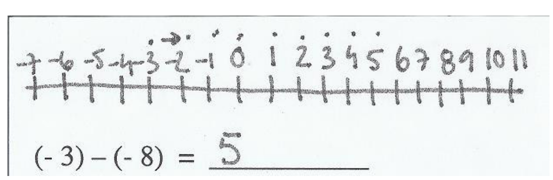
(n=12) *all gave the correct answer*: [5]. Among students using *only formal* reasoning 83% were successful and 17% gave an incorrect answer. Table 1.3 shows a few examples of the different solution strategies.

TABLE 1.2: Students' answers to item 1: (-3) - (-8) = categorized according to solution strategy

% within strategy for solving (-3) - (-8)

		calculate (-3) - (-8) =				
		5 - correct n = 65	-11 n = 24	-5 n = 3	two different answers n = 2	Total
strategy for solving (-3) - (-8)	formal reasoning n = 64	83 %	14 %	3 %		100 %
	only metaphor n = 14		100 %			100 %
	first formal then metaphor n = 12	100 %				100 %
	other or none n = 4		25 %	25 %	50 %	100 %

TABLE 1.3: Examples of solutions strategies for item 1: subtraction.

Formal reasoning correct answer	I think to change the sign when there are two of the same sign. -3 + 8 = 5
Formal reasoning incorrect answer	When there are two minus signs close by it becomes plus, and then there is one minus left that you save. -3 + 8 = -11
Only metaphorical reasoning	I think of the thermometer. It was minus 3º and then it got 8º colder. (-3) - (-8) = -11
	-3 count down 8 steps to -11 
First formal, then metaphorical reasoning	I think that two minus make a plus. I picture a thermometer. -3 + 8 = 5
	Like my drawing. I remember that two minus cancel each other and makes plus, hence -3 + 8 = 5. 

For the subtraction item, a large group of incorrect answers (n=14 out of 33) was found to be by those who only used metaphorical reasoning. Most of them declared that they were rather confident (n=6) or very confident (n=3) about their answer being correct, which indicates that they believed the metaphorical reasoning they used would give them a correct answer. The largest group of correct answers were found among students who transformed the operation (-3) - (-8) into -3 + 8 and then calculated the answer. It is possible that many of these students implicitly used a mental number line, scale, thermometer or other representation when dealing with the operation -3 + 8. Those who explicitly did so arrived at the correct answer. The results suggest that metaphorical reasoning is only helpful when the student is aware of the constraints of the metaphor and is capable of treating numbers formally in order to transform them into something that carries meaning and has similarities to the representation at hand. This is in line with the results of Chiu (2001), claiming that experts know the limitations of the metaphors they use and therefore learn when to use each metaphor. In conclusion, the results indicate that, when it comes to operating with negative numbers, metaphorical reasoning is not sufficient in itself but needs to be supplemented with formal reasoning. Moreover, only formal reasoning does not automatically render correct answers.

Item 2 was a multiplication of two negative numbers: (-2) · (-3) =. On this task 45% of the students gave an incorrect answer, distributed as follows: [6] 55%; [-6] 44%; [-12] 1%. Only 2 students used metaphorical reasoning and they both answered incorrectly: [-6]. Those who used formal reasoning referred to a sign rule indicating that multiplying two negatives gave a positive answer, or stated that it was like ordinary multiplication only "on the negative side" so the answer would also be negative.

On questions about why pupils might find these items difficult, and suggestions of how to teach them, the majority of the students answered that negative numbers were difficult because they were abstract and difficult to explain concretely, and that as teachers they would suggest using concrete models. Some focus on the problem with the minus sign, for example:

> Because it is difficult when there are many [minus] signs and you don't use them in real life. [student 209]
>
> I would show them that two minus can also be a plus if you take one minus and put it crosswise over the other one.[student 129]

The most commonly suggested concrete model was the thermometer, as suggested by student 101 when explaining (-3)-(-8):

> I would show that if it is minus 3 degrees outside and then the temperature went down another -8 degrees, how many degrees would it be then?

Student 235 also suggested a thermometer because it is the everyday experience children have of negative numbers, but then realised that it would be misleading:

> But if you think of a thermometer it is different! And for children a thermometer is what they will think about when they see a negative number. If it is -3ºC and the temperature goes down -8º then the temperature will be minus, not +5ºC !!

No correlation was found in this population between levels of mathematics (number of secondary math courses passed) and achievement on these negative number items, indicating that inappropriate means of reasoning, be they metaphorical or formal, do not disappear automatically but need to be explicitly addressed in elementary as well as secondary school courses.

Thus, this study suggests that knowing the potentials and constraints of a model is necessary if it is to function as a conceptual metaphor and for the learner to be creative in striving to understand. As a contribution to the body of research, these results suggest that the debate should not be concerned with which model to use and why one model is better than another but rather about how to make the connections between models and the mathematics we are trying to model. It is the consequences of our use of models and metaphorical reasoning and how we deal with these consequences that needs to be investigated.

Starting point for the research project

As a consequence of this study, questions arose about how these students acquired their conceptions of negative numbers and why they would prefer to teach them using concrete models although they realized the difficulty to understand these abstract numbers concretely. Therefore, the rest of the research project was focused on finding out about the nature of the way negative numbers is taught in school and conceptualized by pupils when they are initially introduced, and particularly the relation between the abstract numbers and the concrete models and metaphorical reasoning used.

CHAPTER 2

Theoretical Framework

This chapter begins with a description of basic theoretical assumptions that form the foundations of the research presented in the thesis. These include assumptions about the nature of mathematics and the development of mathematical concepts. Learning mathematics is described in terms of change of conceptions. The general framework of learning used in the thesis is a Social Constructivist one, utilizing both an acquisition metaphor and a participation metaphor for learning (Cobb, 1994; Cobb, Jaworski, & Presmeg, 1996; Sfard, 1998). An important notion frequently used in educational literature and curricula is the notion of Number Sense. Making sense of negative numbers can be described in terms of developing number sense for these numbers (Kilhamn, 2009c). A theoretical background to the notion of number sense is given in section 2.4. Three issues concerning teaching and learning negative numbers will be explored throughout the thesis. These issues are; i) the role of metacognitive awareness and metalevel conflicts, ii) the importance of historical influence, and iii) the role of metaphors. The use of metaphors as a tool for understanding and as a tool for instruction is explored by using a theory of Conceptual Metaphors, which is described in section 2.6 and further developed in chapter 3.

The two theoretical notions of Conceptual Metaphors and Number Sense can be seen as two different lenses through which it is possible to view how students make sense of mathematics and how they handle abstract mathematical objects. The study of conceptual metaphors is concerned with what words are used in the discourse about numbers and how these words influence the conception of number, whereas descriptions of number sense can help us relate an individual student's interpretations of what has previously been found to be important aspects of numbers. It is assumed in this work that these two notions in different ways describe the same phenomena and thus complement each other.

2.1 Mathematics as a human invention

> The observable reality of mathematics is this: an evolving
> network of shared ideas with objective properties.
> *Reuben Hersh*

Mathematics can be described as an abstract and general science for problem solving. Thompson and Martinsson (1991, p 278) claim that an established view of mathematics is "the science of numbers, of space and of the many generalizations of these concepts, that have been created by the human intellect"[10]. Mathematics as a scientific discipline has changed greatly through history. In 500 BC it was simply the study of numbers, and 2400 years later it had become "the study of number, shape, motion, change, and space, and of the mathematical tools that are used in this study" (Devlin, 1998, p 2). Today most mathematicians describe it as the *science of patterns* (ibid, p 3) or as the *science of structures of sets* (Thompson & Martinsson, 1991). Patterns or structures in the real world can be studied, but also patterns or structure of the mathematical world itself. "The patterns captured by numbers are abstract, and so are the numbers used to describe them" (Devlin, 1998, p 13).

Mathematics is an aupoietic system – a system that produces the things it talks about (Sfard, 2008, p 161). Mathematical objects are invented, created or constructed by humans for the sake of communicating and solving problems. On the basis of different examples of mathematical experiences, Davis and Hersh (1981) argue that mathematics on the one hand is about ideas in our minds, that mathematical objects are imaginary, and on the other hand that these objects have definite properties that the creator does not necessarily know of and that we may or may not discover. "As mathematicians, we know that we invent ideal objects, and then try to discover the facts about them" (ibid, p 410). This means that we must accept mathematics as fallible, correctible and meaningful.

> Mathematics does have a subject matter, and its statements are meaningful. The meaning, however, is to be found in the shared understanding of human beings, not in an external nonhuman reality. In this respect mathematics is similar to an ideology, a religion, or an art form; it deals with human meanings, and is intelligible only within the context of culture. (Davis & Hersh, 1981 p 410)

Representations and mathematical concepts

Communication of ideas is a fundamentally human activity. The creation of an idea and its representations (verbal, symbolic or other) is an intertwined process where the representation feeds into the idea and the idea in turn generates (or changes) its representations. Understanding the idea becomes tantamount to understanding its representations. A representation is here seen as a communicative device conveying some meaningful idea. Hence the term is used in an inclusive way, encompassing all sorts of representations, but at the same time in a restricted sense as the tools for representing mathematical ideas (Shiakalli & Gagatsis, 2006). Representations can either be internal; i.e. exist only in the mind of the individual; or external, i.e. manifest outside the individuals mind. According to Damerow (2007) there are first and second order representations with quite different properties.

[10] Original in Swedish: *läran om tal, om rummet och de många generaliseringar av dessa begrepp, som skapats av det mänskliga intellektet.*

First order representations are "representations of real objects by symbols or by models composed of symbols and rules of transformation, with which essentially the same actions can be performed as with the real objects themselves". The simplest form of representation is the identification of a concrete object, an attribute, an activity etc. by a name, a word or a sign. These symbols are first order representations. Damerow (ibid) gives as two examples of mathematical first order representations: i) counters as representations of the cardinal structure of sets of objects, and ii) names and symbols for numbers that can be arranged in a succession [1,2,3..] as representations of the ordinal structure of quantities.

Second (or higher) order representations are "symbols or models composed of symbols and rules of transformation which correspond to the operations of the abstract mental model that controls the actions performed with the real objects" (ibid, p 24-25). Second-order representations represent real objects only indirectly. They are independent of the meaning of lower-order representations. A mathematical representation is thus a symbol or model composed of symbols that through a series of links represent real (concrete) objects or real actions but where the meaning of these real objects or actions is lost in higher order representations, as shown in figure 2:1. Interpreting representations (symbols) is the process of making these links.

Each representation has a dual relation. A first order representation relates to on the one hand the concrete object to which it is applied and on the other hand the abstract object it stands for. A second order representation relates to on the one hand the abstract object to which it is applied and on the other hand an abstract object of a metacognitive nature, i.e. an object constructed through reflecting actions of comparison, correspondence, combination and repetition. Mathematical proofs, mathematical structures and formal logico-mathematical concepts are examples of abstract objects of metacognitive nature.

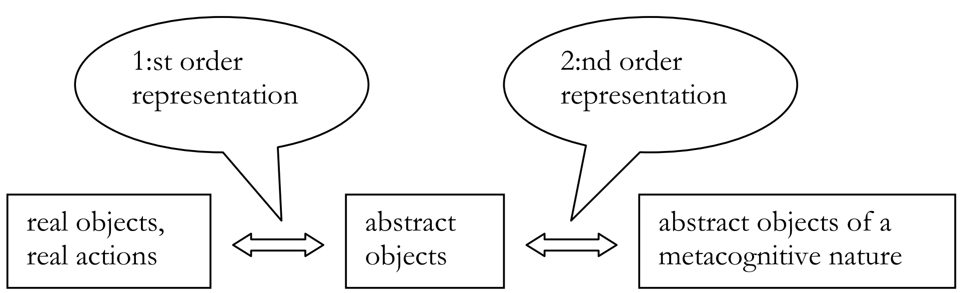


FIGURE 2.1: First order representations link real objects with abstract objects, second order representations link abstract objects with abstract objects.

When counting the fingers on one hand we end up with the word *five*. This word (and later the symbol *5*) represents the action of counting the fingers as well as the five objects that have been counted. *Five* and *5* are here first order representations. They link the real action and the real objects with the abstract number 5. The equation 5+2=7 can be thought of as five fingers on one hand and two on the other and the action of counting them all and ending with the word seven. This is still a first order representation. Now, the mathematical object 5 has many properties and can be manipulated in many ways. The symbols $\sqrt{5}$, 10^5 and -5 are representations linking the abstract object of 5 with other abstract objects which have a metacognitive nature since they are derived from other mathematical objects and ideas about creating structure and coherence among these. The meaning of 5 as the number of fingers and the action of counting them is not present in these second-order representations.

The term *representation* is here to be widely interpreted to include all kinds of symbols and symbolic artefacts such as linguistic manifestations, pictures, sounds, writing, gestures etc. It is impossible to separate a mathematical object from its representations. Sfard (2008, p 173) writes about symbolic artefacts as "far from being but 'early incarnations' of the inherently intangible entities called mathematical objects [they] are, in fact, the very fabric of which these objects are made". Hersh (1997, p 20) uses the mathematical objects called 3- and 4-dimensional cubes as an example to illustrate that a mathematical object "exist only in its social and mental representation". We can imagine a 3-cube as a representation of a 3-dimensional physical object (a box), but due to the constraints of the physical reality there will never exist a mathematically perfect 3-cube in the physical world. The mathematical object is a representation of an ideal cube. A 4-cube is a result of our imagination and does not represent anything in the physical world. It is "a representation without a represented" (Hersh, 1997, p 20) in much the same way as Mickey Mouse is. Mathematical objects very often have multiple representations and no one representation is consummate. Increased knowledge about a mathematical object is often linked to new ways of representing that object, and making connections between these representations.

Interpreting a mathematical object is not a precise process in an algorithmic sense; a finite number of steps or links ending in a 'true interpretation'. There is no 'objective truth' about mathematical objects, only more or less valid interpretations.

> Interpreting a symbol is to associate it with some concept or mental image, to assimilate it into human consciousness. The rules for calculating should be as precise as the operation of a computing machine; the rules for interpretation cannot be any more precise than the communication of ideas among humans. (Davis & Hersh, 1981, p 125)

The building blocks of mathematical ideas are called *concepts*. A mathematical concept is a theoretical construct in the formal universe of mathematics; it is a mathematical object along with all the external representations that make up the essence of the object, as well as its place in a mathematical structure. A concept's counterpart in the universe of human knowing is "the whole cluster of internal representations and associations evoked by the concept", which will be referred to as a *conception* (Sfard, 1991, p 3).

2.2 Learning

> The most important change people can make
> is to change their way of looking at the world.
> *Barbara Ward Jackson*

A theory of learning is on the one hand concerned with *what learning is*, and on the other hand *how learning comes about*. The first is a philosophical and epistemological issue which creates a point of departure for the second. There are many different theories of learning relevant for the study of mathematics education, ranging from constructivism to socio cultural perspectives (cf. Booth, Wistedt, Halldén, Martinsson, & Marton, 1999; Bransford, Brown, & Cocking, 2000; Burton, 1999; Hiebert & Carpenter, 1992; Lave & Wenger, 1991; Marton & Booth, 1997; Steffe, Nesher, Cobb, Goldin, & Greer, 1996). In this thesis learning is seen as the process in which a person in some way changes her way of acting or relating to phenomena she encounters. This includes changes in perception, in conception, in discourse, in ways of thinking as well as changes in ways of acting or participating in a practice. Such an encompassing definition may be criticized as too wide, but the focus in educational research is more concerned with how learning comes about than defining what it is. One important aspect of this definition is that learning cannot be studied without taking into account the phenomena the learner encounters. Learning is always about learning *something*, changing one's way of acting or relating to something. Changes and the process that contributes to these changes can be described in terms of a *learning trajectory*. Some researchers describe learning trajectories for whole groups, for example a learning trajectory of a classroom practice, and distinguish between possible, hypothetical and actual learning trajectories (e.g. Cobb, McClain, & Gravemeijer, 2003; Cobb, Stephan, McClain, & Gravemeijer, 2001). Another way of using the term learning trajectory relates to distributed cognition, where learning trajectories are described as the process in which shared understanding evolves. For instance Melander (2009, p 58) defined a learning trajectory as a collection of interactional sequences that are related to each other through being about the same content. In this thesis the studied learning trajectories are the individual students' changes in ways of relating to mathematical content that, from a mathematical point of view, is considered as the same. The notion of learning trajectory is not normative in itself, but by studying different students, several possible learning trajectories will show up, and depending on the teaching goal some may be more desirable than others.

Sfard's (2008) definition of learning as a change of discourse is here seen in a slightly modified way. In this thesis a change of discourse is seen as an external, and thereby observable, side of cognitive learning[11]. The thesis is concerned with changes of individuals' conceptions and assumes that it can be observed by the student's change of discourse. It is hypothesized that previous knowledge and the discourse of the classroom are two things that greatly influence such a process. Elements of the classroom discourse that are particularly focused are the use of metaphors and the occurrence of metalevel conflicts. The aim of the thesis is to observe and describe on the one hand the changes of individual's conceptions and on the other hand the influence of classroom discourse and previous knowledge on the learning process.

In an often quoted article from 1998, Sfard identified two metaphors for learning (Sfard, 1998). One is *the acquisition metaphor*, where learning is described in terms of the process in which one gains knowledge or learns a skill. Concepts are in this metaphor conceived as units of knowledge that can be accumulated, refined and combined. The learner is someone who constructs meaning. Terms such as knowledge, concept, conception, idea, notion, meaning, sense and representation are part of the acquisition metaphor for learning.

The other metaphor is *the participation metaphor*, where learning is described as the process of becoming a member of a certain community, to communicate in the language of this community and to act according to its particular norms. The learner is someone who participates in activities. Terms such as knowing, discourse, communication, inquiry, cooperation, practice and norms are part of the participation metaphor for learning.

Although some might find these two metaphors contradictory, it could be argued that they are no more contradictory than for example two different metaphors of life; life as a journey and life as a battle. The two metaphors highlight different aspects of life. It also happens that terms from one metaphor are used in the other metaphor, thereby blurring the distinction between them. It could for example be legitimate to talk about acquiring a discourse or making sense of activities in a practice. A different approach to these two perspectives is to say that they are two different phenomena that are brought together under the comprising term "learning" in a process of saming[12].

Sfard (1998) argues that the two metaphors give us two different perspectives on learning that both have something to offer that the other cannot provide. Underlying this research project is the assumption that it is profitable to view

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[11] Cognitive learning as opposed to for example kinaesthetic learning like riding a bike. It is possible to learn how to ride a bike without changing one's discourse.

[12] Using Sfard's (Sfard, p 302) definition of *saming*: assigning one signifier (giving one name) to a number of things previously not considered as 'the same'. This approach was suggested to me by Ola Helenius in one of our many discussions on learning theories.

learning as both participation in mathematical classroom practice and mathematical discourse, and as acquisition of mathematical concepts and representations and the sense making of these. This will be further developed in the next section outlining a social constructivist framework.

2.3 A social constructivist framework

> There are two kinds of truths. First the simple truths, where the opposite always is false, and secondly the great truths where the opposite is also true.
> *Niels Bohr*

A social constructivist approach to learning aims at combining constructivist ideas of knowledge with ideas of knowing as situated in social practices. "Social constructivism, in its various forms, has grown out of the attempt to incorporate an explanation for intersubjectivity into an overall constructivist position" (Lerman, 1996, p 134). Lerman sees these two theories as incompatible, whereas others argue that they are incommensurable (Sfard, 2003) or complementary (Cobb, 1994). In her article about the two metaphors for learning, Sfard (1998) argues that it is sometimes possible to merge seemingly contradictory metaphors if the figurative nature of the metaphors is not forgotten and their use is pragmatically justified.

The research presented here takes on the rather pragmatic view that both the social and the individual perspectives are necessary. Cobb (1994, p 13) argues that "mathematical learning should be viewed as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society". The psychological perspective helps us understand how individual students make sense (or not) of the activities of the mathematics classroom community, and the social perspective helps us understand the conditions for the possibility of learning. "Neither perspective can exist without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective" (Cobb et al., 2001). The individual perspective of the framework is often analysed within an acquisition metaphor where knowledge is a noun and the act of learning is described as an act of gaining knowledge. The social perspective however, is rooted within a participation metaphor, where prominent concepts are: knowing, communication and discourse. A metaphor of coming–to–know underlying both of these perspectives is the construction metaphor. Constructed knowledge could be seen as an object, and as such described as acquired, but it can also be seen as a construction of the world to live in, meaning that it is not something to acquire but something that shapes the conditions of possible actions. Sfard (1998, p 11) proclaims that "the most powerful research is the one that stands on more than one metaphorical leg".

In a comparison of different versions of constructivism, Ernest (1996) characterises social constructivism as having a modified relativist ontology, an epistemology that regards knowledge as that which is socially accepted, and that its associated theory of learning is constructivism, based on a metaphor of carpentry. Some characteristic pedagogical features of all constructivist frameworks are; i) sensitivity towards the learner's previous constructions; ii) attention to cognitive conflicts and meta-cognition; iii) the use of multiple representations of mathematical concepts; and iv) awareness of social contexts (Ernest, 1996, p 346). Particular pedagogical emphases suggested only by *social constructivism* are, according to Ernest; i) quandary about how the mind of the learner is formed by social settings; ii) emphasis on discussion, collaboration, negotiation and shared meanings as a consequence of an awareness of the social construction of knowledge; and iii) understanding of mathematical knowledge as a social construct and as such bound up with texts and semiosis[13]. Social constructivism as depicted by Ernest is well attuned with the theoretical framework of the research project described in this thesis. More details of the framework will be described in the following sections, along with a discussion of the two complementary perspectives of the social and the psychological; participation and acquisition.

Two perspectives: social and psychological

> The social perspective brings to the fore normative taken-as-shared ways of talking and reasoning and the psychological perspective brings to the fore the diversity in students' ways of participating in these practices.
>
> *Paul Cobb*

Over the last two decades Cobb and his colleagues have developed the social constructivist framework and constructed some useful analytical tools, where the social perspective and the psychological perspective are seen as each others foreground and background. Students can be said to actively construct their mathematical ways of knowing as they participate in the classroom mathematical practices. Participation constitutes the possibility of learning; it enables and constrains learning but does not determine it (Cobb & Yackel, 1996). Participation in a mathematical practice is a requirement for developing mathematical knowledge, and the development of mathematical knowledge changes an individual's participation in a mathematical practice. Cobb and his colleagues (2001) suggest that the relationship between the two perspectives is reflexive; that they co-evolve and depend on each other. In a social constructivist framework both the process and the product of mathematical development is considered as social through and through. Social and cultural processes are continually regenerated by cognizing individuals. Interactional routines and patterns are jointly created by individuals in an ongoing process of negotiation

[13] Semiosis is any form of activity, conduct, or process that involves signs, including the production of meaning. The term was introduced by Charles Sanders Peirce (1839-1914.)

and create opportunities for conceptual construction (Cobb et al., 1996; Cobb et al*.,* 2001*)*. In table 2:1 a framework for analysing mathematical learning is outlined showing the two perspectives. This framework is concerned with ways of acting, reasoning and arguing that are normative in a classroom practice. The social perspective brings to the fore normative taken-as-shared ways of talking and reasoning, and the psychological perspective brings to the fore the diversity of student's interpretations, beliefs and ways of participating in these practices.

TABLE 2.1: An interpretive framework for analyzing communal and individual mathematical activity and learning (Cobb et al., 2001, p 119).

Social Perspective	**Psychological Perspective**
Classroom social norms	Beliefs about own role, other's roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical interpretations and reasoning

Social norms are characteristics of the classroom community that describe regularities in classroom activities; they are jointly established by teacher and students. Examples of social norms include how conflicting interpretations are handled, how solutions are explained and justified and by whom, how agreement and disagreements are indicated and resolved. Social norms document regularities of joint activities that are found in classroom settings but not specifically in mathematics classrooms. On the next level the sociomathematical norms describe jointly established regularities of mathematical issues, for instance what counts as a valid mathematical solution, what counts as an acceptable explanation, what kind of problem or mathematical activity is seen as good and relevant. The third level of the social perspective concerns the evolution of the practice itself and tries to describe the learning trajectories of the classroom community. Each of these levels of the social perspective has a counterpart in the psychological perspective, describing psychological features for each individual participating in the practice (see Cobb et al., 2001; Yackel & Cobb, 1996 for a more detailed description). Although the analytical framework is constructed primarily for the benefit of design research some of the components are useful also for a more descriptive and interpretive research approach. For this study the reflexive relation between the two perspectives is an underlying structure of both the research design and the analysis.

The research presented here will focus mainly on the individual and taken-as-shared ways of reasoning, arguing, and symbolizing that are specific to particular mathematical ideas. The unit of analysis is individual students' diverse ways of interpreting and reasoning as well as meanings and ways of doing mathematics that are taken-as-shared in the practice. It is important to point out that taken-as-shared does not imply that individuals' interpretations are equivalent or even overlap. Participants of a practice (e.g. teacher and students) take meanings to be shared if they neglect that they could have different interpretations (Voigt, 1996). The awareness of the two perspectives of mathematical learning and the relation between them as illustrated in table 2:1 does not imply that all these aspects are focused at the same time. The research presented here will foreground the third level of the psychological perspective (mathematical interpretations and reasoning), and describe the social perspective as a necessary and significant background.

Discussion of the participation and acquisition metaphor

In contrast to earlier work (Sfard, 1991, 1994, 1998), Sfard has more recently abandoned the acquisition metaphor in favour of the participation metaphor. Within this metaphor she has created what she calls a theory of Commognition (Ben-Zvi & Sfard, 2007; Sfard, 2007, 2008). The basic assumption of commognition builds on Vygotsky's claim that speech is first a social and external process before it becomes internal (thought). A concept is usually formed through a process of hearing and using the word for the concept in various situations. Thinking is seen as internal communication with oneself[14] mediated by discursive tools (Vygotsky, 1999/1934). In the creation of this theory Sfard rejects, and gives rather harsh criticism to, the acquisition metaphor.

> Traditional educational studies conceptualize learning as the 'acquisition' of entities such as ideas or concepts. Due to the crudeness of these atomic units, those who work within the acquisitionist framework are compelled to gloss over fine details of messy interpersonal interaction within which the individual acquisition takes place. Sfard, 2007, p 567)

In the discursive commognitive framework, Sfard outlines meta-level learning defined as "… a transformation of the discourse: it changes the vocabulary and the ways in which explorations are done" (Sfard 2007, p 126). An irrevocable requirement for meta-level learning in the commognitive framework is active participation in a discourse. Having said this, she concludes later in the article that: "The view that students may, indeed, make a meta-level progress without an initial exposure to the discourse of experts could thus be sustained only by those who adopt the Platonic vision of mathematics …" (ibid, p 137). Without discrediting this conclusion it is close at hand to draw a parallel between Sfard's view of the acquisition metaphor and Platonism, namely that that mathematical objects exist in the world, independent of any human being. From Platonism follows that learning is the process of acquiring knowledge about these objects, and since they exist independently of human beings they can, at least in theory,

[14] Even if I have some doubts about defining every instance of thinking as a type of communication, as Sfard (2008) does, the definition works very well with the particular form of thinking labelled mathematical reasoning. (authors remark)

be learned without human communication. Sfard characterizes aquisitionism as, follows: (based on slides shown at a PhD course at Växjö University, August 2009)

1. *Learning* happens in encounter between the *individual* and things in the *world* (e.g. giraffe, oxygen, number), and
2. In theory, it could occur *without mediation* of other people

If we do believe that communication is essential for learning mathematics an obvious contradiction arises, which is one of the basic tenets of Sfard's critique of acquisitionism. There are two ways to tackle this contradiction. Sfard's way is to totally abandon the acquisition metaphor. A different way would be to redefine 'the world'. Describing the world of mathematics and mathematical objects has been, and still is, a philosophical endeavour with pedagogical implications. Great work in this field has been done by people like Paul Earnest, Hans Freudenthal, Keith Devlin and Reuben Hersh. Even if many mathematicians still, in their practical work[15], encompass a platonic view of mathematical objects as having an existence in the world in parity with physical objects, the humanistic philosophy of mathematics outlined by Hersh (1997) is winning ground. Humanistic views of mathematics see mathematical objects as neither physical nor mental; they are social entities with physical and mental embodiments. Whereas for instance giraffe and oxygen are part of the physical world, numbers are only part of the social world. With such a view of mathematics the first of Sfard's two characteristics of the acquisition metaphor is endorsed, but the second one becomes false. If mathematical objects exist only "at the social-cultural-historical level, in the shared consciousness of people (including retrievable stored consciousness in writing)" (Hersh, 1997, p 19), it is impossible to encounter them without mediation of other people, and so human communication becomes essential. Thus, we are rid of Sfard's most severe critique of the acquisition metaphor. An important feature of Sfard's article from 1998 is that the two perspectives both are needed because they help us answer different questions. This statement is something Sfard still acknowledges although she herself no longer asks question for which she finds the acquisition metaphor useful (Sfard, personal communication 2009).

One of the reasons for choosing a participation framework is that it studies and makes claims only about phenomena which can be observed; about what is actually said and done, about people's participation in an activity. The problem when it comes to educational research is to find ways of researching not solely what is said and done, but also what goes on in the mind of the students, out of reach for direct observation. Many scientists in different fields spend their lives researching things that are impossible to observe directly. Black holes, quarks,

[15] A mathematician may treat mathematical objects in a platonic way in her daily work, but at the same time be aware that the properties of mathematical objects all depend on the basic definitions and axioms that are agreed upon.

wind and social gender are phenomena that have to be observed indirectly. However, it does not prevent scientists from making claims about these things, claims anchored in models of how what is observed relates to the object of research itself. The same could be said about thinking, as expressed by Halldén, Haglund and Strömdahl: "[T]he fact that there is no instrument that we can use to *directly* observe conceptions does not in itself imply that nothing of value can be said about how people conceptualize the world" (Halldén, Haglund, & Strömdahl, 2007, p 26). In the commognitive perspective described by Sfard only participation in interpersonal communication is observable and therefore the only thing about which claims can be made in an educational setting. Change that occurs within an individual, i.e. change in the intrapersonal discourse, is labelled "individualizing the discourse" and "making it a discourse for oneself" (Sfard, 2008). That is a model as good as any, but it is still a model that is created so as to make it possible to claim things about the unobservable intrapersonal communication of the individual, seen as the outcomes of learning, although what is observed is only the visible interpersonal communication. Unfortunately it does not help much since it simply states that it is the individualized version of the interpersonal communication and no more can be said about it since it cannot be observed. The acquisition metaphor creates a different model, describing learning as acquisition of knowledge, as changing conceptions, as building new knowledge structures or as making new connections between inner representations. In the research presented here both these models are seen as productive ways of describing learning in much the same way as the two models for describing light: on the one hand as a particle and on the other hand as a wave motion. Neither of the models is more true to actual facts than the other, but both are useful in our enterprise to understand a phenomenon. In fact, the two learning metaphors help us describe two processes that are mutually dependent; the inner and the outer. A person changes her mathematical conceptions as a result of changing her participation in a mathematical discourse, and she changes her participation in the discourse as a result of changed conceptions.

Both metaphors of learning are, as they are applied in the work of this thesis, based on a metaphor of construction. A person is seen as an active, creative constructer of her knowledge or of her discourse. Knowledge is not something that is simply taken from the outside and put inside the head; it is created. This does not mean that it is invented from scratch. Like a bridge that is constructed, it cannot be constructed without tools and material. Yet, the bridge is something completely different from the sum of the tools and the material. The same set of tools and material in the hands of a different person may result in a quite different bridge. Likewise, constructing mathematical knowledge is impossible without tools and material, and when mathematics is concerned the tools are discursive and the materials used are mathematical concepts and their representations.

Understanding and reasoning in mathematics education

Although the idea of learning mathematics with understanding has for a long time been a widely accepted goal, the notion of understanding is not a straightforward one. A basic assumption for this research is that knowledge is represented internally and that these representations are structured and connected with external representations (Hiebert & Carpenter, 1992, p 66). In the framework proposed by Hiebert and Carpenter connections between internal and external representations are thought to produce networks of knowledge. Understanding can thus be described in terms of internal knowledge structures or internal networks or patterns of knowing. Hence, understanding in mathematics is a result of the process of making connections between ideas, facts, representations and procedures that are defined as mathematical. Understanding is a personal experience; how and when an individual understands a mathematical concept is related to his or her experience of structure and meaning of that concept. This definition of understanding makes it impossible to assess understanding, but possible to promote it. Different researchers have in different ways tried to describe mathematical knowledge and each description reveals some aspect and overlooks others. Ideas that have been found useful here when describing individual students' interpretations are ideas about procedural and conceptual knowledge and ideas about reification.

Mathematical knowledge has often been described as both *procedural* and *conceptual* and the respective role of these two kinds of knowledge in students' learning has been a topic of discussion for the last 25 years. The two kinds of knowledge were defined by Hiebert and Lefevre (1986) relating conceptual knowledge to ideas about a network of concepts and linking relationships, and procedural knowledge to rules and procedures for solving mathematical problems. When procedural and conceptual knowledge are mentioned in this thesis it is accordance with the views of Baroody, Feil and Johnson (2007), who claim that procedural and conceptual knowledge are intertwined and connected, supporting each other to produce expertise. They are mutually dependent and connections with real world situations are essential. Procedural knowledge provides the *how* of mathematics, conceptual knowledge the *what*. In addition to these two aspects Skemp (1976) introduced the term *relational* understanding to indicate understanding of mathematical relations and structure, as opposed to *instrumental* understanding where only ability to use procedures were focused. Structural knowledge provides the *when* and the *why* of mathematics. It is often considered quite easy to learn *how to do* certain things; follow predetermined procedures such as algorithms, use specific rules for arithmetic or adopt special techniques or formulas, as long as it is clearly indicated when these things are to be performed. In mathematical problem solving the tricky part is to figure out what procedures to use when and why. To know what procedures to use and when, is a matter of understanding how different procedures and mathematical concepts are structurally related to each other.

In her theory of *reification* Sfard (1991) elaborates a theory of mathematical concepts that speak of stages in the process of concept formation. Two different kinds of conceptions are distinguished: structural and operational. Structural conceptions are static, instantaneous and integrative whereas operational conceptions are dynamic, sequential and detailed. Structural and operational conceptions are to be seen as a duality, not a dichotomy. There are three statements that serve as assumptions in the theory of reification:

1) Mathematical objects are understood as dualities' both operational and structural,
2) In the learning process the operational comes first and the structural later, and
3) The learning proceeds through a three-phased process of *interiorization*, *condensation* and *reification*.

The first two are gradual processes and r*eification* is the often suddenly occurring, ability to see the new entity as an integrated object-like whole, replacing talk about action with talk about objects. "Processes performed on already accepted abstract objects have been converted into compact wholes, or reified to become a new kind of self-contained static construct" (Sfard, 1991, p 14). When a person makes this instantaneous leap there is an ontological shift in the conception from operational to structural. In most cases this shift is irreversible in the meaning that once you have made the leap you have access to two complementary conceptions. Figure 2:2 illustrates how this iterative three-phased process link different mathematical concepts. The higher level reification and the lower level interiorization are prerequisites for each other, i.e. reification of one object only occurs when it is being used as an object in a new process (Sfard, 1991). The transformation when processes are finally understood to be mathematical objects in their own right involves a paradox: it requires that mathematicians manipulate the processes instrumentally as objects before they are able to mentally grasp them as such.

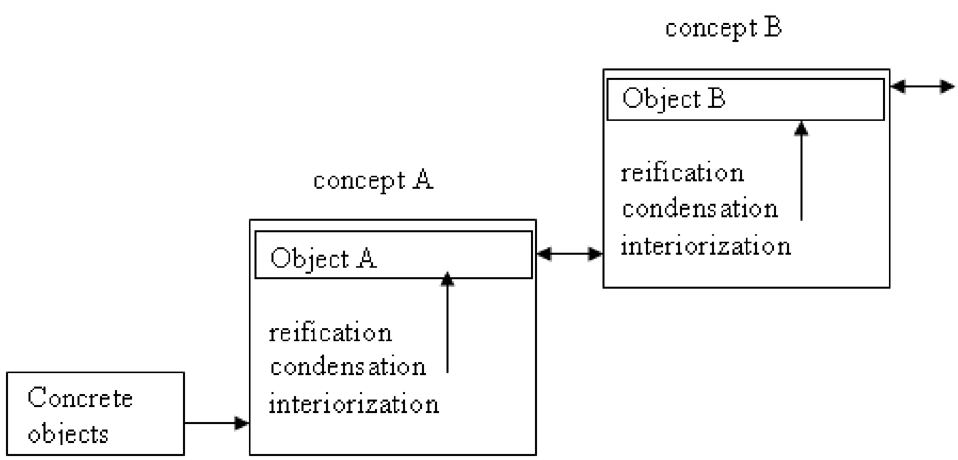

FIGURE 2.2: General model of concept formation (adapted from Sfard, 1991, p 22).

An illustration of reification in learning mathematics is the equation 0.999… = 1. Infinity is a difficult concept. The first interpretation of the equation is always that of a process that goes on without end. 0.999… can be interpreted as a situation where an imaginary frog is placed on a number line. From zero he jumps towards 1, but he only jumps 9 tenths of the way. Next time he jumps 9 tenths of the remaining distance, and so he keeps on. Even if he goes on jumping indefinitely he will never reach 1. Therefore 0.999…≠1. In order to accept the equality as true it is necessary to stop seeing 0.999… as a process that goes on indefinitely and start seeing it as an object-like whole. If infinity is the concept A in figure 2:2 then concept B is perhaps the concept of limits, a concept where infinity is referred to as an object. Accepting infinity as an object is only possible when it is being used as an object in a new process, for instance when working with limits.

Reasoning is an act of thinking and making judgements, it can be described as the art of systematic derivation of utterances, public as speech or private as thinking. It is here treated as the special kind of thinking that occurs consciously towards a particular goal, e.g. to make a decision, produce a convincing argument or solve a problem. Mathematical reasoning is 'mathematics in action', it is the act of doing mathematics. Although an utterance is usually verbal, symbols are so numerous in mathematics that many mathematical utterances are expressed only with symbols. $5(7-3) = (5\cdot 7)-(5\cdot 3) = 35-15 = 20$ is an example of symbolically expressed mathematical reasoning. It is an underlying assumption in this research that there is a link between how we understand mathematics and how we reason mathematically. However, since the nature of that link is obscure, as is the notion of understanding, it is the reasoning as such that is to be studied. Mathematical reasoning can be studied in the doing and saying of the student. English (1997a, p 22) widens the traditional notion of reasoning as being abstract and disembodied to a view of reasoning as being embodied and imaginative. "What is humanly universal about reasoning is a product of the commonalities of human bodies, human brains, physical environments and social interactions". Mathematical reasoning utilizes a number of powerful, illuminating devices described as thinking tools. One of these thinking tools is the metaphor.

Thompson (1993) defines *quantitative* reasoning as an important part of mathematical reasoning but different from *numerical* reasoning. Quantity, he writes, is not the same as number.

> A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measure to reason about them. (Thompson, 1993)

Quantities are more concrete than numbers, since numbers can represent relations between quantities as well as quantities themselves. Mathematical development is here seen as increasingly sophisticated ways of reasoning about mathematics. In classroom data it is often possible to see when development has occurred or seems not to have occurred, although the mechanisms behind are out of view.

Cognitive and Commognitive Conflicts

Attention to cognitive conflicts and meta-cognition has been identified as characteristic issues of a social constructivist framework (Ernest, 1996). Two different theoretical perspectives that pay attention to these issues are briefly presented and discussed in this section: the theory of Commognition and theories of Conceptual Change.

A *commognitive* theory of learning (Sfard, 2007, 2008) is rooted in the assumption that all human skills are products of individualization of historically established collective activities. Different types of communication that bring some people together and exclude others are in this framework called different *discourses*. A discourse is made distinct by its repertoire of permissible actions. Although all discourses originate on the social plane, as interpersonal discourses, a discourse can become individualized and turn into an intrapersonal discourse. Colloquial discourses are visually mediated by concrete material objects existing independently of the discourse. In contrast, literate mathematical discourse makes massive use of symbolic artefacts, invented specifically for the sake of mathematical communication. Mathematical discourse is distinguished by four interrelated features:

- Word use
- Visual mediators (symbols, pictures, gestures, concrete artefacts)
- Routines (procedures, algorithms)
- Narratives (descriptions, statements, definitions, axioms, proofs)

A narrative is neither true nor false: it can only be endorsed and taken as shared by the participants of the discourse. *Commognitive conflict* situations arise when communication occurs across incommensurable discourses; discourses that differ in their use of words and mediators or in their routines. A commognitive conflict is for instance when the negative number -4 is referred to as both larger and smaller than the negative number -3, or when the subtraction 3-7 is said to both have and not have a solution. To be able to accept the statements as either true or false it needs to be clarified within which discourse they are to be used. When seemingly contradictive narratives appear, a commognitive conflict arises. The only way to resolve such contradictions is to become aware of the different discourses and their different uses of words, mediators and routines. See Sfard (2008) for more details about the commognitive framework.

Theories of *conceptual change* focus on the role of prior knowledge in learning. Conceptual change is traditionally taken to mean a *radical change or a clear reorganization of prior knowledge*. (cf. Merenluoto, 2005; Merenluoto & Lehtinen, 2004a, 2004b; Vosniadou & Verschaffel, 2004). When the learner's prior knowledge is incompatible with new conceptualizations, misconceptions appear and systematic errors are made. Sometimes prior knowledge can even hinder the acquisition of new information. In order to overcome this, a radical conceptual change is needed that requires metacognitive awareness. This implies that the learner needs to become aware of her prior conceptions so that she can recognize the difference between her prevailing conception and the new information offered. Two important concepts describing the activity of learning are: *enrichment*, meaning learning by continuous growth, improving existing knowledge structures and assimilating new experiences with prior knowledge, and *reconstruction*, meaning significant reorganization of existing knowledge structures. Prior knowledge is given a crucial role in learning; prior knowledge can promote learning but it can also lead to misconceptions and restrict learning. A misconception is not necessarily to be seen as incorrect or false, only not viable in the context at hand. The way to overcome misconceptions and go through what is deemed as a conceptual change is to create situations of *cognitive conflict*. "[…] a cognitive conflict is to be aware of thinking about the relevant topic in different ways at different points in time, or even of being able to deliberately switch back and forth between two perspectives that are not compatible with each other" (Ohlsson, 2009). Ohlsson points out explicitly that conceptual change is a process of reorganizing knowledge, not a process of falsification (ibid, p 35). This issue is also emphasized by Hatano (1996) who argues that conceptual change can be regarded as a radical form of restructuring in the sense that knowledge systems before and after are incommensurable; that some information in one system cannot easily be translated into the other. However, restructuring knowledge can also take on a milder and more subtle form. The early ideas of radical conceptual change developed within science education have lately been modified by several researchers with a broader interpretation of the term (Vosniadou, 2008). It is acknowledged that misconceptions are often robust and difficult to change radically, and that learners tend to create 'synthetic models' by adding incompatible pieces of knowledge to their prior conceptions (e.g. Vosniadou, 1992; Vosniadou, Vamvakoussi, & Skopeliti, 2008). Such synthetic models, also labelled misconceptions, could be important stages between an intuitive or naïve conception and a more scientific conception. Vosniadou, Vamvakoussi and Skopeliti (2008) conclude that "[T]he process of conceptual change is a gradual and continuous process that involves many interrelated pieces of knowledge and requires a long time to be achieved".

Within a social constructivist framework a conceptual change approach could be useful both for describing the psychological development of each student, but also for looking at how a mathematical concept changes over time and in different social settings. "The conceptual change approach has the potential to enrich a social constructivist perspective and provides the needed framework to systemize … findings and utilize them for a theory of mathematics learning and instruction." (Vosniadou & Verschaffel, 2004). One such example is described by Vlassis (2004) concerning the development of a sense of 'negativity'.

> In the case of negative numbers, we can find evidence of errors in the literature due to students' attempts to assimilate negative numbers with their presuppositions about natural numbers. This kind of conceptual change is related to the fact that what students already know about natural numbers is inconsistent with the new numbers. (Vlassis, 2004)

Sfard (2007) makes a clear distinction between the two related notions of commognitive conflict and cognitive conflict. A comparison between the two is shown in table 2:2 (adapted from Sfard, 2007). The first three columns in the table are copied from Sfard's work. The last column is a reinterpretation of Sfard's description of a cognitive conflict in light of the view of mathematics as socially constructed (described earlier in this chapter), instead of the platonic view on mathematics inherent in Sfard's description. To emphasise the similarity between the two concepts of conflict, a common terminology is used, mainly retrieved from the theory of commognition. One term, interlocutor, is replaced by the more common term participant, meaning that an interlocutor is a participant in the discourse. Having conceptions is referred to as having a personal discourse, i.e. having access to words, representations (mediators), procedures (routines) and narratives connected to a particular mathematical concept. A social discourse is what Sfard labels an interpersonal discourse, i.e. a discourse used in social interaction with other participants.

TABLE 2.2: Comparison of concepts: Commognitive Conflict versus Cognitive Conflict (adapted from Sfard, 2007, p 576) and a Reinterpretation Sfard's description of Cognitive Conflict.

Concept	**Cognitive Conflict**	**Commognitive Conflict**	**Reinterpretation of Cognitive Conflict**
Ontology, conflict is between:	The participant and the world	Incommensurable discourses	The personal discourse and the social discourse
Role in learning:	Is an optional way for removing misconceptions	Is practically indispensable for metalevel learning	Is necessary for changing the personal discourse
How the conflict is resolved?	By rational effort	By acceptance and rationalization (individualization) of the discursive ways of an expert participant	By rational effort to see the difference between two discourses

It is evident from the reinterpretation described in table 2:2 that the distinction between the two concepts of conflict almost vanishes when adjustment is made to the view on mathematics as socially constructed, and that the great difference suggested by the use of different terminology is only illusory. In this thesis the two terms are used interchangeably meaning conflicts between two ways of talking about and conceptualizing mathematics that need metalevel awareness.

2.4 Number sense

> I've dealt with numbers all my life, of course, and after a while you begin to feel that each number has a personality of its own. A twelve is very different from a thirteen, for example. Twelve is upright, conscientious, intelligent, whereas thirteen is a loner, a shady character who won't think twice about breaking the law to get what he wants. Eleven is tough, an outdoorsman who likes tramping through woods and scaling mountains; ten is rather simpleminded, a bland figure who always does what he's told; nine is deep and mystical, a Buddha of contemplation....
>
> *Paul Auster*

The notion of number sense has for the last two decades been much used in educational literature as well as among teachers and teacher educators to describe knowledge about numbers. Originally the notion depicted an innate, intuitive ability to correctly judge numerical quantities. As it became a notion more widely used among researchers and educators it gradually became more and more inclusive. Today, number sense is used in curricula and teaching materials and is included in the mathematical framework underlying the construction of the international tests in TIMSS[16]. In the TIMSS framework concerning grade four and eight is stated that: "students should have developed number sense and computational fluency".[17] The number content domains related to are *whole numbers*, *fractions*, *decimals*, and in grade eight also *integers*. It might seem surprising, considering all the research there is about number sense for positive numbers, that there is no mention of what a developed number sense for integers is supposed to be. It is the ambition of this study to contribute to filling that gap.

The term *number sense* has a built-in multiple meaning that the Swedish translation lacks[18]. *Sense* can mean to have a feeling for, or to be able to understand. The term sense also stands for the powers that give us information about things around us; our senses. When something *makes sense* it means that it has a clear meaning and is easy to understand, it is sensible, it has a good reason or explanation. *To sense* something means to feel that something exists or is true. All these meanings are related to and thus enrich the meaning we put into the term *number sense*. In this research the term *number sense* will be used in an inclusive way,

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[16] Trends in Mathematics and Science Study

[17] Retrieved December 1, 2008, from http://timss.bc.edu/TIMSS2007/PDF/T07_AF_chapter1.pdf

[18] in Swedish *number sense* translates to *taluppfattning*. *Tal* means number, but *uppfattning* only means apprehension, perception or opinion. There is no relation to words indicating that things are reasonable, true or have an explanation.

indicating that it is not only how we perceive numbers but also what sense we make of these numbers.

Number sense is today a widespread concept that, although no more than half a century old, permeates literature on mathematics education for primary school. The term is said to have come from Tobias Danzig in 1954 and was elaborately developed by Dehaene (1997), who considered number sense an end product of the human brain and of a slow cultural evolution. He defines it as an 'imprecise estimation mechanism'. Many animals have a basic number sense according to Dehaene's definition, but what makes humans different is our ability to develop language and symbolic systems that we can use to make intricate and complex plans. Numbers sense therefore has an innate component similar to that of animals but also a culturally mediated component unique to humans. In two of the often used handbooks of research on mathematics teaching and learning, number sense is treated in relation to or as a part of a chapter on estimation. (Sowder, 1992; Verschaffel, Greer, & De Corte, 2007). The original, rather narrow, definition of the concept has by many mathematics educators, curriculum writers and researchers been broadened. (Gay & Aichele, 1997; Gersten & Chard, 1999; Kaminski, 2002; McIntosh, Reys, & Reys, 1992; B. Reys & Reys, 1995; R. Reys, 1998). These authors depict number sense as "an acquired 'conceptual sense-making' of mathematics" (Berch, 2005, p 335). If we are to use the concept when speaking about numbers that are not natural numbers it is necessary to go along with the second view, although also considering features such as intuitions about quantity and magnitude, counting and subitizing (accurate perceptions of small quantities) to be important ingredients in the sense-making process. The number zero and negative integers are not natural in precisely the meaning that they are not intuitively understood.

There are many different views on what comprises number sense. Using the large and inclusive set of components compiled and analysed by Berch (2005) the following subset of five components stand out as especially relevant concerning integers:

1. Elementary abilities or intuitions about numbers.
2. Ability to make numerical magnitude comparisons.
3. Ability to recognize benchmark numbers and number patterns.
4. Knowledge of the effects of operations on numbers
5. A mental number line on which representations of numerical quantities can be manipulated.

The first four components have previously been discussed in a conference paper (Kilhamn, 2009c). Although the components brought up in the paper did not cover the whole concept of Number Sense, data showed that the notion of Number Sense and the components described are applicable to negative numbers. Important to note is that the different components overlap and interrelate and that making them explicit is a way of fulfilling the ambition of creating a well-organized conceptual network that enables a person to relate number and operation, including when the number domain is extended from natural numbers to integers. One of the two empirical studies reported in the second part of this thesis was designed to investigate student's development of number sense. This study is reported in chapter 7.

2.5 Three issues of interest

> So it goes, new mathematics from old, curving back, folding and unfolding, old ideas in new guises, new theorems illuminating old problems. Doing mathematics is like wandering through a new countryside. We see a beautiful valley below us, but the way down is too steep, and so we take another path, which leads us far afield, until, by a sudden and unexpected turning, we find ourselves walking in the valley.
>
> *Rick Norwood*

The theoretical perspectives presented so far in this chapter, although having diverse histories and following different routes, intercept around some important ideas that could be described as junction points. In this section three such junction points that underlie the aim and research questions of this thesis will be outlined.

Metalevel conflict

In both the commognitive and conceptual change perspectives the idea of change on a metalevel is emphasised, and in both perspectives a conflict situation is described when what a student already knows (her prior knowledge or intrapersonal discourse) comes into conflict with what she meets (new information, a new discourse). In both cases a main issue for a resolution of the conflict is to become aware of it on a metalevel. The idea of a conflict and a resolution of that conflict on a metalevel is a useful analytic tool in a social constructivist framework, especially where mathematics is considered a fundamentally human invention of concepts used to communicate. In such a perspective there is no difference between the *real mathematical world* and the *social world of mathematics communication*. A new experience of the mathematical world can be described as a new way of speaking about a mathematical object. In the transition from natural numbers to integers, the cognitive conflict is a conflict between old and new conceptions of number. Vlassis (2004) writes about conceptual change when "becoming flexible in negativity", proposing that meta-cognitive awareness will be determinant for conceptual changes to really occur. She writes: "This can only be developed in discursive practices where students can experience their own symbolizing activities in order to understand the taken-as-shared meaning" (ibid, p 483). If conflict situations are seen as significant, or perhaps indispensable, for the learning of mathematics there is a legitimate reason to look for them in the classroom practice and in student's development of mathematical reasoning and see how they are resolved.

Importance of historical influence

The fact that mathematics is a social and cultural enterprise emphasises the importance of historical influence. "I conceive of mathematics learning as a process of making sense of mathematics as it is brought to us by cultural history" writes van Oers (1996). In a social constructivist perspective this relates to the conceptual history of the individual as the sociomathematical norms and practices of the mathematical community, locally in the classroom, and also in the wider community of mathematical practices at different times in history. Sfard (1991) compares the individual conceptual development with the historical conceptual development and points out that historically an operational conception always precedes a structural conception. In that respect, each individual seems to recapitulate the historical development of a conception.

Drawing on Haeckel's fundamental law of biogeny[19] the so called recapitulation theory for mathematical learning flourished during the late 19th century (Jankvist, 2009a, 2009b; Thomaidis & Tzanakis, 2007). The argument is that the mathematical development of an individual must go through the same stages as the history of mathematics itself. The idea was questioned by many authors for various reasons (cf. Fauvel & van Maanen, 2000) and has later been replaced by arguments of *historical parallelism*. According to Jankvist, it is particularly in relation to single mathematical concepts, what he calls in-issues in mathematics, that historical parallelism is used as an argument for using history as a tool in mathematics teaching. "For example, to learn about the number sets (**N**, **Z**, **Q**, **R**, **C**), their interrelations, their cardinalities etc. is to be considered a study of in-issues" (Jankvist, 2009b, p 23).

Thomaidis & Tzanakis (2007) describe two aspects of parallelism between the world of teaching and learning mathematics and the world of scholarly mathematical activity; a negative parallelism that refers to the difficulties and obstacles found in both worlds, and a positive parallelism referring to the ability to overcome these difficulties. Contrary to the assertion about historical parallelism, Damerow (2007) claims that individual and the historical development of cognition are two fundamentally different processes and there is no reason to believe there are analogies between the two. Mumford (2010) points out, using the evolution of negative numbers as an example, that a concept can evolve differently in different cultures, so there is not always *one* historical evolution to study. According to Hersh (1997); in mathematical development problems come first, axioms later, but then in the formal presentation axioms come early. Once a mathematical concept has become well

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19 The law of biogeny states that ontogeny recapitulates phylogeny, meaning that the evolution of a single organism retraces the evolution of the entire species (For more details see: Haeckel, 1906).

defined and axiomatic, the next generation will meet the concept in a backward way compared to the way it was developed. They meet axioms first and only see problems as examples of these axioms.

Even if the individual and the historical development differ, learning about one might help us better understand the other. Sfard writes: "Being interested in learning, I focus in my analysis on the development of mathematical discourses of individuals, but I also refer to the historical development of mathematics whenever convinced that understanding this latter type of development may help in understanding the former" (Sfard, 2008, p 127). Whatever role the historical development has in relation to the individual student's learning it seems plausible that the study of both and a comparison between them could be enlightening whether the aim is to understand historical or individual development.

Role of metaphors

At the start of a cognitive development there are always fundamental bodily experiences. Experiences from our senses, such as taste, touch, smell, sight, hearing, balance, pain and the kinaesthetic sense are perceived by the body from outside stimuli. These experiences are called embodied because they originate in our physical body. From these experiences the first concepts are socially formed and in a series of linking representations more and more abstract concepts are created. Abstracting, according to Sfard (2008), is a term that refers to the activity of creating concepts that do not refer to concrete objects, and arises from the activity of saming (giving the same name to a number of things previously not considered as being the same) or reification. Hersh (1997) writes:

> The rules of language and of mathematics are historically determined by the workings of society that evolve under pressure of the inner workings and interactions of social groups, and the psychological and biological environment of earth. They are also simultaneously determined by the biological properties and the nervous systems, of individual humans. (Hersh, 1997, p 8)

Studying the process of abstracting and the connections between embodied experiences and abstract objects in terms of reification (Sfard, 1991) and first or higher order representations (Damerow, 2007) could be helpful in order to understand how students make sense of mathematics.

Mathematics, seen as socially constructed, is inevitably a product of language use. In Sfard's (2008) terminology mathematics is a discourse. According to Lakoff and Johnson (1980), one of the strongest linguistic features for meaning making is the use of metaphors. Many researchers emphasize the important role of metaphors in learning and teaching mathematics (cf. Frant Acevedo, & Font, 2005; Lakoff & Núñez, 2000; Sfard, 1994; Williams & Wake, 2007). This was also the topic of a working group at CERME[20] 2005. One of the questions for

[20] The Fourth Congress of the European Society for Research in Mathematics Education.

further research posed in that working group was: What are the characteristic metaphors, in use or possible, for different domains of mathematics? (Parzysz et al., 2005).

In a social constructivist perspective the awareness of the social construction of mathematical knowledge implies that knowledge is irrevocably bound up with texts and semiosis (Ernest, 1996). Consequently, the uses of different linguistic and semiotic tools are expected to influence knowledge construction. Metaphors play an important role as a thinking tool (English, 1997b). They make mathematical interpretation and reasoning about unfamiliar things possible by organizing new experiences in terms of those with which we are already familiar. They are essential to our ability of sense making and a reason why we can participate in a discourse with which we are not familiar. The mechanism of metaphor is known as the action of transplanting words from one discourse to another (Sfard, 2008, p 39). In light of this, studying the use of metaphors in mathematical discourse seems to be an interesting and desirable endeavour. To embark upon the study of metaphors an additional theoretical framework is needed. Such a framework will be described in the next section.

2.6 Conceptual metaphors

> Metaphorical construction is a double-edged sword. On the one hand, it is what brings the universe of abstract ideas into existence in the first place; on the other hand, however, the *metaphors we live by* put obvious constraints on our imagination and understanding. Our comprehension and fantasy can only reach as far as the existing metaphorical structures allow.
>
> *Anna Sfard*

Since the publication of "Metaphors we live by" written by Lakoff and Johnson (1980), the interest in metaphors has had an upswing in mathematics education research. (e.g. Carreira, 2001; Chiu, 2000, 2001; Danesi, 2003; Edwards, 2005; English, 1997b; Font et al., 2010; Frant et al., 2005; Parzysz et al., 2005; Stacey et al., 2001a; Williams & Wake, 2007). These researchers all seem to agree on the importance and power of metaphors in mathematics education, although the term metaphor is not always well defined. At times metaphor is used interchangeably with analogy or model. A clear definition that distinguished one from the other seems difficult to make. Based on the findings presented by Lakoff and Johnson (1980), Sfard (1994) suggested that metaphors can play a role in translating bodily experiences into the abstract realm of mathematical ideas. Metaphors are essential in creating a "universe of mathematics". Lakoff and Núñez (1997, 2000) subsequently developed a theory about the metaphorical origins of mathematics. The conceptual metaphor theory described here is mainly based on that work, but however useful, no new perspective on mathematical thinking goes without critique, and some of the weak points of the theory will therefore be pointed out at the start. One of the main problems with the theory, according to Schiralli and Sinclaire (2003), is that the authors do not recognise that metaphor might function differently depending on whether one is learning, doing or using mathematics. Lakoff and Núñez describe metaphorical pathways to the *concepts* of mathematics, but do not provide evidence that these are the same pathways that an individual will follow in creating a *conception*. Does creating, teaching and learning mathematics follow the same tracks? Another problem is that a strong emphasis on one thing seems to rule out other things, but in fact, although powerful, metaphor is not the only meaning-making strategy used in mathematics.

With such a brief introduction and a small reservation, the theory of conceptual metaphors will now be described and used as an analytical tool for this research. It is to be seen as a useful and enlightening perspective on mathematical thinking, not as a description of any truth about either mathematics or learning. In a study of teaching and learning of graph functions Font et al. (2010) found that a theory of conceptual metaphors was a relevant tool for analyzing teachers' mathematical discourse. The theory itself, in particular those parts that are relevant for arithmetic, is described in this chapter, and in chapter 3 the theory is used as an analytical tool for a metaphorical analysis of models for negative numbers.

Metaphor

According to conceptual metaphor theory (CMT), metaphorical reasoning is fundamental to human thinking and can lead to the construction of mental models (English, 1997a, p.7). Lakoff and Núñez (1997, 2000) see mathematics as a product of inspired human imagination and a product of the embodied mind. According to Lakoff and Johnson (1980) in their book *Metaphors We Live By*, most metaphors are based on experiences that are products of human nature; physical experiences of the body, the surrounding world and interactions with other people.

A metaphor is a semiotic phenomenon based on similarities between two given entities. It can be described as a discursive tool. One entity is described with words from a different domain. "A metaphor is what ties a given idea to concepts with which a person is already familiar" (Sfard, 1994). When we say X is like Y it is an explicit metaphor or a simile. A proper, or implicit, metaphor is when we speak of X as Y and thus understanding X in terms of Y. In CMT a metaphor is seen as a mapping from a source domain to a target domain (Kövecses, 2002; Lakoff & Johnson, 1980). Properties of the target domain (X) are understood in terms of properties of the source domain (Y), which implies that the source domain must be well known. Human experiences of the physical world constitute source domains in conceptual metaphors, whereas the target domains of are abstract concepts. Saying, for instance, that 5 is a larger number than 3, is to speak metaphorically since the abstract mathematical objects 5 and 3 do not have size. They can be visualised as either distances: 5 is larger because it represents a longer distance; or as piles of apples: 5 as larger because it represents a larger pile of apples. Distances and piles have size and this property is transported over to the mathematical objects through metaphor. For some purposes it is useful to distinguish between related linguistic concepts such as metaphor, simile, analogy and metonymy. There is no prevailing agreement as to how these concepts are defined and they are sometimes used interchangeable in literature (Pramling, 2006). In this research project, the term *metaphor* will be treated as a wide term encompassing analogy, simile and metonymy. This treatment of the term goes back to Aristotle (ibid, 2006) and seems to be a general treatment of metaphor in mathematics education research (cf. English, 1997b).

Metaphors make sense of experiences by providing coherent structure, highlighting some things but also hiding others. Different metaphors are used to structure different aspects of a concept, which means that to fully understand a rich concept several different metaphors are needed. A conceptual metaphor is a mapping from entities in one conceptual domain to corresponding entities in another conceptual domain and it is through the similarities that exist between these two domains that the metaphor becomes powerful (see figure 2:3). Metaphors create the universe of abstract ideas and are the source of our understanding, imagination and reasoning.

FIGURE 2.3: Figurative representation of the different parts in a metaphor.

However, properties of the source domain that are not similar to those of the target domain may also be carried over, with contradictions and confusion as a result: "Metaphors are often like Trojan horses that enter discourses with hidden armies of unhelpful entailments" (Sfard, 2008, p 35). Another problem that might arise when using metaphors is the fact that they are used implicitly and differently by different individuals. A metaphor can be seen as a condensed analogy, which means that *A* is talked of *as if it were B* rather than being *like* B. For instance: "life is like a battle" is an analogy, whereas "strategic moves in life" is a metaphorical expression where life is talked of as if it were a battle. Pimm (1981) claims that problems often arise from taking structural metaphors too literally and that "[…] conflicts can arise which can only be resolved [by] understanding the metaphor (which requires its recognition as such), which means reconstructing the analogy on which it is based". Conceptions of time and space have influenced the structure of the natural numbers, how they and the rules of arithmetic are conceptualized. As time passes, one day at a time passes by and every day has a foregoer and a successor. The natural numbers are defined by a unit and its successors and foregoers. In set theory a number is defined as objects (elements) in a set; the more objects there are in a set the larger the number. Bodily experiences of time and space and biologically driven activities form the basis of mental models universal for all time and culture, writes Damerow (2007). Others write about 'innate numerical intuitions' (de Cruz, 2006) or 'self-evident intuitively accepted cognitions' (Fischbein, 1993). Intuitive internal representations and intuitive conceptions are in the theories of conceptual metaphors referred to as the basic or grounding metaphors whose sources are in bodily experiences (English, 1997a; Lakoff & Núñez, 1997, 2000; Sfard, 1994)

Grounding metaphors for arithmetic

Lakoff and Núñes (1997, 2000) argue that basic arithmetic is understood through a set of grounding metaphors that link structures from every-day domains to the domain of mathematics, and more advanced mathematics is understood through linking metaphors, linking one branch of mathematics to another. Basic arithmetic, they claim, is understood through four grounding metaphors:

~ Motion along a Path (numbers as point locations or movements)
~ Object Collection; bringing together, taking away (numbers as collections of objects)
~ Object Construction; combining, decomposing (numbers as constructed objects)
~ Measuring Stick; comparing (numbers as length of segments)

G*rounding* metaphors are fundamental to the mathematical concepts in as much that if you eliminate them, much of the conceptual content of mathematics would disappear. In contrast to the grounding metaphors there are *extraneous* metaphors which, although often useful tools for visualising mathematical structures, can be eliminated without conceptual consequences (Lakoff & Núñez, 2000, p 53). A thermometer could for example be seen as an extraneous metaphor for negative numbers since it is quite possible to understand the negative numbers even without a thermometer, but underpinning the thermometer is a grounding metaphor that is essential for the conceptual content of arithmetic, namely the *motion along a path* metaphor. Lakoff and Núñez (2000) emphasise the fact that mathematics is created out of metaphors, that all mathematical concepts can be traced back to a metaphorical beginning.

The grounding metaphor *Arithmetic as Motions along a Path* will highlight properties of numbers that are similar to properties of locations and movements along a path, such as the ordinality of numbers, whereas other properties will be out of focus. We think of numbers as being close or far from each other and of for instance addition as moving forward. This metaphor transfers experiences of how different physical places are related to each other and how we move between them onto mathematical objects called numbers. The metaphor *Arithmetic as Object Collections* focuses more on the property of cardinality of numbers and answers the question *how many*. Experiences of discrete objects, like that there are more fingers on the two hands together than on one hand, are transferred to numbers so that we speak of 10 as being more than 5. The possibility of partitioning numbers in different ways comes from the metaphor *Arithmetic as Object Constructions*. Experiences of building and constructing things and taking them apart are the source domain for numbers when we say things like: "two halves make a whole", "7 is made up of 3 and 4" or "any whole number can be factorized into prime numbers in a unique way". The last metaphor, *Arithmetic as Measuring Lengths*, answers the question *how much,* i.e. 4 is twice as much as 2, 10 is higher than 8, 8-3 is the distance between 8 and 3.

Number is a rich concept and all four metaphors are needed to structure all features of the concept. Lakoff and Núñez claim that these four metaphors are sufficient for us to understand and make sense of arithmetic with natural numbers, but in order to extend the field of numbers to include zero, fractions and negative numbers the metaphors need to be *extended*, or *stretched* (Lakoff & Núñez, 2000, p 89). However, it is unclear what is meant by stretching or extending a metaphor, and it is not yet empirically shown that the four grounding metaphors are sufficient for the extended number domain. The notion of extending grounding metaphors to incorporate the extended number domain will be examined in chapter 3.

Source and target domains and direction of the mapping.

A metaphor can serve as a vehicle for understanding
a concept only by virtue of its experiential basis.
Lakoff & Johnson

When teachers select concrete material, visualizations, didactical models, or specific teaching materials to represent mathematical concepts they do so because these representations "[…] presumably capture aspects of the concept adults believe to be especially important. There is no guarantee, though, that students see the same relationship in the materials that we do" (Hiebert & Carpenter, 1992, p 72). In other words, the teachers choose a model which they assume to be well known to the students, and which will lend itself as a source domain in a mapping to some mathematical context. There is never a complete one-to-one correspondence between source and target domains in a metaphor. Moreover, the metaphor might have different directions so that source becomes target and vice versa. "The teachers' source domain is mathematics and the target is daily life because they try to think of a common space to communicate with the students*"* (Frant et al., 2005, p 90). To the teacher it is mathematics that is the familiar domain and the chosen model is the target domain, as shown in figure 2:4. The teacher knows that the product of two negative numbers is supposed to result in a positive number and tries to think of a situation in for example a money model that could be mapped onto the multiplication. For the students, the target domain is the still unknown mathematics.

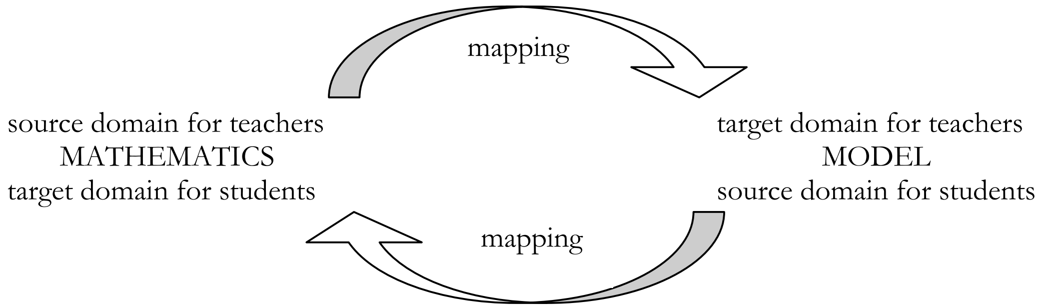

FIGURE 2.4: Figurative representation of the two way metaphor.

Metaphors and models

An often used term that has many uses within mathematics education is the term 'model'. In the instruction theory of Realistic Mathematics Education (RME) (Gravemeijer, 1994) the term is broadly interpreted. "Models are attributed the role of bridging the gap between informal understanding connected to the 'real' and imagined reality on the one side, and the understanding of formal systems on the other side" (van den Heuvel-Panhuizen, 2003, p 13). Metaphors also play this role and could thus be labelled models, and in much of the mathematics education literature no clear distinction between metaphors and models are made. In older literature 'metaphor' is seldom mentioned but many examples that are referred to as metaphors in more recent literature about conceptual metaphors are there referred to as models. For instance the use of an 'objects-in-a-container model' (Fischbein, 1989) being similar to the *Arithmetic as Object Collection* metaphor, or the use of a 'number-line-as-path-model' (Freudenthal, 1983) as similar to the *Arithmetic as Motion along a Path* metaphor.

For most purposes there is no need to differentiate between the two terms, in this work a subtle distinction is made. Metaphors signify the whole system of source domain, target domain and mapping between these, as they are produced in the discourse. Models are treated as source domains in these metaphors and thereby as part of the metaphor. This way of treating the term model makes it possible to talk about and manipulate a model without any connection to that which it is supposed to model. One might object to this by saying that is that case it is not a model. Within RME models are seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the situation, and these representations can *serve as models* (van den Heuvel-Panhuizen, 2003, p 13). When they *do not* serve as models they are still there. It is, for example, possible to do things with Dienes decimal rods, like building towers, when they are not serving as a model for the ten base system. As soon as they serve as a model, and somebody speaks about numbers in terms of these rods, they become the source domain in a metaphor. A metaphor, on the other hand, does not exist without both domains and a mapping between them. Speaking of numbers as cubes and rods is speaking metaphorically, whereas speaking about the Dienes blocks is speaking about the model. Gravemeijer (2005, p 84) points out that one problem with these blocks is that they serve as a model for teachers but for the children they are "nothing but just wooden blocks". That could mean that the metaphorical use the teacher has of the model is lost on the children who only see the rods, and not the mapping to a mathematical target domain. One model can also be the source for several different metaphors. The number line for example, can be used in many different ways and thus be a model of different things, hence a source domain in different metaphors. Speaking of numbers as measured segments or distances on a number line is using the number line model as a source in one metaphor, whereas speaking of numbers as points on a number line is using it as a source in a different metaphor.

All four grounding metaphors have a distinctly spatial nature. Vergnaud (1996, p 234) writes: "Space lends us many metaphorical ideas for thinking about physical and social phenomena, even though they may be nonspatial. It would be a great epistemological mistake to minimize the part played by properties of space in symbolic representation". Among the symbolic representations of space, the number line is for mathematics the most frequently used and most powerful one. However, "it takes many years for students to understand how to read the number line *as a set of numbers*" (ibid, italics not in original). Considering the research about a mental number line reported in chapter 1, it is perhaps surprising that a number line is not so easily understood. The reason for this could be that the number line is not treated as an explicit metaphor for numbers. It is a model that students learn to recognise, read and talk about, but they might not see it as a metaphor for numbers so that they may speak of numbers in terms of the number line.

As a final word about models a *mathematical model* needs to be explained. Much of the mathematical activity that goes on concerns mathematical modelling. Mathematics serves as a model for a realistic or imagined situation. Here the metaphor is turned the other way. We can for example model movements as functions. We speak of these movements in terms of mathematical functions, therefore the mathematical model is a source domain in the metaphor, and the target domain is the problem situation. Which way the metaphor is turned and what constitutes its source and target domains depends on whether we are creating, doing, learning or teaching mathematics. In a teaching-learning situation a model or visual representation is usually intended to serve as a source domain for a mathematical concept or procedure.

The relation between metaphor and model will be further explicated in the following chapter when models for negative numbers are analysed as source domains of metaphors. The theoretical analysis presented in chapter 3 is an attempt to investigate how negative number models can be understood within a theory of conceptual metaphors, and what their affordances and constraints are.

CHAPTER 3

Metaphor Analysis of Negative Number Models

They [negative numbers] lie precisely between the obviously meaningful and the physically meaningless. Thus we talk about negative temperatures, but not about negative width.
Alberto A. Martínez

The aim of this chapter is to analyse some of the models that are frequently used when teaching negative numbers, in order to understand more about why this is difficult and why students often perform poorly on this topic. A collection of models that are common in the teaching of negative numbers in a school context in general and in Swedish schools in particular was made over a number of years in the course of teaching mathematics in teacher training. The collected models are analysed within a theory of conceptual metaphors (Lakoff & Johnson, 1980; Lakoff & Núñez, 1997, 2000). As a complement to the analysed models, a new metaphor is introduced in section 3.4 called *Number as Relation* metaphor. The analysis will focus on addition and subtraction since these operations constitute the bulk of the empirical study. Multiplication and division is briefly commented in section 3.5. The results show that the extension of a grounding metaphor is not a trivial process, and that it may cause as much confusion as clarification.

Contemporary negative number models

A review of some contemporary mathematics textbooks and teacher guides where negative numbers are introduced show a strong focus on negative number models. For example in 'Adding it up' (Kilpatrick, Swafford, & Bradford, 2001) teachers are told that "Students generally perform better on problems posed in the context of a story (debts and assets, scores and forfeits) or through movements on a number line than on the same problems presented solely as formal equations". The models recommended there are: *Apples and Negative Apples* (apples that you owe), *Money Transactions,* and *Arrows on a Number Line*. A discrete model consisting of *Different Coloured Counters* was presented in a special edition on negative numbers from the English net-journal for teachers NRICH[21] in January 2008, and has also been recommended to Swedish teachers in the journal *Nämnaren* (Persson, 2007). In a recent textbook for mathematics teachers (Skott, Hansen, Jess, & Schou, 2010) three types of representations for negative numbers are explained: set representations (black and red objects), position representations (number line, thermometer) and debt representations.

[21] Retrieved February 13, 2008 from: http://nrich.math.org

Negative number models presented in contemporary textbooks[22] and by Swedish in-service teachers during university seminars in Gothenburg in 2004-2008 were collected and compiled. When viewed, many of the models appeared to have similarities and were categorized in two groups according to these similarities. Models in the first group are all related to an extended number line and models in the second group are related to objects of opposite quality.

The first group of models include for example:

> Degrees below and above zero on a thermometer
> An elevator or stairs below and above ground floor
> Meters below and above sea level
> A hot air balloon going up and down
> The hem of a skirt being taken up or down as the fashion changes
> A glacier growing in winter and shrinking in summer
> A car driving back and forth along a road
> Walking or flying north and south

These models all have in common a movement on a path in two directions from a base level or point of reference. A slightly more abstract version of these models is simply moving a pointer on a number line.

The second group of models include for example:

> Money as assets and debts, or profits and losses
> Strokes over and under par (golf)
> People going in and out of a bus or a room
> Happy and sad people moving in and out of a town
> Hot and cold cubes in a witch's kettle
> Magic 'opposite' stones in a pocket
> Deposit and withdrawal of money
> Increase and decrease of price

These models have in common that two opposing objects pair off and thus 'undo' or neutralize each other. They originate from the very first use of negatives in history, namely to keep track of economic transactions. This view of numbers as quantities obstructed the acceptance of negative results, since it was difficult to accept a quantity less than nothing (Beery et al., 2004; Schubring, 2005). A slightly more abstract version of this model is to have counters of two different colours and assigning to each colour the qualities positive and negative (Altiparmak & Özdoan, 2010; Persson, 2007).

Grounding metaphors and their extensions

Lakoff & Núñez (1997, 2000) argue that all mathematics comes from embodied metaphors and that all basic arithmetic can be understood through four grounding metaphors, as described in chapter 2.6. They also describe ways of stretching the metaphors and blending several metaphors so that their

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[22] Mainly textbooks from Sweden, but also some from Australia, USA, and the Netherlands.

entailments together create closure for arithmetic in the enlarged number domain including zero and negative numbers. In particular, they suggest a metaphorical blend with a symmetry metaphor or a mental rotation around a centre. Apart from the idea of opposite numbers related to a mirror metaphor (Stacey et al., 2001a), symmetry and rotation metaphors are scarce in the material collected for this study. This indicates that although rotation around a centre is a "natural cognitive mechanism" (Lakoff & Núñez, 2000, p 92) it does not necessarily follow that it is connected to the extension of the number domain.

This chapter focuses the grounding metaphors and their extensions as they take shape when commonly used models, such as a thermometer, constitute the source domain. As a result of the categorisation of models into two distinctive groups described previously, the analysis will focus mainly on the two grounding metaphors *Arithmetic as Motion along a Path* and *Arithmetic as Object Collection* and explore what happens to them when they are stretched to meet the requirements of negative numbers. The grounding metaphors are basic and intuitive and require little instruction (Lakoff & Núñez, 2000, p 53), whereas the stretched metaphors convey sophisticated ideas that get more contrived the more they are stretched (ibid. p 95).

Method of analysis

To understand a metaphor, the analogy from which it is created needs to be 'unpacked'. The analytical tool chosen here is adapted from Lakoff and Núñez (2000) and Chiu (2001, pp118-119). The metaphor is described in terms of the features of the source domain (i.e. the model) that are used to communicate about the target domain (i.e. the mathematical content) along with the attributes of the target domain to which they correspond. If for example we, say that 'the computer is resting', the metaphor is described like this: Experiences of a human resting (source domain) is mapped onto (→) the situation where there is nothing being processed in the computer (the target domain). Evidently, there are many attributes of a human resting that are not mapped onto corresponding features of the computer, such as lying down instead of standing up, feeling more energetic after the rest etc.

Attributes of the target domain of arithmetic that need to be mapped in order to operate with negative numbers using metaphorical reasoning are: zero, number, sign of number (positive, negative), magnitude, value, addition, subtraction, multiplication, division. The operations sometimes also need to be subdivided into different cases depending on where the negative numbers appear, i.e. for $a - b = c$ either one or several of the numbers a, b and c can be a negative.

First in each of the following sections, a generic extension of the chosen grounding metaphor is made and presented in a table, identifying how different features of the source domain could be mapped onto features of the target domain, and then restrictions of such mappings are analysed. Following the generic extension, different versions of the extended metaphors based on different source domains are analysed.

3.1 Arithmetic as Motion along a Path.

In the *arithmetic as motion along a path* metaphor a line, horizontally as a number line or time line, or vertically as a thermometer, is seen as a *path* and addition and subtraction are seen as *motions* along this path. Sometimes the motions are referred to as the distance the motion covers rather than the motion itself, which can be described as a blend with a *measurement* metaphor, as in table 3:1.

TABLE 3.1: *Motion along a Path* metaphor combined with *Measurement* metaphor extended to include negative numbers (based on: Chiu, 2001p, 118-119; Lakoff & Núñez, 2000, p 68-72). The extension from the original grounding metaphor is highlighted:

Source domain **is mapped onto** →	**Target domain**
No movement	Zero
A *centre location* or starting point	Zero
A location along the path where you start or end up after having moved	A number.
A movement	A number
The length of a movement (distance)	Magnitude[23] of a number
Further to the right or up is higher value	Value of a number
Moving to the right or up (or forward)	Addition
Moving to the left or down (or backward)	Subtraction

The described metaphor is not the only possible version of an extended path metaphor. Nonetheless, it is relevant to analyse it since it is the simplest extension where only the path has been extended and not the interpretations of the operations. How well does the source map onto the target in this extension of the *object collection* metaphor?

- Number: There are two different mappings to number. In the expression $a + b = c$; a, b and c are all numbers. In the source domain only a and c can be understood as locations along a path (points on a line), whereas b is a *movement* or a *distance* on the line, a number of steps to move with a direction indicated by the operation sign. Numbers on the number line have a dual nature: they can be constructed simultaneously as locations and movements. In the target domain all numbers can be understood in terms of both of these properties. In this source domain there is a great difference between a point on a line and a motion along the line; namely that *the motion is always*

[23] *Magnitude* is tantamount to the absolute value of a number, whereas the word *value* is here used to indicate the order value of a number, i.e. …-8 < -7 < -6 .... -1 < 0 < 1 < 2 < 3 …. 7 < 8 < 9 …

positive or zero. You either move or you do not move. There is no such thing as anti-motion. Motion can be in different directions, but the direction is settled by the operation sign. This implies that *b* can only be positive, whereas *a* and *c* can be both positive and negative.

~ Addition: There is nothing in the source domain that maps onto an addition of a negative number since *b* can only be positive.

~ Subtraction: There is nothing in the source domain that maps onto subtraction of a negative number since *b* can only be positive

Walking up and down stairs or travelling along a path.

Mapping the extended *motion along a path* metaphor in the generic version described above has a serious limitation in that it does not allow addition or subtraction of a negative number. Assume that *b* is a negative number. What would that be in the source domain when the idea of moving a negative number of steps is not part of our experiential basis? In models where there is somebody moving along a path, it is possible to impose *direction* and start speaking of motion forward and backward *relative to the person moving*, as shown in table 3:2, or alternatively in table 3:3.

TABLE 3.2: First extension of the *Motion along a Path* metaphor made when dealing with the number line and *direction of movement*, i.e. when the model includes for example a person walking up and down stairs or a travelling vehicle with front and back.

Source domain	**is mapped onto** →	**Target domain**
The first part of the metaphor is the same as in the generic version		
A movement *forward*		A *positive* number
A movement *backward*		A *negative* number
Moving the facing way		Addition
Turning around and moving		Subtraction
Facing right or up		*Default setting*[24]

In this extension the direction of the motion is no longer mapped onto different operations but onto the (polarity) sign of the number. Addition is the implicit action, and *turning around* before moving is mapped onto subtraction. There is a loss of consistency in this new version of the metaphor since there is a simultaneous dual reference to the words forward and backward. In the original metaphor the word *forward* always results in a movement to the right on the number line, but in the extended metaphor *forward relates to the person moving* and will sometimes result in a movement to the left on the number line (when facing left). Furthermore, the metaphor comes into conflict with other metaphors that are deeply rooted in our culture such as *positive is up and forwards*, *adding on is up* (we grow up, we build up etc), *right is forward* (Semitic writing flows from left to

[24] When several operations are involved it is necessary to return to default setting after every operation.

right). Another limitation is the necessity of a default setting. If the calculation involves several operations it is necessary to return to default setting after each operation, cf. 4 - (-3) - (-4) or 4 - 3 - 6. A slightly different version which eliminates the default setting is shown in table 3:3. Instead of turning around for subtractions, it is the facing direction that is mapped onto the operation sign. In this extension the dual reference to the directions is even more prominent, particularly if the original version of the metaphor has been to map motion forward onto addition and backward onto subtraction. When there is no subject with front and back in the model this extension is not possible. A third extension, described in relation to a thermometer model, gives an alternative.

TABLE 3.3: Second extension of the *Motion along a Path* metaphor made when dealing with the number line and *direction of movement*, i.e. when the model includes for example a person walking up and down stairs or a travelling vehicle with front and back.

Source domain	**is mapped onto →**	**Target domain**
The first part of the metaphor is the same as in the generic version		
A movement *forward*		A *positive* number
A movement *backward*		A *negative* number
Facing right or up and moving		Adding a number
Facing left or down and moving		Subtracting a number

Temperatures rising and falling[25]

In table 3:4 the third version of an extended metaphor deals with motions along a path where the path is a thermometer and therefore the movement along the path does not have front and back.

TABLE 3.4: Third extension of the *Arithmetic as Motion along a Path* metaphor when dealing with the *thermometer*. Motion is represented by temperatures rising or falling.

Source domain	**is mapped onto →**	**Target domain**
The first part of the metaphor is the same as in the generic version.		
A movement *up*		Adding a positive number
A movement *down*		Subtracting a positive number
A distance *with direction* between two points		The result of a subtraction
To add a negative number a different metaphor is needed that will make it plausible that $a + (-b) = a - b$ since a temperature can not go up a negative number of degrees.		

[25] This extension is relevant in a Swedish context since all students when they meet negative numbers have experiences of talking about temperatures above and below zero.

Adding a negative number here is interpreted to be the same as subtracting a positive number; 3 + (-2) = 3 - 2, an interpretation which lacks meaning in the source domain. The mapping is no longer internally consistent since addition sometimes is seen as a motion along the line and sometimes need to be transformed into a subtraction. The subtraction (a - b) can be interpreted as the difference between a and b, or the motion needed to get from b to a. Although this is coherent with the mathematical definition of subtraction saying that (a - b) is the number x that solves the equation $b + x = a$, in the definition there is no mention of *from* and *to*. However, in our experience of the difference between two temperatures or two points on a line, we always conceive of that difference as an absolute value. The difference between the two numbers -4 and +6 is always spoken of as 10. If we are to understand that 6 - (-4) and (-4) - 6 yield different answers it is necessary to incorporate into the mapping the aspect of direction. The experience expressed by the phrase: "Today it is 6 but yesterday it was minus 4; it *rises from* minus 4 *to* plus 6 so it is 10 degrees *higher*" is mapped onto the mathematical expression 6 - (-4) = 10. "Today it is (-4) but yesterday it was 6, it *falls from* 6 *to* (-4) so it is 10 degrees *lower*" is mapped onto the mathematical expression (-4) - 6 = (-10). The direction here is *from b to a*, which is from right to left in the expression. This is not coherent with the other uses of the metaphor *from one number to another*; 2 – 8 reads in non-mathematical contexts as *from two to eight*. Direction has been identified as one of the critical features for learning subtraction of negative numbers (Kullberg, 2006). It is, as shown, a feature closely connected to the use of certain metaphors involving the interpretation of subtraction as a difference. This extension of the metaphor is internally inconsistent since for the subtraction ($a - b = c$), a, b and c have different sources depending on whether they are positive or negative. If b is positive then a and c are referred to as temperatures and b is the change, but if b is negative then a and b are referred to as temperatures and c is the change.

To summarize: These extensions of the *motion along a path* metaphor entailed a loss of internal consistency and a loss of coherence with related metaphors.

3.2 Arithmetic as Object Collection

In the metaphor *arithmetic as object collection*, numbers are conceived as collections of objects. In this metaphor the aspect of cardinality of number is prominent. A generic extension of the grounding metaphor is shown in table 3:5.

TABLE 3.5: The *Object Collection* metaphor extended to include negative numbers (based on: Chiu, 2001, p 117; Lakoff & Núñez, 2000, p 55). Extensions from the original grounding metaphor are highlighted.

Source domain	**is mapped onto →**	**Target domain**
No collection (empty container)		Zero
A collection *of one type* of objects		A *positive* number
A collection *of an opposite type* of objects		A *negative* number
More objects in one collection		Magnitude
---------		Value
Combining two object collections		Addition
Combining equal collections of the two types		*Adding opposites to make zero*
Taking part of a collection of objects away from a bigger collection		Subtraction

Every metaphor has its restriction since not all features of a target domain are mapped in a one-to-one correspondence from the source domain. How well does the source map onto the target in this extension of the *object collection* metaphor?

- ~ Value: There is nothing in the source domain that maps onto the value of a number e.g. $-1 > -5$
- ~ Addition: When two collections of opposite types have been combined they pair off and equal collections of each makes zero (empty collection). This is a *new feature* of the object collections that did not exist before the extension. How this is experienced in the source domain depends on what the source domain is. The opposing objects can be said to "create a neutral object", "undo each other", "dissolve each other" etc… It could also be a feature that needs to be linked to a different metaphor: the O*bject Construction metaphor*, where the number zero is seen as constructed of equally big collections of objects of opposite types.
- ~ Subtraction: Only objects of a type that is already there can be taken away, there is nothing in the source domain that maps onto these subtractions: $4 - (-2)$; $(-2) - 4$; $2 - 4$; $(-2) - (-4)$ Only subtractions $a - b$ where a and b are the same type (have the same sign) and $|a| > |b|$ are mapped.

Mapping of the *object collection* metaphor in the generic version described in table 3:5 has limitations as to mapping subtraction and needs to enforce a new interpretation of zero as constructed of combined collections of different types of objects. The necessary requirement being two equally large collections of opposite types that pair off to leave zero. An inconsistency appears as zero thereby is conceptualized in two different ways; as an empty collection and as the result of combining two collections. Another limitation is the lack of mapping to the value of a number. We speak of *small* and *large* collections of negative or positive objects when a *large* collection of negative objects represents a *smaller number* than a *small* collection of negative objects, which is in itself a contradiction in terms. A visual representation of the source domain of this metaphor also gives a defective conception of the value of numbers. If negative numbers are represented by white markers and positive numbers by black markers, a visual representation may look like this:

[●●●●] = 4 [●●] = 2 [○○] = -2 [○○○○] = -4

The representation is coherent with the deeply rooted metaphor *more is bigger* (Lakoff & Johnson, 1980) and with the mathematical idea of absolute value (magnitude), but contradicts the mathematical definition of the order value of numbers where (-4) < (-2) < 2 < 4

Objects of opposite types

The generic extension of the metaphor deals handsomely with additions of all kinds and with subtractions of numbers of the same type when the first term has a greater magnitude, but when it comes to subtracting numbers of different types the source domain fails. We have no experience of what it means to take away objects of one type from a collection of objects of a different type. So the metaphor needs a further extension, which is to create something out of nothing. In order to take away two objects of a certain type, these can be created by adding zero composed of opposite types. Having three cold objects and wanting to take away two hot objects is done by first adding four objects, two hot and two cold, to the collection and thereafter taking away the two hot objects leaving five cold objects. This is sometimes made plausible by statements such as "the kettle is already full of both hot and cold objects, the number given only show the surplus of one kind". In this case the mapping can be described as in table 3:6.

TABLE 3.6: First extension of the *Object Collection* metaphor dealing with *objects of opposite types*, such as models with hot and cold objects, happy and sad people etc

Source domain is mapped onto →	Target domain
Two equally large collections of objects of opposite types in an *infinite* or finite collection	Zero
A *surplus* amount of objects of one type in an *infinite* (or *undefined*) collection	A positive number
A *surplus* amount of objects of an opposite type in an *infinite* (or *undefined*) collection	A negative number
The rest of the extended metaphor is the same as in the generic extension	

A new feature in this extension of the metaphor is that a collection of objects is now infinite or undefined and can consist of both types of objects. This extension of the metaphor contradicts the grounding metaphor where a number was understood as a collection of one type of objects in a finite collection. We have many experiences of the physical world as a source for the conception of zero as an empty container, but no experiences at all of infinite collections of objects of opposite qualities. This extension imposes upon the source domain a property which is not originally there. One way to solve this is to introduce the mathematical idea of multiple representations. The number 2 could then, using the visual illustrations introduced earlier, be represented in either of the following ways:

[●●] or [●●●○] or [●●●●○○] etc.

Depending on the calculations involved a particular representation is chosen. If money is used as a source domain in this metaphor the idea of taking 4 away from 2 results in a debt of 2 only because the 2 to be taken away have to be borrowed before they are taken away. 2 - 4 in thus handled by having 2, adding 2 more and thus generating a debt of 2 and then taking away 4 to leave a debt of 2. The second extension is a different extension dealing specifically with money.

Money

A second way of extending the metaphor used when money is a source domain is to indicate that the minus sign illustrates an inverse, as in table 3:7. If the original number *a* is a profit, then a minus in front it (-*a*) is a loss. But if the original number is a loss (-*a)*, a minus in front of it makes it a profit. -(-*a)* = *a*. A negative loss is the same as a profit. Here the fact that the minus sign has different meanings is neglected, which has been shown to be an important aspect of understanding negative numbers, as described in chapter 1.

TABLE 3.7: Second extension of the *Object Collection* metaphor dealing with *money* and where subtraction has two different mappings.

Source domain	**is mapped onto →**	**Target domain**
The first part of the metaphor is the same as the generic version		
Taking part of a collection of objects away from a larger collection of objects		Subtraction
Changing the type of objects in a collection		Subtraction as addition of the opposite number.

This extension of the metaphor is inconsistent since for the subtraction *a* - *b* the minus sign is conceptualized differently depending on the value and type of *a* and *b*; if $|a|>|b|$ and both are the same type the minus sign is interpreted as a sign of operation, but if $|a|<|b|$ or if *a* and *b* are different types then the minus sign is interpreted as a change of the quality and the operation to be carried out is implicitly addition. For example: if positive numbers are money in the hand and negative numbers are debts then 4 - 2 in interpreted as a set of 2 coins taken away from a set of 4 coins. 2 - 4 on the other hand cannot be understood the same way since you cannot literally take 4 coins away from 2 coins. Instead the minus sign is understood as indicating a change of type so that what we actually have is 2 + (-4). If we have 2 coins and also a debt of 4 coins we can consider our balance to be a debt of 2 coins. 2 - (-4) must consequently be understood as changing the type turning a debt into an asset. What we have is 2 coins and another asset of 4 coins, making a balance of 6 coins. 2 - (-4) = 2 + (-(-4)) = 2 + 4. Here there is an implicit addition as soon as the original interpretation of subtraction as taking away is not applicable. This way of dealing with elements of opposite types is perhaps understandable on a generalized level, and may be used in advanced accountancy, but when using a model with money as a source domain for teaching negative numbers it is far removed from students everyday and familiar experiences of money.

Another way of extending the mapping of the *object collection* metaphor when dealing with money is to interpret subtraction as a difference by comparing two sets of objects, shown in table 3:8. In this extension the act of comparing is mapped onto subtraction rather than taking away or changing the type.

TABLE 3.8: Third extension of the *Object Collection* metaphor: an alternative extended mapping of dealing with *money* where comparison is mapped onto subtraction.

Source domain	**is mapped onto →**	**Target domain**
The first part of the metaphor is the same as the generic version		
Comparing two collections of objects		Subtraction
Directed difference of amounts of objects in two compared collections		A number

In order to differentiate between subtraction *a* - *b* and *b* - *a* the comparison must have a direction and so we run into the same problem as with the temperatures; a difference between two numbers in real life is always referred to by its absolute value. Interpreting 2 - 4 as a comparison between me (first number) and you (second number) is to say that if I have 2 coins and you have 4 coins then I have 2 coins *less* than you, therefore the answer has to be (-2). 2 - (-4) is interpreted as me having 2 coins and you having a dept of 4 coins. A comparison from me to you says that I have 6 coins more than you (which is not obvious since I only have 2 coins). Furthermore, (-2) - (-4) tells me that I have a dept of 2 coins and you have a dept of 4 coins which leaves me with 2 coins more than you (which is not obvious either since I have a debt and therefore no coins). The everyday expression would be to say that I have 2 less in debt than you or your debt is 2 more than mine, but that does not map onto the right calculation. This extension requires us to treat statements about money in ways that are not part of our experiences and thus not part of the original source domain.

To summarize: the first and third extension of the A*rithmetic as Object Collection* metaphor imposed upon the source domain a property which was not originally there; the second extension entailed a loss of internal consistency.

3.3 Number as Relation

> No mathematical concept can be reduced to a physical embodiment, a fraction entails relations, and relations are not palpable, physical objects.
> *David Carraher*

Taking the view of Carraher (1996) in the above quote seriously, not only fractions but also negative numbers can be viewed as relations. In the *motion along a path* metaphor numbers were mainly seen as locations or motions. In the *object collection* metaphor numbers were mainly seen as object collections; i.e. quantities. There were also a few indications of numbers as relations when seen as directed distances or directed differences. If focus is directed towards *Number as Relation*, a completely different metaphor could be introduced A generic version of this metaphor related to both the grounding metaphors analysed in this chapter is described in table 3:9.

In the *number as relation* metaphor there is a big difference between the different meanings of the minus sign. A sign of negativity (a unary sign) always indicates a state; either a specific entity or a relation. A sign of an operation (a binary sign) always indicates an action; a transformation of a relation. For example in $a + b = c$ and $a - b = c$; a and c are relations and b is an entity that transforms the relation a into c, since b comes after an operation sign.

TABLE 3.9: *Number as Relation* metaphor in the extended number domain.

Source domain **is mapped onto** →	**Target domain**
Balance, equilibrium	Zero
A relation between two locations or entities such as collections or movements	A number
More than balance	A positive number
Less then balance	A negative number
Size of inequality	Magnitude
More is higher value	Value
An entity; collection or movement	A number
Collections of opposite type objects *or* movements in opposite directions	Positive and negative numbers
Adding, putting together	Addition
Taking away	Subtraction

Number as Relation when dealing with object collections

If the source domain is object collections we need to have objects of two different and opposite types, as in the extended metaphor described previously. A *relation* is the *balance* between two collections of objects of different types. A transformation is either adding[26] or taking away a collection of one type of objects.

An object example: On the dance floor there are many couples dancing, but sometimes also people standing on the side without partner[27]. All couples consist of one man and one woman. A woman is one type of object called positive, and a man is an opposite type called negative. At one point in time the balance on the dance floor is 2 more women than men, mapped onto the positive number 2. Then 6 men leave the dance floor to get a drink. This transformation of the relation is mapped onto the subtraction of negative 6. The new balance is that 8 women are left without a partner, mapped onto the positive number 8: [2 - (-6) = 8].

A money example: In a bank account money can be put in (deposited) and taken away (withdrawn) and the relation between the amounts deposited and withdrawn is called a balance. On one occasion the balance is such that 100 kronor more has been withdrawn than deposited. This negative balance is mapped onto negative 100. Then another 200 is taken out, which is mapped onto the subtraction of 200 The transaction will result in a new account balance of 300 kronor more being withdrawn, mapped onto negative 300: [(-100) - 200 = (-300)]. The tricky part in this model is to understand what taking away a withdrawal is, as illustrated in the next example: On one occasion the balance is such that 200 kronor more has been deposited than withdrawn, which is mapped onto positive 200. In the balance there had been withdrawals as well as deposits. Then one withdrawal of 500 kronor is taken away; a bill that had been paid was a mistake so that the money that was withdrawn for that bill is returned to the account. This is mapped onto a subtraction of negative 500. The new balance will be that 700 kronor more has been deposited than withdrawn, which is mapped onto positive 700: [200 - (-500) = 700].

With a *number as relation* metaphor it is easy to illustrate the idea that adding one number gives the same result as subtracting the opposite. Say the table is to be set everyday with as many places as there are children in class. If there is an empty place a child is missing, symbolised by -1, and if there is one person who does not get at place there is one too many, symbolized by +1. If one day the children come in and the relation between set places and children is -2 (two empty places) there are two ways of making it even. Either we take away two

[26] In Swedish: *lägga till eller ta bort*. English does not have an everyday expression for adding, equivalent to *taking* away for subtraction, but Swedish does.

[27] The dance floor model was introduced to me by Ingemar Holgersson

empty places (subtract negative 2) or we invite two more children to eat with us (add positive 2). In both cases we end up with an even relation, with equilibrium: [-2 - (-2) = -2 +2 = 0]. See also Linchevski and Williams (1999) who used this metaphor with good results in a model keeping track of how many people entered and left a room.

Number as Relation when walking up and down stairs

If the source domain is a person walking up and down stairs there are two kinds of movements in opposite directions. For the addition $a + b = c$ or the subtraction $a - b = c$, the starting point is the location in relation to zero, mapped onto the first number a, and the new relation to zero after the transformation is the second relation, mapped onto the last number c. The transformation is either adding a movement or taking away a movement, mapped onto $+b$ or $-b$. In this case taking away a movement will be the same as imagining where the person was before that movement was made.

An example: Peter starts this walk on step 4 above ground level (ground level is the equilibrium). This is mapped onto the number positive 4, the relation is 'four above starting point'. Peter then walks down 5 steps. This is mapped onto the addition of negative 5. He ends up on the step 1 below zero, which is mapped onto the new position in relation to zero 'one under starting point'; negative 1: [4 + (-5) = -1]. Fred on the other hand, is also on step 4 above ground level. But now we want to find out where he was before he moved up 5, we want to take away a movement up 5. This is mapped onto the subtraction of positive 5. It turns out that he was on step 1 below ground level before he moved up: [4 - 5 = -1]. These two example illustrate that the calculations 4 + (-5) and 4 - 5 are equivalent inasmuch that they generate the same resulting relation: 'one below starting point', although they are mappings of completely different meaning in the source domain.

A second example: Susie starts walking on step 5 below ground level, 'five below starting point', which is mapped onto the number negative 5. We then take away a movement down 8 steps. This means we have to retrace where Susie was before she moved down 8 steps and reached 5 below ground flour. This is mapped onto the subtraction of the number negative 8. We retrace the movement down 8 by going up 8 steps from 5 and find out that she must have been on step 3 above zero before the move, which is mapped onto the number positive 3: [(-5)-(-8) = 3].

Number as Relation metaphor illustrated on a number line

Similarly, the *number as relation* metaphor can be shown on a number line, as in figure 3.1. The figure is an illustration of two different situations, one addition and one subtraction.

Situation one: the first balance point is found on the number 4 (four above zero). An additional movement of 5 towards the negative side is mapped onto an addition of -5 resulting in a new balance on -1: [4 + (-5) = -1].

Situation two: The first balance point is -1 (one below zero) and from that we want to *take away* a movement of 5 towards the negative side which is mapped onto a subtraction of -5. It is evident that before the movement the balance point must have been positive 4: [(-1) - (-5) = 4]

FIGURE 3.1: Illustration of the two equations: 4 + (-5) = -1 and (-1) - (-5) = 4.

The *number as relation* metaphor is not a completely new metaphor since it utilises elements of the grounding metaphors much like a symmetry metaphor does. It does, however, add a new dimension to the theoretical frame of grounding metaphors constructed by Lakoff and Núñez (2000).

3.4 Multiplication and division of negative numbers

All the models hitherto presented are useful and fairly simple for mapping addition of integers. Consequently, they are all useful for mapping multiplication whenever multiplication can be seen as iterated addition. Iterated addition multiplication differentiates between multiplier and multiplicand and is therefore not a symmetrical operation. If the multiplier is denoted a and the multiplicand m then $a \cdot m = b$ is interpreted as a lots of m, where m can be either an object collection or a movement, but a always has to be a natural number. In the extension from natural numbers to integers, m could still be seen as either object collection or movement, now with the extended source domain that includes objects of opposite types or movements in opposite directions. If the first term, a, is a natural number the multiplication is still an iterated addition, but if a is to be an integer a new dimension needs to be added. The number of times a multiplicand is *either added or subtracted* could be mapped onto a signed number, so that a positive multiplier means an iterated addition and a negative multiplier means an iterated subtraction. The multiplication 3 · (-5) would then be interpreted as for example adding 3 lots of negative 5; adding 3 debts of 5, performing 3 successive movements down 5 steps etc. Multiplication of two negatives like (-3) · (-5) would be interpreted as *taking away* 3 lots of negative 5, taking away 3 debts of 5, taking away (retracing) 3 movements of 5 steps down etc. The original limitations of dealing with subtraction in the different models are carried over as limitation for dealing with multiplication. For division the actions would be the opposite: b/m would mean the number of times m has to be added or taken away to get b.

Apart from the previously mentioned limitation for subtractions, iterated addition multiplication is also asymmetrical, as pointed out by Freudenthal (1983). Mathematically, for $a \cdot b$; a and b are both numbers of the same status and multiplication is commutative so that $a \cdot b = b \cdot a$. The traditional way of teaching this is to abandon the iterated addition multiplication in favour of multiplication as an area. A negative area was never accepted as making any sense in Greek geometry and is never spoken of in everyday contexts, and is therefore not a promising option. Two relations cannot be multiplied.

What remains to be investigated is the metaphorical blend of rotation and symmetry together with numbers as point-locations (Lakoff & Núñez, 2000, p 91). In this metaphor numbers are conceptualised as point-locations on a line with a symmetry axis through zero. Each natural number n has a symmetrical point-location $-n$ on the opposite side of zero. Lakoff and Núñez introduce a specific metaphor of multiplication by (-1) as *rotation*: "Rotation to the symmetry point of n is mapped onto $(-1) \cdot n$". The authors speak of a mental rotation of a point-location 180º around the zero point. They write that "Given this metaphor, $-n \cdot a = (-1) \cdot n \cdot a = -1 \cdot (n \cdot a)$, which is conceptualized as a mental rotation to the symmetrical point of $(n \cdot a)$". It is not said whether the multiplication of two negative numbers is to be conceptualized as an iterated addition of a movement to the left, and thereafter a rotation, or as an iterated addition of a movement to the right and thereafter two rotations. In other words if $-a \cdot -b = -1 \cdot (a \cdot -b)$ or if $-a \cdot -b = -1 \cdot a \cdot -1 \cdot b = ab \cdot -1 \cdot -1$. In the first case there is the same asymmetry as in the previous models, so to be rid of that restriction the second interpretation is to be preferred. The inconsistency in this metaphorical blend appears in the different interpretations of a negative number. If a negative number is to be added, it is seen as a *movement toward the left*, but if it is to be multiplied it is seen as a *movement toward the right and a rotation*. Subtraction of a negative number is seen as a *movement toward the right* but without rotation (ibid, p 91), only it is unclear if it the movement is mapped onto the subtraction or the sign of negativity.

An objection to the metaphorical blend of rotation and symmetry together with numbers as point-locations is that it does not seem to have any historical background. Where the grounding metaphors according to Lakoff and Núñez are intuitive and common to all humans, in a sense innate, and the origin of natural number arithmetic, the described metaphorical blend is consciously and carefully constructed by the authors taking the target domain of mathematics as the point of departure. They write that "a different metaphor is needed for multiplication by negative numbers. That metaphor must fit the laws of arithmetic" (Lakoff & Núñez, 2000, p 91). New metaphors, stretched metaphors and metaphorical blends are constructed not in order to create the laws of arithmetic but to fit laws that are already known. Why laws that apply to natural numbers also must apply to negative numbers is not evident. It may not be inferred that negative numbers and operation on them “come from” this metaphorical blend, but perhaps that they can be understood through it. It could also be relevant to highlight the difference between mathematics that has metaphorical origins and metaphors that have mathematical origins.

3.5 Summary and conclusions

Previous research described in chapter 1 indicated that difficulties in dealing with negative numbers are not easily overcome by any particular model. The result of this analysis shows that it might not be the presence or absence of a particular model that enhances understanding but rather how the model is introduced and related to abstract mathematical ideas, and how explicit the limitations of the model are.

In the analysis, for instance money and money transactions served as a source domain in several metaphors or metaphorical extensions indicating that a combination of these could be present in the classroom discourse at the same time: This might make the participants in the discourse believe they speak about the same thing when using the same money model although they relate to different metaphors. Here, focusing one metaphorical extension at a time has made it possible to see potential limitations and constraints that could cause problems for students who try to make sense of negative numbers through metaphorical reasoning. According to Lakoff and Núñez (2000) not all aspects of a concept will be in focus in each metaphor, which implies that multiple metaphors and metaphorical blends are necessary to get an adequate understanding.

It has been shown that an extension of a grounding metaphor may make the metaphor less functional for three different reasons, all contributing to loss of clarity:

- Loss of consistency within the metaphor itself,
- Loss of coherence with related metaphors or the original grounding metaphor,
- Properties or structure being brought into the source domain that is not part of the experiential basis.

It was assumed that extending the field of natural numbers to integers is not done by a simple extension of the grounding metaphors. On the contrary, metaphors that lend themselves useful in the field of integers, albeit similar to and sprung from the grounding metaphors, are different from, and in some cases even contradictory to, the grounding metaphors. The *extended metaphors* are in fact *new and different metaphors*. Perhaps the differences between a grounding metaphor and its extension are more significant than the similarities. When a metaphor is extended the quality and interpretation of it might change and, even if the structure is carried over, the meaning might change. Pimm (1981) writes: “If the metaphoric quality of certain conceptual extensions in mathematics is not made clear to children, then specific meanings and observations … about the original setting will be carried over to the new setting where they are often inappropriate”. Hence, making the metaphor in use explicit in the public space of shared meanings, discussing similarities and differences between different possible interpretations, and making clear what is to be carried over from one domain to the other, could be didactically profitable or even crucial.

In relation to their work on conceptual change in mathematics Vosniadou, Vamvakoussi and Skopeliti (2008, p 28) come to a similar conclusion concerning the extension of the number concept from natural numbers to rational numbers. They write that students base their initial understandings on naïve but “relatively coherent explanatory systems which are based and continuously re-confirmed by everyday experiences”. As the students employ enrichment mechanisms to add mathematical information to existing knowledge structures they are “destroying their coherence and creating internal inconsistency”.

Lakoff and Núñez (1997, 2000) claim that all mathematics comes from embodied metaphors. This means that mathematical ideas are inherently metaphorical, originating from physical experiences and a result of how our body and our senses work. When a mathematical concept becomes established it is detached from its metaphorical roots and becomes reified, an object in its own right and thereafter enters as source domain in new metaphors. More abstract mathematics is created through a series of metaphors, all with an embodied root. In the end of the process there are many metaphorical layers on different levels of abstraction, and linking metaphors, that link one domain of mathematics to another.

The physical world and the world of mathematics connect to each other in a metaphorical sense when we speak of one in terms of the other. In the original process, here called metaphor M_1, physical experiences are mapped onto mathematics so that mathematical concepts are conceptualised in terms of the physical experiences, see figure 3.2. This processes in enacted in the *creation* of mathematical concepts. For example: a man has 5 coins, he is asked to give 6 coins in return for a product he wants, he is said to become indebted, he now owes 1 coin. To keep track of this activity a new type of number is created and labelled a negative number. The activity is written as 5 - 6 = -1. Negative numbers are henceforth conceptualised in terms of debts and coins or other objects that are ‘yet to be delivered’.

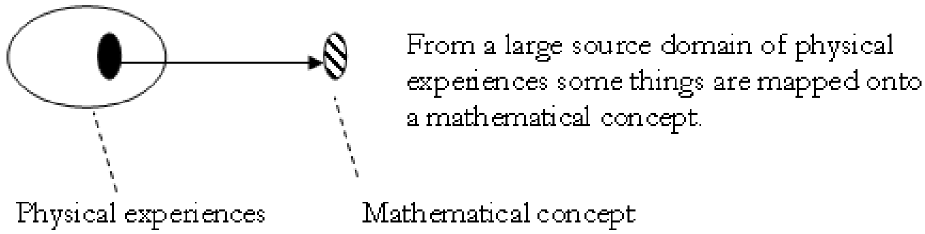

FIGURE 3.2: Metaphor M_1 mapping physical experiences onto mathematics

Once mathematical concepts have been invented, they are incorporated into a consistent mathematical structure, a structure mathematicians strive to make complete.[28] The new mathematical concept, with its metaphorical meaning, starts revealing properties that were not originally there but appear as a result of the mathematical structure. In history, so long as negative numbers were used as representations of negative quantities, there were certain properties, subtracting a negative from a positive, that were not taken to be part of the concept. At some stage however, it was considered useful to do this and the whole concept was reconceptualised. As a result of relating a new concept to older concepts in a consistent structure, new features about both the old and the new concepts are 'discovered'. This process is clearly manifest concerning signed numbers. When negative numbers appear on the scene, the natural numbers need to be reinterpreted as positive numbers. In a sense the mathematical concept 'number' grows, extends beyond the features that were once its origin.

Teaching of mathematics involves a quite different mapping process. Often teachers, who are familiar with the mathematical concepts on a higher level of abstraction, try to map the mathematical concepts onto some parts of the large domain of physical experiences, here called metaphor M_2. For negative numbers the part of the physical world that once was the source of origin for concept (the black domain in figure 3.3), is now too small, since the negative numbers have acquired new features as a result of the mathematical structure they were brought into. There are now features that will not easily map onto the domain of physical experiences. For example mapping 3 - (-8) onto experiences of having collections of coins and giving away collections of coins does not make sense.

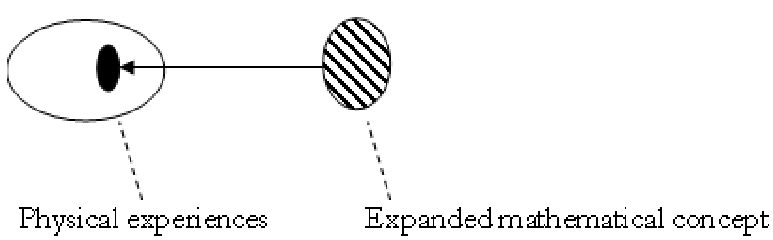

FIGURE 3.3: Metaphor $M_{2:}$ mapping a mathematical concept onto physical experiences:

[28] Although completeness is an aim, it has been shown by Kurt Gödel in 1931 that no consistent system of axioms whose theorems can be listed is capable of proving all facts about the natural numbers. Gödel's incompleteness theorem established inherent limitations of all but the most trivial axiomatic systems. Nevertheless, completeness and consistency is the ideal for a mathematical system.

If metaphor M_1 is employed at this stage, and the mathematical concepts are spoken of in terms of physical experiences, some problems might arise. Finding physical experiences that will map onto all features of the expanded concept is not trivial. Metaphor M_1 needs to be adjusted, extended, stretched or blended with other metaphors to encompass all features of the expanded concept. When physical experiences are not sufficient the source domain can come to include abstract mathematical concepts. Lakoff and Núñes include the laws of basic arithmetic in their source domain when they expand beyond natural numbers. If that is done, the metaphor will not be of any use unless these abstract mathematical concepts actually are reified and no longer talked of as first order representations of physical experiences.

Metaphors are useful tools for understanding important concepts, relations or ideas in mathematics, such as the idea of opposite numbers. Once such an idea is grasped and makes sense to the student it is time to let go of the metaphor and let the abstract mathematical concepts create their own meaning. Lakoff and Núñez claim that natural numbers are understood through the four grounding metaphors. However, negative numbers are not part of the natural numbers, and as the analysis in this chapter shows, the grounding metaphors cannot in a simple way be expanded to fully explain the negative numbers. In contrast, it is the case that they do *not* make sense that is necessary to accept and make use of. Limitations and contradictions that appear in metaphorical reasoning reveal characteristic aspects of negative numbers that distinguishes them from natural numbers. Realizing that 5 - (-3) does not make sense in the metaphor *arithmetic as motion along a path* since it is not possible to move a negative number of steps is a prelude to accepting a solution that is derived from logic or formal reasoning. It is the responsibility of the teacher to help students go beyond the metaphors they use by making them aware of the constraints of their metaphors, and to help them make the transition into the domain of mathematics where formal and algebraic reasoning play a more important role. Baylis and Haggarty (1991, p 28) claim that many students have problems with equivalent expressions and that these problems…

> …arise when we have entities which have many different representations and then attempt to define operations on them by reference to just one such representation. Only if the result of the operation is independent of the representation we use can we call the operation well-defined.

Consequently, since several metaphors are necessary to understand arithmetic with natural numbers, it seems only reasonable that several metaphors are needed to understand operations with integers. Furthermore, the final definition of integers and operations on integers ought to be released from its metaphorical birth to become independent of the different representations.

CHAPTER 4

Research Questions and Research Design

This chapter begins with a presentation of the research questions formulated on the grounds of the theoretical background described in the previous chapters. Following that, is a description of the research project about making sense of negative numbers and the methods used to gather and analyse empirical data. Furthermore, methodological issues such as ethics, research validity and the researcher's role are discussed.

4.1 Research questions

The general aim of the empirical research project is to learn more about how students make sense of negative numbers. This can be expressed as an interest in how students think and how their thinking changes when negative numbers are introduced in mathematics classrooms, and as a consequence, why some students seem to learn to handle negative numbers the way we want them to and others do not. Since it is impossible to study thinking directly it is the external results of thinking, i.e. what students express in actions, words and writing, which make up the researchable data. Based on the pilot study reported in chapter 1.7, the issues of interest presented in chapter 2 and the metaphorical analysis in chapter 3 the following research questions were formulated:

1. How is metaphorical reasoning enacted in a classroom discourse in the context of negative numbers?

To investigate the first question a case study using participant observations and video recordings was designed. The research question was then broken down into three more specific questions:
a) What metaphors are brought into the discourse by the teacher and textbook?
b) What is the rationale for introducing them?
c) When and how are the metaphors used by the teacher / textbook / students?

2. How does the introduction of negative numbers in the mathematics classroom change students' number sense?

The second question was dealt with in a longitudinal interview study over a period of three years. The distinct questions put to this data were:
a) What do different learning trajectories concerning features of number sense relevant for negative numbers look like?
b) How is such a learning trajectory influenced by the use of metaphors?

Based on answers to the first two research questions the discussion in chapter 8 will also try to answer questions about metalevel conflicts and connections between students' learning trajectories and the historical development of negative numbers. More specifically:

3. Do situations of cognitive or commognitive conflict arise in the classroom context when negative numbers are taught and do they pose opportunities for metalevel learning?

4. How can knowledge about the historical development of negative numbers be useful for understanding students' development of number sense?

4.2 Design of the research project

In order to address questions that arose from the results of the pilot study reported in chapter 1.7, a longitudinal case study was designed. The main focus of the study, being the development of students' reasoning and proficiency with negative numbers, places the study in an interpretative paradigm where qualitative case studies are prominent (Cohen et al., 2000, p 181). Some characteristics of a case study are, according to Cohen et al., that it is concerned with a rich and vivid description of events, that it provides a chronological narrative, that it blends descriptions of events with analysis of them and that the researcher is integrally involved in the case. Taking all these into consideration along with the interest in looking at change over time resulted in a decision to choose a case consisting of one class of compulsory school students to follow for a period of three years. Ingredients in the case study were participant observation, recurrent individual interviews and the gathering of written student work, test results and school reports. These are seen as common ingredients in longitudinal case study research (Bryman, 2004, p 52).

The research project involved two separate but interconnected studies: a longitudinal interview study and a video study. The population was the same for the two studies: one mathematics class and one mathematics teacher were followed over a three year period from grade 6 to grade 9, as shown in figure 4.1 (see appendix I for more details). The two studies overlapped in time. The interview study included participant observations and interviews and the video study consisted of video recorded lessons. For the benefit of both studies informal teacher interviews and collections of students' work and students' results were also made throughout the whole period.

1	interviews		interviews		interviews	
	participant observations					
time line:	**grade 6 (spring)**	**grade 7 (autumn)**	**grade 7 (spring)**	**grade 8 (autumn)**	**grade 8 (spring)**	**grade 9 (autumn)**
2				video recorded lessons		one video recorded lesson

FIGURE 4.1: Time line showing data collection activities for study 1 and study 2.

The two studies are to be seen more as methodological triangulation than as two separate studies. Using two different approaches as methodological triangulation, "suitable when a more holistic view of educational outcomes is sought" (Cohen et al., 2000, p 115) makes it possible to focus on both the social and the psychological perspective of learning, which is the intention of the social constructivist framework used in this study.

The design of the research project can best be described as an explorative case study with an interest in individual students' mathematical development, and how that development relates to mathematical achievement in the social setting of a mathematics classroom. The unit of analysis was individuals' actions (including utterances) and the data collected consisted of observations, written work, interviews and video recordings. The video study gave a rich picture of the mathematics classroom practice and was a help to understand the context, constraints and affordances of the situations in which the individual students acted. The interview study gave insight into individual students' ways of reasoning and reactions to the classroom practice. In the following sections the rational and conditions for each of these studies are reported and at the end of this chapter ethical and methodological issues are discussed.

4.3 Study one: a longitudinal interview study

In order to investigate development over time the study was designed to span several years. The grades leading up to and including grade 8 were chosen because that is when negative numbers are formally introduced in Sweden. In higher grades negative numbers appear in algebra and calculus and students are expected to deal with them fluently. It seemed relevant to investigate how students reacted to these numbers when they first encountered them, what they learned about them and how they changed their conceptions of number and ways of reasoning when encountering them.

For pragmatic reasons, a school class was chosen that was located close enough to the university to allow for frequent participation of the researcher and in a socially stable area with little movement in and out. The principal along with the rest of the staff were positive to the presence of the researcher during the three year period. The class was kept together as one group taught by the same mathematics teacher for the three year period and had Swedish as the common spoken language during the school day. Most important was of course the cooperation of the mathematics teacher who was asked to collaborate with the researcher on a weekly basis for three years. The class never had more than 22 students but due to a few changes over the years full data was collected for 20 students and partial data from one student. At the start of the research period they were in the middle of grade 6. Half of the students came from the same class in grades 1-5 and the rest of the students joined the class at the start of grade 6 from other schools in the area. The mathematics teacher was new to them all in grade 6 and followed the class through grade 9. The same teacher also taught natural science to the class.

Participant observations and observation protocols

As the researcher is by profession also a mathematics teacher for the age group in question it was natural to come into the class as a participant observer taking the role of assisting the teacher. That position opened up many opportunities of interacting with the students and thereby observing patterns of reasoning and expressed difficulties. Furthermore, as the students got used to the researcher being there and came to use her as a resource in the classroom asking her as well as the teacher when they needed help, an unconstrained relationship was created that influenced the interview situations. For observations that take place over an extended period of time it is an advantage that a researcher, as a participant observer, can develop more intimate and informal relationships with those they are observing (Cohen et al., 2000, p 188). The researcher participated in approximately one out of three mathematics lesson per week.

Written observation protocols were kept stating how much time was spent on what kind of activity, who was present, what the teacher talked about in the lesson and what was written on the whiteboard. In general the bulk part of the lesson was spent on individual work with textbook exercises. During this time the researcher walked around the classroom talking to individual students. As soon as the researcher had terminated a dialogue with a student that touched upon number sense issues or related question, or overheard a conversation about such issues, a note was made about it in the protocol.

Audio recorded interviews

The method chosen to capture student reasoning was recurrent semi structured interviews with a combination of think-aloud-protocols and open questions about mathematical concepts and representations (cf. Boren & Ramley, 2000; Bryman, 2004; Kvale, 1997). The interviews were held toward the end of each school year. In a first explorative phase of the study and for the purpose of developing interview questions 12 students in other schools were interviewed. Based on previous research a number of mathematical tasks and questions were constructed concerning number sense, subtraction, the minus sign and the number line. A more detailed report of this process and the results from this phase will be published elsewhere (Kilhamn, in press). The resulting interview protocol that was the outcome of the first phase was thereafter used in all the interviews. Small adjustments were made between the different interview sessions; in particular some additional questions were included in the last set of interviews when negative numbers had been introduced. All three sets of interviews included the same questions so that a comparison over time was made possible. The advantage of a structured interview in a longitudinal perspective is that the data can include answers over time to the same question. The third interview (grade 8) also included a stimulated recall sessions when the student was asked to comment on sections of the previous interviews presented by the recordings and also on video recorded episodes from study 2.

As a result of the explorative phase of the study, the interview protocol came to include a mix of mathematical tasks that the student was asked to solve and reflect upon and some open questions of the type: "What is a number?", "What does this sign mean?". To get rich and detailed answers, the structured questions were followed by questions like: "How do you know that?", "How do you work that out?" or "How do you think?" Some issues were present in different questions at different times during the interview in order to give the student several possibilities to talk about them in slightly different contexts. The interviews were structured around seven themes: 1) what a number is, 2) the size of numbers, 3) subtraction; both procedural skill and conceptual meaning, 4) addition, 5) the minus sign, 6) brackets and rules of priority, and 7) the number line. Many of the questions related to several themes and the whole interview protocol was organised around 10 questions labelled Q1–Q10. A full interview protocol is found in appendix II.

The interviews were conducted during school time and took place in a quiet, undisturbed room. Interviews in grade 6 and 7 were approximately 30 minutes and the interviews in grade 8 took between 45 and 60 minutes for each student. All interviews were audio-recorded using an electronic device. Due to technical failure a few interviews were not fully recorded, but in those cases as much as possible of the interview was written down from memory. This happened to one interview in grade 6 and to four interviews in grade 7. Verbatim transcripts were made of all interviews in the first two sets. Based on a tentative analysis of these interviews only relevant parts of the last set of interviews were transcribed verbatim and the rest summarised.

Additional data

In addition to observations and interviews some student work and school reports was collected. This data supplied information about students' procedural knowledge, reasoning in written work and level of achievement in relation to local and national standards. In order not to disrupt the social norms of the mathematics classroom (Cobb et al., 2001) no interventions were made and only tests, homework and a few work sheets that were handed in to the teacher during these years were included in the data collection. These documents where copied and returned to the students.

In this classroom the normal procedure was for students to work by themselves doing exercises in their textbook during the main part of the lesson. The textbook included a section with correct solutions at the back (facit) and a student would commonly do five or six exercises, then check her results and correct those that were incorrect. Since students in Sweden always use pencils rather than ball point pens all incorrect solutions will be rubbed out and no trace of an incorrect solution strategy would generally be visible. Therefore it seemed unnecessary to collect work books.

Test results and homework included were those where number sense issues were prominent and particularly when negative numbers appeared. Tests in other content areas such as geometry and statistics were not included. The students received their first graded school reports in December of grade 8[29]. The marks used were: IG = not pass, G = pass, VG = pass with distinction, and MVG = pass with special distinction. These reports were collected from December and June of grade 8 and December of grade 9. In April of grade 9 the students participated in a standardized national test in mathematics, which showed that the marks given by the teacher corresponded well with national standards.

4.4 Study two: a video study

Closely connected to, and in time intertwined with the interview study, was a case study focusing on the teaching and teacher-students interactions during the lessons when negative numbers was the content matter taught. Since the focus of attention was slightly different from that in study 1 it is here treated as a separate study although for the participants it appeared as a connected whole.

Video recorded lessons

The aim for this part of the project was to study the teaching of negative numbers and students' direct responses to the instruction given. Using video recordings opens up many opportunities for fine grained analysis of what goes on in a classroom. The opportunity of seeing an episode over and over again makes other things accessible than those noted in observation protocols, things that would have passed unnoticed in the current flow of actions, gestures and facial expressions as well as detailed wordings of utterances in interactions. The particular interest for this collection of data was:

[29] Autumn semester ends in December, spring semester ends in June.

- ~ What the teacher offered the student in her teaching on negative numbers
- ~ What the students asked questions about and how the teacher answered these questions
- ~ How the students expressed themselves when they were learning about negative numbers

One camera was placed in a stationary spot in the classroom but always followed and focused the teacher and whoever she was interacting with. The teacher carried a wireless microphone. Several trial videos were made with different approaches such as leaving the camera steady on a few focus students or setting the camera to capture the whole classroom and moving around a table microphone. Since most of the lesson time in this mathematics classroom was spent doing individual work, not much data concerning the learning of mathematics was captured with such approaches. The interference of the camera and the microphone was also much greater when they were moved around during the lesson. Once a place for the camera was settled and the students knew that only those close to the teacher would be heard in the microphone they relaxed and stopped talking about the taping. During these video recorded lessons the researcher stayed behind the camera and did not participate as she did during the observed lessons in study 1. However, observation protocols were kept as a backup to the video recordings.

Trial taping was done on two occasions at the end of grade 7. In August of grade 8 the video recording started and a total of 12 lessons out of 14 consecutive mathematics lessons were video recorded. On three occasions there was a substitute teacher. Two of these occasions were not video recorded due to the lack of opportunity to ask the substitute teacher for informed consent. Because the researcher was more familiar with the students than the substitute teacher the students looked to her for authority, which undermined her role as a researcher on these occasions, making the lessons inappropriate for analysis even if they had been recorded. The lessons captured on video treated different contents under the text book topic "More about numbers". The different topics and activities included in the lessons are shown in table 4:1, where it is clear that seven of the videos contain work with negative numbers: tapes 8.4; 8.5; 8.6; 8.7; 8:8; 8.11 and 8.13. These seven tapes are used in the analysis.

TABLE 4.1: Video recorded lessons

Date	Data	Content / Activity
26/8	Tape 8.1	Introduction to the new chapter about numbers
29/8	Tape 8.3	Decimal numbers
4/9	**Tape 8.4**	**Introduction to negative numbers: dice game**
5/9	**Tape 8.5**	**Negative numbers addition and subtraction**
9/9	**Tape 8.6**	**Negative numbers addition and subtraction**
11/9	**Tape 8.7**	**Negative numbers, multiplication**
12/9	observations	Individual work and home work follow up. (substitute)
16/9	**Tape 8.8**	**Powers**
18/9	Tape 8.9	Decimal numbers and fractions
19/9	Tape 8.10	Diagnostic test and homework follow up
23/9	**Tape 8.11**	**Group work constructing problems** (substitute)
25/9	observations	Concluding and handing in group work (substitute)
30/9	Tape 8.12	Students show group work at the board: decimals
7/10	**Tape 8.13**	**Students show group work at the board: negative numbers**

4.5 Methods of analysis

The social constructivist framework applied in this thesis and described in chapter 2 requires a method of analysis that attempts to account for students' mathematical learning as it occurs in a social context. Cobb and Whitenack (1996) developed such an approach for analysis of longitudinal case studies including class room video recordings and interview data. Their approach was based on a method of constant comparison described by Glaser and Strauss (1967, in Cobb et al., 2001; Cobb & Whitenack, 1996). Glaser and Strauss argue that the development of theoretical constructs should occur simultaneously with data collection and analysis and the theoretical constructs that emerge in a constant comparison process are empirically grounded. The method developed by Cobb and Whitenack is suitable for analysing students' construction of particular mathematical conceptions and involves assumptions about the relationship between individual psychological processes and classroom social processes. Their method of analysis goes through three phases where the first is an episode-by-episode chronological analysis by four themes: i) children's expectations and obligations, ii) mathematical meanings the children gave to their own activity and the task at hand, iii) learning opportunities, and iv) conceptual reorganizations of each child. The second and third phase involves a meta-analysis to develop a final chronology over each individual child's mathematical activity and learning (Cobb & Whitenack, 1996).

Since this project consisted of data well suited for the kind of analysis developed by Cobb and Whitenack their method was taken as a point of departure for the development of the analytical method used. The analytic process was also inspired by a procedure described by Hycner (1985, in Cohen et al., 2000), where emphasis is put on delineating units of meaning and identifying general and unique themes. When the empirical interview and classroom data was analysed it was done by looking at entities described by the terms *interpretation*, *reasoning* and *conception* as defined below:

- ~ *Interpretation*: an individual's expressed meanings and ideas associated with the interpreted concept or representation
- ~ *Instances of reasoning*: chunks of utterances (oral, written or through gestures) that appear to occur consciously towards a particular goal, e.g. to make a decision, produce an argument or solve a problem
- ~ *Conception*: the whole cluster of internal representations and associations evoked by a concept

Questions that were asked of the data were for instance: How does the student interpret a mathematical symbol or task? What is the character of the student's reasoning? and What seems to be the nature of the student's conception?

Analytic process

Data from both studies were analyzed in similar ways, although the video recordings were more selectively transcribed than the interviews. The process of analyzing the interviews started with three quite distinct steps:

1. The interviews were transcribed verbatim and the transcripts were used as data for analysis.
2. For each student excerpts within the seven themes that were built into the interview questions were collected and ordered chronologically as a way of condensing the material and making patterns and changes over time visible. The seven themes were: *what a number is*, *size of numbers*, *subtraction*, *addition*, *the minus sign*, *brackets and rules of priority*, *the number line*. Furthermore, the theme *use of metaphor* was added. All excerpts that seemed to express some specific interpretation, way of reasoning or way of acting mathematically were included in this initial collection, giving a broad picture of students' conceptions.
3. A more detailed analysis of the specific themes followed, focussing on number sense aspects of the themes that differed in one of four ways: i) among the students, ii) between different interviews with the same student, iii) between what the student expressed and a mathematically correct idea or iv) between what the students said and what the researcher had expected.

At this stage the analytical process came to evolve in a heuristic and iterative way, what Kvale (1997) refers to as "creating meaning ad hoc" (p 184). Whenever an aspect of negative numbers seemed to show up an interesting pattern of change or diversity of interpretations, the data was again looked through with that particular aspect in focus. Students' utterances were examined in detail, looking for variations in use of words or routines. All analysis at this stage was done in Swedish so that significant variations would not be lost. A translation of the transcripts for the sake of reporting at conferences, seminars and in this thesis was done after the analysis. Revision of the analytical results always meant returning to the original Swedish transcript and when necessary also to the audio recording.

The process of analysing the videos differed in some ways. Only interactions concerning the relevant mathematical themes were transcribed, and very little of the non verbal actions in the classroom was included in the transcripts. However, the analytic process was mainly carried out on the basis of the videos themselves rather than on the transcripts. Instead of collecting transcribed excerpts thematically, small sections of the videos were brought together using the software Transana. Transcripts were mainly used as a means of clarifying what was actually said and later as a means of reporting the findings.

Transcripts

A transcript is a translation from oral to written media with two different sets of rules, the former being interpersonal and the latter decontextualized (Cohen et al., 2000, p 281). Transcribing audio recorded data is therefore the first step of data selection and interpretation. Even more so with video recorded data where most of what is visible on the video never appear in the written text (ibid, p 282). Initially the interviews were transcribed in great detail including all kinds of data such as tone of voice, length of pauses etc, but as the analytical process evolved the use of words became the dominant feature. Discourse content was given priority over discourse structure, which, as pointed out by Bucholtz (2007), influenced the transcribed representation. The resulting transcripts are reliable as representations of discourse content which is the requirement for the analysis carried out here, but less reliable as representations of discourse structure. The wording was carefully transcribed to be faithful to the original data. Repeated and interrupted words and colloquial expressions were retained but spelling was done in a literate rather than spoken approach to enhance readability. The transcription orthography used in both studies includes the following symbols:

…	A short pause or silence
(…)	A longer pause or silence, usually more than one second
//	Overlapping speech
[…]	Removed words or utterances
[word]	Written words or symbols that the conversation is about
(word)	Clarification or description of an action
word	Word uttered with emphasis
T	Identity of the speaker is indicated in the margin: I is interviewer, T is teacher
8-128	Serial number: for interviews the first digit in the serial number indicates the grade of the interview (6, 7 or 8), for videos there is only a serial number

Translating a transcript, as in this case from Swedish to English, involves many difficulties. Bucholtz (2007) points out that a colloquial or a formal translation style imply different attitudes toward the speaker. The participants in this study used many colloquial expressions, unfinished sentences and words with implicit meanings known only in the local culture. These, along with discourse particles such as "ju" or "typ" are very difficult to translate (cf. Spenader, 2004). In this research project the translations aimed at being as verbatim as possible without becoming incomprehensible. It was also seen as important to convey an impression of an authentic classroom situation which sometimes involved adjustments of the spoken utterances. For comments on translation of specific mathematical terminology see chapter 5.3. One great advantage with translating transcripts was that the translation process itself implied a renewed deep insight into the interaction, and often involved a new comparison with the original recordings.

4.6 Methodological issues

Every research method entails methodological problems concerning issues such as ethics, validity, reliability and relevance. This section describes how these issues were considered in the research project.

Ethics

This project followed the ethical principles of the Swedish Council of Scientific Research. Both the participants and their parents were informed about the project and the long time presence of the researcher. Information was given in written form as well as face-to-face at a parent's group meeting. All parents were positive and gave full consent to all parts of the study. Half way through the project the parents again met the researcher and had opportunity to ask about the project. The students themselves were informed as a group and also individually in short introductory interviews. All students were willing to participate and accepted the researcher's presence in the classroom. The record of data collection has been reported to the University of Gothenburg. Every student was initially given a false name which was used throughout the data collection. The false names chosen followed the gender of the student but were otherwise assigned without any system.

Deciding how much information to reveal concerning the aim of the study without influencing the result and still be trustworthy turned out to be a delicate matter. The students were told that what was interesting was their mathematical thinking and their mathematical reasoning, and that the aim of the project was to find ways of improving mathematics teaching based on better knowledge of student understanding. Nothing was said about the specific focus on negative numbers, and during the first two years prior to when the topic was treated in class the researcher never explicitly asked about negative numbers unless the student brought up the topic first. The students were also told that for the sake of research, "correct" answers were not important. One reason being that their difficulties and mistakes were of great research interest, and the other being that sometimes different answers can be correct. To illustrate this statement the students were told to add 11 and 3. One answer is 14, but a different answer is 2, correct if you speak about time (so called clock arithmetic, or counting modulo 12). This was a strange way for these students to talk about mathematics, they were much influenced by the norm that each task always had one and only one correct answer. This issue was brought up solely in connection to the interviews.

Many of the students found the video recordings to be the most interesting part of the project. As a means of feedback every student was given the opportunity to watch episodes where they were in focus and comment on them during the last interview. This stimulated recall session in parts also served as respondent validation.

Researchers role

According to Kvale (1997) the researcher's relation to the participants in a study is an important ethical issue. In this study the relation between the researcher and the teacher could be described as friend and colleague with a mutual understanding of the situation as one in which both could learn from the other. The teacher was informed about the detailed aim of the study but was asked to keep teaching as she would have done if the researcher had not been there. The teacher did not always know when the researcher was to participate and did not, according to what she said, adjust the lesson in any way as a result of the researcher's presence. Although the researcher appeared on the scene as an expert concerning mathematics education research, the expertise of the teacher to teach her own students was never questioned. As a result of the long term cooperation, a mutual understanding and trust developed where the researcher never judged or expressed opinions about the teacher's work. At times mathematical or didactical discussions occurred and the researcher balanced smoothly on the edge between sharing her knowledge and not expressing opinions about what the teacher ought to be doing or saying.

Concerning interview results no data was shared with the teacher during the course of the project due to the confidentiality principle. When all interviews were completed the teacher and the researcher had a feedback session watching some parts of the video recorded lessons where the teacher was actively teaching, discussing what happened during these lessons and how that might be related to the students' test results. At the termination of the project, which was after the students had finished grade 9 and left the school to go on to upper secondary school, the researcher and the teacher had a respondent validation session where the teacher read through the manuscript of the result part of this thesis and made comments. Her overall impression was that it gave a fair picture of the mathematics classroom observed, but that the narrow focus naturally did not give full justice to the complexity of classroom activities and teacher responsibilities. Research always implies a narrowing down and simplification of a complex reality. The teacher also pointed out that she herself had learned a lot during the cause of the project and by reading the manuscript. She acknowledged that as a teacher it is always possible to learn more and improve your teaching. Any teacher, when continuously observed, will find that s/he says and does things of which s/he is unaware. Throughout the study the teacher was respected as a professional, acting within the frames created by the current curriculum, the social norms of the school, the classroom, the expectations of her students and their parents, the restrictions of the chosen textbook, and so forth.

The researcher's relation to the students was one of mutual respect but asymmetrical since the researcher was assisting the teacher. The students were fully informed about the researcher's role as a researcher and the reasons for her participation, but also about her competence as a mathematics teacher. It was made clear that all important decisions about what and when to study and about the discipline and social order of the classroom were made by the teacher. The researcher made a point of not getting involved in questions of authority and not becoming a link between the teacher and the students. It was also made clear that none of the conversations students had with the researcher would influence school reports. The researcher always tried to answer questions from the students either by prompting them to find an explanation themselves or by referring to what the teacher or the textbook said. At times when the researcher would have preferred a different explanation she chose to pass the question on to the teacher in order not to give different, and perhaps confusing, input.

The research project was not intended to be experimental and the researcher's role was not to introduce specially designed learning opportunities. However, the presence of a participant observer as well as recurrent interviews obviously influenced the educational setting. To minimize the interview effect the interview questions were never discussed outside the interview and the researcher was careful not to use the interview as a teaching situation. A few times students would say they remembered an interview question from the previous year, and the question itself might have initiated a development of the student's conception, but just as often students would show surprise when told during the last interview that the questions had been the same each year. A structured interview protocol helped to insure that learning opportunities were the same for all students in the study. The researcher never told students what a "correct" answer to an interview question was. On the contrary, all answers were judged as interesting and correct if they could be explained. During the interviews the researcher encouraged students to explain more and gave positive feedback through phrases such as: "Well explained", "You seem certain", "That was quick", etc.

Validity and relevance

The interviews were structured in order to make it possible to compare answers over the years. Posing the same questions on different occasions in order to study change increases the validity of the data since the interview group is the same and therefore serve as their own control, according to Cohen et al. (2000, pp 108-109). They also claim that being regularly present in the mathematics classroom as a participant observer increases the internal validity by reducing observer and interviewer effects. Staying long enough as a participant observer creates a situation where the researcher's presence and questions are taken for granted. Regular observation will also make it possible to give a rich description of the context in order to increase transferability. The long term design of this research project complies well with these conditions since the researcher came to be a natural part of the mathematics classroom practice.

The interpretation of empirical data in this project was theoretically driven in that the research questions directed what has been observed. To increase validity, a theoretical triangulation was done where students' sense making of negative numbers were investigated both through their use of conceptual metaphors and through their development of number sense. However, claims made about students' reasoning and students' conceptions are to be seen and validated as interesting examples, not as objective truths about the students. It is important to point out that one interpretation of a concept or representation never excludes another interpretation. It is often acknowledged that one person can embrace several interpretations, in fact a full understanding of mathematical concepts often implies several qualitatively different ways of experiencing one phenomenon (cf. Bentley, 2008; Ekeblad, 1996; Gelman & Gallistel, 1978; Marton & Booth, 1997; Neuman, 1987). Only interpretations that are expressed can be analysed, not others that may also be there. A student's conception is always richer and more diversified than what is expressed in the data, no matter how extensive a data collection is. This means that any claim about a student's interpretations or a student's conceptions must be seen as instances of what that student expressed on that occasion, not as a true picture of the student's mind.

Halldén et al. (2007) point out that when a student, in a classroom situation or an interview, gives a wrong answer to a question it often means that s/he is answering a different question. Intentional analysis involves searching for the interviewees' way of perceiving the world and way of reasoning within their own frames of reference, and finding the logic in their ways of making sense of what they are experiencing. It is essential to find out what the interviewees are actually talking about, rather than what we think they should be talking about. They write:

> It is very difficult, if not almost impossible, to ascertain what knowledge an individual *does not* possess: a seeming lack of knowledge can always be an artefact of the kind of questions we are asking or of the way we are arranging the setting of a test. However, what we can do, is to describe the knowledge an individual utilises when trying to solve a problem or handle a situation. (Halldén et al., 2007)

Although the interviews were structured in predetermined themes and followed a set protocol, there were never any objectively right or wrong answers to the questions. Responses were categorised as mathematically incorrect or mathematically correct within different number domains. The interviewees were always assumed to answer seriously and in accordance to a logic that in some way made sense to them. At times that could include answering what the student thought the interviewer was expecting rather than what s/he thought made best sense mathematically. This aspect was also taken into consideration during the analysis.

Although the researcher worked alone, different parts of the data and the analysis were discussed with colleagues to validate the researcher's interpretations. To increase reliability the analysis was kept very close to the data and many excerpts from the empirical data are included in the result part of the thesis.

The analytic approach in this thesis is what Kilpatrick (1993) calls a systematic approach with an interpretivist view, an approach with "the purpose to capture and share the understanding that teachers and pupils have of their educational encounter" (ibid, p 16). Such an approach has strong authenticity. Freudenthal (1991) brought up the question of relevance for mathematics education research in his China lectures by combining the two questions *What is the use of it?* and *For whom?* It is the aim of this research to be relevant for other researchers as an in depth study of a theoretical concept, but also relevant for teachers as a tool through which they can reflect on their own situation and develop their teaching. Case study data carefully collected and rigorously described may shed light on patterns of regularity recognised by teachers.

In an article about relating classroom teaching to student learning Nuthall (2004) claims that the most significant knowledge needed to improve the quality of teaching is knowledge about the ways in which classroom activities, including teaching, affect the changes taking place in the minds of students. Among criteria for research on the teaching-learning relationship Nuthall emphasises complete, continuous data on individual student experiences, classroom activities and learning processes, and analysis based on connections between these. These criteria have guided the research design and data analysis of this research project with the hope of contributing to bridging the gap between research and practice.

CHAPTER 5

Classroom Practices and Sociomathematical Norms

The results part of this thesis begins with a background chapter setting the scene for the coming video and interview based analysis of classroom discourse and student's development of numbers sense. The results presented in this chapter are based on observation data collected over a period of three years. In compliance with the social constructivist framework described in chapter 2 learning is viewed from two perspectives, social and psychological, that co-evolve and depend on each other. Therefore, to study learning implies studying not only the classroom discourse and individual students' development, but also the social setting wherein it takes place. Sociomathematical norms and classroom mathematical practices (Cobb et al., 2001) will in this chapter be described with the aim of creating an understanding of the background for the activities and interactions described and analysed in chapter 6 and 7.

5.1 Mathematical classroom practices

In general four different types of activities occurred in this mathematics classroom;

1. Whole class instruction: generally this is done at the beginning of a lesson and is called a *go-through*[30]. It can include instruction on new content as well as recapitulations of previous lessons.
2. Individual desk work: during this activity the students work individually with exercises either in their textbooks or on separate worksheets, and the teacher generally walks around the classroom helping those who want help or need encouragement, monitoring or support. Sometimes the teacher might choose to sit at the front doing correction work. On such occasions students who need help come to the teacher. Individual work is to be done quietly but as they sit in pairs it is accepted and encouraged to talk quietly within the pair to help each other. Although a few students utilize this possibility, as a whole it is not done much since students work at their own pace in the book and therefore seldom work on the same exercise. On the contrary, it is popular to have earplugs with music playing during desk work. When a task is done the student usually checks the answer section at the back of the book, and if the answer is incorrect

[30] In Swedish: *genomgång*. In Swedish school mathematics practices this term is commonly used, utilized by teachers as well as students and parents and an expected part of the lesson.

or s/he does not understand it s/he will call the teacher for help. The answer section is also used by some students when they do not know how to work out the answer; they look first at the answer and then try to work "backwards" to see how they are supposed to work it out.

3. Other mathematical activities: this includes handing back and going through a mathematics test, playing mathematical games, and correcting each others homework.
4. Non mathematical activities: this includes all kinds of information about practical things, exams, homework, leave of absence, other subjects or school activities, changes of placing in the classroom etc.

Counting from the first classroom observation on February 21, 2007 until the last on October 14, 2009, 61 lessons were observed. Out of these 61 lessons; 52 were quite similar in their structure; two had no time slots recorded; five were spent on group work; one was spent on a geometrical hands-on activity and on the last observation the class was split into two groups who did quite different things. The 52 similar lessons could be described as "typical" and therefore give a fair picture of the classroom practices. A quantitative analysis of the time spent on different types of activities during the 52 typical lessons is shown as mean values in table 5.1. As seen in the table the bulk part of each lesson was spent on individual desk work.

TABLE 5.1: Mean values of time spent on different activities in the observed class on 52 observed lesson.

Total time of lesson	Whole class instruction	Individual desk work	Other mathematical activities	Non mathematical activities
57 min	9 min	37 min	5 min	6 min

The students are always placed in pairs, mostly by the teacher (from now on called T). Usually T will place students who need a lot of help at the front. Working together is encouraged in the pairs but only if it can be done quietly. The mathematics textbook structures and dominates the content of the mathematics lessons. The book includes introductory sections for each topic, rules, examples and a variety of tasks on different ability levels, as well as a section with homework tasks, a 'toolbox' containing the most important facts and rules related to the concepts brought up in the book, and finally an answer section[31] at the very back.

Students are held responsible for doing, checking and correcting their work themselves. Each week in grade 6, one homework assignment is handed in and corrected by T. In grade 7 and 8 a homework assignment is handed in but often

[31] In Swedish: *facit*.

distributed and corrected by a classmate. In grade 9 the responsibility for doing and correcting homework is handed over to the individual student. Two or three times every term, students' knowledge and competence is assessed through a written test. Grades given each term in grade 8 and 9 are based on the results of these tests as well as on the students' performance during lessons and on their homework.

5.2 Sociomathematical norms

Sociomathematical norms describe regularities in classroom activities related to mathematical issues. It is important to point out that all norms are jointly established, dynamic, and specific to a certain combination of students and teacher, i.e. a certain classroom practice. Often norms are invisible and implicit until they are changed or violated. Norms can only be noticed if observations occur over time, in this case observations were made over a period of three years.

Data exposed four dominant sociomathematical norms in this class. Firstly, the mathematical activities and mathematical narratives are *mainly procedural* and the most common type of question asked by the students begins with the words: "How do I do this[32]…?" Secondly, mathematical problems are expected to have *one correct answer* and not to be ambiguous. Thirdly, the *teacher is given authority over mathematical truths* and ways of thinking. Fellow students and the book can be questioned, but not the teacher. Fourthly, a general belief is that when you do not understand you do *more of the same.* These four sociomathematical norms are exemplified and explained below, followed by a discussion of the character of interactions in the mathematics classroom.

Focus on procedures

The teacher quite frequently writes procedural rules on the whiteboard during the go-through that the students are asked to copy into their notebook. Here are some typical examples of such rules that illustrate the focus on procedures:

> [081209] Rules for simplifying expressions with brackets
> If there is a minus sign in front of ( ) the sign inside ( ) is changed
> + becomes –
> – becomes +
> If there is a number directly in front of ( ) multiply it with all the terms inside ( )
>
> [081017] (working with area units)
> Rule: For each 'ordinary' step you move two steps when you have area units.
> Example: 1,7 m2 = 17000 cm2 ;
> 4 steps because there are two ordinary steps from metre to centimetre.

[32] Hur ska jag göra…? Hur gör man…?

A different example is given in a dialogue between Olle and T about a mathematical pattern displayed in the textbook as an introduction to opposite numbers. The pattern is shown in table 5.2 and the interaction in excerpt 5.1. T is focussing her attention on the procedures needed to answer the questions rather than on what mathematical insight the table might have offered.

TABLE 5.2: Illustration from the textbook of addition as subtraction of the opposite and vice versa. (Carlsson et al., 2002, p 18, original in Swedish).

[opposite numbers]	
2 + 3 = 5	2 - 3 = (-1)
2 + 2 = 4	2 - 2 = 0
2 + 1 = 3	2 - 1 = 1
2 + 0 = 2	2 - 0 = 2
2 + (-1) = 1	2 - (-1) = 3
2 + (-2) = 0	2 - (-2) = 4
2 + (-3) = (-1)	2 - (-3) = 5
2 + (-3) = 2 - 3 = (-1)	**2- (-3) = 2 + 3 = 5**
Adding a negative number is the same as subtracting the opposite number	Subtracting a negative number is the same as adding the the opposite number.

EXCERPT 5.1: Video 8.6.Olle 1. Olle and T discuss the number pattern in table 5.2.

Olle	How about this, what is this?
T	These things here? Below?
Olle	Yes
T	well you could ask that…they put…if they put…probably they want to…then…what they want to show is…you could ask that. But it was a bit unclear I think. but they possibly, I think they want to show this which they have written underneath, that 2 + -2 equals 0, and that is 2 - 2 it's also equal to 0. Different signs give minus.
Olle	mhm
T	and here, over here in this corner we have 2 - (-3) makes 5 and that is the same as 2 + 3 is 5. So two of the same signs give plus but, it was not very pedagogical, I think
Olle	mm?
T	hm, but you have learned the rule so you know it anyway. So this strange, well…but you understood this? (points at the heading "*opposite numbers*")
Olle	mm
T	but that is all you need on this one. And then these, you already know the rule. You can use the rule.

From the students' point of view the norm is constituted by the way they ask question when they need help. The dominant type of question concerns procedures. Students say for instance "What am I supposed to do here?" and "How do I work that out?"

One correct answer

The formulation of a problem should be clear so that the student would know what to do to in order to get the correct answer given in the answer section. On several occasions T articulates a dislike about some tasks in the book because of their ambiguity:

> [070221] Task 33 in the book shows a map of Tanzania, and the question asked is: *About how far in Swedish miles is the distance between the north and south borders?* (Carlsson, Liljegren, & Picetti, 2004b, p 43)
>
> T: This task is not good, it can be interpreted in different ways, it is not evident which distance to measure… It ought to be clear which distance the answer section has intended.

On some mathematical topics, like working with percentages, T brings up different ways of solving a task encouraging different strategies. Different solutions are accepted, but whatever strategy is used, there is always one correct answer to the task. Tasks needing some interpretation or additional conditions in order to be solvable are considered bad problems because they cause frustration and confusion. T explains that students want to feel they know what to do and that they can get the answer that is in the answer section of the book. For individual student work the answer section plays a very important role.

This norm was violated by the researcher when the aim of the interviews was described at the beginning of the research period. When the researcher said that "depending on how you think, tasks can have different answers that are correct" the students reacted strongly, that was not their view of mathematics. To show her point the researcher asked what 11 plus 3 is. All students said that it is 14, but the researcher claimed that if you think about time a correct answer is also 2. This episode was related to on a few occasions during the interviews; when the interviewer asked if there was another way of thinking or if something could have a different meaning some students answered "perhaps the time". Quite often when an interview was terminated the students wanted to know if their answers were correct.

Teacher's authority

For the students there is no question on T's authority over mathematical truths, correct procedures and correct answers. Whenever a student gets stuck, s/he asks T and trusts that T will explain how to think and what to do. Students do not question what T says concerning mathematics. Excerpt 5.2 illustrates this.

EXCERPT 5.2: [090123] Ove is doing a worksheet about fractions. He starts with the first task and writes: 1/8 + 1/4 = 2/6. T sees what he is doing and points out that he has to have a common denominator.

Ove	what do you mean, common denominator?
T	they are different types. You cannot combine fourths and eighths to get sixths.
Ove	Why not?
T	Look here at Elke. She has solved the task by expanding one fourth to make it into eighths.

Ove then crosses over his work and copies what Elke has written. T moves away. A while later the researcher asks Ove how he got his first answer: 2/6.

Ove	I pulled these together [points at 8 and 4] and that gave 6, in between. But that wasn't allowed.
Researcher	do you understand why it wasn't allowed?
Ove	no but T said so.

In this case Ove did not question T or even make an effort to understand what T said, because by telling him to look at Elke T had shown him the right way of doing it. Although T expresses that her aim is to teach for understanding, she often refers to students who do not understand a specific idea with the words: "he hasn't bought it yet". Understanding is seen as the same as accepting the explanation given.

When metaphors are used in the teaching about negative numbers T has authority also over what metaphor to use. Explanations involving metaphorical reasoning all come from T, often as suggestions of ways of thinking presented with the words "think like this". Students rarely share their own metaphors and are seldom asked to do so. The preferences of T become clear in comments such as: "it's tiresome to draw number lines all the time, it's better to think mathematically. Think like this…"

More of the same

The general view of mathematics is that it is a set of truths, facts and fixed procedures that each student has to get into his head. If it isn't there yet, then s/he needs more practice, and has to accept it as it is, without questioning it. The following quote is an example of how this is expressed. [081017]

> T: now you practice this on page 49 in the book so that you get it into you head.

The layout of the textbook also emphasizes the idea of repetition. First everybody does the green section of a chapter, followed by a diagnostic test. After the test, a student who made many mistakes goes on to do the blue section which usually presents the same things in the same way all over again. There is also a more advanced red section which is optional, and which presents some new ideas and other types of problems connected to the mathematical ideas of the first section. Advanced students do the red section instead of the blue section.

T is very ambitious concerning tests and feedback. After every test each student is given a matrix where every mathematical content or skill that was assessed in the test is clarified at different levels of ability and different colours mark what each individual student had shown knowledge of or not. The purpose of this feedback is to let the student know what areas in the textbook he needs to go back to and do again, and thus handing over the responsibility for repetition to the student.

Interactions

Another classroom norm that is less related to mathematical issues but nonetheless worth mentioning is the character of interactions in the classroom. During whole class instructions, most of the interaction between teacher and students follow an IRE-pattern (Initiation – Response – Evaluation) (Mehan, 1979). Almost all questions asked have known responses of a type that is either right or wrong. Here is a typical excerpt:

[070226] 350 g = 0,350 kg (T is teacher, Stud is student)

T	What does kilo mean?
Stud	thousand
T	How many zeros?
Stud	three
T	so, that's why we move tree steps.

The evaluation part of the IRE-sequence is conveyed through small gestures and facial expressions, and by the fact that T continues with a new question. An incorrect response would be followed by the same question being repeated.

Since the major part of the lesson is spent on desk work most of the interaction in the classroom is face-to-face interaction between T and a student. On those occasions the conversation is more diverse. Usually the student asks and T answers and explains. T tries to get the student to be very specific about what the problem is and what it is that s/he does not understand, but rarely does T ask the student to explain his solution or elaborate on his reasoning. "Explain what it is you don't understand" is a more common question from T than "explain to me why you think it is like that". Students express in the interviews that they expect an explanation from T; they are not accustomed to justify their own narratives. T makes a point of finding out what the problem is, but having found that out she starts explaining to the students what to do and how to think. Whilst explaining, T guides the student through a solution using an IRE-pattern, in a manner Swedish educators refer to as piloting[33] (Löwing, 2004; Löwing & Kilborn, 2002). The outcomes of a piloting dialogue is that by answering very distinct isolated questions, the student is guided through the solution to reach a correct answer even if s/he only understands each separate step and not the

[33] In Swedish: *lotsning*

general structure of the problem. This type of interaction is illustrated in excerpt 5.3; a dialogue between T, Elke, and Martina (M) who is sitting next to Elke:

EXCERPT 5.3: Video 8.6 time: 27:12

> Elke has raised her hand to ask about task 81: (-18)-(-6). She has solved it and found that her solution was not the same as the answer given in the answer section of the book. She wanted 24, or perhaps -24, which is the result of adding 18 and 6 after simplifying the expression (-18)-(-6) to -18+6. The answer section says -12, but Elke does not pay attention to the minus sign, only the magnitude.

Elke	hey look, now they say here that you should add, since there are two of the same. But the answer is 12?
M	no the answer is minus 12
T	it is minus 12 because
Elke	yes yes there is a little minus sign there but, that's not what you get if you add.
T	no, but these (T circles the two minus signs between the numerals) you simplify into a plus sign. These two minus signs can be simplified into a plus sign.
Elke	yes but then it doesn't make 12.
T	no
M	it makes minus 12
Elke	but it said so in facit
T	did it say plus 12 in facit?
Elke	well, minus
T	yes but that's what it is.
M	minus 12
T	because now they are different. That one is minus 18
Elke	yes that's right, you count upwards, you don't count… but hang on…
T	you know they are sort of different
Elke	yes
T	one is positive and the other is negative
Elke	ye…esss… okay. (writes -12)
T	ok

> Elke now has the correct answer in her book and T walks off. Elke does not look convinced.

In this excerpt T is directed toward traditional telling of new content and correct procedures, rather than eliciting, i.e. drawing out students' mathematical ideas to share, discuss, justify, reflect upon and refine (Lobato, Clarke, & Ellis, 2005). When combined, the two norms *focus on procedures* and *one correct answer* indicate a corresponding psychological perspective concerning mathematical beliefs and values (Cobb et al., 2001, see also chapter 2.3). In this classroom, the underlying view on mathematics is that mathematics is about learning a set of procedures that are utilised to get correct and predetermined answers when solving mathematical problems, what Skemp (1976) would call an *instrumental* understanding of mathematics. School mathematics is about doing exercises in a book and getting correct answers. There is no hint of mathematics as a creative process and a science of patterns in the prevalent norms. The main questions for this study are concerned with mathematical interpretations and reasoning, by Cobb described as the bottom level of the psychological perspective. Mathematical beliefs and values could be studied in more detail, how norms are constituted, sustained, challenged and as a result how they change. However, in this study the sociomathematical norms serve as a background, against which it will be possible to understand mathematical interpretations and reasoning.

5.3 Remarks on translation and mathematical conventions

Transcripts from recordings in this study are translated as closely to the original wording as possible. For a reader not used to the language of a Swedish mathematics classroom, some things may seem strange. Mathematical symbolization is to a large extent a matter of convention, but although the language of mathematics is worldwide, some conventions differ from country to country. In a metaphor the choice of words is significant, and when the corresponding English word has a different metaphorical underpinning to the original Swedish, it is specifically pointed out in the analyses. Some particular Swedish conventions and common phrases that appear in the data are here listed to facilitate the reading of the analyses. A few interesting aspects of the concepts studied have shown up in the differences between Swedish and English school mathematical discourse, and have become visible as a result of the analyses and the fact that it is reported in English instead of Swedish. Without the English translations these things would have gone unnoticed. Table 5.3 summarizes the differences.

TABLE 5.3: An overview of some Swedish words found in the data, their English translation in the transcript and their corresponding English words.

Swedish original	**Translation**	**English correspondent**
positiva och negativa tal	positive and negative numbers	integers / signed numbers
plustal och minustal	plus-numbers and minus-numbers	positive and negative numbers
minustre	minus3	negative 3
att plussa, att lägga ihop	to plus, to put together	to add
att minusa, att ta bort	to minus, to take away	to subtract
blir	makes, comes to	is equal to
är	is	is equal to
större (tal)	bigger (number)	greater, larger (number)
mindre (tal)	smaller (number)	smaller (number)
större än, mindre än	bigger than, smaller than	greater than, less than
räkna ut	calculate, work it out	calculate, compute
ställa upp	write it up	use a vertical algorithm
decimalkomma; 3,8	decimal point; 3,8	decimal point; 3.8

The Swedish term for *integer* is *whole number*[34] which is rarely used in school mathematics, perhaps because the term itself emphasizes the difference between wholes and parts more than the difference between unsigned and signed numbers. The term does not appear in the empirical data reported here. There is no Swedish term for the word *signed number* or *directed number*, meaning the set of all positive and negative numbers. This set of numbers is referred to as: *the positive and negative numbers.*

The terms *positive* and *negative number* are used in textbooks. Students and teachers also talk about these numbers as *plus-numbers* and *minus-numbers*, and in the data reported here that is the common terminology. 5 + (-3) = reads: "*fem plus 'minustre' blir…*" which literally means: "*five plus 'minusthree' becomes...*" In the translated transcripts (-3) is written as *minus3* when it is pronounced as one word emphasising that it is a negative number, and as *minus 3* when it is pronounced as two words, which could mean either subtract three or negative three.

A negative number is usually written inside brackets, e.g. 5 - (-8), except when it appears as a first term. The minus sign indicating negativity looks the same as the subtraction sign. A positive number is generally written without a plus sign.

To *add* is commonly referred to as *to plus* or *to put together.* Contrary to English, where the mathematical term *to add* in frequently used in everyday language, in Swedish the term *att addera* is only used in mathematical contexts. To *subtract* is commonly referred to as *to minus* or *to take away.*

The equal sign is either referred to as *blir* (becomes), which evokes a dynamic understanding, or as the static *är lika med* (is equal to), or simply *är* (is).

Concerning the size of numbers the word *större* (bigger) and *mindre* (smaller) are commonly used in this data, but sometimes also *högre* (higher) and *lägre* (lower) appear. The sign < is called *mindre än* (smaller than) and > is *större än* (bigger than). There is no Swedish counterpart to the terms *greater than* and *less than*.

When asked to deliver an answer to a task the student is commonly asked to *räkna ut det* (calculate it, work it out). If the calculation is difficult to do mentally the students are told to *ställa upp* which literally means to *put up*, a phrase used for doing the calculation using a traditional vertical algorithm. In the translated transcripts this is labelled *writing it up*.

A decimal point is written as a decimal comma in Swedish. In the data the written symbols follow the Swedish convention with a comma, but in the translated utterances the English word point is used. The decimal number 3.8 will thereby be written as 3,8 but talked about as 3 point 8.

[34] The English term *whole number* as {N, 0} does not have a counterpart in Swedish. The Swedish term *naturliga tal* is alternately used to mean {N} and {N, 0}

CHAPTER 6

The Role of Metaphors in Classroom Discourse

In the previous chapter the classroom practice was described in terms of organization of activities and sociomathematical norms. In this chapter the classroom discourse is more closely scrutinized, particularly concerning the role of metaphors and metaphorical reasoning. The overall aim of the chapter is to study how metaphors can be understood as a means for making sense of negative numbers. After a short presentation of the findings, the chapter begins with a description in general terms of the observed process of teaching and learning negative numbers in the studied classroom. After that a detailed analysis of the metaphors used in the process is carried out, building on a method of metaphor analysis described in chapter 3, and illustrated with several instances of metaphorical reasoning. At the end of the chapter one particular metaphorical expression that seemed to create a lot of confusion among the students is focused on in order to investigate what impact a metaphor can have on the students' sense making. Empirical data collected for this part of the study consisted of classroom video recordings, interviews with the teacher, and textbook analysis. The questions put to the data were:

a) What metaphors are brought into the discourse by the teacher and the textbook?
b) What is the rationale for introducing them?
c) When and how are metaphors used by the teacher / the textbook / the students?

Questions about how students appropriate the metaphors present in the discourse, and how these influence the development of number sense concerning negative numbers will be posed in the next chapter.

Findings

Viewing the process of teaching and learning as it unfolds in the classroom activities highlights that the teacher, and to some extent the textbook, seem to have a clear teaching goal of helping students make sense of tasks through metaphorical reasoning. In most of the teacher-student interactions the teacher supplies the metaphorical expressions and does the metaphorical reasoning. Students generally say things like "plus five" and "do minus", thus speaking mainly in mathematical terms. The teacher 'translates' into metaphorical terms. Although the teacher appears to be persistently striving towards the goal of teaching for metaphorical reasoning, the final goal seems to be to make students fluent in calculating with negative numbers using the rule "same signs makes plus, different signs make minus". The debt-and-gain context is predominant, but the teacher tries changing context when a student does not understand, and also makes frequent use of the sign rule as a last resort when the metaphors seem to fail.

Three of the grounding metaphors identified by Lakoff and Núñez (2000) are found to be part of the mathematical discourse of this classroom and extended to the new number domain. These are:

- Numbers as object collections
- Numbers as locations on a path and distances (measurement) between locations
- Numbers as movements along a path

Different mappings of these metaphors in the extended number domain are not clear and explicit, and different versions appear. Conditions of use of the different metaphors are never made a topic of discussion, and no comparisons between them are explicitly made, although there are several situations with that potential. Whenever a metaphor is no longer useful the students are encouraged to apply the sign rule *minus minus make plus*. The sign rule itself is justified by referring to a situation with money that is not part of the source domain for the students except under very specific conditions ("taking away a debt is the same as earning money"). Generalizing from these specific conditions is left for the students to do themselves.

One metaphorical expression stood out as particularly ambiguous; the expression "difference between numbers". A close analysis of diverse uses of the word 'difference' shows that its metaphorical underpinnings make an impact on interpretation and sense making. The vagueness of the mathematical discourse and its underlying analogies seems to create confusion between the minus sign as a sign of operation (binary) and sign of polarity (unary) as well as between magnitude and value of negative numbers.

The chosen contexts serve two purposes. One is to justify the use of negative numbers, but this does not work very well since almost all contextualized problems are easy to solve without involving negative numbers. The second purpose is to create source domains for metaphorical reasoning. This works quite well for specific isolated tasks, but the students are given no guidance to judge when to use which metaphor and to see conditions of use of the different metaphors. Most of the metaphorical reasoning appears in the process of making sense of isolated tasks, not for making sense of mathematical properties that underlie the concept and the sign rules.

6.1 A teaching–learning process for metaphorical reasoning

In the process of viewing the data many times with a specific focus on the use of metaphors, a pattern for the rationale and sequencing of activities emerged.

Whenever a metaphor appeared in the classroom discourse it was analysed as described in chapter 3; finding source domain, target domain and the mapping in between the two domains. The empirical findings were then synthesising into an account of what happened during the lessons where negative numbers were taught, with respect to the research questions. That account is here described as *the teaching-learning process for metaphorical reasoning*.

Teaching mathematics involves many different goals. One can be referred to as *teaching for understanding*. The aim of such teaching is that students learn new mathematical concepts and procedures in a way that makes them feel that the concepts and procedures make sense and are meaningful. One way of doing this, which stands out as distinct in the empirical data, is to help the students make metaphorical mappings between well known domains and new ideas, representations and procedures, i.e. giving metaphorical meaning to mathematical concepts and procedures. There are many other goals in mathematics education as a whole, and many of the things teachers say and do have a different aim than to teach metaphorical reasoning. Although it is metaphorical reasoning that is viewed in this study, it does not follow that it is the most important or predominant goal. At times when the teacher seems to lose track of that goal it is probably because some other goal is given priority in that situation, something outside the boundaries of this research.

The *teaching-learning process for metaphorical reasoning* that was found in the data can be illustrated in five steps. The first two steps show the preparation work of the teacher and textbook author. Students are brought into the process in step three. In this section a short description and discussion of the different steps of the process is made with a few examples, and in the section 6.2 a deeper analysis of data is done in relation to each step of the process. The two main sources of influence over the instruction in the mathematics classroom, apart from the students themselves, are the teacher and the textbook. On that ground the analysis incorporates both of these sources. The process described in these five steps is not a method of instruction; it is a description of the empirical findings viewed from a certain perspective focusing on the role of metaphors.

The different steps of the process

The metaphors in mathematics can have different directions depending on which of the two domains is the source and the target. Is, for example, mathematics spoken of in terms of objects and movements, or are objects and movements spoken of in terms of mathematics? The relation between these two domains is visualized in figure 6.1. In the first process, here called the *representation process*, it is the mathematics that is mapped onto a real world context, i.e. a real world context is spoken and conceptualized in mathematical terms, whereas in the *symbolization process* it is a real world context that is mapped onto mathematics.

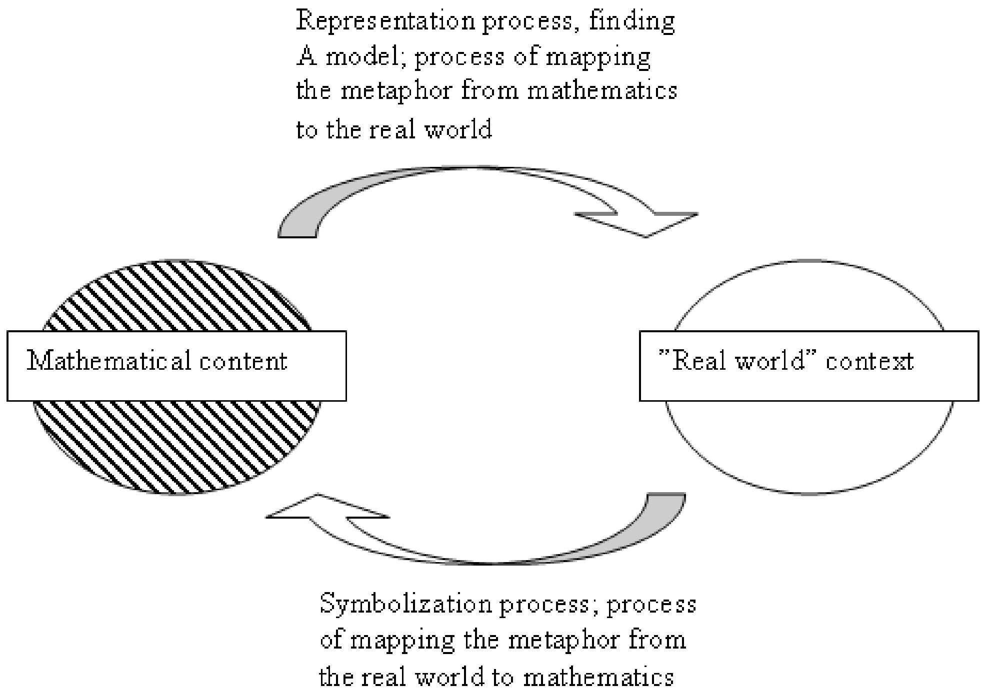

FIGURE 6.1: The relation between two domains and the different processes of mapping a metaphor.

Figure 6.1 is quite similar to the figurative representation of the two ways metaphor described in chapter 2.6 In the representation process the source domain is mathematics and the target domain is a model or a well known concrete context, whereas in the symbolization process the metaphor is reversed and the model or concrete context makes up the source domain and mathematics is the target domain (see figure 2:4 in chapter 2.6).

The five steps of the *teaching-learning process for metaphorical reasoning* identified in the data can be related to the two domains as shown in figure 6.2. The first two steps in this data only involve the teacher and the textbook author. Students are part of the process from step 3.

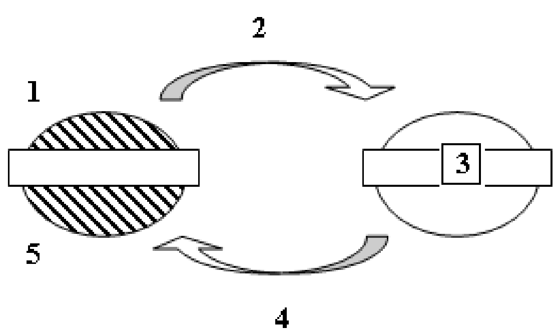

FIGURE 6.2: Five steps of the teaching-learning process for metaphorical reasoning.

Step 1

Both the teacher (T) and textbook author (TA) start in the world of mathematics. A mathematical concept or procedure that is new to the students is to be introduced. The context in which this is done and the rationale for doing it at this particular moment is a result of various circumstances and considerations, ranging from curriculum issues and textbook layout to individual student's needs. No mathematical content is totally new; it will always connect to the previously taught contents. When analysing one specific teaching sequence there are many variables influencing the choices made by T/TA that the researcher lacks insight into. One of the variables is T/TA's own understanding of the mathematical concept. What metaphors underlie the meaning T/TA herself sees in the concept, and how T/TA connects the concept to other parts of mathematics. If we assume that metaphors play an important role in the sense-making of mathematics, then T/TA is also restricted in her interpretation of the concept by the metaphors she relates to. Some ways of speaking about the concept will be consciously chosen, others will be there implicitly or intuitively.

Step 2

A real world situation, experience, visual representation or concrete material (henceforth called *contexts*[35]) is chosen to represent the intended mathematics. When T/TA makes a choice there are many things to take into consideration such as the students' previous knowledge, the metaphors frequently used, representations that are well-known to the students, the overall context the new mathematical content fits into, the structure of the textbook, etc. T might know a lot about her students and adapt her choices to that knowledge whereas TA knows nothing about the individual students that will come to read the book. On the other hand T might adapt to the structure of the textbook without much reflection once the textbook has been chosen.

When the choice of context is made the representation process starts. The metaphor at work in this process is one where *the mathematics is the source* domain and *the context is the target* domain. In the case of negative numbers T/TA will choose a context that in some way represents negative and positive numbers. Mathematical objects are mapped onto (i.e. represented by, talked about, thought of in terms of) the context. Instead of saying "negative 3 plus positive 4 equals what?" the phrase "put together a debt of 3 and a gain of 4, how much money will you have?" could be used. T/TA will also consider whether one context or several are to be preferred. In the case of TA, these things can be carefully thought through, but then no adjustments to the students can be made once the textbook is printed. T on the other hand, can make many decisions on the spur

[35] The term *context* is chosen in order to incorporate a variety of possible referents. Often these contexts are models, but sometimes they are simply situations or visual aids. Another term could have been embodiment, which is not used since it could be confused with the ideas of embodied metaphors.

of the moment as a result of how her students react to what she is saying, or to what is said in the textbook.

Step 3

This is when the students meet the context and are asked to solve word problems. One example taken from the textbook used in the study is this:

> Emperor Augustus was born the year 63 BC. That could be written as year -63. He died in the year 14 AC. How old was he when he died?

If a student is focused on finding a solution to the problem she will probably solve it within the context and without employing negative numbers, by simply saying that Augustus lived 63 years before Christ was born and another 14 years after that, which makes 77 years all together. If mapped onto mathematics it would be 63 + 14 = 77. The teacher, however, wants students to map this problem onto the mathematical expression 14 – (-63). In this phase, at least when it comes to negative numbers, it is often easier to solve the problem within the real world context than engage in the process of mapping the metaphor in the intended way (cf. Sfard, 2007; Thompson, 1993). Another example is a task made up by one of the students:

> The emperor is <u>given</u> cows from a farmer. How much did the farmer lose if each cow would have cost 13 dollar? The emperor gets 128 cows.

This task only involves natural numbers since it is in the choice of words that a loss rather that a gain is indicated. Instead of writing this problem as 128 · (-13) = (-1864) students seem to prefer 128 · 13 = 1864 to work out that the farmer lost 1864 dollar. Viewing the contextualised problems as a starting point for the process of mapping it onto operations with negative numbers (step 4) would need some scaffolding in these cases. It is a question of introducing meta-level reasoning; of making students aware that the role of the context and word problem is to create a mapping to a new mathematical context.

Step 4

The mapping of the metaphor is negotiated and made explicit. Here *the source domain is the chosen context* and *the target domain is mathematics*. This process is closely related to the mathematisation process in problem solving. Mathematical terms are used to talk about the context and the context is symbolized, i.e. represented by mathematical symbols. Questions such as "how would you write this mathematically: I have 5 kronor and buy a cake for 8 kronor?" are posed. In this step different ways of mathematising the same problem can be discussed.

Step 5

Metaphorical reasoning is used to manipulate and solve abstract mathematical tasks, in Sweden called 'naked tasks', i.e. tasks without context such as 2+(-5)=_.

What happens in this step depends very much on the student. Some students may feel comfortable with the mapping that was negotiated and appropriate the metaphor so that it becomes a way for them to reason about and make sense of mathematical tasks. Other students may choose to bring in their own metaphors. Yet others may disregard the metaphors and choose a procedural or formal, intra-mathematical, way of reasoning. Some might give up trying to make any sense of it and try solving the tasks by applying rules[36]. In the empirical data it is when a student feels she does not understand, or thinks she understands but gets an incorrect answer, and the teacher intervenes that we have an opportunity to study how the student seems to appropriate the metaphor and in what way she or the teacher uses metaphorical reasoning.

Since the goal of the instruction process was to introduce metaphorical representations to give meaning to a mathematical content the process ends here, there is no separate contextualisation of solutions to problems solved. The metaphor has been introduced as a "thinking tool" for doing abstract mathematics (cf. English, 1997b), and the process is ended when the student can meet mathematically represented problems and make sense and solve them with the help of metaphorical reasoning. The metaphors have then become an integrated part of the mathematical concept.

In the following section, a deeper analysis of data is done in relation to each step of the teaching-learning process for metaphorical reasoning. The different steps of the process structure the analysis and empirical examples of each of the steps are chosen to illuminate how they are enacted in the classroom, and what is being offered to the students of this particular classroom concerning metaphorical reasoning. The chosen examples have been found to be relevant because they take up a large amount of the lesson time or because they generate many of the questions asked by the students.

6.2 Analysis of the teaching–learning process for metaphorical reasoning

The five steps presented in the previous section are analysed as follows: *Step 1and 2* give a description of what contexts are chosen and why, based on a close reading of the textbook and unstructured interviews with the teacher. *Step 3* describes how the contexts are presented to the students based on a close reading of the textbook and a video recorded whole class introduction lesson.

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[36] *Formal mathematical reasoning* is here taken to mean reasoning with logical deductions, finding patterns, using mathematical structures and ideas of mathematical consistency. This might include mathematical rules, but only in cases where these rules make sense for the user as being part of a mathematical structure. *Applying rules*, on the other hand, could also be referred to as 'procedural mimicry'. It is when many disconnected mathematical rules are learned in order to direct procedures. Such rules are often of the type: 'if X do Y'.

The last two steps; 4 and 5, are based on video recorded lessons including both whole class instruction and desk work. Step 4 gives analysis of how the mapping is done, what the conditions of use are and what is made explicit. *Step 5* describes instances of metaphorical reasoning; instances where the metaphor is helpful and instances where the metaphor fails as a tool for sense making.

Since the excerpts in this chapter are video transcripts they are presented in a table with a right hand column containing things that are happening in the classroom and written on the whiteboard, and in some cases an analysis of the situation, alongside the dialogue presented in the two left columns.

Step 1 and 2 –Preparation

In the textbook (Carlsson et al., 2002), the topic of negative numbers is introduced as part of a chapter called "More about numbers", under the heading "Numbers smaller than zero". Previously the students have met the expression "very small numbers" as a label for small quantities represented by decimals or fractions. The students will have met negative temperatures as part of the everyday discourse each winter, but negative numbers have not been brought up as a topic in the mathematics classroom before. In the interview during the time she is planning this topic T expresses that she finds negative numbers, subtraction and minus rather tricky. It is difficult, she says, "to separate negative number from subtraction and be clear about it". T considers using the terms *subtraction* and *subtract* as an alternative to *minus*, but expresses a concern that the students would find it more difficult to use terms they are not confident with.

Before the summer holidays T has given instructions about a pre algebraic procedure that relates to negative numbers. Using the metaphor 'to tidy up' she has instructed the students to rearrange long expressions so that all additions come first followed by subtractions, e.g. 2-8-7+15 = 2+15-8-7 = 17-16 = 1. This procedure is intended to introduce the idea of "collecting same terms", and is explained as "collecting all the pluses first and then all the minuses". In the case of algebra the same metaphor of 'tidy up' is used for a similar procedure, but now it is not the terms with the same signs but rather the terms with the same variables that are brought together, e.g. 2b-8a-7b+15a = -8a+15a+2b-7b. This way of speaking about numbers that are to be subtracted as if they were *minusnumbers*, i.e. negative numbers, is precisely how these numbers were spoken of before the concept of number was freed from the concept of quantity (see chapter 1.1). There is no distinction made between a negative number and a number to be subtracted.

Teacher's comments on negative number contexts

The textbook introduces negative numbers in a money context and an account balance. T believes the students understand a lot when money is concerned, relating plus with gaining money and minus with buying things. However, the idea of having a negative balance on your account is perhaps new to the students she says. She anticipates that some students will say that it is impossible to buy something for 8 kronor if you only have 2 kronor. T also expects the students to be confident with a thermometer as long as it is drawn vertically; that they can read it and compare temperatures. She is more doubtful about changes of temperature. It might not be obvious to all students for example, that if the temperature starts on -20 and gets 5 degrees warmer it will end up on -15. In the textbook the thermometer is drawn horizontally to resemble a number line but T thinks that will confuse the students since it's not what a thermometer looks like. T expresses reluctance towards the number line saying things like: "I don't like the number line" and "The thermometer is easier than the number line. The thermometer you recognise, you see it everywhere, but the number line exists only in the maths book". It is clear that T associates the term *number line* only with the very specific number line used in maths books for specific *number line tasks*, and does not see it as a term for a general number structure. For example, she does not consider measuring scales and rulers as number lines. T says: "When I was a child there weren't any number lines and we learned anyway. It's difficult to work with number lines, the students can't draw them properly, they find it difficult to get all the little lines exactly right, it becomes messy and they feel discouraged." Talk about distances above and below sea level T finds too removed from negative numbers. There is no intuitive connection between 10 meters below sea level and the number -10, according to her. The big problem, says T, is the connection between symbol and concept. From an everyday semantic point of view the students understand, but they lack strategies for interpreting symbolic expressions.

In her planning T has decided to predominantly use money exchange situations as a context for negative numbers. Since the thermometer and the number line are introduced by the textbook the students will meet them as well and "if they like them better that's fine", she says.

Looking through the textbook makes it clear that multiplication and division is introduced in the advanced part of the book, the red section that is optional, which means that all students would not encounter these operations in their individual work. T decides to include it in her whole class instruction. She says that $3 \cdot (-4)$ and $(-4) \cdot 3$ are easy to understand as owing 3 people 4 kronor, which means that you have a total debt of 12 kronor. Both expressions mean the same, according to T, that is something the students have learned when learning their multiplication tables. However, T finds $(-5) \cdot (-8)$ more difficult to understand so there she says you just have to rely on the rule "minus and minus make plus".

The textbook introduces multiplication and division with negative numbers by way of looking at patterns, generating the rules "the product of a negative number and a positive number is negative" and "the product of two negative numbers is positive" (Carlsson et al., 2002, p 34).

$$2 \cdot 3 = 6$$
$$1 \cdot 3 = 3$$
$$0 \cdot 3 = 0$$
$$(-1) \cdot 3 = (-3)$$
$$(-2) \cdot 3 = (-6)$$

The product of a negative number and a positive number is negative

$$2 \cdot (-3) = (-6)$$
$$1 \cdot (-3) = (-3)$$
$$0 \cdot (-3) = 0$$
$$(-1) \cdot (-3) = 3$$
$$(-2) \cdot (-3) = 6$$

The product of two negative numbers is positive

When T looks at this she comments: "I think that just looks strange. … It looks a bit like a number line, like a thermometer with positive up here and negative down here" She points at the number sequence 6, 3, 0, -3, -6, but discards it by saying: "you shouldn't have to recite the whole three times table to see the solution".

Interpretation of the choices made

The teacher prefers money exchange situations and temperatures as contexts for negative numbers because she thinks these are more familiar to the students than a number line, levels above and below zero, or number patterns. When it comes to things that are difficult to understand she believes it is necessary to rely on the rule "minus and minus make plus", the same rule applied for all operations.

The textbook starts off introducing negative numbers in the context of an account balance, then relates to the thermometer, the number line, and finally justifies the sign rules by means of number patterns. The sign rules in the textbook are more distinct than the one the teacher uses. e.g. "Subtracting a negative number is the same as adding the opposite number" and "The product of two negative numbers is positive"

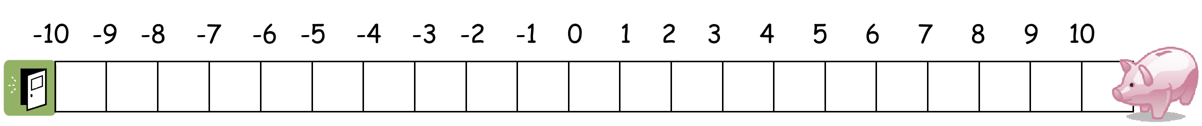
Step 3 - Real world contexts presented to the students

There are three main contexts are presented to the students. These are: a dice game, a bank account including money transactions, and a thermometer.

The dice game

During the very first lesson on negative numbers a dice game is used as an introduction to the topic. The rules of the game are as follows:

> You throw 2 dice, one is green and one is red. Green dice represents money you gain from Aunt Greta. Red dice represents money you pay to the Lolly Man when buying sweets. The Lolly Man keeps track of your incomes and expenses by moving a counter on a board. Throw the two dice, move a counter along a number line (ranging from -10 to +10) to the right as many steps as the green dice shows and to the left as many steps as the red dice shows. Where do you end up? The winner is the person who first comes to +10 (if you come to -10 you're out).



In this context two different metaphors for numbers are at work:

Numbers as Object Collections (dots on the dice) of two different colours; there are *two different types of objects*. These are spoken of as

- ~ money you gain = claims[37] = plusmoney
- ~ money you pay = debts = minusmoney

The operation included in the metaphor is one of seeing the *difference* between the two object collections. For example 6 green dots and 4 red dots will result in 2 green dots because there are 2 more of the green dots than the red dots.

Numbers as Motions Along a Path (a counter is moved along a number line): there are *two different motions* (directions):

- ~ Movement to the right
- ~ Movement to the left

The operation in this metaphor is one of *combining* two different movements that are *carried out one after the other*.

The dice game is quite a complex context since it links together two different metaphors; *object collections* and *motion along a path*. The links between the two metaphors used in the dice game are described in the table 6.1. These mappings are labelled **Version A** (see also chapter 3, table 3.1 and 3.5)

TABLE 6.1: Version A: Links between the *Object Collection* metaphor and the *Motion along a Path* metaphor in the dice game context

Dice **is linked to→ (object metaphor)**	***Movement*** **is mapped onto→ (path metaphor)**	***Mathematics***
Dots on green dice (claims)	Movement to the right	A positive number
Dots on red dice (debts)	Movement to the left	A negative number
More dots	Longer move	Magnitude
Same amount of green and red dots	Moving one way and back again	Zero
Difference between amount of dots on the two dice	Combining two movements	Addition

[37] The Swedish word *fordring* (claim) is used although this is not quite a situation where there is a claim. The teacher uses this word consistently through the whole teaching sequence when she is talking about money that is represented with a positive number. In the first mapping from mathematics to real world context, positive numbers are mapped onto the two different expressions "money you get" and "a claim", thus obscuring the difference between an addition and a positive number. The term claim is not a commonly used word in everyday Swedish.

The bank account,

The topic of negative numbers is introduced in the textbook under the heading "numbers smaller than zero". The first context is a bank account where stories about money being put into the account or used up are described as a *change* of the amount on the account. In this context only one metaphor for numbers appears:

Numbers as Object Collections (money). There are *two different types of money*:

- ~ Money that you owe = minusmoney
- ~ Money that you have = plusmoney

Two operations are included in the metaphor:

- ~ money that you have can be *put into* the account
- ~ money that you have can be *taken out* of the account

The balance on the account is not necessarily an object collection, but could be seen as a relation, showing the balance between money that has been removed or added. It is referred to as "what it says on the account information". In Swedish the word for account balance is *kontobesked,* meaning literally *account information*, which does not invoke a metaphor of balance in itself. The conception of this as a relation rather than money in the account is not clear in Swedish.

The thermometer

The book also introduces the thermometer as a context for the topic of negative numbers. It is displayed in both a horizontal and a vertical position, and on the same page a number line is drawn as well. The metaphor for number present here is related to the number as *motion along a path* metaphor. Along the path, not only motions but also locations and distances can all be mapped onto numbers.

Thermometer:

Number line:

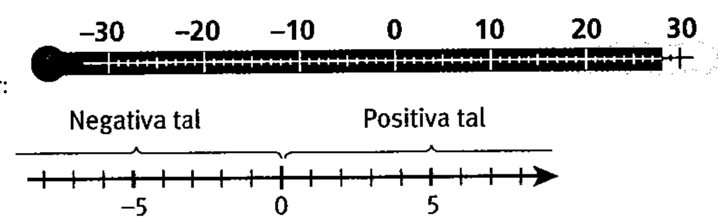


FIGURE 6.3: Thermometer and Number line from the textbook. (Carlsson et al., 2002, p 17)

Numbers as Motions Along a Path: A *path* (a scale or a line) along which *locations* (points) appear in a certain order. The points represent temperatures and can be talked of as either temperatures or points.

- ~ One point is special, it is marked zero and is a *reference* point for the others
- ~ Points to the right of zero = plus degrees[38]
- ~ Points to the left of zero = minus degrees

[38] In Swedish, temperatures above and below zero are referred to as *plus-degrees* and *minus-degrees*.

Two operations are included in the metaphor: a coloured bit of the path or a pointer can *move in two directions*:

- ~ Temperatures can rise (go up, to the right)
- ~ Temperatures can fall (go down, to the left)

Step 4 - Mathematisation phase: analysing the mapping of metaphors in the dice game

After spending some time playing the dice game the students are asked to try to write their actions mathematically and some of the tasks are then discussed by the whole class. This phase involves symbolizing. The teacher says that "writing it up" is a better way than just doing it in your head or thinking it on the number line. This phase could be described as *making the mapping of the metaphor explicit* or *negotiating the meaning of the mathematical symbols*. Three excerpts of whole class teaching where the teacher takes the dice game as a point of departure will here illustrate how this mapping is done. The data shows that although the context is the same (uses the same model), and the metaphorical source and target domains are the same; the mappings between the domains are quite different at different times, described here as different versions of the extended metaphors. The first version, version A, of the two metaphors *Object Collection* and *Motion along a Path* were illustrated in table, 6.1. Each version has its conditions of use, but these are never brought to attention in the lesson.

Two green and five red: addition or subtraction?

In the first excerpt the teacher (T) has suggested the following situation: 2 on the green dice and 5 on the red dice. Erik and Axel have suggested two different ways of writing this. In the excerpt the metaphorical expressions are highlighted. The right hand column shows what T writes on the whiteboard.

EXCERPT 6.1: Episode video 8.4. time 23:33. Introductory lesson, whole class teaching.

1	T:	this is how Axel wrote	2 + -5 = -3
2		and this is how Erik wrote	2 - 5 = -3
3		Both of them got the correct answer.	
4		It is ok, ssh ssh ssh, it is ok to write it in any way actually	
5		Axel he **added together, his debt and his claim**	
6		**the money he received** and the **money he had to pay**.	
7		he simply added them together, **put together** his **debts**	
8		30 seconds talk about writing brackets	
9	T	Like this, it looks really good. It is super what you wrote	2 + (-5) = -3
10		He has written these two dice, **put together these two dice**.	
11		One was the **negative red dice**	2 + (- 5) = - 3 ☝ T maps the **red five** onto the symbol (-5).

12	the other one was **the positive green dice** ssh ssh ssh, Paula.	2 + (- 5) = - 3 ☝ T points to 2, mapping the **green two** onto the symbol 2
13	but when you calculate it	
14	you just do exactly, what Erik did here	2 - 5 = -3
15	you **take the difference** of these two	
16	That's what you did all the time you know	
17	you **walked back and forth**	T indicates **movements left and right** in the air
18	**this way,** and then **back**	
19	**forward** and then **back**	
20	and you **landed on** minus since there were **more minus**	
21	the **debts were more** than	2 - 5 = -3 ☝ T points to 5, mapping the **debt of five** onto the symbol 5
22	the **money Aunt Greta gave you** weren't they?	
23	you bought more money, **bought more sweets** than	
24	what you were **given money** for	
25	so you **landed** on minus	T refers to the **end point of the movements.**

In this excerpt the mapping of an *object collection* metaphor expressed by T is slightly different from the above described Version A (see table 6.1). Table 6.2 illustrates **Version B** of the *object collection* metaphor. The difference from Version A is that there now is a mapping onto subtraction as well. Implicitly there is also a mapping to zero. For each version of a metaphor the conditions of use are described. These conditions of use outline the restrictions and limitations of the metaphor and indicate where difficulties could appear should they not be known.

TABLE 6.2: Version B of the *Object Collection* metaphor.

Source domain is mapped onto→	**Target domain**	**Data**
A collection of green objects plusmoney (claims, money you gain)	A positive number	2
A collection of red objects minusmoney (debts, money you pay)	A negative number	-5
More objects in one collection	Magnitude	Line 21
Combining two collections into one	Addition	Erik; 2+(-5)
Difference between two collections	Subtraction	Axel; 2 – 5
No objects	Zero	
One object of each 'type' combined	Zero	

Conditions of use:

- ~ Number: There is nothing in the source domain that maps onto the value of a number. 2 > -5 the magnitude is emphasized (lines 20, 21, 23).
- ~ Subtraction: Subtraction is only concerned with magnitudes; there is nothing that maps onto subtraction with negative numbers.

T associates the *difference* to the difference between the magnitudes (number of objects in each collection) and links it to the *path metaphor*. In Version A of the *path metaphor* (table 6.1), movements were mapped onto numbers and all movements were added, there was no subtraction. But here T speaks of movements when she points at the subtraction 2 - 5. She creates a slightly different mapping which is described as **Version B** of the *path metaphor* in table 6.3. In version B the direction of the movement (left or right) is mapped onto the operation sign rather than the sign of polarity.

TABLE 6.3: Version B of the *Path* metaphor.

Source domain	**is mapped onto→**	**Target domain**	**Data**
A point along the path		A number	Lines 20, 25
A movement		A number	Lines 17–19
The length of a movement		Magnitude	
Direction of movement to the right		Addition	2
Direction of movement to the left		Subtraction	-5

Conditions of use:

- ~ Number: There is nothing in the source domain that maps onto polarity. All numbers in an expression need to be natural numbers. The first number in an expression is always interpreted as a movement from zero (or another starting point). 2 + 4 is interpreted as 0 + 2 + 4 and - 4 - 5 in interpreted as 0 - 4 - 5. A starting point is implied, as well as a plus sign in front of the first number if there is no minus.
- ~ Addition and subtraction: There is nothing in the source domain that maps onto addition and subtraction when the second term is negative, i.e. $a \pm b$ where b is negative.

The path metaphor as it was introduced in the playing of the dice game (Version A) has no mapping onto subtraction. The minus sign is attributed to negativity. Addition is the combination of two movements. Movement to the right 2 steps followed by movement to the left 5 steps will leave you 3 steps to the left (of the starting position). The path metaphor as it is used the second time (Version B) has a different mapping. The minus sign is no longer attributed to negativity but to subtraction. Using the path metaphor in an inconsistent way could perhaps create confusion between these two meanings.

Two movements in opposite directions would leave you in a position that actually is the result of *combining* the two movements, first moving one way and then the other. However, the *difference* between a movement 2 to the right and 5 to the left can also be interpreted as a distance the length of 7. *Combining* and taking the *difference* are in the domain of natural numbers mapped onto opposite operations whereas here they are used interchangeably depending on the underlying metaphor in use. This potential complication will be further investigated in section 6.3.

Taking away a debt

In the next two excerpts T has suggested the following situation: three red dice show 2, 4 and 5 respectively. In the first part, excerpt 6.2, the question discussed is how to write this mathematically. There is a negotiation of the mapping of the metaphor. In the second part, excerpt 6.3, the task is to eliminate one debt to make “the debt as small as possible”. In the excerpt the metaphorical expressions are highlighted. The mathematical symbols in the right hand column show what T writes on the whiteboard. T is teacher, Stud is an unidentified student.

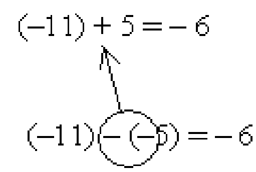
EXCERPT 6.2: Episode Video 8.5 time 06:20. Whole class teaching. Three red dice are drawn on the board showing 2, 4 and 5 dots. Part I: Adding a debt.

1	Viktor	11	
2	T	11? Mm, explain your thinking.	
3	Viktor	plus them all **together**	
4	T	you plussed them all **together**. 2 plus 4 plus 5.	
5		and then you knew it was all about **negative money**	
6		so to speak, that **it was about debts**.	
7		Then you know it was **a debt of 11 kronor** or	2 + 4 + 5 = 11
		whatever it was. In other words minus 11	**debt** of 11 kr
8		(…) do you all follow? If you want to write	-11
9		that you add **together** this **money.**	
10		If you add together you do plus. So if we want to	
11		write that we add together the **debts**,	
12		as I have done there	
13		But if we want to add together them as <u>**debts**</u>,	
14		and **debts are minusmoney, negative money**.	
15		How do we write that? If we want to	
16	T	**add a debt with another debt with another debt**?	
17	Stud	well, you might do minus 2 minus 4 minus 5	
18	T	minus 2 minus 4 minus 5	(-2) (-4) (-5) =
19		and these **all together** we want to add, which means?	
20	Stud	plus	
21	T	we plus. Ok, and that makes a total of?	(-2) + (-4) + (-5) =
22	Stud	11	
23	T	minus 11, mm, good.	(-2) + (-4) + (-5) = -11

In this excerpt the *object collection* metaphor is very clear. T talks about these collections of red dots (that represent debts) as a particular kind of object. You can add them either as natural numbers and mark the type of object only in the result [lines 4 to 9], or you can write them as signed numbers all the way through [line 13 to 23]. In the first case the numbers are treated as natural numbers and there is no extension of the metaphor, in the second case there is an extension from natural numbers to integers. In the next excerpt, 6.5, the discussion continues with the question of taking one of the three debts away.

EXCERPT 6.3: Episode Video 8.5. Whole class teaching. Three red dice are drawn on the board showing 2, 4 and 5 dots. Part II: Taking away a debt.

24	T	Now if I had these, and then wanted	(-2) + (-4) + (-5) = -11
25		as the one you worked with on the worksheet	
26		where you had to **take away one of the debts**	
27		You would **be rid of** one of these **debts**	
28		Which **debt** would you prefer to **take away**	
29		if you wanted the **smallest debt** possible left?	
30		to **save as much money** as possible?	
31	Stud	the 5	
32	T	the 5, since it is **the biggest debt**	T crosses over the dice
33		so if I **take it away**	with five dots, see picture
34		Then we have this and **take away that debt**.	6.3
35		what do you do when you **take away**?	
36		we have minus 11 and then we, **take away**......	-11 (-5)
37		… a **debt** of 5 kronor	
38		What operation do we normally use when we **take away**? … Linda?	
39	Linda	subtraction	
40	T	Precisely! So then	-11 - (-5)
41		how **much money**, or how **much debt** do I have if I take away that **debt**?	
42		we see it here	T points to the dice as
43		Sean how **much debt** do we have if we **take away the 5-debt**?	shown in figure 6.4
44	Sean	well then we have 6 in **debt**	
45	T	mm, then we have 6 in **debt**	
46		Okay, so this should make 6. It looks a bit strange.	
47		How could that be?	
48		But it's quite right what he says,	
49		we know we have a **debt** of 6 kronor since we've **taken away the debt** of 5.	
50		how can you get, if you work this out	
51		how can you calculate this	
52		so that you really understand that it makes minus 6?	
53	Sean	you, well I usually, you could count up from 5 to 11, and then you see that it's 6	
54	T	from 5 to 11 it's 6? So 6 is the **difference**? If you think of the **number line**?	
55	Sean	yea	
56	T	mm. But if you want to calculate it mathematically?	
57		how should you write, if you want to rewrite it?	
58		you know what we had this, I had this before	
59		[…] T spends 30 s repeating the rule that a plus and a minus make a minus	

60		Can we do something with this?	-11 - (-5)
61		Can we simplify it? When you're thinking	
62		So that the answer actually is right, so that it makes minus 6 there?	
63	Sean	plus	
64	T	you can turn them into a plus sign.	
65		Then we have minus 11 plus 5 makes minus 6.	
66		Minus 11	T **moves** her hand **to the left**
67		plus 5	T **moves** her hand **to the right**
68		you **end up on** minus 6.	
69		so then we can say there is a rule that says	
70		that two same signs becomes a plus	
71	Sean	Always?	
72	T	Always. When they are close together	
73	Sean	but what if there are two pluses?	
74	T	well if there are two plusses it's plus	
75		if you get 10 kronor from him, plus you get 20 kronor from him, then you get 30 kronor don't you?	
76		so similar signs always becomes plus and different signs always becomes minus	T writes the rule on the whiteboard and the students copy it into their books

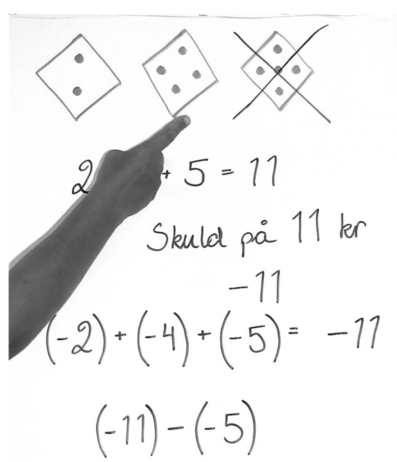


FIGURE 6.4: Illustration of taking away (-5) from (-11) on the board.

In this new version of the extended *object collection* metaphor debts are referred to as money and are mapped onto negative numbers. T refers to the size of the debts depending on how much money there is in the collection [lines 29, 30, 32, 41, 43]. Like in Version A, it is the negativity that is mapped onto the minus sign, but the extension now includes a mapping also onto subtraction. The action *taking away* objects is explicitly mapped onto subtraction [lines 26 to 39]. The new extension of the *object collection* metaphor, **Version C**, is illustrated in table 6.4, with its conditions of use listed below.

TABLE 6.4: Version C of the *Object Collection* metaphor.

Source domain	is mapped onto→	Target domain	Data
A collection of red objects, a collection of minusmoney (a debt)		A negative number	-11, -5, -6 on the board
More objects in one collection than in another		Magnitude	Lines; 29, 30, 32, 41, 43,
Taking away a part of a collection of objects from a bigger collection		Subtraction	(-11) - (-5) on the board

Conditions of use:

- Number: There is nothing in the source domain that maps onto the value of a number, (-1 > -5), it is the magnitude of numbers that relates to size.
- Subtraction: You can only take away objects that you have. There is nothing in the source domain that maps onto subtractions of the type $a - b$ *where* $a > b$, such as 4 - (-2) , (-2) - 4 , 2 - 4 or (-2) - (-4). Only subtractions of the type $a - b$ where a and b are the same type (have the same sign) and $|a| > |b|$ are mapped.

Twice a *motion along a path* metaphor is mentioned. The first time it is done by Sean who suggests a counting up strategy to solve a subtraction [line 53]. It is clear in his statement that he only focuses on the magnitude because he suggests counting up from 5 to 11. In the domain of integers the corresponding suggestion would be to count *down* from -5 to -11, or a counting *up* from -11 to -5. T vaguely acknowledges the metaphor but decides to let the suggestion be, it's not the point she wants to make [lines 54 to 56]. The second time a reference is made to a path metaphor it is done mainly by gestures. T uses this metaphor to justify the calculation -11+5 = -6 [lines 66 to 68].

Step 4 - Mathematisation phase; analysing the mapping of the thermometer

The thermometer appears in the textbook on the second page of the chapter on negative numbers, where it forms the context and visual representation for tasks about ordering temperatures (and later numbers). On the same page there are word problems of a "real world character", for example:

> The temperature in Bydalen was -18ºC in the morning. Two hours later the temperature had gone up 6 degrees. What was the temperature at that time?

The textbook does not instruct the student to write a calculation, it only requests an answer. The only feature of negative numbers that becomes visible if the student does not write a calculation is the value (order) of the negative numbers. As for the rest of the mapping of operations onto these numbers it is left to the students to do. The problems can all be solved by looking at the thermometer and counting up and down. *If* students attempt a mapping of the metaphor, **Version C** of the extended *path* metaphor, illustrated in table 6.5, is an extension coherent with the textbook presentation. The limitations and constraints of the extended metaphor are described below the table as its conditions of use. The mapping itself or its conditions of use are never made explicit in the book or during the lessons.

TABLE 6.5: Version C of the *Path* metaphor where the path is a temperature scale:

Source domain **is mapped onto** →	**Target domain**
A point on the scale	A number
A (centre) point on the scale	Zero
A movement	A number (unsigned)
The length of a movement	Magnitude
The further up a on the scale the larger	Value
Movement upwards on the scale	Addition
Movement downwards on the scale	Subtraction

Conditions of use:

- Addition and Subtraction: There is nothing in the source domain that maps onto addition or subtraction when the second term is negative, i.e. $a \pm b$ where b is negative. A temperature cannot rise or fall with a negative number of degrees.

Step 5 - Metaphorical reasoning phase

In the metaphorical reasoning phase, students encounter naked tasks and try to solve them and make sense of them. In this section some examples of the kind of reasoning teacher and students develop during this phase are given. The analysis shows that in some cases metaphorical reasoning is helpful; temperatures make sense of the value of numbers and movements along a path make sense of subtraction. In other cases the metaphors in use are less helpful or even counterproductive, for instance temperatures do not make sense of adding or subtracting a negative number. "Minusmoney" is found to be a complicated metaphorical expression that makes sense to some but not to others. The data suggests that if the target domain is already known (as it is for the teacher) and accepted on a formal or intra-mathematical level, then the idea of taking away a debt being the same as earning money is accepted and makes sense. However, students who first try to understand the source domain in order to create a mapping onto mathematics that is unknown to them, they seem to react to the source domain with defiance. These students focus their attention on the source domain of the metaphor rather than the abstract mathematical ideas the metaphor is meant to map onto.

In the following excerpts the metaphorical expressions are highlighted and the right hand column contains comments on the role played by the metaphor in the sense making of the task. Below each excerpt there is a summary of the analysis. The excerpts in this section are all transcripts of teacher-student dialogues that occurred during desk work. Most commonly a student has signalled for the teacher to come and help.

Temperatures make sense of the value (order) of numbers

Viktor is ordering signed numbers, seeing them as temperatures help him out.

EXCERPT 6.4: Episode 8.6. Viktor 1, time 18:23 – 19:07. Task nr 70: place these numbers in order of size with the smallest number first: 0,5 17,9 (-32) (-4,5)

Victor has written: 0,5 4,5 17,9 32

1	T	What have you done here?	Viktor has only looked at the magnitude of the numbers and treated them as natural numbers ignoring brackets and minus signs.
2	Viktor	What? That's number 70	
3	T	(…) it says write the smallest number first	
4	Viktor	That's a 32.	
5	T	Yes but you have minus signs and things like that as well don't you	
6	Viktor	y…yeeess	
7	T	Yes, can you just ignore those can you?	
8	Viktor	No but	
9	T	No. So **which is the coldest**?	T uses a **temperature metaphor,** talking of the numbers as if they were **markings on a scale**
10	Viktor	Just change places there	
Viktor writes: 32 4,5 17,9 0,5			
11	T	And then what? Then you have to put the minus signs there as well, otherwise you don't know that it's minus do you.	T uses the **temperature metaphor** to get Viktor to see that the temperature is mapped to a signed number; a natural number becomes a positive number.
12	Viktor	no	
13	T	Otherwise you would think it was **32 degrees warm that's the coldest**, and that would be strange wouldn't it.	
Viktor writes -32			
14	That's right.		
15	Viktor	4 point…	
16	T	And then 4 point 5 there. Minus 4 point 5 that one too. Yes. Finished.	

What role does the metaphor play here? By speaking about the numbers metaphorically as temperatures Viktor can relate to his own experiences and focus on the value (order) of the numbers rather than the magnitude, which makes him able to order them in a mathematically correct way.

Temperatures do not make sense of adding and subtracting a negative number

The first naked tasks students are asked to solve in the textbook after the thermometer has been introduced appear on a page following the introduction and contextualised problems. On this page the sign rules are also introduced. The first and last of the tasks on that page are:

76a) 2 + (-4) =   and   81 c) (-89) - (-12)

If the student had made a mapping of the path metaphor related to the thermometer, as suggested in version C of the path metaphor above, there is nothing in that metaphor that can be mapped onto these tasks. The students are expected to make use of the sign rules to solve these tasks. It is only possible to make sense of these tasks with this metaphor if the tasks are simplified according to the sign rules first. There is, however, no mention of these restrictions anywhere.

Movement along a path makes sense of subtraction

Thinking in terms of movements along a number line helps Lina to make sense of subtracting 6, but not of adding negative 6.

EXCERPT 6.5: Episode 8.7. Lina.1, time 30:56 – 31:35. Task nr 86b: Fill in the missing number: ( ) + (-6) = (-2)

1	Lina	I don't understand this	
2	T	you don't?	
3	Lina	no	
4	T	Then we'll have to guess. Suggest something that might work there	
5	Lina	This	
6	T	mm, well let's start by simplifying. That will make it much easier. Simplify those into a minus sign.	T decides to simplify first.
7		Now there's something minus 6 that should make minus 2.	
8	Lina	I don't know (sigh)	
9	T	(draws a number line)	T draws a number line and talks about **numbers as points** on that line.
10		Look. **Here** is the zero. **There** is minus 2.	
		We want to go, take away 6. **Take away** 6. 1 2 3 4 5 6 Where did that arrow start?	T speaks about moving, relating it to taking away and therefore **moving to the left.**
11		(T draws an arrow 6 places long ending on -2)	
12	Lina	Now I get it. 4	
13	T	mm, precisely. Plus 4.	

What was the role played by the metaphor here? A path metaphor is used to make sense of the operation 4 - 6 = (-2). Here the metaphor is only partially extended into the new number domain because in this metaphor the second term can not be negative (see Version B of the Path metaphor). T does not explain this, it is implicit in the fact that T first simplifies the expression + (-6) to -6 [line 6]. Lina says that she "gets it" which indicates that the explanation was helpful for her in solving the task. However, the metaphor makes sense of a subtraction of 6, not of an addition of -6, since the simplification is carried out without any explanation.

Does 'minusmoney' make sense?

In this section three examples will illustrate the use of the terms *plusmoney* and *minusmoney* when speaking about operating with positive and negative numbers. In the first example with Olle, the metaphor does not seem obvious to him although he accepts it. In the second example Fia and Elke try using the metaphor and the teacher leads them on, and in the third example Freddy is struggling to understand the calculation 2 - (-4) = 6 and the metaphor does not seem to make sense to him.

When using an objects collection metaphor about debts and gains T is inconsistent in what *a debt* is mapped onto. Sometimes a debt is mapped onto a negative number and sometimes it is mapped onto a subtraction of a positive number (something to be taken away). T says it is "a number with a minus sign in front of it". At times this is not clear to the students as in excerpt 6.6 where T and Olle are discussing the task 650 - 320 - 350. T speaks of -320 and -350 as debts, which is obviously not clear to Olle.

EXCERPT 6.6: Episode 8.5.Olle 1, time 31:49 - 33:33. Task: 650 - 320 - 350

T	or you could look at how much debt you have in total
Olle	so these here are debts?
T	mm, yes they are, they are minuses both of them.

Many students get the task -250 + 75 wrong. They add the numbers disregarding the minus sign and get 325 (or -325 by attaching the sign again at the end). Since there are no two signs close to each other in this task the students can not make use of a sign rule. T uses an *object collection* metaphor to make sense of this by mapping a situation of having a debt of 250 and a gain of 75 onto the expression -250 + 75. In the next episode, Fia, Elke and Lotta are together struggle with this task.

EXCERPT 6.7: Episode 8.5. Fia 1, time 23:53. Task: -250 + 75 =__

1	Fia	Plus is that the **debt** or what?	Fia is wondering if it is all about a debt due to the first minus sign
2	T	No it	
3	Fia	so he pays what?	
4	T	no that's **money he gets** you see	T uses **debt and gain metaphor** where numbers are **objects** that you can have and get.
5	Fia	yes	
6	T	First he has a **debt** of 250 kronor And then he is **given** 75 kronor by Aunt Greta	
7	Fia	mm	
8	T	How much **money does he have** then, in total?	
9	Fia	//or are you supposed to take	Fia is asking for a procedure
10	T	// so then he had a **smaller debt** didn't he?	Smaller here refers to the **magnitude** of the negative number.
11	Fia	Yes, so you take 250 minus	
12	T	That's right You can write it up. If you can't do it in your head. But perhaps you can do it in your head.	

Treating this as a debt of 250 and a gain of 75 is to think of the numbers as objects of different types. To add them is to find the difference between the amounts in the collections, but you do that by subtracting the magnitudes. Write -250+75 but think 250-75 is what the teacher suggests. The teacher leaves Fia to work it out but is called back again some minutes later to help with another task. The teacher then looks at this task and realises that Fia and Elke both have written -250 + 75 = 175.

EXCERPT 6.8: Episode 8.5.Fia 2, time 29:46. Task: -250 + 75 =

Fia and Elke have written -250 + 75 = 175

1	T	What's wrong here? If you have minus 250, That means **you owe me** 250 kronor	T speaks of the number -250 as a **debt**, and the addition of 75 as an **activity of getting** something.
2	T	// and then **you get**	
3	Lotta	// you just have to write minus	
4	T	Then you need to write a minus sign, don't you? Because it was **a debt**.	

Fia and Elke write -250+75= -175

Elke changes also the next task: -62 + 100 = 38 to: -62 + 100 = -38

5	T	No not there! There you didn't have **a debt**. there you hade **more plusmoney**.	T uses an **objects** metaphor speaking of **plusmoney** and **minusmoney** as **two kinds of objects.**
6		so, the kind you have most of, here's **more plusmoney**, and there is **more minusmoney**,	
7		and there is **more minusmoney** so the answer has to be minus.	
8	Fia	okidoki	

In excerpt 6.8 the teacher maps "getting money" onto adding 75 [line 3] but she also speaks of positive numbers as *plusmoney* [line 4]. In doing so, there is a subtle shift in meaning, implicitly addressing two plus signs although only one is written. *Getting* something is mapped onto addition and getting *plusmoney* is mapped onto a positive number, which should be symbolised as (-250) + (+75). It is a case of simultaneously viewing +75 as an addition and as a positive number.

In the next excerpt Freddy is trying to make sense of the task 2 - (-4) = 6. He has seen the correct answer in the answer section of the book but has a different opinion about it believing the answer to be -6. In excerpt 6.9 Olle is sitting next to Freddy and tries to join in the conversation but is brushed aside by Freddy.

EXCERPT 6.9: Episode 8.6. Freddy 3, time: 54:17 – 56:07. Task: 2 - (-4) = 6

1	Freddy:	I don't really understand this	
2	L:	mm?	
3	Freddy:	2 minus minus 4 shouldn't that be minus 6?	Freddy speaks in mathematical terms and focuses on the 2 whole
4		it is after all 2 whole minus	
5	Olle	you take away a de // bt	Olle and T uses a **debt and gain metaphor as an object metaphor** where taking away one type of objects is seen as the same as adding another type of objects.
6	Freddy:	// wait	
7	L:	Mm?	
8	Freddy:	I don't get any of this	
9	T	just as Olle said you **take away a debt**, if you **have** 2 kronor anth-	
10	Freddy	but then it should be 2 plus minus 4, then you just **add, eh away the debt**	Freddy speaks within the source domain where you get rid of a debt by adding money. This also relates to previous addition tasks where **a debt of 4 and a gain of 4 were added to result in zero,** which is the same as not having a debt anymore.
11	T	no but you add **a debt** then what you do is you **add on** a debt	T refers to **adding a debt to another debt**; a different situation
12		then what you get is like **one more debt**	
13		(…)	
14	Freddy	but that's what I have done on all my other tasks and it hasn't been wrong	Freddy indicates that he has been thinking about **adding debts or gains** in his other tasks; they have all been addition tasks. In these tasks the object metaphor has worked well.

15	T	but in that case you haven't had any tasks like that	T indicates that this task is different, but does not specify that it is because it is a subtraction
16		before you see that's it, now come some different	
17		(…)	
18	F	yes obviously	

T talks about addition tasks, starting with 8 + (-6) = 2

19	T	here you figured right because here you **added a debt**	T talks through the addition tasks using metaphorical reasoning and there seems to be no problem.
20		you **had money** and then you **added a debt** so **what you had became a little less**	
21		eight plus minus6 makes plus2	
22	Freddy	mm	The words large and small refer to the **magnitude** of the numbers; a big debt, a large debt.
23	T	mm that was perfectly right (…) and here (…)	
24		**here your debt was a bit too big** so that made it minus in the result	
25		(…) and here you had **two debts that you added together**	
26		that made a **very large debt** that's very good perfect (…)	

Freddy points to the task 2- (-5) =

27	Freddy	well then but, well then what does that make	T maps **take away** onto subtraction and a **debt** onto a negative number.
28	T	but here like we said you, **take away, a debt**	
29		because then it is as if you	
30		if I say "oh you don't **have to pay me those 5 kronor"**	T connects "not having to pay" with "earning".
31		then you could say that you **earned 5 kronor**	
32	Freddy	what so it makes m, 10 then	
33	T	no yea 2 plus, 5, is what it is when you **earn**	
34		//them	
35	Freddy	//does it make minus 7 then? or, or does it just make 7?	Freddy does not conceive of a positive number as directed (signed), but as a normal, natural number
36	T	just 7	

L writes 2 - (-5) = 2 + 5 = 7

37		2, plus, 5, so you have 2 kronor	
38	Freddy	mm	
39	T	and then you **earned** 5 kronor since **you did not have to pay a debt**	
40	Freddy	mm	
41	T	so then you can say that you **have** 7 kronor	
42	Freddy	yes mm	

Freddy is not content with the explanation that he "**earned** 5 kronor since **you did not have to pay a debt"** [line 39]. In his *object collection* metaphor there is nothing in the source domain (his experiences with money) that map onto subtraction of negative numbers from positive numbers. He cannot take one type of objects away from a collection of a different type of objects, he cannot take debts away from money he has. Only in a situation where he has debts to start off with can he take a debt away, which is the example the teacher used when mapping the metaphor. In such a situation they would be the same, but only if you used the money you earned to pay off the debt. Transferring the idea of taking away a debt as the same as earning money to a situation where there is no money to start off with is to generalise a mathematical equivalence, which is not a trivial task. The two situations are definitely not the same in the source domain. T intends to justify the mathematical equivalence -(-5) = +5 through a linguistic mapping, saying that taking away a debt is the same as earning money. What she in fact does, is justify the statement "taking away a debt is the same as earning money" by referring to the mathematical equivalence -(-5) = +5, T knows the mathematics so her metaphor maps mathematics onto money experiences, whereas for the students the metaphor maps money experiences onto mathematics. The representation process and the symbolization process are two different processes. In the whole class teaching T has given two examples to justify her metaphor. The first example is (-11) - (-5), described above in excerpt 6.3. The second example is (-79) - (-7), illustrated in excerpt 6.10. The process of generalising from this example is explained below the excerpt.

EXCERPT 6.10: Episode 8.6.2, time 04:25. Whole class teaching. T is teacher. The right hand column shows what is written on the board.

1	T	if we have a debt of 79 kronor and we **take <u>away</u> a debt** of 7 kronor?	-79
2		how do I write that? Tomas?	
3	Tomas	plus seven	
4	T	you <u>count</u> plus seven yes. and I but if I write it then I **take away**	-79 -
5		**a debt** of seven kronor, huh?	-79 -(-7) =
6		but, it is just as you say the same as if we count it like this	-79 + 7 =

In both these examples, excerpts 6.5 and 6.12, the source domain supplies a situation where there is a larger debt from which to take away a smaller debt, and it is suggested by the students that in order to take the debt away you add money. The calculations end up like this:

-79 - (-7) = (-72)

-79 + 7 = (-72)

Taking away a (smaller) debt from a (larger) debt is written as -79 - (-7) = (-72)
Adding money to a debt is written as -79 + 7 = (-72)
The process of generalising this idea can be described as follows:

1. Both results are equal, (-72), so the mathematical expression -(-7) and +7 must be equal, i.e. *they have the same value.*
2. In the metaphor *taking away a debt from a debt* is mapped onto -79 - (-7) which means that *taking away a debt* is mapped onto -(-7). In the metaphor the *adding of money* is mapped onto +7.
3. From the produced example the *taking away of a debt* as being the same as *adding money* needs to be generalized and decontextualized in order to be applied to situations where there is no debt to start off with. e.g. 3 - (-7). This phase is left for the students to do for themselves. Excerpt 6.9 shows how Freddy does not manage to do that.

Gestures as part of the metaphorical discourse

Although metaphors are generally a linguistic phenomenon there are other ways of communication, such as gestures, that become part of the metaphorical discourse. During the lessons on negative numbers T makes use of gestures as a means of illustration or emphasis several times. Some of the gestures are illustrations of motions along a path, moving the hand to the right and left to illustrate these movements. In all such cases T uses a horizontal path despite the fact that she expressed a preference of a vertical thermometer if the students were to understand. Two particular instances of gesture use stand out as significant in the way T tries to create metaphorical mappings for the students. The first instance is a situation when T is explaining the notion of opposite numbers, illustrated in excerpt 6.11 where Lotta has asked what opposite numbers are, because it is mentioned in the textbook.

EXCERPT 6.11: Episode 8.6 Lotta 1, time 20:33 – 20:48. Lotta and T are talking about the question "What is the sum of two opposite numbers?" The right hand column describes the teacher's gestures.

1	T	Opposite numbers they mean, kind of like, plus 7 and minus 7 they are opposite numbers	T places her two hands at **equal distances** to the **right** and to the **left of her body centre**
2	Lotta	ok, so it's 0 then?	
3	T	yes precisely!	T **slams her hands together** in the **middle**.
4	Lotta	yea ok	
5	T	if you have as many **debts** as you have **claims** it makes zero	T connects to the object collection metaphor using **debts and claims**.
6	Lotta	yea ok	

The metaphor underlying the gesture is a *path metaphor* where a centre point is mapped onto zero, a point on the right is mapped onto a positive number and a point on the left is mapped onto a negative number. The 'embodied feeling' that two points located at the same distance from zero not only represents two points but also two movements, is connected by T to the idea of opposite types of objects (debts and claims) counterbalancing each other, so that when brought together they will meet at the centre point (zero).

The second instance of gesture use is quite different and appears in a whole class discussion in video 8.8 when the topic has moved on from negative numbers to powers. In order to work with powers of negative numbers, e.g. $(-2)^3$, it becomes necessary to repeat the sign rule for multiplication. T uses the short version of the rule saying "two minus signs becomes a plus sign" indicating with her index fingers two horizontal lines and then bringing them together into a cross, as shown in figure 6.5.

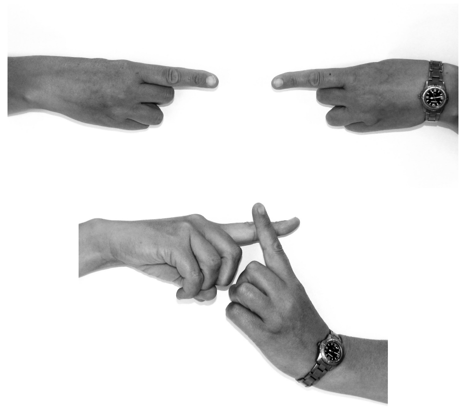


FIGURE 6.5: Illustration of how two minus signs make one plus sign.

This gesture serves more as a reminder of the rule than being a conceptual metaphor, but it does evoke a sense of two signs becoming one sign. The embodied and visual transformation of two horizontal lines becoming a cross is mapped onto the idea that two minus signs are transformed into one plus sign. For multiplication this could be seen as the two signs of the numbers to be multiplied are transformed into the one sign of the number of the product; (-2) · (-3) = (+6). The two minus signs in the task become a plus sign in the result. However, since the same sign rule is used for subtraction it would need a different interpretation. For subtraction the two minus signs *next to* each other are transformed into a plus sign *before the calculation is made*; (-3) - (-2) = (-3) + 2. The second term is now left without a sign (so it is implicitly understood as positive) and the sign of the result is not involved. Mathematically, the *two minus signs* next to each other are equivalent to *two plus signs* next to each other: (-3) - (-2) = (-3) + (+2), since a subtraction of an integer is equivalent to an addition of the opposite integer, but the conventional way of writing an addition of a positive number is to write only one plus sign. As shown, the metaphorical meaning the gesture entails is not mathematically correct.

Comments

The presented results contribute to previous research by attempting to answer a question posed at CERME 2005: What are the characteristic metaphors in use for different domains of mathematics? (Parzysz et al., 2005). The identified teaching-learning process for metaphorical reasoning shows what metaphors that were used and what role they played in the domain of signed numbers in the studied classroom.

The instruction about negative numbers given in this classroom offers contexts that bring up properties of number coherent with three of the grounding metaphors. The metaphors are isolated and students need to find out themselves when to make use of each metaphor. When no metaphor seems to work there is always a sign rule to apply. Figure 6.6 illustrates this. However, data supports the claim that to understand the whole concept several metaphors are needed since each metaphor only highlights some aspects of the concept (Lakoff & Johnson, 1980; Lakoff & Núñez, 2000).

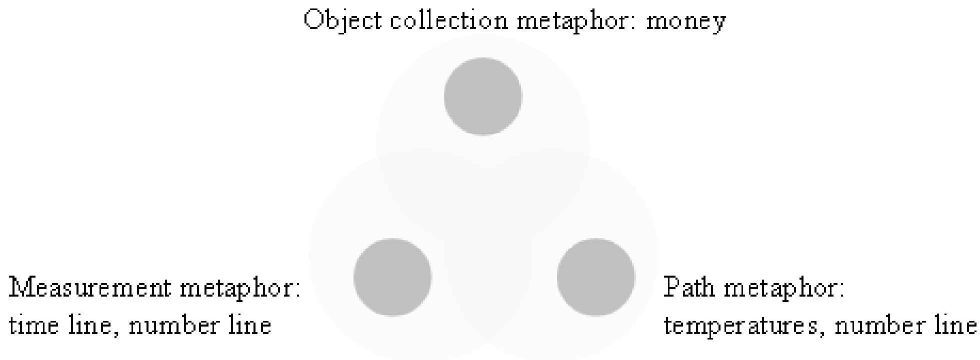


FIGURE 6.6: Illustration of the metaphors offered in the classroom instruction.

Participants in the studied classroom tend to associate to different aspects of number, understood through different grounding metaphors, when trying to make sense of a single task, suggesting a different approach to teaching for metaphorical reasoning. Making use of the same metaphors, but in a more holistic way with the goal not only of understanding isolated tasks but of understanding the mathematical properties of the concept of signed numbers, could be to focus on how the different metaphors relate to each other, when they overlap and when they do not, as shown in figure 6.7.

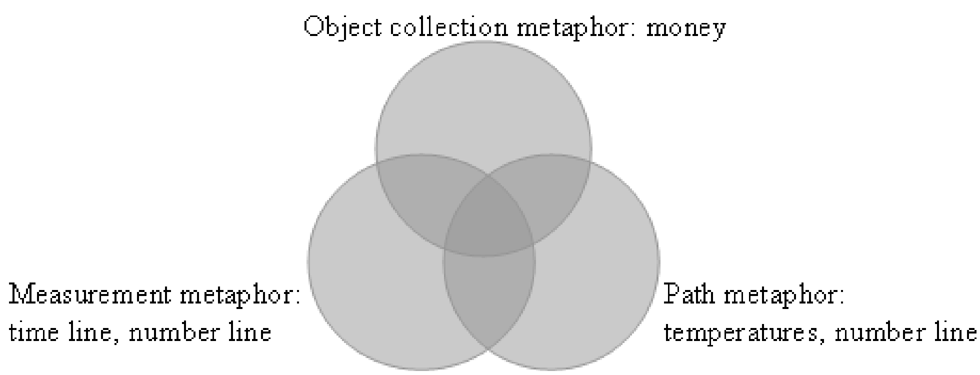


FIGURE 6.7: Illustration of possible interrelations between the metaphors of the classroom instruction.

Naked tasks could be solved using different metaphors, discussing how the task could be understood in terms of different metaphors for number, and when doing so properties underlying the sign rules could become apparent. For example the dice game played during the introduction lesson supplied possibilities for a metalevel discussion and a comparison between two metaphors. However, although the teacher made a few connections, she quickly left the movements along a path and focused on the debts and gains.

New questions:

Questions raised by the observations reported here are for instance how students appropriate these metaphors, and how the metaphors structure their developing number sense with regard to negative numbers. The following section will in detail analyse one of the major obstacles that appeared many times in the teacher-student interactions; the metaphorical meaning of the phrase "taking the difference between two numbers". Thereafter we shall turn to the interview data to look at the development of students' number sense in chapter 7.

6.3 Different differences

In this section the main research question of how we can understand metaphor as a means for making sense of negative numbers is taken one step further. Having seen which metaphors were brought into the classroom discourse, the goal behind them of making sense of negative numbers through metaphorical reasoning, and the way in which the metaphors were implemented, focus is now shifted towards the students. Questions asked in this section are:

- ~ In what different ways do students make sense of metaphors introduced in the classroom discourse, and
- ~ How do these metaphors help them make sense of negative numbers?

To answer these questions both classroom data and student interview data is used. One of the major difficulties observed in the classroom discourse about negative numbers is the use of the phrase "find the difference between two numbers". It is defined as a major difficulty because it is discussed in many of the teacher-students interactions and takes up a large part of the time spent on negative numbers. This chapter will bring to the surface a few situations where this phrase seems to have different meanings depending on what metaphor it relates to. These situations could be thought of as obstacles for the students, or as opportunities to bring a cognitive conflict to the surface. Before looking at the discourse of the classrooms two different, in some ways contradictory, metaphors of difference between numbers are described and analysed. These two metaphors, with several different mappings in the extended number domain, are frequently referred to in the classroom discourse but never made explicit or compared.

After the description of the different metaphors follows a number of excerpts that illustrate how these metaphors are used in the classroom discourse, and problems and misunderstandings that arise as a result. Different students with different mathematical ability are represented. At the end of the section different ways of understanding "difference" is related to students' achievement on written test questions.

The excerpts in this section are presented in tables with a right hand column containing an analysis of the situation, alongside the dialogue presented in the two left hand columns. Most of the interactions are desk work situations where a student has called the teacher's attention. Data suggests that although teacher, textbook and student all use the same words (difference, number) they mean different things. Problems arise when they are unaware of this and meanings are taken as shared although they differ. A commognitive conflict (Sfard, 2008) is there but it is not realized and therefore not resolved.

"Difference" in the two metaphors: an introductory analysis

In the following text it is necessary to distinguish between the two different values a number can have. The term "magnitude" refers to the magnitude of a number, in mathematical symbols often written as the absolute value; $| a |$ for the magnitude of any number *a*. The signed numbers (+8) and (-8) are said to have the same magnitude 8. Each number also has another value, here termed as the "value", that relates to the order of numbers in the number system, such that… $(-3) < (-2) < (-1) < 0 < 1 < 2 < 3$ … and so $-8 < +8$ but $|-8|=|+8|$

Object Collection metaphor

T uses the word difference for all situations she identifies as comparison situations. In the *object collection* metaphor, or the *measurement* metaphor, *difference is mapped onto subtraction* when two collections, or two segments, are compared and the *difference between them is in itself a (smaller) object collection* or segment. In the domain of natural numbers this could be visualised as in figure 6.8. The result (the difference) is what you get if you *subtract the smaller number from the larger number*. In the domain of natural numbers the magnitude and the value cohere and many students learn as a rule to always subtract the smaller from the larger.

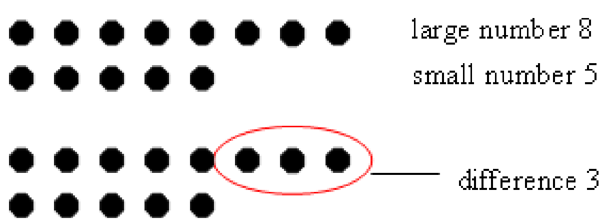


FIGURE 6.8: *Object Collection* metaphor in the domain of natural numbers. The difference is what you get if you calculate the **subtraction** 8 - 5 = 3.

In the domain of integers this mapping is the same *if* both the collections are of the same type (both are mapped to either positive or negative numbers). This situation can be visualised as in figure 6.9. However, it is now important to realise that it is the *magnitudes* that are considered when calculating the subtractions. It is the number with *the smallest magnitude that is to be subtracted* from the number with *the largest magnitude*. If the numbers are negative, the magnitude is the *opposite* from the value. Finding the difference between the two numbers (-8) and (-5) as done in this metaphor, is to find the difference between a debt of

8 and a debt of 5. The difference is of course a debt of 3 and is mapped onto the subtraction (-8) - (-5) = (-3). The difference is also considered a *magnitude* (with a sign only indicating the type of object) and can, as in the metaphor in the domain of natural numbers, be said to be *smaller* than the original collection, i.e. a debt of 3 is a smaller debt than a debt of 8, even if, as in this case, the value of (-3) is larger than that of (-8); (-3) > (-8).

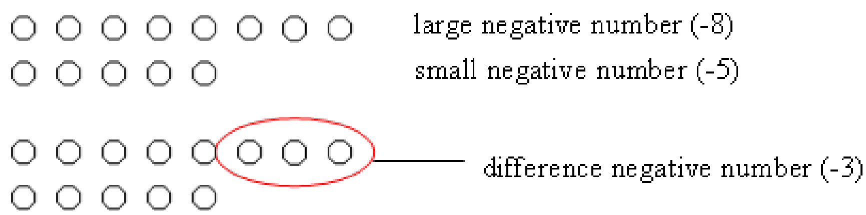


FIGURE 6.9: Extended O*bject Collection* metaphor in the domain of integers. The difference is the result of the **subtraction** (-8) - (-5) = (-3).

In this metaphor the *difference is a collection of objects*, and the smallest collection of objects is the empty collection. This means that the difference is always a magnitude, unless of course you change the metaphor and incorporate the notion of directed differences (Kullberg, 2010). In that case it is another metaphor and it is not an object collection that is mapped onto the answer but the *relation* between the two object collections, where ideas of *more than* and *less than* are mapped onto the sign, rather than the type of objects in the collection. It is a different metaphor, a different extension of the *object collection* metaphor known from the domain of natural numbers, which does not appear in the empirical data collected for this study.

Returning to the extended *object collection* metaphor described above, a situation with two objects of different types becomes a special case. In figure 6.10, the situation of finding the difference between a debt of 8 and a gain of 5 is visualised. The inconsistent feature is that *difference* is mapped onto the *addition* of (-8) and (+5), *not the subtraction*; i.e. (-8) + (+5) = (-3) *not* (-8) - (+5) = (-3).

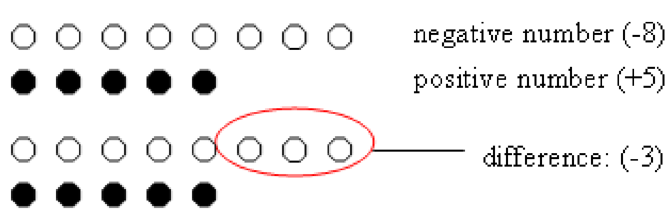


FIGURE 6.10: Extended *Object Collection* metaphor in the domain of integers. The difference is the result of the **addition** (-8) + (+5) = (-3).

The metaphor just described is embedded in the sign rules formulated in the *Brahmasphuta-siddhanta* year 638, (quoted in Mumford, 2010, p 123) where it is written concerning addition of integers: "[*The sum*] of two positives is positive, of two negatives, negative; of a negative and a positive [*the sum*] is their *difference*…"

and concerning subtraction: "[if] a larger from a smaller, their *difference* is reversed – negative becomes positive and positive negative." [chapter 18, verses 30, 31; emphasis not in original]

The word difference can in this way come to be associated with both addition and subtraction. However, as we shall see in the excerpts, the teacher tries to bridge this inconsistency by saying that they should "think subtraction" (of magnitudes) i.e. 8 - 5 = 3 in this case, whenever they "write the addition" (-8) + 5 = (-3). They will know what sign to put on the answer by considering which type of objects there were more of from the start.

Path / Measurement metaphor

Before analysing the empirical data we shall also look at how *difference* is mapped in the *path / measurement* metaphor. In the domain of natural numbers, a difference on a path is mapped onto the *distance* between two locations (points) on the path, as visualised in figure 6.11. This difference is, like in the *object collection* metaphor, mapped onto the *subtraction of the smallest number from the largest number*, but the difference itself is a distance, or a number of steps between two locations, and can as such only be a magnitude. The source domain does not (normally) include experiences of negative distances. In the expression 7 - 3 = 4 there are two possible metaphorical mappings. Either all three numbers are interpreted as distances: a distance of 3 (from 0 to the point at 3) is subtracted from the longer distance of 7, and the result is a distance of 4. Or, the first two numbers in the expression are interpreted as points and the result is a distance between these points. Both metaphors lead to the same calculation and the same result.

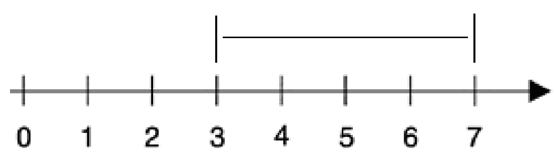


FIGURE 6.11: *Path/measurement* metaphor in the domain of natural numbers. The difference is what you get if you calculate the **subtraction** 7 - 3 = 4.

In figure 6.12, the number domain is extended to include integers. The difference is still referred to as the distance between two points along the path and is mapped to the *subtraction of the smaller number* (value) *from the bigger number* (value) as in the well known metaphor for natural numbers. This can for example be written as 3 - (-4) = 7. Contrary to the metaphor in the natural number domain the distance here could *also* be mapped onto *addition of the magnitudes*, i.e. 4 + 3 = 7.

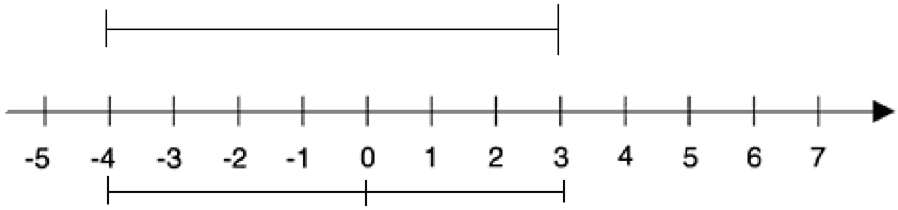


FIGURE 6.12: *Path* metaphor in the domain of integers. The difference is a result of the **subtraction** 3 - (-4) = 7 *or* the **addition** 3 + 4 = 7.

A visualisation could be used to illustrate the correspondence and equivalence between the two expressions, and so it is used in the textbook, see figure 6.13. In these two illustrations the model is the same, the two domains are the same, but the metaphorical mapping is different. In the expression 3 - (-4) = 7 the numbers 3 and -4 are interpreted as points along the line and the resulting number is the distance between them. In the expression 3 + 4 = 7 all the three numbers are interpreted as distances: two distances added together becomes a longer distance.

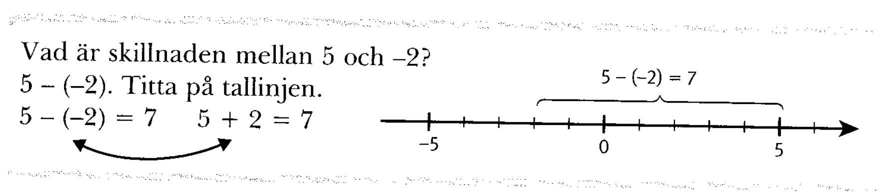


FIGURE 6.13: Illustration from the textbook. The text says: What is the difference between 5 and -2? 5 - (-2). Look at the number line. (Carlsson et al., 2002, p 30)

The condition of use for this metaphor is that a distance cannot be negative, so you must know to *subtract the smallest number (*value) from *the largest number* (value) and get the answer as a *magnitude*. There is nothing in the source domain that maps onto a subtractions such as (-10) - 6 = (-16). In fact, by referring to the difference as a distance between two points along a path, the calculations (-10) - (6) and (6) - (-10) would be the same. The distance between the numbers is 16 in both cases, not 16 in one case and (-16) in the other. In the textbook the instruction in figure 6:13 is followed by six naked tasks of the pattern [largest value] – [smallest value] and nothing is said about the restrictions connected to the use of the metaphor.

Summary

In the *object collection* metaphor the difference between two object collections of different types is a special case. It is mapped onto the difference between two numbers of opposite signs as either the subtraction [larger magnitude] – [smaller magnitude] where the answer is a magnitude and the sign needs to be considered separately, or the addition [value] + [value]. This means that to use this metaphor in metaphorical reasoning about abstract mathematical tasks involving integers it is an *addition of two numbers of opposite signs* that is to be identified as a situation of finding the "difference between the two numbers".

In the *path /measurement metaphor* the distance between two points along the path is also a special case. It is mapped onto the difference between two numbers of opposite signs as either the subtraction [larger value] – [smaller value], or the addition [magnitude] + [magnitude]. In both cases the answer is a magnitude, i.e. does not have a sign. This means that to use this metaphor in metaphorical reasoning about abstract mathematical tasks involving integers, it is a *subtraction of a negative number from a positive number* that is to be identified as a situation of finding the "difference between the two numbers", and that can also be rewritten as the addition of the two magnitudes.

It is not certain that different participants in a discourse make the same reference to the same metaphor when using the phrase "finding the difference between the two numbers", and confusion as to whether to use addition or subtraction and magnitudes or values could be the result.

"Difference between two numbers" in the discourse of the classroom

This section contains episodes where the idea of "finding the difference between two numbers" surfaces as a main issue in the discourse. The difference between two numbers is sometimes associated with the difference between two magnitudes in an addition with a negative and a positive number, and sometimes associated with the difference in value in a subtraction of two numbers. Mappings that are made in the extended grounding metaphors are illustrated and analysed in the same way as previously in this chapter. Highlighted words are the author's emphasis.

Whole class instruction about adding a positive and a negative number

In the following episode the teacher is discussing with the whole class how to work out the sum of 4 on the green dice and 6 on the red dice. One student has suggested to write this as 4 + (-6), but then Ove suggests 4 - 6. T makes a distinction between how to think and how to write.

EXCERPT 6.12: Video 8.5, time 05:53. Whole class instruction: T is teacher. ☝ marks where T is pointing on the board.

1	T	and when you work this out, how were you supposed to work this out?	4 + (-6)
2		it looks rather strange when there are two, Ove,	
3		when there is both a plus and a minus next to each other, it looks a bit strange	
4		So how do you explain your thinking?	
5		How are you supposed to **think** when you work it out?	
6	Ove	you **think 4 minus 6**	4 - 6
7	T	well, 4 minus 6	
8		and how then, do you **think** when you work out 4 minus 6?	
9	Ove	you think, eh, just **4 take away 6**, you have a **debt** then	
10	T	you get a **debt** then, ok, of 2.	4 - 6 = -2
11		The question is, we know **they are different so to speak**	☝☝
12		is **something which is positive and something which is negative** is what we have here	4 + (-6) ☝ ☝
13		and then we must **work out the difference between them**	
14		The **difference between 6 and 4 is 2**, yes	
15		and **there were more debts**, there were more on the red dice that are debts	

T tries to make a distinction between how to write and how to think [line 1-5]. Ove seems to think about the numbers as the same kind of objects: money. You have 4 of them and take away 6. After taking away 4 you are 2 short and that's what we call a debt of 2. He maps the situation onto subtraction [line 9]. T on the other hand sees the situation as *an addition of money and debts*, *two opposite types of objects* [lines 11 to 15]. The difference she gets is a magnitude and she only considers the sign afterwards.

TABLE 6.6: Ove's mapping of the *Object Collection metaphor*

Source domain	**is mapped onto →**	**Target domain**
A collection of objects		A number
A collection of objects that are yet to be taken away (missing, lacking)		A negative number
Combining two collections of objects into one		Addition
Taking a part of a collection of objects away from another collection		Subtraction

Ove's mapping exhibits only a slight beginning of an extension from natural numbers to integers. In his metaphor the mappings onto the operations are the same as in the domain of natural numbers. The problem with this metaphor is that the negative numbers are not seen as objects in their own right, they are not fully objectified, or reified, and cannot be operated on since they only represent the lack of something. It is the same problem mathematicians of the past had for hundreds of years, before the concept of number was freed from the concept of quantity (Schubring, 2005).

There is a subtle difference between the way Ove and T talk about this expression. T objectifies negative numbers by mapping red dots or 'minusmoney', rather than the lack of objects or the 'objects yet to be taken away', onto negative numbers. Doing so, she opens up for the possibility of operating on these numbers, even though her metaphor also interprets number as quantity. As a result of this modification of the source domain the mapping onto subtraction changes slightly.

TABLE 6.7: T's mapping of the *Object Collection* metaphor (a combination of Versions B and C described in section 6.2.)

Source domain	**is mapped onto →**	**Target domain**
A collection of green objects, plusmoney (claims, money you gain)		A positive number
A collection of red objects, minusmoney (debts, money you pay)		A negative number
More objects in one collection than in another		Magnitude
No objects		Zero
One object of each "type" combined		Zero
Combining two collections of objects into one		Addition
Difference in amount of objects in two collections		Subtraction (of magnitudes)
Taking a part of a collection of objects away from a bigger collection		Subtraction

Conditions of use:

- ~ Number: There is nothing in the source domain that maps onto the value of number.
- ~ Subtraction: Subtraction is concerned with magnitudes, so the smallest magnitude needs to be subtracted from the largest magnitude. There is no experience in the source domain to map onto subtractions such as $a - b$ when $|a|<|b|$. The special case of combining two collections of different types, although mapped onto an addition of two numbers with different signs is *also* mapped onto a subtraction of magnitudes.

George is solving (-2) - (-7)

George is a student who does quite well in mathematics. He works mostly by himself and does not often ask T for help. During the fourth lesson on negative numbers he has become uncertain about some tasks and wants to check with T if they are right. He starts out saying "a negative number minus another negative number will always be a positive number" and a long discussion follows where T gives counterexamples to his conjecture and talks about situations of taking away a smaller debt from a larger debt and still ending up with a debt. These are examples that fit well into T's mapping of the objects collection metaphor. Then George asks about the task (-2) - (-7) = 5 which is a subtraction of a type that doesn't correspond to anything in the source domain of T's metaphor. T's line of reasoning involves simplifying this into an addition and then interpreting that addition metaphorically as "finding the difference between the two numbers" and working it out by subtracting the smaller magnitude from the larger, as shown in excerpt 6:13.

EXCERPT 6.13: Episode 8.7.George.1, time: 47:24 – 48:10. The task was to write two negative numbers in the brackets making the equivalence true. ( ) - ( ) = 5

George has suggested (-2) - (-7) = 5

1	George	How about this, is it right?	George has a solution but is not sure if it is correct
2		Minus 2 minus minus7 is 5?	
3	T	Minus 2 minus minus7 becomes plus7, doesn't it?	T uses the sign rule to rewrite the expression
4	George	yes	
5	T	And **minus 2 plus 7,**	
6		The **difference between** 2 and 7 is 5	T interprets (-2) + 7 within an **object metaphor**, she sees the **numbers as objects of different kinds,** the **"difference" is mapped onto addition**. It is the **magnitude** of the numbers that is compared and the sign is determined by the type there is more of.
7		**Which were there more of**, positive or negative?	
8	George	Positive	
9	T	Precisely, so therefore it's 'plus5'.	

Implicit in T's reasoning is the fact that she needs to simplify the expression (-2) - (-7) into (-2) + 7 because it doesn't make sense as it is in her metaphor. By the application of a sign rule she gets an expression that she can make sense of.

Lina and Tina are solving -12,8 + 7,88

Another situation where an addition is interpreted as a comparison between two object collections is illustrated in excerpt 6.14, an episode from the third lesson in the sequence about negative numbers.

EXCERPT 6.14: Episode 8.6. Lina 1, time: 28:17 – 30:47. Task: -12,8 - (-7,88)

Tina and Lina work together, they have simplified and come up with -12,8 + 7,88
They ask T how to solve this and T shows them on the calculator, she presses 12,8 - 7,88.

1	T	You need to take the **difference**, between, you have **minus 12** point 8 and **plus 8** point, eh 7 point 88. So you need to work out the **difference** between them.	T interprets this addition as a situation of finding the **difference between two numbers of opposite types in an object collection metaphor. It is the magnitude** of the numbers that she compares. The difference of magnitudes is mapped onto subtraction. The biggest collection decides what type of objects there are in the answer.
2		You get the **difference** by pressing 12 point 8 minus 7 point 88.	
3		And you get 4 point 92, and then **you know that the answer is minus**. Because you saw, **12 is bigger than 7.**	
4	Tina	But did you take minus now?	
5	T	Now I just took the **normal**, 12 point, I took the **difference** between them, 12 point 8 and 7 point 88, equals, and then I got 4 point 92	
6		And then, I know the answer is minus 492, 4 point 92. Because **there were more minuses**, weren't there?	
7	Tina	yes	
8	T	mm	
9	Tina	You sort of need to	
10	T	You **don't have to bother about this minus sign**, it just mixes it up for you. It's better to use your brain. Like that.	
11		No **they are different**, so you just take the **difference**.	
12	Lina	yes… (tries doing it on the calculator)	
13	T	oops, you plussed them	
14	Lina	But wasn't I supposed to plus, there were two minus signs?	Lina wants to add the numbers since she has changed two minuses into a plus indicating addition.
15	T	That one is minus. And that one is plus. And **they should be plussed together.** There you are.	T is **not consistent** on whether to map onto addition or subtraction.
16		But now it's minus 12 point, and then they are **different signs. Then it's minus**.	
17	Lina	I don't understand anything.	
18	T	You **owe me** 12 kronor	T uses an **object collection metaphor relating to debts and gains,** in order to show that you get the difference by **subtracting the smaller (magnitude) from the larger** when you add them.
19	Lina	yes	
20	T	and you **are given** 7 kronor	
21	Lina	yes	
22	T	Well, then what you get, is a **difference**	
23	Lina	How do you calculate it?	Lina is still unsure about which operation to use.
24	T	Minus	
25	Lina	You do minus?	

In lines 1-6 T is talking metaphorically about the task in resonance with what she did in the whole class instruction. The addition of two numbers with opposite signs is interpreted as a comparison situation of two collections of different types of objects. This comparison is then mapped onto a subtraction of magnitudes. T uses the term *normal difference* [line 5] to show that she now considers only the magnitudes and treats the numbers as quantities. In lines 10 and 11, T emphasises that the minus sign only serves the task of telling which type of objects there are, in the process of calculation it is easier to disregard it. Lina wants to do an addition since there was a plus sign [line 14] and despite the persistent demonstration from T she is still surprised in the end that T calculates a subtraction [line 25]. In lines 15-16 T is inconsistent about whether to map onto addition or subtraction. At other times she distinguishes between *writing* it as an addition and *thinking* of it as a subtraction. Tina and Lina do not use any metaphorical reasoning on their own accord indicating that the metaphor is not part of their discourse.

Olle is solving -12,8 + 3,02

During the fourth lesson on negative numbers Olle is having trouble with the task (-12,8) - (-3,02). He has used a sign rule to simplify it to the expression (-12,8) + 3,02 but has difficulty making sense of that calculation (excerpt 6.15).

EXCERPT 6.15: Episode 8.7.Olle.2, time: 44:10 – 45:09. Task: (-12,8) - (-3,02) = -12,8 + 3,02

1	Olle	I know two minus signs make plus	Olle uses a sign rule but feels that if you create a plus you should **add** the numbers, he focuses on **magnitudes.**
2	T	yes	
3	Olle	But if you do plus it will make, like, 15	
4	T	yes but it's minus	T indicates the negative 12,8 and the positive 3,02. She sees them as **objects of two different kinds** that cannot be put together in one collection
5	Olle	yea?	
6	T	So **you can't add these two together**	
7		You need to have 'minus12'. Minus 12,8, plus 3,02	
8	Olle	but then	
9	T	Then they are **different**, yes but it's **minusnumbers** you know so it can't be 15,	T sees numbers as objects of different types. The **"difference" is mapped onto addition**. But to work out the difference is to **think subtraction of the magnitudes.**
10		because they are **different** so you need to work out the **difference** don't you	
11	Olle	mm… so I should **use minus** there? … that's what it is	Olle shows something on the calculator
12	T	That's right	
13	Olle	So every time there is a minus in front of it, you	Olle tries to find a rule, focusing on the minus sign

14	T	Well every time there are **<u>different</u> signs** in front of them you mean, then we have to **work out the difference between them**. Then you have **a debt and a claim so to speak.**	T shifts the focus to the fact that the two numbers have different signs and speaks of them in terms of **objects of different types.** She maps the two signs onto the types of objects, leaving the expression without an operation.
15	Olle	**Take away the minus sign** and take	Olle explains the procedure of doing the subtraction of magnitudes
16	T	no don't take away the minus sign	
17	Olle	no I mean, well… so I take that one minus that one (starts writing 3,02 – 12,8 in a vertical algorithm)	
18	T	Yes, that is, you can't write it up, putting it upside down so to speak. You can't write 3 on top and -12 underneath. In that case you need to write 12 – 3	In going from using a calculator to using an algorithm you need to use **the magnitudes of the numbers** to get the difference and **subtract the smaller number from the larger.** The answer you get is a **magnitude.** The **type of number** is decided by which type there were **more** of to start with.
19	Olle	Mm	
20	T	But in that case keep in mind that it's negative, since **there was most negative**.	

Olle only talks about numbers and plus and minus. He does not explicitly use any metaphorical reasoning at all. T refers to these numbers as collections of objects of different types, mapping difference onto addition, i.e. interpreting the addition of numbers with different signs as a comparison situation. The difficulty here seems to be for Olle to interpret the numbers in the expression -12,8 + 3,02 as object collections, he is more focused on finding a general procedure. In the *object collection* metaphor it is the difference between quantities, between amounts, that is relevant, and in order to use an algorithm the numbers all need to be treated as magnitudes where the smaller magnitude is subtracted from the larger magnitude. The difference is mapped onto addition when it is written, but onto subtraction of magnitudes when it is calculated mentally. What type of objects you end up with is decided by what type you had most of from the start. The situation makes possible a comparison between two metaphors where (-12,8) - (-3,02) could be understood within the *object collection* metaphor as taking 3,02 negative things away from 12,8 negative things, and (-12,8) + 3,02 could be interpreted in a *motion along a path* metaphor as moving 3,02 steps up from the point -12,8. Both interpretations would have given the same result. As it is done here it is uncertain if Olle is comfortable with the metaphorical reasoning, being uncertain about magnitudes or values and whether to add or to subtract. The appearance of both calculator and vertical algorithm in the same task could also be taken as a point of departure to explore the difference and sameness in the underlying metaphors.

Anna is solving -250 + 75

Excerpt 6.16 illustrates an episode where Anna is having trouble with many of the negative number tasks. T has spent several minutes talking about different tasks when she sees that Anna has solved the following task incorrectly: -250 + 75 = 350. When talking through this task T involves several metaphors, gliding furtively between them, to make sense of different features of the problem as they come up along the way.

EXCERPT 6.16: Episode 8.5.Anna 1, time: 38:39. Task: -250 + 75.
Anna's solution: -250 + 75 = 350

1	T	This one I'm a bit worried about. What does it say, minus 250? And then plus 75? (…)	
2		well we know there was minus 250, it was **on the account,** and then you **received** 75 kronor.	T talks within the **account metaphor** introduced by the textbook. It is unclear if T sees the -250 on the account as a relation (a balance) or if she sees it as objects, as "negative money"
3		Then you can't have 350 on the account can you?	
4	Anna	So it, so that makes it minus then or what?	
5	T	You need to think, if there's minus 250 on the account, and then you get 75 kronor from me and put it into the account	
6	Anna	mm	
7	T	What will the account say? (…) Will it be on minus or on plus?	
8	Anna	minus	
9	T	Mm, because you had such **a big debt** and you **get some money** from me, right?	Here T specifies that she speaks of **object collections**, collections of different types of money
10	Anna	mm	
11	T	So it has to be minus. But you **had some debt**, and **you got some plusmoney** from me,	
12		then they are different, they are **red and green** so to speak right?	T relates to the context of the dice game. In the **object collection metaphor** a **difference between two collections of different types is mapped onto an addition** so T interprets this addition as a situation of "finding the difference between two numbers"
13		So we need to calculate the difference between them. (…)	
14		The difference between 250 and 75. How much is the **difference** between 250 and 75?	T **compares the magnitudes**
15	Anna	25	T refers to a **path metaphor** indicating that the **difference** is about how many steps you need to take **going from one point** (75) to another (250), stopping at various points on the way, such as 100.
16	T	more, 25 that only **gets us up to** 100. And we want to get up to 250.	
17	Anna	But that makes it 75 there as well doesn't it?	
18		(…)	

Anna mumbles something

19	T	You can write it up with 250 minus 75	When the **path metaphor** does not seem to help Anna, T suggests an algorithm, which, by words such as "**borrow**" builds on some kind of **object metaphor.**
Anna does the vertical algorithm and gets 175			
20	Anna	that's 5	
21	T	mm	
22	Anna	4 minus 7, so I need to **borrow** there too	
23	T	mm, that's right. now what did you get, plus or minus, what did we say?	When the **difference between the magnitudes** has been found the **sign** needs to be attached.
24	Anna	minus	
25	T	mm, perfect.	

In this episode T uses a whole range of metaphorical resources starting by identifying the problem as a problem about money, relating to two kinds of money (debts and gains) and therefore identifying the task as a problem of finding the difference between two *collections of objects*. Once that is done she starts treating it as a problem of comparing quantities and can rely on natural number metaphors, such as the *motion along a path* metaphor where the difference between two numbers is the movement needed to get from the smallest to the largest number. During the whole process of finding the difference between the numbers it is treated as a natural number problem where the metaphors are consistent. At the very end, they need to refer back to the types of objects to decide about the sign of the number.

Tomas interprets difference as more than zero

In the third interview with Tomas a short episode from one of the classroom videos was viewed as a starting point for discussing difference. The video showed a situation from video 8:13, the last lesson on the topic of negative numbers, where the students in pairs came to the whiteboard to present a negative number problem they had constructed along with its solution. Tomas and Hans presented the problem:

> It is 22º inside and -13º outside. What is the temperature difference?

This is what their solution looked like:

> 22 - (-13) = 35. The difference is 35º

When solving the task they used the words "difference" and "degrees" from the context of the task. They said: "You need to take the difference between 22 degrees and minus 13. And so that is 35 degrees. So it makes plus, two minus (pointing to the two minus signs), and so you get the difference".

EXCERPT 6.17: Videotape 8:13, time 04:20. Tomas: interview G8, stimulated recall. I. stands for interviewer.

499	I	How do you know that you should take 22 minus 'minus13'?	Tomas knows from the domain of natural numbers that **difference is the bigger number minus the smaller number** (irrespective of what metaphorical sense the numbers have)
500	Tomas	Because, that's what you need to do to work out the **difference**	
501	I	yes?	
502	Tomas	Then you need to take minus, I mean, the **big** number minus, the, **smaller**	
503	I	yes?	
504	Tomas	and so you've got minus 13 as the **smallest** number	In Tomas' example -13 is smaller than +22 in both magnitude and value.
505	I	yes ok, is it always the bigger number first?	
506	Tomas	eh… or yes, **otherwise you get a negative number**, that can… that's **not** a … well… yes that's how it is, you need to take the big	
507	I	mm, so if this was the case, eh if I had to work out the difference between the outside minus 35, and the inside… is 20, plus 20 we could even say here. And I was to work out the difference between them, how would you do it?	By writing: outside -35º, inside +20º, I introduces a situation where the magnitude and the value diverge; $(-35) < (+20)$ but $\lvert 20 \rvert < \lvert 35 \rvert$
508	Tomas	Then you take eh, that, plus 20 minus 'minus35'	Tomas identifies the **value of the numbers as indication of size.**
509	I	mm? and so that's biggest?	
510	Tomas	yes	
511	I	If I make it the opposite, if I take… the smaller first, how would I do that?	I writes 20 - (-35)
512	Tomas	eh minus…	When asked to take the smallest number first. Tomas is confused, suggesting -25 + 20 (possibly miss-saying -25 in place of -35)
513	I	<u>this</u> is what you said first	
514	Tomas	well… that would be, minus 25… eh, and then it makes… what does it make wait… it probably makes plus 20 in that case.	
515	I	is that the difference between them?	
516	Tomas	(…) eh… no that's not right. hrm. or (…) yes it is. That's what it is. because… well…	
517	I	If I calculate this 20 minus 'minus35. What does that make?	Tomas calculates 20-(-35) correctly.
518	Tomas	That's ehm, 55	
519	I	mm… and how about if I work this out minus 35 plus 20, what does that make? [writes -35 + 20]	I asks about the addition (-35)+20 **coherent with the object metaphor: a difference between two types of numbers is mapped onto addition.**
520	Tomas	well that makes ff… minus 15	Tomas calculates (-35)+20 correctly.

521	I	So then, which, which is the difference between these numbers, is it 55 or is it 15?	Tomas identifies 55 as the difference rather than 15, i.e. he focuses on the **difference in value (between points on a number line) rather than magnitude (amount of objects in collection)**
522	Tomas	eh 55	
523	I	Well, so is it important to know in which order to write them?	
524	Tomas	yes	
525	I	And which operation to use, How do you know that you need to put the biggest number first?	
526	Tomas	…you know… well, but you always kind of something to… if you've got something, some difference	Tomas conceives of a **difference as a positive number (a magnitude)** and connects it to a well known procedure.
527	I	mm?	
528	Tomas	then you need to have one higher and one lower	
529	I	mm?	
530	Tomas	And eh, if you want to work out, a eh…. positive number which is what you need to get	
531	I	mm?	
532	Tomas	If you want to work out the difference that is, the only way is, take the biggest…	
533	I	mm, good explanation. Why does the difference have to be a positive number?	
534	Tomas	Because, ehm, that's smaller. It always has to be… a… you know it always has to be… something, smaller than… bigger. So there kind of has to be a… difference between them	
535	I	mm?	
536	Tomas	and so it's positive	

In Tomas' interpretation, difference is a magnitude, never a negative number. The difference between two numbers is associated with a procedure coherent with a *path metaphor*. He can work out -35+20 correctly but does not consider it a difference so his mapping of a difference is more in line with that of the textbook than the teachers way of reasoning associated with an *object collection* metaphor.

Comments concerning the classroom discourse

Two things about the discourse can be noted in these results. First of all T conveys that there are two different discourses: what you think and what you write e.g. thinking subtraction but writing addition. Secondly, most of the metaphorical reasoning comes from T. The students more often use mathematical terminology suggesting that they are attempting an intra-mathematical discourse. According to Sfard (2007) it is necessary to change the mathematical metarules in order to make sense of negative numbers. The new metarule justifies negative numbers and operations by showing how they are part of a mathematical structure whereas the old metarules justify through the use of concrete models (see also chapter 1.6). However, T guides the students into metaphorical reasoning by only using models and metaphorical justifications, thereby staying completely within the old meta-rule. The models T uses are only "concrete" in an imagined sort of way, she does not use money and thermometers to show her reasoning "hands on". This makes the "concrete" models abstract without a connection to the abstract mathematical concepts. Furthermore, T leaves the "concrete" models only when she has formulated a sign rule, but the examples show that even when students apply the rule correctly they have problems making sense of the calculation and getting a correct answer. The most problematic expression in relation to the idea of "difference between two numbers" is of the type $-a+b$ ($a,b \in \mathbf{N}$) which is spoken of as the difference between the numbers *a* and *b* with the sign considered separately.

Although some models are explicitly used metaphorically by T, at other times metaphors appeared implicitly and T might not even have been aware of them. The example of *difference* showed that the underlying metaphors created confusion. The observation is similar to that of Font et al. (2010, p 148) who observed that "There was no control over metaphors while teachers were unaware of using them". In their study, as in this study, the teachers' discourse did not include any mention of inferences that could have been made by students, but which were not mathematically accepted.

Another important finding in these results is the confusion about whether to consider magnitudes or values of the numbers. Since there is no explicit terminology used in the classroom discourse to distinguish between these two different aspects of a number, and since they diverge for negative numbers, the students become confused. Difference is for most of the students associated with magnitudes and with subtraction, and a difference between two numbers must be more than zero, as Tomas expressed in excerpt 6.21. It was described in chapter 1.3 how various researchers have emphasised the importance and difficulties of distinguishing between the magnitude and direction of negative numbers (cf. Altiparmak & Özdoan, 2010; Ball, 1993). These findings support that claim and suggests the use of a terminology that makes it possible to distinguish between the two meanings of size. The last section of this chapter illustrates how the confusion about magnitudes and values and about whether "difference" refers to addition or subtraction makes an impact on student achievement.

Mathematical achievement

So far, different ways of interpreting the phrase "taking the difference between two numbers" by the teacher and the textbook and by students has been described. The phrase has different connotations depending on what metaphorical underpinnings it has, and it has been shown that different participants in the discourse make different metaphorical connections and end up with different interpretations. For a teacher of mathematics an important question is whether this metaphorical reasoning has any impact on students' achievement.

On the test at the end of the teaching sequence a question appeared that could have been trivial to solve had the students been looking at a thermometer. The student is asked specifically for the difference between two temperatures. However, the task is not simply to find an answer to the question but to mathematise it in an appropriate way, knowing that it was a task about negative numbers. The different solutions indicate confusion about whether to add or subtract and whether to use magnitudes or values. Table 6.8 shows that correct answers are given by 10 students who model the difference as an *addition of magnitudes* and by 5 students who model it as a *subtraction of values*. 7 students suggest incorrect calculations or solutions, i.e. 32% of the students in this class did not achieve what was expected of them.

Question: The temperature in the freezer was -14ºC.[39]
The temperature in the room was +20ºC.
How many degrees difference was there between the freezer and the room?

TABLE 6.8: Students' solutions to the question about temperature differences.

Students' solutions consistent with mapping the *difference onto addition:*	
Correct solutions (9)	**Incorrect solutions (3)**
Axel, Olle, Ove, Hans: *14 + 20 = 34* with a clear reference to a visual representation, e.g. drawing a number line and marking the two distances added.	Anna and Freddy : *-14 + 20 = 6* Consistent with the mapping onto addition but without considering the fact that it is the magnitudes that need to be added, not the values
Lina, Viktor, Elke, Malin, Petra *14 + 20 = 34* no comment given	Linda: *(-14) + 20 = 34* The first temperature is written as a signed number but in the calculation the magnitude is used
Students' solutions consistent with mapping the *difference onto subtraction:*	
Correct solutions (5)	**Incorrect solutions (2)**
George, Erik, Fia, Martina, Tomas: *20 - (-14) = 34*	Paula, Lotta: 20 - 14 = 6 Consistent with the mapping onto subtraction but using magnitudes rather than values

[39] Two test versions were used, one had the numbers -12 and +20 and the other -14 and +20. For comparison reasons they are here all referred to as if they were -14 and +20.

TABLE 6.8: Continued…

Other: **Incorrect solutions: (2)**
Tina : *14 - 20 = 34* (writing a subtraction but calculating an addition) Sean : *-14 and up to 20 steps = 5 C. Answer: +5C between freezer and room.* Sean is possibly counting up from 14 to 20, counting only the five numbers in between: 15,16,17,18,19

In this task, students with incorrect solutions choose a wrong combination of operation and magnitude or value, suggesting that their problems could be a result of the use of different metaphors where these things vary without ever making this variation explicit and a part of the mathematical discourse.

Another task on the test was a multiple choice task with subtraction of a positive number from a negative number, i.e. a subtraction $a - b$ where $a < b$ and where a and b have different signs.

The task: (-12) - 15 = ___ circle the right answer: 3 (-3) 27 (-27)

The different alternatives could be analysed as having the following features:

- ~ 3 is the *difference between the magnitudes* of the two numbers, or it is the *distance between* them on the number line if they are interpreted as -12 and -15
- ~ (-3) is the *difference between the magnitudes* of the two numbers *with a minus sign attached*
- ~ 27 is the *sum of the magnitudes* of the two numbers, or it is the *distance between* them if they are interpreted as -12 and +15
- ~ (-27) is the *difference between the values*, or the *motion needed* to get from the second point (+15) to the first point (-12) on a number line.
  (-27) is the mathematically correct answer.

Incorrect answers were given by 7 students as follows: [3] was chosen by Paula and Freddy, [(-3)]) was chosen by Lina, Fia and Malin, [27] was chosen by Erik and Sean

Since this is a multiple choice task we know nothing of the students' reasoning. Incorrect answers were given by 36% of the students and varied between all the possible answers. This could indicate that the students' were simply picking one randomly, but it could also reveal a confusion about which combination of operation and magnitude or value that would be appropriate. Notable is that 5 out of 7 who failed on this task solved the temperature task correctly, which suggests that mathematising a contextualised task (which was the case in the temperature task) and metaphorical reasoning about a naked task are two quite different processes.

The findings indicate that the rather imprecise discourse of this mathematics classroom could have influenced these students achievement. References were made to the two different notions of size as either magnitude or value without making this difference clear in the discourse. Great uncertainty prevails among the students concerning the meaning of the phrase “take the difference between the numbers” caused by the different contexts where this phrase has been used and the different metaphors these contexts become source domains for.

The following chapter will look more closely at the interview data to see examples of the development of number sense in relation to the introduction of integers and the use of metaphors. In doing so, focus shifts even more towards the psychological perspective where classroom discourse becomes a background to individual cognitive development.

CHAPTER 7

Development of Number Sense

This chapter takes a psychological perspective on the main research question of how students make sense of negative numbers. Having seen the use of metaphorical reasoning in the classroom discourse, and then touched upon the impact this might have on students' sense making, focus is now directed towards individual students' development of number sense, in search of answers to the following questions:

a) What do different learning trajectories concerning features of number sense relevant for negative numbers look like?
b) How is such a learning trajectory influenced by the use of metaphors?

In different sections of this chapter different features of number sense are discussed. The empirical data, although restricted by the construction of the interview questions, was too vast to be fully included in this thesis. In the process of transcribing and reading through all the interviews, and making presentations at several conferences, some parts of the interviews were put aside for future analysis. These parts include, for example, the use of brackets and conceptions of infinity. The rest of the interviews were then scrutinized again in search for information about students' conceptions of, and reasoning with and about signed numbers. The different questions in the interviews were compared and new entities came up and crystallized as features of number sense that seemed important for making sense of negative numbers. The different sections of this chapter will treat these different features. First, the extension of the number domain from unsigned to signed numbers is described and discussed along with different kinds of change in number sense that can be detected in the data. Thereafter, four separate features are discussed: i) *conceptualizing negative numbers, zero and the size of numbers*, ii) *subtraction with a negative difference*, iii) *the different meanings of the minus sign*, and iv) *the number line*. Finally, a restructuring of the interview data is done to make different learning trajectories and patterns of change visible.

In the following sections empirical data from the three consecutive interviews are labelled G6, G7 and G8 to show in which grade the interview was made. The period of teaching about negative numbers occurred at the beginning of grade 8, which means half way between the interviews labelled G7 and G8. Interviews have numbers *a-bcd* where *a* is the grade of the interview (6, 7 or 8) and *bcd* is a serial number. An excerpt label *interview G6, Q8* indicates interview grade 6, question 8. The letter I stands for interviewer.

Extending the number domain from unsigned to signed numbers.

The questions posed in the yearly interviews were constructed on the basis of what aspects of negative numbers earlier research had found to be critical, described in chapter 1. See appendix II for the full interview protocol and chapter 4 for a closer description of the research design. All audio recorded data was transcribed verbatim in Swedish and carefully analysed. The aim of the initial analysis was to find topics where change occurred in the students' answers, ways of reasoning, narratives and justifications, and to describe these changes in terms of learning trajectories. It took many readings and several restructurings of the empirical data before some interesting aspects of change became clear. When the task involved a mathematical calculation the produced answer could change, but so could also the student's justifications and ways of reasoning. To describe how students change their number sense when the number domain was extended from unsigned to signed numbers, the responses were categorised into three different levels[40]:

0: mathematically incorrect or invalid answer
1: mathematically correct in $\mathbf{R}^+$ (the domain of unsigned numbers)
2: mathematically correct in $\mathbf{R}$ (the domain of signed numbers)

When this was done it was found that, also many of the more interpretative questions could also be categorised into these levels. See table 7.1 for examples.

TABLE 7.1: Examples of the three different levels of categorizing students' responses.

Level	Task Q5: calculate 2-5	Task Q7c: what does -2 mean?
0	3	2, nothing, I don't know
1: $\mathbf{R}^+$	It can't be done	Subtract 2; there must be a number in front (x-2)
2: $\mathbf{R}$	-3	Negative 2, 2 less than zero

In order to focus attention on change, a response that was categorized at a different level compared to the previous year's response was highlighted in the tables. An *up level change* was given a light shade and a *down level change* a dark shade. An empty space indicates missing data.

up level change
down level change

[40] It is the difference between responses placed within a number domain of unsigned numbers and a domain of signed numbers that is focused, not the difference between **N**, **Z**, **Q** and **R**. Most of the questions and responses only treat whole numbers (i.e. the extension from **N** to **Z**), and some treat decimals and fractions (i.e. the extension from $\mathbf{Q}^+$ to **Q**). No responses explicitly treat irrational numbers, but since the number line is included the responses are all labelled as correct or incorrect within **R**. The domain labelled $\mathbf{R}^+$ in this text represents the domain of unsigned numbers. All responses correct within **N** and $\mathbf{Q}^+$ are automatically also correct within $\mathbf{R}^+$

One finding when studying these changes was a more fine grained characterization of change of conception than those of *enrichment* and *reconstruction* described in theories of conceptual change (see chapter 2.3 for more details). For a conception to change, the lower level conception sometimes needs to be *reconstructed*, when the interpretation could be seen as a misconception. Interpreting 2 - 5 as 3 could be a misconception about subtraction as a commutative operation in par with addition, and would in that case need a reconstruction. However, it could also be a misunderstanding of how subtraction is written symbolically. It is not necessarily a misconception about subtraction, but about the connection between the symbolic representation of subtraction and subtraction as a mathematical concept. A student might be uncertain about whether 2 - 5 means take 2 from 5 or take 5 from 2, but sure about the different meanings of the two actions. However, a student might also see 2 - 5 as the distance between 2 and 5 on a number line, and a distance is always positive. Changing to a level 1 interpretation here is not automatically an improvement. Interpreting subtraction as a difference could be a good starting point for introducing directed differences so that a change from level 0 straight to level 2 is made possible. In the data some students go from level 0 to level 1 and others from level 0 straight to level 2 on this task. From a metaphor perspective the trajectory could depend on which metaphor for subtraction that is being extended. This will be discussed later in the chapter.

In some cases the interpretation appropriate at a lower level needs to be *specified*, as in the change from level 1 to level 2 in the example 2 - 5. Subtracting a larger number from a smaller number cannot be done in $\mathbf{R}^+$, but it can be done in $\mathbf{R}$. The students need to change their prior knowledge that was based on experiences in $\mathbf{R}^+$. It is not a misconception though, since it is true in $\mathbf{R}^+$. Going through a change of conception here does not involve deconstructing or abandoning the previous concept, only specifying it and making clear the circumstances when the old conception is valid and when it is not valid, and constructing a new and different conception in the new number domain.

In the second example about the interpretation of -2, a level 0 response does not pay any attention to the minus sign at all. Changing into a level 1 interpretation involves *becoming aware* of the sign as something relevant for the interpretation of what is there. Changing from level 1 to level 2 involves the *incorporation of a new idea* without abandoning the old interpretation. The minus sign is given a new function of polarity, of relating the number to one side of zero, or to an opposite number. This new idea does not change the old interpretation, i.e. the minus sign can still be interpreted as a subtraction sign.

> There are many instances where leading mathematicians have tried to dramatically change or restrict mathematics, but instead have succeeded only in the creation of new mathematics, not in replacing the old. (Martínez, 2006, p 173)

To summarise; four different kinds of changes in conceptions that occurred when extending the domain of numbers from $\mathbf{R}^+$ to $\mathbf{R}$ were found. These were:

- ~ Reconstructing a misconception.
- ~ Specifying conditions for a previous conception and constructing a new and different conception in the new domain.
- ~ Becoming aware of new features.
- ~ Incorporating new ideas to make the concept richer and more complex.

When all the interview data that was possible to categorise was put into a large table and all instances of change were highlighted, some interesting patterns appeared. First of all, the patterns of change over the years were quite different for different students, and could be described as different learning trajectories. Secondly, change did not occur only up levels but also down levels. Thirdly, different questions or topics in the interview showed very different patterns of change. Only some questions showed an expected pattern of change where most students change up levels, other topics seemed very resistant to change, and yet others exposed a lot of change down levels. These specific topics were then looked into in more detail and a deep analysis of the reasoning involved was done in search of explanations for these different patterns of change. The use of metaphors in particular and their possible contribution to the change was in focus.

7.1 Conceptualising negative numbers and zero: the size of numbers.

There are two questions in the interviews that treat the size[41] of numbers, particularly around zero. These are question 1 (Q1) and question 8 (Q8). In Q1 it is possible to only consider numbers as quantities but in Q8 negative numbers are included, and the question implies a comparison, a relation, between the numbers. The interviewer only talks about *numbers* and does not use the terms *positive number*, *negative number* or *minusnumber*[42] unless it is first used by the student. Symbolically, however, negative numbers appear in the questions. Typically the time spent between Q1 and Q8 is about 15 minutes. The effect of the different formulations of these tasks is that although a student in Q1 does not think of or relate to negative numbers, she is faced with them in Q8.

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[41] Here the term size is used instead of value since in Swedish the word *size* (storlek), rather than *value* (värde), along with the related words *small* (liten) and *big* (stor), are the words used in school mathematics. In this thesis the term value is used to distinguish between two aspects of size: value and magnitude.

[42] In Sweden it is common to say *minus-numbers* instead of *negative numbers*, which is also the case in the class studied here.

***Q1*)** Give the student a sheet of paper
Ask the students to write a number, any number. Is it a small or a large number?
Write a smaller number, write an even smaller number.
Write the smallest number you know.
Which is the smallest number there is? Is there no smaller number than that?

***Q8*)** Show two cards with [-1] and [2].
Which of the two is the largest/smallest number?
Show a card with [-4] Can you tell me a number that is smaller than this number?
Show these five numbers one at a time and ask if it is a number. When all are on the table, ask the student to order them according to size. 20, -5, 0, 8, -16, in grade 8 also 0,02 is included.

In grade 7 the textbook has introduced a number line above zero and defined size of numbers with the words: "The further to the right a number is placed on the number line the bigger it is"[43]. The size of a number is only related to the value, or the order of the number relative to zero, and not at all to quantities. In grade 8 the chapter on negative numbers has the title: "Numbers that are smaller than zero"[44]. In this book the size of a number is related to the bank account and "having and owing money" as well as to an extended number line. In the owing money metaphor it is not evident that owing more money is represented by a smaller number. It turns out that although the term used by the researcher is *a small number*, when students were asked to explain why for example -16 is smaller than -5 they start talking about *a number below zero* or *far away from zero*, thus using terms with a different metaphorical underpinning. Embodied experiences of quantities can be referred to as small, whereas distances and places along a path can be referred to as below or far away. Numbers incorporate both of these features.

Findings

Several aspects of number sense stand out from the interviews as important but not obvious to the students when they speak about the size of numbers:

~ There are numbers that are smaller than 1
~ Zero is a number
~ There are numbers that are smaller than zero
~ Decimals and halves are not smaller than negative wholes (they are if we speak quantity or magnitude, but not if we speak numbers or value)
~ The size, or value, of a number can be seen as a numbers relation to zero
~ Negatives are ordered in an opposite way to positives in relation to zero
~ Counting negatives is done in an opposite way to counting positives (the larger the quantity the smaller the number)

[43] In Swedish: *Ju längre till höger på tallinjen ett tal ligger desto större är det* (Carlsson et al., 2001, p 14).
[44] In Swedish: *Tal som är mindre än noll* (Carlsson et al., 2002, p 16).

- ~ The duality of size:
  - Size of number (value) in not the same as size of quantity (magnitude)
  - The size of a number can be described in two different ways illustrating value and magnitude, and for negative numbers these diverge.
  - Zero is very useful as a benchmark number. It is easier to order signed numbers correctly in **R** if zero is included.

The above listed aspects of negative numbers are the same as those Ball (1993) described that her 8 and 9 year olds struggled with. Some students use metaphorical reasoning as a means of describing and making sense of numbers. Two things can be seen in the excerpts:

- ~ An extended *object collection* metaphor (including negative quantities) is an aid to see negative numbers as numbers and not as subtractions but does not make it easy to see the value of numbers. Within this metaphor zero can only be viewed as nil-zero i.e. as 'nothing', no quantity.
- ~ A *numbers as locations along a path* metaphor embodied in the extended number line on both sides of zero makes it possible to see negative numbers and relate their size to zero. Within this metaphor zero is a location (point) just like other numbers although it has a special position. The unified number line, where numbers grow bigger in one direction and smaller in the other direction (the increase direction typically shown by an arrow on the positive side) serves as a metaphor for value of numbers, whereas the divided number line is more complex, incorporating more of the magnitude of numbers. Many of the students express a conception of the divided number line, preferring a number line with a strongly marked zero point, "the middle", and an arrow on both sides indicating that the line and the numbers "keep going" in both directions (see also section 7.4 for further results concerning the number line).

Empirical data

Students' responses to questions Q1 and Q8 are shown in table 7.2, categorised into the three levels in the corresponding legend below. Emphasis is put on when change occurs. The legend shows a summary of the characteristic responses and how they were graded according to level as either; incorrect, correct within $\mathbf{R}^+$ or correct within **R**. Up a level and down a level changes are emphasised.

TABLE 7.2: Overview of categories of all students' responses to Q1 and Q8, according to number domain levels described in the legends below.

	Q1: smallest number			Q8: 1 or -2 smallest			Number smaller than -4			Order according to size of number		
	G6	G7	G8	G6	G7	G8	G6	G7	G8	G6	G7	G8
Anna	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**		$\mathbf{R}^+$	**R**	**R**	**R**	**R**
Axel	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Elke	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Erik	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Fia	$\mathbf{R}^+$	$\mathbf{R}^+$	0	**R**	**R**	**R**	0		**R**	**R**	0	0
Freddy	$\mathbf{R}^+$	$\mathbf{R}^+$	$\mathbf{R}^+$	$\mathbf{R}^+$	$\mathbf{R}^+$, **R**	**R**	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	**R**	**R**
George	**R**	**R**	**R**	**R**		**R**	**R**		**R**	**R**		**R**
Hans	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Lina	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	**R**	**R**	**0**	**R**	**R**	$\mathbf{R}^+$	**R**	**R**
Linda	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	$\mathbf{R}^+$	**R**	**R**
Lotta	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	$\mathbf{R}^+$	**R**	**R**	**R**	0	**R**	**R**	**R**	**R**
Malin	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	0	0	**R**	0	**R**	**R**
Martina	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Olle	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	**R**	**0**	**R**	**R**	$\mathbf{R}^+$	**R**	**R**
Ove	$\mathbf{R}^+$	**R**	**R**	**R**	**R?**	**R**	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	**R**	**R**
Paula	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	
Petra	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Sean	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	**R**		$\mathbf{R}^+$	**R**		0	**R**	
Tina	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Tomas	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Viktor	$\mathbf{R}^+$	$\mathbf{R}^+$	**R**	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	**R**	**R**	$\mathbf{R}^+$	**R**	**R**

level	answer
$\mathbf{R}^+$	positive
$\mathbf{R}^+$	0
R	negative

level	answer
$\mathbf{R}^+$	1
R	-2
R	depends

level	answer
0	-3
$\mathbf{R}^+$	3
R	-5
R	depends

level	answer
0	20 8 0 -16 -5
$\mathbf{R}^+$	20 16 8 5 0
$\mathbf{R}^+$	impossible
R	20 8 0 -5 -16

Number – Task

The very first thing the students are asked in the interviews is to write a number, any number they choose. On that request, more than half of the students in the class write an expression; e.g. 1+1 [Freddy G6], $12 \cdot 3$ [Anna G6] or 6 + 5 = 11 [Fia G6]. This meaning of the Swedish word for number, *tal*, is frequently used in the mathematics classroom since it is the way students and teacher talk about the tasks they are working on, how many tasks they have done, which task is a difficult one etc. Freddy shows how strong this interpretation is when asked to compare 1 and -2, as seen in excerpt 7.1.

EXCERPT 7.1: Freddy interview G6, Q8

6-271	Freddy	yes, but, that's a minus number [-2]
6-272	I	yes?
6-273	Freddy	but, only, but you know, that well that's not a number now
6-274	I	no?
6-275	Freddy	so well, the minus sign is really, unnecessary.
6-276	I	yes?
6-277	Freddy	So the two, is like a whole two. And the two is bigger than the one.

In G6 Freddy refers to -2 as "not a number now" [lines 6-273 to 6-274]. In his interpretation it would have been a "minusnumber", i.e. a subtraction task, only if there had been a number to subtract from, e.g. 6-2. In G7 his first interpretation of -2 is that it is only "half a number", i.e. half a subtraction task, before he also interprets it as coming "after 0".

Decimals and negatives on a number line: Fia

In Q1 when asked to write the smallest number, 16 of the students answer in R^+ in G6, 9 in G7, and still 6 out of 20 in G8. The comparisons in Q8 show a far larger proportion of answers in R all over the three interviews, suggesting that some students include negative numbers in their number domain when faced with them but do not change their conception of number to include them when thinking about numbers in general.

In Q8 the students were asked to relate different numbers to each other. The table shows clearly that all students except Fia can correctly order positive and negative numbers. Fia's learning trajectory is shown in excerpts 7.2 and 7.3.

EXCERPT 7.2: Fia interview G6, Q1

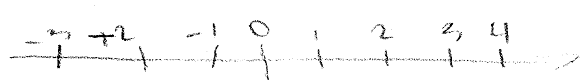
6-01	I	a number smaller than 1?
6-02	Fia	don't suppose there is any. Or there is…
6-03	I	yes is there?
6-04	Fia	...well...zero point something, no...
6-05	I	yes? What do you think? Yes, yes write down what you thought of…(Fia writes 0,5) do you think that's a number
6-06	Fia	no. or
6-07	I	not really?
6-08	Fia	or well
6-09	I	why not? Or how did you think?
6-10	Fia	I don't know. In fact.
6-11	I	no? How did you think?
6-12	Fia	It's like a half. It's kind of like not a whole number

As seen, in G6 Fia does not consider numbers smaller than 1 as numbers since she relates the word number to wholes, not parts [line 6 to 12]. In spite of that she does order the numbers in Q8 correctly in R. In G7, Fia only orders the two numbers 20 and 8, declaring about the others that "they aren't numbers" so "they don't have size". In G8 she treats them all as numbers but does not know where to place the decimal number, as seen in excerpt 7.3.

EXCERPT 7.3: Fia interview G8, Q8: Fia puts the numbers like this:: -16 -5 0,02 0 8 20

8-801	Fia	I don't know where to put that one (points at 0,02)
8-802	I	//0,02?
8-803	Fia	// that feels a bit wrong
8-804	I	yes? Why does it feel wrong?
8-805	Fia:	because it is zero point, those are like minus, that's much easier, but this one…
8-806	I	where else could it be placed?
8-807	Fia	(...) I don't know
8-808	I	no?
8-809	Fia	I'll place it here
8-810	I	so, is it bigger, is it smaller than 0?
8-811	Fia	yes. I think of the number line all the time. Then this is how it is.

When Fia orders according to R in G6, she says that -16 is smaller than -5 because it is "more minus". She is very uncertain about decimal numbers as shown in her response to Q1 in G6, where she seems to connect the word *number* to natural numbers [excerpt 7.2 line 6 to 12]. In the same interview she also suggests -3 as smaller than -4, indicating that she is not quite certain about the size of negative numbers. However, the fact that she orders them correctly when faced with five numbers including negatives, positives and zero suggests that zero is helpful as a benchmark number. In G7 she refuses to order the negative numbers altogether. In G8 she orders all the integers correctly in **R**, but has problems with the size of decimal numbers, stating in Q1 that 0.000…1 is smaller than 0 and placing 0.02 between -5 and 0, as shown in excerpt 7.3. She explains this by relating to a number line [excerpt 7.3 line 8-811], but when viewing her answers to the number line questions in the interviews (Q10), there is great uncertainty about the number line. Fia says she does not know how to draw a number line in G6 or G7. In G8 she draws a correct number line exhibiting positive and negative integers but she is still uncertain about where to place decimals.



Fia's number line in G8

These responses indicate that a number line is not something that Fia makes use of as a number metaphor in a general sense; that her experiences of number lines on rulers, scales etc are not connected to her sense of size and order of numbers.

Decimals and negatives: Anna

Two of the low achieving students in the class are Anna and Freddy. Both of them give responses mostly within R^+ in G6, and expand their number domain slowly. In excerpt, 7.4, Anna is asked about the smallest number.

EXCERPT 7.4: Anna interview G6, Q1

6-32	I	Can you write a number that is smaller than 1?
6-33	Anna	...there is 0 but eh…maybe it doesn't count…
6-34	I	no? ...why wouldn't it count?
6-35	Anna	I don't know. Should I write it then?
6-36	I	You sound doubtful whether it is a number or not?
6-37	Anna	yes
6-38	I	yes. Is there none other smaller than 1? Is it small the smallest number?
6-39	Anna	mm, only if you count a half.
6-40	I	mm?
6-41	Anna	but that's not much to count with in many calculations [45]
[...]		
6-46	I	ok. So if I say, if I ask you one more time to tell me which is the smallest number there is, what would you answer? … what do you think?
6-47	Anna	1

In G6 Anna only thinks of natural numbers as proper numbers. She mentions 0 which doesn't really count as a number [line 6-33], and a half [line 6-39], but settles on 1 as the smallest number. In G7 Anna mentions decimals and suggests 0,000….2 as a very small number and in G8 she suggests a negative number. However, as shown in excerpt 7.5, the relationship between decimals and negative wholes is not evident for Anna.

EXCERPT 7.5: Anna interview G8, Q1. Anna has suggested -1 and -3 to be smaller than 1.

8-51	I	ok. Can you write a number that is very very small?
8-52	Anna	hmm (...) this will look strange (writes 0,000000000000002)
[…]		
8-58	I	mm. Is that smaller than, minus 3?
8-59	Anna	... I think so.
8-60	I	mm…how about 0? Is that a number?
8-61	Anna	... hmm... I don't think so
8-62	I	no? is this what you wrote here is it smaller or bigger than 0
8-63	Anna	it's smaller than 0 I think
8-64	I	mm?
8-65	Anna	no, it's bi, noo… it's probably bigger than 0.
8-66	I	...ok?... it's bigger than 0?
8-67	Anna	yes
8-68	I	yes... are there numbers smaller than 0?
8-69	Anna	well then you have this thing with minus sss

[45] In Swedish: *men det finns ju inte så mycket att räkna med så mycket i tal.* This is a difficult sentence to translate. The last word, *tal,* is here interpreted to mean task or calculation rather than number.

8-70	I	yes?
8-71	Anna	something
8-72	I	ok? So minus 3, is smaller than 0?
8-73	Anna	mm
8-74	I	but that [-3] is still bigger than that? [0,000000000000002] or? You said before that this was smaller than 0
8-75	Anna	that [0,000000000000002] is bigger than 0, and that [-3] is smaller than 0.
8-76	I	yes? So how about that, compared with that then? It this, 0 point?
8-77	Anna	that [0,000000000000002] is bigger than, that [-3]. It must be
8-78	I	ok, so you change your mind??
8-79	Anna	yes
[…]		
8-85	Anna	because I didn't think, well, minus then it's below 0
8-86	I	mm
8-87	Anna	but you know this, this could be as long as you like but still be above 0.

Anna is still in G8 uncertain about the size of decimals and negatives [lines 8-51 to 8-58] until she is told to compare them with zero. The comparison makes her relate the numbers correctly, although she still does not consider 0 a number [line 8-60 to 8-61]. When numbers are referred to as small or big, Anna relates to quantities, but when she compares the numbers with 0 she shifts to a *path* metaphor describing numbers as below or above 0 rather than smaller or bigger.

Concerning the comparison task in Q8, in G6 and G7 Anna does not quite know how to compare a natural number with what she interprets as an operation. Something smaller than -4 is not understood at all in G6 and in G7 Anna suggests "a half, because a half is always smaller". The negative numbers -5 and -16 are not accepted as numbers until she is asked to order them, when she suddenly exclaims that they are temperature numbers and it is like a thermometer, and so she orders them correctly in **R**. In G8 Anna speaks directly about the numbers relation to zero and can thereby correctly explain their size, as shown in excerpt 7.6. Again, zero works as a benchmark number.

EXCERPT 7.6: Anna interview G8, Q8. Anna has ordered the numbers -16 -5 0 8 20 and is explaining why -16 is the smallest.

8-325	Anna	well it's biggest and smallest in its own way. It's biggest on its minus side, but it's smallest if you count from the plus side
8-326	I	yes? ok, yes that I think was well explained
8-327	I	But how did you know that minus should, -16 should come before -5?
8-328	Anna	because the zero is kind of like a centre
8-329	I	yes?
8-330	Anna	and then you get minus 1, minus 2 and so on all along

Initially, it is not clear to Anna that -2 is a number, she interprets it as an operation. When she can think of the numbers metaphorically as temperatures she starts relating them to zero and can do two things: first she accepts negative numbers as numbers, although initially only as temperature numbers, and secondly she orders them correctly in size in relation to zero. For Anna the duality of size is found within the number line metaphor and she does not need to relate to quantity to see it. She interprets the numbers as if they were on a unified number line, where numbers grow smaller in one direction and bigger in the other, and at the same time as a divided number line where values grow bigger the further away from zero they are.

Duality of the size: Tomas sees only the numerical size

Concerning the duality of size of a number, its value and magnitude, Tomas is a student who focuses on the value of a number and its relation to zero rather than its magnitude. Even when he is prompted to compare with a quantity (debt) and he acknowledge its magnitude, he maintains his understanding of value.

EXCERPT 7.7: Tomas interview G8, Q8

8-239	I	can you say a number that is smaller than this? [-4]
8-240	Tomas	minus 5
8-241	I	mm, how do you know that minus 5 is smaller than minus 4
8-242	Tomas	because it is, further away from 0, that is, downwards. Than minus 4
8-243	I	mm but if you think of minus 4 as a debt?
8-244	Tomas	mm
8-245	I	then minus 5 would be a bigger debt
8-246	Tomas	yees, eh,… yees, but eh… yees…
8-247	I	but it's still a smaller number?
8-248	Tomas	yee…es
8-249	I	or? ... are you sure?
8-250	Tomas	yes

Tomas is a student who seems to have a good sense of numbers in **R** already in G6. He acknowledges zero as a number and negative numbers as smaller than zero. He relates clearly to numbers as points located along a vertical path (up-down). In G7 he suggests -20 as smaller than -4 and explains it by saying: "because after zero comes minus 1, and then, the lower it gets, the more minus it gets". In G8 [excerpt 7.7] he repeats the same explanation relating the numbers to zero as a benchmark number [line 8-242] but is then provoked to think about numbers as debts, where the magnitude of a number is highlighted. He dismisses this aspect of size as irrelevant [lines 8-246 to 8-250]. This could be seen as an example of a situation where the unified number line has taken over as the mathematically valid conception.

Duality of the size: Malin has two different notions of size

Some students answer differently when comparing two numbers on opposite sides of zero as opposed to comparing two negative numbers. On the latter several students suggest -3 as smaller than -4, which could indicate that they think of negative numbers as quantities or measures (i.e. 3 m < 4 m; a debt of 3 is smaller than debt of 4). Here the metaphorical interpretation of number connects numbers with quantities, as in collections of objects or measures, and correctly expresses the magnitude of the number. In contrast, an interpretation of numbers as points along a path (number line) would be more useful in order to compare the value of numbers; to see that -5 is smaller than -4. Malin is a student who gives two different interpretations of the size of number. In G6 Malin suggests -3 as a smaller number than -4 and orders -5 < -16 < 0, indicating that she focuses on the magnitude even though she is aware that negative numbers are considered smaller than 0. In G7 she suggests -2 < -4 but can now manage to order the five integers correctly in **R**, as shown in excerpt 7.8, ordering -16 < -5. She is content with the two different interpretations of size but without distinguishing between them.

EXCERPT 7.8: Malin interview G7, Q8. Malin has ordered the numbers: 20, 8, 0, -5, -16.

7-803	I	mm, how do you know that minus 16 is supposed to be smaller than minus 5?
7-804	Malin	eh because minus 5 is closer to 0 than what minus 16 is
7-805	I	mm. if we go back to the previous question where you had minus 4, and then you said something smaller than minus 4 and you said minus 2.
7-806	Malin	yes
7-807	I	do you hold on to that?
7-808	Malin	yes, I do.

Duality of the size: Freddy mixes two metaphors

Freddy sees a minus sign as an operation and initially only relates size to magnitude. In G7 he starts relating numbers to zero but because he speaks of negative numbers as debts, his focus is on the quantity and not the position relative zero. He explains that -16 is smaller than -5 by saying "because, when you have minus then, the bigger the number is the smaller it gets, I think". In G8 Freddy expresses the duality of the size of negatives by mixing two metaphors.

EXCERPT 7.9: Freddy interview G8, Q8

8-238	I	here are two numbers, which is the biggest? [1] [-2]
8-239	Freddy	1
8-240	I	why?
8-241	Freddy	because minus 2 is like a debt. It's like eh, after 0
8-242	I	mm? Can you say a number smaller than this? [-4]
8-243	Freddy	yes, minus 5
8-244	I	mm, how do you know minus 5 is smaller than minus 4?
8-245	Freddy	because it's a bigger debt, if that's how you think of it

Minus 4 is a debt [line 8-241] and therefore has a bigger value than minus 2 [line 8-245], but at the same time it is located after 0 [line 8-241]. Freddy indicates that it is a special way of thinking but lacks a discourse that makes the distinction between the two aspects of size.

Duality of the size: Lina shows one, then another and finally a dual sense of size of number

Lina's learning trajectory goes through three distinct phases: from an interpretation of size connected to magnitude in G6, she connects to value by the relation to zero in G7, and then explicitly expresses the duality in G8, as shown in excerpt 7.10. However, she does not have a mathematical discourse to explain the duality and is uncertain of which aspect of size she is being asked about.

EXCERPT 7.10: Lina interview G8, Q8. Lina has chosen 1 as larger than -2, after long consideration.

8-801	I	mm... you are a bit doubtful?
8-802	Lina	yees, because I think, or eh, if you think like this, the number line
8-803	I	yes?
8-804	Lina	then we have 1 ... like, bigger in a positive way, or do you understand what I mean?
8-805	I	mm
8-806	Lina	and 2 is like minus and that's always smaller so
8-807	I	mm?
8-808	Lina	but the 2 itself is still bigger than the 1.

Comments about conceptualizing negative numbers

Many words in a language have different meanings and if participants are expected to make the same associations, it is necessary to make clear to all which meaning is the currently relevant one. A main issue seems to be the need for clarity in the discourse. The importance of distinguishing between number and task when talking about integers is an empirical example of problems and misunderstandings that arise when the discourse is unclear.

A main issue for developing number sense concerning the conceptualization of negative numbers and zero suggested by these results is awareness of the duality of size of numbers. Again, clarity of the discourse would be profitable. Words that help students talk about the difference between value and magnitude of a number would probably make it easier for them to see the difference and connect to prior ideas about size and order. Since many students make use of an embodied *path metaphor* with zero as a point of reference on the path when comparing numbers, it would probably be useful to connect to that by explicitly using a number line with zero as a benchmark number. That is in line with recommendations by Freudenthal (1983) and Bruno and Martinón (1999) but contradicts those of Küchemann (1981) and Gallardo (1995). However, as

Galbraith (1974) expressed, conceptualising these numbers is one thing, and operating with them is something else. A lot of work could be done in early grades to conceptualise these numbers so that by the time it is necessary to operate with them they could have become reified mathematical objects.

7.2 Subtraction 1: accepting a negative difference[46]

As shown in chapter 1, subtraction is closely related to the concept of negative numbers. Knowledge of the effects of subtraction on numbers are important aspects of number sense, and Sfard (1991) claims that it is when a person becomes skilful in performing subtractions that negative numbers are interiorized. When extending the number domain from unsigned to signed numbers one of the first conceptions of a negative number is to accept a negative difference. In the historical evolution of signed numbers, a negative difference between two positive numbers was an early feature. It could be interpreted as missing objects, or as going below a certain point of reference. This acceptance of a negative difference is also fundamental in the definition of negative numbers. In a sense, negative numbers were invented so that we could solve subtractions of a larger number from a smaller number. It will be shown in this section that students learn to accept a negative difference along different learning trajectories.

In order to investigate whether students always try to make sense of their calculations or if they sometimes rely on intuitions, procedures and number facts, two very similar subtraction questions Q5 and Q7f were asked in different ways. About 10 minutes into the interview six quick response questions (Q5) were asked as a contrast to the open ended questions and think aloud protocols of the rest of the interview.

> ***Q5***) Show these on flash cards. No follow-up questions.
> Do some quick calculations. Only say the answers.
> 4 + 2 =
> 6 - 3 =
> 7 + 11 =
> 2 - 5 =
> 12 + 23 =
> 6 - 27 =
>
> ***Q7f***) Show [3 - 7] on a card. Ask: What does it mean?

The first two tasks in Q5 were intentionally made simple in order to infuse self confidence and make the students comfortable with the new type of question.

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[46] In this context the *difference* is used strictly in the mathematical sense: as the result of a subtraction. In Swedish the mathematical term *differens* would be used, not the everyday word *skillnad*, which was the main focus in chapter 6.3. In English the mathematical word is the one used in everyday situations.

As expected the students answered the first two tasks quickly and correctly. This result suggests that they have a procedural knowledge of addition and subtraction and know their number facts for single digit addition and subtraction; i.e. they know which number fact in the part-part-whole relationship they are supposed to retrieve when asked to add or subtract. The only exception is Victor who added both tasks.

At the very end of the interview some questions were asked about the number line. In G7 and G8 the students were asked in question Q10 if the calculations 9 + 2 and 5 - 8 could be shown on the number line. Students' interpretations of 5 - 8 on the number line is related to the subtractions 2 - 5 and 3 - 7 and is therefore included in this comparison.

Findings

Two aspects of number sense stand out from the student interviews as important to develop in the process of accepting a negative difference but not evident for all students. These are:

~ Subtraction is not commutative. If subtraction is misconceived as commutative, a negative difference will never appear.
~ Magnitude and direction of a difference are two things that can be attended to separately or simultaneously.

Concerning the use of metaphorical reasoning the data showed that the use of metaphors was not always helpful for understanding subtraction within the extended the number domain:

~ *Numbers as locations and movements along a path*, represented on a number line of any kind (number line, thermometer, ruler…), was not very common in these students' ways of reasoning. Referring to negative numbers as "under 0" (position along a path) and as "smaller than 0" (object collection or object construction) seemed equally frequent, but more common was just the mathematical statement that it was "minus".
~ The metaphorical meaning of *subtraction as taking away a collection of objects* seemed difficult to extend to negative differences since taking away a larger collection of objects than the one you start with usually doesn't make sense. The only context in which this metaphor seemed to make sense was a context of money where there is a metaphorical concept of owing or becoming/being indebted, to represent the objects taken away that are not there to be taken away.

Empirical data

Students' responses to the questions above are shown in table 7.3, categorised into the three levels in the corresponding legend below. Emphasis is put on when change occurs. The legend shows a summary of the characteristic responses and how they were graded according to level as either; incorrect, correct within $\mathbf{R}^+$ or correct within $\mathbf{R}$.

TABLE 7.3: Overview of categories of all students' responses to questions Q5, Q7f and Q10 according to the number domain levels described in the legends below.

	Q5: 2 - 5 = (quick response)			Q7f: what does 3-7 mean?			Q10: 5-8 on the number line	
	G6	G7	G8	G6	G7	G8	G7	G8
Anna	0	**R**$^+$	**R**		**R**$^+$	**R**	**R?**	**R**
Axel	0	0	**R**	**R**$^+$	**R**$^+$	**R**	0	**R?**
Elke	0	**R**	**R**	**R**$^+$	**R**$^+$	**R**	**R**	**R**
Erik	0	**R**	**R**	0	**R**	**R**	**R**	**R**
Fia	0	0	**R**	**R**$^+$	**R**$^+$	**R**	**R**	**R**
Freddy	**R**$^+$	0	**R**$^+$	**R**$^+$	**R**$^+$	**R**	0	0
George	0		**R**	**R**		**R**		**R**
Hans	0	**R**	**R**	**R**	**R**	**R**	**R**	**R**
Lina	0	**R**	**R**	**R**$^+$	**R**$^+$	**R**	**R**	**R**
Linda	0	0	**R**	**R**$^+$	**R**	**R**	0	**R**
Lotta	0	**R**	**R**	**R**$^+$	**R**$^+$	**R**	0	**R**
Malin	0	**R**	0	**R**$^+$	**R**	**R**	**R**	**R**
Martina	R	**R**	**R**	**R**$^+$	**R**	**R**	**R**	**R**
Olle	0	0	**R**	0	**R**	**R**	0	**R**
Ove	0	0	0	**R**$^+$	**R**	**R**	0	0
Paula	0	0	**R**	**R**	**R**	**R**	**R**	0
Petra	0	0, **R**	**R**	**R**	**R**	**R**	**R**	**R**
Sean	0	0		**R**$^+$	**R**$^+$		0	
Tina	**R**	**R**	**R**	**R**	**R**	**R**	0	**R**
Tomas	**R**	**R**	**R**	**R**	**R**	**R**	0	**R**
Viktor	0	0	0	0	0	**R**$^+$	0	0

level	answer
0	3
R$^+$	impossible
R	-3

level	answer
0	4
0	3 to 7
R$^+$	impossible
R$^+$	less than 0
R	debt of 4
R	-4

The categorization of quick responses in Q5 was very simple. Responses that focussed on the absolute difference 3 were categorised as level 0. Responses of the type: "you can't subtract 5 from 2" or "you can't do 2 minus 5, it has to be the largest number first" were categorized as correct in **R**$^+$. For Q7f, where students were asked to reason about the meaning, another interpretation was also categorized as level 0; the interpretation of 3 - 7 as "from 3 to 7". In this interpretation the minus sign is seen as a linguistic symbol rather than a mathematical symbol. In writing, a horizontal line similar to the minus sign is often used with that meaning, relating it to the difference between the two numbers when ordered. This could be seen as a metaphorical interpretation of numbers as positions along a path and as measures (distance between two positions), only it is the wrong way. Mathematically, 8 - 2 is a representation of the number you must add to 2 to get 8, i.e. from 2 to 8. So mathematically; 3 - 7 would mean from 7 to 3 rather than the semantic interpretation from 3 to 7. Yet another meaning of the expression 3 - 7 was given by several students in addition to other meanings, but not noted separately in the table; that of 3 - 7 as a goal difference in a soccer game. Such an interpretation could be related to subtraction as a difference followed by a discussion about the different meanings of a goal difference of 3 - 7 and 7 - 3. However, in this data that interpretation never came up in a classroom situation or to a question in the interview when a solution was asked for. A goal difference is more commonly referred to with the expression including both numbers (3 - 7) rather than the factual difference (-4). Responses that were categorised as correct in $\mathbf{R}^+$ were either those who insisted that a subtraction has to have the largest number first or those that said in a general sense that it would be "less than nothing" or "less than zero". Finally, the interpretation of 3 - 7 as having 3, paying 7 and therefore having a debt of 4, or saying simply that it is "4 less than zero" were categorized as correct in **R**. This categorization could be discussed since such a meaning of negative numbers might indicate that the student still conceptualizes numbers as unsigned. However, a number less than zero, a missing quantity or a debt could also be seen as an early conception of a negative number. All responses including the words "negative 4" or "minus 4" were also categorized as correct in **R**.

As shown in table 7.3, in G6 most students gave a level 0 answer on the quick response question but a level $\mathbf{R}^+$ answer on the reasoning question. These results suggests that when stressed by time students retrieved number facts and interpreted subtraction as a difference in quantity, i.e. the number asked for is the missing part in the part-part-whole relation. When given time to reflect, most of the students were thinking within the domain of natural numbers, interpreting subtraction as taking away a quantity, and thus finding it impossible to take away a larger quantity than the one you have to start with. On the quick response question many students went straight from level 0 to level **R**, whereas on the reasoning question most students passed through the stage of $\mathbf{R}^+$. Surprisingly, illustrating subtraction as movements or arrows on the number line did not stand out as a well known representation; 10 out of 21 students could not think of a way of showing 5 - 8 on the number line in G7, and one student (Anna) did it but with great uncertainty (categorized as **R**?). In G8 more students showed 5 - 8 as a movement from the location 5, and 8 steps (or a distance of 8) towards "the minus side" or "down", but 4 out of 20 students could still not think of a way of illustrating it.

It is clear from the table that the teaching sequence on negative numbers had some impact on the extension from unsigned to signed numbers concerning the acceptance of a negative difference. On each question seven students made the change up to level **R** in G8, after the teaching about negative numbers (not the same students on each question though). However, on Q5 and Q7f, this change was made for almost as many students already in G7, before the teaching sequence, indicating that other matters influence this extension of the number domain than explicit teaching of negative numbers. In the following sections some of the different learning trajectories are illustrated.

A negative difference accepted in the procedure: Lotta and Elke

Three students (Elke, Lina and Lotta) make the up level change to give a response in the domain of signed numbers (**R**) earlier on the quick response question than on the reasoning question, indicating that they first change their procedure of calculation, accepting a negative answer, and only later encompass a conceptual understanding of a negative difference. A typical response is that of Lotta, who in G7 correctly answers 2 - 5 = -3 but is very uncertain about 3 - 7 when asked about the meaning of it. She is on the way of extending her number domain to include numbers smaller than zero but can not express how to make sense of them, as shown in except 7.11.

EXCERPT 7.11: Lotta interview G7, Q7f: 3 - 7 (categorised as correct in $\mathbf{R}^+$)

7-701	Lotta	eh, 3 minus 7
7-702	I	mm, and what does it mean?
7-703	Lotta	well, that it is, is 3 minus 7, I don't know…
7-704	I	could you write like that?
7-705	Lotta	yes, you could I guess
7-706	I	what does it mean, if that's what is written?
7-707	Lotta	(...) I don't know, but it gets, smaller than 0 kind of (giggles)

Lotta relates the meaning of 3 - 7 to something smaller than 0, followed by an exclamation of uncertainty and a giggle [line 7-707]. A year later she seems more certain, as shown in excerpt 7.12.

EXCERPT 7.12: Lotta interview G8, Q7f: 3 - 7 (categorised as correct in **R**)

8-703	Lotta	no but look here, haha, well, you… you know you borrow from someone for example
8-704	I	yes
8-705	Lotta	and then perhaps … you can't pay back so you couldn't really afford it so then maybe it's minus 4, I don't know or, I don't know
8-706	I	no?
8-707	Lotta	it's hard to explain
8-708	I	could you use the way you talked about it first, to take away…?
8-709	Lotta	yes
8-710	I	what do you have in that case, and what do you take away?
8-711	Lotta	you take away 7, from 3
8-712	I	yes, could you do that?
8-713	Lotta	yees, but then it becomes minus
8-714	I	yes?
8-715	Lotta	but yes, of course you could…
8-716	I	eh, could you use that way you talked about difference?

8-717	Lotta	mm, yes
8-718	I	how would that be?
8-719	Lotta	but that becomes strange to take the difference! Because then it is… well the difference is 3 whole … eh … no it's, what is it? Will it be minus? Yes, yes minus4 is the difference! Silly me … eh yes
8-720	I	is that right? You sound doubtful?
8-721	Lotta	yes, no, but …
8-722	I	the difference between 3 and 7?
8-723	Lotta	well, ehh ... well the difference between those two numbers if you didn't have a minus would be just 4
8-724	I	yes?
8-725	Lotta	and then it would have been 4. But, when you take minus, because it goes kind of, these 3 that go down to 0, they are whole, and then, it gets … well I don't know
8-726	I	are these 7 not whole?
8-727	Lotta	well, but you kind of take them minus, like as if they come below 0
8-728	I	mm
8-729	Lotta	so I don't know

In G8, Lotta is more certain about some things; she starts by saying that 3 - 7 means "minus 4". When asked about a meaning she relates to borrowing money but cannot clearly relate the idea of borrowing to the negative number [line 8-705]. The interviewer prompts her by repeating the suggestion she had made earlier about the meaning of the *minus as taking away* [line 8-708 to 8-710] at which she answers that of course you can take away 7 from 3 [line 8-711 to 8-715]. This suggests that it is easier for Lotta to accept the discourse about taking away when talking about numbers as reified mathematical objects than it is when talking about numbers as collections of real objects (money), although the idea of subtraction as taking away is generated from the *numbers as objects collection metaphor*. When the interviewer introduces *minus as difference*, also previously suggested by Lotta, she is more concerned [line 8-719]. Can a difference be negative? She uses the term, "3 whole" instead of "3 positive" [lines 8-719 and 8-725 to 8-727], an indication that she doesn't see positive numbers as signed numbers and thus have not fully extended her number domain from unsigned to signed numbers[47]. She concludes that a difference can be negative when it "comes under 0" [lines 8-719 and 8-727] but is not certain about it [line 8-729]. A possible interpretation is that Lotta uses a counting down strategy and zero as a benchmark number.

Lotta's G8 response is categorized as correct in **R** since she is certain about the fact that 3 - 7 means -4 (mathematically it has the same value) but she is not much helped by her metaphorical reasoning in justifying or explaining her answer. It could be said that Lotta shows a better procedural than conceptual

[47] A similar, commonly used phrase in the classroom discourse is to compare negative numbers with "ordinary numbers".

knowledge concerning subtraction with a negative difference. However, it could also be suggested that she has by now reified the concept of subtraction to such an extent that it does not need to be verified metaphorically anymore. Taking away a larger number from a smaller number is not a problem in the world of numbers; it is only a problem in the world of quantities. 3 - 7 is simply another way of writing the number -4. In Lotta's case this insight came first in the quick response question (2-5 = -3) in G7, before any teaching about negative numbers. Lotta does not relate to a number line or *motion along a path* metaphor when solving subtractions. When explicitly asked to show 5-8 on the number line in G7 she cannot think of a way to do it. In G8 she has incorporated the number line in her discourse and can suggest that you start on 5 and move 8 steps down, although she denies that she would solve a task that way. When asked how she would solve the subtraction 5 - 8 she reveals that she uses 0 as a benchmark number, first subtracting 5 to get to 0, and then subtracting the rest, 3, from 0 to get a negative number. That is a way of reasoning that could be related to a number line but just as well to an *object collection* metaphor. For Lotta, no metaphor is needed as these numbers in G8 seem to make sense to her as reified mathematical objects. Zero subtract three (0-3) is just another way of representing negative three (-3).

Elke's learning trajectory is similar to Lotta's, except Elke's uncertainty in G7 is strictly related to her use of metaphor, as shown in excerpt 7.13.

EXCERPT 7.13: Elke interview G7, Q7f: 3 - 7

7-234	I	would this be ok to write?
7-235	Elke	...it would, but it gets, small, it will get smaller than 0 then so it makes, well you need to count below
7-236	I	yes? Could you do that?
7-237	Elke	ehm...yees but not if want to work out this, if there are 7 people and then 3 go away, and then how many are left. Then you couldn't count like that, then you would have to take 7 minus 3 instead
7-238	I	yes?
7-239	Elke	because in that case, otherwise the result is that there aren't any people at all
7-240	I	no?
7-241	Elke	or like minus 4
7-242	I	yes? Could that be?
7-243	Elke	no (giggles)

In G7 Elke answers correctly on the quick response question: 2 - 5 = -3. On the reasoning question she first says that it will be smaller than 0 [excerpt 7.13, line 7-235] She talks about number as a position along a vertical path stating that "you will need to count below", and then explains that it is impossible if you think about object collections like collections of people [line 7-237]. When subtracting a larger object collection from a smaller one, there will be none left [line 7-239]. For Elke, it is not possible to be a negative number of people [lines 7-241 to 7-243], and so the subtraction does not make sense.

The following year Elke reflects on the meaning of 3 - 7 by referring to money, saying that you can owe money. It seems easier for Elke to accept a lack of money than a lack of people as a collection of objects represented by a negative number. Perhaps this is because we have a special word in the vocabulary for a missing amount of money, but no such word for a missing amount of people. It could also be an effect of the teaching sequence where debts were frequently used to illustrate negative numbers. However, although Elke's response in G8 was categorized as correct in **R**, she is still somewhat uncertain about her own justification, as shown in excerpt 7.14.

EXCERPT 7.14: Elke interview G8, Q7f: 3 - 7

8-182	I	would this be ok to write?
8-183	Elke	mm
8-184	I	yes, so what would it mean?
8-185	Elke	that you eh, well you could say that if you have 3 kronor … well no I, I can't explain it ehm … you take 7 away from 3 but it will be a negative number anyway. So you could say that you, like, owe 4

Elke says that when you take 7 away from 3, you will get a negative number anyway. She knows that the answer will be a negative number, the idea of interpreting it as owing money is only a figure of speech, indicated by the words "you could say that you, like, owe 4" [line 8-185].

A negative difference conceptually grasped first: Hans and Olle

Eight students (Freddy, George, Hans, Linda, Olle, Ove, Paula, Petra) make the up level change to give an answer in the domain of signed numbers earlier on the reasoning question than on the quick response question, suggesting that they first extend their number domain conceptually and only later change their established procedure for calculation.

George, Hans, Paula and Petra all answer correctly in **R** for the reasoning task already in G6 although they say that 2-5=3 in the quick response question. George and Petra conceptualise 3 - 7 as having 3 and taking away 7 ending up with a debt of 4. For Hans, already in G6 a year and a half before the teaching of negative numbers, it seems unproblematic to take away 7 from 3 ending up with a negative number, without any metaphorical reference at all, see excerpt 7.15.

EXCERPT 7.15: Hans interview G6, Q7f: 3 - 7

7-701	I	would this be ok to write?
7-702	Hans	...nooo...or...yes I think so, but it will turn out to be more, minus. Or, if you have 3 and then you take away 7. That will make, minus 4.
7-703	I	mm? would that be all right?
7-704	Hans	it depends what you are counting with.
7-705	I	mm? If you're counting with things and such would it be all right then?
7-706	Hans	yes.

Olle's learning trajectory describes a clear development of mathematical abstraction concerning the reasoning question, although he replies incorrectly on the quick response question in both G6 and G7. In G6, Olle interprets 3 - 7 as having 7 apples and eating 3 of them, i.e. an interpretation 3 - 7 as meaning the same as 7 - 3. Excerpt 7.16 shows Olle's response in G7 and excerpt 7.17 his response on G8.

EXCERPT 7.16: Olle interview G7, Q7f: 3-7

7-702	I	would this be ok to write?
7-703	Olle	well yes it would but it would become negative
7-704	I	yes? Could it mean something?
7-705	Olle	...nn…
7-706	I	can you think of a situation where you'd write like that?
7-707	Olle	no...

In G7 Olle replies that 3 - 7 becomes negative [line 7-703], but he cannot think of a situation where this has any meaning [lines 7-706 to 7-707].

EXCERPT 7.17: Olle interview G8, Q7f: 3-7

8-701	Olle	that's 3 minus 7, and then it becomes negative
8-702	I	mm?
8-703	Olle	if you take … the 7 we know is bigger than the 3. And when you take, subtract them, then it becomes negative
8-704	I	mm?
8-705	Olle	so if there is a 3 in front
8-706	I	yes... could you, interpret it somehow, or some situation when you would write it?
8-707	Olle	(...) yes if someone had 3 kronor and buys something for 7 kronor
8-708	I	mm? ok. So there you would use it? Could you say that you had 3 apples and ate 7?
8-709	Olle	no. You have minus 2 apples[48].

In G8 Olle states that 3 - 7 will be negative [line 8-01] because the second term is larger than the first term [lines 8-703 to 8-705]. When asked for a contextual meaning he suggests money; having 3 and buying something for 7. Olle

48 Olle's mistake of saying *minus 2 apples* instead of *minus 4 apples* is interpreted as carelessness rather than a conceptual mistake since the focus of the dialogue is the interpretation of the sign.

conceptualises subtraction of a larger number without any explicit metaphorical justification. Furthermore, when specifically asked to contextualise he makes use of the numbers as collections of objects when concerning money but when other objects are brought in (apples) he leaves the context indicating that you cannot take away more than you have because you end up with "minus" objects [lines 8-708 to 8-709]. Minus (negative) objects do not make sense with apples, but they do with money.

A negative difference as self evident: Tina, Tomas and Martina

Tina and Tomas both answer correctly in **R** on these subtraction questions in all interviews, indicating that they accept a negative difference both procedurally and conceptually already in G6, long before any teaching of negative numbers. Martina claims in G6 that the subtraction can not be done (therefore categorised as correct in $\mathbf{R}^+$) but adds that in mathematics you can write like that, "it will be less than 0". In the quick response question in G6 she shows, along with Tina and Tomas, that she can also state how much less than zero it would be. The question in search of an answer is what makes these three students develop such an advanced number sense concerning negative numbers at such an early stage? One answer could be that they have a strong sense of numbers as locations along a path and movements along this path, a metaphor that is easily extended to the left of zero in the encounter with for example a thermometer. This is not clearly the case, since neither Tina nor Tomas can think of a way in G7 to show 5 - 8 on a number line, although they know that the result will be -3. On the contrary, in all their responses the three students simply state that "3 minus 7 is minus 4" or "5 minus 8 is minus 3". As an alternative explanation they suggest a goal difference in a soccer game or the interpretation of 3 - 7 as "from 3 to 7". When provoked by the interviewer to think of a situation where one would write 3 - 7 a typical answer is that from Tomas in G6, excerpt 7.18.

EXCERPT 7.18: Tomas interview G6, Q7f: 3 - 7

6-701	Tomas	...m...it could mean 3 to 7
6-702	I	mm?
6-704	Tomas	and also 3 minus 7
6-705	I	mm?...and 3 minus 7 feels eh, does it feel like it makes sense?
6-706	Tomas	noo, well it makes minus 4.
6-707	I	yees? Could you think of a situation, when you would actually have such a calculation?
6-708	Tomas	(...) eh...I can't think of any right now
6-709	I	no? but it still feels quite ok to make it?
6-710	Tomas	yes

Tomas is sure about the answer [line 6-706], using the little word *ju*[49] to signal that 3 - 7 = -4 is an agreed true statement, whether or not it makes sense. He then fails to come up with any context where this calculation would apply [lines 6-707 to 6-708] but does not think it necessary in order to solve it [lines 6-709 to 6-710].

The responses from these three students suggest that they already in G6 operate within the domain of signed numbers without any need to justify their statements metaphorically. Their responses are all given within a mathematical discourse; explaining it to the interviewer in contextualised terms seems to be an extra complication. After the teaching about negative numbers in G8, they have more contextual references but it does not seem to clarify anything for them, as shown in the following three excerpts.

EXCERPT 7.19: Tina interview G8, Q7f: 3 - 7

8-701	I	mm? what does it mean?
8-702	Tina	that it's 3 minus 7, and so, that makes minus 4.
[…]		
8-707	I	mm? does that answer tell you anything? If it is minus 4 here, what does that answer mean?
8-708	Tina	nothing
8-709	I	nothing?
8-710	Tina	or, well like the difference between, 3 minus 7
8-711	I	ok? like, the difference between 3 and minus 7?
8-712	Tina	... yes
8-713	I	the difference between 3 and, minus 7? Not the difference between 3 and 7?
8-714	Tina	(...) I don't know, no it's just the difference between 3 and… no I don't know

Tina reflects upon subtraction as a difference, but is not clear on whether 3 - 7 is the difference between 3 and minus 7 or the difference between 3 and 7 [lines 8-7010 to 8-714]. Tina does not make use of a number line to justify this but does not seek justification. For Tina, the answer -4 does not have any meaning outside the mathematical context [lines 8-707 to 8-708]. Having an extra-mathematical meaning is not tantamount to making sense for Tina, she has been certain about the calculation 3 - 7 = -4 for several years, so presumably it makes sense to her in an intra-mathematical context.

[49]In Swedish line 6-706 reads: *det är ju minus 4*. The term *ju* is a small Swedish word frequently used in everyday speech and very difficult to translate. It can have many different functions in a text, where a sense of mutual agreement is the one that seems most relevant here (Spenader, 2004).

EXCERPT 7.20: Martina interview G8, Q7f: 3 - 7

8-701	Martina	but it does seem strange when you say it but say you have 3 and take away 7
8-702	I	yes? Why does it sound strange?
8-703	Martina	well if you say it like that, you know you can't take away 7 if you, have 3
8-704	I	no?
8-705	Martina	but it could be that somebody becomes gets to owe 4 so to speak

Martina refers to numbers as collections of objects and subtraction as taking away collections of objects when trying to contextualise 3 - 7. She has no problem with the fact that 3 - 7 = -4, but finds the idea of taking 7 away from 3 strange. It is impossible to take 7 away from 3 [line 8-703], but it is perhaps possible if you accept the idea of becoming indebted [line 8-705].

EXCERPT 7.21: Tomas interview G8, Q7f: 3 - 7

8-701	Tomas	it, eh, well it means eh … well that you, that there are 3 and then you take away 7. And then, like, it gets smaller than 0.
8-702	I	mm
8-703	Tomas	what you have left
8-704	I	could you do that?
8-705	Tomas	yees, or well, I don't know if you could do it in real life but, so to speak. In maths you can.

Tomas states explicitly that you might not be able to take away 7 from 3 in real life, but you can in mathematics [line 8-705]. These three students seem to be comfortable in a mathematical discourse where numbers are related to each other more than to extra-mathematical contexts. They make a clear distinction between the world of mathematics and the real world, and speak of mathematical objects such as numbers and subtractions as second order representations, no longer directly connected to the embodied experiences of their origin.

Not accepting a negative difference: Viktor, Ove and Freddy.

Viktor, Ove and Freddy were the only three students (not counting missing data from Sean in G8) who did not adjust their quick response answers to the new number domain even after the teaching of negative numbers. Freddy and Ove did answer correctly in **R** on the reasoning question in G8, but Viktor only changed from level 0 to level $\mathbf{R}^+$. A common feature among these three students was that they could not think of a way to show the subtraction 5 - 8 on a number line in any of the interviews, not even after the teaching about negative numbers. This is not evidence that a frequent use of a number line is essential, but it suggests that it might have been helpful in the process of enlarging the number domain.

Viktor interprets 3 - 7 as 7 - 3. In G6 he associates it with having 7 of something and losing 3 of them, and in G7 he describes it as taking 3 away from 7. In G8 he does not associate with collections of objects but with a counting strategy, that could be relating numbers to positions along a path or numbers as collections of objects where counting down is taking one object away at a time. He first counts 7-6-5 and answers 5, but when asked if he is sure he counts 6-5-4 and answers 4. Although Viktor now uses a different metaphor for numbers, his basic problem is the same: his misconception of subtraction. At the end of interview G8, in the stimulated recall part of the interview, Viktor gets to listen to his response from G7 to Q7f and reflect upon it to see if he now, after the teaching about negative numbers, has changed his interpretation. As seen in excerpt 7.22, Viktor still interprets 3 - 7 as 7 - 3.

EXCERPT 7.22: Viktor, interview G8; discussion about Viktor's answer to Q7f in G7, where he has said that 3 - 7 meant that you take 3 from 7.

8-001	Viktor	yes, it means you take away
8-002	I	but does it mean you take away 3 from 7 or does it mean you take away 7 from 3?
8-003	Viktor	(...) it's probably both, or it could of course (...) no it's 3, and then the answer
8-004	I	it says 3 minus 7
8-005	Viktor	yes
8-006	I	(writes 3-7) Do I take away 3 from 7 or do I take away 7 from 3 there?
8-007	Viktor	3 from 7
8-008	I	you take away 3 from 7? ...yes, and then you have, how many did you have to start off with?
8-009	Viktor	7 kronor. And then I take away 3. And that makes 4
8-010	I	ok. How about if I want to take away 7 from 3? Then you have, 3
8-011	Viktor	yes then it will become minus
8-012	I	yes? And then you have 3 to start with?
8-013	Viktor	mhm, yes
8-014	I	could you take away 7 then?
8-015	Viktor	no you can't

Subtraction for Viktor is "taking away", and the order in subtraction doesn't matter [line 8-003], but since it is impossible to take 7 away from 3 [line 8-015] he interprets it as taking 3 from 7. Considering that he is consistent in interpreting 3-7 as taking 3 away from 7, and that none of the examples he gives in any of the interviews suggests that he could be thinking of taking 7 away from 3, it is not utterly true to assert that he believes subtraction to be commutative. A more plausible interpretation is that it is the connection between his conception of subtraction and the symbols with which we write subtraction that are confused or misunderstood. When working in the domain $\mathbf{R}^+$ he meets subtractions such as 6 - 2 = 4. This symbolic expression reads "*six minus two* makes four" or "*two from six* makes four". It is not uncommon in the observed classroom discourse that these two expressions are mixed so that students say "*two minus six* makes four" when they read the 6 - 2 = 4. Particularly when the calculation appears as a partial calculation in an algorithm like:

$$\begin{array}{r} 356 \\ \underline{-22} \end{array}$$

Within the context of $\mathbf{R}^+$ this way of speech is perfectly well understood, it is only when the domain is extended to **R** that the direction becomes essential. Furthermore, when only concerned with a difference in quantities and measures subtraction actually is commutative: $|3-7| = |7-3|$.

Procedures for subtracting with a negative answer: Petra; Erik and Tina

Although the quick response questions were not supposed to be dwelt upon, the very last of the six tasks was discussed with many of the students in the last interview (G8). This proved to be worthwhile, since in the discussion a solving procedure was revealed, a procedure that turned out to be commonly used by all students who were asked. The task 6-27 involves two separate procedures; finding the difference between the magnitude of the numbers 6 and 27 and deciding the sign for the answer. All students who were asked about this task did these two things separately, most commonly deciding the sign first. This solving procedure can be anticipated when listening to the audio recorded quick response questions where the students often start by saying minus, then make a very short pause, and then say the number, e.g. 2 - 5 = "minus … 3". Petra describes how she solved 6 - 27: "First I figured it would be a minus number, and then I calculated what it would be if I switched it around and then, I put a minus sign in front" [Petra, G8, Q5]. Excerpt 7.23 shows Erik's similar procedure. First he states that the answer will be negative because the subtracted term is larger than the first term [line 8-502], then he decides on the difference between 27 and 6 being 21 [lines 8-508 to 8-510]. Without explicitly saying so he separates the magnitude from the direction, treating them separately.

EXCERPT 7.23: Erik interview G8, Q5: 6 – 27

8-501	I	this last one, when you said it was minus 21, did you see straight away that it would be minus?
8-502	Erik	yes because it is, well yes, if it is, a small number minus something bigger then that's it sort of
8-503	I	mm, so how did you work out that it was minus 21? Do you know it?
8-504	Erik	no I worked it out
8-505	I	so how did you work it out?
8-506	Erik	but, if you count down to … or if you take, minus .. or
8-507	I	yes?
8-508	Erik	... you just take 27 minus 6
8-509	I	yes?
8-510	E	then, you have 21 left.

Comments about the acceptance of a negative difference

Metaphorical reasoning as a means of explaining the meaning or calculation or justifying a negative difference was rarely used by the students except when the interviewer explicitly asked for a context. More often the students spoke in mathematical terms. When used, metaphorical interpretations of numbers and subtraction sometimes confused the students or held them back in the domain of unsigned numbers instead of helping them to extend their number domain. Treating numbers and operations as second order representations, and thus in a sense separating the world of mathematics from the real world, was a common feature among those students who found it easy to accept a negative difference. The students related the "meaning" of numbers to the numbers numerical meaning, in for example saying that 3-7 means -4.

Concerning student's learning trajectories, the results indicate that procedural knowledge does not necessarily precede conceptual knowledge or vice versa. Some students change their procedures before they change their conceptual meaning making, some do the opposite, and yet others seem to change them simultaneously.

7.3 Subtraction 2: the different meanings of the minus sign

Understanding the different meanings of the minus sign was brought up in chapter 1 as a critical feature of negative numbers. Knowing if the minus sign symbolises the operation subtraction, thus having a binary function appearing between two numbers, or if it symbolises polarity, having a unary function of characterizing a number, is highlighted as important in many of the research reports concerning negative numbers. In order to investigate how students interpret the different meanings of the minus sign, question Q7 was designed as an open ended question, taking a written symbol or symbolic expression (strings of symbols) as a point of departure. The symbols were shown on cards and the students were asked to interpret them.

Q7) Q7g: -6-2 Q7h: (-3)-1-2

Is it ok to write like this?	What does it mean do you think?
What is the use of it?	What does it show?
Could it mean something else?	Could it be used for something else?

The two last expressions in Q7 included both a negative number and a subtraction, in both cases the negative number was followed by a subtraction of a positive number; Q7g: -6-2, and Q7h: (-3)-1-2. No subtraction of a negative number was included but in both cases the negative number appears as the first term. In the last interview in G8 such a subtraction with a negative number was included and discussed in the interview with some of the students. Most students solved that subtraction correctly in a very procedural manner making a circle around the two minus signs and exchanging it for a plus sign saying "two minuses becomes a plus". Figure 7.1 shows an extract from Anna's exercise book illustrating how this procedure was taught during the lessons.

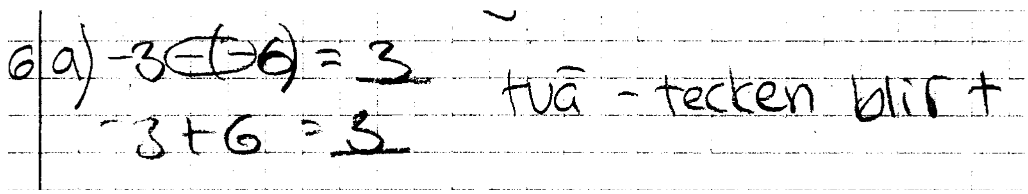

FIGURE 7.1: Procedure for replacing two minus signs with a plus sign. The text says: *two - signs becomes* +. [Anna's work]

However, cases where a positive number was to be subtracted from a negative number exhibited interesting patterns of change. The two expressions in Q7 were are also compared with a similar task, Q4, involving only natural numbers and only categorised as either 0 or **R** since there was no interpretation that was considered correct in $\mathbf{R}^+$ but not in **R**. In this section, students' responses to the three questions Q4, Q7g and Q7h are reported and analysed

Q4) 31-12-2 . Calculate keeping a think aloud protocol.

Findings:

In this data most of the students interpret a minus sign only as a sign of subtraction in grade 6. Subsequently, for some students as a result of the explicit teaching about negative numbers but for others as part of the continuous extension of their mathematical knowledge, they distinguish also the meaning of the sign as marking a negative number. In many situations it is unclear which of the interpretations is "correct", particularly if the negative number appears as a first term. In the expression -6-2 the first minus sign marks a negative number and the second is a subtraction sign, but in the expression x-6-2 both minus signs are subtractions. The metaphorical interpretation of subtraction and negative number has great influence on students' understanding of the two meanings of the minus sign.

~ Interpreting subtraction within an *object collection* metaphor as taking away makes it difficult to make sense of taking away a positive number from a negative number. For some students this is resolved by interpreting for instance -6-2 as x-6-2 and thus interpreting both signs as subtractions and taking away from an arbitrary collection of objects.

~ Interpreting subtraction within a *measurement* metaphor as the distance between two numbers leaves out the direction. It is also unclear at times which of the three numbers in a subtraction $a - b = c$ is the distance and which are the positions. In a measurement metaphor a and b are positions and c is the distance between them, whereas in a *motion along a path* metaphor a and c are positions and b is the motion from a to c.

~ Once the interpretation of the minus sign as indicating a negative number has been introduced the *motion along a path* metaphor is left out, perhaps due to the fact that a negative number is conceptualised as a debt in this classroom.

Empirical data

Students' responses to the questions above are shown in table 7.4, categorised into the three levels in the corresponding legend below. Emphasis is put on when change occurs. The legend shows a summary of the characteristic responses and how they were graded according to level as either: incorrect, correct within **R**$^+$ or correct within **R.**

TABLE 7.4: Overview of categories of all students' responses to questions Q4, Q7g and Q7h, according to number domain levels described in the legends below.

	Q4: 31-12-2 =			Q7g: -6-2			Q7h: (-3)-1-2		
	G6	G7	G8	G6	G7	G8	G6	G7	G8
Anna	**R**	**R**	**R**	**R**$^+$	**R**$^+$	**R**	**R**$^+$	**R**$^+$	**R**$^+$
Axel	**R**	**R**	**R**	**R**$^+$	**R**	0	**R**$^+$	**R**$^+$	**R**$^+$
Elke	**R**	**R**	**R**	**R**$^+$	**R**$^+$	**R**$^+$		**R**$^+$	**R**
Erik	**R**	**R**	**R**	**R**$^+$	**R**	0	**R**$^+$	**R**	0
Fia	**R**	**R**	0	**R**$^+$	**R**$^+$	**R**	**R**$^+$	**R**$^+$	0
Freddy	**R**	**R**	**R**	**R**$^+$	**R**$^+$	**R**		**R**$^+$	**R**
George	**R**	**R**	**R**	**R**$^+$		**R**	**R**$^+$		
Hans	**R**	**R**	**R**	**R**$^+$	**R**	**R**	**R**$^+$	**R**$^+$	**R**
Lina	0	**R**	**R**	0	0	0	**R**$^+$	**R**$^+$	0
Linda	0	**R**	**R**	**R**$^+$	**R**	**R**	**R**$^+$	0	**R**
Lotta	0	**R**	0	**R**$^+$	**R**$^+$	**R**	**R**$^+$	**R**$^+$	**R**
Malin	0	0, **R**	**R**	**R**$^+$	**R**$^+$	**R**	**R**$^+$	**R**$^+$	
Martina	0	**R**	**R**	**R**	**R**	**R**	**R**$^+$	**R**	**R**
Olle	**R**	**R**	0	**R**$^+$	0	0	**R**$^+$	0	**R**
Ove	0	0	0	**R**$^+$	0	**R**			
Paula	**R**	**R**	0	**R**$^+$		**R**	0	**R**	**R**
Petra	**0**	**R**	**R**	0	**R**	**R**		**R**$^+$	**R**
Sean	**R**	0		0	0		**R**$^+$	0	
Tina	**R**	**R**	**R**	**R**$^+$	0	**R**		0	**R**
Tomas	**R**	**R**	**R**	**R**$^+$	**R**	**R**	**R**$^+$	**R**	**R**
Viktor	**R**	**R**	0	**R**$^+$	**R**$^+$	**R**	**R**$^+$		**R**

level	answer
0	21
0	11, 18, 16, -21
R	17

level	answer
0	-4
0	from -6 to 2
0	-4 or -8
R$^+$	x-6-2
R	-8

level	answer
0	6, 1
R$^+$	x-3-1-2
R$^+$	impossible
R	-6

Two consecutive subtractions of natural numbers

Although one third of the students misinterpret subtraction in G6, mostly by treating it as associative taking 31-12-2 = 31-10 = 21, by G7 all but two students (Ove and Sean) make the correct calculation on this task. Surprisingly, in G8 after the teaching about negative numbers, five students who solved it correctly in G6 and G7 now fail. A possible interpretation is that something in the teaching of negative numbers has made students inclined to abandon a previously accurate procedure of calculating each operation in turn from left to right in favour of some less accurate procedure. None of the students uses any metaphorical language when "thinking aloud", they all say things like "31 minus 12 is 19, and 19 minus 2 is 17" [Tomas G6] or "2 minus 12 is 10, and 31 minus 10 is 21" [Lina G6]. However, when scrutinizing the responses, it turns out that Olle and Paula make incorrect mental calculations (mixes up their number facts) but treat subtraction correctly. Fia and Lotta choose to start with the subtraction 12-2, explaining it with the words "it's easier". Only Viktor shows signs of confusion related to the treatment of negative numbers on this task.

EXCERPT 7.24: Viktor interview G8, Q4: 31-12-2=__

8-401	Viktor	well 31 minus 12, that makes … 19. And then minus … again that will be, that makes 21 again (...)
8-402	I	could you tell me again, how did you how did you see how did you work out so quickly that 31 minus 12 would be 19?
8-403	Viktor	well no I thought, first 10, whatever minus or
8-404	I	yes?
8-405	Viktor	(inaudible) that makes 20 there, yes 21, and then minus 2
8-406	I	ok, good. And then, you had 19?
8-407	Viktor	yes, and then minus 2, well since it's minus, then it's 2, more, and then it's 21
8-408	I	ok, but you just wrote 21 you didn't write minus 21 there
8-409	Viktor	well but it says
8-410	I	oh I see it does say minus 21 there? Ok ok now I see, it says 21, minus 21.

Excerpt 7.24 shows how Viktor, after having done the first calculation correctly, thinks he is left with a negative number. Why he suddenly interprets 31-12 as a negative difference is hard to say. As a result of the teaching about negative numbers he now, in grade 8, knows that a difference can be negative, but maybe he is still uncertain about the direction of the symbols in a subtraction. (i.e. if 31 - 12 is a subtraction of 12 from 31 or 31 from 12). After the first subtraction he has to subtract again, and since he believes he is on the negative side subtracting 2 from 19 makes 21 [line 8-401 and 8-407]. The analysis leads to the conclusion that most of the students correctly interpret two consecutive subtractions in G7 and G8, but their interpretations are procedural and have no longer any explicit connection to underlying metaphors. Both a metaphor of *taking away collections of objects* and a metaphor of *moving to the left along a path* could underlie the meaning

they give this subtraction task, 31-12-2, and be used to make sense of the procedure of subtracting.

A negative number followed by a subtraction

A subtraction where the first term is negative and the subtracted term is positive is in G6 interpreted by all the students either incorrectly or correctly in $\mathbf{R}^+$**.** No one recognises a first term with a minus sign in front of it as a negative number, with or without brackets. The common response in $\mathbf{R}^+$ is that it is only possible, or only makes sense, if there is a term in front of the first minus sign, preferably one with a greater magnitude that the rest of the terms together. For instance they would suggest to insert 8 in the expressions to get 8-6-2, or (8-3)-1-2. A typical example of such a response is Freddy G7, as shown in Excerpt 7.25. Once again Freddy refers to a task when he says number [line 7-232], which was discussed in section 7.1. For Freddy, the minus sign only has the meaning of a subtraction sign.

EXCERPT 7.25: Freddy interview G7, Q7g: -6-2

7-230	Freddy	yes so all that's missing is a number there[50]
7-231	I	mm? in front?
7-232	Freddy	mm. because if you take for example 8 minus 6 minus 2, then it would be a number, so it would be 0 then
7-233	I	mm, could you have 4 there?
7-234	Freddy	no
7-235	I	why not?
7-236	Freddy	because, eh, then it wouldn't work out. Then it would be like, strange.

During the following two years many students start interpreting the first number as a negative, either by saying "you have minus 6", or "a debt of -6"[51] or "start on minus 6". Some reach a correct solution in **R**, others an incorrect solution. Many changes are made in G7, *before* the teaching of negative numbers, so it is not necessarily a result of the teaching, although in some cases it could be.

Two of the fairly high achieving students, Erik and Axel, both get confused on these tasks in G8 and make a down level change. In G6 Erik suggests to write 10-6-2 in order to make sense of -6-2. When faced with the suggestion 5-6-2 he responds that it would "make a minus result". The following transcripts exhibit how Erik's reasoning changes from one year to the next. Excerpt 7.26 shows an up level change in G7 compared with G6.

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[50] Freddy uses the word *siffra*, which literally means numeral or digit, but which is often also used to mean number.

[51] The phrase a *debt of -6* is really a tautology since the minus signs is put there to indicate that it is a debt. The proper phrase should be *a debt of 6*. This tautology is quite common in the classroom discourse. Likewise a phrase like *he owes me -6* when the meaning is that *he owes me 6*.

EXCERPT 7.26: Erik interview G7, Q7g -6-2

7-701	I	How about this, what does this mean? Or is it ok to write like this is the first question?
7-702	Erik	yes
7-703	I	in that case what does it mean?
7-704	Erik	minus 6 minus 2, that makes minus 8
7-705	I	mm? How do you know it makes minus 8?
7-706	Erik	if you have, given away, 6, if you don't have any and then you give away another 2, then you have given away 8, that you don't have

In G7 Erik's answer is correct in **R**, but he interprets both minus signs in the same way; as giving away [line7-706]. There is, however, a subtle different in the tense of the verb that could be interpreted as an indication that he sees this as an example of a situation of state – change – state. Erik first states that "you had given away 6" (state, negative number), then "you give away another 2" (change, subtraction) and finally you end up "having given away 8" (new state, negative number).

If the signs represent the same thing, does it represent subtraction or negative number? In both cases the expression would be written slightly different in formal mathematics. The interpretation of both signs as subtractions would formally be written as 0-6-2 = -8. There is 0 to start off with, and then 6 and 2 are subtracted. In an *object collection* metaphor in R^+, the activity of taking away is usually mapped onto a minus sign, and collections of objects are mapped onto numbers (see metaphor analyses in chapter 3). In that interpretation there are no negative numbers except for the result, and the result is a collection of objects (money) that has been given away although they are not there, that is mapped onto a negative number. The second interpretation of both signs as signs of negativity would be written (-6) + (-2) = (-8). Here the money you give away (not the activity of giving them) is mapped onto a negative number, and the act of adding two of these lots of money that you give away is mapped onto the operation addition. The result is another collection of money. Erik's interpretation in G7 is ambiguous, not clarifying if the minus sign represents an operation or a type of number.

Excerpt 7.26 above showed how Erik treated subtraction of a positive number from a negative number in the extended number domain **R** by treating the two minus signs as representing the same thing, i.e. not clearly differentiating between the unary and the binary function of the sign. After the teaching on negative numbers he has a different way of reasoning, where he distinguishes between negative numbers and subtraction, and relates them to having a debt and taking away money. As a result, his answer is incorrect since he interprets the act of taking away from a debt as a decrease of the debt. In excerpt 7.27 Erik exhibits this reasoning and his confusion as a result of it.

EXCERPT 7.27: Erik interview G8, Q7h: (-3)-1-2

8-120	I	if you were to translate this into a situation, what could it stand for?
8-121	Erik	well it's the same as before with money
8-122	I	mm?... so describe the money, what you did with money there?
8-123	Erik	I have a debt of 3, so, then I take it away, so the money, or, well I take away a debt, so I get a money[52]. And then I get, take away another debt so I get 2 money …
8-124	I	and then you have?
8-125	Erik	0. or I don't have any debt at all.

In this episode Erik explicitly states that he has a debt from which he takes away, and that taking away a debt is the same as gaining money [line 8-123]. It seems as if the clear distinction between the two different meanings of the minus sign makes Erik focus on one as *a type of object collection* and on the other as *an activity of taking away*. Furthermore, in combination with the debt, the act of taking money away from a debt is incomprehensible and is reinterpreted as taking away some of the debt. For Erik, the ambiguity of interpretation he expressed in G7 indicated a higher level of abstraction than his reasoning in G8 where he explicitly differentiated between the two meanings of the minus sign and tried to make metaphorical sense of them.

Axel is taking away from a debt

Axel, like Erik above, develops his sense making of the expression -6-2 from $\mathbf{R}^{+}$ in G6 to R in G7, but gets confused in G8 as a result of the teaching about debts. In excerpt 7.28 Axel displays two different interpretations of the task -6-2.

EXCERPT 7.28: Axel interview G8, Q7g: -6-2

8-701	I	mm, is it ok to write like this do you think?
8-802	Axel	yes, I think so
8-703	I	mm? so what would it mean in that case?
8-704	Axel	... ehm ... think it could be minus4
8-705	I	yes? How do you figure that?
8-706	Axel	or it's minus8
8-707	I	yes...
8-710	Axel	yes, think minus8
8-711	I	yes? How did you work out minus4?
8-712	Axel	no but, you have minus 6, and then you take away 2 from that. Then you kind of think that it will be, smaller number
8-713	I	yes?
8-714	Axel	minus4, instead
8-715	I	but then you changed your mind and said minus 8, so how did you work that out?

[52] The Swedish word *pengar* means money but can be enumerated as: one money; two money… meaning some unit of money.

8-716	Axel	I don't know I think 4 now because, or, because if you have minus and add minus, it should become, if it is the same unit, it should in any case become, minus8 like here
8-717	I	yes? ... so which one will you choose, minus 4 or minus 8?
8-718	Axel	minus...4 I think
8-719	I	ok. When you, if you, get a task like this on a maths lesson you, it wouldn't help you to think about debts or a number line or something like that?
8-720	Axel	no, no I don't know
8-721	I	it's not what you usually do? No? … what would it mean if you tried to interpret it as money?
8-722	Axel	... well if it was the one with minus8, then it could be that you owe somebody 6, and then what do you call it, you get to owe somebody else another 2 kronor, and then you owe 8
8-723	I	mm
8-724	Axel	could be. Or if you have 6 something, and then you owe someone 6 of them, and you give away, or pay 2, then you have minus 2 of what you owe
8-725	I	mm
8-726	Axel	and then there's only minus4 left

One of Axel's interpretations distinguishes between the two different meanings of the minus sign. Axel thinks of taking 2 away from 6, and 6 is a negative number, a debt [lines 8-712 to 8-714, and 8-724 to 8-726]. Axel relates to the magnitude of the numbers. In the second interpretation Axel treats both minus signs as signs of polarity, or type of number. Axel says "unit", and the operation is one of adding these numbers [line 8-716]. In the money context Axel maps the state of being indebted (owing), as well as the activity of becoming indebted, onto negative numbers. [line 8-722].

Although Axel can reason about both of the interpretations he cannot make up his mind which of them gives the correct answer. Axel's response to Q4 in G8 shows that he calculates 31-12-2 as 31-12 = 19 and 19-2 = 17. When asked why he did it that way (since he had just solved 16-4+2 = 16-6 = 10) he responds that it is "because they are both minus". This indicates that Axel is not totally confident about how to treat subtraction, or if addition should be taken before subtraction, but in a situation with two consecutive subtractions he takes them one after the other. What is the big difference between the meaning of *a-b-c* and *-b-c*? If *-b-c* is interpreted as two subtractions from an (arbitrary) starting number, or as an addition of two negative numbers, it would be much easier to connect to experiences from the domain of unsigned numbers. Treating them as having the same meaning would be easier than distinguishing between the two different meanings.

A common feature in the responses from both Axel and Erik on question Q10 where the number line is discussed is that they cannot in G8 think of a way to show the calculation -3-5+2 on a number line with a correct solution. (Axel wants to move 3, 5 and 2 steps but does not know in which direction, Erik starts on 7 and moves -3 down to reach 4). This suggests that they do not have a mental number line and a *motion along a path* metaphor readily accessible as a thinking tool when making sense of these kinds of tasks.

When metaphors do not help: Olle

Scrutinizing the learning trajectory of Olle gives a picture of a student who tries to make use of metaphorical reasoning but falls back on procedural knowledge when the metaphor fails. In G6 Olle suggests to put 8-6-2 to make any sense of -6-2, and when faced with the suggestion 7-6-2 he rejects it although he exhibits a vague feeling of negative numbers, as shown in excerpt 7.29.

EXCERPT 7.29: Olle interview G6, Q7g: -6-2

6-701	Olle	mm... perhaps for example put an 8 here
6-702	I	mm? 8 minus 6 minus 2?
6-703	Olle	yes
6-704	I	would that be ok?
6-705	Olle	mm
6-706	I	could it just as well be a 7?
6-707	Olle	no
6-708	I	why not?
6-709	Olle	then it would be minus … no. Don't know

Excerpt 7.30 shows how Olle in G7 runs into the same problem as Erik and Axel when interpreting the subtraction sign as 'taking away'.

EXCERPT 7.30: Olle interview G7, Q7g: -6-2

7-701	Olle	minus, yes but it must become, minus 6, minus 2, comes to minus 8
7-702	I	mm, how did you think there, to make minus 8?
7-703	Olle	because it already says minus 6 and then you just take 6 minus 2, or perhaps it comes to minus 4
7-704	I	mm, which do you think it comes to?
7-705	Olle	(...) well minus 2 then you take away something, then it comes to minus 4 actually.
7-706	I	mm? because you want to take, you want to take away the 2 there from the 6?
7-707	Olle	mm, because you sort of take minus 5, then you take minus 2, take away 2 of them from the 6.[53]

For Olle the problem of taking something away from a debt arises before the teaching and is therefore not a direct result of the metaphors introduced in the whole class teaching, but more likely a result of the metaphors he has for numbers in the domain of unsigned numbers. He is clearly speaking about *numbers as object collections* by referring to the 2 as '2 pieces', i.e. a collection of 2 objects [line 7-707]. This is an example where the metaphors Olle makes use of

[53] Olle's mistake of saying 5 is probably a slip of the tongue, since he mentions 6 later in the sentence.

to conceptualise numbers (*object collections*) and subtraction (*taking away object collections*) do not help him to correctly interpret the expression -6-2.

In G8, Olle has had teaching about negative numbers. When encouraged to think about a context to interpret the expression -6-2 he relates to money, the dominant context of the classroom discourse on negative numbers.

EXCERPT 7.31: Olle interview G8, Q7g: -6-2

8-701	Olle	well yes, if you go with the kronor again
8-702	I	yes? What would it mean in that case?
8-703	Olle	you have a, if you have minus 6 kronor, in you account for instance, then you get 2 kronor
8-704	I	yes?
8-705	Olle:	then that will make, then you get more, then you … no that was wrong … no I don't know!
8-706	I	this here would mean you had minus 6?
8-707	Olle	yes
8-708	I	and then you get 2 is that what it means?
8-709	Olle	no
8-710	I	no? so what would it mean?
8-711	Olle	so then they take 2 kronor ...
8-712	I	yes?
8-713	Olle	I think...
8-714	I	how will you know if it means that you get or they take?
8-715	Olle	I don't know

The account balance is not seen as a relation, but as an amount of negative money in the account, i.e. a quantity, or possibly as money missing from the account [line 8-703]. Olle does not know how to make sense of subtracting 2 if there is -6 on the account. It could either mean adding 2 to decrease the debt [line 8-703] or taking 2 away from the debt [line 8-711].

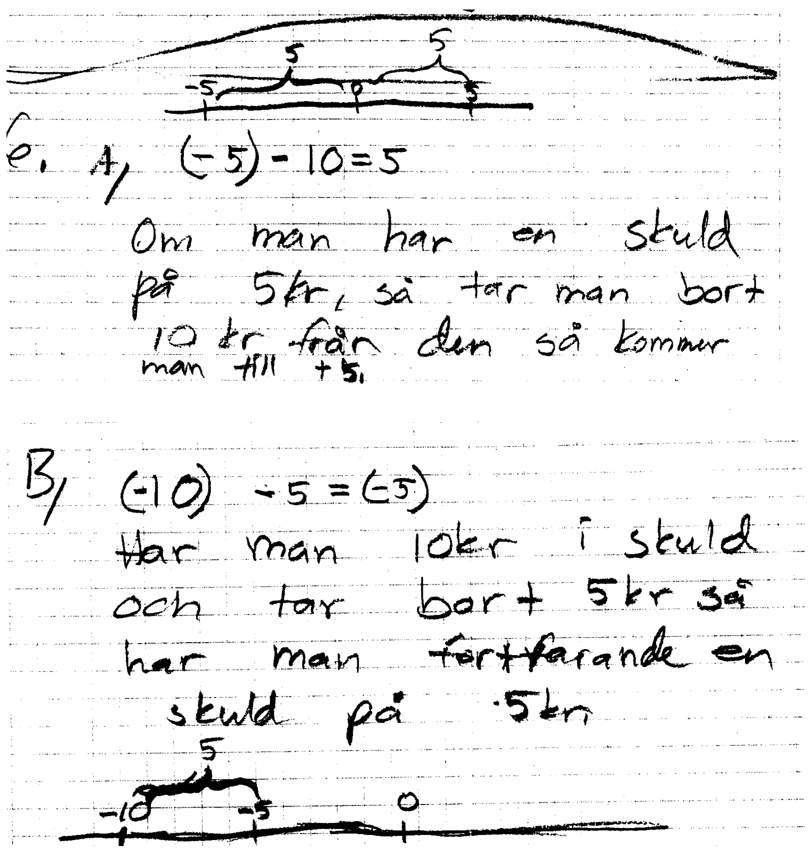
At the end of the teaching sequence about negative numbers in grade 8 Olle has problems with this type of calculation on the written test. He calculates for instance (-12)-15 = 3. Problem 6 on the test asks for an explanation and reveals some of his reasoning. The problem reads:

6) Write **a negative number** and **a positive number** to make the equality true:

a) _____ – _____ = 5 Explain your answer

b) _____ – _____ = (-5) Explain your answer

Olle's response to problem 6 is shown in figure 7.2. In the written motivation Olle uses an *object collection* metaphor mapping *taking away* onto subtraction, but in neither of the examples can he take away (positive) money from a debt, interpreted as a collection of negative money or a missing collection of money. The sign rule of replacing a subtraction of a negative with an addition of the opposite has been frequently formulated in class as "taking away a debt is the same as earning money". In this case Olle uses that rule when he has in fact written a subtraction of a positive number rather than a negative number. Olle also combines his explanation with a drawing of a number line indicating a distance, relying on a *measurement* metaphor, mapping *distance between two locations along a path* onto subtraction.



If you have a debt of 5 kr, and then you take away 10 kr from it you will get to +5.

If you have 10 kr as a debt and take away 5 kr you will still have a debt of 5 kr.

FIGURE 7.2: Olle's solutions to problem 6 on the negative number test in grade 8.

In both cases Olle illustrates the distance between the starting number and the final number. However, in the *measurement* metaphor the distance between two locations is mapped onto the result of the subtraction (the difference), not the distance between the first number and the final number. One interpretation of Olle's reasoning is that he mixes up the *measurement* metaphor with a *motion along a path* metaphor, where a movement from the first location (first term) to the final location (resulting number) is mapped onto the second term, except that he does not indicate a movement, only a distance.

There are ways of illustrating these subtractions within all three metaphors in a comprehensible way, but all such illustrations would need an adjustment of the metaphors to the new number domain. The embodied metaphors that make up Olle's interpretation of number and subtraction in $\mathbf{R}^+$ need to be changed and in a sense become different metaphors in the new number domain. In chapter 3 such extensions of the metaphors were explained. Olle's failure to reach a mathematically correct answer to these problems suggests that he has not made those adjustments to his metaphors, and therefore relies on memories of rules from the classroom discourse and illustrations he has seen in the textbook. The sign rule of replacing two minus signs with a plus sign has in the teaching been translated into metaphorical language (taking away a debt is the same as earning money) without actually incorporating a metaphorical meaning.

The evasive distinction between a noun and an adverb

The subtle difference between the activity of borrowing and the objects that have been borrowed is not always important in a verbal context. (cf. I have borrowed 5 kronor from you; I have a debt of 5 kronor to you). Is it important in mathematics? Previous research reported in chapter 1 has identified the ability to see the different meanings of the minus sign as an important aspect of negative numbers. More specifically, it might be of vital importance to realise *that there are* two different meanings, but less important to state as a fact which of the meanings is 'correct' in each mathematical expression. One of the benefits of using the same sign is that it opens up the possibility to vary the interpretation. When for example solving the task in Q4: 31-12-2, it is possible to first add 12 and 2 and then subtract the sum from 31. Formally, this is possible because the expression is treated as an addition of three signed numbers: (+31)+ (-12)+ (-2), and addition is associative.

In this class there is a lot of talk about debts, and in Swedish it is possible to say "en skuld" (a debt), but also "vara skyldig" (to be indebted [to]) and "bli skyldig" (to become indebted [to]), where the word for debt is an adverb instead of a noun. When this is used in the mathematical discourse of the classroom, the difference between the interpretations of the minus sign as an operation and a negative number becomes obscure. The following episode shows how Petra expresses this in G8:

EXCERPT 7.32: Petra, interview G8, Q12 (stimulated recall part discussing -6-2)

8-1	Petra	if there are two signs then one sign is what you are supposed to calculate and the other is, what kind of number it is.
8-2	I	mm? so when you have this one that we spoke about before where it said minus 6 minus 2, do you remember we had that on a card before?
8-3	Petra	mm
8-4	I	then I asked you about that sign there, is that subtraction, or is it the 2 that is negative?
8-5	Petra	that is subtraction …because something has got to be that sort of. And otherwise it would be …
8-6	I	but before you said that it was like, two negative numbers that you put together
8-7	Petra	yes it can, mm you could think that as well, because if you switch it around, if you move the 6 there then it kind of becomes ehm … then the 2 that is negative … because the

		minus sign goes with it, that's this one, the sign in front always goes with ehm… the number behind. And then, if you take away the 6 so it's not there anymore then what you've got, it means that it's negative, but if the 6 is there then it means that you have to take away from a positive number.
8-8	I	ok? And you think that's fine, to jump between these interpretations a bit as you want?
8-9	Petra	yees, well, not really as you want but you can… switch it around

The students have been taught that in the procedure of moving around numbers in an expression the number must be attached to the sign in front of it. -6-2 can therefore without problem for Petra be rewritten as -2-6. Formally, this can only be done by interpreting it as an addition of two negative numbers.

Martina, who is a high achieving student, expresses how she interprets these tasks in excerpts 7.33 and 7.34. Like Petra, she speaks about numbers as if they were reified mathematical objects, she does not explicitly speak in terms of object collections or movements. The following episode is from interview G7, before the whole class teaching about negative numbers.

EXCERPT 7.33: Martina interview G7, Q7g: -6-2

7-701	I	What could this, can it, can it be like this, does it look strange?
7-702	Martina	no it can be like that
7-703	I	yes so what does it mean?
7-704	Martina	minus 8. or that it comes to minus 8 sort of it
7-705	I	mm?
7-706	Martina	yes
7-707	I	how do you know?
7-708	Martina	because you, first you take minus 6, and then you take another minus 2 and that makes minus 8
7-709	I	yes? How did you work that out?
7-710	Martina	...I don't really know
7-711	I	no? why doesn't it make minus 4?
7-712	Martina	because, it already is minus, or you don't have anything to, take it away from

Martina is very certain about all the tasks given in both G7 and G8. Her interpretation of -6-2 as first taking -6 and then taking -2 is coherent with an *object collection* metaphor in that she talks about them as being *taken* [line 7-708], suggesting an interpretation of the expression as (-6) + (-2). She also states that 2 cannot be taken from -6 because there is nothing to take from [line 7-712] indicating that she accepts two consecutive actions of taking away from some arbitrary or implicit starting number but not taking away a positive number from a negative number. However, her interpretation could also be a version of a *motion along a path* metaphor; starting on an arbitrary starting point and then moving first 6, then 2. What she means when saying "first you take, minus 6" is not apparent [line 7-708], perhaps not even for her. Excerpt 7.34 illustrates how Martina interprets the expression (-3)-1-2 in the same manner as -6-2, i.e. as an addition of negative numbers.

EXCERPT 7.34: Martina interview G7, Q7h: (-3)-1-2

7-730	Martina	no you, that you first have, minus 3
7-731	I	yes?
7-732	Martina	and then, you can put these together. Then you have another minus 3.
7-733		Or you just take minus 1 and minus 2. It doesn't really matter. But yes, it comes to minus 6.

Seeing (-3)-1-2 as three negative numbers that are added enables Martina to add them in the reverse order: (-1) + (-2) + (-3). She makes clear that this is only *a way of thinking about it* by suggesting another interpretation which is coherent with a *motion along a path* metaphor, starting on -3 and moving 1 and then 2 in the minus direction [line 7-733]. Martina does not need these metaphors explicitly, and might not think about them; they are incorporated into her well developed sense of number and have become the fabric of numbers and operations as reified mathematical objects.

In the expression -6-2 there is a missing (implicit) plus sign. Convention states that a number without a sign is a positive number, which means that the implicit plus sign is to be placed in front of the number 2: (-6) - (+2). What Martina shows is that if such an interpretation fails to make sense, like taking 2 positive away from 6 negative [excerpt 7.33, line 7-712] it is also possible to place the implicit plus sign between the two numbers: (-6) + (-2) [excerpt 7.33, line 7-708]. This experience could make a fruitful starting point for a discussion about exchanging an addition of a negative with a subtraction of a positive or vice versa, depending on if what is desired is an interpretation that makes sense or an easy calculation. As a contrast an episode from the classroom shows how Sean tries to understand the rule that "one minus and one plus make a minus".

EXCERPT 7.35: Video 8:8, time 10:23. T is teacher

1	Sean	why is it, if there are like plus minus why can't it be plus? Why does it have to be minus? It's kind of, is like minus, stronger than plus?
2	T	no but since
3	Sean	but you know we say, plus minus, then it's always minus
4	T	mm
5	Sean	why is it? Why can't it be plus even if, it kind of
6	T	because you add on like a debt you see
7	Sean	yes but it's kind of, unfair

Sean tries to make sense of these signs metaphorically as a question of which is the strongest in a struggle [line 1], whereas the teacher talks of the numbers as debts. In the teachers metaphor, the adding of a debt [line 6] can be seen as in a sense the same as subtracting money, i.e. a + (-b) = a - (+b). If the implicit plus sign is made explicit in the discourse, both verbally and symbolically, the rule could more easily be understood. Sean shows that a student might bring in completely different metaphors when the ones presented are too vague or incomprehensible.

A note about wording

Concerning the use of money as a context for metaphorical reasoning, the subtle differences in the verbal phrases do not correspond in a trivial manner to the different meanings of the minus sign, which implies that a mapping from negative numbers and subtraction to expressions about debts and owing money is not simple. The metaphor described here is a mapping from mathematics to a context of money. Subtraction is an operation and a negative number is a state or an object. In what way do states and operations map onto these phrases? There are four common phrases that express debts and owing money[54]:

1. X is indebted (X owes) — *X är skyldig*
2. X becomes indebted — *X blir skyldig*
3. X has a debt — *X har en skuld*
4. X gets a debt — *X får en skuld*

All four phrases express relations (Halliday, 2004, pp 219-222); 1 and 2 are intensive relations whereas 3 and 4 are possessive relations. Therefore, it might seem logical to map subtraction onto 1 and 2 where the attribute is a quality, something that describes X; and negative numbers onto 3 and 4 where the attribute is an entity, something that can be possessed by X. On the other hand, 2 and 4 express something that is about to happen, whereas 1 and 3 express a state, suggesting that subtraction be mapped onto 2 and 4 and negative numbers onto 1 and 3. These different mappings are illustrated in table 7.5

TABLE 7.5: Metaphorical mappings of a negative numbers and subtraction onto debts

Source domain	**Target domain**	**Description**
Subtraction	1: X is indebted 2: X becomes indebted	The attribute is a **quality**; describes X
Negative number	3: X has a debt 4: X gets a debt	The attribute is an **entity**; can be possessed by X
Subtraction	2: X becomes indebted 4: X gets a debt	The relation is **phased**; something will ensue
Negative number	1: X is indebted 3: X has a debt	The relation is **neutral**; static, a state

[54] All these phrases are frequently used in the classroom data and in Swedish everyday language.

To make things even more complex, all the phrases are **relational**; indicating a relation between X and an attribute Y. This suggests that subtraction could be mapped onto all four phrases since the minus sign in subtraction has a binary function, in between two numbers x and y. Or, negative numbers could be mapped onto all fours phrases in a *number as relation* metaphor.

The above mappings show that if the metaphor is introduced with the intention of making a clear distinction between the two different meanings of the minus sign, it would need a very careful and explicit mapping. The use of these phrases in the discourse might blur the distinction more than clarify it. However, if the intention is to illustrate the interchangeability of the signs, the fact that a subtraction of a positive number can also be interpreted as an addition of the opposite negative number, then maybe these interchangeable phrases could be useful and should in that case be elaborated on.

Comments about the two meanings of the minus sign

In accordance with previous research, this data shows that distinguishing between the two meanings of the minus sign is sometimes difficult and often a cause of misinterpretation and miscalculation. However, the analysis shows that problems connected to the two different meanings of the minus sign appear in situations that would be easy to interpret and calculate correctly if the distinction is *not* made. It is when the distinction is made that the metaphors for arithmetic, as they are known from the domain of unsigned numbers, fail to give meaning to the tasks involving signed numbers. The results suggest that the two different meanings of the minus sign need to be discerned as different, but at the same time as interchangeable. Moreover, metaphors known from the domain of unsigned numbers need an explicit and carefully made extension to open up for the distinction between the meanings of the minus sign. As shown in the historical review in chapter 2 this distinction was not made as long as number was tantamount to quantity. A negative quantity was seen as a quantity to be subtracted. Likewise, a geometrical illustration of the law of distribution used to "prove" that multiplication of two negatives is positive is in fact an illustration of a multiplication of 'two numbers to be subtracted'.

Furthermore, it was found that, in parallel with the different meanings of the minus sign, the different meanings of the plus sign is often neglected but would need to be made explicit. For instance that equity of adding a debt and giving away money should not be symbolised as $a+(-b) = a-b$ but as $a+(-b) = a-(+b)$. Only after this equity is understood should the convention of not marking positive numbers with a plus sign be obeyed.

The strong focus on negative numbers as debts in this class confined the students to an *object collection* metaphor when negative numbers appeared and they could no longer make use of a *motion along a path* metaphor to make sense of expressions. In some situations this metaphor would have been a useful thinking tool. The following section reports on the students' responses to question explicitly about the number line, showing that conceptions about the number line are not highly developed among these students, nor is the number line a frequently used representation of numbers in the classroom discourse.

7.4 The number line

The students use of, or sometimes lack of associations to a number line, seems to be related to their conception of negative numbers. Previously in this chapter it has been shown that a *movements along a path* metaphor is useful for understanding the structure of the extended number domain. In this section questions and responses concerning the number line are described in more detail. The number line was discussed at the end of the interview, typically after at least 20 minutes conversation about numbers. Q10 begins with a request to draw a number line, and afterwards the students own number line was compared with other number lines (see appendix II for more details).

> ***Q10***) Do you know what a number line is?
> Draw a number line. Tell me about your number line.
>
> Could you use the number line to find out the answer to these tasks?
> 9+2 = and 5-8=
> This question was included in G7 and G8. Students who could show these on a number line in G8 were also asked to show 3 - (-4) and -3-5+2.

Number line conventions

After having drawn their own version of a number line the students were shown a variety of number lines and asked to comment on them. Faced with a normal looking mathematical number line all students recognized it and remembered having seen such a line, even if they initially did not know how to draw one. This exercise in G6 could have influenced the result of the subsequent interviews in as much as the students might remember the number lines they had seen. Such an influence is suggested by the fact that many of the students in G7 and G8 drew number lines very similar to the ones they had been shown the year before (ranging from -5 to +5 with 0 placed in the middle). Figure 7.3 shows two of the number lines the students were asked to interpret and comment on.

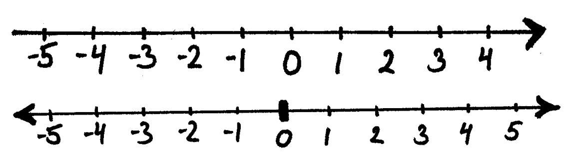


FIGURE 7.3: Two number lines from Q10: the Conventional and the American.

The first of these lines is drawn according to the Swedish convention: the line is horizontal and the value of the numbers increases to the right, which is marked by an arrow on the right. This type of number line is here named a *Conventional number line*. In some other countries both ends of the line are marked with arrows indicating that the line continues in both directions. This is common in American literature and is therefore here named the *American number line*. One detail in this drawing differs from the American convention and that is the highlighted point at the location of zero. This emphasis was made to see if the students found it important to mark zero as a starting point.

Findings

Concerning number sense issues, it was found that when students draw extended number lines they prefer to have zero in the middle and symmetry around zero. They also put arrows on both sides in accordance with an American number line although all number lines in their book are of the conventional type. This shows that zero is seen as an important benchmark number and that a divided number line is the dominant conception.

Although many students use metaphorical expressions related to numbers as places or movements along a path (in expressions such as "higher up" below zero", "after zero") they do not in these results spontaneously use a number line as a metaphorical embodiment of numbers and many of them do not know how to draw a 'normal' number line when asked to do so in grade 6. Using a number line to show addition and subtraction is not a known procedure to all the students. Concerning the use of metaphors: *number as point on a line* (position along a path) is present in the students reasoning; *number as a movement on the line* (*motion along a path*) is less common and does not come into mind for some of the students even after the teaching about negative numbers in grade 8; and *number as distance between points* (measurement) only occurs in connection to the number line on a few occasions.

Empirical data

In this section students' responses to question Q10 in the interview are explored and related to observation of how the number line was presented in the textbook and by the teacher (see also chapter 6 for more observations about number lines in the classroom). Table 7.6 shows how the students responded to Q10.

TABLE 7.6: Students responses to Q10, according to the levels described in the legend below.

	Q10: number line		
	G6	G7	G8
Anna	**R**$^{+}$	**R** +	**R** + -
Axel	0	**R**$^{+}$	**R** + -
Elke	0	**R**$^{+}$ + -	**R** + -
Erik	**R**$^{+}$	**R** + -	**R** + -
Fia	0	0 +	**R** + -
Freddy	0	0 +	**R**
George	**R**	**R**	**R**$^{+}$ + -
Hans	**R**	**R**	? + -
Lina	**R**$^{+}$	**R** + -	**R** ..+ -
Linda	0	**R**$^{+}$ +	**R**$^{+}$ + -
Lotta	**R**$^{+}$	0	**R** + -
Malin	0	**R**$^{+}$ -	**R** + -
Martina	**R**	**R** + -	**R** + -
Olle	**R**$^{+}$	**R** +	**R** + -
Ove	0	**R**	**R**
Paula	**R**$^{+}$	0 + -	**R**$^{+}$
Petra	0	**R**$^{+}$ + -	**R** + -
Sean	0	**R**$^{+}$ +	no data
Tina	**R**$^{+}$	**R**$^{+}$	**R** + -
Tomas	**R**$^{+}$	**R**	**R** + -
Viktor	0	**R**$^{+}$	**R**

level	drawn number line
0	no 'normal' number line
R^{+}	positive number line
R	extended number line
+	can show 9 - 2 on the number line
-	can show 5 - 8 on the number line

Improper number lines

Concerning the number line, the first surprising result is that several students were not able to draw a number line when asked to do so in G6. Three students (Freddy, Ove and Viktor) did not associate the term *number line* to mathematics. In order to understand their associations it is necessary to explain the different meanings of the Swedish word *tal*: apart from meaning *number*, and being used to speak about *tasks*, it also means *speech*. *Att tala* is the common verb meaning *to speak*; *att hålla tal* means to *give a speech*; *på tal om* means *speaking about*. When asked to draw a number line in G6, Ove draws a line representing volume when somebody is speaking louder and louder, Freddy draws a dash followed by an indication of writing, thus representing speech as in a written dialogue, and Viktor draws a line with a speaking man standing on it, see figure 7.4.

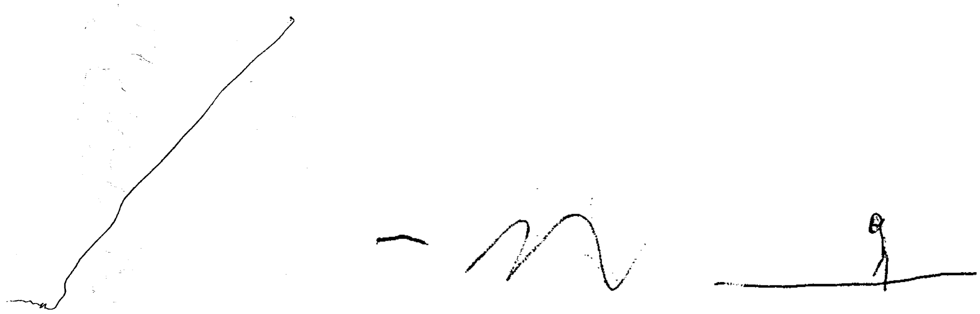


FIGURE 7.4: Ove's, Freddy's and Viktor's number lines, G6.

Some number lines were associated with numbers in an unconventional way; e.g. the line at the bottom of a vertical algorithm (Axel), a line with some random numbers on (Petra), and a sequence of numbers (Linda, Malin). Elke started on a number line but did not place any numbers on it and could not finish it, see figure 7.5.

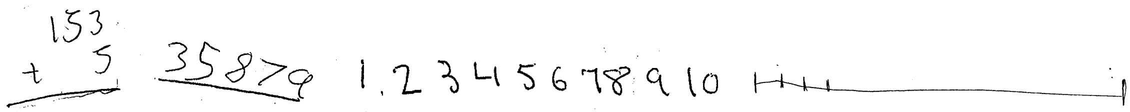


FIGURE 7.5: Axel's, Petra's, Linda's and Elke's number lines, G6.

Unified or divided number line: what the arrow stands for

All the number lines in the textbook are drawn according to the Swedish convention. In Q10 the students are asked about the arrows, and the common interpretation is that the arrow shows that "it goes on in that direction". Despite the fact that all the number lines the students have ever encountered only have one arrow, most of them state that two arrows is the proper way because "the line goes on in both directions"[55]. The students' reactions to questions about the arrow suggest that it has not been brought to their attention before and they have never really reflected about what it means and why it is there.

When the arrow is related to the size of numbers it is the magnitude that is focused, as shown in excerpt 7.36. Anna has preferred the American number line with a highlighted 0 and arrows in both directions to the conventional number line with an arrow only on the positive side.

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[55] In the pilot study one student connected to her knowledge of geometry, stating that a line always goes on indefinitely, otherwise you need to mark it as a segment showing where it ends, so the arrow must mean something else. In this class no students makes that comment. One student in this class interprets the arrows as end markings rather than arrows.

EXCERPT 7.36: Anna interview G6, Q10: comments on presented number lines.

6-424	Anna	yes that one is much better than that, other one was
6-425	I	mm? Why?
6-426	Anna	because here it goes on higher and higher up (points at the conventional number line), but here the 0 is all the time in the middle (points at the American number line)
6-427	I	yes? And so it goes on higher...? higher here...?
6-428	Anna	yes, from 1, and then up and then here from mi, minus 1 and up

Anna describes the increase as ranging "from 1 and up" and "from minus 1 and up" [line 6-428] which is consistent with the magnitude of numbers but not the value of numbers. Anna also points out the importance of having 0 in the middle [line 6-426]. Anna's conception of the number line as described in this excerpt is a divided number line, suggesting that she has not yet conceptually unified the number line; that is to see it as one line, one axis, instead of two semi-lines opposite one another with different symbols. This reconceptualization was described by Glaeser (1981) as one of the important landmarks in the historical evolution of the concept of negative numbers. Anna's description is representative for most of the students in this class.

Number lines outside the maths book

When asked if they have ever seen a number line anywhere outside of the mathematics textbook most of the students reply no, or associate with a thermometer or a time line. In G8 some also associate with a diagram or the axes in a Cartesian coordinate system, which is also an artefact of the mathematics classroom. Nobody mentions rulers or other types of measuring scales apart from the thermometer.

Extending the number line

In G7 most of the students draw proper mathematical number lines, and eight students have already included the negative numbers although they have not yet appeared in the classroom discourse. This could be an effect of the repeated interview since the interview one year earlier (G6) also included some extended numbers lines in Q10. In G8 all the students draw a number line, most of them representing **R**. Although the period between the two interviews has not involved any explicit teaching about the number line, it has appeared in the textbook in the chapter about negative numbers, and on a few occasions been used by the teacher to illustrate movements when teaching about negative numbers. The interviews themselves could also have contributed to the development of student's conceptions of a number line, but it cannot be ruled out that it is a development and an extension of a grounding metaphor that appears as a consequence of the extension of the number domain even if it is not made a prominent part of the teaching.

Movements on the number line

The results in this study do not support the idea that a *motion along a path* metaphor is a grounding metaphor necessary for understanding numbers. The interviews indicate that using a number line to conceptualize addition and subtraction as movements along the line is an unknown procedure for many of the students. Paula, Freddy, Ove and Victor can not think of a way of illustrating either 9+2 or 5-8 on the number line in G8. Ove and Viktor do not do it in G7 either, which strengthens the interpretation that they do not make use of a *motion along a path* metaphor when thinking about numbers. In these results, motion along a number line can be seen as a teaching tool, not explicitly used before the content of negative numbers was introduced.

In the mathematics textbook for grade 6 the number line appears explicitly[56] in a chapter about decimals (Carlsson et al., 2004a). There are seven examples and 22 tasks with number lines, all similar to the one in figure 7.6. On task 34 the student is supposed to write what number the arrows point to, and on task 35 the text says: "Copy the number line and draw arrows pointing to the numbers,". There are no examples of operations as motion along a number line in the textbook. In the course of teaching negative numbers the teacher, on a few occasions, draws number lines and interprets addition and subtraction on the number line as an alternative to a money metaphor. When this is done it is commonly accompanied with a comment about the inefficiency of always drawing number lines, emphasizing "mathematical thinking" instead. On one occasion the teacher tries to represent a subtraction of a negative number with an arrow directed to the left on the number line but is left in a state of confusion when it does not work, and so tells the students that they need to apply the rule "two minus make a plus" instead, before showing it on the number line.

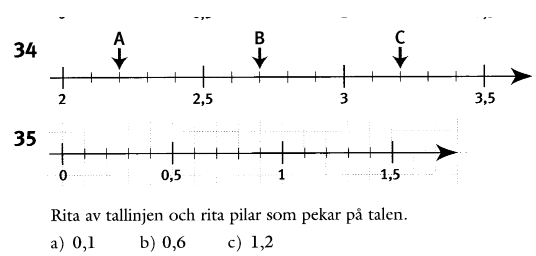


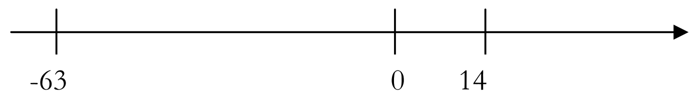
FIGURE 7.6: Number line task from the textbook. (Carlsson et al., 2004a, p 13).

Viktor's somewhat confused conception of the number line comes to be expressed in a teacher-student interaction when Viktor is trying to solve the problem of how old Emperor Augustus was if he was born in the year -63 and died in the year 14. Without success he has tried the subtraction 14 - 63 to get a result. The teacher draws a number line, naming it a *year line*[57]. She marks the year

[56] Number lines also appear implicitly in chapters about measures and statistics but in those parts of the textbook the term *number line* is never mentioned.
[57] In Swedish: *årslinje*, not a common word, more common is *tidslinje* (time line)

of the birth of Christ, year 0, and asks Viktor to point out where the years 14 after Christ and 63 before Christ are on the line. She writes 14 and -63 as he points them out. She then says that Augustus lived all the way from -63 to 14. How long is that? Viktor is quiet for a while and then the exchange in Excerpt 7.37 takes place.

EXCERPT 7.37: video 8.7.Viktor.1



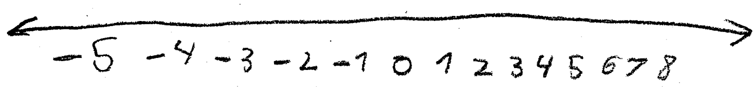
1	T	how long did he live before Christ?
2	Viktor	eh, that was, 37 years
3	T	aha, that's what you think! Now I understand. But it's like this, you know where, where is year, eh if we say year 25 before Christ where is that?
4	Viktor	… well, I guess it's there in the middle (points between -63 and 0)
5	T	there in the middle yes that was good. There is year, minus 25 isn't it? Mm, ok. And year 63 before Christ was, 60 before Christ where is that?
6	Viktor	… well that will be… that will be ehm, that will be ehm … further away there (points to the left of -63)
7	T	will it be on that side?
8	Viktor:	yes it will…
9	T	there is 25 you said?
10	Viktor:	ah yes then it's there (points to the right of -63)

Even if Viktor in a broader perspective knows that the negative numbers are placed to the left of 0 and the positive numbers are to the right of 0, it seems as if he specifically in relation to each number, thinks of the numbers as increasing to the right. Increasing is for Viktor tantamount to increasing in magnitude. In other words, he 'counts up' to the right on the number line and disregards the minus sign when doing so. To get to 0 from 63 he counts up 37, thus thinking of 100 in the place of 0 [line 2]. The teacher then asks about 25 B.C., and Viktor correctly points to a place in between -63 and 0, thus again taking the broader perspective [line 4]. Then the teacher points to -63 and asks about the year 60 before Christ, which brings Victor back to the local perspective around the number and he points incorrectly to the left of -63 [lines 5 and 6]. When the teacher relates it to 25 he points to the right of -63, since 60 must be in between 25 and 63 [line 9 and 10]. This confusion about the order of the numbers on the number line is related to the understanding of magnitude and value of numbers.

Paula and Freddy follow a slightly different learning trajectory since in G7 they do describe movements on the number line, but in G8 they do not. It could be speculated that the examples of movements along the number line that have been brought up in class in connection with the chapter on negative numbers have confused these students. In G8 Freddy explains that 9+2 can be shown as starting on 9 and going up 2 steps. When asked about 5-8 he responds as in excerpt 7.38.

EXCERPT 7.38: Freddy interview G7, Q10: showing 5 - 8 on the number line

7-402	Freddy	...yes that's a bit difficult, because it's 5 minus 8, but...I don't know if it's like this minus 5, then like this … wait (draws a number line)



7-403	Freddy	Ok. But eh, I don't know, if you take 5 minus 8, but it will be a bit difficult to work out, I would rather take 8 minus 5.
7-404	I	mm. If it had been 8 minus 5, how would you in that case have done it on the number line?
7-405	Freddy	I would just have taken, like this, then it would have been 3 left over. 8 to 5. There is 3 left over. So you know it.

Since Freddy draws a number line starting on -5 and ending on 8, he seems to connect the minus sign in the expression 5 - 8 to the number -5 rather than to a movement [line 7-402]. He knows how to do 8 - 5 and expresses it as a movement from 8 to 5 but blends this *motion along a path* metaphor with an *object collection* metaphor by stating that "there is 3 left over" instead of saying that he ends up on 3 [line 7-404]. Different aspects of numbers, grounded in different metaphors, are combined in the one explanation. Alternatively, Freddy is using a *measuring stick* metaphor seeing 8 and 5 and 3 as distances. In this case the introduction of the number line by the interviewer forces Freddy to use a metaphor he does not seem to be very accustomed to, using it more to justify a result he already knows: "So you know it." [line 7-404]

Comments about the use of a number line

In these results it is clear that many of the students do not see the number line as a general representation of numbers. They see it rather as a specific picture in some maths book tasks, possibly associated with a thermometer. Since it is a highly useful representation of the order aspect of numbers and the structure of our number system, useful in all number domains from **N** to **R,** a better knowledge of and confidence with the number line could be of help. Particularly since many students express an implicit *numbers as positions along a path* metaphor when speaking about numbers. All they need is for this implicit metaphor to become explicit in the number line. This, however, is not just a case of showing the students a number line, which was illustrated by the fact that although a number line had appeared in the text book and all students had completed tasks placing numbers on the number line, the representation did not become a spontaneously used mathematical tool. For that to happen it also needs to be frequently used in the public discourse of the classroom.

A divided number line was pointed out by Glaeser (1981) as one of the major epistemological obstacles in the evolution of the concept of negative numbers. A divided number line supports the magnitude aspect of number whereas a unified number line supports the value aspect. Since these two aspects cause a lot of confusion among the students in this study, and since they predominantly conceptualise a divided number line, the results supports Greasers claim and suggests that the process of unifying the number line is made a teaching content. A conceptual change is necessary, and in this case a metalevel awareness of the two different conceptions of a number line would make the change possible.

Having now shown examples of how students responded to some of the interview questions, and described results concerning their development of number sense and use of metaphor in relation to negative numbers, an attempt will be made to summarise by describing different learning trajectories. By comparing a variety of learning trajectories, patterns of change may become visible. The following section will conclude this chapter in an attempt to describe such patterns of change.

7.5 Patterns of change

In order to make patterns of change found in the data clear to the reader the results were synthesized into features that stood out as important when extending the number domain from $\mathbf{R}^+$ to $\mathbf{R}$. The data is not sufficient to support a claim of these being the only important features, or that they are essential. The features are brought out as important in light of the fact that they build on features identified in previous research, have showed up in the data as instances where students change their number sense, and display a variety of learning trajectories for different students. They might in many cases seem to be hierarchical, i.e. building on each other, but that is not necessarily so. The aim of this section is to describe these features and students' diverse conceptions and then to bring together all the students' different learning trajectories so that patterns of change can be visible.

Features and levels of interiorization

Table 7.7 presents an overview of features of arithmetic in the domain of signed numbers related to interview questions where the features became visible in students' expressed conceptions. As seen in the second column of the table some of the features are based on a cluster of questions from different parts of the interview. When a student's responses are in accordance with the new number domain, **R**, the student is said to have interiorized the feature (Sfard, 1991), meaning that the student has incorporated the new number domain into her understanding in such a way as to make this feature part of her mathematical discourse in a mathematically correct manner. Two levels of interiorization have been distinguished for most of the features, where the lower level can be seen as vague or incipient, and the higher level as fully adequate[58]. Using Sfard's (2008) more recent terminology, the lower level indicates that the student is aware of the new extended number domain discourse mainly as a "discourse-for-others", and at the higher level the student has made it a "discourse-for-herself". The different features are labelled A to G. The lower level of interiorization of a feature is marked by a lower-case letter and the higher level by a capital letter, for instance g and G for the two levels of feature G. Additionally; there are two features in the table named NL_1 and NL_2, showing students' use of the number line, based on the interview question Q10. For NL_1, the lower level of interiorization is marked $\mathbf{R}^+$ and indicates that the student drew a positive number line starting at 0. Hence, a positive number line is taken to be part of the student's mathematical discourse. **R** indicates a number line including numbers on both sides of 0. For NL_2, the symbols + and - indicate that the student could show 9+2 and 5-8 respectively as movements along a number line in a mathematically intelligible way. After the presentation of this table, students' individual learning trajectories will be displayed in table 7.8.

[58] In this work there is no clear distinction made between interiorization and condensation (Sfard, 1991) since the idea of condensation was found to be difficult to discern among student's responses. Instead, interiorization is characterized as evolving, in this material showing up in two levels.

TABLE 7.7: Features of negative numbers found in the data, related interview questions, and a description of two levels of interiorization of each feature.

Feature		**Interview question**	**Lower level of interiorization**	**Higher level of interiorization**
A	*Negative numbers exist* in relation to zero, or as temperatures, and their value is correctly conceived.	*Q1*:smallest number and *Q8*: relative size of numbers.	**a**: At least two of the four instances of describing size of numbers are given a correct answer in **R**, the rest in $\mathbf{R}^+$.	**A** : All four instances of describing size of numbers are given a correct answer in **R**.
B	Acceptance of a *negative difference*, i.e. the result of a subtraction can be less than zero.	*Q5*: quick response to 2-5 and 6-27 and *Q7*: 3 - 7	**b**: Correct answer within **R** is given for the interpretive question but not for the quick response.	**B**: Correct answers within **R** are given for all questions.
C	*Subtraction* is treated correctly, i.e. not commutative, not associative	*Q2, Q4, Q5, Q7*	**c**: Subtraction is treated correctly in all calculations but the quick response.	**C**: Subtraction is treated correctly in all calculations.
D	*Subtraction*: a positive number can be subtracted from a negative number	*Q7*: -6-2 and (-3)-1-2	**d**: One of the two tasks is correctly interpreted and solved.	**D**: Both tasks are correctly interpreted and solved.
E	*Operations and size*: in **R**, the result of both addition and subtraction can be larger or smaller than the first term.	*Q3*: addition and subtraction	**e**: one of the operations is interpreted within **R**	**E**: both addition and subtraction is interpreted within **R**
F	A negative number can be *subtracted* by rewriting it as an addition	*Q11*: (-3) - (-8) = [only in G8]	**f**: Correct answer, but student indicates uncertainty.	**F**: Correct answer (-3) - (-8) = 5 student certain
G	Negative numbers can be correctly *multiplied*	*Q11*: (-2) · (-3) = [only in G8]	**g**: Correct answer, but student indicates uncertainty.	**G**: Correct answer (-2) · (-3) = 6 student certain
NL_1	Number line: what it looks like	*Q10*: Draw a number line	$\mathbf{R}^+$: Draws a correct number line starting on zero	**R** : Draws a correct number line with both positive and negative numbers
NL_2	Number line: using it to represent addition and subtraction.	*Q10*: Show me on the number line 9+2 and 5-8	+ or – One of the operations is shown as movements on the number line	+ – Both operations are shown as movements on the number line

Learning trajectories and patterns of change

In table 7.8 features of the extension of the domain of numbers from $\mathbf{R}^+$ to $\mathbf{R}$ are listed vertically, and for each student the level of their answers to the corresponding questions in the interviews are noted according to table 7.7 above. Each student is represented in three horizontal rows, one for each interview G6, G7 and G8. When all answers from a student are in $\mathbf{R}^+$, or mathematically incorrect, and no mention of negative numbers is made during the whole interview the level of interiorization for all features A-G is labelled 0. In the table there are also two letters underneath each student that refer to school marks. The fist letter indicates the mark given to that student on the test on negative numbers at the end of the teaching sequence early in grade 8, and the second letter refers to the school mark in mathematics given at the end of grade 8, soon after interview G8. The possible marks are: IG = not pass; G = pass; V = pass with distinction; M = pass with special distinction. Students are ordered in the table in accordance with how many features they had interiorized in G8, with the highest number of interiorized features at the top of the table.

As an example, look at the student Erik at the bottom of the first section of the table. The first of his three rows tells us that during the interview made in grade 6 he drew a proper positive number line. He had interiorized feature A (existence of a negative numbers) at the higher level, but none of the other features. His second row tells us that in grade 7 he drew a number line with positive and negative numbers and could show both addition and subtraction as movements on the number line. Ha had by then interiorized features A, B, C and D at the higher level. His third row is similar to the second but for some reason he is no longer sure about feature D. He shows feature E at the higher level and F at the lower level of interiorization. The letters V - V indicate his marks; he had pass distinction (V) on the test on negative numbers as well as on the end of year mark in mathematics. Erik's learning trajectory shows that he made great progress about negative numbers before the teaching sequence on negative numbers but fell back on feature D as a result of the formal introduction of these numbers.

The table can also be viewed column wise to see the learning trajectory of the class concerning specific features. For instance look at column C. The first four students have interiorized this feature (proper treatment of subtraction) long before the teaching of negative numbers, the following six students do it in grade 8, perhaps as a result of the teaching about negative numbers, but among the rest of the students this feature is almost absent.

TABLE 7.8: Students' levels of interiorization of each of the features described in table 7.7, as expressed in the three interviews in grade 6, 7 and 8 (* indicates missing data).

NL	NL	**Student**	**Grade**		A	B	C	D	E	F	G
R		Martina	6		A		C	d			
R	[+ -]		7		A	B	C	D			
R	[+ -]	M - M	8		A	B	C	D	E	F	G
		Petra	6		a	b					
R+	[+ -]		7		a	B	C	d			
R	[+ -]	V - M	8		A	B	c	D	E	F	G
R+		Tomas	6		A	B	C				
R			7		A	B	C	D			
R	[+ -]	M - V	8		A	B	C	D	E	F	G
R		George	6		A	b	c	d			
R	*		7		A	B	C	D			
R+	[+ -]	M - M	8		A	B	C	D	E	F	
R		Hans	6		a	b	c				
R			7		a	B	C	d	e		
*	[+ -]	V - M	8		A	B	C	D	E	F	
R+		Lina	6	0							
R	[+ -]		7		A						
R	[+ -]	M - V	8		A	B	C		E	F	G
R+		Olle	6	0							
R	[+]		7		A	b	c				
R	[+ -]	G - V	8		A	B	C	d	E		G
		Linda	6	0							
R+	[+]		7		a	B	c	d			
R+	[+ -]	G - V	8		A	B	C	D		F	g
R+		Erik	6		A						
R	[+ -]		7		A	B	C	D			
R	[+ -]	V - V	8		A	B	C		E	f	
R+		Tina	6		A	B					
R+			7		a	B					
R	[+ -]	V - V	8		a	B	C	D		F	

continued on the next page…

Table 7.8: continued…

NL	NL	**Student**	**Grade**		A	B	C	D	E	F	G
		Fia	6		a						
	[-]		7								
R	[+ -]	G - G	8		A	B	C	d		F	
R+		Paula	6		a	b		d			
	[+ -]		7		a	b		D			
R+		G - G	8		A	B		D		F	
		Freddy	6	0							
	[+]		7		A						
R		IG - G?	8		A	B	C	D			
		Elke	6		a						
R+	[+ -]		7		a						
R	[+ -]	G - G	8		A	B		d	e	F	
		Malin	6	0							
R+	[-]		7		a	B					
R	[+ -]	G - V	8		a	b	c	d		F	g
		Axel	6		a						
R+			7		A			d			
R	[+ -]	G - V	8		A	B			e	F	
		Ove	6	0							
R			7		A	b					
R		G - G	8		A	b		d	E	f	
R+		Anna	6	0			c				
R	[+]		7		a				e		
R	[+ -]	IG - G	8		A	B			e	F	
R+		Lotta	6		a						
			7		a						
R	[+ -]	G - V	8		a	B		D		f	
		Viktor	6	0							
R+			7		a						
R		G - G?	8		a			D		F	
		Sean	6	0							
R+	[+]		7		a						
*	*	IG - IG	8	*	*	*	*	*	*	*	*

Findings and comments

There is a lot of interesting information to be extracted from table 7.8. In this section findings concerning a few areas of interest are discussed.

Prior knowledge

The first three students in the table are high achievers who also receive high marks. They have appropriated the first four features already in G7 (A: existence of negative numbers and value of signed numbers, B: negative difference, C: subtraction, D: subtraction of a positive number from a negative number). When this is done in G7, the extension of the mathematical discourse to include also subtraction and multiplication of negative numbers is possible to master in the time frame of the few weeks that was spent on teaching negative numbers. Most of the other students are uncertain about one or more of the first four features when the teaching of negative numbers starts. The picture suggests that the level of prior knowledge of a concept greatly influences a learning trajectory.

Subtraction

Many of the students do not treat subtraction correctly, even when no negative numbers are included in the calculation. This is seen in column C; very few of the students in the bottom half of the table have interiorized this feature at all. Proper treatment of subtraction does not imply knowledge of negative numbers so this feature could be fully interiorized as prior knowledge. However, Sfard (1991) points out that in the process of reification the higher level reification and the lower level interiorization are prerequisites for each other. The reification of one process only occurs when it is being used as an object in a new process. Applied to the case of negative numbers it would mean that reification of subtraction is a prerequisite for negative numbers, but interiorization of negative numbers is also a prerequisite for the reification of subtraction. The conclusion of that argument is that in the teaching of negative numbers proficiency in subtraction cannot be taken for granted but needs to be discussed and highlighted when negative numbers are introduced. That is certainly the case in this class.

Subtracting and multiplying negative numbers

Many of the students showed great success in interiorizing feature F; the capability to correctly subtract a negative number: (-3) - (-8). This is perhaps surprising, but could be explained by the teaching that strongly emphasized the procedure of replacing two minus signs with a plus sign and thus rewriting the task as (-3) + 8. What has been interiorized in relation to this feature is to a great extent a procedural knowledge, a routine. The frequency of correct solutions to this task is 90% (although some of them with uncertainty). This can be compared to the results of the same task in a study of 99 prospective pre school teachers (Kilhamn, 2009a) where the frequency of correct answers was 70%. On the multiplication task illustrated by feature G: (-2) · (-3), this class had a correct solution frequency of 40% compared to the prospective pre school teachers 55%. It could be said about the teaching of negative numbers in this class that much time and effort was put into gaining procedural knowledge about subtraction of negative numbers, and very little on multiplication of negative numbers. This observation coheres well with the results of feature F and G. Furthermore, it must be noted that features F and G were not measured by a cluster of questions as were the features A–E, but by one single question. If more tasks had been included the results might have been different.

Change of conception

Feature E concerns the understanding of operations and what effect they have on numbers. Students need to go through a change of conception from that of addition as "making larger" and subtraction as "making smaller", which is an interpretation consistent with their experiences of unsigned numbers representing quantities and thus correct in $\mathbf{R}^+$. In the interviews in G6 and G7 most of the students express this conception very steadily, with self confidence. In G8, after they have had experience in class of adding and subtracting negative numbers, some of the students change their conception of these operations, interiorizing feature E; that a result of an addition or a subtraction can either be smaller or larger than the first term, depending on the sign of the second term. However, eight students do not interiorize this feature at all and four students only at the lower level. Hence the old conception of subtraction as making smaller still dominates and might need to be replaced or at least specified to be true only under certain circumstances.

Number line

Table 7.8 shows that in grade 7, before the teaching of negative numbers, two of the first nine students drew positive number lines and seven drew number lines representing **R**, whereas among the remaining twelve students only two drew number lines representing **R** and four did not draw any proper number line at all. This suggests that the level of interiorizing different features of negative numbers is related to the image of an extended number line. The results presented do not give any indication as to the cause and effect of using a number line and interiorizing features of negative numbers, but they do suggest a connection between the two. It could be argued that they are prerequisites for each other. The results show that this class is not very familiar with the number line as a thinking tool before the introduction of negative numbers, something they would most probably have profited from.

CHAPTER 8

Conclusions and Discussion

This research project set out to investigate how students make sense of negative numbers, and more specifically what role models and metaphors play in that process. Largely it can be characterized as an explorative study and it has not produced any simple answers. On the contrary, the study has illuminated the complexity of mathematical thinking and the richness of the concept of negative numbers. Although background and results of the study are separated in the thesis, much of the background also belongs to the result. What was learned about the concept of negative numbers through the historical and didactical reviews in chapter 1 and the analysis of metaphors in chapter 3 made possible the insights made in chapter 5, 6 and 7. In this last chapter the results are discussed and related to the research questions posed in chapter 4.1. Implications for teaching are made in the hope that practice will benefit from the research presented here.

8.1 The role of metaphors

In this section the role of metaphors in the studied classroom is taken as the point of departure for a discussion of research question 1: *How is metaphorical reasoning enacted in a classroom discourse in the context of negative numbers?*

Revisiting the grounding metaphors

Lakoff and Núñez (2000) describe four grounding metaphors and claim that they, together with a metaphor of rotation, are sufficient to understand arithmetic with signed numbers. In the empirical material in this study, three of the grounding metaphors mainly appear, and there is no indication of a rotation metaphor. Historically, some models presented during the 19th century as a result of the invention of the extended number line used directions of forward and backward and reversing direction that could be interpreted as a rotation. However, it is unclear if they ever referred to rotation. Going from forward to backward does not necessarily entail a rotation. This suggests that the rotation metaphor is not an innate metaphor for signed numbers, nor is it the origin for the existence of negative numbers. Negative numbers did not "come from" this metaphor, which does not rule out that it could be a didactically useful metaphor. Historically, and in the empirical material, a mirror metaphor is present in terms of opposite sides, opposite directions and opposite numbers.

Are the four grounding metaphors described in the theory consistent with the findings of this study? When summarising what was found in previous literature and which metaphors were present in the empirical data, it became evident that the grounding metaphors identified by Lakoff and Núñez (2000) need to be described in more detail. In chapter 3 a theoretical analysis was made showing that when the number domain is extended the grounding metaphors are changed and in fact become *different* metaphors. Also a *number as relation* metaphor was described and suggested as a complement to the four grounding metaphors.

When doing arithmetic, there are numbers and operations to deal with. A basic arithmetic equation includes three numbers and one operation ($a * b = c$). Very often the three numbers have different metaphorical underpinnings. For example a number can be either a *location along the path* (point) or a *movement between two locations*, either an *object collection* or a *relation between two object collections*, and either a *point* or a *measured length between two point* (distance) or a *comparison relation between two measures*. In a similar way the operations can have different metaphorical meanings. It could of course be discussed whether these different metaphorical meanings are to be seen as belonging to the same grounding metaphors, or if they are to be described as different metaphors. When several metaphors are combined Lakoff and Núñes speak of metaphorical blends.

Findings presented in chapter 6 and 7 showed that three of the grounding metaphors were often used in the classroom discourse. The following tables (8.1 to 8.3) show how the different metaphors and metaphorical blends present in this study can make sense, with different meanings for numbers in three types of subtractions.

Motion along a Path metaphor

Table 8.1 shows four different versions of the *motion along a path* metaphor. The first two metaphors in the table are very similar: the subtle difference lies in the wording. It was found that some students, and sometimes the teacher, made use of a metaphor of movements along a number line (often represented by arrows or jumps on a number line) to make sense of subtraction of positive numbers and addition of negative numbers in either of the two variations described as nr 1 and nr 2. However, when a subtraction of a negative number appeared they did not blend the metaphor with a rotation metaphor to accommodate two successive indications of direction but chose to simply rewrite the double negation as an addition in accordance with a procedural rule, and then think of the addition as a movement along a number line. Metaphor nr 4 did not appear in the data but was present in a metaphor blend with a *measurement* metaphor, particularly when temperatures and time lines were involved, see metaphor blend nr 10 in table 8.3.

TABLE 8.1: Doing subtractions using a *Motion along a Path* metaphor.

Metaphor		**-6-2 = -8**	**(-6)-(-2) = -4**	**6-(-2) = 8**
1:	Two movements without direction and a total movement without direction, all minus signs represent subtraction indicating direction; an implicit starting point.	6, 2 and 8 are all movements; all three minus signs are subtraction signs indicating direction towards the minus side. Moving first 6 then 2 to the left is the same as moving 8 to the left.	*Does not make sense,* would need to be blended with a rotation metaphor.[59]	
2:	Two movements with direction and a total movement with direction, implicit addition.	-6, -2 and -8 are all movements with direction; there is no subtraction but an implicit addition. Two movements of 6 and 2 to the left is the same as one movement of 8 to the left.	*Does not make sense,* would need to be blended with a rotation metaphor.	
3:	Starting point and movement results in end point, subtraction indicates direction of movement.	-6 and -8 are points, 2 is a movement and the subtraction sign indicates direction towards the minus side.	*Does not make sense,* would need to be blended with a rotation metaphor.	
4.	End and starting point, subtraction indicates movement in between, result is movement with direction.	-6 and 2 are points, -8 is the movement with direction needed to get from 2 to -6.	-6 and -2 are points, -4 is the movement with direction (-) needed to get from -2 to -6.	6 and -2 are points, 8 is the movement with direction (+) needed to get from -2 to 6

[59] The motion along a path and rotation metaphor blend is not further elaborated here since it did not appear in the study, see chapter 3 for more details.

Object Collection metaphor

Table 8.2 shows different versions of the *object collection* metaphor (nr 5, 6, 7) along with an *object collection and relation* metaphor blend (nr 8).

TABLE 8.2: Doing subtractions using an *Object Collection* metaphor and an *Object Collection and Relation* metaphor blend.

Metaphor		**-6-2 = -8**	**(-6)-(-2) = -4**	**6-(-2) = 8**
5:	Three object collections, minus signs indicate taking away, an implicit arbitrary starting collection.	An arbitrary starting collection; 6, 2 and 8 are all object collections All minus signs are subtractions. Taking away first 6 and then 2 from any collection is the same as taking away 8.	*Does not make sense,* would need to be blended with a metaphor for taking away what was taken away.	
6:	Three object collections, implicit addition.	-6, -2 are -8 are all objects collections there is no subtraction but an implicit addition.	*Does not make sense,* would need to be blended with a metaphor for taking away what was taken away.	
7:	Two object collections, subtraction indicates taking the second collection away from the first collection.	*Does not make sense,* would need to be blended with a relation metaphor, see nr 8.	-6, -2 and -4 are all object collections, subtraction indicates taking a collection of 2 negatives away from a collection of 6 negatives.	*Does not make sense,* would need to be blended with a relation metaphor, see nr 8.
8:	*Object collection and relation metaphor blend*: the first number is a relation and the second number is a collection, subtraction indicates taking away and thus altering the relation (balance).	-6 and -8 are balance relations, 2 is an object collection and the subtraction sign indicates taking away 2 positives and thus altering the balance from -6 to -8.	-6 and -4 are balance relations, -2 is an object collection, the subtraction sign indicates taking away 2 negatives and thus altering the balance from -6 to -4.	6 and 8 are balance relations (understood as +6[60] and +8), -2 is an object collection, the subtraction sign indicates taking away 2 negatives and thus altering the balance from +6 to +8.

[60]Whenever a relation metaphor is introduced the relation numbers always need to be interpreted as signed numbers. There is no 8; only 8 more (+8) or 8 less (-8).

Metaphors nr 5 and 6 require a meaning for a double negation "taking away what was taken away"; introduced in the data as "taking away a debt is the same as earning money". That was rejected by some of the students, mainly because they were thinking within metaphor nr 7, where taking away means taking away *from* something. This is an example of the difference between metaphors and models. Both teacher and students use the same money model, but since they have different metaphors for which the model is a source domain they do not agree on the narratives. However, the procedure of exchanging the two minus signs into a plus sign implied rewriting the double negation as an addition, and subsequently using a different metaphor to make sense of the addition. Metaphor nr 8 was implicitly present in tasks connected to the money account model in the textbook, but the model came to be spoken of more in terms of metaphor nr 7, i.e. as object collections rather than a balance relation.

Measurement metaphor

Table 8.3 shows a *measurement* metaphor (nr 9) along with two different *measurement* metaphor blends (nr 10 and 11).

TABLE 8.3: Doing subtractions using a M*easurement* metaphor, a *Measurement and Motion along a Path* metaphor blend, and a *Measurement and Relation* metaphor blend.

metaphor		**-6-2 = -8**	**(-6)-(-2) = -4**	**6-(-2) = 8**
9:	Two points on a line, subtraction indicates the measured distance between them. The distance is an absolute value.	Only makes sense as an absolute value: $\|-6-2\| = 8$	Only makes sense as an absolute value: $\|(-6)-(-2)\| = 4$	Only makes sense as an absolute value: $\|6-(-2)\| = 8$
10:	*Measurement and motion along a path metaphor blend*:: two points on a line, subtraction indicates a directed difference or a movement from the second point to the first.	-6 and 2 are points, -8 is the directed difference from 2 to -6 (down 8)	-6 and -2 are points, -4 is the directed difference from -2 to -6 (down 4).	6 and -2 are points, +8 is the directed difference from -2 to 6 (up 8).
11:	*Measurement and relation metaphor blend*: two measures described as values and a comparison between the two.	-6 and +2 are values and -8 is a comparison relation between these values indicating that -6 is 8 less than +2.	-6 and -2 are values and -4 is a comparison relation between these values indicating that -6 is 4 less than -2.	6 and -2 are values and +8 is a comparison relation between these values indicating that -6 is 8 more than +2.

Numbers as measurements often occurred in the data, for instance when signed numbers were given meaning as temperatures, or years before and after year 0. Metaphor nr 9 was used only when tasks involved the last type of subtraction where the absolute value gave a mathematically correct (positive) difference. Since no differences with a negative direction appeared the students had no opportunity to realise that the distance between two numbers was in fact an absolute value. The notion of absolute value was not introduced and the limitation of the metaphor was not made explicit, which caused some problems.

Metaphor blend nr 10 was present in word problems with for example temperatures rising or falling. It was also implicit in the introductory task when a counter was moved along a number line to keep track of economic values. Conceptually it blends ideas of length measures with ideas of value, representing all kinds of values as lengths. A directed difference can be referred to as a *vector*, yet this analogy does not appear in the data of this study. The metaphor blend is also similar to the geometric model recommended by Freudenthal (1983), where all numbers are interpreted as *arrows*. In the data it was exclusively the number operated on that was referred to as a directed difference or a movement and drawn as an arrow. Metaphor blend nr 11 did not appear in the data but was dormant in tasks involving temperatures and time lines, for example in the task about the age of Emperor Augustus.

A different application of this metaphor blend was shown by Kullberg (2010) who reported a learning study where the teachers interpreted economy as a discrete measure which is also an object collection: an economy of -6 is the same as owing somebody 6, and the comparison -6-2 = -8 is when I owe 6 and you have 2 which means that *I have 8 less* than you. The comparison is from the first to the second number. This is not quite straightforward: it would be more clear to say that if *you give away 8* we become equal, which is a transaction from the second to the first number that can be described as "down 8", as in the *motion along a path* metaphor nr 4 in table 8.1, and similar to metaphor nr 10.

Object Construction metaphor

The fourth grounding metaphor, arithmetic as object construction and numbers as constructed objects, did not come up in the work with integers in this data, except for the idea that zero can be seen as constructed of equal amounts of opposite objects. Certainly there were instances when rational numbers with several decimals were given meaning as constructed by smaller parts but that was not relevant for them as signed numbers.

Implications concerning theory of conceptual metaphors

The analysis has shown that conceptual metaphor theory is useful to analyse teaching and learning about negative numbers. The claims Lakoff & Nùñez (2000) make about the four grounding metaphors seems valid but not sufficiently elaborated. The claim that non signed numbers (natural and rational) need these four metaphors to be fully conceptualised is strengthened by the results of this study that students make use of several metaphors in their process of making sense of negative numbers. Although the teacher tried to stick to one metaphor (numbers as object collections using a money model) it was not always accepted by the students who connected numbers and operations to other metaphors as best they could.

The tables 8.1 to 8.3 show eleven different metaphors and metaphorical blends for subtracting negative numbers. The best ones as far as coverage goes are the metaphorical blends nr 11, nr 10, and nr 8, and the path metaphor nr 4 which is closely related to blend nr 10. It seems as if the complex structure of signed numbers needs complex metaphorical structures. Even if there are metaphorical blends that sufficiently cover and make sense of all additions and subtractions, choosing only one of these metaphors would greatly restrain the learner simply by not relating to his/her conceptions of non signed numbers.

Since mathematical objects change over time as they are incorporated into and adapted to a mathematical structure, there is not a one-to-one relation between the emergence of mathematical objects, the nature of these objects and the learning of these objects. In accordance with other authors (e.g. Font et al., 2010; Schiralli & Sinclair, 2003), the theory of conceptual metaphors and the methodology for analysing mathematical ideas and mathematical discourse that have emerged from the works of Lakoff and Núñez is seen as useful for describing the emergence of mathematical objects, such as negative numbers, both historically and for individuals learning them, but not sufficient for describing the full nature and structure of these objects. Mathematical objects have both metaphorical meaning and intra-mathematical meaning.

Rationale for introducing metaphors

One of the research questions (question 1b) concerned the rationale for introducing metaphors when teaching negative numbers. Answers to that question were reported in chapter 6 and described as the teaching-learning process for metaphorical reasoning. That description concerned the classroom investigated in this study and may not be representative for other classrooms. However, the textbook used and the teaching style of the participating teacher has been described in a way that readers may recognise the practice and make judgements about external validity. In this particular case it seems as if the rationale for introducing metaphors by using real world contexts and models was based on a belief that abstract mathematical concepts need to be made concrete to be understood. That same rationale was also expressed by the students participating in the pilot study reported in chapter 1.7. The final goal for teaching the topic did not appear to be to understand the abstract aspects of signed numbers but simply to learn how to handle these numbers in tasks presented in class. The object of learning was the capability of solving arithmetic operations with signed numbers. Heading for such a goal, metaphors served as tools for making sense of specific mathematical expressions, and metaphorical reasoning became a teaching goal. Another tool to use when metaphors failed to make sense was a sign rule. Consequently, the students were given a set of thinking tools consisting of disconnected instances of metaphorical reasoning and sign rules.

The question is whether metaphorical reasoning could be a teaching tool instead of a teaching goal, which touches upon the importance of the '*why'* of teaching mathematics. Do we teach the topic of negative numbers in order to give students the capability of solving additions and subtractions with negative numbers, or is it in order to help the students engage in the creative work of a mathematician, to discover how signed numbers work as part of an algebraic system, and to realise that mathematics is a human invention, a set of rules that are commonly agreed upon and sometimes altered? Most students will not need the capability of adding and subtracting negative numbers once they leave school. If they ever encounter such operations they will have machines that carry out the procedures efficiently. Students who will need this capability are the ones who pursue a career which includes university mathematics, where abstract thinking and mathematical creativity are even higher valued capabilities. The terms *relational* and *instrumental* understanding were introduced by Skemp (1976) as a means of describing these two quite different goals for mathematics education. An instrumental view of mathematics describes a view where understanding mathematics is tantamount to being able to perform mathematical calculations and using mathematical tools for problem solving. This, according to Skemp, is probably the most common view among students, parents, and even among school teachers, of what understanding mathematics is about. Contrary to this is the view of a mathematician, for whom understanding mathematics is about seeing the structures and patterns of the number system and the relations between and the potential of different mathematical concepts. In this second view procedures and calculations become important as tools, but are not the goal itself.

The introductory lesson for the topic of negative numbers in the studied class started with a dice game. The game included counting positive and negative dice and moving a counter up and down a number line. Inherent in the game is a potential to explore the connections between two metaphors for number which could lead to the conclusion that (-11) + 5 is just another way of writing (-11) - (-5). Having 11 negatives and taking away 5 of them is kept track of by moving the counter 11 steps down and then back up 5 steps. Were metaphors used as tools to understand the mathematical equivalence (-11) - (-5) = (-11) + 5, it would the connection between these representations which was important and should be emphasised.

Küchemann (1981) described essentially two types of teaching approaches for the content of negative numbers: either an abstract approach or an approach that relies on the use of concrete models to make sense of integers. If the model approach is used teachers seem to often ask for a single model that will cover all operations with negative numbers. If the abstract model is chosen the content is placed very late in the curricula when students are expected to miraculously have reached a stage where they are capable of abstract reasoning. Instead of making this choice, a third approach could be suggested as a result of this study: an approach where models are used as a way into the abstract structures. Models can be used to make sense of this structure rather than for making sense of each separate object or expression. The findings of this study supports the results of Galbraith (1974) and Linchevski and Williams (1999) who suggested that a multitude of models be used to conceptualize negative numbers, but that a more algebraic approach is necessary when it comes to operating with these numbers.

At the start of this project it was believed that subtraction with a negative number and multiplication of two negative numbers would cause most difficulties. However it was found that many of the students had great difficulties making sense of operations with negative numbers at earlier stages, operations that could be easily understood metaphorically, such [-3+4] and [-6-2]. Since many different metaphors are linked together in the different models used, and since no metaphor covers all aspects of the concept of numbers, what is missing is a metalevel discussion about the fact that numbers and operations can be interpreted metaphorically in different ways. Metaphors can serve as tools for thinking only when we become aware of them as tools.

Many of the features that seem to be important for making sense of negative numbers could be part of the curriculum and mathematical discourse long before the formal introduction of operations with signed numbers. Proficiency in subtraction is one of these features. Understanding the extended number line and the difference between magnitude and value for the numbers below 0, i.e. the difference between a divided and a unified number line, is another.

8.2 Developing number sense

In this section the results are discussed in relation to research question 2: *How does the introduction of negative numbers in the mathematics classroom change students' number sense?*

The findings presented and discussed in chapter 7 showed different learning trajectories for the process of learning about negative numbers. Some important necessary changes of conception were described. Historically, in Western mathematics, conceptual changes such as accepting zero as a number, and then acknowledging numbers smaller than zero, took a very long time to be established. In this classroom some students are expected to go through that change in the same lesson they start adding and subtracting these new numbers. As a result, many of the students learned to subtract integers in a procedural way without making these conceptual changes. It would not be surprising if some of these students forgot the procedures quite readily, which was the conclusion drawn by Küchemann (1981) who found that 15-year-olds solved operations with integers less accurately than 14-year-olds. He attributed this to the fact that the rules on which they based the procedures on lacked meaningful support and were therefore difficult to check for consistency. That conclusion is also strengthened by the results of the pilot study reported in chapter 1.7. Among the pre service teachers who used a rule to calculate (-3)-(-8) as many as 17% arrived at an incorrect answer. They remembered the rule "two minuses make a plus", but not the correct procedure to follow when applying the rule.

If pre-knowledge is seen as decisive for learning, then the ground should be better prepared when the time comes to start operating with integers. Introducing zero as a number, numbers below zero on a number line, and negative differences could be done at a much earlier stage than the formal introduction of operations with integers. The results here are in accordance with Galbraith (1974) who claimed that conceptualising integers is a process in its own right. Students who had gone through this process already in grade 6 had no problems operating with integers when they were introduced. This was also shown by Ball (1993) whose work with 9-year-olds showed that they could conceptualise negative numbers and make sense of them in first and third position of an addition, but not as a number to be added or subtracted, which would need a more abstract kind of reasoning.

In a model of concept formation described by Sfard (1991), subtraction would be a concept A that needs to be at the level of reification before it can serve as an object in the process of interiorizing the new concept B of zero and negative numbers, which in turn must reach the stage of reification before it can serve as an object for operations with integers (see chapter 2.3). Concept A must be at the level of reification to be a requisite for the interiorization of concept B and so on. In plain words: if subtraction is only just at the first stage of interiorization, the concept of zero and negative numbers can not be interiorized, and if these numbers are not at the stage of reification they can not be operated on.

The findings of this study support that description, suggesting that although the same thing is said to all students, only those who have interiorized subtraction and come to the stage of reification of zero and negative numbers will be able to follow and understand the reasoning of the teacher when she talks about operations with these numbers. However, as pointed out by the theory, it is only when an object is used in the process of interiorizing a new concept that it can become fully reified. Which means that one should not wait to introduce negative numbers until subtraction is fully reified. The essential thing is to give the students time for each process to settle, for each interiorized concept to condensate. Conceptualisation of signed numbers is one thing, operating with these is another.

The introduction of negative numbers generally entailed a development of number sense from $\mathbf{R}^+$ to $\mathbf{R}$, but not necessarily as much as desired. Students with a poorly developed number sense in $\mathbf{R}^+$ in this data did generally not develop their number sense sufficiently in $\mathbf{R}$. The findings also showed a few instances of backward development of numbers sense, where students showed uncertainty about numbers or subtractions as a result of the introduction of negative numbers. This particularly concerned subtraction since the minus sign for subtraction got mixed up with the new meaning of polarity. On the other hand, some students with a well developed numbers sense for $\mathbf{R}^+$ showed an extension of the number domain before integers were formally introduced. The latter indicates that there are many opportunities for such an extension to be made since signed numbers, unlike in the 17th century, are an integrated and unquestioned part of the contemporary mathematical discourse. Although it is uncommon to operate with them in everyday life, their existence is often taken for granted. Children can encounter them as, for example, 'minus- and plus-points' in games, on the thermometer, as polarity in chemistry and on account balances.

Implications concerning theories of conceptual change

The findings of this study suggest that the idea of conceptual change needs to be extended to include not only enrichment and reconstruction, but also the awareness of multiple conceptions. Very often one conception is seen as incorrect when it has been replaced by another, for instance when a heliocentric world view replaced a geocentric view. In the process of changing from one to the other a number of hybrids may appear, so called "synthetic models" (Vosniadou et al., 2008). However, there are many examples, particularly in mathematics, when different conceptions need to exist side by side. Learning to specify the different conceptions and the conditions for when they are judged as true is a different kind of conceptual change.

The concept of number is different in $\mathbf{N}$ and in $\mathbf{R}$. The conception of numbers as discrete quantities does not become untrue when a conception of numbers as signed, continuous and infinitely small is introduced. Whilst agreeing with Ohlsson's (2009) claim that conceptual change is a reorganizing of knowledge and not a process of falsification it is also crucial to point out that one conception need not necessarily be abandoned when a new one is accepted. A prior conception does not need to be a misconception. An expert mathematician knows that mathematical truths are contextually dependent, for instance properties of geometrical objects are different in Euclidean geometry and in non-Euclidian geometry. The principle of ratio holds in the domain of natural numbers and not in the domain of real numbers. To understand this the necessary process is not only a construction of a new conception, perhaps as an end result of several synthetic models that in some way combines the principle of ratio with the real numbers, nor is it simply a restructuring of prior knowledge to abandon the principle of ratio as a misunderstanding about numbers: instead it is an awareness of the two number domains and their different conditions that is needed. As Vosniadou, Vamvakoussi and Skopeliti (2008) writes: it is important to develop in students the necessary "metaconceptual awareness" and "epistemological sophistication".

With evidence from many empirical studies a number of misconceptions about real numbers have been described as synthetic models of number created when learners try to combine new ideas with an initial conception of number based on natural numbers (Vosniadou et al., 2008). Although the natural numbers are commonly considered a subset of the integers it is not a true subset, since some of the properties that apply to natural numbers do not apply to integers and vice versa. Integers are signed; they have polarity. Natural numbers are not signed. They do not become positively signed until they are included in the set of integers. The set of integers does not have a smallest number but the set of natural numbers does. In the set of integers the minus sign has several functions, in the set of natural numbers only one. In the set of integers the size of number is twofold (magnitude and value) but in the set of natural numbers these coincide to form only one interpretation of size. The number domains are usually presented as inclusive. However, each set of numbers ($\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}^+$, $\mathbf{Q}$, $\mathbf{R}^+$, $\mathbf{R}$) have unique properties and when one set is included in another set some of the properties change. An expert mathematician can move with flexibility between the different number domains. In order to teach students to become expert in this sense it is necessary to introduce these domains as different, giving them names and acknowledging that although the concept of number changes to fit the larger domain the old conception still fits in the smaller domain.

8.3 Conflict situations and metalevel learning

This section discusses the issue of conflict situations asked in research question 3: *Do situations of cognitive or commognitive conflict arise in the classroom context when negative numbers are taught and do they pose opportunities for metalevel learning?*

There were certainly many situations were conflicting discourses and conflicting conceptions of number were present in the mathematics classroom. However, the sociomathematical norms of the classroom did not allow these conflicting discourses to become a topic for discussion and comparison. One of the predominant norms in the class was that a mathematical problem always has one and only one correct answer. With such a norm, it is not possible to acknowledge different discourses. For instance, statements such as “1 is the smallest number”, “subtraction makes smaller”, “subtraction of a larger number from a smaller number is impossible” or “-4 is larger than -2” are deemed incorrect, although they are undoubtedly correct within some discourses. To turn these conflict situations into opportunities for metalevel learning would entail making them explicit and acknowledging that mathematical statements are only true in relation to prevalent definitions and axioms. One is the smallest of the natural numbers, subtracting a larger number from a smaller number is impossible within the domain of $\mathbf{R}^+$, and -4 has a larger absolute value than -2. When that has been recognized, a discussion can be introduced about changing conditions and introducing new definitions.

Another inherently commognitive conflict was the difference between metaphorical reasoning and intra-mathematical reasoning, what Sfard (2007) calls a necessary change of meta-rules for justifying mathematical statements. Most (perhaps all) mathematical statements students meet before they encounter signed numbers are justified by concrete models and can be expressed by first order representations (Damerow, 2007). The next step in developing mathematical proficiency is to use second order representations and accept mathematical statements as true because they are justified by the mathematical structure itself or by the rules of logic and algebraic reasoning. In this study the students often used an intra-mathematical discourse. The teacher however, consistently made use of metaphorical reasoning to justify statements, and when she did not succeed she simply supplied a rule. It could be argued that a revision of what it means to understand a mathematical statement is necessary.

The theoretical framework of this thesis described in chapter 2 defines understanding in mathematics as the result of a process of making connections between ideas, facts, representations and procedures. That is a definition quiet different from only seeing understanding as the result of making connections between mathematical symbols and physical reality. Instead of connecting negative numbers to concrete models one could take knowledge about natural numbers as a point of departure and connect negative numbers to natural numbers and the students’ existing connections between ideas, facts, representations and procedures concerning these numbers. For such an approach to be fruitful the two different discourses must be made explicit. Students would need to know that when they are talking about numbers metaphorically, they are using a specific discourse, but the mathematical concept is a general one and any specific discourse will only give a limited understanding of the general. Metaphors are still necessary, because the general is always understood through specific examples, but each specific metaphorical discourse is only one node in the network of connections necessary for understanding a concept. A teacher needs to make the metaphor in use explicit in the public space of shared meanings, discussing similarities and differences between different possible interpretations and thus helping the students to create meaningful connections.

If the teaching–learning process for metaphorical reasoning described in chapter 6 is an established practice, the outcome might change by simply taking the students into the process from step 1. If an introduction to the topic starts with a subtraction like 2 - 5 a new number will appear. Classroom discussions could follow concerning this new strange number and what would happen if it was added, subtracted, multiplied and divided. That would establish an intra-mathematical discourse from the start, and when metaphors are brought in they could be discussed as alternative ways of reasoning about these numbers, be compared and deemed more or less useful. As a result the different metalevel rules could become part of the discourse, and students could learn to move flexibly between different metaphors, different representations and different number domains.

Sociomathematical norms

What is to be considered as good teaching depends highly on the learning goals. Good teaching is teaching that helps students reach set learning goals. If a teacher, a school or a community does not reach the set goals they might consider changes and reforms, and if change is needed, there are many different things that can be changed. Besides discussing the cognitive perspective on numbers, number sense and the use of metaphors, this study also sheds a little light on the sociomathematical norms of the classroom. Norms like the ones found in the classroom in this project can be changed only if the participants become conscious of them. Only when a norm is brought out into the open can it be valued and revised. It is not an easy task since the sociomathematical norms in a classroom are influenced not only by the different participants in the classroom but also by more general social norms and beliefs about the nature of mathematical activity in society at large. If mathematical activity is tantamount to acquiring procedural skills to solve model type tasks, that will result in one type of learning goals and sociomathematical norms. However, if mathematics is considered an activity characterized by creative thinking and problem solving, it should result in different learning goals and sociomathematical norms, where procedural skills serve as tools rather than goals. The distinction is not between procedural and conceptual understanding but between different goals set up for teaching procedures and concepts.

Among the norms described in chapter 5 is the strong focus on procedures in mathematics. In this data there was no indication of procedural knowledge preceding conceptual knowledge. On the contrary, the findings corresponds with Baroody, Feil & Johnsson (2007) who claim that procedural and conceptual knowledge are mutually dependent and connected. However, it was evident that for some students the strong emphasis on procedures conveyed the idea that knowing mathematics is tantamount to knowing the correct procedures. That is a statement to be questioned, much in line with the ideas of relational and instrumental understanding discussed by Skemp (1976). The sociomathematical norms present in the classroom of this study support an instrumental view on mathematics learning and teaching, but as stated by Goldin and Shteingold (2001, p 5): even a high level of ability to perform arithmetic and algebraic computations "[does] not imply an understanding of mathematical meanings, the recognition of structures, or the ability to interpret results".

As mentioned earlier, the norm that a mathematical problem always has only one correct answer was prevalent. Such a norm is understandable since mathematics is known to be a science of true statements. It is, however, possible to view mathematics as a science of true statements under well defined and restricted conditions. A change of conditions may change the truthfulness of a statement. If this has not been brought to students' attention before it would fit very well into the chapter on signed numbers. When the number domain is extended from unsigned to signed numbers the conditions are changed so that some things that were true before now become false. Küchemann (1981) writes: "..since the integers are not simply an extension of the natural numbers they should not be presented as if they were". One of the main arguments for those who resisted negative numbers before the 19th century was the fact that the principle of ratio that governs multiplication in $\mathbf{R}^+$ would no longer be true. Only when it could be accepted that this principle was not true in the new number domain were the numbers accepted.

The sociomathematical norm that gives the teacher authority over mathematical truths and ways of reasoning can also be questioned. Although it is important that a teacher and a textbook try to supply students with mathematical tools and ways of reasoning it is also possible that students themselves have insightful associations and ways of reasoning. It was discussed in chapter 3.6 that the teacher's perspective when using metaphorical reasoning is not the same as the students' perspective. What seems to be true or easy to understand for the teacher might not be the same from the perspective of the student.

8.4 The importance of history

In this section the importance of knowledge about mathematical history is discussed in an attempt to answer research question 4: *How can knowledge about the historical development of negative numbers be useful for understanding student's development of number sense?*

One might ask if there are analogies between how students today develop their understanding of the concept of negative numbers and how the concept developed through history. Is there a historical parallelism found in the data?

Ball (1993, p 393) poses the question like this: "Just because it took hundreds of years for mathematicians to accept negative numbers does not necessarily imply that third graders must also struggle endlessly to incorporate them into their mathematical domain." Damerow (2007) shows that the individual development of cognition and the historical development of cognition are two fundamentally different processes. It is evident that Ball and Damerow have a point: if learning a concept should involve all the stages of its historical evolution, mathematics would not advance very much, and certainly not at the speed it has during the last century. Once a concept has become integrated in the discourse of expert mathematicians along with symbols and definitions, it is already something quite different from the original concept. Furthermore, the cross-cultural comparison of the history of negative numbers done by Mumford (2010), showed that the concept evolved differently in different cultures and particularly that is was constrained in Western mathematics by the strong influence of Euclidian geometry. There is not *one* evolution of the concept, but several. It is also clear in this data that some students extend their number domain without any problems even before it is taught formally in the classroom. It could be assumed that they do this merely as a result of participating in a discourse in and outside of school where signed numbers naturally occur.

However, findings of this study show that for those students who *do* find negative numbers difficult, many of the problems they have are similar to the problems that surfaced during the evolution of the concept. This finding is supported by others authors, for example Lybeck (1981), who studied students' conceptions of the concepts of density, proportion and proportionality. Lybeck suggested making students' conceptions and their parallel historical conceptions explicit as a content of the lesson. Going back to the "obstacles" identified by Glaeser (1981) and the epistemological changes that occurred in the evolution of the concept (see chapter 1.1), it is clear that some aspects, such as difficulties making sense of negative quantities, accepting two conceptions of zero, and unifying the number line, are common obstacles for students of this study. Even more noteworthy is that some of the obstacles seem to be obstacles also for teachers and textbook authors, as showed both in the empirical data and in chapter 1.3. These are for example stagnation in the phase of concrete operations and a wish for a unified model that will cover all arithmetic operations. Algebraic reasoning was historically the key to the acceptance of negative numbers, which suggests that when it comes to operating with these numbers focus should shift from concrete models to algebraic reasoning and the mere fact that understanding these numbers is an algebraic issue.

A study of the history of negative numbers could help teachers to discover aspects of these numbers that might be problematic for students: when students show the same opposition to negative numbers that characterized the evolution of these numbers, teachers might find in the study of history the keys to how such opposition could be overcome. The results of this study are in line with conclusion drawn by Thomaidis and Tzanakis (2007) who found similarities between obstacles encountered in history and students' difficulties in understanding the ordering of negative numbers (in the present study described as the differentiation between magnitude and value of number) and in overcoming the conception of numbers as quantities. They also found that didactic factors intervene and constrain students' approaches. They write that:

> [T]hese similarities could be exploited didactically: To foresee possible persistent difficulties of the students; and to make teachers more tolerant towards their students' errors, by increasing their awareness that these errors and difficulties do not simply mean that 'the student has studied enough', but may have deeper epistemological roots which should be explored and understood thoroughly. However … *this exploitation should be done with caution*, in view of the importance of the intervening didactic factors. (Thomaidis & Tzanakis, 2007, p 180)

In the results of this study an intervening didactic factor is for example the existing sociomathematical norm that the teacher has the authority over mathematical truths and mathematical ways of thinking, hence students are not usually given a chance to develop alternative ideas, as it happened historically. This norm constrains the possibilities of a positive parallelism. Other intervening didactic factors are the existence of modern algebraic symbols that facilitates algebraic thinking, and the existence of visual number lines in many different contexts. These artefacts of a modern culture shape very different conditions for students of today compared with mathematicians from a different era.

Students could benefit from a study of the history of mathematics, for instance by learning that mathematicians have struggled with strong feelings about these numbers, not making any sense of them and calling them absurd. It was not because they make physical sense that negative numbers were finally accepted but because they were so useful. This is what Jankvist (2009a) calls the motivating argument for history as a tool for mathematics education. The frustration shown by some of the students in this study indicates that affective and motivational issues need to be considered by the teacher. The history of negative numbers certainly provides us with opportunities to deal with these issues. If it is a fruitful approach or not is a good question to pursue in future experimental research.

The historical review presented in chapter 1 brought to the fore that it was only as a result of algebraization and a paradigmatic change from dealing with numbers as quantities to numbers as algebraic entities that mathematicians were able to overcome the conceptual conflict created by negative numbers. With that hindsight, it is surprising that school mathematics treat negative numbers as a prerequisite for algebra and not vice versa. It could be suggested that algebraic reasoning, alongside metaphorical reasoning, be used as a teaching tool when it comes to operating with negative numbers.

8.5 Implications for teaching

As an outcome of the described research project and the discussion held in this chapter, I will here venture to give some suggestions for future teaching practice. Concerning the role of *metaphors* I suggest teachers make use of the metaphors students already have for numbers, make these explicit and discuss them with their students so that it becomes evident that each metaphor is only one way of thinking about numbers, highlighting some aspects and obscuring some, and that each metaphor brings with it affordances and constraints and only function under certain conditions of use. Conditions of use for different metaphors need to be part of the discourse.

The aim should be to use metaphors as teaching tools rather than a teaching goal. It is also important to discuss connections between the metaphors. For students who very tightly hold on to only one metaphor for number (most probably the *object collection* metaphor), the teacher needs to actively bring in the others. When speaking about numbers using metaphorical reasoning it is essential that teachers are sensitive to different interpretations of what they are saying. The example of "difference between two numbers" elaborated in chapter 6.3 showed how different underlying metaphors gave different interpretations to what the teacher said. Metaphors essential for conceptualizing signed numbers are:

Numbers as Object Collections,
Numbers as Positions along a Path (points on a number line),
Numbers as Movements along a Path,
Numbers as Relations.

In particular I would like to highlight the N*umbers as Relations* metaphor since relations are more easily symbolised using signed numbers than using natural numbers. In many cases the same models can be used (e.g. money, elevators, stairs). The difference is all in the wording, in the fact that it is the relation that is mapped onto a number. Making use of a *number as relation* metaphor could also serve to bridge the gap described by Vergnaud (1982) between the structure of problems and the mathematical concepts being taught. If negative numbers are introduced as an effective way of symbolizing relations and a help in solving relational problems, children might be more motivated to incorporate them in their discourse. These described grounding metaphors can be blended with other metaphors such as:

Mirror or Reflection metaphor (the aspect of opposite numbers),
Rotation metaphor

A *Measuring Stick* metaphor where a number is seen as a measured distance is also useful but for the extension of the number domain this metaphor needs to be linked to the mathematical idea of absolute value (magnitude).

However, although metaphors are in use, and they always are when we encounter new concepts according to the theory of conceptual metaphors described in this thesis, students also need to reify the negative numbers so that zero and signed numbers become objects in their own right, given meaning as part of an algebraic structure. Teachers as well as students need to overcome the last two obstacles described by Gleaser (see chapter 1.1); they need to let go of the wish to give everything meaning through concrete operations and the wish to find a unifying model for all operations with negative numbers. Obscure concrete explanations might leave negative numbers as meaningless. Learning a sign rule does not give signed numbers meaning, but understanding the logical reasoning behind the rules might. It is not too difficult to give a plausible metaphorical justification of some specific situations when subtracting a negative number gives the same result as adding the opposite positive number. The essential step is to take such an instance of reasoning as a point of departure for generalising this idea, arguing for the existence of a general rule.

Concerning the development of *number sense,* my recommendations very much relate to the use of language. According to Sfard (2008), a necessary requirement for a change of discourse is the participation in a discourse with experts. One problem with the mathematics classroom in this study and with the mathematics classroom discourse in Swedish mathematics textbooks is that the discourse offered is not clearly an expert discourse. There are many critical features of signed numbers that are either not brought up at all or that lack a useful Swedish terminology. Thus, these features are not included in the discourse, and the appropriate number sense may not be developed. It was shown in the result chapters how what was said become blurred and created confusion as a consequence of the poorly developed mathematical language. In order to create a more expert mathematics classroom discourse in Sweden I suggest the following:

- Make explicit the different number domains. When asked for the smallest number it is not incorrect to say, as for example Anna did in grade 6, that the smallest number is 1. In the domain of natural numbers the smallest number is 1. The challenge is to realize that there are different number domains and that the term ‘number’ incorporates more than natural numbers. To do this it could be suggested that the different types of numbers and the different number domains are given explicit names in the classroom discourse.
- In order to distinguish between the different number domains they need to be given different names. It would therefore be useful to invent a Swedish term for *Signed Numbers*, so that it becomes clear that in the domain of signed numbers all numbers are either positive or negative[61]. It is not just the case that negative numbers suddenly appear alongside ‘ordinary numbers’. The natural numbers change character when they are brought into the set of integers.

[61] Possible candidates are: tecken-tal, tal med tecken, laddade tal, polära tal, riktade tal.

- ~ Use the terms *positive number* and *negative number* instead of plusnumber and minusnumber and the word *subtraction* instead of minus for the operation, to make it possible for students to distinguish between the two different meanings of the minus sign.
- ~ When the two meanings of the minus sign have been introduced, discuss in connection with various examples the possibility of flexibly moving between the two interpretations.
- ~ Introduce the concept of *absolute value* together with signed numbers to make it possible to distinguish between the dual meanings of size of signed numbers: value and magnitude[62].
- ~ Make explicit the different meanings of the Swedish word for number (tal) to make sure that students give the word a correct mathematical meaning in the mathematics classroom.

The classroom studied in this project also showed a need to put more emphasis on the use of a number line. I therefore suggest that teachers and textbooks make explicit use of the *Number Line* as a representation for numbers, and explain how it is constructed. Use 0 as a benchmark number and compare a divided number line with a unified number line. In the interview results a path metaphor (as in numbers being further away, higher up, below etc) was the most commonly used metaphor in the students own references to negative numbers, even though it was very rarely referred to in class and the students were uncertain about how to talk about or draw mathematically correct number lines. There was, in other words, little connection between the students embodied metaphor and the mathematical number line model.

Since the epistemological issues are of great importance when children develop their number sense, I suggest signed numbers are introduced at a much earlier stage in the school curricula than grade 8. This suggestion is in accordance with recommendations by Goldin and Shteingold (2001) who investigated representations of negative numbers among first graders. They found that many children already at that early age could give meaning to and construct representations of negative numbers in ordinal and/or cardinal contexts, and therefore suggested that such representations should become part of the mathematical content matter.

If children are given time and support during earlier years to conceptualize negative numbers and develop a good number sense, then operating with these numbers in grade 8 would be just one new thing to tackle, instead of a complete change of conception all at once. The results showed that many students were not sufficiently proficient in subtraction when they were faced with negative

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[62] In a preliminary investigation of around ten currently available mathematics textbooks only one (Johansson, Renlund, & Risberg, 2004) brought up the concept of absolute value in the chapter about signed numbers.

numbers, indicating that this operation needs more time and explicit teaching already in the domain of natural numbers.

Finally I draw upon the results concerning metalevel conflicts to say that sociomathematical norms in a classroom may need to change to make room for such conflicts. Norms are not easily changed but the first step is always to become aware of them. Making use of the history of negative numbers, highlighting the cognitive conflicts past mathematicians struggled with, might be a way to help students reveal their conceptions so that conflicting conceptions can surface and be discussed.

8.6 Critical reflections and suggestions for further research

The results presented in this study are neither totally new nor unexpected. The research project has had a wide scope and an ambition to combine different perspectives in order to penetrate the complex activity of making sense of an abstract mathematical concept. In that respect it is different from other related studies and can be seen as a contribution to the field. Some researchers might object to the pragmatic use of theories. There are two reasons for that pragmatic choice. Firstly, the research questions could not have been answered using only one theory, since the holistic approach to the classroom situations demands both cognitive and social theories. Secondly, the whole research project is based on the foundations of its basic assumptions about the nature of mathematics as a social and as an abstract science and the research result must be judged in the light of those assumptions. Metaphors are seen as specifically human communicative and discursive tools. Metaphors therefore serve as a link between the social and the cognitive. If these assumption about mathematics and about metaphors were to be left aside, then the two perspectives symbolised by what Sfard (1998) called the acquisition and participation metaphor for learning might not both have been necessary.

A weak point in the study is the lack of results about multiplication and division of negative numbers. At the start of the project the intention was to study students' sense making of these operations as well. As it were, in this class these operations passed by quite unnoticed; students either learned them by acquiring and applying a sign rule and never questioned the rule, or they did not learn them at all, which was accepted because they were presented in the advanced part of the textbook. Some teachers might object to the relevance of the study because of this. A teacher might say that addition and subtraction is no problem because there are models to explain those, but multiplication is impossible to explain. However, it was found was that students seemed to have more problems understanding and accepting subtractions with negative numbers than multiplication. That was also found in Küchemann's study (1981) and was explained by the fact that multiplication of two negatives more easily than subtraction conforms to the rule of "minus minus makes plus". In relation to the grounding metaphors it could be an example where numbers are seen as object constructions. If a negative number $-a$ is seen as constructed by the two factors (-1) and a, and (-1) is interpreted as a rotation or a shift of polarity, then

$$(-a) \cdot (-b) = (-1) \cdot a \cdot (-1) \cdot b = ab \cdot (-1) \cdot (-1)$$

which would mean that the two negative numbers are multiplied and the polarity is changed twice. This sounds easy but of course, like all the other extended grounding metaphors, would show up limitations and inconsistencies if closely analysed. For instance; a two times change of polarity from what? Studying a class who do spend time trying to make sense of multiplication and division of signed number would be a welcome complement to the present study.

Another suggestion for further research is to design a teaching experiment where metaphorical reasoning is brought in as a teaching tool rather than a teaching goal and where time in class is spent on examining the difference between intra-mathematical reasoning and metaphorical reasoning. Another interesting teaching experiment would be to investigate what happens to students' sense making if a substantial portion of the history of mathematics and the evolution of the concept of negative numbers was included in the lessons.

The whole project included a rather large collection of data and it might be questioned whether it was necessary to collect all that data. Could a different method or a more limited data collection have been chosen? In a sense this research has been explorative. It was not known at the start of the project what would turn out to be the interesting parts of the data collection. The initial research question only concerned development of number sense and the study was planned as a longitudinal interview study. As the project took shape, the research questions expanded, partly due to new theoretical angles of approach. Participant observations and video recordings were included and the body of data came to be large and somewhat scattered. However, the time spent as a participant observer in the class improved the reliability of the interviews and the different types of data came to serve as a mutual internal validation of each other. Even if there are many things in the data that did not find its way into this thesis, those that did were strongly influenced by the complexity of the data. It could be a suggestion for further research to look more deeply into those questions that were put aside or only slightly touched upon here.

CHAPTER 9

Sammanfattning på Svenska

Utgångspunkten för denna avhandling är intresset för hur elever lär sig och förstår matematik, i det specifika fallet inom området negativa tal. Arbetet är ett bidrag till forskningsfältet inom "pedagogical content knowledge" (Shulman, 1987), dvs. kunskap kring ett matematiskt innehåll som är relaterat till undervisningen om detta innehåll i en skolkontext. Området negativa tal är valt för att det är ett innehåll i matematiken vilket ofta upplevs som svårbegripligt. Det är ett innehåll som behövs för algebran och som därför ingår i den obligatoriska skolans kursplan. Samtidigt är det ett område av matematiken som sällan används i vardagslivet. De flesta vardagliga eller konkreta problem som kan lösas med negativa tal kan också lösas med positiva tal. Exempelvis är uppgiften om hur gammal Kejsar Augustus blev om han föddes år 63 f.kr. och dog år 14 e.kr. (Carlsson et al., 2002, s 19) enkel att lösa genom beräkningen 63 + 14 = 77. När läraren vill att det ska lösas med beräkningen 14 - (-63) = 77 kan det upplevas av elever som tillkrånglat. Negativa tal är ett område som förutsätter en övergång från intuitiv till formell matematik.

Avhandlingens övergripande syfte är att undersöka hur elever förstår och skapar mening med de negativa talen[63]. Ett grundantagande för meningsskapandet är att de metaforer som används när man talar om tal har en avgörande betydelse för hur man uppfattar tal. Metaforers roll i matematikundervisningen har på senare år lyfts fram av en mängd forskare (exempelvis Danesi, 2003; English, 1997b; Parzysz et al., 2005). Lakoff och Núñez (2000) beskriver matematiken som ett system av metaforer. Alla abstrakta begrepp, hävdar de, beskrivs i metaforiska termer, och metaforer för den grundläggande aritmetiken är alla kroppsliga, fysiska metaforer. Matematiska begrepp omtalas som om de vore fysiska objekt; till exempel är 4 *större än* 3 och 6 *längre från* 3 än 4. Storlek och avstånd är då fysiska erfarenheter som används för att beskriva de abstrakta talen.

Som inledning på avhandlingsprojektet genomfördes en pilotstudie med 99 lärarstudenter från en kurs om matematik för förskola och skolans första år (Kilhamn, 2009a). Av studiens deltagare gav 70 % ett korrekt svar på uppgiften (-3)-(-8) medan 30 % inte kunde lösa uppgiften korrekt. Bland dem som enbart använde sig av ett metaforiskt resonemang; exempelvis "det är minus 3 grader och sen blev det 8 grader kallare" återfanns alla i gruppen med felaktigt svar. De som först skrev om uppgiften till (-3)+8 och sedan använde sig av ett metaforiskt resonemang erhöll samtliga korrekta svar. Bland dem som använde

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[63] Engenska: *make sense of.*

sig enbart av formella resonemang och regler hade 83 % korrekta och 17 % felaktiga svar. Resultaten indikerar att metaforiska resonemang är användbara under förutsättning att man vet när de är tillämpbara och vilka begränsningar de har, samt att även inommatematiska resonemang behövs. Utfallet av pilotstudien gav avhandlingsprojektet en inriktning mot metaforiska resonemang, och avhandlingens huvudstudie planerades som en longitudinell fallstudie innehållande observationer, intervjuer och videoinspelade lektioner.

Tidigare forskning

Studier av de negativa talens historiska utveckling påvisar några viktiga evolutionära steg. (jfr. Arcavi & Bruckheimer, 1983; Beery et al., 2004; Cajori, 1991; Glaeser, 1981; Katz, 1993; Mumford, 2010; Schubring, 2005, m.fl.). Avgörande hinder eller svårigheter för de negativa talens utveckling och acceptans kan sammanfattas som följer:

~ Negativa tal sågs tidigt som motsatta kvantiteter men begränsades då till situationer där negativa kvantiteter är meningsfulla.

~ Geometriska bevis spelade en stor roll, men somliga teckenregler för operationer med negativa tal blev inte geometriskt meningsfulla eftersom negativa sträckor saknar mening.

~ Ett negativt tal beskrevs länge som "ett tal som ska subtraheras", vilket gjorde det svårt att särskilja minustecknets två betydelser: subtraktion och negativitet.

~ Talet 0 var länge enbart ett uttryck för "ingenting" och inte ett tal med samma status som andra tal.

~ När tallinjen infördes för negativa tal var det en delad linje där noll var utgångspunkten för två linjer åt motsatt håll, inte en enad linje där alla tal har samma status.

Det var först när talen frigjordes från den starka kopplingen till kvantiteter och sträckor och istället sågs som delar i en algebraisk struktur som negativa tal blev helt accepterade som matematiska objekt. I en algebraisk struktur får talen mening genom sina relationer till andra tal. Glaeser (1981) menar att två av de största hindren för de negativa talens utveckling och acceptans var fokuseringen på konkreta operationer, och det ihärdiga sökandet efter en enhetlig modell som täcker alla operationer med negativa tal. En forskningsöversikt över studier kring undervisning om negativa tal visar att många av de historiska problemen återfinns bland elevernas svårigheter, men också bland lärare och forskare. Flera av de studier som gjorts är experimentella studier där en specifik konkret modell testas. Ibland har tallinje-modeller ställts mot diskreta modeller och ibland har det hävdats att en mångfald av modeller måste finnas. (jfr. Altiparmak & Özdoan, 2010; Bruno & Martinón, 1999; Gallardo, 1995; Schorr & Alston, 1999 m.fl.). Sfard (2007) förespråkar istället att man måste överge de konkreta modellerna till förmån för en mer formell diskurs. Det saknas dock empiriska studier som beskriver hur detta kan göras. Ett antal studier tar upp problem och missuppfattningar som relateras till att elever lär sig teckenregler såsom "minus minus blir plus" utan att förstå eller kunna härleda reglerna, vilket leder till att reglerna används felaktigt. Empiriska exempel på felaktigt tillämpade teckenregler är: $^{-}2$ - $^{+}3$ = $^{+}5$ med motiveringen att "två minus blir plus" (Küchemann, 1981) eller (-3) - (-8) = (-11) med motiveringen att "där är tre minus varav två blir plus och ett flyttas över" (Kilhamn, 2009a).

Aspekter av negativa tal som lyfts fram i olika studier såsom betydelsefulla för förståelsen är: *acceptans av 0 och tal mindre än 0*, *skillnaden mellan decimaltal och negativa tal*, *nollans dubbla betydelse som dels noll-mängd och dels summan av två motsatta tal*, *skillnaden mellan magnitud och riktning för ett negativt tal*, *de negativa talens lokalisering till vänster på tallinjen*, *minustecknets dubbla betydelse*, samt *operationen subtraktion* (bl.a. Ball, 1993; Gallardo, 1995; Gallardo & Hernández, 2007; Kullberg, 2010; Küchemann, 1981; Stacey et al., 2001a; Vlassis, 2004). Trots att många studier har utforskat olika modeller för negativa tal, såsom exempelvis tallinjer eller diskreta positiva och negativa objekt, råder det ingen samsyn bland forskare över vilken modell som är fördelaktigast eller huruvida man bör använda en eller flera modeller.

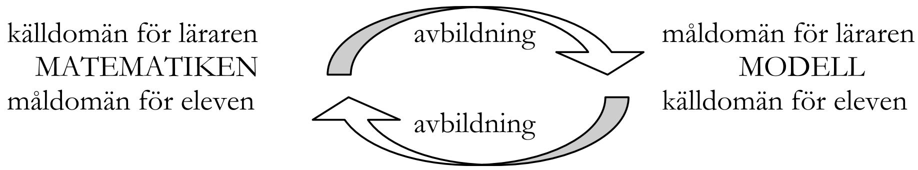
Metaforer

I undervisning om negativa tal är det vanligt att man använder olika modeller såsom termometern, skulder och tillgångar, vektorer, hissar etc. I den föreliggande avhandlingen betraktas modellerna utifrån ett metaforperspektiv, vilket innebär att de ses som del av ett system innehållande en källdomän, en måldomän och en avbildning från källdomänen till måldomänen. Användandet av metaforer studeras i avhandlingen utifrån teorin att metaforer används för att tala om allt som inte kan erfaras genom sinnena (Lakoff & Johnson, 1980). Eftersom matematiken inte kan erfaras direkt byggs den upp av metaforer (Lakoff & Núñez, 2000). Aritmetiken förstås, enligt Lakoff och Núñez, huvudsakligen genom fyra grundläggande metaforer:

~ *Vägmetaforen*: tal som platser eller rörelser längs en väg
~ *Objektsmetaforen*: tal som en samling objekt, ett antal i en grupp
~ *Konstruktionsmetaforen*: tal som ett konstruerat objekt; är del av, består av
~ *Mätmetaforen*: tal som uppmätta enheter, t.ex. längder som kan jämföras

Varje metafor framhäver vissa aspekter av ett begrepp och sätter andra aspekter i skymundan. Därför behövs det alltid flera metaforer för att belysa ett komplext begrepp. Lakoff och Núñez hävdar att de fyra grundläggande metaforerna täcker all aritmetik med positiva hela och rationella tal och att dessa metaforer endast behöver "tänjas", eller utvidgas något för att också täcka negativa tal.

I metaforen utgör modellen oftast källdomänen för eleverna, men kan för lärarna ibland utgöra måldomänen, enligt figur 9.1. Läraren, som är väl förtrogen med matematiken, avbildar matematiska objekt på situationer och objekt i den fysiska verkligheten. För eleven är det den fysiska verkligheten som är välbekant och som i undervisningen avbildas på matematiska objekt.



FIGUR 9.1: Illustration av metaforens två riktningar.

Som en del i avhandlingen finns en detaljerad analys av metaforer verksamma i modeller hämtade från dagens undervisning om negativa tal. Analysen visar att det inte är en enkel process att utvidga en metafor så att den täcker in negativa tal och att de utvidgade metaforerna i själva verket är nya metaforer. Den nya metaforen kan begränsas av att den tappar intern konsistens, tappar samstämmighet med andra närliggande metaforer, eller genom att den delvis bygger på andra och annorlunda erfarenheter än utgångsmetaforen. Analysen gav upphov till introduktionen av ytterligare en användbar metafor för talområdets utvidgning; en *Relationsmetafor*, där ett tal ses som en relation mellan antingen objektsmängder, sträckor eller rörelser. -3 kan då betyda att det är *tre fler* negativa än positiva, eller 3 steg *längre åt vänster* än åt höger. Utifrån denna analys ställdes frågor om metaforiskt resonerande i diskursen kring negativa tal. Med metaforiskt resonerande menas att man talar och resonerar om matematiken i termer av något annat.

Teoretiskt ramverk och forskningsfrågor

En teoretisk utgångspunkt för avhandlingen är synen på matematik som en social konstruktion (Davis & Hersh, 1981; Hersh, 1997). Matematiska objekt såsom tal och operationer är å ena sidan socialt konstruerade diskursiva entiteter, å andra sidan blir de inlemmade i en matematisk struktur vilket ger dem egenskaper som inte var uppenbara från början. Matematikens byggstenar kallas *begrepp* och individens tolkning av dessa begrepp, med alla de associationer begreppet väcker till olika representationer och till andra begrepp, benämns *begreppsuppfattning* (Sfard, 1991). Lärande i matematik är i detta sammanhang beskrivet som en förändring av individens begreppsuppfattning, vilket studeras genom elevernas tolkning och användning av olika begrepp. I avhandlingens huvudstudie fokuseras talbegreppet inklusive andra, till tal relaterade, begrepp såsom operationer med tal och tals värde. Detta betecknas *taluppfattning* och har i relation till negativa tal en vid betydelse (Berch, 2005) som inkluderar bland annat uppfattningar om olika sorters tal, storleksjämförelse av tal, referens-punkter, operationers inverkan på tal, och referenser till en mental tallinje (Kilhamn, 2009c).

Avhandlingen tillämpar ett socialkonstruktivistiskt ramverk (Cobb, 1994; Cobb et al., 2001) där hänsyn tas till både den sociala och den psykologiska dimensionen. Det sociala perspektivet ser på rådande klassrumsnormer, speciellt sociomatematiska normer, medan det psykologiska perspektivet fokuserar individens tolkningar, uppfattningar, värderingar och resonemang. De båda perspektiven förutsätter varandra och påverkar varandra ömsesidigt. Den föreliggande avhandlingen undersöker främst förändringar i den psykologiska dimensionen och studerar den mot bakgrund av rådande sociomatematiska normer, dvs. normer relaterade specifikt till skolämnet matematik. Då matematik ses som socialt konstruerad blir betydelsen av den matematiska diskursen stor. Med diskurs menas här de ord, begrepp, besättelser, procedurer och visuella hjälpmedel som utgör det sätt en grupp människor kommunicerar med vandra i ett visst sammanhang (Sfard, 2008).

Huvudfrågan för avhandlingsstudien är hur elevernas taluppfattning förändras då talområdet utvidgas, och vilken roll metaforer spelar i undervisningen. Utifrån denna huvudfråga formulerades fyra mer specifika forskningsfrågor:

1. På vilket sätt förekommer metaforiskt resonerande i klassrumsdiskursen? Vilka metaforer förs in i diskursen av läraren och läroboken? Med vilken avsikt införs metaforerna och hur används de?

2. Hur påverkas elevernas taluppfattning av introduktionen av negativa tal i undervisningen? Vilka olika banor tar lärandet för olika elever då tal-området utvidgas och hur påverkas dessa lärandebanor[64] av de metaforer som används?

3. Uppstår kognitiva/kommognitiva[65] konflikter i klassrumsdiskursen i samband med negativa tal?

4. Vilken betydelse skulle kunskap om den historiska utvecklingen av negativa tal kunna ha för förståelsen av elevernas lärande om negativa tal?

Forskningsdesign

Avhandlingens huvudstudie är en longitudinell intervjustudie i en svensk skolklass. En matematiklärare och 21 elever följdes under 3 års tid med årliga individuella intervjuer och deltagande observationer en gång i veckan. Totalt observerades 61 lektioner från och med vårterminen i årskurs 6 fram till höstterminen i årskurs 9. Som komplement och metodologisk triangulering gjordes även en videostudie då 7 lektioner som behandlade negativa tal videofilmades under höstterminen i årskurs 8. En kamera följde läraren hela lektionstiden och fångade lärarens undervisning och alla situationer då läraren interagerade med elever. De årliga intervjuerna följde ett fast intervjuprotokoll

[64] Engelska: *learning trajectories.*

[65] Kommognitiva konflikter (Sfard, 2008) uppstår när två olika och delvis motstridiga diskurser möts. I relation till matematik som socialt konstruerad ses kognitiva konflikter (Ohlsson, 2009; Vosniadou & Verschaffel, 2004) i detta arbete som liktydigt med kommognitiva konflikter eftersom en kognitiv konflikt uppstår när ny information motsäger tidigare kunskap. Då matematiken är socialt konstruerad är ny information om matematik liktydigt med en ny matematisk diskurs.

innehållande både öppna frågor och uppgifter att beräkna under det att eleven ombads "tänka högt" (Boren & Ramley, 2000; Bryman, 2004; Kvale, 1997). Protokollet konstruerades utifrån de aspekter av negativa tal som belysts av tidigare forskning och i den historiska genomgången och innehöll följande sju teman: *vad är ett tal*, *storlek på tal*, *subtraktion*, *addition*, *minustecknets betydelse*, *parenteser och prioriteringsregler*, och *tallinjen*. I analysprocessen inkluderades även temat *metaforanvändning*.

Som komplement till intervjuer, observationer och videofilmer samlades en även elevarbeten och provresultat in. Läroboken spelade en stor roll i undervisningen och analyserades därför som en del av den undervisning eleverna erbjöds.

Datamaterialet transkriberades och analyserades kvalitativt genom ett iterativt tillvägagångssätt, där elevers och lärares sätt att uttrycka sig tolkades i relation till matematikens nutida och historiska definitioner, samt i relation till tidigare forskning kring elevers uppfattningar om negativa tal.

Resultat och diskussion

Klassrumspraktiken och sociomatematiska normer

Matematiklektionerna som studerats innehåller fyra typer av aktiviteter, som under 52 av de 61 observerade lektionerna följer en karakteristisk struktur. Av den i genomsnitt 57 minuter långa lektionen ägnas 9 minuter åt genomgång i helklass, 37 minuter åt enskilt arbete i matteboken, 5 minuter åt övriga matematiska aktiviteter (tester, läxrättning, spel), samt 6 minuter åt icke-matematiska aktiviteter.

Sociomatematiska normer beskriver regelbundenheter i klassrumsaktiviteter relaterat till matematiska frågor. En norm skapas och upprätthålls av alla parter i klassrummet. I datamaterialet framträder fyra tydliga sociomatematiska normer, dessa var:

- Matematik handlar om procedurer som ska läras och nötas in.
- Den som ännu inte lärt sig gör mer av samma sak.
- Det finns endast ett korrekt svar på en bra matematikuppgift.
- Läraren har tolkningsföreträde och auktoritet över den matematiska kunskapen.

De rådande sociomatematiska normerna i klassrummet bidrog till att de kognitiva/kommognitiva konflikter som blev synliga i elevintervjuerna aldrig gjordes till del av klassrumsdiskursen och därför aldrig kunde lösas tillsammans.

Språkbruk

I samband med att transkript och klassrumsbeskrivningar översatts till engelska har en del språkliga brister i den svenska matematiska skoldiskursen framträtt. Enligt Sfard (2008) handlar lärande om att förändra sin diskurs vilket förutsätter att novisen får chans att delta i expertens diskurs. I flera fall framstår inte matematikdiskursen i läromedlet och i klassrummet som en expertdiskurs, delvis på grund av brister i det svenska språket. Ett par exempel:

- Den svenska diskursen saknar en motsvarighet till det engelska "signed numbers", vilket betyder ungefär "tecken-tal". Matematik-området "signed numbers" heter i den svenska diskursen "negativa tal", vilket får till följd att eleverna pratar om "vanliga tal" kontra "negativa tal" och inte inser att de vanliga talen genom införandet av negativa tal också fått tecken och blivit positiva. Regeln "minus minus blir plus" är för subtraktion av negativa tal helt missvisande då subtraktion av ett negativt tal skrivs om som en addition av det motsatta positiva talet, dvs. två minus blir två plus, även om endast det ena av konvention skrivs ut.
- Åtskillnaden mellan minustecknets olika betydelser, subtraktion och negativitet, blir svår att urskilja om endast ordet minus används, såsom: "att ta minus" eller "att minusa" (eng. to subtract, to take away), "minustal" (eng. negative number)
- Likhetstecknet får en operationell betydelse när det utläses "blir" istället för "är lika med" (eng. is equal to).
- Ordet "tal" har på svenska en mängd olika betydelser (att prata, en räkneuppgift, ett numeriskt tal) och för elever är det ibland oklart vilken betydelse som avses.
- Eleverna uppvisar svårigheter att skilja på de negativa talens två olika storleksaspekter: magnitud och riktning. Införandet av begreppet absolutbelopp i diskursen skulle troligen göra det möjligt att särskilja aspekterna.

Metaforers roll i klassrummet

Under lektionerna om negativa tal förekom *objektsmetaforer* (skulder), *vägmetaforer* (förflyttning på tallinjen, temperaturer som sjunker) och *mätmetaforer* (avstånd på tallinjen). En analys av metaforerna visar att de antar olika skepnad för olika individer och i olika situationer. Trots samma källdomän och måldomän blir avbildningarna olika. Det är också skillnad mellan avbildningarna i den grundläggande metaforen och den utvidgade metaforen. Varje metafor har sina begränsningar för när och hur de är tillämpbara men detta togs aldrig upp explicit under lektionerna. Inte heller jämfördes olika metaforer med varandra.

Metaforer för negativa tal införs i klassrumsdiskursen av läraren/läroboken (LL) genom en *undervisningsprocess för metaforiskt resonerande*. Denna process kan beskrivas i fem steg: först tar LL avstamp i ett matematikinnehåll, här negativa tal. Steg två innebär att LL väljer ut kontexter (modeller) som kan representera de negativa talen och gör en avbildning från matematiken till kontexten. Exempelvis genom att fundera på hur 3 - (-4) ska kunna beskrivas i termer av skulder och tillgångar. I detta steg är matematiken källdomän för LL och kontexten är måldomän. I steg tre involveras eleverna i processen genom att kontexten presenteras för dem. I steg fyra matematiseras (symboliseras) kontexten genom att kontexten avbildas på matematiken. I femte och sista steget möter eleverna matematiskt skrivna uttryck och förväntas kunna resonera om dem metaforiskt, dvs. förstå matematiken i termer av den tidigare presenterade kontexten. Resultaten pekar på att eleverna föredrar en mer inommatematisk diskurs än den metaforiska som erbjuds. En sådan diskurs finns delvis i läroboken men lyfts inte i klassrummet där målet förefaller vara att dels kunna föra metaforiska resonemang, och dels kunna tillämpa teckenregler. Metaforerna presenteras som åtskiljda, utan integrering och relationer sinsemellan. Resultatet blir att eleverna får en bild av att man för olika uppgifter kring negativa tal ska ta till olika sätt att tänka, och ibland också att man ska tänka på ett sätt och skriva matematiskt på ett annat. När och varför olika metaforer ska användas problematiseras inte.

Ett uttryck som är problematiskt för eleverna är uttrycket "beräkna skillnaden mellan två tal". Analysen visar att uttrycket används på olika sätt i olika situationer beroende på de ingående talen och den underliggande metaforen. Det kan vara talens magnitud eller värde som åsyftas, och beräkningen kan vara en addition eller en subtraktion. Särskilt problematiskt i sammanhanget är additioner av två tal med olika tecken ($-a+b$). Den *objektsmetafor* som dominerar klassrumsdiskursen innebär att additionen löses genom att först beräkna skillnaden mellan de två talens magnitud och sedan fundera på tecknet. Eleverna förväntas således tänka subtraktion trots att de skriver addition.

Metaforanalysen visar att teorin om fyra grundläggande metaforer för aritmetik beskriven av Lakoff & Núñez (2000) är användbar för att analysera den matematiska diskursen i ett klassrum, men att teorin behöver förtydligas och byggas ut vad gäller utvidgningen av talområdet. Negativa tal är ett så komplext begrepp att flera metaforer samspelar för att belysa alla aspekter. Metaforanalyserna i föreliggande studie visar att en blandning, och i viss mån en sammansmältning[66], av metaforer kan vara användbar, och att de fyra grundläggande metaforerna för aritmetik bör kompletteras med en *Relationsmetafor*. Det är dock av vikt att framhålla att matematiska objekt både har metaforisk mening och inommatematisk mening såsom delar av en logisk-matematisk struktur.

Taluppfattning

I intervjumaterialet kan elevers sätt att tala om samma uppgifter eller matematiska begrepp följas över tid och visa på förändringar av deras taluppfattning. I analysen av intervjuerna framträder exempel på fyra olika sorters förändringar i begreppsuppfattning:

1. Rekonstruktion av en missuppfattning. Den nya uppfattningen ersätter den gamla.
2. Specificering av villkoren för den rådande uppfattningen och konstruktion av en ny och annorlunda uppfattning i den nya domänen. Båda uppfattningarna samexisterar
3. Uppmärksamhet på nya drag eller kännetecken som tidigare inte varit relevanta
4. Inlemmande av nya idéer som gör begreppsuppfattningen rikare och mer komplex

[66] Engelska: *blend*

Detta resultat är en utveckling av den beskrivning av begreppsförändring som finns inom teorier om *Conceptual Change*, där begreppsförändringen antingen ses som ett berikande (3 och 4) eller en rekonstruktion av en tidigare uppfattning (1) där den gamla uppfattningen måste överges (Merenluoto & Lehtinen, 2004a, 2004b).

Elevernas uppfattning om **tals storlek** förändras över tid. Aspekter av tal som Ball (1993) beskriver att elever i årskurs 3 brottas med när hon inför negativa tal i undervisningen visar sig inte heller vara självklara för alla elever i årskurs 6 i denna studie. Intervjuresultaten visar att eleverna förändrar sin taluppfattning genom att inkorporera dessa aspekter men att förändringarna sker i olika takt för olika elever och att somliga ännu i årskurs 8 uppvisar osäkerhet. De aspekter av tal som framträder är:

~ att begreppet tal utvidgas från att omfatta enbart hela tal till att även innefattar 0, delar (tal mindre än 1), samt tal mindre än 0

~ att tals storlek kan ses som talets relation till referenspunkten 0 och att negativa tal är ordnade i motsatt ordning till de positiva

~ att decimaltal och "halvor" inte är mindre än hela negativa tal (vilket de är om man talar om magnitud men inte om värde), dvs. tal storlek har en dubbel innebörd: magnitud och värde

En *objektsmetafor* visade sig vara användbar för att urskilja negativa tal som tal snarare än som subtraktioner, men metaforen underlättade inte urskiljandet av den dubbla betydelsen av storlek. Inom den metaforen ses 0 som "ingenting" och 2 negativa är fler än 1 positiv. En *vägmetafor* underlättade förmågan att relatera tal till 0 och acceptera 0 som ett tal. Även om metaforer förekom i elevers diskurs var det mycket vanligare att eleverna höll sig till en inommatematisk diskurs där talen gavs mening genom sin relation till andra tal. När metaforer användes blev resultatet ofta att eleven hölls kvar i det snävare talområdet. Resultaten antyder att de grundläggande metaforerna är hjälpsamma för att utveckla taluppfattningen men att de till viss del också har begränsningar, samt att eleverna själva initierar en mer inommatematisk diskurs som skulle kunna utvecklas.

Subtraktion behandlas på ett flertal sätt i intervjuerna och materialet visar på svårigheter att acceptera en negativ differens. Ibland beror det på att subtraktion behandlas som kommutativt vilket skulle kunna hänga samman med en underliggande *objektsmetafor* som gör det svårt att ta bort objekt man inte har. Många av eleverna behandlade subtraktion som kommutativt och/eller associativt under alla intervjuerna, vilket visar att kunskap om subtraktion inte kan tas för given. Få elever använde en *vägmetafor* i dessa sammanhang, inte ens för att beskriva exempelvis 3-7 som blir tydlig i termer av rörelse; att man står på 3 och går 7 steg bakåt. Magnitud och riktning är två aspekter av en negativ differens som kan bearbetas var för sig eller simultant. Vanligt i materialet är att eleverna först bestämmer riktningen (tecknet) och sedan magnituden.

Angående **minustecknets olika betydelser** visar resultaten att vissa elever håller fast vid subtraktionsbetydelsen även efter att negativa tal introducerats, då de betraktar ett negativt tal som ett tal som ska subtraheras. Andra elever anammar de två olika betydelserna men är osäkra på vilken tolkning som är korrekt i enskilda fall. Mycket problem vållar uttryck såsom -6-2 där det första minustecknet av konvention markerar ett negativt tal och det andra en subtraktion, medan båda tecknen är subtraktioner om det står x-6-2. Vissa elever tänker -6-2 som en addition av två negativa tal, exempelvis att en skuld på 6 och ytterligare en skuld på 2 ger en skuld på 8. Olika metaforer ger olika tolkning av minustecknen och därmed olika möjligheter att skapa mening i uttrycken. En elev med väl utvecklad taluppfattning är medveten om minustecknets olika betydelser men är också flexibel i sin tolkning.

Tallinjen förekommer i klassrumsdiskursen men ses som en begränsande visuell bild snarare än som en matematisk struktur. Intervjumaterialet visar att många elever som inte associerar till en vanlig numerisk tallinje när de i årskurs 6 ombeds rita en tallinje. När elever ritar eller talar om en utvidgad tallinje väljer de en som är symmetrisk runt 0 med en pil i varje ända som markerar att "talen fortsätter", trots att alla tallinjer i svenska läroböcker har en pil endast på den positiva sidan som markerar att talen ökar i värde. Eleverna ser alltså 0 som en tydlig referenspunkt i vilken två tallinjer möts, snarare än att tallinjen är en enad linje (jfr Glaeser, 1981). Resultatet visar också att trots att eleverna använder många *vägmetaforiska* uttryck (t.ex. högre upp, under noll, efter noll) så är det sällan de använder en tallinje i sina resonemang. Tal som en punkt på en linje/väg förekommer, speciellt när två tal relateras till varandra, medan tal som en rörelse längst vägen eller en sträcka mellan två punkter är ovanligt i elevernas resonerande.

I analysen av olika elevers individuella **lärandebanor** framkommer att existerande kunskaper och taluppfattning influerar elevens lärandebana. När negativa tal introduceras i undervisningen har elever med en väl utvecklad taluppfattning inga svårigheter att lära sig operera med negativa tal. En väl utvecklad taluppfattning kan innebära att de internaliserat följande drag: existensen av negativa tal (-3 är ett tal), värde av tal till skillnad från kvantitet eller magnitud ($-3 < -2$), möjligheten av en negativ differens (2 - 5 går att beräkna), subtraktion som icke kommutativ ($4 - 5 \neq 5 - 4$) och icke associativ ($31 - 12 - 2 \neq 31 - 10$), samt subtraktion av ett positivt tal från ett negativt tal (-6 - 2). Resultaten visar också att eleverna inte nödvändigtvis förändrar sin beräkningsprocedur först och sitt konceptuella meningsskapandet senare, eller tvärtom, utan följer olika lärandebanor i detta (jfr. Baroody et al., 2007; Hiebert & Lefevre, 1986).

Under intervjuerna uppdagades ett flertal potentiella kognitiva/kommognitiva konflikter i samband med negativa tal, men på grund av de rådande sociomatematiska normerna kom dessa inte fram eller löstes i klassrummet.

Slutdiskussion

Genom sin holistiska och longitudinella ansats belyser avhandlingen komplexiteten i matematiskt tänkande och den intrikata relationen mellan fysiska erfarenheter och abstrakta matematiska begrepp. Resultaten visar hur metaforer som används i samband med negativa tal konstituerar talens innebörd men i många fall också ger upphov till motsägelser och skapar förvirring hos elever. Då varje metafor endast belyser vissa aspekter av begreppet blir slutsatsen att flera olika metaforer, och därmed flera olika representationer behövs (jfr. Chiu, 2001; Linchevski & Williams, 1999). Samma slutsats drar också Ball (1993) men ställer ändå frågan om inte många olika representationer i sig förvirrar eleverna. Den här studien visar att eleverna grundar sin taluppfattning på flera olika metaforer som skulle kunna utnyttjas för att utvidga talområdet. För att det ska ske krävs dock att eleverna blir varse sina olika sätt att tänka om tal, och flexibla i att byta mellan olika metaforer. Det innebär att metaforernas möjligheter och begränsningar är ett viktigt meta-innehåll och att olika metaforer bör lyftas fram och jämförs. En sådan undervisning förutsätter dock en sociomatematisk norm som innebär intresse för och tilltro till elevernas tolkningar och resonemang.

I det studerade klassrummet introducerades metaforiskt resonerande som ett undervisningsmål. Eleverna erbjöds metaforer som skulle användas för att resonera om enskilda uppgifter i syfte att förstå och kunna lösa uppgifterna. Istället skulle metaforiska resonemang kunna fungera som ett *redskap* i undervisningen med syfte att belysa de matematiska samband som omger negativa tal. Genom att göra metaforerna med sina begränsningar explicita skulle fokus komma att förskjutas från hur man ska tänka för att lösa en viss uppgift till olika sätt att tänka för att göra tal och samband meningsfulla. Detta kräver dock ett skifte från en instrumentell till en relationell syn på matematiken (jfr. Skemp, 1976). Därmed väcks frågan om undervisningen syfte. Är undervisningens syfte endast att eleverna ska kunna lösa räkneuppgifter med negativa tal, eller är syftet att engagera eleverna i kreativt matematisk arbete där de kan upptäcka hur det utvidgade talområdet fungerar som en del av ett algebraiskt system, av människor konstruerat och i ständig förändring? I det senare fallet är förmågan att utföra beräkningar med negativa tal ett medel snarare än ett mål i sig.

Många av svårigheterna som elever i huvudstudien uppvisar när de stöter på negativa tal liknar dem som varit historiska stötestenar, och finns med bland de historiska hindren för utveckling av negativa tal som lyfts fram av Glaeser (1981) Framförallt gäller detta synen på tal som kvantiteter, synen på tallinjen som delad istället för enad, fokuseringen på konkreta operationer och önskan om en enhetlig modell för operationer med negativa tal. Enligt Sfard (2007) och i samstämmighet med den historiska utvecklingen, kan inte negativa tal förstås så länge den rådande föreställningen är att matematiska sanningar ska bevisas konkret. Vissa egenskaper hos negativa tal är ett resultat av talens inpassning i en algebraisk struktur och behöver därför bevisas genom matematiska resonemang.

Eleverna i den här studien uppvisar en vilja att använda matematisk terminologi, vilket skulle kunna fångas upp och utvecklas till inommatematiska resonemang. . Bättre kunskaper om den historiska utvecklingen skulle kunna ge lärare förståelse för vilka svårigheter deras elever har och hur svårigheterna skulle kunna övervinnas

Sammanfattningsvis har den föreliggande avhandlingen visat hur rikt och omfattande begreppet *negativa tal* är, vilket borde ge det större utrymme i skolans kursplan än ett kort avsnitt i boken i årskurs 8. Många av de aspekter av negativa tal som studien lyft fram skulle kunna behandlas i undervisningen långt före årskurs 8 för att hjälpa eleverna till en god taluppfattning som grund för att kunna förstå operationer med negativa tal. Vad gäller språkets roll har studien belyst den ibland fattiga matematiska klassrumsdiskursen. En slutsats blir att ett mer precist och tydligt matematiskt språk skulle kunna minska svårigheterna.

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Appendix I: Time line of data collection

In the following chronological description of the data collection process the grades six through nine are labelled G6, G7, G8 and G9 respectively.

Nov-Dec 2006: G6	First visits
Jan-June 2007: G6	Meeting with parents, informed consents collected
	22 introductory interviews á 5 min
	4 lessons observed
	22 audio recorded individual interviews á 30 min
	Test results collected in April
Aug-Dec 2007: G7	1 audio recorded individual interview
	10 lessons observed
Jan-June 2008: G7	11 lessons observed (video recorded lessons included)
	2 lessons video recorded (test recording)
	21 audio recorded individual interviews á 30 min
Aug-Dec 2008: G8	Planning session with teacher
	21 lessons observed (video recorded lessons included)
	13 lessons video recorded
	Test result collected in October and November
	Meeting with parents
	Negative number worksheets and homework collected
	School reports collected
Jan-June 2009: G8	10 lessons observed
	20 audio recorded individual interviews á 45–60 min.
	Test results collected in February
	School reports (grades) collected
	Audio recorded feed back session with teacher
Aug-Dec 2009: G9	4 lessons observed
	1 lesson video recorded (repetition neg. numbers)
	Test results collected in October
	School reports collected.
May-June 2010: G9	National test results collected.
	Respondent validation session with teacher

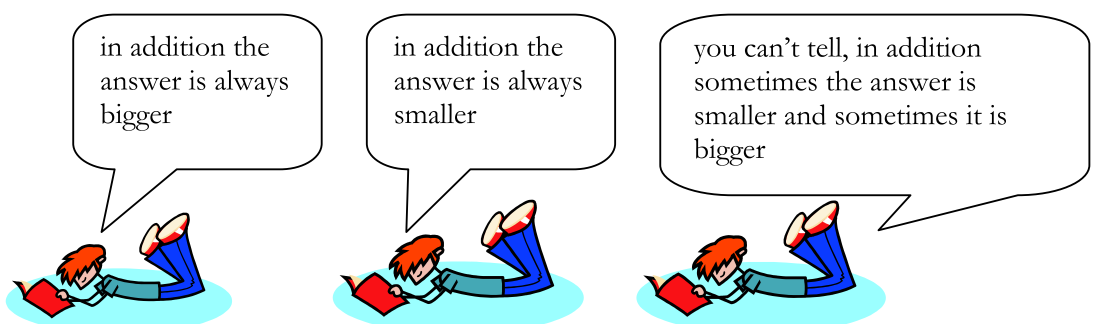
Appendix II: Interview protocol; grade 6, grade 7, grade 8

***Q1*)** Give the student a sheet of paper
Ask the student to write a number, any number
Is it a small or a large number?
Write a smaller number; write an ever smaller number ...
(Keep going until you see where the student is going to end up)
Write the smallest number you can.
Which is the smallest number there is?
Is there no smaller number than that?

***Q2*)** **S**how cards with these equivalences on.
Ask is it true or false? Why do you think so? How do you know?
5 + 4 = 4 + 5
5 – 4 = 4 – 5
10 – 3 + 2 = 10 – 2 + 3
8 + (5 – 2) = 8 + 5 – 2
8 - (5 – 2) = (8 - 5) – 2

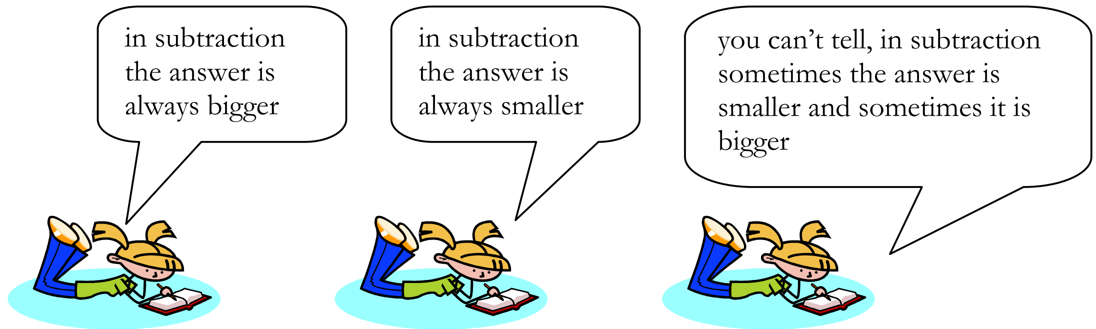
***Q3*)** **(**Addition is discussed after Q2, subtraction just before Q6)
Show the pictures and ask the student who s/he agrees with. Why ?
What do you think the others mean?
If the student is doubtful about the words addition and subtraction, then ask specifically if s/he can explain these words.

<u>Addition</u>



Kim Leo Max

Subtraction



Rayen Sara Tina

Q4) (Q4 is split up so that Q5 is done half way through)
Give the student these strings of operations to solve, one at a time on different papers.
Ask her/him to "think aloud" and to write down whatever s/he needs or wants, as well as the answer.

	grade 6	grade 7	grade 8
8 + 6 – 2 =	x	x	
16 – 4 + 2 =	x	x	x
31 – 12 – 2 =	x	x	x
30 + 12 – 5 + 5 – 12 =	x		
30 – 8 + 5 – 5 + 3 + 8 =	x	x	x
16 – (4 + 2) =	x		
6 – 8 + 3 =		x	x
8 – 12 – 5 + 8 =		x	x

Q5) Show these on flash cards. No follow-up questions.
Do some quick calculations. Only say the answers.

4 + 2 =
6 – 3 =
7 + 11 =
2 – 5 =
12 + 23 =
6 – 27 =

Q6) (included only in grade 6)
Ask the student to make a calculation using an algorithm, and to "think aloud" and say out loud all the different steps of the calculation.
Read the numbers to the student : 253 minus 28

***Q7*)** Use flash cards. Show these strings of symbols one at a time. Ask:
Is it okay to write like this?
What does it mean / could it mean, do you think?
What is the use of it? What does it show?
Could it mean something else? Could it be used for something else?
First symbol is a minus sign: [-] , then go on with:

a) 8 - 2
b) 2-
c) -2 (in grade 8 show b and c simultaneously)
d) (-2) (only in grade 6)
e) + (-2) (only in grade 6)
f) 3 - 7
g) - 6 - 2
h) (-3) - 1 - 2

***Q8*)** Show two cards with [-1] and [2].
Which of the two is the largest/smallest number?
Show a card with [-4]
Can you tell me a number that is smaller than this number?

Show these five numbers one at a time and ask if it is a number:
[20] [-5] [0] [8] [-16] (in grade 8 also [0,02] is included)
When all are on the table,
ask the student to order them according to size.

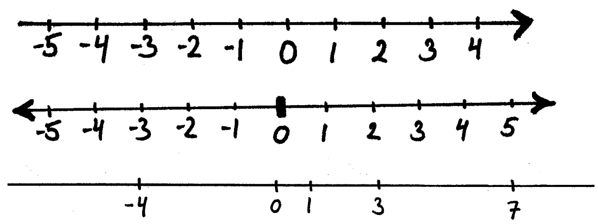
***Q9*)** Ask the student to name a number, any number.
Perhaps the number you had from the start? Then say:
Imagine that I come from another planet
(or that I do not know anything about mathematics)
You say this is a number.
"What is a number?" I ask.
What is a number? What would you tell me?
Can you describe what a number is?

Follow up questions:
What about 0,5; is that a number?
Does your description of what a number is include 0,5?
How about -7, is that a number?
Does your description of what a number is include -7?
How many numbers are there?
If you think of <u>all the numbers</u>, can you picture them? Draw if you like.

In grade 8 this question is not given much attention in the interview,
for some students it is completely excluded.

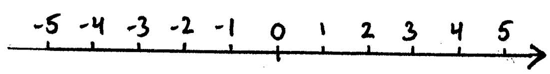
Q10) Do you know what a number line is?
Draw a number line. Tell me about your number line.

<u>Number line pictures and questions grade 6 and grade 7</u>:
Show three different number lines, one at a time.
Here is a number line. Tell me about it, what does it show?
Follow-up questions:
What does the arrow mean?
Do you think it is a good number line?
What could this number line be used for?

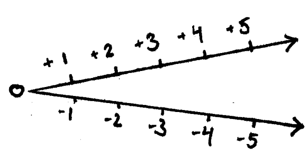


<u>Number line pictures and questions grade 8</u>:
Show four different number lines all at once.
Tell me about these number lines. What do they show?
Do you think they are good number lines?
Have you seen number lines like this?
Which one would you prefer?

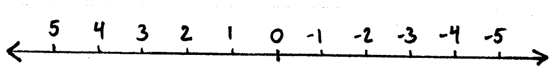
A



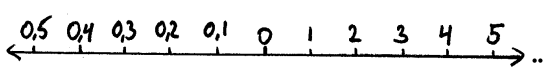
B



C



D



<u>In grade 7 and grade 8 this task is also included</u>:
Could you use a number line to find out the answer to these tasks?
9+2 = and 5-8=
If the student could show 9+2 and 5-8 on the number line
ask also if s/he can use a number line to work out 3 - (-4) and -3-5+2

Q11) (only in grade 8)

a) Put + or – in the squares so that the value becomes as high a possible: - 5 □ - 6 □ +3 □ - 9

b) Compute: (- 3) - (- 8) = ___________
How certain are you that your answer is correct? (choose one answer)
◇ very confident
◇ rather confident
◇ a bit uncertain
◇ very uncertain
There are different ways of thinking to reach an answer to a question like this. Try to describe your way of thinking.

c) Compute (- 2) · (- 3) = ___________
How certain are you that your answer is correct? (choose one answer)
◇ very confident
◇ rather confident
◇ a bit uncertain
◇ very uncertain
There are different ways of thinking to reach an answer to a question like this. Try to describe your way of thinking.

d) Here is a pattern:
3 - 3 = 0
3 - 2 = 1
3 - 1 = 2
3 - 0 = 3
What goes on the next line in the pattern?

***Q12*)** (only in grade 8)
Stimulated recall:
Replay up to five audio and video recordings of episodes where the student is active. Ask the student to comment.
Ask follow up questions about the student's way of answering and reasoning in the episodes.